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Dualizing sup-preserving endomaps of a complete lattice

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It is argued in [5] that the quantale $[L, L]_{\vee}$ of sup-preserving endomaps of a complete lattice L is a Girard quantale exactly when L is completely distributive. We have argued in [16] that this Girard quantale structure arises from the dual quantale of inf-preserving endomaps of L via Raney's transforms and extends to a Girard quantaloid structure on the full subcategory of \mathbf{SLatt} (the category of complete lattices and sup-preserving maps) whose objects are the completely distributive lattices.

It is the goal of this talk to illustrate further this connection between the quantale structure, Raney's transforms, and complete distributivity. Raney's transforms are indeed mix maps in the isomix category \mathbf{SLatt} and most of the theory can be developed relying on naturality of these maps. We complete then the remarks on cyclic elements of $[L, L]_{\vee}$ developed in [16] by investigating its dualizing elements. We argue that if $[L, L]_{\vee}$ has the structure a Frobenius quantale, that is, if it has a dualizing element, not necessarily a cyclic one, then L is once more completely distributive. It follows then from a general statement on involutive residuated lattices that there is a bijection between dualizing elements of $[L, L]_{\vee}$ and automorphisms of L . Finally, we also argue that if L is finite and $[L, L]_{\vee}$ is autodual, then L is distributive.

1 Lattice structure of the homsets in \mathbf{SLatt}

The homset $[X, Y]_{\vee}$ in \mathbf{SLatt} . The category \mathbf{SLatt} of complete lattices and sup-preserving functions is a well-known $*$ -autonomous category [1, 9, 8, 5]. For complete lattices X, Y , we denote by $[X, Y]_{\vee}$ the homset in this category. The two-element Boolean algebra $\mathbf{2}$ is a dualizing element and $[X, \mathbf{2}]_{\vee}$ is, as a lattice, isomorphic to the dual lattice X^{op} . More generally, the functor $(\cdot)^* = [\cdot, \mathbf{2}]_{\vee}$ is naturally isomorphic to the functor $(\cdot)^{op}$ where, for $f : Y \rightarrow X$, $f^{op} : X^{op} \rightarrow Y^{op}$ is the right adjoint of f , noted here by $\rho(f)$ (the left adjoint of an inf-preserving function $g : Y \rightarrow X$ shall be denoted by $\ell(g) : X \rightarrow Y$).

Let us describe the internal structure of the homset $[X, Y]_{\vee}$ as a complete lattice. For $x \in X$ and $y \in Y$, we define the following elements of $[X, Y]_{\vee}$:

$$c_y(t) := \begin{cases} y, & t \neq \perp, \\ \perp, & t = \perp, \end{cases} \quad a_x(t) := \begin{cases} \top, & t \not\leq x, \\ \perp, & t \leq x, \end{cases}$$

$$(y\overline{\otimes}x)(t) := \begin{cases} \top, & t \not\leq x, \\ y, & \perp < t \leq x, \\ \perp, & t = \perp, \end{cases} \quad e_{y,x}(t) := \begin{cases} y, & t \not\leq x, \\ \perp, & t \leq x. \end{cases}$$

Lemma 1.1. *For each $f \in [X, Y]_{\vee}$, $x \in X$, and $y \in Y$, $f(x) \leq y$ if and only if $f \leq y\overline{\otimes}x$. Consequently, for each $f \in [X, Y]_{\vee}$,*

$$f = \bigwedge_{x \in X} \{y\overline{\otimes}x \mid f(x) \leq y\} = \bigwedge_{x \in X} f(x)\overline{\otimes}x,$$

and the sup-preserving functions of the form $y\overline{\otimes}x$ generate $[X, Y]_{\vee}$ under arbitrary meets.

The tensor notation arises from the canonical isomorphism $[X, Y]_{\vee} \simeq (Y^{op} \otimes X)^{op}$ of $*$ -autonomous categories. That is, $[X, Y]_{\vee}$ is dual to the tensor product $Y^{op} \otimes X$ and the functions $y\overline{\otimes}x$ correspond to elementary tensors of $Y^{op} \otimes X$. For $f \in [X, Y]_{\vee}$, $g \in [Y, Z]_{\vee}$, and $h \in [X, Z]_{\vee}$, let us recall that there exists uniquely determined maps $g \setminus h \in [X, Y]_{\vee}$ and $h / f \in [Y, Z]_{\vee}$ satisfying

$$g \circ f \leq h \text{ iff } f \leq g \setminus h \text{ iff } g \leq h / f.$$

The binary operations \setminus and $/$ are known under several names: they yield left and right Kan extensions and (often when $X = Y = Z$) they are named residuals or division operations [6] or right and left implication [14, 5]. With the division operations at hand, let us list some elementary relations between the functions previously defined:

Lemma 1.2. *The following relations hold: (i) $y\overline{\otimes}x = c_y \vee a_x = c_y / c_x = a_y \setminus a_x$, (ii) $e_{y,x} = c_y \wedge a_x = c_y \circ a_x$, (iii) $c_y = y\overline{\otimes}\top = c_y \circ a_{\perp}$, (iv) $a_x = \perp\overline{\otimes}x = c_{\top} \circ a_x$.*

Proof. The relations $y\overline{\otimes}x = c_y \vee a_x$ and $e_{y,x} = c_y \wedge a_x$ are well known, see e.g. [19], and (iii) and (iv) are immediate consequences of these relations.

Let us focus on $y\overline{\otimes}x = c_y / c_x = a_y \setminus a_x$ and notice that, in order to make sense of these relations, we need to assume $c_x : X' \rightarrow X$, $c_y : X' \rightarrow Y$, $a_x : X \rightarrow Y'$, and $a_y : Y \rightarrow Y'$. Observe that $f(x) \leq y$ if and only if $f \circ c_x \leq c_y$ and therefore, in view of Lemma 1.1 and of the definition of $/$, the relation $y\overline{\otimes}x = c_y / c_x$. Next, the condition $f \leq a_y \setminus a_x$ amounts to $a_y \circ f \leq a_x$, that is, for all $t \in X$, if $t \leq x$ then $f(t) \leq y$. Clearly this condition is equivalent to $f(x) \leq y$, and therefore to $f \leq y\overline{\otimes}x$. Finally, the relation $e_{y,x} = c_y \circ a_x$ is directly verified. However, let us observe the abuse of notation, since for the maps a_x and c_y to be composable we need to assume either $X = Y$ or $a_x : X \rightarrow \mathbf{2}$ and $c_y : \mathbf{2} \rightarrow Y$. \square

Inf-preserving functions as tensor product. Let $[X, Y]_{\wedge}$ denote the poset of inf-preserving functions from X to Y , with the pointwise ordering. Observe that, as a set, $[X, Y]_{\wedge}$ equals $[X^{op}, Y^{op}]_{\vee}$. Yet, as a poset or a lattice, the equality $[X, Y]_{\wedge} = [X^{op}, Y^{op}]_{\vee}^{op}$ is the correct one. As a matter of fact, we have $f \leq_{[X, Y]_{\wedge}} g$ iff $f(x) \leq_Y g(x)$, all $x \in X$, iff $g(x) \leq_{Y^{op}} f(x)$, all $x \in X$, iff $g \leq_{[X^{op}, Y^{op}]_{\vee}} f$. Using standard isomorphisms of $*$ -autonomous categories, we have

$$[X, Y]_{\wedge} = [X^{op}, Y^{op}]_{\vee}^{op} \simeq Y \otimes X^{op}.$$

That is, the set of inf-preserving functions from X to Y can be taken as a concrete realization of the tensor product $Y \otimes X^{op}$. This should not come as a surprise, since it is well-known that the set of Galois connections from X to Y —that is, pairs of functions $(f : X \rightarrow Y, g : Y \rightarrow X)$ such that $y \leq f(x)$ iff $x \leq g(y)$ —realizes the tensor product $Y \otimes X$ in SLatt , see e.g. [18, 12] or [5, §2.1.2]. Such a pair of functions is uniquely determined by its first element, which is an inf-preserving functions from X^{op} to Y .

Notice now that

$$[X, Y]_{\vee}^{op} \simeq Y^{op} \otimes X \simeq X \otimes Y^{op} \simeq [Y, X]_{\wedge}, \quad (1)$$

from which we derive the following principle:

Fact 1.3. *There is a bijection between sup-preserving functions from $[X, Y]_{\vee}^{op}$ to $[X, Y]_{\vee}$ and sup-preserving functions from $[Y, X]_{\wedge}$ to $[X, Y]_{\vee}$.*

The map yielding the isomorphism in equation (1) is ρ , the operation of taking the right adjoint. The bijection stated in Fact 1.3 is therefore obtained by precomposing with ρ .

We exploit now the work done for $[X, Y]_{\vee}$ to recap the structure of $[X, Y]_{\wedge}$ as a tensor product. Consider the maps

$$\gamma_y(t) := \begin{cases} \top, & t = \top, \\ y, & \text{otherwise,} \end{cases} \quad \alpha_x(t) := \begin{cases} \top, & x \leq t, \\ \perp, & \text{otherwise,} \end{cases}$$

$$y \otimes x(t) = \begin{cases} \top, & t = \top, \\ y, & x \leq t, \\ \perp, & \text{otherwise.} \end{cases}$$

By dualizing Lemma 1.1, we observe that the relation $y \otimes x = \gamma_y \wedge \alpha_x$ holds, the maps $y \otimes x$ realize the elementary tensors of the (abstract) tensor product $Y \otimes X^{op}$, $[X, Y]_{\wedge}$ is join-generated by these maps, and every $g \in [X, Y]_{\wedge}$ can be canonically written as $g = \bigvee_{x \in X} g(x) \otimes x$.

Recall that a bimorphism $\psi : Y \times X^{op} \rightarrow Z$ is a function that is sup-preserving in each variable, separately. This in particular means that infs in X are transformed into sups in Z . The universal property of $[X, Y]_{\wedge}$ as a tensor product can be therefore stated as follows:

Fact 1.4. *Given a bimorphism $\psi : Y \times X^{op} \rightarrow Z$, there exists a unique sup-preserving functions $\tilde{\psi} : [X, Y]_{\wedge} \rightarrow Z$ such that $\tilde{\psi}(y \otimes x) = \psi(y, x)$. For $g \in [X, Y]_{\wedge}$, $\tilde{\psi}(g)$ is defined by*

$$\tilde{\psi}(g) := \bigvee_{x \in L} \psi(g(x), x).$$

2 Raney's transforms

For $g \in [X, Y]_{\wedge}$ and $f \in [X, Y]_{\vee}$, define

$$g^{\vee}(x) := \bigvee_{x \not\leq t} g(t), \quad f^{\wedge}(x) := \bigwedge_{t \not\leq x} f(t).$$

It is easily seen that g^{\vee} has a right adjoint, so $g^{\vee} \in [X, Y]_{\vee}$, and that f^{\wedge} has a left adjoint, so f^{\wedge} belongs to $[X, Y]_{\wedge}$. We call the operations $(\cdot)^{\vee}$ and $(\cdot)^{\wedge}$ the Raney's transforms, even if Raney defined these transforms on Galois connections. (In [16] we explicitly related these maps to Raney's original way of defining them). Notice that $g^{\vee} \leq f$ if and only if $g \leq f^{\wedge}$, so $(\cdot)^{\wedge}$ is right adjoint to $(\cdot)^{\vee}$. Raney's transforms have been the key ingredient allowing us to prove in [17] that $[C, C]_{\vee}$ is a Girard quantale if C is a complete chain and, lately in [16], that the full-subcategory of SLatt whose objects are the completely distributive lattices is a Girard quantaloid.

Consider the bimorphism $e : Y \times X^{op} \rightarrow [X, Y]_{\vee}$ sending y, x to $e_{y,x} = c_y \circ a_x \in [X, Y]_{\vee}$ and its extension

$$\tilde{e}(f) = \bigvee \{ c_{f(t)} \circ a_t \mid t \in X \}.$$

By evaluating $\tilde{e}(f)$ at $x \in L$, we obtain

$$\tilde{e}(f)(x) = \bigvee_{x \not\leq t} \{ (c_{f(t)} \circ a_t)(x) \mid t \in L \} = \bigvee_{x \not\leq t} f(t) = f^{\vee}(x),$$

that is, $\bar{e}(f) = f^\vee$. Remark now that $e : Y \times X^{op} \rightarrow [X, Y]_\vee$ is the (set-theoretic) transpose of the trimorphism

$$\langle y, x, t \rangle = \begin{cases} \perp, & t \leq x \\ y, & \text{otherwise.} \end{cases}$$

Consequently, Raney's transform $(\cdot)^\vee$ is the transpose of the map

$$Y \otimes X^{op} \otimes X \xrightarrow{\cong} Y \otimes [X, \mathbf{2}]_\vee \otimes X \xrightarrow{Y \otimes eval} Y \otimes \mathbf{2} \xrightarrow{\cong} Y. \quad (2)$$

In the category \mathbf{SLatt} , $\mathbf{2}$ is both the unit for the tensor product $Y \otimes X$ and its dual $(Y^{op} \otimes X^{op})^{op} \simeq [X, Y^{op}]_\vee$. $*$ -autonomous categories with this property are examples of isomix categories in sense of [3, 4, 2] where the transpose of the map in (2) is named *mix*. That Raney's transforms are mix maps was recognized in [8] where also the nuclear objects—i.e. those objects whose mix maps are invertible—in the category \mathbf{SLatt} were characterized (using Raney's Theorem) as the completely distributive lattices. The importance of this characterization stems from the fact that the nucleus of a symmetric monoidal closed category—that is, the full subcategory of nuclear objects—yields a right adjoint to the forgetful functor from the category of compact closed categories to that of symmetric monoidal closed ones, as mentioned in [15]. In particular, the full subcategory of \mathbf{SLatt} whose objects are the completely distributive lattices is more than symmetric monoidal closed or $*$ -autonomous, it is compact closed [10].

A key property of Raney's transforms, importantly used in [17, 16], is the following. For $g \in [X, Y]_\wedge$ and $f \in [X, Y]_\vee$, the relations

$$\rho(g^\vee) = \ell(g)^\wedge, \quad \ell(f^\wedge) = \rho(f)^\vee, \quad (3)$$

hold. This property might be directly verified, as we did in [17, 16]. It might also be inferred from the commutativity of each square in the diagram below:

$$\begin{array}{ccc} [X, Y]_\wedge & \xrightarrow{(\cdot)^\vee_{X,Y}} & [X, Y]_\vee \\ \downarrow \ell & \searrow & \downarrow \rho \\ & Y \otimes X^{op} & \nearrow \text{mix}_{Y,X} \\ & \downarrow \sigma & \\ & X^{op} \otimes Y & \searrow \text{mix}_{X^{op}, Y^{op}} \\ [Y^{op}, X^{op}]_\wedge & \xrightarrow{(\cdot)^\vee_{Y^{op}, X^{op}}} & [Y^{op}, X^{op}]_\vee \\ \parallel & & \parallel \\ [Y, X]_\vee^{op} & \xrightarrow{(\cdot)^\wedge_{Y,X}} & [Y, X]_\wedge^{op} \end{array}$$

Naturality of Raney's transforms. Observe now that, for $g : X' \rightarrow X$ and $f : Y \rightarrow Y'$, we have

$$f \circ c_y = c_{f(y)}, \quad a_x \circ g = a_{\rho(g)(x)}, \quad \text{and} \quad f \circ e_{y,x} \circ g = e_{f(y), \rho(g)(x)},$$

implying that the following diagram commutes:

$$\begin{array}{ccc}
Y \otimes X^{op} & \xrightarrow{(\cdot)_{Y,X}^\vee} & [X, Y]_\vee \\
\downarrow f \otimes g^{op} & & \downarrow [g, f]_\vee \\
Y' \otimes X'^{op} & \xrightarrow{(\cdot)_{Y',X'}^\vee} & [X', Y']_\vee
\end{array}$$

That is, Raney's transform $(\cdot)^\vee$ is natural in both its variables. Let us remark on the way the following:

Proposition 2.1. *There are exactly two natural arrows from $Y \otimes X^{op}$ to $[X, Y]_\vee$, the trivial one and Raney's transform.*

In order to simplify reading, we use ψ both for a bimorphism $\psi : Y \times X^{op} \rightarrow Z$ and for its extension to the tensor product $\tilde{\psi} : Y \otimes X^{op} \rightarrow Z$.

Proof. If ψ is natural, then

$$\psi(y, x) = \psi(c_y(\top), \gamma_x(\perp)) = c_y \circ \psi(\top, \perp) \circ a_x,$$

since $\rho(a_x) = \gamma_x$. Let $f = \psi(\top, \perp)$. If $f = \perp$, then ψ is the trivial map. Otherwise, $f \neq c_\perp$ and $f(\top) \neq \perp$. Then, observing that $f \circ a_x = c_{f(\top)} \circ a_x$ and that $c_y \circ c_z = c_y$ for $z \neq \perp$, it follows that

$$\psi(y, x) = c_y \circ f \circ a_x = c_y \circ c_{f(\top)} \circ a_x = c_y \circ a_x. \quad \square$$

Remark 2.2. Similar considerations can be developed if naturality is required in just one variable. For example, if the bimorphism $\psi : Y \times X^{op} \rightarrow [X, Y]_\vee$ is such that $\psi(y, x) \circ g = \psi(y, \rho(g)(x))$, then $\psi(y, x) = \chi(y) \circ a_x$ for some $\chi : Y \rightarrow [X, Y]_\vee$. \diamond

For $f : X \rightarrow Y$ in the category SLatt , let $j = \rho(f) \circ f$ and $o = f \circ \rho(f)$. Denote by X_j (resp. Y_o) the set of fixed points of j (resp., of o). Then, we have a standard (epi, iso, mono)-factorization

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow j & & \uparrow \\
X_j & \xrightarrow{\cong} & Y_o
\end{array}$$

Thus, f is mono if and only if $j = id_X$ and f is epic if and only if $o = id_Y$. Notice that Y_o is the image of X under f , while X_j is the image of Y under $\rho(f)$. We apply this factorization to Raney's transforms.

Definition 2.3. A sup-preserving function $f : X \rightarrow Y$ is *tight* if $f^{\wedge\vee} = f$, or, equivalently, if it belongs to the image of $[X, Y]_\wedge$ via the Raney's transform $(\cdot)^\vee$. We let $[X, Y]_\vee^t$ be the set of tight functions from X to Y .

By its definition, $[X, Y]_\vee^t$ is the sub-join-semilattice of $[X, Y]_\vee$ generated by the $c_y \circ a_x$. Moreover, it is easily seen that $w \overline{\otimes} z \in [X, Y]_\vee^t$, for each $w \in Y$ and $z \in X$, and that $c_y \circ a_x \leq w \overline{\otimes} z$ if and only if $y \leq w$ or $z \leq x$. From these relations, $[X, Y]_\vee^t$ yields a concrete representation of Wille's tensor product $Y \widehat{\otimes} X^{op}$, see [19], which, for finite lattices, coincides with the Box tensor product of [7].

Next, we list some immediate consequences of naturality of Raney's transforms:

Proposition 2.4. *The following statements hold:*

- (i) $[X, Y]_\vee^t$ is a bi-ideal of $[X, Y]_\vee$.
- (ii) For a complete lattice L , the transform $(\cdot)^\vee : [L, L]_\wedge \rightarrow [L, L]_\vee$ is surjective if and only if $id_L \in [L, L]_\vee^t$, that is, if $id = id^{\wedge\vee}$.

(iii) For each complete lattice L , the pair $([L, L]_{\vee}^{\natural}, \circ)$ is a quantale.

Let us recall that Raney's Theorem [13] characterizes completely distributive lattices as those complete lattices satisfying the identity

$$z = \bigvee_{z \not\leq x} \bigwedge_{y \not\leq x} y.$$

This identity is exactly the identity $id = id^{\wedge \vee}$ or, as we have seen in Proposition 2.4, the identity $f = f^{\wedge \vee}$ holding for each $f \in [L, L]_{\vee}$. Since complete distributivity is autodual (at least in a classical context), we derive that Raney's transform $(\cdot)^{\vee} : [L, L]_{\wedge} \rightarrow [L, L]_{\vee}$ is surjective if and only if it is injective.

We conclude this section with a glance at the quantale $([L, L]_{\vee}^{\natural}, \circ)$ of tight maps, where L is an arbitrary complete lattice L . We pause before for a technical lemma needed end the section and later on as well. Recalling the equations in (3), let us define

$$f^* := \ell(f^{\wedge}) \quad (= \rho(f)^{\vee}),$$

and observe the following:

Lemma 2.5. For each $x \in X$, $y \in Y$, and $f \in [X, Y]_{\vee}$, the following conditions are equivalent: (i) for all $t \in X$, $x \leq t$ or $y \leq f(t)$, (ii) $c_y \circ a_x \leq f$ (iii) $y \otimes x \leq f^{\wedge}$ (iv) $y \leq f^{\wedge}(x)$ (v) $f^*(y) \leq x$ (vi) $f^* \leq x \overline{\otimes} x$.

Proof. (i) \Leftrightarrow (ii) : direct verification. (ii) \Leftrightarrow (iii) : since $c_y \circ a_x = (y \otimes x)^{\vee}$, $c_y \circ a_x \leq f^{\wedge}$ and by the adjunction $(\cdot)^{\vee} \dashv (\cdot)^{\wedge}$. (iii) \Leftrightarrow (iv) : by the dual of Lemma 1.1. (iv) \Leftrightarrow (v) : since $f^* \dashv f^{\wedge}$. (v) \Leftrightarrow (vi) : by Lemma 1.1. \square

Proposition 2.6. Unless L is a completely distributive lattice (in which case $[L, L]_{\vee}^{\natural} = [L, L]$), the quantale $([L, L]_{\vee}^{\natural}, \circ)$ is not unital.

Proof. Let u be unit for $([L, L]_{\vee}^{\natural}, \circ)$ and write $u = \bigvee_{i \in I} c_{y_i} \circ a_{x_i}$. For an arbitrary $x \in L$, evaluate at x the identity

$$a_x = a_x \circ u = \bigvee a_x \circ c_{y_i} \circ a_{x_i}$$

and deduce that, for each $i \in I$, $\perp = a_x \circ c_{y_i} \circ a_{x_i}(x)$. This happens exactly when $x \leq x_i$ or $y_i \leq x$, that is, when $c_{y_i} \circ a_{x_i} \leq x \overline{\otimes} x$. Since $x \in L$ and $i \in I$ are arbitrary, we have, within $[L, L]$, $u = \bigvee_{i \in I} c_{y_i} \circ a_{x_i} \leq \bigwedge_{x \in L} x \overline{\otimes} x = id_L$. Again, for $y \in L$ arbitrary, evaluate at \top the identity

$$c_y = u \circ c_y = \bigvee c_{y_i} \circ a_{x_i} \circ c_y.$$

and deduce that $y = \bigvee_{y \not\leq x_i} y_i$. Considering that $c_{y_i} \circ a_{x_i} \leq id_L$, then we have $y_i \leq id^{\wedge}(x_i)$ and therefore

$$y = \bigvee_{y \not\leq x_i} y_i \leq \bigvee_{y \not\leq x_i} id^{\wedge}(x_i) \leq \bigvee_{y \not\leq t} id^{\wedge}(t) = id^{\wedge \vee}(y).$$

Since this holds for any $y \in L$, $id \leq id^{\wedge \vee}$ and since the opposite inclusion always holds, then $id = id^{\wedge \vee}$. By Raney's Theorem, L is a completely distributive lattice. \square

Recall that a *dualizing element* in a quantale (Q, \circ) is an element $0 \in Q$ such that $0/(x \setminus 0) = (0/x) \setminus 0 = x$, for each $x \in Q$. As consequences of Proposition 2.6, we obtain:

Corollary 2.7. *Unless L is a completely distributive lattice,*

- (i) *the quantale $([L, L]_{\vee}^{\natural}, \circ)$ has no dualizing element,*
- (ii) *the interior operator $(\cdot)^{\wedge\vee}$ obtained by composing the two Raney's transform is not a conucleus on $[L, L]_{\vee}$.*

Proof. (i) If 0 is dualizing, then $0 \setminus 0$ is a unit of the quantale. (ii) We argue that the inclusion $(g \circ f)^{\wedge\vee} \leq g^{\wedge\vee} \circ f^{\wedge\vee}$, required for $(\cdot)^{\wedge\vee}$ to be a conucleus on $[L, L]_{\vee}$, does not hold, unless L is completely distributive. The opposite inclusion $g^{\wedge\vee} \circ f^{\wedge\vee} \leq (g \circ f)^{\wedge\vee}$ holds since $g^{\wedge\vee} \circ f^{\wedge\vee}$ belongs to $[L, L]_{\vee}^{\natural}$, $g^{\wedge\vee} \circ f^{\wedge\vee} \leq g \circ f$, and $(g \circ f)^{\wedge\vee}$ is the greatest element of $[L, L]_{\vee}^{\natural}$ below $g \circ f$. If $(g \circ f)^{\wedge\vee} = g^{\wedge\vee} \circ f^{\wedge\vee}$ for each $f, g \in [L, L]_{\vee}$, then $1^{\wedge\vee}$ is a unit for $[L, L]_{\vee}^{\natural}$ and L is completely distributive. \square

3 Dualizing elements of $[L, L]_{\vee}$

We investigate in this section dualizing elements of $[L, L]_{\vee}$. Proposition 2.6.18 in [5] states that if id_L^{\vee} is dualizing, then L is completely distributive. Recall that a *cyclic element* in a quantale (Q, \circ) is an element $0 \in Q$ such that $0/x = x \setminus 0$, for each $x \in Q$. Trivially, the top element of a quantale is cyclic. Our work [16] proves that if $[L, L]_{\vee}$ has a non-trivial cyclic element, then this element is id_L^{\vee} and, once more, cyclicity of id_L^{\vee} implies that L is completely distributive. It was still open the possibility that $[L, L]_{\vee}$ might have dualizing elements and no non-trivial cyclic elements. This is possible in principle, since the tool Mace4 [11] provided us with an example of a quantale where the unique dualizing element is not cyclic. The quantale is built on the modular lattice M_5 (with atoms u, d, a, b, c) and has the following multiplication table:

	\perp	u	d	a	b	c	\top
\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp
u	\perp	u	d	a	b	c	\top
d	\perp	d	\top	\top	\top	\top	\top
a	\perp	a	\top	\top	\top	d	\top
b	\perp	b	\top	d	\top	\top	\top
c	\perp	c	\top	\top	d	\top	\top
\top	\perp	\top	\top	\top	\top	\top	\top

It is verified that d is the only non-cyclic element and that, at the same time, it is the only dualizing element. For the quantale $[L, L]_{\vee}$ we shall see that existence of a dualizing element again implies complete distributivity of L (and therefore existence of a cyclic and dualizing element).

As this might be of more general interest, we are going to investigate how divisions $\cdot \setminus f$ and f / \cdot by an arbitrary $f \in [L, L]_{\vee}$ act on the $e_{y,x}$ and $y \overline{\otimes} x$. To this end, we start remarking that Raney's transforms intervene in the formulas for computing left adjoints of the maps c and a .

Lemma 3.1. *The functions $c : Y \rightarrow [X, Y]_{\vee}$ and $a : X^{op} \rightarrow [X, Y]_{\vee}$ have both a left and a right adjoint. Namely, for each $x \in X$, $y \in Y$, and $f \in [X, Y]_{\vee}$, the following relations hold:*

$$\begin{aligned}
 c_y \leq f & \text{ iff } y \leq f^{\wedge}(\perp), & f \leq c_y & \text{ iff } f(\top) \leq y, \\
 a_x \leq f & \text{ iff } f^*(\top) \leq x, & f \leq a_x & \text{ iff } x \leq \rho(f)(\perp).
 \end{aligned}$$

Using the relations stated in Lemma 3.1 computing divisions becomes an easy task.

Lemma 3.2. For each $x \in X$, $y \in Y$, and $f, g \in [X, Y]_{\vee}$, we have

$$\begin{aligned} f/a_x &= f^\wedge(x) \overline{\otimes} \top & (= c_{f^\wedge(x)}), & & c_y \setminus f &= \perp \overline{\otimes} f^*(y) & (= a_{f^*(y)}), \\ f/c_y &= f^\wedge(\perp) \overline{\otimes} y, & & & a_x \setminus f &= x \overline{\otimes} f^*(\top). \end{aligned}$$

Proof. We compute as follows:

$$g \leq c_y \setminus f \text{ iff } c_y \circ g \leq f \text{ iff } c_y \circ a_\perp \circ g \leq f \text{ iff } c_y \circ a_{\rho(g)(\perp)} \leq f \text{ iff } f^*(y) \leq \rho(g)(\perp) \text{ iff } g \leq a_{f^*(y)}.$$

where we have used Lemma 2.5. Verification that $f/a_x = c_{f^\wedge(x)}$ is similar. The other two identities are verified as follows:

$$h \leq f/c_y \text{ iff } h \circ c_y \leq f \text{ iff } c_{h(y)} \leq f \text{ iff } h(y) \leq f^\wedge(\perp) \text{ iff } h \leq f^\wedge(\perp) \overline{\otimes} y,$$

and, dually,

$$h \leq a_x \setminus f \text{ iff } a_x \circ h \leq f \text{ iff } a_{\rho(h)(x)} \leq f \text{ iff } f^*(\top) \leq \rho(h)(x) \text{ iff } h(f^*(\top)) \leq x \text{ iff } h \leq x \overline{\otimes} f^*(\top). \quad \square$$

The relations in the following proposition are then easily derived.

Proposition 3.3. The following relations hold:

$$\begin{aligned} f/(c_y \circ a_x) &= f^\wedge(x) \overline{\otimes} y, & (c_y \circ a_x) \setminus f &= x \overline{\otimes} f^*(y), \\ f/(y \overline{\otimes} x) &= f^\wedge(x) \overline{\otimes} \top \wedge f^\wedge(\perp) \overline{\otimes} y, & (y \overline{\otimes} x) \setminus f &= \perp \overline{\otimes} f^*(y) \wedge x \overline{\otimes} f^*(\top). \end{aligned}$$

From Proposition 3.3, it follows that

$$c_{f^\wedge(x)} \circ a_y \leq f/(y \overline{\otimes} x), \quad c_x \circ a_{f^*(y)} \leq (y \overline{\otimes} x) \setminus f.$$

If f^* is invertible, then its inverse is also its right adjoint. From the uniqueness of the right adjoint it follows that f^* is inverted by its right adjoint f^\wedge . Thus, f^* is invertible if and only if f^\wedge is invertible, in which case we have $f^*(\top) = \top$ and $f^\wedge(\perp) = \perp$, since both f^* and f^\wedge are bicontinuous. In some important case, the expressions exhibited in Proposition 3.3 simplify:

Corollary 3.4. If $f^\wedge(\perp) = \perp$ (resp., $f^*(\top) = \top$), then

$$f/(y \overline{\otimes} x) = c_{f^\wedge(x)} \circ a_y \quad (\text{resp., } (y \overline{\otimes} x) \setminus f = c_x \circ a_{f^*(y)}).$$

These relations hold as soon as either f^\wedge or f^* is invertible.

Example 3.5. If L is a completely distributive lattice, then the relation $id_L^{\wedge \vee} = id_L$ holds, by Raney's Theorem [13]. Let $o = id_L^{\vee}$, then o^\wedge is invertible, since it is the identity. Necessarily, we also have $o^* = id_L$ and therefore:

$$o/(c_y \circ a_x) = (c_y \circ a_x) \setminus o = x \overline{\otimes} y, \quad o/(y \overline{\otimes} x) = (y \overline{\otimes} x) \setminus o = c_x \circ a_y. \quad \diamond$$

Theorem 3.6. If $f \in [L, L]_{\vee}$ is dualizing, then f^\wedge and f^* are inverse to each other and L is completely distributive.

Proof. If f is dualizing, then, for all $x, y \in L$,

$$c_y = f/(c_y \setminus f) = f/a_{f^*(y)} = c_{f^\wedge(f^*(y))}, \quad a_x = (f/a_x) \setminus f = c_{f^\wedge(x)} \setminus f = a_{f^*(f^\wedge(x))},$$

and since both c and a are injective, then $y = f^\wedge(f^*(y))$ and $x = f^*(f^\wedge(x))$. Thus, f^\wedge and f^* are inverse to each other. Remark now that $f^* \in [L, L]_{\vee}^{\natural}$, since $f^* = \ell(f^\wedge) = \rho(f)^\vee$. Then $id_L = f^\wedge \circ f^* \in [L, L]_{\vee}^{\natural}$, since $[L, L]_{\vee}^{\natural}$ is an ideal of $[L, L]_{\vee}$. Then L is completely distributive by Raney's Theorem. \square

It is not difficult to give a direct proof of the converse, namely that if f^* is invertible, then f is dualizing. We prove this as a general statement about involutive residuated lattices. Notice that the map sending f to f^* is definable in the language of involutive residuated lattices, since $f^* = f \setminus o$, where $o = id_L^\vee$ is the canonical cyclic dualizing element of $[L, L]_{\vee}$ (if L is completely distributive). The statement in the following Proposition 3.7 is implicit in the definition of a (symmetric) compact closed category in [10] (see [20, §5] for the non symmetric version of this notion).

Proposition 3.7. *In every involutive residuated lattice Q , f is dualizing if and only if f^* is invertible.*

Proof. If $f \in Q$ is dualizing, then $f/(x^* \setminus f) = (f/x^*) \setminus f = x^*$, for each $x \in Q$. Using well known identities of involutive residuated lattices, compute as follows:

$$x = x^{**} = (f/(x^* \setminus f))^* = (x^* \setminus f) \circ f^* = (x/f^*) \circ f^*.$$

Letting $x = 1$, then $1 = (1/f^*) \circ f^*$. We derive $1 = f^* \circ (f^* \setminus 1)$ similarly, from which it follows that f^* is inverted by $1/f^* = f^* \setminus 1$.

Conversely, suppose that f^* is invertible, say $f^* \circ g = g \circ f^* = 1$. It immediately follows that $g = 1/f^* = f^* \setminus 1$. Again, for each $x \in Q$,

$$x^* = x^* \circ (1/f^*) \circ f^* \leq (x^*/f^*) \circ f^*,$$

and then, dualizing this relation, we obtain

$$x = x^{**} \geq ((x^*/f^*) \circ f^*)^* = f/(x \setminus f).$$

Since $x \leq f/(x \setminus f)$ always hold, we have $x = f/(x \setminus f)$. The identity $x = (f/x) \setminus f$ is derived similarly. \square

Example 3.8. If $L = [0, 1]$, then f is dualizing if and only if it is invertible. Indeed, f is dualizing iff f^* is invertible iff f^\wedge is invertible. Now, if f^\wedge is invertible, then it is continuous and $f^\wedge = f$; therefore f is invertible. Similarly, if f is invertible, then it is continuous and $f^\wedge = f$; therefore f^\wedge is invertible. \diamond

Example 3.9. Consider a poset P , the complete lattice $\mathcal{D}(P)$ of downsets of P , and recall that $\mathcal{D}(P)$ is completely distributive. The quantale $[\mathcal{D}(P), \mathcal{D}(P)]_{\vee}$ is isomorphic to the quantale of weakening relations (profunctors/bimodules) on the poset P . These are the relations $R \subseteq P \times P$ such that $yRx, y' \leq y$, and $x \leq x'$ imply $x'Ry'$ (for all $x, x', y, y' \in P$). Thus, weakening relations are downsets of $P \times P^{op}$ and the bijection between $[\mathcal{D}(P), \mathcal{D}(P)]_{\vee}$ and $\mathcal{D}(P \times P^{op})$ goes along the lines described in previous sections, since

$$[\mathcal{D}(P), \mathcal{D}(P)]_{\vee} \simeq \mathcal{D}(P) \otimes \mathcal{D}(P^{op}) \simeq \mathcal{D}(P \times P^{op}).$$

Explicitly, this bijection, sending f to R_f , is such that

$$(y, x) \in R_f \text{ iff } y \in f(\downarrow x) \text{ iff } c_{\downarrow y} \circ a_{\downarrow x} \leq f, \quad (4)$$

where, for $x \in P$, $\downarrow x := \{y \in P \mid y \leq x\}$. We have seen that dualizing elements of $[\mathcal{D}(P), \mathcal{D}(P)]_{\vee}$ are in bijection with automorphisms of $\mathcal{D}(P)$ which in turn are in bijection with automorphisms of P . Given such an automorphism, we aim at computing the dualizing weakening relation corresponding to this automorphism. To this end, let us recall that, when L is completely distributive, $o = id^{\vee} = \ell(id^{\wedge})$ is the unique non-trivial cyclic element. Observe that since $o = id^*$, o is also dualizing and that

$$(y, x) \in R_o \text{ iff } x \not\leq y.$$

For $f \in [\mathcal{D}(P), \mathcal{D}(P)]_{\vee}$, recall that $f^* = f \setminus o$. We use the relations in (4) to compute the dualizing element of $\mathcal{D}(P \times P^{op})$ corresponding to an invertible order preserving map $g : P \rightarrow P$, as follows:

$$\begin{aligned} (y, x) \in R_{\mathcal{D}(g)^*} &\text{ iff } c_{\downarrow y} \circ a_{\downarrow x} \leq \mathcal{D}(g)^* = \mathcal{D}(g) \setminus o \\ &\text{ iff } \mathcal{D}(g) \circ c_{\downarrow y} \circ a_{\downarrow x} = c_{\mathcal{D}(g)(\downarrow y)} \circ a_{\downarrow x} = c_{\downarrow g(y)} \circ a_{\downarrow x} \leq 0 \\ &\text{ iff } x \not\leq g(y). \end{aligned} \quad \diamond$$

4 Further bimorphisms, bijections, directions

Even when Raney's transforms are not inverse to each other, it might still be asked whether there are other isomorphisms between $[L, L]_{\wedge}$ and $[L, L]_{\vee}$. By Fact 1.3, this question amounts to understand whether $[L, L]_{\vee}$ is autodual.

Let us discuss the case when L is a finite lattice. We use $J(L)$ for the set of join-irreducible elements of L and $M(L)$ for the set of meet-irreducible elements of L . The reader will have no difficulties convincing himself of the following statement:

Lemma 4.1. *A map $f \in [L, L]_{\vee}$ is meet-irreducible if and only if it is an elementary tensor of the form $m \otimes j$ with $m \in M(L)$ and $j \in J(L)$.*

The following statement might instead be less immediate:

Lemma 4.2. *For each $j \in J(L)$ and $m \in M(L)$, the map $e_{j,m}$ is join-irreducible.*

Proof. Let $m \in M(L)$ and $j \in J(L)$, and let us use m^* to denote the unique upper cover of m . Suppose that $e_{j,m} = \bigvee_{i \in I} f_i$. By evaluating the two sides of this equality at m^* , we obtain $j = \bigvee_{i \in I} f_i(m^*)$ and therefore $j = f_i(m^*)$ for some $i \in I$. If $t \leq m$, then $f_i(t) \leq e_{m,j}(t) = \perp$. Suppose now that $t \not\leq m$, so $m < t \vee m$ and $m^* \leq t \vee m$. Observe also that $f_i(t) \leq e_{j,m}(t) = j$, since $e_{j,m} = \bigvee_{i \in I} f_i$. Then $j = f_i(m^*) \leq f_i(m \vee t) = f_i(m) \vee f_i(t) = \perp \vee f_i(t) = f_i(t)$, so $j = f_i(t)$. We have argued that $f_i(t) = e_{j,m}$, for all $t \in L$, and therefore that $f_i = e_{j,m}$. \square

Theorem 4.3. *If L is a finite lattice and $[L, L]_{\vee}$ is autodual, then L is distributive.*

Proof. If $\psi : [L, L]_{\vee}^{op} \rightarrow [L, L]_{\vee}$ is invertible, then ψ restricts to a bijection $M([L, L]_{\vee}) \rightarrow J([L, L]_{\vee})$, so these two sets have same cardinality. For $m \in M(L)$ and $j \in J(L)$, the $e_{j,m}$ as well as the elementary tensors $m \otimes j$ are pairwise distinct. Therefore, we have

$$|M(L)| \times |J(L)| \leq |J([L, L]_{\vee})| = |M([L, L]_{\vee})| = |M(L)| \times |J(L)|$$

and $|M(L)| \times |J(L)| = |J([L, L]_{\vee})|$. That is, the elements $e_{j,m}$ are all the join-irreducible elements of $[L, L]_{\vee}$ and therefore the set $\{e_{j,m} \mid j \in J(L), m \in M(L)\}$ generates $[L, L]_{\vee}$ under joins. It follows that $[L, L]_{\vee} = [L, L]_{\vee}^{\dagger}$ and that L is distributive. \square

We do not know yet if the theorem above can be generalized to infinite complete lattices or whether there is some fancy infinite complete lattice L that is not completely distributive and such that $[L, L]_{\vee}$ is autodual. It is clear, however, that in order to construct such a fancy lattice, properties of bimorphisms $\psi : L \times L^{op} \rightarrow [L, L]_{\vee}$ need to be investigated. What are the properties of a bimorphism ψ forcing L to be completely distributive when $\tilde{\psi}$ is surjective? Taking the bimorphism e as example, let us abstract part of Raney's Theorem:

Proposition 4.4. *Let $\psi : L \times L^{op} \rightarrow [L, L]_{\vee}$ be a bimorphism such that for each $x, y \in L$, the image of L under $\psi(y, x)$ is a finite chain. If id_L belongs to the image of $\tilde{\psi} : [L, L]_{\wedge} \rightarrow [L, L]_{\vee}$, then L is a completely distributive lattice.*

Proof. Let $z \in Z$ such that $z = \bigwedge_{i \in I} \bigvee_{j \in J_i} z_j$. We aim at showing that $z \leq \bigvee_s \bigwedge_{i \in I} z_{s(i)}$, with the index s ranging on choice functions $s : I \rightarrow \bigcup_{i \in I} J_i$ (s is a choice function if $s(i) \in J_i$, for each $i \in I$). Since $id_L = \bigvee \{ \psi(y, x) \mid \psi(y, x) \leq id_L \}$, we also have $z = \bigvee \{ \psi(y, x)(z) \mid \psi(y, x) \leq id_L \}$ and therefore, in order to achieve our goal, it will be enough to show that for each $y, x \in L$, if $\psi(y, x) \leq id_L$, then $\psi(y, x)(z) \leq \bigvee_s \bigwedge_{i \in I} z_{s(i)}$. Let y, x be such that $\psi(y, x) \leq id_L$, fix $i \in I$, and observe then that

$$\psi(y, x)(z) \leq \psi(y, x)\left(\bigvee_{j \in J_i} z_j\right) = \bigvee_{j \in J_i} \psi(y, x)(z_j),$$

since $z \leq \bigvee_{j \in J_i} z_j$. Since the set $\{ \psi(y, x)(z_j) \mid j \in J_i \}$ is finite and directed (it is a finite chain), it has a maximum: there exists $j(i) \in J_i$ such that $\bigvee_{j \in J_i} \psi(y, x)(z_j) = \psi(y, x)(z_{j(i)})$. It follows that

$$\psi(y, x)(z) \leq \psi(y, x)(z_{j(i)}) \leq z_{j(i)},$$

since $\psi(y, x) \leq id_L$. By letting i vary, we have constructed a choice function $j : I \rightarrow \bigcup_{i \in I} J_i$ such that $\psi(y, x)(z) \leq \bigwedge_{i \in I} z_{j(i)}$, and consequently $\psi(y, x)(z) \leq \bigvee_s \bigwedge_{i \in I} z_{s(i)}$. \square

Bimorphisms satisfying the conditions of Proposition 4.4 might be easily constructed by taking $f \in [L, [L, L]_{\vee}]_{\vee}$, $g \in [L^{op}, L^{op}]_{\vee}$ (resp., $f \in [L, L]_{\vee}$ and $g \in [L^{op}, [L, L]_{\vee}]_{\vee}$), and defining then

$$\psi(y, x) := f(y) \circ a_{g(x)} \quad (\text{resp., } \psi(y, x) := c_{f(y)} \circ g(x)).$$

These bimorphisms satisfy the conditions of Proposition 4.4, since they only take two values. As a consequence of the proposition, they cannot be used to construct a fancy dual isomorphism of $[L, L]_{\vee}$.

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