



HAL
open science

An accurate iterative method for ordinary differential equations with classical and Caputo-Fabrizio derivatives

Abdon Atangana, Seda Igret Araz

► To cite this version:

Abdon Atangana, Seda Igret Araz. An accurate iterative method for ordinary differential equations with classical and Caputo-Fabrizio derivatives. 2023. ⟨hal-03956673⟩

HAL Id: hal-03956673

<https://hal.science/hal-03956673v1>

Preprint submitted on 25 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

An accurate iterative method for ordinary differential equations with classical and Caputo-Fabrizio derivatives

Abdon ATANGANA^{1,a}, Seda İĞRET ARAZ^{1,2}

¹AtanganaA@ufs.ac.za, Institute for Groundwater Studies,
Faculty of Natural and Agricultural Sciences, University of the Free State,
South Africa

^aDepartment of Medical Research, China Medical University Hospital,
China Medical University, Taichung, Taiwan

²Siirt University, Department of Mathematics Education, Siirt, Turkey

January 25, 2023

Abstract

The powerful mathematical equations known as nonlinear ordinary differential equations are used in both theoretical and practical applications. Since they are used to simulate problems in the actual world, their answers are quite significant. However, not all of them can be analyzed to find the precise solution. To acquire and approximative solution, researchers rely on several alternative methodologies. However, there are several methods that can be utilized to guarantee the circumstances in which a precise and singular answer exists. To ensure existence and uniqueness, one of them involves defining a sequence that converges toward a solution and combining this with the fixed-point theory. In recent decades, names like Picard, Tonelli, and numerous more have been proposed. We have successfully proposed an iteration that converges more quickly than the Picard iteration in this study. We demonstrate that the analysis of this iteration of the ordinary differential equation uses the conventional Caputo-Fabrizio derivative and one or more orders. We found precise solutions to a few linear and nonlinear equations.

Keywords: Existence and uniqueness; nonlinear differential equations; New iteration, Picard iteration, convergence.

1 Introduction

The study of nonlinear ordinary differential equations spans both theoretical and practical areas of engineering, physics, and pure mathematics. The characteristics of differential equations of different sorts are a topic that is addressed by all these professions. While applied mathematics emphasizes the thorough explanation of the methods for approximating solutions, pure mathematics emphasizes the existence and uniqueness of solutions. Virtually every physical, technological, or biological

process ranging from celestial motion to bridge construction to interactions between neurons can be modeled using nonlinear ordinary differential equations. It's possible that nonlinear ordinary differential equations employed to address real-world issues are not always immediately solvable or may not have closed form solutions. Numerical techniques can be used to approximate the solutions instead. Differential equations are not solved in the same way as algebraic equations. Their answers are frequently ambiguous, but whether they are unique or even exist at all is a major subject of inquiry. The Peano existence theorem[1] provides one set of conditions under which a solution exists for first order initial value problems. An ordinary differential equation's condition of existence and uniqueness can be determined using a variety of methods. One method is to design a sequence and demonstrate that it converges to a solution. In addition, the Lipschitz condition and boundness are utilized. In particular, the Picard-Lindelöf theorem in differential equations provides a set of circumstances in which an initial value problem has a singular solution[2]. The existence and uniqueness theorem, Cauchy-Lipschitz theorem, and Picard's existence theorem are some other names for it[[3]-[6]]. This iteration has been employed with great success for the past ten years, but several other iterations have been proposed and found to be more suitable for supplying a condition of existence and uniqueness, such as the Tonelli sequence[7], which has been employed even within the context of fractional calculus. In this study, we propose a Picard-like iteration and use it to demonstrate that a particular Cauchy problem with a classical Caputo-Fabrizio derivative has a unique solution[8]. The study is organized as follows: in section 1, we offer some analysis and then compare the suggested sequence to the Picard version. We solve several fundamental linear and nonlinear ordinary differential equations in section two, compare our findings to those of Picard and even Taylor series[9], and note that the comparisons are based on iterations. In Section 3, we focus on sequence analysis using the growth linear property. In the final section, we offer an analysis for the situation of Cauchy problems with the Caputo-Fabrizio derivative. The first four sections will be devoted to Cauchy problems with high orders.

2 Derivation of the iteration and comparison with Picard

We consider the following Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases} . \quad (1)$$

Here $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be Lipschitz and bounded. We wish to prove that the differential equation (1) has a unique solution. One of the most used iterative methods is from Picard who suggested that, the equivalent of equation (1)

$$y(t) = y(t_0) + \int_{t_0}^t f(s, y(s)) ds \quad (2)$$

could be used to obtain the following sequence.

$$y'_n(t) = f(t, y_{n-1}(t)) \quad (3)$$

or equivalently

$$y_n(t) = y(t_0) + \int_{t_0}^t f(s, y_{n-1}(s)) ds. \quad (4)$$

It is clear that by imposing this, we use the previous functions to calculate the next. While this iteration has gained interest in many fields due to its accuracy, it is alleged that in the definition some terms may be making the iteration less accurate as could be. In this paper, we suggest an iterative or sequence and compare it with the Picard version. To construct the sequence, we start with

$$\begin{aligned} \frac{dy_1}{dt} &= f(t, y_1), & y_1(t_0) &= y_0, \\ &\vdots \\ \frac{dy_n}{dt} &= f(t, y_n), & y_n(t_0) &= y_0. \end{aligned} \quad (5)$$

Indeed, this is a bit complex owing the fact that $y_n(t)$ is not known. Therefore, we replace $y_n(t)$ on the right hand side by $\bar{y}_n(t)$ to have

$$\frac{dy_n(t)}{dt} = f(t, \bar{y}_n(t)). \quad (6)$$

$\bar{y}_n(t)$ will be defined using the terms of the past, here, we borrow the Picard

$$\frac{d\bar{y}_n(t)}{dt} = f(t, y_{n-1}(t)). \quad (7)$$

Indeed, the key part of this process is to show that, the suggested iteration converges and may be accurate than the Picard version. To differentiate between the Picard iteration and the suggested one, we put $y_n^P(t)$ for Picard and $y_n^A(t)$ for the suggested one. Then, we write

$$y_n^P(t) = y(t_0) + \int_{t_0}^t f(s, y_{n-1}(s)) ds, \quad (8)$$

and

$$y_n^A(t) = y(t_0) + \int_{t_0}^t f(s, \bar{y}_n(s)) ds. \quad (9)$$

We now evaluate

$$\begin{aligned} |y_n^A(t) - y_n^P(t)| &= \left| \int_{t_0}^t (f(s, y_{n-1}(s)) - f(s, \bar{y}_n(s))) ds \right| \\ &\leq \int_{t_0}^t |f(s, y_{n-1}(s)) - f(s, \bar{y}_n(s))| ds. \end{aligned} \quad (10)$$

Using the Lipschitz condition of f , we get

$$\begin{aligned}
|y_n^A(t) - y_n^P(t)| &\leq L \int_{t_0}^t |\bar{y}_n(s) - y_{n-1}(s)| ds & (11) \\
&\leq L \int_{t_0}^t \left| y_0(s) + \int_{t_0}^s f(s, y_{n-1}(s)) ds - y_{n-1}(s) \right| ds \\
&\leq L \int_{t_0}^t |\bar{y}_n(s) - y_{n-1}(s)| ds \\
&\leq L \int_{t_0}^t \left| \int_{t_0}^s |(f(l, y_{n-1}) - f(l, y_{n-2}))| dl \right| ds \\
&\leq L^n \int_{t_0}^t \int_{t_0}^{\tau} \dots \int_{t_0}^{\lambda} |y_1 - y_0| d\lambda \dots ds \\
&\leq L^n M \frac{(t - t_0)^n}{n!}
\end{aligned}$$

Thus, we get

$$|y_n^A(t) - y_n^P(t)| \leq L^n M \frac{(t - t_0)^n}{n!}. \quad (12)$$

If f is a contraction, then

$$\lim_{n \rightarrow \infty} |y_n^A(t) - y_n^P(t)| = 0. \quad (13)$$

Moreover,

$$\begin{aligned}
\sum_{n=0}^{\infty} |y_n^A(t) - y_n^P(t)| &\leq M \sum_{n=0}^{\infty} \frac{[L(t - t_0)]^n}{n!} & (14) \\
&\leq M \exp(L(t - t_0)).
\end{aligned}$$

Now, an analysis of $y_n^A(t)$ is presented. We shall present the well-defined

$$\begin{cases} y_n(t) = y_0(t) + \int_{t_0}^t f(s, \bar{y}_n(s)) ds \\ \bar{y}_n(t) = y_0(t) + \int_{t_0}^t f(s, y_{n-1}(s)) ds \end{cases} \quad (15)$$

or

$$\begin{cases} \frac{d}{dt} y_n(t) = f(t, \bar{y}_n(t)) \\ \frac{d}{dt} \bar{y}_n(t) = f(t, y_{n-1}(t)) \end{cases} \quad (16)$$

Then, we have

$$\begin{aligned}
|y_n(t) - y_0| &= \left| \int_{t_0}^t f(s, \bar{y}_n(s)) ds \right| \\
&\leq \int_{t_0}^t |f(s, \bar{y}_n(s))| ds \\
&\leq M(t - t_0) \\
&\leq M\delta < \varepsilon
\end{aligned} \tag{17}$$

where

$$\delta = \min \left\{ \frac{1}{2L}, \frac{\varepsilon}{M} \right\}. \tag{18}$$

We now evaluate the difference of two consecutive terms

$$\begin{aligned}
\zeta_n(t) &= y_n(t) - y_{n-1}(t) \\
\frac{d}{dt}\zeta_n(t) &= \frac{d}{dt}y_n(t) - \frac{d}{dt}y_{n-1}(t) \\
&= f(t, \bar{y}_n(t)) - f(t, \bar{y}_{n-1}(t)).
\end{aligned} \tag{19}$$

We now evaluate

$$\begin{aligned}
\left| \frac{d}{dt}\zeta_n(t) \right| &= |f(t, \bar{y}_n(t)) - f(t, \bar{y}_{n-1}(t))| \\
&\leq L|\bar{y}_n - \bar{y}_{n-1}| \\
&\leq L|\zeta_{n-1}| \\
&\leq L\Pi_{n-1}
\end{aligned} \tag{20}$$

where

$$\Pi_n = \max_{t_0 - \delta < t \leq t_0 + \delta} |\zeta_n(t)|. \tag{21}$$

Here indeed $|\zeta_n(t_0)| = 0$, then $\forall t \in (t_0 - \delta, t_0 + \delta)$, we have

$$|\zeta_n(t)| < L\delta\Pi_{n-1}. \tag{22}$$

Indeed from the definition of $\zeta_n(t)$ the maximal value $|\zeta_n(t)|$ satisfies

$$\Pi_n \leq L\delta\Pi_{n-1}. \tag{23}$$

If we choose $\delta \leq \frac{1}{2L}$, thus $L\delta \leq \frac{1}{2}$ and then

$$\Pi_n \leq \frac{1}{2}\Pi_{n-1} \leq \left(\frac{1}{2}\right)^2 \Pi_{n-2} \leq \dots \leq \left(\frac{1}{2}\right)^{n-1} \varepsilon. \tag{24}$$

We note that $\sum y_n(t)$ converges, the sequence $\zeta_n(t)$ also converges because

$$|\zeta_n(t)| \leq \Pi_n(t) \leq \left(\frac{1}{2}\right)^{n-1} \varepsilon. \quad (25)$$

We have that

$$\sum_{n=1}^{\infty} |\zeta_n(t)| \leq \sum_{n=1}^{\infty} \Pi_n \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \varepsilon = 2\varepsilon \quad (26)$$

which is achieved via geometric series. Since the series $\sum_{n=0}^{\infty} |\Pi_n|$ converges by the property of absolutely convergence $\sum_{n=0}^{\infty} \Pi_n$ also converges. Therefore

$$y(t) = \lim_{n \rightarrow \infty} y_n(t) = y_0(t) + \sum_{n=1}^{\infty} \zeta_n(t) \quad (27)$$

exists, the sequence of $y_n(t)$ converges.

$$y(t) - y_n(t) = \sum_{l=n+1}^{\infty} y_l(t). \quad (28)$$

We evaluate

$$|y(t) - y_n(t)| \leq \left| \sum_{l=n+1}^{\infty} y_l(t) \right| \leq \left| \sum_{l=n+1}^{\infty} \left(\frac{1}{2}\right)^{l-1} \varepsilon \right| = \left(\frac{1}{2}\right)^{n-1} \varepsilon \quad (29)$$

Then, we write

$$\begin{aligned} \lim_{n \rightarrow \infty} y_n(t) &= y_0(t) + \lim_{n \rightarrow \infty} \int_{t_0}^t f(s, y_n(s)) ds \\ &= y_0(t) + \int_{t_0}^t \lim_{n \rightarrow \infty} f(s, y_n(s)) ds \\ &= y_0 + \int_{t_0}^t f(s, y(s)) ds \\ &= y(t). \end{aligned} \quad (30)$$

2.1 Application of the iteration for solving some ODE

We shall solve some ordinary differential equations with the suggested iteration and the Picard version.

Example 1 *We consider a simple decay equation*

$$\begin{cases} y'(t) = -\lambda y(t) \\ y(0) = 1 \end{cases}. \quad (31)$$

We recall that the Picard iteration and suggested one are given as

$$y_n^P(t) = y(t_0) + \int_{t_0}^t f(s, y_{n-1}(s)) ds, \quad (32)$$

and

$$\begin{cases} y_n^A(t) = y(t_0) + \int_{t_0}^t f(s, \bar{y}_n(s)) ds, \\ \bar{y}_n(s) = y_n^P(s) \end{cases}. \quad (33)$$

We use the initial condition as initial guess

$$y_n^P(t_0) = y_n^A(t_0) = 1. \quad (34)$$

Then, we write

$$\begin{aligned} y_1^P(t) &= y(0) + \int_0^t f(s, y_0(s)) ds \\ &= 1 - \lambda t. \end{aligned} \quad (35)$$

However, we write

$$\begin{aligned} y_1^A(t) &= y(0) + \int_0^t f(s, y_1^P(s)) ds \\ &= 1 - \lambda \left(t - \frac{\lambda t^2}{2} \right) \\ &= 1 - \lambda t + \frac{\lambda^2 t^2}{2} \end{aligned} \quad (36)$$

Next iteration can be done as

$$\begin{aligned} y_2^P(t) &= 1 + \int_0^t f(s, y_1^P(s)) ds \\ &= 1 - \lambda t + \frac{\lambda^2 t^2}{2}. \end{aligned} \quad (37)$$

and

$$\begin{aligned} y_2^A(t) &= y(0) + \int_0^t f(s, y_2^P(s)) ds \\ &= 1 - \lambda \left(t - \frac{\lambda t^2}{2} + \frac{\lambda^2 t^3}{6} \right) \\ &= 1 - \lambda t + \frac{\lambda^2 t^2}{2} - \frac{\lambda^3 t^3}{6}. \end{aligned} \quad (38)$$

Using the iterations $\forall n \geq 1$, we have the following:

$$y_n^P(t) = \sum_{j=0}^n \frac{(-\lambda t)^j}{j!} \quad (39)$$

while

$$y_n^A(t) = \sum_{j=0}^{n+1} \frac{(-\lambda t)^j}{j!}. \quad (40)$$

Therefore

$$|y_n^A(t) - y_n^P(t)| = \left| \frac{(-\lambda t)^{n+1}}{n!} \right|. \quad (41)$$

The Cauchy sequence comparison test between Picard and our iteration is presented in Figure 1 and 2.

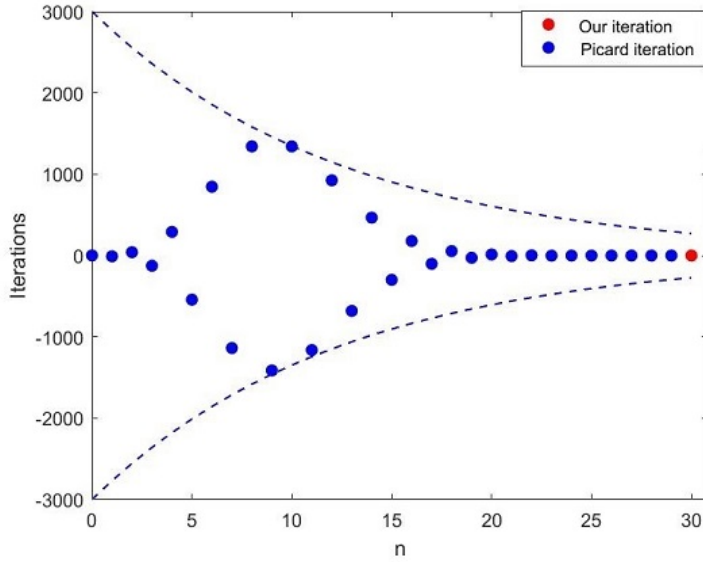


Figure 1. Cauchy sequence comparison test for Picard and our sequences.

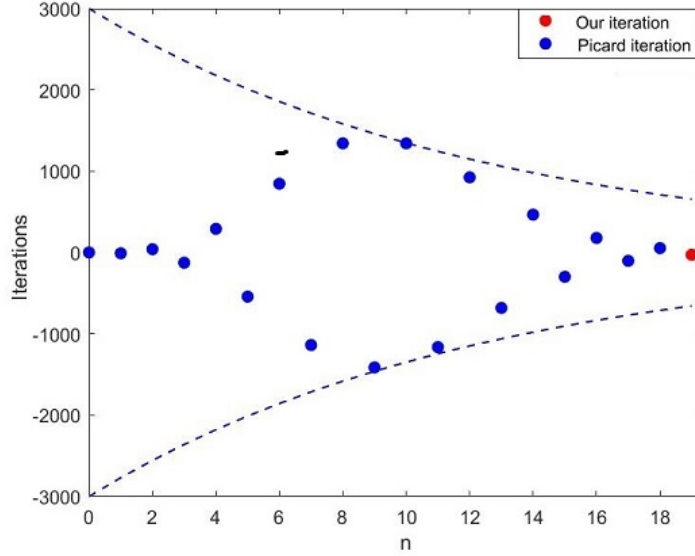


Figure 2. Cauchy sequence comparison test for Picard and our sequences.

The red dot in the above figures shows that the suggested iteration converges than the Picard.

Example 2 We consider the following equation

$$\begin{cases} y'(t) = 1 + y(t) \\ y(0) = y_0 \end{cases} \quad (42)$$

For $y(0) = 1$, the exact solution is

$$y(t) = 2e^t - 1. \quad (43)$$

The initial guess is as follows:

$$y_n^P(t_0) = y_n^A(t_0) = 1. \quad (44)$$

We get iterations

$$\begin{aligned} y_1^P(t) &= y(0) + \int_0^t f(s, y_0(s)) ds \\ &= y(0) + \int_0^t (1 + y_0(s)) ds \\ &= 1 + \int_0^t 2 ds = 1 + 2t. \end{aligned} \quad (45)$$

while

$$\begin{aligned}
y_1^A(t) &= y(0) + \int_0^t f(s, y_1^P(s)) ds \\
&= 1 + \int_0^t (1 + y_1^P(s)) ds \\
&= 1 + \int_0^t (1 + 2t + 1) ds \\
&= 1 + 2t + \frac{2t^2}{2}.
\end{aligned} \tag{46}$$

For next step, we have

$$\begin{aligned}
y_2^P(t) &= y(0) + \int_0^t (1 + y_1^P(s)) ds \\
&= 1 + 2t + \frac{2t^2}{2}
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
y_2^A(t) &= y(0) + \int_0^t f(s, y_2^P(s)) ds \\
&= 1 + \int_0^t (2 + 2s + s^2) ds \\
&= 1 + 2t + \frac{2t^2}{2} + \frac{2t^3}{6}.
\end{aligned} \tag{48}$$

For $\forall n \geq 1$, we have the following:

$$y_n^P(t) = 1 + 2 \sum_{j=1}^n \frac{t^j}{j!} = 2 \sum_{j=0}^n \frac{t^j}{j!} - 1 \tag{49}$$

while

$$y_n^A(t) = 1 + 2 \sum_{j=1}^{n+1} \frac{t^j}{j!} = 2 \sum_{j=0}^{n+1} \frac{t^j}{j!} - 1. \tag{50}$$

It can be easily seen that

$$\lim_{n \rightarrow \infty} y_n^P(t) = 2e^t - 1 \tag{51}$$

and

$$\lim_{n \rightarrow \infty} y_n^A(t) = 2e^t - 1. \tag{52}$$

Therefore

$$|y_n^A(t) - y_n^P(t)| = \frac{2t^{n+1}}{(n+1)!}. \tag{53}$$

Example 3 We consider the following equation

$$\begin{cases} y'(t) = 1 + y^2(t) \\ y(1) = y_0 \end{cases} . \quad (54)$$

By choosing $y(1) = 0$, we have the exact solution

$$y(t) = \tan(t - 1). \quad (55)$$

We write

$$\begin{aligned} y_1^P(t) &= 0 + \int_1^t f(s, y_0^2(s)) ds \\ &= t - 1. \end{aligned} \quad (56)$$

while

$$\begin{aligned} y_1^A(t) &= 0 + \int_0^t f(s, [y_1^P(s)]^2) ds \\ &= \int_1^t (1 + (s - 1)^2) ds \\ &= \frac{t^3}{3} - t^2 + 2t - \frac{4}{3}. \end{aligned} \quad (57)$$

Note that

$$(t - 1) + \frac{(t - 1)^3}{3} = \frac{t^3}{3} - t^2 + 2t - \frac{4}{3}. \quad (58)$$

Therefore

$$y_2^A(t) = (t - 1) + \frac{(t - 1)^3}{3}. \quad (59)$$

Then, we have

$$\begin{aligned} y_2^P(t) &= (t - 1) + \frac{(t - 1)^3}{3}, \\ y_2^A(t) &= (t - 1) + \frac{(t - 1)^3}{3} + \frac{2(t - 1)^{15}}{15}. \end{aligned} \quad (60)$$

or $\forall n \geq 1$, we have the following:

$$y_n^P(t) = \sum_{j=1}^n \frac{(-1)^{j-1} 2^{2j} (2^{2j} - 1) B_{2j} (t - 1)^{2j-1}}{(2j)!}, |t - 1| < \frac{\pi}{2} \quad (61)$$

where B_{2j} is Bernoulli number. For suggested one, we can have the following:

$$y_n^A(t) = \sum_{j=1}^{n+1} \frac{(-1)^{j-1} 2^{2j} (2^{2j} - 1) B_{2j} (t-1)^{2j-1}}{(2j)!}, |t-1| < \frac{\pi}{2}. \quad (62)$$

Therefore, we get

$$\begin{aligned} |y_n^A(t) - y_n^P(t)| &= \frac{(t-1)^{2n+1}}{1.3.5\dots(2n+1)} \\ &= \frac{2n!(t-1)^{2n+1}}{(n-1)\left(2^{\frac{n-1}{2}}\right)!}. \end{aligned} \quad (63)$$

In the next example, we shall compare Taylor' series method, Picard and our iteration.

Example 4 We now consider the following problem

$$\begin{cases} y'(t) = t^2 - y(t) \\ y(0) = 1 \end{cases}. \quad (64)$$

Taylor series formula is achieved as

$$\begin{aligned} y' &= t^2 - y, & y'(0) &= -1 \\ y'' &= 2t - y', & y''(0) &= 1 \\ y''' &= 2 - y'', & y'''(0) &= 1 \\ y^{iv} &= -y''', & y^{iv}(0) &= -1 \\ y^v &= -y^{iv}, & y^v(0) &= 1. \end{aligned}$$

Then, we have

$$y(t) = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}.$$

By choosing $y(1) = 0$, we have the exact solution

$$y(t) = \tan(t-1). \quad (65)$$

We write

$$\begin{aligned} y_1^P(t) &= 0 + \int_1^t f(s, y_0^2(s)) ds \\ &= t - 1. \end{aligned} \quad (66)$$

while

$$\begin{aligned}
y_1^A(t) &= 0 + \int_0^t f\left(s, [y_1^P(s)]^2\right) ds \\
&= \int_1^t \left(1 + (s-1)^2\right) ds \\
&= \frac{t^3}{3} - t^2 + 2t - \frac{4}{3}.
\end{aligned} \tag{67}$$

Note that

$$(t-1) + \frac{(t-1)^3}{3} = \frac{t^3}{3} - t^2 + 2t - \frac{4}{3}. \tag{68}$$

Therefore

$$y_2^A(t) = (t-1) + \frac{(t-1)^3}{3}. \tag{69}$$

Then, we have

$$\begin{aligned}
y_2^P(t) &= (t-1) + \frac{(t-1)^3}{3}, \\
y_2^A(t) &= (t-1) + \frac{(t-1)^3}{3} + \frac{2(t-1)^{15}}{15}.
\end{aligned} \tag{70}$$

or $\forall n \geq 1$, we have the following:

$$y_n^P(t) = \sum_{j=1}^n \frac{(-1)^{j-1} 2^{2j} (2^{2j} - 1) B_{2j} (t-1)^{2j-1}}{(2j)!}, |t-1| < \frac{\pi}{2} \tag{71}$$

where B_{2j} is Bernoulli number. For suggested one, we can have the following:

$$y_n^A(t) = \sum_{j=1}^{n+1} \frac{(-1)^{j-1} 2^{2j} (2^{2j} - 1) B_{2j} (t-1)^{2j-1}}{(2j)!}, |t-1| < \frac{\pi}{2}. \tag{72}$$

Therefore, we get

$$\begin{aligned}
|y_n^A(t) - y_n^P(t)| &= \frac{(t-1)^{2n+1}}{1.3.5\dots(2n+1)} \\
&= \frac{2n! (t-1)^{2n+1}}{(n-1) \left(2^{\frac{n-1}{2}}\right)!}.
\end{aligned} \tag{73}$$

In the next example, we shall compare Taylor' series method, Picard and our iteration.

Example 5 We now consider the following problem

$$\begin{cases} y'(t) = t^2 - y(t) \\ y(0) = 1 \end{cases} . \quad (74)$$

By choosing $y(0) = 1$, we have the exact solution

$$y(t) = t^2 - 2t + 2 - e^{-t}. \quad (75)$$

Taylor series formula is achieved as

$$\begin{aligned} y' &= t^2 - y, & y'(0) &= -1 \\ y'' &= 2t - y', & y''(0) &= 1 \\ y''' &= 2 - y'', & y'''(0) &= 1 \\ y^{iv} &= -y''', & y^{iv}(0) &= -1 \\ y^v &= -y^{iv}, & y^v(0) &= 1. \end{aligned}$$

Then, we have

$$y(t) = 1 - t + \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \frac{t^5}{5!}. \quad (76)$$

Using Picard and our iteration, we write

$$\begin{aligned} y_1^P(t) &= 1 + \int_0^t (s^2 - y_0(s)) ds \\ &= 1 - t + \frac{t^3}{3}. \end{aligned} \quad (77)$$

while

$$\begin{aligned} y_1^A(t) &= 1 + \int_0^t f(s, y_1^P(s)) ds \\ &= 1 + \int_0^t \left(s^2 - 1 + s - \frac{s^3}{3} \right) ds \\ &= 1 - t + \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{12}. \end{aligned} \quad (78)$$

Then, we have

$$\begin{aligned} y_2^P(t) &= 1 - t + \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{12}, \\ y_2^A(t) &= 1 - t + \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{12} + \frac{t^5}{60} \end{aligned} \quad (79)$$

and

$$\begin{aligned} y_3^P(t) &= 1 - t + \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{12} + \frac{t^5}{60}, \\ y_3^A(t) &= 1 - t + \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24} + \frac{t^5}{60} - \frac{t^6}{360}. \end{aligned} \quad (80)$$

3 Investigation of Picard and our iteration under linear growth condition

In this section, we examine the convergence of Picard iteration and the suggested iteration. If the function $f(t, y(t))$ satisfies the linear growth condition that is $\forall t \in [t_0, T]$, then, by recursive, we have

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}. \quad (81)$$

With Picard iteration, we have equivalently

$$\begin{aligned} |y_1(t)|^2 &= \left| y(t_0) + \int_{t_0}^t f(s, y_0(s)) ds \right|^2 \\ &\leq 2|y(t_0)|^2 + 2 \left| \int_{t_0}^t f(s, y_0(s)) ds \right|^2 \\ &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t |f(s, y_0(s))|^2 ds. \end{aligned} \quad (82)$$

Using the linear growth condition of f yields

$$\begin{aligned} |y_1(t)|^2 &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K(1 + |y_0(s)|^2) ds \\ &\leq 2|y(t_0)|^2 + 2K(t - t_0) + 2K \int_{t_0}^t |y(t_0)|^2 ds \\ &\leq \left[2|y(t_0)|^2 + 2K(t - t_0) + 2K(t - t_0)|y(t_0)|^2 \right] \\ &\leq \left[\begin{array}{l} 2|y_0|^2(1 + 2K(t - t_0)) \\ + (1 + 2K(t - t_0)) \end{array} \right] \end{aligned} \quad (83)$$

We now evaluate y_2

$$\begin{aligned} |y_2(t)|^2 &= \left| y(t_0) + \int_{t_0}^t f(s, y_1(s)) ds \right|^2 \\ &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t |f(s, y_1(s))|^2 ds. \end{aligned} \quad (84)$$

Using the linear growth condition of f yields

$$|y_2(t)|^2 \leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K \left(1 + |y_1(s)|^2\right) ds \quad (85)$$

$$\begin{aligned} &\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t \left[\frac{2|y_0|^2(1+2K(t-t_0))}{1+2K(t-t_0)} \right] ds \\ &\leq \left[\begin{array}{c} \left[\begin{array}{c} 2|y(t_0)|^2 + 2K(t-t_0) \\ +4K^2 \frac{(t-t_0)^2}{2} + 4K(t-t_0)|y_0|^2 \\ +2K \left[(t-t_0) + 2K \frac{(t-t_0)^2}{2} \right] \end{array} \right] \\ \left[\begin{array}{c} 2|y_0|^2 \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \right) \\ + \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \right) \end{array} \right] \end{array} \right] \end{aligned} \quad (86)$$

We next evaluate y_3

$$\begin{aligned} |y_3(t)|^2 &= \left| y(t_0) + \int_{t_0}^t f(s, y_2(s)) ds \right|^2 \\ &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t |f(s, y_2(s))|^2 ds. \end{aligned} \quad (87)$$

Using the linear growth condition of f yields

$$\begin{aligned} |y_3(t)|^2 &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K \left(1 + |y_2(s)|^2\right) ds \\ &\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t |y_2(s)|^2 ds \\ &\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t \left[\frac{2|y_0|^2 \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \right)}{1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2}} \right] ds \\ &\leq \left[\begin{array}{c} 2|y_0|^2 \left(\begin{array}{c} 1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \\ + 8K^3 \frac{(t-t_0)^3}{6} \end{array} \right) \\ + \left(\begin{array}{c} 1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \\ + 8K^3 \frac{(t-t_0)^3}{6} \end{array} \right) \end{array} \right] \end{aligned} \quad (88)$$

When we proceed for n , we can get the following

$$\begin{aligned}
|y_n(t)|^2 &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K \left(1 + |y_{n-1}(s)|^2\right) ds \\
&\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t |y_{n-1}(s)|^2 ds \\
&\leq \left[2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t |y_{n-1}(s)|^2 ds \right] \\
&\leq \left[2|y(t_0)|^2 + 2K(t-t_0) + 4K^2 \int_{t_0}^t |y_{n-1}(s)|^2 ds \right]
\end{aligned} \tag{89}$$

After arranging, we can get

$$\begin{aligned}
|y_n(t)|^2 &\leq 2|y_0|^2 \sum_{m=0}^n \frac{(2K)^m (t-t_0)^m}{m!} + \sum_{m=0}^n \frac{(2K)^m (t-t_0)^m}{m!} \\
&\leq \left[\exp \left[2K(t-t_0) \frac{\Gamma(n+1, 2K(t-t_0))}{\Gamma(n+1)} \right] |y_0|^2 \right. \\
&\quad \left. + \exp \left[2K(t-t_0) \frac{\Gamma(n+1, 2K(t-t_0))}{\Gamma(n+1)} \right] \right].
\end{aligned} \tag{90}$$

Taking the limit $n \rightarrow \infty$, we have

$$\begin{aligned}
|y_n(t)|^2 &\leq \left[\exp \left[2K(t-t_0) \frac{\Gamma(n+1, 2K(t-t_0))}{\Gamma(n+1)} \right] |y_0|^2 \right. \\
&\quad \left. + \exp \left[2K(t-t_0) \frac{\Gamma(n+1, 2K(t-t_0))}{\Gamma(n+1)} \right] \right] \\
&\leq \exp(2K(t-t_0)) + \exp(2K(t-t_0)) |y_0|^2 \\
&\leq \exp(2K(t-t_0)) (1 + |y_0|^2).
\end{aligned} \tag{91}$$

If $y_0 = 0$, then

$$|y_n(t)|^2 \leq \exp(2K(t-t_0)). \tag{92}$$

We analyze the suggested sequence holds the linear growth condition. If the function $f(t, y(t))$ satisfies the linear growth condition that is $\forall t \in [t_0, T]$, then, by recursive, we have

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}. \tag{93}$$

or equivalently

$$\begin{aligned}
|y_1(t)|^2 &= \left| y(t_0) + \int_{t_0}^t f(s, \bar{y}_1(s)) ds \right|^2 \\
&\leq 2|y(t_0)|^2 + 2 \left| \int_{t_0}^t f(s, \bar{y}_1(s)) ds \right|^2 \\
&\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t |f(s, \bar{y}_1(s))|^2 ds.
\end{aligned} \tag{94}$$

Using the linear growth condition of f yields

$$\begin{aligned}
|y_1(t)|^2 &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K \left(1 + |\bar{y}_1(s)|^2\right) ds & (95) \\
&\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t \left(|y(t_0)| + \int_{t_0}^s |f(l, y_0(l))| dl \right)^2 ds \\
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 4K \int_{t_0}^t [y(t_0)]^2 ds \\ &+ 2K \int_{t_0}^t \int_{t_0}^s [f(l, y_0(l))]^2 dl ds \end{aligned} \right] \\
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 4K(t-t_0)[y(t_0)]^2 \\ &+ 4K \int_{t_0}^t \int_{t_0}^s K \left(1 + |y_0(s)|^2\right) dl ds \end{aligned} \right] \\
&\leq \left[\begin{aligned} &2|y_0|^2 + 2K(t-t_0) + 4(t-t_0)K|y_0|^2 \\ &+ 4K^2 \left(1 + |y_0|^2\right) \frac{(t-t_0)^2}{2} \end{aligned} \right] \\
&\leq \left[\begin{aligned} &2|y_0|^2 + 2K(t-t_0) + 4(t-t_0)K|y_0|^2 \\ &+ 4K^2 \frac{(t-t_0)^2}{2} + 4K^2|y_0|^2 \frac{(t-t_0)^2}{2} \end{aligned} \right] \\
&\leq \left[\begin{aligned} &2|y_0|^2 \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2}\right) \\ &+ \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2}\right) \end{aligned} \right] \\
&\leq \left[\begin{aligned} &|y_0|^2 \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2}\right) \\ &+ \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2}\right) \end{aligned} \right]
\end{aligned}$$

We now evaluate y_2

$$\begin{aligned}
|y_2(t)|^2 &= \left| y(t_0) + \int_{t_0}^t f(s, \bar{y}_2(s)) ds \right|^2 & (96) \\
&\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t |f(s, \bar{y}_2(s))|^2 ds.
\end{aligned}$$

Using the linear growth condition of f yields

$$\begin{aligned}
|y_2(t)|^2 &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K \left(1 + |\bar{y}_2(s)|^2\right) ds & (97) \\
&\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t \left(y(t_0) + \int_{t_0}^s f(l, y_1(l)) dl \right)^2 ds \\
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t 2|y(t_0)|^2 ds \\ &+ 2K \int_{t_0}^t 2 \left(\int_{t_0}^s f(l, y_1(l)) dl \right)^2 ds \end{aligned} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 4K(t-t_0)|y(t_0)|^2 \\ &+ 4K \int_{t_0}^t \int_{t_0}^s K \left(1 + |y_0|^2 \left(1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \right) \right) dl ds \end{aligned} \right] \\
&\leq \left[\begin{aligned} &\left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) \\ &+ 2K^2(t-t_0)^2 + 4K(t-t_0)[y(t_0)]^2 \end{aligned} \right] \\ &+ 4K^2 \left[|y_0|^2 \left(\frac{(t-t_0)^2}{2} + 2K \frac{(t-t_0)^3}{6} + 4K^2 \frac{(t-t_0)^4}{4} \right) \right] \end{aligned} \right] \\
&\leq \left[\begin{aligned} &|y_0|^2 \left(\begin{aligned} &1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \\ &+ 8K^3 \frac{(t-t_0)^3}{6} + 16K^4 \frac{(t-t_0)^4}{4} \end{aligned} \right) \\ &+ \left(\begin{aligned} &1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \\ &+ 8K^3 \frac{(t-t_0)^3}{6} + 16K^4 \frac{(t-t_0)^4}{4} \end{aligned} \right) \end{aligned} \right].
\end{aligned}$$

We next evaluate y_3

$$\begin{aligned}
|y_3(t)|^2 &= \left| y(t_0) + \int_{t_0}^t f(s, \bar{y}_3(s)) ds \right|^2 \\
&\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t |f(s, \bar{y}_3(s))|^2 ds.
\end{aligned} \tag{98}$$

Using the linear growth condition of f yields

$$\begin{aligned}
|y_3(t)|^2 &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K \left(1 + |\bar{y}_3(s)|^2 \right) ds \\
&\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t \left(y(t_0) + \int_{t_0}^s f(l, y_2(l)) dl \right)^2 ds \\
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 4K \int_{t_0}^t [y(t_0)]^2 ds \\ &+ 2K \int_{t_0}^t 2 \left(\int_{t_0}^s f(l, y_2(l)) dl \right)^2 ds \end{aligned} \right] \\
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 4K(t-t_0)[y(t_0)]^2 \\ &+ 4K^2 \int_{t_0}^t \int_{t_0}^s \left(1 + |y_0|^2 \left(\begin{aligned} &1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} \\ &+ 8K^3 \frac{(t-t_0)^3}{6} + 16K^4 \frac{(t-t_0)^4}{4} \end{aligned} \right) \right) dl ds \end{aligned} \right] \\
&\leq \left[\begin{aligned} &|y_0|^2 \left(\begin{aligned} &1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} + 8K^3 \frac{(t-t_0)^3}{6} \\ &+ 16K^4 \frac{(t-t_0)^4}{24} + 32K^5 \frac{(t-t_0)^5}{5!} + 64K^6 \frac{(t-t_0)^6}{6!} \end{aligned} \right) \\ &+ \left(\begin{aligned} &1 + 2K(t-t_0) + 4K^2 \frac{(t-t_0)^2}{2} + 8K^3 \frac{(t-t_0)^3}{6} \\ &+ 16K^4 \frac{(t-t_0)^4}{24} + 32K^5 \frac{(t-t_0)^5}{5!} + 64K^6 \frac{(t-t_0)^6}{6!} \end{aligned} \right) \end{aligned} \right]
\end{aligned}$$

When we proceed for n , we can get the following

$$\begin{aligned}
|y_n(t)|^2 &\leq 2|y(t_0)|^2 + 2 \int_{t_0}^t K \left(1 + |\bar{y}_n(s)|^2\right) ds \\
&\leq 2|y(t_0)|^2 + 2K(t-t_0) + 2K \int_{t_0}^t \left(y(t_0) + \int_{t_0}^s f(l, y_{n-1}(l)) dl\right)^2 ds \\
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 4K \int_{t_0}^t [y(t_0)]^2 ds \\ &+ 2K \int_{t_0}^t 2 \left(\int_{t_0}^s f(l, y_{n-1}(l)) dl\right)^2 ds \end{aligned} \right] \\
&\leq \left[\begin{aligned} &2|y(t_0)|^2 + 2K(t-t_0) + 4K(t-t_0) [y(t_0)]^2 \\ &+ 4K^2 \int_{t_0}^t \int_{t_0}^s \left(1 + |y_{n-1}(s)|^2\right) dl ds \end{aligned} \right]
\end{aligned} \tag{100}$$

After arranging, we can get

$$\begin{aligned}
|y_n(t)|^2 &\leq |y_0|^2 \sum_{m=0}^{2n} \frac{(2K)^m (t-t_0)^m}{m!} + \sum_{m=0}^{2n} \frac{(2K)^m (t-t_0)^m}{m!} \\
&\leq \left[\begin{aligned} &\exp \left[2K(t-t_0) \frac{\Gamma(2n+1, 2K(t-t_0))}{\Gamma(2n+1)} \right] |y_0|^2 \\ &+ \exp \left[2K(t-t_0) \frac{\Gamma(2n+1, 2K(t-t_0))}{\Gamma(2n+1)} \right] \end{aligned} \right].
\end{aligned} \tag{101}$$

Taking the limit $n \rightarrow \infty$, we have

$$\begin{aligned}
|y_n(t)|^2 &\leq \left[\begin{aligned} &\exp \left[2K(t-t_0) \frac{\Gamma(2n+1, 2K(t-t_0))}{\Gamma(2n+1)} \right] |y_0|^2 \\ &+ \exp \left[2K(t-t_0) \frac{\Gamma(2n+1, 2K(t-t_0))}{\Gamma(2n+1)} \right] \end{aligned} \right] \\
&\leq \exp(2K(t-t_0)) + \exp(2K(t-t_0)) |y_0|^2.
\end{aligned} \tag{102}$$

4 High order ordinary differential equations

In this section, we examine the case of high order ODE. We consider the following IVP

$$\begin{cases} y^{(n)}(t) = f(t, y(t)) \\ y^j(t_0) = y_0^j, 0 \leq j \leq n-1. \end{cases} \tag{103}$$

The above equation using the fundamental theorem of calculus is equivalent to

$$\begin{cases} y(t) = \sum_{j=0}^{n-1} y^j(t_0) \frac{(t-t_0)^j}{j!} + \int_{t_0}^t \int_{t_0}^{\tau_1} \int_{t_0}^{\tau_2} \dots \int_{t_0}^{\tau_{n-1}} f(\tau_n, y(\tau_n)) d\tau_n \dots d\tau_1 \\ y^j(t_0) = y_0^j, 0 \leq j \leq n-1. \end{cases} \tag{104}$$

Assuming that f satisfies the condition under Cauchy formula of repeated integral, the above yields

$$\begin{cases} y(t) = \sum_{j=0}^{n-1} y^j(t_0) \frac{(t-t_0)^j}{j!} + \frac{1}{(n-1)!} \int_{t_0}^t (t-\tau)^{n-1} f(\tau, y(\tau)) d\tau \\ y^j(t_0) = y_0^j, 0 \leq j \leq n-1. \end{cases} \quad (105)$$

Let $\tilde{y}_0(t) = \sum_{j=0}^{n-1} y^j(t_0) \frac{(t-t_0)^j}{j!}$. Thus, we have

$$\begin{cases} y(t) = \tilde{y}_0(t) + \frac{1}{(n-1)!} \int_{t_0}^t (t-\tau)^{n-1} f(\tau, y(\tau)) d\tau \\ y^j(t_0) = y_0^j, 0 \leq j \leq n-1. \end{cases} \quad (106)$$

To start our iterative method, we choose

$$y_0(t) = \tilde{y}_0(t), \quad (107)$$

$$\begin{cases} y_m(t) = \tilde{y}_0(t) + \frac{1}{(n-1)!} \int_{t_0}^t (t-\tau)^{n-1} f(\tau, \bar{y}_m(\tau)) d\tau, \\ \bar{y}_m(t) = \tilde{y}_0(t) + \frac{1}{(n-1)!} \int_{t_0}^t (t-\tau)^{n-1} f(\tau, y_{m-1}(\tau)) d\tau \\ y^j(t_0) = y_0^j, 0 \leq j \leq n-1. \end{cases} \quad (108)$$

We shall prove that $(y_m(t))$ is well-defined. However for simplicity, we show the proof when $t \in [t_0, t_0 + a]$

$$\begin{aligned} |y_m(t) - \tilde{y}_0(t)| &= \frac{1}{(n-1)!} \left| \int_{t_0}^t (t-\tau)^{n-1} f(\tau, \bar{y}_m(\tau)) d\tau \right| \\ &\leq \frac{1}{(n-1)!} \int_{t_0}^t (t-\tau)^{n-1} |f(\tau, \bar{y}_m(\tau))| d\tau \end{aligned} \quad (109)$$

Providing that f is bounded

$$\begin{aligned} |y_m(t) - \tilde{y}_0(t)| &\leq \frac{M}{(n-1)!} \int_{t_0}^t (t-\tau)^{n-1} d\tau \\ &\leq \frac{M}{n(n-1)!} (t-\tau)^n \Big|_{t_0}^t \\ &\leq \frac{M}{n!} (t-t_0)^n \leq \frac{M}{n!} a^n \\ &\leq \frac{M}{n!} a^n < b, a \leq \left(\frac{bn!}{M} \right)^{\frac{1}{n}}. \end{aligned} \quad (110)$$

We shall show that $(y_m(t))$ is uniformly bounded.

$$\begin{aligned}
|y_m(t)| &= \left| \tilde{y}_0(t) + \frac{1}{(n-1)!} \int_{t_0}^t (t-\tau)^{n-1} f(\tau, \bar{y}_m(\tau)) d\tau \right| \\
&\leq |\tilde{y}_0(t)| + \frac{1}{(n-1)!} \left| \int_{t_0}^t (t-\tau)^{n-1} f(\tau, \bar{y}_m(\tau)) d\tau \right| \\
&\leq |\tilde{y}_0(t)| + \frac{1}{(n-1)!} \frac{(t-t_0)^n}{n} M \\
&\leq |\tilde{y}_0(t)| + \frac{M}{n!} a^n.
\end{aligned} \tag{111}$$

Indeed $\tilde{y}_0(t)$ is continuously differentiable

$$\tilde{y}_0(t) = \sum_{j=0}^{n-1} y^j(t_0) \frac{(t-t_0)^j}{j!}, \tag{112}$$

then we apply absolute value on both sides

$$\begin{aligned}
|\tilde{y}_0(t)| &= \left| \sum_{j=0}^{n-1} y^j(t_0) \frac{(t-t_0)^j}{j!} \right| \\
&\leq \sum_{j=0}^{n-1} |y^j(t_0)| \frac{a^j}{j!} \\
&\leq \max_{0 \leq j \leq n-1} |y^j(t_0)| \sum_{j=0}^{n-1} \frac{a^j}{j!} \\
&\leq \max_{0 \leq j \leq n-1} |y^j(t_0)| e^a.
\end{aligned} \tag{113}$$

Replacing all above together yields

$$\begin{aligned}
|y_m(t)| &\leq \max_{0 \leq j \leq n-1} |y^j(t_0)| e^a + \frac{M}{n!} a^n \\
&\leq e^a \left(\max_{0 \leq j \leq n-1} |y^j(t_0)| + M \right).
\end{aligned} \tag{114}$$

Therefore, $(y_m(t))$ is uniformly bounded. We now show that $(y_m(t))$ is equicontinuous. Then, we write the following such that $t_1 > t_2 \in [t_0, t_0 + a]$

$$\begin{aligned}
|y_m(t_1) - y_m(t_2)| &= \left| \begin{aligned} &\sum_{j=0}^{n-1} y^{(j)}(t_0) \frac{(t_1 - t_0)^j}{j!} - \sum_{j=0}^{n-1} y^{(j)}(t_0) \frac{(t_2 - t_0)^j}{j!} \\ &+ \frac{1}{(n-1)!} \int_{t_0}^{t_1} (t_1 - \tau)^{n-1} f(\tau, y_m(\tau)) d\tau \\ &- \frac{1}{(n-1)!} \int_{t_0}^{t_2} (t_2 - \tau)^{n-1} f(\tau, y_m(\tau)) d\tau \end{aligned} \right| \quad (115) \\
&\leq \left| \sum_{j=0}^{n-1} y^{(j)}(t_0) \frac{(t_1 - t_0)^j}{j!} - \sum_{j=0}^{n-1} y^{(j)}(t_0) \frac{(t_2 - t_0)^j}{j!} \right| \\
&\quad + \frac{1}{(n-1)!} \left| \begin{aligned} &\int_{t_0}^{t_2} (t_1 - \tau)^{n-1} f(\tau, y_m(\tau)) d\tau \\ &- \int_{t_0}^{t_2} (t_2 - \tau)^{n-1} f(\tau, y_m(\tau)) d\tau \\ &+ \int_{t_1}^{t_2} (t_1 - \tau)^{n-1} f(\tau, y_m(\tau)) d\tau \end{aligned} \right| \\
&\leq \left| \sum_{j=0}^{n-1} y^{(j)}(t_0) \frac{(t_1 - t_0)^j}{j!} - \sum_{j=0}^{n-1} y^{(j)}(t_0) \frac{(t_2 - t_0)^j}{j!} \right| \\
&\quad + \frac{1}{(n-1)!} \int_{t_0}^{t_2} \left| (t_1 - \tau)^{n-1} - (t_2 - \tau)^{n-1} \right| |f(\tau, y_m(\tau))| d\tau \\
&\quad + \frac{1}{(n-1)!} \left| \int_{t_1}^{t_2} (t_1 - \tau)^{n-1} f(\tau, y_m(\tau)) d\tau \right| \\
&\leq \left| \sum_{j=0}^{n-1} \max_{0 \leq j \leq n-1} |y^{(j)}(t_0)| \left\{ \frac{(t_1 - t_0)^j}{j!} - \frac{(t_2 - t_0)^j}{j!} \right\} \right| \\
&\quad + \frac{1}{(n-1)!} \left| \frac{(t_1 - t_0)^n}{n} - \frac{(t_1 - t_2)^n}{n} - \frac{(t_2 - t_0)^n}{n} \right| M \\
&\quad + \frac{1}{(n-1)!} \frac{(t_1 - t_2)^n}{n} M.
\end{aligned}$$

Noting that the function $(t - t_0)^j$ is differentiable, therefore, by the Mean value theorem, we can find $c \in [t_1 - t_0, t_2 - t_0]$ such that $j(c - t_0)^{j-1}(t_1 - t_2) = (t_1 - t_0)^j - (t_2 - t_0)^j$. Therefore

$$\begin{aligned}
|y_m(t_1) - y_m(t_2)| &\leq \left| \sum_{j=0}^{n-1} \max_{0 \leq j \leq n-1} |y^j(t_0)| \left\{ \frac{j(c-t_0)^{j-1}}{j!} (t_1 - t_2) \right\} \right| \\
&\quad + \frac{1}{(n-1)!} \left| \frac{(t_1 - t_0)^n}{n} - \frac{(t_1 - t_2)^n}{n} - \frac{(t_2 - t_0)^n}{n} \right| M \\
&\quad + \frac{1}{(n-1)!} \frac{(t_1 - t_2)^n}{n} M \\
&\leq \left| \sum_{j=0}^{\infty} \max_{0 \leq j \leq n-1} |y^j(t_0)| \left\{ \frac{(c-t_0)^{j-1}}{(j-1)!} (t_1 - t_2) \right\} \right| \\
&\quad + \frac{(c-t_0)^{n-1}}{(n-1)!} (t_1 - t_2) M + \frac{(t_1 - t_2)^n}{n!} M \\
&\leq \left\{ \max_{0 \leq j \leq n-1} |y^j(t_0)| e^{(c-t_0)} (t_1 - t_2) \right\} + \frac{(c-t_0)^{n-1}}{(n-1)!} (t_1 - t_2) M + \frac{(t_1 - t_2)^n}{n!} M.
\end{aligned}$$

If $t_1 - t_0 \leq 1$, then

$$|y_m(t_1) - y_m(t_2)| \leq \left(\left\{ \max_{0 \leq j \leq n-1} |y^j(t_0)| e^{(c-t_0)} \right\} + \frac{(c-t_0)^{n-1}}{(n-1)!} M + \frac{M}{n!} \right) (t_1 - t_2) < \varepsilon, \quad (116)$$

implies

$$\delta < \frac{\varepsilon}{\left(\left\{ \max_{0 \leq j \leq n-1} |y^j(t_0)| e^{(c-t_0)} \right\} + \frac{(c-t_0)^{n-1}}{(n-1)!} M + \frac{M}{n!} \right)}. \quad (117)$$

If $t_1 - t_2 > 1$, then

$$|y_m(t_1) - y_m(t_2)| \leq \left(\left\{ \max_{0 \leq j \leq n-1} |y^j(t_0)| e^{(c-t_0)} \right\} + \frac{(c-t_0)^{n-1}}{(n-1)!} M + \frac{M}{n!} \right) (t_1 - t_2)^n < \varepsilon, \quad (118)$$

implies

$$\delta < \left[\frac{\varepsilon}{\left(\left\{ \max_{0 \leq j \leq n-1} |y^j(t_0)| e^{(c-t_0)} \right\} + \frac{(c-t_0)^{n-1}}{(n-1)!} M + \frac{M}{n!} \right)} \right]^{1/n}. \quad (119)$$

Under the above condition, uniform continuity holds. By the Arzela-Ascoli theorem, some subsequence $(y_{m_k}(t))$ of $y_m(t)$ converges uniformly to y [[10],[11]]. The function $f(\cdot, y(\cdot))$ is uniformly limit of $f(\cdot, y_{m_k}(\cdot))$ on the $[t_0, t_0 + a]$. Let $\varepsilon > 0$, since f is continuous on the compact set, we conclude that it is also uniformly continuous. Thus, by taking $\delta_1 > 0$ such that

$$|f(\tau, y_1) - f(\tau, y_2)| < \varepsilon \text{ if } |y_1 - y_2| < \delta. \quad (120)$$

But we have that $(y_{m_k}(t))$ converges uniformly to y , there exists $M_1 \in M$ such that

$$|y_{m_k}(\tau) - y(\tau)| < \delta_1, \forall \tau \in [t_0, t_0 + a], k \geq M_1. \quad (121)$$

Thus if $k > M$, then

$$|y_{m_k}(\tau) - y(\tau)| < \varepsilon. \quad (122)$$

Since f is uniformly continuous, we have $k > M$

$$\left| f(\tau, y_{m_k}) \frac{(t-\tau)^{n-1}}{(n-1)!} - f(\tau, y) \frac{(t-\tau)^{n-1}}{(n-1)!} \right| < \varepsilon. \quad (123)$$

We now fix $t \in [t_0, t_0 + a]$. For $t = t_0$, we obtain the exact solution. If $t > t_0, \forall k$ large enough

$$y_{m_k}(t) = \tilde{y}(t_0) + \frac{1}{(n-1)!} \int_{t_0}^{t_1} (t-\tau)^{n-1} f(\tau, y_{m_k}(\tau)) d\tau. \quad (124)$$

Taking limit

$$\lim_{k \rightarrow \infty} y_{m_k}(t) = \tilde{y}(t_0) + \lim_{k \rightarrow \infty} \frac{1}{(n-1)!} \int_{t_0}^{t_1} (t-\tau)^{n-1} f(\tau, y_{m_k}(\tau)) d\tau \quad (125)$$

and then

$$y_m(t) = \tilde{y}(t_0) + \frac{1}{(n-1)!} \int_{t_0}^{t_1} (t-\tau)^{n-1} \lim_{k \rightarrow \infty} f(\tau, y_{m_k}(\tau)) d\tau. \quad (126)$$

By uniform convergence theorem and uniform of $f(\cdot, y_{m_k}(\cdot))$ the integral converges to

$$\frac{1}{(n-1)!} \int_{t_0}^{t_1} (t-\tau)^{n-1} f(\tau, y_m(\tau)) d\tau. \quad (127)$$

Therefore

$$y_m(t) = \tilde{y}(t_0) + \frac{1}{(n-1)!} \int_{t_0}^{t_1} (t-\tau)^{n-1} f(\tau, y_m(\tau)) d\tau. \quad (128)$$

Therefore the solution exists. We shall next show the uniqueness.

Example 6 We next consider the following problem

$$\begin{cases} y''(t) = -4y \\ y(0) = 0, y'(0) = 2. \end{cases} \quad (129)$$

The exact solution is

$$y(t) = \sin(2t). \quad (130)$$

Using Cauchy formula yields

$$\left\{ \begin{array}{l} y_1(t) = \sum_{j=0}^{n-1} y^j(t_0) \frac{(t-t_0)^j}{j!} + \int_0^t (t-\tau) f(\tau, \bar{y}_1(\tau)) d\tau \\ \quad = 2t + \int_0^t (t-\tau) - 8\tau d\tau \\ \quad = 2t - \frac{4}{3}t^3 \\ \bar{y}_1(t) = y(0) + 2t + \int_0^t (t-\tau) 0 d\tau \\ \quad = 2t \end{array} \right. \quad (131)$$

$$\left\{ \begin{array}{l} y_2(t) = 2t - 4 \int_0^t (t-\tau) \left(2\tau - \frac{4}{3}\tau^3 \right) d\tau \\ \quad = 2t - \frac{4}{3}t^3 + \frac{4}{15}t^5 \\ \bar{y}_2(t) = \tilde{y}_0(t) + \int_0^t (t-\tau) f(\tau, \bar{y}_1(\tau)) d\tau \\ \quad = 2t + \int_0^t (t-\tau) (-8\tau) d\tau \\ \quad = 2t - \frac{4}{3}t^3 \end{array} \right. \quad (132)$$

$$\left\{ \begin{array}{l} y_3(t) = 2t - 4 \int_0^t (t-\tau) \left(2\tau - \frac{4}{3}\tau^3 + \frac{4}{15}\tau^5 \right) d\tau \\ \quad = 2t - \frac{4}{3}t^3 + \frac{4}{15}t^5 - \frac{8t^7}{315} \\ \bar{y}_3(t) = \tilde{y}_0(t) + \int_0^t (t-\tau) f(\tau, \bar{y}_2(\tau)) d\tau \\ \quad = 2t - 4 \int_0^t (t-\tau) \left(2\tau - \frac{4}{3}\tau^3 \right) d\tau \\ \quad = 2t - \frac{4}{3}t^3 + \frac{4}{15}t^5 \end{array} \right. \quad (133)$$

For our iteration, we can have the following:

$$y_n(t) = \sum_{j=0}^{n+1} \frac{(-1)^j 2^{2j+1} t^{2j+1}}{(1+2j)!}. \quad (134)$$

It can be easily seen that

$$\lim_{n \rightarrow \infty} y_n(t) = \sin(2t). \quad (135)$$

The Cauchy sequence comparison test between Picard and our iteration is performed in Figure 3 and 4.

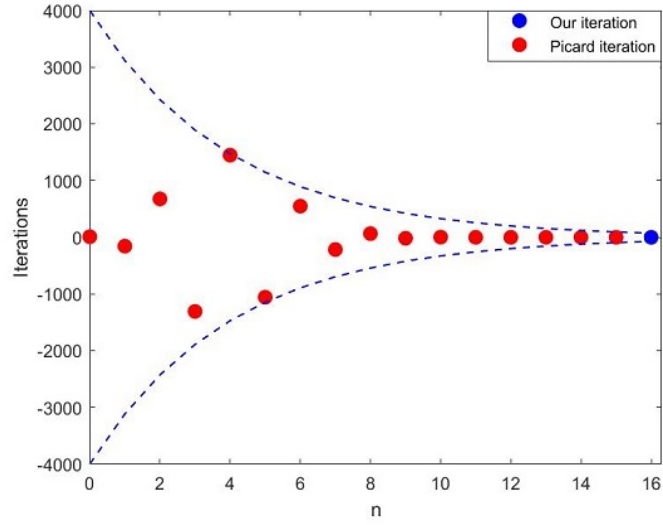


Figure 3. Cauchy sequence comparison test between Picard and our iteration.

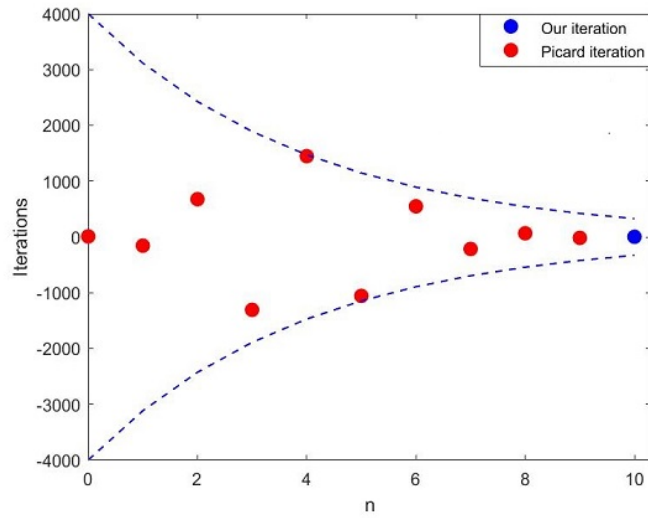


Figure 4. Cauchy sequence comparison test between Picard and our iteration.

5 Initial value problems with Caputo-Fabrizio derivative

In this section, we extend the proposed sequence to the Cauchy problem with the Caputo-Fabrizio fractional derivative. We consider the following nonlinear ordinary differential equation

$$\begin{cases} {}_0^{CF}D_t^\alpha y(t) = f(t, y(t)), t > 0 \\ y(0) = y_0, t = 0 \end{cases} . \quad (136)$$

Integrating above yields

$$\begin{cases} y(t) = y(0) + (1 - \alpha) f(t, y(t)) + \alpha \int_0^t f(\tau, y(\tau)) d\tau, t > 0 \\ y(0) = y_0, t = 0 \end{cases} . \quad (137)$$

We create the following iteration

$$\begin{cases} y_n(t) = y(0) + (1 - \alpha) f(t, y_n(t)) + \alpha \int_0^t f(\tau, y_n(\tau)) d\tau, t > 0 \\ y(0) = y_0, t = 0 \end{cases} . \quad (138)$$

To avoid explicit case, we replace for the right hand side $y_n(t)$ by $\bar{y}_n(t)$ to obtain

$$\begin{cases} y_n(t) = y(0) + (1 - \alpha) f(t, \bar{y}_n(t)) + \alpha \int_0^t f(\tau, \bar{y}_n(\tau)) d\tau, t > 0 \\ y(0) = y_0, t = 0 \end{cases} . \quad (139)$$

Here

$$\bar{y}_n(t) = y(0) + (1 - \alpha) f(t, y_{n-1}(t)) + \alpha \int_0^t f(\tau, y_{n-1}(\tau)) d\tau. \quad (140)$$

Therefore, the iteration is given as

$$\begin{cases} y_n(t) = y(0) + (1 - \alpha) f(t, \bar{y}_n(t)) + \alpha \int_0^t f(\tau, \bar{y}_n(\tau)) d\tau, t > 0 \\ \bar{y}_n(t) = y(0) + (1 - \alpha) f(t, y_{n-1}(t)) + \alpha \int_0^t f(\tau, y_{n-1}(\tau)) d\tau \\ y(0) = y_0, t = 0 \end{cases} . \quad (141)$$

We should present few step to reach the existence and uniqueness of the equation.

Theorem 7 (Cauchy-Peano): *Let us assume that $f : [t_0 - c, t_0 + c] \times \bar{B}(\alpha, \beta) \rightarrow \mathbb{R}$ is continuous and bounded by $M > 0$ then the Cauchy problem with Caputo-Fabrizio derivative has a solution.*

Proof. Without loss of generality, we consider the analysis within $[t_0, t_0 + \lambda]$. For each $n \in N$, we want to show that (y_n) converges toward (y) which is the solution of our equation. For a fixed $n \in N$,

$$\begin{aligned} |y_n(t) - y(0)| &\leq (1 - \alpha) |f(t, \bar{y}_n(t))| + \alpha \int_0^t |f(\tau, \bar{y}_n(\tau))| d\tau \\ &\leq (1 - \alpha) M + \alpha M t \\ &\leq (1 - \alpha) M + \alpha M a \\ &\leq M [(1 - \alpha) + \alpha a] \\ &\leq M [1 + \alpha a]. \end{aligned} \quad (142)$$

Then, we get

$$\begin{aligned}
|y_n(t_1) - y_n(t_2)| &\leq (1 - \alpha) |f(t_1, y_n(t_1)) - f(t_2, y_n(t_2))| \\
&\quad + \alpha \left| \int_0^{t_1} f(\tau, y_n(\tau)) d\tau - \int_0^{t_2} f(\tau, y_n(\tau)) d\tau \right| \\
&\leq (1 - \alpha) L |y_n(t_1) - y_n(t_2)| + \alpha \int_{t_2}^{t_1} |f(\tau, y_n(\tau))| d\tau \\
&\leq (1 - \alpha) L |y_n(t_1) - y_n(t_2)| + \alpha M (t_1 - t_2).
\end{aligned} \tag{143}$$

Arranging above leads to

$$(1 + (\alpha - 1)L) |y_n(t_1) - y_n(t_2)| \leq \alpha M (t_1 - t_2). \tag{144}$$

Therefore, we write

$$\begin{aligned}
|y_n(t_1) - y_n(t_2)| &\leq \frac{\alpha M}{(1 + (\alpha - 1)L)} (t_1 - t_2) < \varepsilon \\
\delta &< \varepsilon \frac{(1 + (\alpha - 1)L)}{\alpha M}.
\end{aligned} \tag{145}$$

Here, we needed f to be bounded and Lipschitz with Lipschitz constant L . Therefore, the sequence is uniform equicontinuous. By the Arzela-Ascoli theorem, there exists some sequence (y_{n_i}) of (y_n) that converges uniformly to $\bar{y}(t)$ on $[t_0, t_0 + \lambda]$. Indeed, the function $f(\cdot, y_n(\cdot))$ is the uniform limit of $f(\cdot, y_{n_i}(\cdot))$. Let fix $[t_0, t_0 + \lambda]$. $\forall l$ sufficiently large

$$y_{n_i}(t) = y(0) + (1 - \alpha) f(t, y_{n_i}(t)) + \alpha \int_0^t f(\tau, y_{n_i}(\tau)) d\tau. \tag{146}$$

Applying the limit on both sides

$$\lim_{l \rightarrow \infty} y_{n_i}(t) = y(0) + (1 - \alpha) \lim_{l \rightarrow \infty} f(t, y_{n_i}(t)) + \alpha \int_0^t \lim_{l \rightarrow \infty} f(\tau, y_{n_i}(\tau)) d\tau. \tag{147}$$

Then we get

$$y(t) = y(0) + (1 - \alpha) f(t, y(t)) + \alpha \int_0^t f(\tau, y(\tau)) d\tau. \tag{148}$$

For uniqueness, we define the following norm

$$\|\varphi\|_{\Omega} = \sup \left\{ e^{-L(t-t_0)} |\varphi(t)| \mid t \in [t_0 - \lambda, t_0 + \lambda] \right\}. \tag{149}$$

Let Λ be the set of continuous function from $[t_0 - \lambda, t_0 + \lambda]$ to $\bar{B}(\alpha, \beta)$. The above norm is equivalent to the common supremum norm $\|\cdot\|_{\infty}$ on $C[[t_0 - \lambda, t_0 + \lambda]]$ since

$$M_1 \|\varphi\|_{\infty} \leq \|\varphi\|_{\Omega} \leq M_2 \|\varphi\|_{\infty}. \tag{150}$$

Ω is therefore equipped with

$$d(y_1, y_2) = \|y_1 - y_2\|_{\Omega} \quad (151)$$

is complete metric space. We next define the following mapping

$$(\Pi_1 y)(t) = y(0) + (1 - \alpha) f(t, y(t)) + \alpha \int_0^t f(\tau, y(\tau)) d\tau. \quad (152)$$

Indeed if $y \in \Lambda$ then $(\Pi_1 y)$ is valid. Since $y \in \Lambda$, then obviously $(\Pi_1 y)$ is continuous $\forall t \in [t_0 - \lambda, t_0 + \lambda]$

$$\begin{aligned} |(\Pi_1 y)(t) - y(0)| &= (1 - \alpha) |f(t, y(t))| + \alpha \int_0^t |f(\tau, y(\tau))| d\tau \\ &\leq (1 - \alpha) M + \alpha M \lambda < \beta, \\ \lambda &< \frac{\beta}{\alpha M} + \frac{\alpha - 1}{\alpha}. \end{aligned} \quad (153)$$

Therefore, with the above condition $(\Pi_1 y)(t) \in \overline{B}(\alpha, \beta)$, therefore $(\Pi_1 y)(t) \in \Lambda$. Let $y_1, y_2 \in \Lambda$

$$\begin{aligned} \|\Pi_1 y_1 - \Pi_1 y_2\|_{\Omega} &= \sup \left\{ \begin{aligned} &e^{-Lt} |(1 - \alpha) (f(t, y_1(t)) - f(t, y_2(t)))| \\ &+ \alpha \int_0^t \{f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))\} d\tau \end{aligned} \right\} \\ &\leq \sup \left\{ \begin{aligned} &e^{-Lt} (1 - \alpha) L \|y_1 - y_2\| \\ &+ \alpha e^{-Lt} \int_0^t \{f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))\} d\tau \end{aligned} \right\} \\ &\leq (1 - \alpha) L \|y_1 - y_2\|_{\Omega} + \alpha e^{-Lt} L \int_0^t \|y_1 - y_2\|_{\Omega} e^{L\tau} d\tau \\ &\leq (1 - \alpha) L \|y_1 - y_2\|_{\Omega} + \alpha \|y_1 - y_2\|_{\Omega} (1 - e^{-Lt}) \\ &\leq (1 - \alpha) L \|y_1 - y_2\|_{\Omega} + \alpha \|y_1 - y_2\|_{\Omega} \\ &\leq ((1 - \alpha) L + \alpha) \|y_1 - y_2\|_{\Omega}. \end{aligned} \quad (154)$$

To have a contraction, we can write

$$((1 - \alpha) L + \alpha) < 1 \Rightarrow L < 1. \quad (155)$$

Therefore, our equation has a unique solution $y(t)$. ■

Notice that all the presented proof presented above, the formula of $y_n(t)$ on the right hand side was not considered. In the next section, we present the proof by taking into account the definition of $y_n(t)$ on the right hand side. However, note that this will effect mostly equicontinuity of the

defined sequence

$$\begin{aligned}
|y_n(t_1) - y_n(t_2)| &= \left| \begin{aligned} &(1 - \alpha)(f(t_1, \bar{y}_n(t_1)) - f(t_2, \bar{y}_n(t_2))) \\ &+ \alpha \int_0^{t_1} f(\tau, \bar{y}_n(\tau)) d\tau - \int_0^{t_2} f(\tau, \bar{y}_n(\tau)) d\tau \end{aligned} \right| \quad (156) \\
&\leq (1 - \alpha) |f(t_1, \bar{y}_n(t_1)) - f(t_2, \bar{y}_n(t_2))| + \alpha \int_{t_1}^{t_2} |f(\tau, \bar{y}_n(\tau))| d\tau \\
&\leq (1 - \alpha) L |\bar{y}_n(t_1) - \bar{y}_n(t_2)| + \alpha M |t_1 - t_2| \\
&\leq (1 - \alpha) L \left| \int_0^{t_1} f(\tau, y_{n-1}(\tau)) d\tau - \int_0^{t_2} f(\tau, y_{n-1}(\tau)) d\tau \right| + \alpha M |t_1 - t_2| \\
&\leq (1 - \alpha) L \int_{t_1}^{t_2} |f(\tau, y_{n-1}(\tau))| d\tau + \alpha M |t_1 - t_2| \\
&\leq (1 - \alpha) L \int_{t_1}^{t_2} |f(\tau, y_{n-1}(\tau))| d\tau + \alpha M |t_1 - t_2| \\
&\leq (1 - \alpha) LM |t_1 - t_2| + \alpha M \\
&\leq |t_1 - t_2| ((1 - \alpha) LM + \alpha M) < \varepsilon.
\end{aligned}$$

Then

$$\delta < \frac{\varepsilon}{M(L + \alpha)}. \quad (157)$$

6 Application of the linear growth condition for Picard and our iteration

In this section, we examine the convergence of Picard iteration and the suggested iteration. If the function $f(t, y(t))$ satisfies the linear growth condition that is $\forall t \in [t_0, T]$, then, by recursive, we have

$$\begin{cases} {}^{CF}D_t^\alpha y(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}. \quad (158)$$

or equivalently

$$\begin{aligned}
|y_1(t)|^2 &= \left| y(t_0) + (1 - \alpha) f(t, y_0(t)) + \alpha \int_{t_0}^t f(s, y_0(s)) ds \right|^2 \quad (159) \\
&\leq 3|y(t_0)|^2 + 3(1 - \alpha)^2 |f(t, y_0(t))|^2 + 3\alpha^2 \int_{t_0}^t |f(s, y_0(s))|^2 ds.
\end{aligned}$$

Using the linear growth condition of f yields

$$\begin{aligned}
|y_1(t)|^2 &\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K \left(1 + |y_0(t)|^2\right) + 3\alpha^2 \int_{t_0}^t K \left(1 + |y_0(s)|^2\right) ds & (160) \\
&\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K \left(1 + |y_0(t)|^2\right) + 3\alpha^2 K(t-t_0) + 3\alpha^2 K \int_{t_0}^t |y(t_0)|^2 ds \\
&\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K \left(1 + |y_0(t)|^2\right) + 3\alpha^2 K(t-t_0) + 3\alpha^2 K(t-t_0) |y(t_0)|^2 \\
&\leq 3 \left[\begin{array}{c} |y(t_0)|^2 \left(1 + (1-\alpha)^2 K + \alpha^2 K(t-t_0)\right) \\ + 1(1-\alpha)^2 K + \alpha^2 K(t-t_0) \end{array} \right] \\
&\leq \left[\begin{array}{c} 3|y(t_0)|^2 \left(1 + 3(1-\alpha)^2 K + 3\alpha^2 K(t-t_0)\right) \\ + \left(1 + 3(1-\alpha)^2 K + 3\alpha^2 K(t-t_0)\right) \end{array} \right] \\
&\leq \left[\begin{array}{c} 3|y(t_0)|^2 (B_1 + B_1(t-t_0)) \\ + (B_1 + B_1(t-t_0)) \end{array} \right]
\end{aligned}$$

where $B_1 = \max \left\{ 1 + 3(1-\alpha)^2 K, 3\alpha^2 K \right\}$. We now evaluate y_2

$$\begin{aligned}
|y_2(t)|^2 &= \left| y(t_0) + (1-\alpha) f(t, y_1(t)) + \alpha \int_{t_0}^t f(s, y_1(s)) ds \right|^2 & (161) \\
&\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 |f(t, y_1(t))|^2 + 3\alpha^2 \int_{t_0}^t |f(s, y_1(s))|^2 ds.
\end{aligned}$$

Using the linear growth condition for the function $f(s, y_1(s))$ yields

$$\begin{aligned}
|y_2(t)|^2 &\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K \left(1 + |y_1(t)|^2\right) + 3\alpha^2 K \int_{t_0}^t \left(1 + |y_1|^2\right) ds & (162) \\
&\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K \left(1 + \left[\begin{array}{c} 3|y(t_0)|^2 \left(1 + 3(1-\alpha)^2 K + 3\alpha^2 K(t-t_0)\right) \\ + \left(1 + 3K(1-\alpha)^2 + 3\alpha^2 K(t-t_0)\right) \end{array} \right] \right) \\
&\quad + 3\alpha^2 K \int_{t_0}^t \left(1 + \left[\begin{array}{c} 3|y(t_0)|^2 \left(1 + 3(1-\alpha)^2 K + 3\alpha^2 K(t-t_0)\right) \\ + \left(1 + 3K(1-\alpha)^2 + 3\alpha^2 K(t-t_0)\right) \end{array} \right] \right) ds
\end{aligned}$$

$$\begin{aligned}
&\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K + 3(1-\alpha)^2 K \left[\begin{aligned} &3|y(t_0)|^2 \left(1 + 3(1-\alpha)^2 K + 3\alpha^2 K(t-t_0)\right) \\ &+ \left(1 + 3K(1-\alpha)^2 + 3\alpha^2 K(t-t_0)\right) \end{aligned} \right] \\
&\quad + 3\alpha^2 K(t-t_0) + 3\alpha^2 K \left[\begin{aligned} &3|y(t_0)|^2 \left(\left(1 + 3(1-\alpha)^2 K\right)(t-t_0) + 3\alpha^2 K \frac{(t-t_0)^2}{2}\right) \\ &+ \left(\left(1 + 3(1-\alpha)^2 K\right)(t-t_0) + 3\alpha^2 K \frac{(t-t_0)^2}{2}\right) \end{aligned} \right] \\
&\leq \left[B_2 + B_2(t-t_0) + B_2 \frac{(t-t_0)^2}{2} \right] + 3|y(t_0)|^2 \left[B_2 + B_2(t-t_0) + B_2 \frac{(t-t_0)^2}{2} \right] \\
&\leq B_2 \left(1 + (t-t_0) + \frac{(t-t_0)^2}{2} \right) + 3B_2 |y(t_0)|^2 \left(1 + (t-t_0) + \frac{(t-t_0)^2}{2} \right)
\end{aligned}$$

where $B_2 = \left\{ 1 + 6(1-\alpha)^2 + 9(1-\alpha)^4 K, 3\alpha^2 K \left(1 + 3(1-\alpha)^2 K \right), 9\alpha^4 K^2 \right\}$ When we proceed for n , we can get the following

$$\begin{aligned}
|y_n(t)|^2 &\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 |f(t, y_{n-1}(t))|^2 + 3\alpha^2 \int_{t_0}^t |f(s, y_{n-1}(s))|^2 ds \quad (163) \\
&\leq 3|y(t_{n-1})|^2 + 3(1-\alpha)^2 K \left(1 + |y_{n-1}(s)|^2 \right) + 3\alpha^2 K \int_{t_0}^t \left(1 + |y_{n-1}(s)|^2 \right) ds \\
&\leq 3|y(t_{n-1})|^2 + 3(1-\alpha)^2 K + 3\alpha^2 K(t-t_0) \\
&\quad + 3(1-\alpha)^2 K |y_{n-1}(s)|^2 + 3\alpha^2 K \int_{t_0}^t \left(|y_{n-1}(s)|^2 \right) ds
\end{aligned}$$

After arranging, we can get

$$\begin{aligned}
|y_n(t)|^2 &\leq \left[3|y_0|^2 \sum_{m=0}^n B_{\alpha,K} \frac{(t-t_0)^m}{m!} + \sum_{m=0}^n B_{\alpha,K} \frac{(t-t_0)^m}{m!} \right] \quad (164) \\
&\leq 3B_{\alpha,K} \exp[K(t-t_0)] |y_0|^2 + B_{\alpha,K} \exp[K(t-t_0)].
\end{aligned}$$

where $B_{\alpha,K}$ is the maximum value of the coefficients. Taking the limit $n \rightarrow \infty$, we have

$$|y_n(t)|^2 \leq 3B_{\alpha,K} \exp[K(t-t_0)] |y_0|^2 + B_{\alpha,K} \exp[K(t-t_0)]. \quad (165)$$

We can verify convergence for our sequence. If the function $f(t, y(t))$ satisfies the linear growth condition that is $\forall t \in [t_0, T]$, then, by recursive, we have

$$\begin{cases} {}^C D_t^\alpha (t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}. \quad (166)$$

or equivalently

$$\begin{aligned}
|y_1(t)|^2 &= \left| y(t_0) + (1-\alpha)f(t, y_0(t)) + \alpha \int_{t_0}^t f(s, \bar{y}_1(s)) ds \right|^2 \\
&\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K \left(1 + |\bar{y}_1(t)|^2\right) + 3\alpha^2 \int_{t_0}^t |f(s, \bar{y}_1(s))|^2 ds.
\end{aligned} \tag{167}$$

Using the linear growth condition of f yields

$$\begin{aligned}
|y_1(t)|^2 &\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K \left(1 + |\bar{y}_1(t)|^2\right) + 3\alpha^2 \int_{t_0}^t K \left(1 + |\bar{y}_1(s)|^2\right) ds \\
&\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 K + \left(3(1-\alpha)^2 K \left(|y(t_0)| + \int_{t_0}^s |f(l, y_0(l))| dl\right)^2\right) \\
&\quad + 3\alpha^2 K(t-t_0) + 3\alpha^2 K \int_{t_0}^t \left(|y(t_0)| + \int_{t_0}^s |f(l, y_0(l))| dl\right)^2 ds \\
&\leq 3(1-\alpha)^2 K + \left(3\alpha^2 K + 6(1-\alpha)^2 K\right) (t-t_0) + 3|y(t_0)|^2 \\
&\quad + 6(1-\alpha)^2 K |y(t_0)|^2 + 6(1-\alpha)^2 K (t-t_0) |y(t_0)|^2 \\
&\quad + 6\alpha^2 K |y(t_0)|^2 (t-t_0) + 6\alpha^2 K^2 \frac{(t-t_0)^2}{2} + 6\alpha^2 K^2 |y_0(t)|^2 \frac{(t-t_0)^2}{2} \\
&\leq 3 \left(1 + 2(1-\alpha)^2 K + \left(2\alpha^2 K + 2(1-\alpha)^2 K\right) (t-t_0) + 2\alpha^2 K^2 \frac{(t-t_0)^2}{2}\right) \\
&\quad + 3|y(t_0)|^2 \left(1 + 2(1-\alpha)^2 K + \left(2\alpha^2 K + 2(1-\alpha)^2 K\right) (t-t_0) + 2\alpha^2 K^2 \frac{(t-t_0)^2}{2}\right).
\end{aligned} \tag{168}$$

We next proceed for n

$$\begin{aligned}
|y_n(t)|^2 &\leq 3|y(t_0)|^2 + 3(1-\alpha)^2 |f(t, \bar{y}_{n-1}(t))|^2 + 3\alpha^2 \int_{t_0}^t |f(s, \bar{y}_{n-1}(s))|^2 ds \\
&\leq 3|y(t_{n-1})|^2 + 3(1-\alpha)^2 K \left(1 + |\bar{y}_{n-1}(s)|^2\right) + 3\alpha^2 K \int_{t_0}^t \left(1 + |\bar{y}_{n-1}(s)|^2\right) ds \\
&\leq 3|y(t_{n-1})|^2 + 3(1-\alpha)^2 K + 3\alpha^2 K(t-t_0) \\
&\quad + 3(1-\alpha)^2 K |y_{n-1}(s)|^2 + 3\alpha^2 K \int_{t_0}^t \left(|y_{n-1}(s)|^2\right) ds
\end{aligned} \tag{169}$$

After arranging, we can get

$$\begin{aligned} |y_n(t)|^2 &\leq \left[3|y_0|^2 \sum_{m=0}^{n+1} \bar{B}_{\alpha,K} \frac{(t-t_0)^m}{m!} + \sum_{m=0}^{n+1} \bar{B}_{\alpha,K} \frac{(t-t_0)^m}{m!} \right] \\ &\leq 3\bar{B}_{\alpha,K} \exp[K(t-t_0)] |y_0|^2 + \bar{B}_{\alpha,K} \exp[K(t-t_0)]. \end{aligned} \quad (170)$$

7 Conclusion

The class with a classical derivative and the class with a fractional derivative based on the Caputo-Fabrizio derivative were the two types of initial value issues for ordinary nonlinear differential equations that were taken into consideration. We offered an iterative approach for resolving nonlinear differential equations. For both situations, we have demonstrated that, under some circumstances, the sequence converges toward a solution. By resolving a few linear and nonlinear equations, we were able to compare our findings to those of Picard. The comparison's findings demonstrated that the recommended iteration converges more quickly than the Picard iteration.

References

- [1] Peano, G. (1890). Demonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*. 37 (2): 182–228.
- [2] Lindelöf, E. (1894). Sur l'application de la méthode des approximations successives aux équations différentielles ordinaires du premier ordre. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*. 118: 454–7.
- [3] Rach R. (1987). On the Adomian Decomposition method and comparisons with Picard's method. *J. Math. Anal. Appl.* 128, 480-483.
- [4] Bellomo N. and Sarafyan D. (1987). On Adomian Decomposition method and some comparisons with Picard's method. *J. Math. Anal. Appl.* 123, 389-400.
- [5] Robin WA., Solving differential equations using modified Picard iteration, *International Journal of Mathematical Education in Science and Technology*, 41(5), 2010.
- [6] Kirkwood JR., *Introduction to Analysis*, Chapman and Hall/CRC, 3 rd edition, 2021, 206-212.
- [7] Tonelli, L., Suffe equazio»i 6u»zio»afi def tipo de Vofte~~a, *Bull. Calcutta Math. Soc.* 20 (1928), 31-48.
- [8] Caputo, M., Fabrizio M., A New Definition of Fractional Derivative Without Singular Kernel, *Progress in Fractional Differentiation and Applications*, 1(2015),2, pp.73-85.

- [9] Taylor, Brook (1715). *Methodus Incrementorum Directa et Inversa* [Direct and Reverse Methods of Incrementation] (in Latin). London. p. 21–23 (Prop. VII, Thm. 3, Cor. 2).
- [10] Arzelà, Cesare, 1883, Un’osservazione intorno alle serie di funzioni, *Rend. Dell’ Accad. R. Delle Sci. dell’Istituto di Bologna*: 142–159.
- [11] Ascoli, G. 1884, Le curve limite di una varietà data di curve, *Atti della R. Accad. Dei Lincei Memorie della Cl. Sci. Fis. Mat. Nat.*, 18 (3): 521–586.