



HAL
open science

A geometric description of the set of almost-fuchsian representations

Samuel Bronstein, Graham Andrew Smith

► **To cite this version:**

Samuel Bronstein, Graham Andrew Smith. A geometric description of the set of almost-fuchsian representations. 2023. hal-03956297

HAL Id: hal-03956297

<https://hal.science/hal-03956297>

Preprint submitted on 25 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

A geometric description of the set of almost-fuchsian representations

SAMUEL BRONSTEIN, GRAHAM ANDREW SMITH

ABSTRACT. We prove that the set of proper minimal disks in the real hyperbolic 3-space whose principal curvatures belong in a closed set of $(-1, 1)$ is parametrized by a convex set of Hopf differentials. We give a quantitative estimate of this convex set in terms of sup norms of the quadratic differentials. Applying this result to disks equivariant under some surface group action, we get a parametrization of almost-fuchsian representations by a fiberwise convex open set in the cotangent bundle to Teichmüller space.

CONTENTS

1. Context and Statement of Results	1
2. introduction and notation	2
3. Proof of main theorem	6
References	13

1. CONTEXT AND STATEMENT OF RESULTS

In this paper we study the geometry of the space of complete almost-fuchsian minimal disks in \mathbb{H}^3 . In the special case of disks equivariant under the action of some cocompact surface group in \mathbb{H}^3 , this naturally gives a description of the moduli space of almost-fuchsian representations by a fiberwise convex subset of the cotangent bundle of Teichmüller space. The same discussion yields a description of minimal almost-fuchsian disks invariant under some fixed group $\Gamma < \mathrm{PSL}(2, \mathbb{R})$.

Let Σ_g be a closed hyperbolic surface of genus $g \geq 2$ and let $\Gamma = \pi_1(\Sigma_g)$ denote its fundamental group. In her seminal work on minimal surfaces in hyperbolic 3-space [Uhl83], K. Uhlenbeck defines the notion of *almost-fuchsian representation*: it is a discrete and faithful representation ρ of Γ into $\mathrm{PSL}(2, \mathbb{C})$ such that there is a ρ -equivariant immersion $f : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ with minimal image and principal curvatures in $(-1, 1)$. Uhlenbeck showed that those representations are always convex-cocompact, and that they form an open neighborhood of the fuchsian locus inside $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$. In \mathbb{H}^3 , convex-cocompact representations are quasi-fuchsian representations, hence the denomination of almost-fuchsian for this stronger notion.

Considering the minimal surface problem in \mathbb{H}^3 , almost-fuchsian representations are of special interest because they admit a unique invariant minimal disk with small principal curvatures. This is not the case for arbitrary convex-cocompact representations, see for instance [HW15].

Krasnov–Schlenker [KS07] have extended the notion to almost-fuchsian immersions or almost-fuchsian disks, not necessarily invariant under a surface group. Seppi [Sep16] used it to show that quasi-circles with quasisymmetry constant close to 1 bound minimal disks.

We will work with the following notion of almost-fuchsian immersion.

Definition 1.1. An immersion $f : \mathbb{D} \rightarrow \mathbb{H}^3$ is said to be almost-fuchsian if it is proper, conformal with minimal image, and has principal curvatures in $(-1 + \epsilon, 1 - \epsilon)$, for some $\epsilon > 0$.

A subdisk of H^3 is said to be an almost-fuchsian subdisk if it is the image of an almost-fuchsian immersion.

Let $\mathcal{Q}(\mathbb{D})$ denote the space of holomorphic quadratic differentials over \mathbb{D} . The main result we will prove in this paper is the following:

Theorem 1.2. *Let $C \subseteq \mathcal{Q}$ denote the set of Hopf differentials of almost fuchsian immersions. Then*

- (1) *2 disks with same Hopf differentials are identified by an isometry of \mathbb{H}^3 .*
- (2) *C is a bounded, open and convex subset of \mathcal{Q} , and*
- (3) *C contains all holomorphic quadratic differentials such that $|\phi|_\infty \leq \frac{1}{2}$.*
- (4) *For any $\phi \in C$, $|\phi|_\infty \leq 1$.*

Krasnov–Schlenker [KS07] already consider this parametrization for the case of holomorphic quadratic differentials invariant under a closed surface subgroup of $\mathrm{PSL}(2, \mathbb{C})$. In that case, it was already known that the set is open and star-shaped around 0, see for instance [Tra19].

The almost-fuchsian moduli space is an example of moduli space endowed with a hyperkähler structure, see [Don03] and [Hod05]. The hyperkähler structure actually coincides with the Feix-Kaleidin structure when considering C as a neighborhood of the fuchsian moduli space, [Tra19].

This description of C has 2 main consequences. First, it extends the notion of almost-fuchsian representation to any infinite discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Also, considering holomorphic quadratic differentials invariant under a surface group representation, we prove the following result.

Theorem 1.3. *There exists a Finsler metric p on the Teichmüller space \mathcal{T}_g such that*

$$(1.1) \quad \forall \phi \in T^*\mathcal{T}_g, p(\phi) < 1 \text{ if and only if } \phi \in C .$$

This Finsler metric is uniformly equivalent to the Finsler metric dual to the unit L^∞ balls under the identification of the cotangent bundle with the space of holomorphic quadratic differentials.

The authors thank Nicolas Tholozan for his insights and discussions on this topic.

2. INTRODUCTION AND NOTATION

This section is a classical survey of the Gauss-Codazzi equation for almost-fuchsian disks. The results are well-known, we explain them here to make our notation clear. The reader comfortable with Gauss-Codazzi theory may go directly to the next section.

In the following, Σ is a connected, oriented surface, with $\chi(\Sigma) < 0$. Let (M, h) is a complete, 3-dimensional hyperbolic manifold, and let $f : \Sigma \rightarrow M$ be a smooth immersion.

Let g_0 denote the induced metric over S , let \mathbb{I} denote the second fundamental form of f , and let A denote its shape operator. Recall that f is said to be **minimal** whenever its mean curvature $H := \text{Tr}(A)$ vanishes.

Let κ denote the intrinsic curvature of g_0 . The Gauss equation is

$$(2.1) \quad \kappa = \det_{g_0}(\mathbb{I}) - 1 .$$

where the determinant of the second fundamental form \mathbb{I} is taken with respect to g_0 . The determinant of \mathbb{I} is also the determinant of the shape operator A .

Definition 2.1 (Hopf differential). The Hopf differential ϕ of f is the $(2, 0)$ -component of \mathbb{I} . That is, ϕ is the symmetric bilinear form given by

$$(2.2) \quad \phi(\xi, \nu) := \frac{1}{4} \left(\mathbb{I}(\xi, \nu) - i\mathbb{I}(J\xi, \nu) - i\mathbb{I}(\xi, J\nu) - \mathbb{I}(J\xi, J\nu) \right) .$$

In particular, when f is minimal we readily obtain:

$$(2.3) \quad \det_{g_0}(\mathbb{I}) = -|\phi|_{g_0}^2 .$$

In the following, we will denote g a fixed hyperbolic metric on a Riemann surface, we call the pair (u, ϕ) Gauss-Codazzi data for the immersion, where $e^{2u}g$ is the induced metric and ϕ is the Hopf differential of the corresponding immersion.

Lemma 2.2. *Let (Σ, g_0) be a riemannian surface and let $g_1 = e^{2u}g_0$ be a metric conformal to g_0 . Let Δ^i and κ^i denote respectively the laplacian and curvature of the metric g_i for $i \in \{0, 1\}$. Then*

$$(2.4) \quad \kappa^1 e^{2u} = \kappa^0 - \Delta^0 u$$

$$(2.5) \quad \kappa^0 = (\kappa^1 + \Delta^1 u) e^{2u} .$$

Proof. Denote Ric^i the Ricci tensor of g_i . The following equation in Besse's book, 1.159 d) [Bes07], relates the Ricci curvatures of g_1 and g_0 :

$$(2.6) \quad \text{Ric}^1 = \text{Ric}^0 - (\Delta^0 u)g_0$$

The remaining equations follow from the fact that, for a riemannian surface (Σ, g_i) , $\text{Ric}^{g_i} = \kappa^i g_i$. Note that Besse uses the analysts' laplacian, whose sign is the inverse of that of the geometer's laplacian. \square

Combining these equations yields the Gauss-Codazzi equation.

Proposition 2.3 (Gauss-Codazzi). *Let $f : \Sigma \rightarrow (M, h)$ be a minimal immersion of a surface into a hyperbolic manifold. Let g_0 denote the induced metric and let $g = e^{-2u}g_0$ denote the unique, complete hyperbolic metric in its conformal class. Then u satisfies*

$$(2.7) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|_g^2 .$$

Proof. By lemma 2.2,

$$(2.8) \quad \kappa_0 = (-1 - \Delta u)e^{-2u} .$$

By Gauss equation,

$$(2.9) \quad \kappa_0 = -1 - e^{-4u}|\phi_0|_g^2 .$$

The result follows upon combining these two equations. \square

By the fundamental theorem of surface theory, for every pair (u, ϕ) , where $u : \Sigma \rightarrow \mathbb{R}$ and ϕ is a holomorphic quadratic differential on Σ satisfying the Gauss-Codazzi equation (2.7), there exists an immersion of $\tilde{\Sigma}$ into \mathbb{H}^3 having this Gauss-Codazzi data. As the surface considered is hyperbolic $\tilde{\Sigma}$ identifies with the open disk \mathbb{D} , and the immersion is equivariant with regard to some representation $\rho : \pi_1 \Sigma \rightarrow \text{Isom}(\mathbb{H}^3)$. Furthermore, this immersion is unique up to post composition by an isometry of \mathbb{H}^3 .

We will require the following result of Li-Mochizuki [LM20].

Proposition 2.4 (Sub-supersolution method). *Let (Σ, g) be a riemannian surface, and let F be a smooth function defined on $\Sigma \times \mathbb{R}$. Consider the equation (E):*

$$(2.10) \quad \Delta_g u = F(x, u)$$

Assume there are $u_-, u_+ : \Sigma \rightarrow \mathbb{R}$ satisfying:

$$(2.11) \quad u_- < u_+$$

$$(2.12) \quad \Delta_g u_- \geq F(x, u_-)$$

$$(2.13) \quad \Delta_g u_+ \leq F(x, u_+)$$

Then the equation (E) admits at least one solution u satisfying $u_- \leq u \leq u_+$.

First remark that almost-fuchsian subsurfaces of H^3 are necessarily uniformized by the Poincaré disk, and that the conformal factor between the induced metric and the hyperbolic one is bounded:

Lemma 2.5. *Let M be a complete hyperbolic 3-manifold. Let $f : \Sigma \rightarrow M$ be an almost-fuchsian immersion, then its induced metric g_0 is conformal to a complete hyperbolic metric g . Denote $g_0 = e^{2u}g$. Then u satisfies.*

$$(2.14) \quad -\frac{\ln 2}{2} \leq u \leq 0 .$$

Proof. Consider the prescribed curvature equation on the universal cover :

$$(2.15) \quad \Delta^{g_0} v = e^{2v} - \kappa_{g_0} .$$

For any solution v , the corresponding metric $e^{2v}g_0$ will be hyperbolic. As f is almost fuchsian, the curvature κ_{h_0} satisfies $-2 \leq \kappa_{h_0} \leq -1$. Upon applying the sub-supersolution method (Prop 2.4) to (2.15) with subsolution $v_- = 0$ and supersolution $v_+ = \frac{\ln 2}{2}$, we get a solution v satisfying

$$(2.16) \quad 0 \leq v \leq \frac{\ln 2}{2} .$$

Since v is bounded, $(\Sigma, e^{2v}g_0)$ is quasi-isometric to (Σ, g_0) . As f is almost-fuchsian, it is proper. Thus g_0 is complete, and $e^{2v}g_0$ is also complete. By the Ahlfors-Schwarz-Pick lemma [Ahl38], the complete hyperbolic metric. is unique in its conformal class, hence $g = e^{2v}g_0$ is complete hyperbolic. Denote $u = -v$, we check:

$$(2.17) \quad -\frac{\ln 2}{2} \leq u \leq 0 ,$$

As asserted. □

Remark.

- With the same proof, one can prove that this is a general property of complete pinched Hadamard disks, due to Bland and Kalka [BK86]. It is also stated in Benoist-Hulin [BH20], Prop. 3.1. : A complete pinched hadamard disk is uniformly quasiconformal to the complete hyperbolic disk.
- In [AS20], Alvarez and Smith show a more general version of this: The conformal factor is actually bounded in $C^{k+2,\alpha}$.

Theorem 2.6. *Let g be a complete hyperbolic metric on Σ , $u : \Sigma \rightarrow \mathbb{R}$ a bounded continuous map, and ϕ a holomorphic quadratic differential on Σ satisfying*

$$(2.18) \quad \Delta^g u = -1 + e^{2u} + e^{-2u} |\phi|^2, \text{ and}$$

$$(2.19) \quad \sup_{z \in \Sigma} (e^{-2u(z)} |\phi(z)|) < 1.$$

Then there is an almost-fuchsian immersion of Σ into a hyperbolic manifold such that $e^{2u}g$ is the induced metric and ϕ is the $(2, 0)$ -part of the second fundamental form. Conversely, for any almost-fuchsian immersion the associated data satisfies these relations.

Proof. By the fundamental theorem of surface theory, we get an immersion into a hyperbolic manifold. Since u is bounded, the induced metric is complete and it follows from (2.19) that the principal curvatures lie in $(-1, 1)$. Hence f is almost-fuchsian.

Conversely, if f is almost-fuchsian, consider (u, ϕ) the associated Gauss-Codazzi data. Then Gauss-Codazzi is verified, the bound on the principal curvatures yields (2.19) and the boundedness of u follows from the previous lemma. \square

Note that almost-fuchsian subdisks of \mathbb{H}^3 are always properly embedded and bound a quasicircle at infinity. Indeed, this readily follows from the original work [Uhl83] of Uhlenbeck and the argument of Theorem 1.3 of [HW13]. For the reader's convenience, we state and prove this result here.

Theorem 2.7. *Let $e : \mathbb{D} \rightarrow \mathbb{H}^3$ be an almost-fuchsian disk. Then the disk $e(\mathbb{D})$ is properly embedded and bounds a quasi-circle at infinity. Furthermore, if k denotes the supremum of the principal curvatures of e , the quasisymmetry constant of the quasi-circle is bounded by $\frac{1+k}{1-k}$.*

Proof. Let g be the Poincaré metric of \mathbb{D} . Let $g_0 = e^{2u}g$ denote the induced metric of e on the disk \mathbb{D} . Consider ν the unit normal vector field, and define \tilde{e} by

$$(2.20) \quad \tilde{e} : \begin{cases} \mathbb{D} \times \mathbb{R} & \rightarrow & \mathbb{H}^3, \text{ and} \\ (x, t) & \mapsto & \exp_x(t\nu_x). \end{cases}$$

where \exp denotes the exponential map of \mathbb{H}^3 . Let A denote the shape operator of f . The same computation as of lemma 2.1. of [HW13] gives the following expression of the metric induced by \tilde{e} :

$$(2.21) \quad h^\nu(x, t) := h_t(x) \oplus dt^2,$$

where, for all t ,

$$(2.22) \quad h_t(\cdot, \cdot) = e^{2u}g((\cosh(t)I_0 + \sinh(t)A)\cdot, (\cosh(t)I_0 + \sinh(t)A)\cdot).$$

Consider $0 < k < 1$ such that the operator norm of A is less than k . Since $|A| \leq k < 1$, h^ν is complete and thus \tilde{e} is a diffeomorphism. In particular, e is properly embedded.

Let $f : \mathbb{D} \rightarrow \mathbb{H}^3$ be an isometric parametrization of a totally geodesic plane, let ν_f denote its unit normal vector field, and define $\tilde{f} : \mathbb{D} \times \mathbb{R} \rightarrow \mathbb{H}^3$ by

$$(2.23) \quad \tilde{f}(x, t) = \exp_{f(x)}(t\nu_f(x)) .$$

The metric induced by \tilde{f} is

$$(2.24) \quad \tilde{g}(x, t) = \cosh^2(t)g \oplus dt^2 .$$

It follows from (2.24) and (2.22) that $\alpha := e \circ f^{-1}$ is bilipschitz. It therefore extends (see Gehring [Geh62] or Mostow [Mos68]) to a quasiconformal map $\tilde{\alpha}$ from $\partial_\infty \mathbb{H}^3$ to itself. It follows that $\Gamma := \partial_\infty e(\mathbb{D})$ is a quasicircle.

Note that, for all t , the restrictions of h^0 and \tilde{g} to the slice $\mathbb{D} \times \{t\}$ are K -quasiconformal, with

$$(2.25) \quad K = \frac{1+k}{1-k} .$$

Letting t tend to $+\infty$, it follows that $\tilde{\alpha}$ is K -quasiconformal. As the boundary of the disk is the image by $\tilde{\alpha}$ of an equator in \mathbb{S}^2 , this completes the proof. \square

3. PROOF OF MAIN THEOREM

As in this paper we are mainly interested in the description of conformal almost-fuchsian disks, let us define the space we want to describe:

Definition 3.1. We denote \mathcal{AF} the set of almost-fuchsian immersions $\mathbb{D}^2 \rightarrow \mathbb{H}^3$ up to postcomposition by an isometry of \mathbb{H}^3 .

The set \mathcal{AF} , equipped with its quotient topology, can be identified with the set of almost-fuchsian immersions with prescribed image and differential of the origin o . As such, it can be viewed as a subspace of the space of immersions from the disk into \mathbb{H}^3 .

Remark. The space \mathcal{AF} does not parametrize almost-fuchsian subdisks of \mathbb{H}^3 . However, the set of almost-fuchsian subdisks up to isometry can be identified to the quotient of \mathcal{AF} by a $\mathrm{PSL}(2, \mathbb{R})$ action by precomposition, which is not a properly discontinuous action.

In order to prove our theorems in a noncompact setting, we will use the following version of Omori-Yau's theorem [Omo67, Yau75]. It can be found in Li-Mochizuki [LM20], we reproduce it here for the reader's convenience.

Theorem 3.2 (Omori-Yau). *Let (M, h) be a complete manifold with Ricci curvature bounded from below. Let $u : M \rightarrow \mathbb{R}$ be a map bounded from above and let $\varepsilon > 0$. Then there is $x \in M$ such that*

$$(3.1) \quad u(x) \geq \sup u - \varepsilon ,$$

$$(3.2) \quad |\nabla_h u(x)| \leq \varepsilon , \text{ and}$$

$$(3.3) \quad \Delta_h u(x) \leq \varepsilon .$$

The first step is to show that the moduli space of almost-fuchsian disks injects into the space of bounded holomorphic quadratic differentials.

Proposition 3.3. *Let ϕ be a holomorphic quadratic differential on the disk. Then there is at most one bounded solution to*

$$(3.4) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2$$

such that $e^{-2u}|\phi| < 1$ everywhere.

Proof. Suppose (u, ϕ) and (v, ϕ) are two bounded solutions of (3.4) such that $e^{-2u}|\phi| < 1$ and $e^{-2v}|\phi| < 1$. Then $u - v$ is bounded and satisfies

$$(3.5) \quad \Delta(u - v) = (e^{2v} - e^{-2v}|\phi|^2)(e^{2(u-v)} - 1) + |\phi|^2 e^{-2v}(e^{2(u-v)} + e^{2(v-u)} - 2).$$

Using the positivity of the second term of the right hand side and the almost-fuchsian property of v , one gets

$$(3.6) \quad \Delta(u - v) \geq (e^{2v} - e^{-2v}|\phi|^2)(u - v).$$

Applying Omori-Yau's maximum principle gives $u - v \leq 0$. Exchanging u and v , one has $u = v$, thus the corresponding almost-fuchsian disks are conjugated. \square

Remark. The same reasoning actually shows that if $(e^{2u}g, \phi)$ is almost-fuchsian, then any upper bounded v solution to the Gauss-Codazzi equation with quadratic differential ϕ must satisfy:

$$(3.7) \quad v \leq u.$$

In [HLT21], Huang, Lucia and Tarantello study the solutions of the Gauss-Codazzi equation on closed surfaces, of which the maximality of almost-fuchsian solutions is a consequence.

Moreover, we will show that the image of the almost-fuchsian locus in the space of bounded holomorphic quadratic differentials on the disk is actually open. Combining Proposition 3.3 with the fundamental theorem of surface theory, we get :

Corollary 3.4. *For $f : \mathbb{D}^2 \rightarrow \mathbb{H}^3$ an immersion, denote ϕ_f its Hopf differential. Then the map:*

$$(3.8) \quad \Phi : \begin{cases} \mathcal{AF} & \rightarrow & QD(\mathbb{D}^2) \\ f & \mapsto & \phi_f \end{cases}$$

is injective.

Remark. The statement of this corollary in the equivariant case is the Lemma 2.11 of Krasnov–Schlenker [KS07].

Proof. We already know that the Gauss-Codazzi data of an immersion characterizes it up to isometry. The prop (3.3) shows that if two almost-fuchsian immersions have same Hopf differentials, then they have the same Gauss-Codazzi data, hence the injectivity of our map. \square

Theorem 3.5. *The image of the almost-fuchsian locus is open in the space of bounded holomorphic quadratic differentials.*

Proof. Consider ϕ in $\mathfrak{S}(\mathcal{AF})$, u and $0 < \delta < 1$ satisfying

$$(3.9) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2 \text{ and}$$

$$(3.10) \quad \sup_{z \in \mathbb{D}} e^{-2u}|\phi| \leq 1 - \delta.$$

We will prove that the linearized operator of (3.9) $L_u : v \mapsto -\Delta v + 2(e^{2u} - e^{-2u}|\phi|^2)v$ is an isomorphism between the Banach spaces $C^{2,\alpha}(\mathbb{D})$ and $C^{0,\alpha}(\mathbb{D})$, where here \mathbb{D} is equipped with its Poincaré metric, hence, using the local inversion principle, we will be able to find a solution \tilde{u} uniformly close to u on a neighborhood of ϕ , which then has to be almost-fuchsian.

At first, as u is bounded, L_u trivially defines a bounded linear map from $C^{2,\alpha}(\mathbb{D})$ towards $C^{0,\alpha}(\mathbb{D})$. The almost-fuchsian condition implies

$$(3.11) \quad \frac{\delta}{2} \leq e^{2u} - e^{-2u}|\phi|^2 \leq 1.$$

Hence L_u is strictly elliptic. We now show that L_u is injective. Indeed, choose $v \in \ker L_u$. Since v is bounded, Omori-Yau's maximum principle implies

$$(3.12) \quad v \leq 0.$$

The same reasoning to $-v$ shows that $v = 0$, hence L_u is injective, as asserted.

It remains only to prove surjectivity. First consider a compact set K in \mathbb{D} with smooth boundary. It is a standard consequence of elliptic theory and of the strict ellipticity of L_u that L_u is an isomorphism from $C_0^{2,\alpha}(K)$ into $C^{0,\alpha}(K)$. Now choose $u \in C^{0,\alpha}(\mathbb{D})$. and a compact exhaustion (K_n) of \mathbb{D} such that each K_i has smooth boundary. For each i there is a unique $v_i \in C_0^{2,\alpha}(K_i)$ such that $L_u v_i = w \mathbf{1}_{K_i}$. We claim that the sequence $(|v_n|_\infty)$ is bounded. Indeed, by the maximum principle, for all i ,

$$(3.13) \quad |u_i|_\infty \leq \max\{2\delta, 1\}|w \mathbf{1}_{K_i}|_\infty \leq 2|w|_\infty.$$

Fix a radius $r > 0$ and $x \in \mathbb{D}$. For i big enough such that $B(x, 2r) \subset K_i$,

$$(3.14) \quad |L_u v_i|_{C^{0,\alpha}(B(x,2r))} \leq |w|_{C^{0,\alpha}}, \text{ and}$$

$$(3.15) \quad |v_i|_{C^0(B(x,2r))} \leq 2|w|_{C^0}.$$

Hence, by the Schauder estimates, there is a $C > 0$ depending only on r such that

$$(3.16) \quad |v_i|_{C^{2,\alpha}(B(r,x))} \leq C|w|_{C^{0,\alpha}}.$$

By the Arzela-Ascoli theorem, up to extracting a subsequence, we can assume there exists $v_\infty \in C_{\text{loc}}^{2,\alpha}$ such that $v_n \rightarrow v_\infty$ on every compact set in the $C^{2,\beta}$ topology for $\beta < \alpha$. Furthermore

$$\begin{aligned} |v_\infty|_{C^{2,\alpha}} &\leq C' \sup_{x \in \mathbb{H}^2} |v_\infty|_{C^{2,\alpha}(B(r,x))} \\ &\leq C' \sup_{x \in \mathbb{H}^2} \limsup_{i \rightarrow \infty} |v_i|_{C^{2,\alpha}(B(r,x))} \\ &\leq C' C |w|_{C^{0,\alpha}}, \end{aligned}$$

so that $v_\infty \in C^{2,\alpha}(\mathbb{H}^2)$. By continuity of L_u ,

$$(3.17) \quad L_u v_\infty = w.$$

Thus L_u is surjective.

By the Closed Graph Theorem, L_u is a linear isomorphism. It follows by the inverse function theorem that the image of the almost-fuchsian locus is open in the space of bounded quadratic differentials, as desired. \square

Equipped with this theorem, we will consider a perturbation ϕ_t of ϕ in the image of the almost-fuchsian locus and we will consider the family (u_t) of bounded maps satisfying the Gauss-Codazzi equation. In particular, we will need the derivatives \dot{u}_t and \ddot{u}_t to be bounded.

The following is actually a straightforward consequence of the implicit function theorem we used.

Proposition 3.6. *Consider (e^{2u}, ϕ) an almost-fuchsian disk. Let f_t be a $C^{0,\alpha}$ perturbation of $|\phi|^2$. Denote by (u_t) the smooth branch of solutions to :*

$$(3.18) \quad \Delta u_t = -1 + e^{2u_t} + e^{-2u_t} f_t$$

for t in a neighborhood of 0. Then \dot{u}_t and \ddot{u}_t are bounded.

Proof. Consider $F : C^{2,\alpha}(\mathbb{D}) \times [0, 1] \rightarrow C^{0,\alpha}(D)$ defined by $F(v, t) = \Delta v + 1 - e^{2v} - e^{-2v} f_t$. Thanks to the previous lemma, we know the local existence of a smooth $G : [0, 1] \rightarrow C^{2,\alpha}(\mathbb{D})$ such that $F(G(t), t) = 0$. It follows by smoothness of G that $\dot{u}_t, \ddot{u}_t \in C^{2,\alpha}(\mathbb{D})$ for all t , and the result follows. \square

We are now able to show that the almost-fuchsian locus retracts to the set of almost-fuchsian immersions with zero quadratic differentials.

Theorem 3.7. *Let $\phi \in \mathfrak{S}(\mathcal{AF})$ and let ψ be a holomorphic quadratic differential on the disk satisfying*

$$(3.19) \quad \forall z \in \mathbb{D}, |\psi(z)| \leq |\phi(z)| .$$

Then $\psi \in \mathfrak{S}(\mathcal{AF})$.

Proof. Let $u : \mathbb{D} \rightarrow \mathbb{R}$ be such that (u, ϕ) is the Gauss-Codazzi data of an almost-fuchsian disk. Let $f_t : \mathbb{D} \rightarrow [0, \infty[$ be a smooth map such that $f_0 = |\psi|^2$ and $f_1 = |\phi|^2$. As $|\psi| \leq |\phi|$, one can choose f so that the maps $t \mapsto f_t(z)$ are nondecreasing.

We will show that $\psi \in \mathfrak{S}(\mathcal{AF})$, i.e. that there is a solution v to

$$(3.20) \quad \Delta v = -1 + e^{2v} + e^{-2v} f_0, \text{ and}$$

$$(3.21) \quad \sup_{z \in \mathbb{D}} e^{-4v} f_0 < 1 .$$

To do so, consider for $t \in [0, 1]$ the equation.

$$(3.22) \quad \Delta u_t = -1 + e^{2u_t} + e^{-2u_t} f_t .$$

Let $I \subseteq [0, 1]$ denote the set of all s such that, for all $t \geq s$, this equation has a bounded solution u_s satisfying $|e^{-4u_s} f_s|_\infty < 1$, for some $\epsilon > 0$. By Theorem 2.6, $1 \in I$. By Theorem 3.5 applied to a $C^{0,\alpha}$ perturbation, I is open. It thus remains only to show that I is closed.

At first, consider $t_0 = \inf I$ and $(t_n) \in I$ a monotonous sequence such that $t_n \rightarrow t_0$. By Lemma 2.5, the sequence (u_{t_n}) is uniformly bounded, from which it follows that (Δu_{t_n}) is also uniformly bounded. By Schauder's estimates, the sequence (u_{t_n}) is uniformly bounded in $C^{1,\alpha}$ on any compact set. Hence we may assume that u_{t_n} converges towards a map $u_{t_0} \in C^{2,\alpha}(\mathbb{H})$ in the $C^{2,\beta}$ topology for all $\beta < \alpha$. Trivially u_{t_0} solves (3.22).

It remains only to check that $|e^{-4u_{t_0}} f_{t_0}|_\infty < 1$.

Differentiating the above equation with respect to t yields,

$$(3.23) \quad \Delta \dot{u}_t = 2(e^{2u_t} - e^{-2u_t} f_t) \dot{u}_t + e^{-2u_t} \dot{f}_t .$$

By Lemma 2.5, there exists $B > 0$ such that $|u_t|_\infty < B$ for all t . It follows by the Omori-Yau maximum principle that, over I , $\dot{u}_t \leq 0$, so that, for all $t \in I$,

$$(3.24) \quad |e^{-4u_t} \dot{f}_t|_\infty \leq |e^{-4u_1} \dot{f}_1|_\infty < 1 .$$

Closedness of I now follows so that, by connectedness, $0 \in I$. At $t = 0$, this means that (u_0, ψ) is the Gauss-Codazzi data of an almost-fuchsian disk. We have shown that $\psi \in \mathfrak{S}(\mathcal{AF})$, as desired. \square

Corollary 3.8. *Let $\phi \in \mathfrak{S}(\mathcal{AF})$ and let $u : \mathbb{D} \rightarrow \mathbb{R}$ be the associated bounded map such that*

$$(3.25) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2, \text{ and}$$

$$(3.26) \quad \sup_{z \in \mathbb{D}} e^{-2u}|\phi| \leq 1 - \delta.$$

Denote $\phi_t = t\phi$ for $t \in [0, 1]$. Then for any $t \in [0, 1]$, there is a $u_t : \Sigma \rightarrow \mathbb{R}$ bounded such that $(e^{2u_t}g, \phi_t)$ satisfies the Gauss-Codazzi equation. Furthermore, for all $t \in [0, 1]$, $(e^{2u_t}g, \phi_t)$ is almost-fuchsian.

Remark. For the equivariant case, this computation can already be found in a survey of Trautwein [Tra19], step 3 of proof of theorem 5.13.

Corollary 3.9. *The moduli space of almost-fuchsian disks up to isometry is contractible. The set of almost-fuchsian immersions retracts onto the set of geodesic immersions of disks in \mathbb{H}^3 .*

We continue with the proof of our main theorem:

Theorem 3.10 (Convexity). *Let $(e^{2u_0}g, \phi_0)$ and $(e^{2u_1}g, \phi_1)$ be two pairs corresponding to almost-fuchsian disks. For $t \in [0, 1]$, denote $\phi_t := (1-t)\phi_0 + t\phi_1$. Then for any t , there is $(e^{2w_t}g, \phi_t)$ satisfying the Gauss-Codazzi equation, and it corresponds to an almost-fuchsian disk.*

Proof. First, we write down Gauss-Codazzi and we differentiate it twice in t

$$(3.27) \quad \Delta w_t = -1 + e^{2w_t} + e^{-2w_t}|\phi_t|^2,$$

$$(3.28) \quad \Delta \dot{w}_t = 2\dot{w}_t e^{2w_t} - 2\dot{w}_t e^{-2w_t}|\phi_t|^2 + 2e^{-2w_t}g(\phi_t, \phi_1 - \phi_0), \text{ and}$$

$$(3.29) \quad \begin{aligned} \Delta \ddot{w}_t &= 2\ddot{w}_t(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 4\dot{w}_t^2(e^{2w_t} + e^{-2w_t}|\phi_t|^2) \\ &\quad - 8\dot{w}_t e^{-2w_t}g(\phi_t, \phi_1 - \phi_0) + 2e^{-2w_t}|\phi_1 - \phi_0|^2. \end{aligned}$$

We first show that as long as $(e^{2w_t}g, \phi_t)$ is almost-fuchsian,

$$(3.30) \quad \ddot{w}_t + 4\dot{w}_t^2 \leq 0.$$

This is a consequence of the Omori-Yau maximum principle and the following standard formula.

$$(3.31) \quad \Delta(\dot{w}_t^2) = 2\dot{w}_t \Delta \dot{w}_t + 2|\nabla \dot{w}_t|^2 \geq 2\dot{w}_t \Delta \dot{w}_t.$$

Then, using equations (3.28) and (3.29), we obtain

$$(3.32) \quad \begin{aligned} \Delta(\ddot{w}_t + 4\dot{w}_t^2) &\geq 2\ddot{w}_t(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 20\dot{w}_t^2 e^{2w_t} - 12\dot{w}_t^2 e^{-2w_t}|\phi_t|^2 \\ &\quad + 8\dot{w}_t e^{-2w_t}g(\phi_t, \phi_1 - \phi_0) + 2e^{-2w_t}|\phi_1 - \phi_0|^2. \end{aligned}$$

Judiciously rearranging the terms, we obtain

$$(3.33) \quad \begin{aligned} \Delta(\ddot{w}_t + 4\dot{w}_t^2) &\geq 2(\ddot{w}_t + 4\dot{w}_t^2)(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 12\dot{w}_t^2(e^{2w_t} - e^{-2w_t}|\phi_t|^2) \\ &\quad + 8\dot{w}_t^2 e^{-2w_t}|\phi_t|^2 + 8e^{-2w_t}g(\dot{w}_t \phi_t, \phi_1 - \phi_0) + 2e^{-2w_t}|\phi_1 - \phi_0|^2. \end{aligned}$$

As $8A^2 + 8AB + 2B^2 = 2(2A + B)^2$ is nonnegative, we obtain

$$(3.34) \quad \Delta(\ddot{w}_t + 4\dot{w}_t^2) \geq 2(\ddot{w}_t + 4\dot{w}_t^2)(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 12\dot{w}_t^2(e^{2w_t} - e^{-2w_t}|\phi_t|^2) .$$

Hence, provided the pair is almost-fuchsian, we get

$$(3.35) \quad \Delta(\ddot{w}_t + 4\dot{w}_t^2) \geq 2(\ddot{w}_t + 4\dot{w}_t^2)(e^{2w_t} - e^{-2w_t}|\phi_t|^2) .$$

Thus we deduce that

$$(3.36) \quad \ddot{w}_t + 4\dot{w}_t^2 \leq 0 .$$

We have shown that as long as $(e^{2w_t}g, \phi_t)$ is almost-fuchsian, $\ddot{w}_t + 4\dot{w}_t^2 \leq 0$, as asserted.

Consider now the family of quadratic holomorphic differentials

$$(3.37) \quad \phi_{s,t} := s(1-t)\phi_0 + st\phi_1$$

For all s , let \mathcal{J}_s denote the set of all $t \in [0, 1]$ for which there exists a function $w_{s,t}$ making $(e^{2w_{s,t}}g, \phi_{s,t})$ into an almost Fuchsian pair. Let \mathcal{S} denote the set of all $s \in [0, 1]$ for which \mathcal{J}_s equals the interval $[0, 1]$. Trivially \mathcal{S} contains 0. By Theorem 3.5, \mathcal{S} is open. We now show that \mathcal{S} is closed.

First, if $s_0 \in \overline{\mathcal{S}}$, the equicontinuity of all functions $(u_{s,t})$ for $s \in \mathcal{S}$, $t \in [0, 1]$ allows us to get a limiting family of functions which are solutions to Gauss-Codazzi equation. We claim that, for all s , the function $t \mapsto e^{-4w_{s,t}}|\phi_{s,t}|^2$ is convex over \mathcal{J}_s . Indeed,

$$\begin{aligned} e^{4w_{s,t}}\partial_t^2(e^{-4w_{s,t}}|\phi_{s,t}|^2) &= -4(\ddot{w}_{s,t} + 4\dot{w}_{s,t}^2)|\phi_{s,t}|^2 + 32\dot{w}_{s,t}^2|\phi_{s,t}|^2 - 16\dot{w}_{s,t}g(\phi_{s,t}, \phi_{s,1} \\ &\quad - \phi_{s,0}) + 2|\phi_{s,1} - \phi_{s,0}|^2 . \\ &= -4(\ddot{w}_{s,t} + 4\dot{w}_{s,t}^2)|\phi_{s,t}|^2 + 2|4\dot{w}_{s,t}\phi_{s,t} - (\phi_{s,1} - \phi_{s,0})|^2 . \end{aligned}$$

This is nonnegative, and convexity follows. By convexity, $e^{-4w_{s,t}}|\phi_{s,t}|^2$ is smaller than the maximal value for $t = 0$ and $t = 1$, and we deduce that it is smaller than $1 - \delta$, for $\delta > 0$ depending only on ϕ_0 and ϕ_1 . It now readily follows that \mathcal{S} is closed, so that, by connectedness, $1 \in \mathcal{S}$. In other words, $\mathcal{J}_1 = [0, 1]$, and the result follows. \square

We also have the following quantitative estimate for this convex set.

Theorem 3.11. *The convex set $\mathfrak{S}(\mathcal{AF})$ satisfies:*

$$(3.38) \quad \left\{ \phi : \sup_{z \in D} |\phi(z)| < \frac{1}{2} \right\} \subset \mathfrak{S}(\mathcal{AF}) \subset \left\{ \phi : \sup_{z \in \mathbb{D}} |\phi(z)| < 1 \right\}$$

Proof. Take ϕ with $|\phi|_\infty \leq \frac{1}{2}$. Then $-\frac{\ln 2}{2}$ and 0 are respectively sub and supersolutions of

$$(3.39) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2 .$$

Hence by Proposition 2.4, we have a bounded solution satisfying Gauss-Codazzi, and necessarily $e^{-2u}|\phi| \leq 2|\phi|_\infty < 1$. Hence $\phi \in \mathfrak{S}(\mathcal{AF})$. Conversely, if $\phi \in \mathfrak{S}(\mathcal{AF})$, u the solution to Gauss-Codazzi satisfies $u \leq 0$, and $|\phi|_\infty \leq |e^{-2u}\phi|_\infty < 1$. \square

Fixing now a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$, one can look at the subset \mathcal{AF}_Γ of elements of \mathcal{AF} equivariant under some representation of Γ . The uniqueness properties shown imply that this set can be identified with a subset of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))/\mathrm{PSL}(2, \mathbb{C})$. In the case of a closed surface group, this subset are known as *almost-fuchsian representations* considered by Uhlenbeck [Uhl83].

Fix a hyperbolic surface $\Sigma_{g,n}$, and consider Γ its fundamental group. Considering now holomorphic quadratic differentials equivariant under the action of Γ , identifying holomorphic quadratic differentials with the cotangent bundle to Teichmüller space of genus (g, n) , we get a smooth family of open bounded sets in each leaf. As these sets are isotropic (if $(e^{2u}g, \phi)$ is an almost-fuchsian disk, so is $(e^{2u}g, z\phi)$ for any z of modulus one), it induces a Finsler metric on Teichmüller space, which is quasi-isometric to the Finsler metric dual to the L^∞ unit balls in $T^*\mathcal{T}_g$.

Corollary 3.12. *For $g \geq 2$ and \mathcal{T}_g the Teichmüller space of genus g , there is a Finsler metric p on \mathcal{T}_g such that:*

$$(3.40) \quad p(\phi) < 1 \iff \phi \in \mathfrak{S}(\mathcal{AF})$$

This Finsler metric is quasi-isometric to the Finsler metric dual to the L^∞ unit balls in $T^\mathcal{T}_g$.*

Remark. It is interesting to compare this Finsler metric to the metrics dual to L^p -unit balls in $T^*\mathcal{T}_g$. For $p = 2$, one gets Weil-Petersson's metric, see Wolpert [Wol09]. For $p = 1$, one gets the Teichmüller metric, see Papadopoulos-Su [PS15].

Remark. With this discussion, we have a description of the fibration map from $\mathcal{AF}_\Gamma \rightarrow \mathcal{T}_{g,n}$, which is simply the conformal class of the induced metric on $\Sigma_{g,n}$.

REFERENCES

- [Ahl38] Lars V Ahlfors. An extension of Schwarz’s lemma. *Transactions of the American Mathematical Society*, 43(3):359–364, 1938.
- [AS20] Sébastien Alvarez and Graham Smith. Prescription de courbure des feuilles des laminations: retour sur un théorème de candel. *arXiv preprint arXiv:2009.04594*, 2020.
- [Bes07] Arthur L Besse. *Einstein manifolds*. Springer Science & Business Media, 2007.
- [BH20] Yves Benoist and Dominique Hulin. Harmonic quasi-isometries of pinched hadamard surfaces are injective. *arXiv preprint arXiv:2012.08307*, 2020.
- [BK86] J Bland and Morris Kalka. Complete metrics conformal to the hyperbolic disc. *Proceedings of the American Mathematical Society*, 97(1):128–132, 1986.
- [Don03] Simon K Donaldson. Moment maps in differential geometry. *Surveys in differential geometry*, 8(1):171–189, 2003.
- [Geh62] Frederick W Gehring. Rings and quasiconformal mappings in space. *Transactions of the American Mathematical Society*, 103(3):353–393, 1962.
- [HLT21] Zheng Huang, Marcello Lucia, and Gabriella Tarantello. Bifurcation for minimal surface equation in hyperbolic 3-manifolds. In *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, volume 38, pages 243–279. Elsevier, 2021.
- [Hod05] Thomas Wolf Stephen Hodge. *Hyperkähler geometry and Teichmüller space*. PhD thesis, Imperial College London, 2005.
- [HW13] Zheng Huang and Biao Wang. On almost-fuchsian manifolds. *Transactions of the American Mathematical Society*, 365(9):4679–4698, 2013.
- [HW15] Zheng Huang and Biao Wang. Counting minimal surfaces in quasi-fuchsian three-manifolds. *Transactions of the American Mathematical Society*, 367(9):6063–6083, 2015.
- [KS07] Kirill Krasnov and Jean-Marc Schlenker. Minimal surfaces and particles in 3-manifolds. *Geometriae Dedicata*, 126(1):187–254, 2007.
- [LM20] Qionglng Li and Takuro Mochizuki. Complete solutions of toda equations and cyclic higgs bundles over non-compact surfaces. *arXiv preprint arXiv:2010.05401*, 2020.
- [Mos68] George D Mostow. Quasi-conformal mappings in n -space and the rigidity of hyperbolic space forms. *Publications Mathématiques de l’IHÉS*, 34:53–104, 1968.
- [Omo67] Hideki Omori. Isometric immersions of riemannian manifolds. *Journal of the Mathematical Society of Japan*, 19(2):205–214, 1967.
- [PS15] Athanase Papadopoulos and Weixu Su. On the finsler structure of teichmüller’s metric and thurston’s metric. *Expositiones Mathematicae*, 33(1):30–47, 2015.
- [Sep16] Andrea Seppi. Minimal discs in hyperbolic space bounded by a quasicircle at infinity. *Commentarii Mathematici Helvetici*, 91(4):807–839, 2016.
- [Tra19] Samuel Trautwein. The hyperkähler metric on the almost-fuchsian moduli space. *EMS Surveys in Mathematical Sciences*, 6(1):83–131, 2019.
- [Uhl83] Karen K Uhlenbeck. Closed minimal surfaces in hyperbolic 3-manifolds. In *Seminar on minimal submanifolds*, volume 103, pages 147–168, 1983.
- [Wol09] Scott A Wolpert. The weil-petersson metric geometry. *Handbook of Teichmüller theory*, 2:47–64, 2009.
- [Yau75] Shing-Tung Yau. Harmonic functions on complete riemannian manifolds. *Communications on Pure and Applied Mathematics*, 28(2):201–228, 1975.