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# A geometric description of the set of almost-fuchsian representations

SAMUEL BRONSTEIN, GRAHAM ANDREW SMITH

ABSTRACT. We prove that the set of proper minimal disks in the real hyperbolic 3-space whose principal curvatures belong in a closed set of  $(-1, 1)$  is parametrized by a convex set of Hopf differentials. We give a quantitative estimate of this convex set in terms of sup norms of the quadratic differentials. Applying this result to disks equivariant under some surface group action, we get a parametrization of almost-fuchsian representations by a fiberwise convex open set in the cotangent bundle to Teichmüller space.

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## 1. CONTEXT AND STATEMENT OF RESULTS

In this paper we study the geometry of the space of complete almost-fuchsian minimal disks in  $\mathbb{H}^3$ . In the special case of disks equivariant under the action of some cocompact surface group in  $\mathbb{H}^3$ , this naturally gives a description of the moduli space of almost-fuchsian representations by a fiberwise convex subset of the cotangent bundle of Teichmüller space. The same discussion yields a description of minimal almost-fuchsian disks invariant under some fixed group  $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ .

Let  $\Sigma_g$  be a closed hyperbolic surface of genus  $g \geq 2$  and let  $\Gamma = \pi_1(\Sigma_g)$  denote its fundamental group. In her seminal work on minimal surfaces in hyperbolic 3-space [Uhl83], K. Uhlenbeck defines the notion of *almost-fuchsian representation*: it is a discrete and faithful representation  $\rho$  of  $\Gamma$  into  $\mathrm{PSL}(2, \mathbb{C})$  such that there is a  $\rho$ -equivariant immersion  $f : \mathbb{H}^2 \rightarrow \mathbb{H}^3$  with minimal image and principal curvatures in  $(-1, 1)$ . Uhlenbeck showed that those representations are always convex-cocompact, and that they form an open neighborhood of the fuchsian locus inside  $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ . In  $\mathbb{H}^3$ , convex-cocompact representations are quasi-fuchsian representations, hence the denomination of almost-fuchsian for this stronger notion.

Considering the minimal surface problem in  $\mathbb{H}^3$ , almost-fuchsian representations are of special interest because they admit a unique invariant minimal disk with small principal curvatures. This is not the case for arbitrary convex-cocompact representations, see for instance [HW15].

Krasnov–Schlenker [KS07] have extended the notion to almost-fuchsian immersions or almost-fuchsian disks, not necessarily invariant under a surface group. Seppi [Sep16] used it to show that quasi-circles with quasisymmetry constant close to 1 bound minimal disks.

We will work with the following notion of almost-fuchsian immersion.

**Definition 1.1.** An immersion  $f : \mathbb{D} \rightarrow \mathbb{H}^3$  is said to be almost-fuchsian if it is proper, conformal with minimal image, and has principal curvatures in  $(-1 + \epsilon, 1 - \epsilon)$ , for some  $\epsilon > 0$ .

A subdisk of  $H^3$  is said to be an almost-fuchsian subdisk if it is the image of an almost-fuchsian immersion.

Let  $\mathcal{Q}(\mathbb{D})$  denote the space of holomorphic quadratic differentials over  $\mathbb{D}$ . The main result we will prove in this paper is the following:

**Theorem 1.2.** *Let  $C \subseteq \mathcal{Q}$  denote the set of Hopf differentials of almost fuchsian immersions. Then*

- (1) *2 disks with same Hopf differentials are identified by an isometry of  $\mathbb{H}^3$ .*
- (2)  *$C$  is a bounded, open and convex subset of  $\mathcal{Q}$ , and*
- (3)  *$C$  contains all holomorphic quadratic differentials such that  $|\phi|_\infty \leq \frac{1}{2}$ .*
- (4) *For any  $\phi \in C$ ,  $|\phi|_\infty \leq 1$ .*

Krasnov–Schlenker [KS07] already consider this parametrization for the case of holomorphic quadratic differentials invariant under a closed surface subgroup of  $\mathrm{PSL}(2, \mathbb{C})$ . In that case, it was already known that the set is open and star-shaped around 0, see for instance [Tra19].

The almost-fuchsian moduli space is an example of moduli space endowed with a hyperkähler structure, see [Don03] and [Hod05]. The hyperkähler structure actually coincides with the Feix-Kaleidin structure when considering  $C$  as a neighborhood of the fuchsian moduli space, [Tra19].

This description of  $C$  has 2 main consequences. First, it extends the notion of almost-fuchsian representation to any infinite discrete subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ . Also, considering holomorphic quadratic differentials invariant under a surface group representation, we prove the following result.

**Theorem 1.3.** *There exists a Finsler metric  $p$  on the Teichmüller space  $\mathcal{T}_g$  such that*

$$(1.1) \quad \forall \phi \in T^*\mathcal{T}_g, p(\phi) < 1 \text{ if and only if } \phi \in C .$$

*This Finsler metric is uniformly equivalent to the Finsler metric dual to the unit  $L^\infty$  balls under the identification of the cotangent bundle with the space of holomorphic quadratic differentials.*

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## 2. INTRODUCTION AND NOTATION

This section is a classical survey of the Gauss-Codazzi equation for almost-fuchsian disks. The results are well-known, we explain them here to make our notation clear. The reader comfortable with Gauss-Codazzi theory may go directly to the next section.

In the following,  $\Sigma$  is a connected, oriented surface, with  $\chi(\Sigma) < 0$ . Let  $(M, h)$  is a complete, 3-dimensional hyperbolic manifold, and let  $f : \Sigma \rightarrow M$  be a smooth immersion.

Let  $g_0$  denote the induced metric over  $S$ , let  $\mathbb{I}$  denote the second fundamental form of  $f$ , and let  $A$  denote its shape operator. Recall that  $f$  is said to be **minimal** whenever its mean curvature  $H := \text{Tr}(A)$  vanishes.

Let  $\kappa$  denote the intrinsic curvature of  $g_0$ . The Gauss equation is

$$(2.1) \quad \kappa = \det_{g_0}(\mathbb{I}) - 1 .$$

where the determinant of the second fundamental form  $\mathbb{I}$  is taken with respect to  $g_0$ . The determinant of  $\mathbb{I}$  is also the determinant of the shape operator  $A$ .

**Definition 2.1** (Hopf differential). The Hopf differential  $\phi$  of  $f$  is the  $(2, 0)$ -component of  $\mathbb{I}$ . That is,  $\phi$  is the symmetric bilinear form given by

$$(2.2) \quad \phi(\xi, \nu) := \frac{1}{4} \left( \mathbb{I}(\xi, \nu) - i\mathbb{I}(J\xi, \nu) - i\mathbb{I}(\xi, J\nu) - \mathbb{I}(J\xi, J\nu) \right) .$$

In particular, when  $f$  is minimal we readily obtain:

$$(2.3) \quad \det_{g_0}(\mathbb{I}) = -|\phi|_{g_0}^2 .$$

In the following, we will denote  $g$  a fixed hyperbolic metric on a Riemann surface, we call the pair  $(u, \phi)$  Gauss-Codazzi data for the immersion, where  $e^{2u}g$  is the induced metric and  $\phi$  is the Hopf differential of the corresponding immersion.

**Lemma 2.2.** *Let  $(\Sigma, g_0)$  be a riemannian surface and let  $g_1 = e^{2u}g_0$  be a metric conformal to  $g_0$ . Let  $\Delta^i$  and  $\kappa^i$  denote respectively the laplacian and curvature of the metric  $g_i$  for  $i \in \{0, 1\}$ . Then*

$$(2.4) \quad \kappa^1 e^{2u} = \kappa^0 - \Delta^0 u$$

$$(2.5) \quad \kappa^0 = (\kappa^1 + \Delta^1 u) e^{2u} .$$

*Proof.* Denote  $\text{Ric}^i$  the Ricci tensor of  $g_i$ . The following equation in Besse's book, 1.159 d) [Bes07], relates the Ricci curvatures of  $g_1$  and  $g_0$ :

$$(2.6) \quad \text{Ric}^1 = \text{Ric}^0 - (\Delta^0 u) g_0$$

The remaining equations follow from the fact that, for a riemannian surface  $(\Sigma, g_i)$ ,  $\text{Ric}^{g_i} = \kappa^i g_i$ . Note that Besse uses the analysts' laplacian, whose sign is the inverse of that of the geometer's laplacian.  $\square$

Combining these equations yields the Gauss-Codazzi equation.

**Proposition 2.3** (Gauss-Codazzi). *Let  $f : \Sigma \rightarrow (M, h)$  be a minimal immersion of a surface into a hyperbolic manifold. Let  $g_0$  denote the induced metric and let  $g = e^{-2u}g_0$  denote the unique, complete hyperbolic metric in its conformal class. Then  $u$  satisfies*

$$(2.7) \quad \Delta u = -1 + e^{2u} + e^{-2u} |\phi|_g^2 .$$

*Proof.* By lemma 2.2,

$$(2.8) \quad \kappa_0 = (-1 - \Delta u) e^{-2u} .$$

By Gauss equation,

$$(2.9) \quad \kappa_0 = -1 - e^{-4u} |\phi_0|_g^2 .$$

The result follows upon combining these two equations.  $\square$

By the fundamental theorem of surface theory, for every pair  $(u, \phi)$ , where  $u : \Sigma \rightarrow \mathbb{R}$  and  $\phi$  is a holomorphic quadratic differential on  $\Sigma$  satisfying the Gauss-Codazzi equation (2.7), there exists an immersion of  $\tilde{\Sigma}$  into  $\mathbb{H}^3$  having this Gauss-Codazzi data. As the surface considered is hyperbolic  $\tilde{\Sigma}$  identifies with the open disk  $\mathbb{D}$ , and the immersion is equivariant with regard to some representation  $\rho : \pi_1 \Sigma \rightarrow \text{Isom}(\mathbb{H}^3)$ . Furthermore, this immersion is unique up to post composition by an isometry of  $\mathbb{H}^3$ .

We will require the following result of Li-Mochizuki [LM20].

**Proposition 2.4** (Sub-supersolution method). *Let  $(\Sigma, g)$  be a riemannian surface, and let  $F$  be a smooth function defined on  $\Sigma \times \mathbb{R}$ . Consider the equation (E):*

$$(2.10) \quad \Delta_g u = F(x, u)$$

Assume there are  $u_-, u_+ : \Sigma \rightarrow \mathbb{R}$  satisfying:

$$(2.11) \quad u_- < u_+$$

$$(2.12) \quad \Delta_g u_- \geq F(x, u_-)$$

$$(2.13) \quad \Delta_g u_+ \leq F(x, u_+)$$

Then the equation (E) admits at least one solution  $u$  satisfying  $u_- \leq u \leq u_+$ .

First remark that almost-fuchsian subsurfaces of  $H^3$  are necessarily uniformized by the Poincaré disk, and that the conformal factor between the induced metric and the hyperbolic one is bounded:

**Lemma 2.5.** *Let  $M$  be a complete hyperbolic 3-manifold. Let  $f : \Sigma \rightarrow M$  be an almost-fuchsian immersion, then its induced metric  $g_0$  is conformal to a complete hyperbolic metric  $g$ . Denote  $g_0 = e^{2u}g$ . Then  $u$  satisfies.*

$$(2.14) \quad -\frac{\ln 2}{2} \leq u \leq 0 .$$

*Proof.* Consider the prescribed curvature equation on the universal cover :

$$(2.15) \quad \Delta^{g_0} v = e^{2v} - \kappa_{g_0} .$$

For any solution  $v$ , the corresponding metric  $e^{2v}g_0$  will be hyperbolic. As  $f$  is almost fuchsian, the curvature  $\kappa_{h_0}$  satisfies  $-2 \leq \kappa_{h_0} \leq -1$ . Upon applying the sub-supersolution method (Prop 2.4) to (2.15) with subsolution  $v_- = 0$  and supersolution  $v_+ = \frac{\ln 2}{2}$ , we get a solution  $v$  satisfying

$$(2.16) \quad 0 \leq v \leq \frac{\ln 2}{2} .$$

Since  $v$  is bounded,  $(\Sigma, e^{2v}g_0)$  is quasi-isometric to  $(\Sigma, g_0)$ . As  $f$  is almost-fuchsian, it is proper. Thus  $g_0$  is complete, and  $e^{2v}g_0$  is also complete. By the Ahlfors-Schwarz-Pick lemma [Ahl38], the complete hyperbolic metric. is unique in its conformal class, hence  $g = e^{2v}g_0$  is complete hyperbolic. Denote  $u = -v$ , we check:

$$(2.17) \quad -\frac{\ln 2}{2} \leq u \leq 0 ,$$

As asserted. □

*Remark.*

- With the same proof, one can prove that this is a general property of complete pinched Hadamard disks, due to Bland and Kalka [BK86]. It is also stated in Benoist-Hulin [BH20], Prop. 3.1. : A complete pinched hadamard disk is uniformly quasiconformal to the complete hyperbolic disk.
- In [AS20], Alvarez and Smith show a more general version of this: The conformal factor is actually bounded in  $C^{k+2,\alpha}$ .

**Theorem 2.6.** *Let  $g$  be a complete hyperbolic metric on  $\Sigma$ ,  $u : \Sigma \rightarrow \mathbb{R}$  a bounded continuous map, and  $\phi$  a holomorphic quadratic differential on  $\Sigma$  satisfying*

$$(2.18) \quad \Delta^g u = -1 + e^{2u} + e^{-2u} |\phi|^2, \text{ and}$$

$$(2.19) \quad \sup_{z \in \Sigma} (e^{-2u(z)} |\phi(z)|) < 1.$$

*Then there is an almost-fuchsian immersion of  $\Sigma$  into a hyperbolic manifold such that  $e^{2u}g$  is the induced metric and  $\phi$  is the  $(2,0)$ -part of the second fundamental form. Conversely, for any almost-fuchsian immersion the associated data satisfies these relations.*

*Proof.* By the fundamental theorem of surface theory, we get an immersion into a hyperbolic manifold. Since  $u$  is bounded, the induced metric is complete and it follows from (2.19) that the principal curvatures lie in  $(-1, 1)$ . Hence  $f$  is almost-fuchsian.

Conversely, if  $f$  is almost-fuchsian, consider  $(u, \phi)$  the associated Gauss-Codazzi data. Then Gauss-Codazzi is verified, the bound on the principal curvatures yields (2.19) and the boundedness of  $u$  follows from the previous lemma.  $\square$

Note that almost-fuchsian subdisks of  $\mathbb{H}^3$  are always properly embedded and bound a quasicircle at infinity. Indeed, this readily follows from the original work [Uhl83] of Uhlenbeck and the argument of Theorem 1.3 of [HW13]. For the reader's convenience, we state and prove this result here.

**Theorem 2.7.** *Let  $e : \mathbb{D} \rightarrow \mathbb{H}^3$  be an almost-fuchsian disk. Then the disk  $e(\mathbb{D})$  is properly embedded and bounds a quasi-circle at infinity. Furthermore, if  $k$  denotes the supremum of the principal curvatures of  $e$ , the quasisymmetry constant of the quasi-circle is bounded by  $\frac{1+k}{1-k}$ .*

*Proof.* Let  $g$  be the Poincaré metric of  $\mathbb{D}$ . Let  $g_0 = e^{2u}g$  denote the induced metric of  $e$  on the disk  $\mathbb{D}$ . Consider  $\nu$  the unit normal vector field, and define  $\tilde{e}$  by

$$(2.20) \quad \tilde{e} : \begin{cases} \mathbb{D} \times \mathbb{R} & \rightarrow & \mathbb{H}^3, \text{ and} \\ (x, t) & \mapsto & \exp_x(t\nu_x). \end{cases}$$

where  $\exp$  denotes the exponential map of  $\mathbb{H}^3$ . Let  $A$  denote the shape operator of  $f$ . The same computation as of lemma 2.1. of [HW13] gives the following expression of the metric induced by  $\tilde{e}$ :

$$(2.21) \quad h^\nu(x, t) := h_t(x) \oplus dt^2,$$

where, for all  $t$ ,

$$(2.22) \quad h_t(\cdot, \cdot) = e^{2u}g((\cosh(t)I_0 + \sinh(t)A)\cdot, (\cosh(t)I_0 + \sinh(t)A)\cdot).$$

Consider  $0 < k < 1$  such that the operator norm of  $A$  is less than  $k$ . Since  $|A| \leq k < 1$ ,  $h^\nu$  is complete and thus  $\tilde{e}$  is a diffeomorphism. In particular,  $e$  is properly embedded.

Let  $f : \mathbb{D} \rightarrow \mathbb{H}^3$  be an isometric parametrization of a totally geodesic plane, let  $\nu_f$  denote its unit normal vector field, and define  $\tilde{f} : \mathbb{D} \times \mathbb{R} \rightarrow \mathbb{H}^3$  by

$$(2.23) \quad \tilde{f}(x, t) = \exp_{f(x)}(t\nu_f(x)) .$$

The metric induced by  $\tilde{f}$  is

$$(2.24) \quad \tilde{g}(x, t) = \cosh^2(t)g \oplus dt^2 .$$

It follows from (2.24) and (2.22) that  $\alpha := e \circ f^{-1}$  is bilipschitz. It therefore extends (see Gehring [Geh62] or Mostow [Mos68]) to a quasiconformal map  $\tilde{\alpha}$  from  $\partial_\infty \mathbb{H}^3$  to itself. It follows that  $\Gamma := \partial_\infty e(\mathbb{D})$  is a quasicircle.

Note that, for all  $t$ , the restrictions of  $h^0$  and  $\tilde{g}$  to the slice  $\mathbb{D} \times \{t\}$  are  $K$ -quasiconformal, with

$$(2.25) \quad K = \frac{1+k}{1-k} .$$

Letting  $t$  tend to  $+\infty$ , it follows that  $\tilde{\alpha}$  is  $K$ -quasiconformal. As the boundary of the disk is the image by  $\tilde{\alpha}$  of an equator in  $\mathbb{S}^2$ , this completes the proof.  $\square$

### 3. PROOF OF MAIN THEOREM

As in this paper we are mainly interested in the description of conformal almost-fuchsian disks, let us define the space we want to describe:

**Definition 3.1.** We denote  $\mathcal{AF}$  the set of almost-fuchsian immersions  $\mathbb{D}^2 \rightarrow \mathbb{H}^3$  up to postcomposition by an isometry of  $\mathbb{H}^3$ .

The set  $\mathcal{AF}$ , equipped with its quotient topology, can be identified with the set of almost-fuchsian immersions with prescribed image and differential of the origin  $o$ . As such, it can be viewed as a subspace of the space of immersions from the disk into  $\mathbb{H}^3$ .

*Remark.* The space  $\mathcal{AF}$  does not parametrize almost-fuchsian subdisks of  $\mathbb{H}^3$ . However, the set of almost-fuchsian subdisks up to isometry can be identified to the quotient of  $\mathcal{AF}$  by a  $\mathrm{PSL}(2, \mathbb{R})$  action by precomposition, which is not a properly discontinuous action.

In order to prove our theorems in a noncompact setting, we will use the following version of Omori-Yau's theorem [Omo67, Yau75]. It can be found in Li-Mochizuki [LM20], we reproduce it here for the reader's convenience.

**Theorem 3.2** (Omori-Yau). *Let  $(M, h)$  be a complete manifold with Ricci curvature bounded from below. Let  $u : M \rightarrow \mathbb{R}$  be a map bounded from above and let  $\varepsilon > 0$ . Then there is  $x \in M$  such that*

$$(3.1) \quad u(x) \geq \sup u - \varepsilon ,$$

$$(3.2) \quad |\nabla_h u(x)| \leq \varepsilon , \text{ and}$$

$$(3.3) \quad \Delta_h u(x) \leq \varepsilon .$$

The first step is to show that the moduli space of almost-fuchsian disks injects into the space of bounded holomorphic quadratic differentials.

**Proposition 3.3.** *Let  $\phi$  be a holomorphic quadratic differential on the disk. Then there is at most one bounded solution to*

$$(3.4) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2$$

such that  $e^{-2u}|\phi| < 1$  everywhere.

*Proof.* Suppose  $(u, \phi)$  and  $(v, \phi)$  are two bounded solutions of (3.4) such that  $e^{-2u}|\phi| < 1$  and  $e^{-2v}|\phi| < 1$ . Then  $u - v$  is bounded and satisfies

$$(3.5) \quad \Delta(u - v) = (e^{2v} - e^{-2v}|\phi|^2)(e^{2(u-v)} - 1) + |\phi|^2 e^{-2v}(e^{2(u-v)} + e^{2(v-u)} - 2).$$

Using the positivity of the second term of the right hand side and the almost-fuchsian property of  $v$ , one gets

$$(3.6) \quad \Delta(u - v) \geq (e^{2v} - e^{-2v}|\phi|^2)(u - v).$$

Applying Omori-Yau's maximum principle gives  $u - v \leq 0$ . Exchanging  $u$  and  $v$ , one has  $u = v$ , thus the corresponding almost-fuchsian disks are conjugated.  $\square$

*Remark.* The same reasoning actually shows that if  $(e^{2u}g, \phi)$  is almost-fuchsian, then any upper bounded  $v$  solution to the Gauss-Codazzi equation with quadratic differential  $\phi$  must satisfy:

$$(3.7) \quad v \leq u.$$

In [HLT21], Huang, Lucia and Tarantello study the solutions of the Gauss-Codazzi equation on closed surfaces, of which the maximality of almost-fuchsian solutions is a consequence.

Moreover, we will show that the image of the almost-fuchsian locus in the space of bounded holomorphic quadratic differentials on the disk is actually open. Combining Proposition 3.3 with the fundamental theorem of surface theory, we get :

**Corollary 3.4.** *For  $f : \mathbb{D}^2 \rightarrow \mathbb{H}^3$  an immersion, denote  $\phi_f$  its Hopf differential. Then the map:*

$$(3.8) \quad \Phi : \begin{cases} \mathcal{AF} & \rightarrow & QD(\mathbb{D}^2) \\ f & \mapsto & \phi_f \end{cases}$$

is injective.

*Remark.* The statement of this corollary in the equivariant case is the Lemma 2.11 of Krasnov–Schlenker [KS07].

*Proof.* We already know that the Gauss-Codazzi data of an immersion characterizes it up to isometry. The prop (3.3) shows that if two almost-fuchsian immersions have same Hopf differentials, then they have the same Gauss-Codazzi data, hence the injectivity of our map.  $\square$

**Theorem 3.5.** *The image of the almost-fuchsian locus is open in the space of bounded holomorphic quadratic differentials.*

*Proof.* Consider  $\phi$  in  $\mathfrak{S}(\mathcal{AF})$ ,  $u$  and  $0 < \delta < 1$  satisfying

$$(3.9) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2 \text{ and}$$

$$(3.10) \quad \sup_{z \in \mathbb{D}} e^{-2u}|\phi| \leq 1 - \delta.$$

We will prove that the linearized operator of (3.9)  $L_u : v \mapsto -\Delta v + 2(e^{2u} - e^{-2u}|\phi|^2)v$  is an isomorphism between the Banach spaces  $C^{2,\alpha}(\mathbb{D})$  and  $C^{0,\alpha}(\mathbb{D})$ , where here  $\mathbb{D}$  is equipped with its Poincaré metric, hence, using the local inversion principle, we will be able to find a solution  $\tilde{u}$  uniformly close to  $u$  on a neighborhood of  $\phi$ , which then has to be almost-fuchsian.

At first, as  $u$  is bounded,  $L_u$  trivially defines a bounded linear map from  $C^{2,\alpha}(\mathbb{D})$  towards  $C^{0,\alpha}(\mathbb{D})$ . The almost-fuchsian condition implies

$$(3.11) \quad \frac{\delta}{2} \leq e^{2u} - e^{-2u}|\phi|^2 \leq 1.$$

Hence  $L_u$  is strictly elliptic. We now show that  $L_u$  is injective. Indeed, choose  $v \in \ker L_u$ . Since  $v$  is bounded, Omori-Yau's maximum principle implies

$$(3.12) \quad v \leq 0.$$

The same reasoning to  $-v$  shows that  $v = 0$ , hence  $L_u$  is injective, as asserted.

It remains only to prove surjectivity. First consider a compact set  $K$  in  $\mathbb{D}$  with smooth boundary. It is a standard consequence of elliptic theory and of the strict ellipticity of  $L_u$  that  $L_u$  is an isomorphism from  $C_0^{2,\alpha}(K)$  into  $C^{0,\alpha}(K)$ . Now choose  $u \in C^{0,\alpha}(\mathbb{D})$ . and a compact exhaustion  $(K_n)$  of  $\mathbb{D}$  such that each  $K_i$  has smooth boundary. For each  $i$  there is a unique  $v_i \in C_0^{2,\alpha}(K_i)$  such that  $L_u v_i = w \mathbf{1}_{K_i}$ . We claim that the sequence  $(|v_n|_\infty)$  is bounded. Indeed, by the maximum principle, for all  $i$ ,

$$(3.13) \quad |u_i|_\infty \leq \max\{2\delta, 1\} |w \mathbf{1}_{K_i}|_\infty \leq 2|w|_\infty.$$

Fix a radius  $r > 0$  and  $x \in \mathbb{D}$ . For  $i$  big enough such that  $B(x, 2r) \subset K_i$ ,

$$(3.14) \quad |L_u v_i|_{C^{0,\alpha}(B(x,2r))} \leq |w|_{C^{0,\alpha}}, \text{ and}$$

$$(3.15) \quad |v_i|_{C^0(B(x,2r))} \leq 2|w|_{C^0}.$$

Hence, by the Schauder estimates, there is a  $C > 0$  depending only on  $r$  such that

$$(3.16) \quad |v_i|_{C^{2,\alpha}(B(r,x))} \leq C|w|_{C^{0,\alpha}}.$$

By the Arzela-Ascoli theorem, up to extracting a subsequence, we can assume there exists  $v_\infty \in C_{\text{loc}}^{2,\alpha}$  such that  $v_n \rightarrow v_\infty$  on every compact set in the  $C^{2,\beta}$  topology for  $\beta < \alpha$ . Furthermore

$$\begin{aligned} |v_\infty|_{C^{2,\alpha}} &\leq C' \sup_{x \in \mathbb{H}^2} |v_\infty|_{C^{2,\alpha}(B(r,x))} \\ &\leq C' \sup_{x \in \mathbb{H}^2} \limsup_{i \rightarrow \infty} |v_i|_{C^{2,\alpha}(B(r,x))} \\ &\leq C' C |w|_{C^{0,\alpha}}, \end{aligned}$$

so that  $v_\infty \in C^{2,\alpha}(\mathbb{H}^2)$ . By continuity of  $L_u$ ,

$$(3.17) \quad L_u v_\infty = w.$$

Thus  $L_u$  is surjective.

By the Closed Graph Theorem,  $L_u$  is a linear isomorphism. It follows by the inverse function theorem that the image of the almost-fuchsian locus is open in the space of bounded quadratic differentials, as desired.  $\square$

Equipped with this theorem, we will consider a perturbation  $\phi_t$  of  $\phi$  in the image of the almost-fuchsian locus and we will consider the family  $(u_t)$  of bounded maps satisfying the Gauss-Codazzi equation. In particular, we will need the derivatives  $\dot{u}_t$  and  $\ddot{u}_t$  to be bounded.

The following is actually a straightforward consequence of the implicit function theorem we used.

**Proposition 3.6.** *Consider  $(e^{2u}, \phi)$  an almost-fuchsian disk. Let  $f_t$  be a  $C^{0,\alpha}$  perturbation of  $|\phi|^2$ . Denote by  $(u_t)$  the smooth branch of solutions to :*

$$(3.18) \quad \Delta u_t = -1 + e^{2u_t} + e^{-2u_t} f_t$$

for  $t$  in a neighborhood of 0. Then  $\dot{u}_t$  and  $\ddot{u}_t$  are bounded.

*Proof.* Consider  $F : C^{2,\alpha}(\mathbb{D}) \times [0, 1] \rightarrow C^{0,\alpha}(D)$  defined by  $F(v, t) = \Delta v + 1 - e^{2v} - e^{-2v} f_t$ . Thanks to the previous lemma, we know the local existence of a smooth  $G : [0, 1] \rightarrow C^{2,\alpha}(\mathbb{D})$  such that  $F(G(t), t) = 0$ . It follows by smoothness of  $G$  that  $\dot{u}_t, \ddot{u}_t \in C^{2,\alpha}(\mathbb{D})$  for all  $t$ , and the result follows.  $\square$

We are now able to show that the almost-fuchsian locus retracts to the set of almost-fuchsian immersions with zero quadratic differentials.

**Theorem 3.7.** *Let  $\phi \in \mathfrak{S}(\mathcal{AF})$  and let  $\psi$  be a holomorphic quadratic differential on the disk satisfying*

$$(3.19) \quad \forall z \in \mathbb{D}, |\psi(z)| \leq |\phi(z)| .$$

Then  $\psi \in \mathfrak{S}(\mathcal{AF})$ .

*Proof.* Let  $u : \mathbb{D} \rightarrow \mathbb{R}$  be such that  $(u, \phi)$  is the Gauss-Codazzi data of an almost-fuchsian disk. Let  $f_t : \mathbb{D} \rightarrow [0, \infty[$  be a smooth map such that  $f_0 = |\psi|^2$  and  $f_1 = |\phi|^2$ . As  $|\psi| \leq |\phi|$ , one can choose  $f$  so that the maps  $t \mapsto f_t(z)$  are nondecreasing.

We will show that  $\psi \in \mathfrak{S}(\mathcal{AF})$ , i.e. that there is a solution  $v$  to

$$(3.20) \quad \Delta v = -1 + e^{2v} + e^{-2v} f_0, \text{ and}$$

$$(3.21) \quad \sup_{z \in \mathbb{D}} e^{-4v} f_0 < 1 .$$

To do so, consider for  $t \in [0, 1]$  the equation.

$$(3.22) \quad \Delta u_t = -1 + e^{2u_t} + e^{-2u_t} f_t .$$

Let  $I \subseteq [0, 1]$  denote the set of all  $s$  such that, for all  $t \geq s$ , this equation has a bounded solution  $u_s$  satisfying  $|e^{-4u_s} f_s|_\infty < 1$ , for some  $\epsilon > 0$ . By Theorem 2.6,  $1 \in I$ . By Theorem 3.5 applied to a  $C^{0,\alpha}$  perturbation,  $I$  is open. It thus remains only to show that  $I$  is closed.

At first, consider  $t_0 = \inf I$  and  $(t_n) \in I$  a monotonous sequence such that  $t_n \rightarrow t_0$ . By Lemma 2.5, the sequence  $(u_{t_n})$  is uniformly bounded, from which it follows that  $(\Delta u_{t_n})$  is also uniformly bounded. By Schauder's estimates, the sequence  $(u_{t_n})$  is uniformly bounded in  $C^{1,\alpha}$  on any compact set. Hence we may assume that  $u_{t_n}$  converges towards a map  $u_{t_0} \in C^{2,\alpha}(\mathbb{H})$  in the  $C^{2,\beta}$  topology for all  $\beta < \alpha$ . Trivially  $u_{t_0}$  solves (3.22).

It remains only to check that  $|e^{-4u_{t_0}} f_{t_0}|_\infty < 1$ .

Differentiating the above equation with respect to  $t$  yields,

$$(3.23) \quad \Delta \dot{u}_t = 2(e^{2u_t} - e^{-2u_t} f_t) \dot{u}_t + e^{-2u_t} \dot{f}_t .$$

By Lemma 2.5, there exists  $B > 0$  such that  $|u_t|_\infty < B$  for all  $t$ . It follows by the Omori-Yau maximum principle that, over  $I$ ,  $\dot{u}_t \leq 0$ , so that, for all  $t \in I$ ,

$$(3.24) \quad |e^{-4u_t} \dot{f}_t|_\infty \leq |e^{-4u_1} \dot{f}_1|_\infty < 1 .$$

Closedness of  $I$  now follows so that, by connectedness,  $0 \in I$ . At  $t = 0$ , this means that  $(u_0, \psi)$  is the Gauss-Codazzi data of an almost-fuchsian disk. We have shown that  $\psi \in \mathfrak{S}(\mathcal{AF})$ , as desired.  $\square$

**Corollary 3.8.** *Let  $\phi \in \mathfrak{S}(\mathcal{AF})$  and let  $u : \mathbb{D} \rightarrow \mathbb{R}$  be the associated bounded map such that*

$$(3.25) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2, \text{ and}$$

$$(3.26) \quad \sup_{z \in \mathbb{D}} e^{-2u}|\phi| \leq 1 - \delta.$$

Denote  $\phi_t = t\phi$  for  $t \in [0, 1]$ . Then for any  $t \in [0, 1]$ , there is a  $u_t : \Sigma \rightarrow \mathbb{R}$  bounded such that  $(e^{2u_t}g, \phi_t)$  satisfies the Gauss-Codazzi equation. Furthermore, for all  $t \in [0, 1]$ ,  $(e^{2u_t}g, \phi_t)$  is almost-fuchsian.

*Remark.* For the equivariant case, this computation can already be found in a survey of Trautwein [Tra19], step 3 of proof of theorem 5.13.

**Corollary 3.9.** *The moduli space of almost-fuchsian disks up to isometry is contractible. The set of almost-fuchsian immersions retracts onto the set of geodesic immersions of disks in  $\mathbb{H}^3$ .*

We continue with the proof of our main theorem:

**Theorem 3.10** (Convexity). *Let  $(e^{2u_0}g, \phi_0)$  and  $(e^{2u_1}g, \phi_1)$  be two pairs corresponding to almost-fuchsian disks. For  $t \in [0, 1]$ , denote  $\phi_t := (1-t)\phi_0 + t\phi_1$ . Then for any  $t$ , there is  $(e^{2w_t}g, \phi_t)$  satisfying the Gauss-Codazzi equation, and it corresponds to an almost-fuchsian disk.*

*Proof.* First, we write down Gauss-Codazzi and we differentiate it twice in  $t$

$$(3.27) \quad \Delta w_t = -1 + e^{2w_t} + e^{-2w_t}|\phi_t|^2,$$

$$(3.28) \quad \Delta \dot{w}_t = 2\dot{w}_t e^{2w_t} - 2\dot{w}_t e^{-2w_t}|\phi_t|^2 + 2e^{-2w_t}g(\phi_t, \phi_1 - \phi_0), \text{ and}$$

$$(3.29) \quad \begin{aligned} \Delta \ddot{w}_t &= 2\ddot{w}_t(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 4\dot{w}_t^2(e^{2w_t} + e^{-2w_t}|\phi_t|^2) \\ &\quad - 8\dot{w}_t e^{-2w_t}g(\phi_t, \phi_1 - \phi_0) + 2e^{-2w_t}|\phi_1 - \phi_0|^2. \end{aligned}$$

We first show that as long as  $(e^{2w_t}g, \phi_t)$  is almost-fuchsian,

$$(3.30) \quad \ddot{w}_t + 4\dot{w}_t^2 \leq 0.$$

This is a consequence of the Omori-Yau maximum principle and the following standard formula.

$$(3.31) \quad \Delta(\dot{w}_t^2) = 2\dot{w}_t \Delta \dot{w}_t + 2|\nabla \dot{w}_t|^2 \geq 2\dot{w}_t \Delta \dot{w}_t.$$

Then, using equations (3.28) and (3.29), we obtain

$$(3.32) \quad \begin{aligned} \Delta(\ddot{w}_t + 4\dot{w}_t^2) &\geq 2\ddot{w}_t(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 20\dot{w}_t^2 e^{2w_t} - 12\dot{w}_t^2 e^{-2w_t}|\phi_t|^2 \\ &\quad + 8\dot{w}_t e^{-2w_t}g(\phi_t, \phi_1 - \phi_0) + 2e^{-2w_t}|\phi_1 - \phi_0|^2. \end{aligned}$$

Judiciously rearranging the terms, we obtain

$$(3.33) \quad \begin{aligned} \Delta(\ddot{w}_t + 4\dot{w}_t^2) &\geq 2(\ddot{w}_t + 4\dot{w}_t^2)(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 12\dot{w}_t^2(e^{2w_t} - e^{-2w_t}|\phi_t|^2) \\ &\quad + 8\dot{w}_t^2 e^{-2w_t}|\phi_t|^2 + 8e^{-2w_t}g(\dot{w}_t \phi_t, \phi_1 - \phi_0) + 2e^{-2w_t}|\phi_1 - \phi_0|^2. \end{aligned}$$

As  $8A^2 + 8AB + 2B^2 = 2(2A + B)^2$  is nonnegative, we obtain

$$(3.34) \quad \Delta(\ddot{w}_t + 4\dot{w}_t^2) \geq 2(\ddot{w}_t + 4\dot{w}_t^2)(e^{2w_t} - e^{-2w_t}|\phi_t|^2) + 12\dot{w}_t^2(e^{2w_t} - e^{-2w_t}|\phi_t|^2) .$$

Hence, provided the pair is almost-fuchsian, we get

$$(3.35) \quad \Delta(\ddot{w}_t + 4\dot{w}_t^2) \geq 2(\ddot{w}_t + 4\dot{w}_t^2)(e^{2w_t} - e^{-2w_t}|\phi_t|^2) .$$

Thus we deduce that

$$(3.36) \quad \ddot{w}_t + 4\dot{w}_t^2 \leq 0 .$$

We have shown that as long as  $(e^{2u_t}g, \phi_t)$  is almost-fuchsian,  $\ddot{w}_t + 4\dot{w}_t^2 \leq 0$ , as asserted.

Consider now the family of quadratic holomorphic differentials

$$(3.37) \quad \phi_{s,t} := s(1-t)\phi_0 + st\phi_1$$

For all  $s$ , let  $\mathcal{J}_s$  denote the set of all  $t \in [0, 1]$  for which there exists a function  $w_{s,t}$  making  $(e^{2w_{s,t}}g, \phi_{s,t})$  into an almost Fuchsian pair. Let  $\mathcal{S}$  denote the set of all  $s \in [0, 1]$  for which  $\mathcal{J}_s$  equals the interval  $[0, 1]$ . Trivially  $\mathcal{S}$  contains 0. By Theorem 3.5,  $\mathcal{S}$  is open. We now show that  $\mathcal{S}$  is closed.

First, if  $s_0 \in \overline{\mathcal{S}}$ , the equicontinuity of all functions  $(u_{s,t})$  for  $s \in \mathcal{S}$ ,  $t \in [0, 1]$  allows us to get a limiting family of functions which are solutions to Gauss-Codazzi equation. We claim that, for all  $s$ , the function  $t \mapsto e^{-4w_{s,t}}|\phi_{s,t}|^2$  is convex over  $\mathcal{J}_s$ . Indeed,

$$\begin{aligned} e^{4w_{s,t}}\partial_t^2(e^{-4w_{s,t}}|\phi_{s,t}|^2) &= -4(\ddot{w}_{s,t} + 4\dot{w}_{s,t}^2)|\phi_{s,t}|^2 + 32\dot{w}_{s,t}^2|\phi_{s,t}|^2 - 16\dot{w}_{s,t}g(\phi_{s,t}, \phi_{s,1} \\ &\quad - \phi_{s,0}) + 2|\phi_{s,1} - \phi_{s,0}|^2 . \\ &= -4(\ddot{w}_{s,t} + 4\dot{w}_{s,t}^2)|\phi_{s,t}|^2 + 2|4\dot{w}_{s,t}\phi_{s,t} - (\phi_{s,1} - \phi_{s,0})|^2 . \end{aligned}$$

This is nonnegative, and convexity follows. By convexity,  $e^{-4w_{s,t}}|\phi_{s,t}|^2$  is smaller than the maximal value for  $t = 0$  and  $t = 1$ , and we deduce that it is smaller than  $1 - \delta$ , for  $\delta > 0$  depending only on  $\phi_0$  and  $\phi_1$ . It now readily follows that  $\mathcal{S}$  is closed, so that, by connectedness,  $1 \in \mathcal{S}$ . In other words,  $\mathcal{J}_1 = [0, 1]$ , and the result follows.  $\square$

We also have the following quantitative estimate for this convex set.

**Theorem 3.11.** *The convex set  $\mathfrak{S}(\mathcal{AF})$  satisfies:*

$$(3.38) \quad \left\{ \phi : \sup_{z \in D} |\phi(z)| < \frac{1}{2} \right\} \subset \mathfrak{S}(\mathcal{AF}) \subset \left\{ \phi : \sup_{z \in \mathbb{D}} |\phi(z)| < 1 \right\}$$

*Proof.* Take  $\phi$  with  $|\phi|_\infty \leq \frac{1}{2}$ . Then  $-\frac{\ln 2}{2}$  and 0 are respectively sub and supersolutions of

$$(3.39) \quad \Delta u = -1 + e^{2u} + e^{-2u}|\phi|^2 .$$

Hence by Proposition 2.4, we have a bounded solution satisfying Gauss-Codazzi, and necessarily  $e^{-2u}|\phi| \leq 2|\phi|_\infty < 1$ . Hence  $\phi \in \mathfrak{S}(\mathcal{AF})$ . Conversely, if  $\phi \in \mathfrak{S}(\mathcal{AF})$ ,  $u$  the solution to Gauss-Codazzi satisfies  $u \leq 0$ , and  $|\phi|_\infty \leq |e^{-2u}\phi|_\infty < 1$ .  $\square$

Fixing now a subgroup  $\Gamma$  of  $\mathrm{PSL}(2, \mathbb{R})$ , one can look at the subset  $\mathcal{AF}_\Gamma$  of elements of  $\mathcal{AF}$  equivariant under some representation of  $\Gamma$ . The uniqueness properties shown imply that this set can be identified with a subset of  $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))/\mathrm{PSL}(2, \mathbb{C})$ . In the case of a closed surface group, this subset are known as *almost-fuchsian representations* considered by Uhlenbeck [Uhl83].

Fix a hyperbolic surface  $\Sigma_{g,n}$ , and consider  $\Gamma$  its fundamental group. Considering now holomorphic quadratic differentials equivariant under the action of  $\Gamma$ , identifying holomorphic quadratic differentials with the cotangent bundle to Teichmüller space of genus  $(g, n)$ , we get a smooth family of open bounded sets in each leaf. As these sets are isotropic (if  $(e^{2u}g, \phi)$  is an almost-fuchsian disk, so is  $(e^{2u}g, z\phi)$  for any  $z$  of modulus one), it induces a Finsler metric on Teichmüller space, which is quasi-isometric to the Finsler metric dual to the  $L^\infty$  unit balls in  $T^*\mathcal{T}_g$ .

**Corollary 3.12.** *For  $g \geq 2$  and  $\mathcal{T}_g$  the Teichmüller space of genus  $g$ , there is a Finsler metric  $p$  on  $\mathcal{T}_g$  such that:*

$$(3.40) \quad p(\phi) < 1 \iff \phi \in \mathfrak{S}(\mathcal{AF})$$

*This Finsler metric is quasi-isometric to the Finsler metric dual to the  $L^\infty$  unit balls in  $T^*\mathcal{T}_g$ .*

*Remark.* It is interesting to compare this Finsler metric to the metrics dual to  $L^p$ -unit balls in  $T^*\mathcal{T}_g$ . For  $p = 2$ , one gets Weil-Petersson's metric, see Wolpert [Wol09]. For  $p = 1$ , one gets the Teichmüller metric, see Papadopoulos-Su [PS15].

*Remark.* With this discussion, we have a description of the fibration map from  $\mathcal{AF}_\Gamma \rightarrow \mathcal{T}_{g,n}$ , which is simply the conformal class of the induced metric on  $\Sigma_{g,n}$ .

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