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Hindered Settling of Well-Separated Particle Suspensions

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Abstract

We consider N identical inertialess rigid spherical particles in a Stokes flow in a domain $\Omega \subseteq \mathbb{R}^3$. We study the average sedimentation velocity of the particles when an identical force acts on each particle. If the particles are homogeneously distributed in directions orthogonal to this force, then they hinder each other leading to a mean sedimentation velocity which is smaller than the sedimentation velocity of a single particle in an infinite fluid. Under suitable convergence assumptions of the particle density and a strong separation assumption, we identify the order of this hindering as well as effects of small scale inhomogeneities and boundary effects. For certain configurations we explicitly compute the leading order corrections.

1 Introduction

The sedimentation velocity of a single inertialess rigid sphere in an infinite fluid follows immediately from Stokes' law for the drag force. This law entails that the sphere falls parallel to the direction of the force acting on the particle (say gravity) with amplitude:

$$V^{\text{St}} := \frac{|F|}{6\pi\mu R}, \quad (1.1)$$

where F is the force acting on the particle, R its radius and μ the fluid viscosity. When several particles fall in the flow, the possible interactions between the particles through the fluid make however the situation much more complicated as soon as there are more than 3 particles, see [GM12, Section 6.1].

When F is gravity, computing the mean sedimentation velocity of a cloud of particles in a Stokes flow is then a classical problem that has been studied in many previous references [Bat72; Bur38; Feu84; GM88; Has59; Saf73], to mention a few. We refer to the review [DA85]

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and to the introduction of [DG22] for a historical perspective. In these works it has been observed (mostly on a formal level) that the mean sedimentation velocity of a cloud of N particles in the whole space remains parallel to F and that its magnitude \bar{V}_N^{sed} behaves in fundamentally different ways dependent on the particle distribution.

- (Dil) There is a characterization of *diluteness* of suspensions for which the settling particles behave as if they were alone in the fluid [JO04].
- (MF) If the particles are less dilute and not homogeneously distributed in directions orthogonal to gravity, a *macroscopic fluid flow* is created which enhances sedimentation: for sufficiently regular particle distributions, where not too much clustering occurs, the mean sedimentation velocity is of order

$$\bar{V}_N^{sed} \sim \max \left\{ V^{St}, \frac{NF}{\mu L} \right\} \quad (1.2)$$

where N is the number of particles and L is the typical length scale of the particle cloud [Höf18; Mec19]. The additional term $\frac{NF}{\mu L}$ is precisely the parameter that characterizes diluteness in the above sense for such regular distributions and can be much larger than V^{St} .

- (HS) If the particles are closer and homogeneously distributed in directions orthogonal to gravity, the incompressibility of the fluid prevents the onset of a macroscopic fluid flow that enhances sedimentation. Instead, a small fluid backflow is created that *hinders the particle sedimentation*. The order of this hindering is again sensitive to the particle distribution:

- a) If the particles are *periodically* distributed, then

$$\bar{V}_N^{sed} = V^{St} (1 - a_{per} \phi^{\frac{1}{3}} + o(\phi^{\frac{1}{3}})) \quad (1.3)$$

for some $a_{per} > 0$, where ϕ is the particle volume fraction inside the fluid [Has59].

- b) If the particles are distributed according to *hardcore Poisson process* with hardcore distance $2R$, then

$$\bar{V}_N^{sed} = V^{St} (1 - a_{uni} \phi + o(\phi)) \quad (1.4)$$

for some $a_{uni} > 0$ [Bat72].

The expansion (1.3) has been rigorously shown in [Has59] on the torus. In this contribution we show that it persists to hold asymptotically for large N if the particles are placed in a container $\Omega \subseteq \mathbb{R}^3$ such that

- The particles respect a separation distance of order $N^{-1/3}$.
- The container Ω is bounded in directions orthogonal to the direction of the acting force and the particles are sufficiently close to a macroscopic density n which is constant in directions orthogonal to the acting force.

Although we are mainly interested in the (HS) situation, we complement the analysis in the case (MF) when the orthogonality assumption is not satisfied.

The influence of the container on the sedimentation has been studied on a formal level in several works, see *e.g.* [BM85; GM88; Bru+96]. In these works, the particles are distributed according to a hardcore Poisson process as in [Bat72]. However, in contrast to [Bat72] where the whole space is considered, a nonoverlapping condition with the boundary $\partial\Omega$ restricts the particle centers to lie in $\Omega_R = \{x \in \Omega : \text{dist}(x, \partial\Omega) > R\}$. Since the particles are spherical, this leads to a lower mean volume concentration of particles in $\Omega \setminus \Omega_R$ than in Ω_R (where this concentration is constant). This discrepancy leads to a macroscopic fluid flow v_f just like in (MF). However, since the inhomogeneity only occurs in the small region $\Omega \setminus \Omega_R$, this macroscopic fluid flow, called *intrinsic convection*, is much smaller than in (MF). The authors in [BM85; GM88; Bru+96] obtain $v_f = O(\phi V^{\text{St}})$. Moreover v_f decreases the sedimentation speed of particles close to the boundary of the container while it increases the sedimentation speed of particles in the bulk. In the present paper, we rigorously identify a related but quantitatively different effect. Namely, for particle configurations satisfying both items above, we analyze perturbations of the particle distributions on the $N^{-1/3}$ -scale that occur in the bulk rather than at the boundary of the container. This leads to macroscopic fluid velocities $v_f = O(N^{1/3}\phi^{1/3}V^{\text{St}})$. The contribution of this macroscopic fluid velocity to the average sedimentation velocity is much lower though, namely of order $\phi^{1/3}V^{\text{St}}$.

All these approaches to the computation of sedimentation velocity (including the present contribution) are based on a similar construction of the many-particle Stokes solution. Acting a force on each particles entails a microscopic disturbance in the flow around the particle that decays very slowly to zero at infinity. Summing the microscopic disturbances of all the particles cloud on one particle then creates a macroscopic disturbance that modifies its sedimentation velocity. A key-difficulty is then to prove that, despite the slow decay of the microscopic disturbances, the macroscopic disturbance remains bounded, motivating many of the previous references on the topic. If the particles are sufficiently far one from the other then the macroscopic disturbance can be shown to be neglectible and we recover [JO04]. While, if the particles are closer, it turns out that the macroscopic disturbance can be proved to be bounded only because of a backflow due to the fluid incompressibility. For instance, in the case of particles on cubic lattices, Hasimoto mimicks the backflow on the torus by imposing the constraint that the total fluid flow (after extending the fluid flow inside of the particles) vanishes. By Fourier analysis, he then explicitly computed the expansion (1.3) [Has59].

In this contribution, we show that the boundaries make the macroscopic disturbance converge: they induce naturally a normalization of the pressure that makes the backflow explicit and the microscopic disturbances due to each particle decay faster. This improves the simplicity of the analysis.

1.1 Setting

Let $\Omega \subseteq \mathbb{R}^3$ be of class \mathcal{C}^2 and contained in an infinite cylinder with an orientation ξ , *i.e.*,

$$\exists C_1 > 0, \quad \text{s.t.} \quad \Omega \subseteq \{x \in \mathbb{R}^3 : \text{dist}(x, \text{span}\{\xi\}) < C_1\}. \quad (\text{H0})$$

We point out that Ω might be bounded as well as unbounded. For $N \in \mathbb{N}$ and $r > 0$, let $R_N := N^{-1/3}r$ and $X_i^N \in \Omega$ such that $B_i^N := B_{R_N}(X_i^N) \Subset \Omega$ and $\overline{B}_i^N \cap \overline{B}_j^N = \emptyset$ for all $1 \leq i \neq j \leq N$. We will write R , X_i and B_i instead of R_N , X_i^N and B_i^N in the following. We

assume throughout the paper that the distribution of particles is regular in the following sense. Firstly, we have the following separation assumptions:

$$\exists c > 0 \quad \min_{i \neq j} |X_i - X_j| \geq cN^{-1/3}, \quad \min_{i=1, \dots, N} \text{dist}(X_i, \partial\Omega) \geq cN^{-1/3}. \quad (\text{H1})$$

The key information here is that the constant c does not depend on N . Secondly, we assume that the empirical measure

$$\rho_N = \frac{1}{N} \sum_{i=1}^N \delta_{X_i} \quad (1.5)$$

is close to a density $n \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$ where $\mathcal{P}(\Omega)$ denotes the space of probability measures on Ω . For this, we impose the following control on the infinite Wasserstein distance:

$$\mathcal{W}_\infty(\rho_N, n) \leq C_0 N^{-1/3}. \quad (\text{H2})$$

Again, the key information here is that the constant C_0 is independent of the number of particles. For simplicity, we assume that the cloud of particles is uniformly bounded, i.e.,

$$\exists K \Subset \bar{\Omega}, \quad \forall i \in \{1, \dots, N\}, \quad X_i \in K. \quad (1.6)$$

Our goal in this paper is to derive information on the mean sedimentation velocity of the particles when they are submitted to a given force $F \in \mathbb{R}^3$. Since we restrict to a linear Stokes problem, we assume without restriction that F is directed along the third vector e_3 of the canonical basis and we normalize its amplitude to $N^{-\frac{1}{3}}$. In this way, the Stokes velocity (cf. (1.1)) is independent of N , namely,

$$V^{\text{St}} = V_r^{\text{St}} = \frac{1}{6\pi r}. \quad (1.7)$$

We consider then the problem

$$\left. \begin{aligned} -\Delta u_N + \nabla p_N &= 0 && \text{in } \Omega \setminus \bigcup_{i=1}^N \bar{B}_i, \\ \text{div } u_N &= 0 && \text{in } \Omega \setminus \bigcup_{i=1}^N \bar{B}_i, \\ u_N &= 0 && \text{on } \partial\Omega, \\ u_N(x) &= V_i + \Omega_i \times (x - X_i) && \text{in } \bar{B}_i \quad \text{for all } 1 \leq i \leq N, \\ - \int_{\partial B_i} \sigma[u_N, p_N] \nu &= N^{-\frac{1}{3}} e_3 && \text{for all } 1 \leq i \leq N, \\ - \int_{\partial B_i} (x - X_i) \times \sigma[u_N, p_N] \nu &= 0 && \text{for all } 1 \leq i \leq N, \\ \lim_{|x| \rightarrow \infty} u_N(x) &= 0 && \end{aligned} \right\} \quad (1.8)$$

In this system, we recall that ν is the normal to ∂B_i (directed inwards B_i). The symbol σ stands for the fluid stress tensor given by Newton law:

$$\sigma[u, p] = 2D(u) - p\mathbb{I}_3 = (\nabla u + \nabla^\top u) - p\mathbb{I}_3.$$

Note that the first equation in (1.8) reads also:

$$\operatorname{div}(\sigma(u_N, p_N)) = 0$$

where the operator div acts rowwise on the matrix $\sigma(u_N, p_N)$. The symbols V_i and Ω_i stand respectively for the linear and angular velocities of particle B_i . We emphasize that these velocities together with (u_N, p_N) are the unknowns in (1.8). The system is then algebraically well-posed, the velocities (V_i, Ω_i) being the Lagrange multipliers of the two last equations in (1.8). In particular, these velocities depend on N but we skip the dependencies for legibility. The last condition in (1.8) is needed in the case when Ω is unbounded in order to rule out Poiseuille type flows. We will in the following not write this condition explicitly. We will only consider velocity fields in $\dot{H}^1(\Omega)$ though, and Poiseuille type flows are not contained in this space.

We are interested in the average particle velocity

$$\bar{V}_N := \frac{1}{N} \sum_{i=1}^N V_i. \quad (1.9)$$

for large N under the assumption:

$$\operatorname{curl}(ne_3) = \nabla n \times e_3 = 0. \quad (\text{Hom})$$

This assumption is reminiscent of (HS). We recall that, as mentioned in introduction, if the limit density n is not constant in the directions perpendicular to e_3 (namely, in case (MF)), the particles create a collective fluid velocity proportional to the number of particles N and the magnitude of \bar{V}_N scales differently in N . The importance of this assumption can be observed as follows. If the particles are small and their distribution dilute, the force acting on the particles is seen reciprocally by the fluid as a forcing term f concentrated in the particles:

$$f \sim \sum_{i=1}^N 6\pi N^{-1/3} e_3 \delta_{X_i} \sim 6\pi N^{2/3} ne_3.$$

For large N we expect then that the leading term in the velocity-unknowns behaves like $N^{2/3}(u, p)$ with (u, p) solution to

$$\left. \begin{aligned} -\Delta u + \nabla p &= 6\pi ne_3 && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \right\} \quad (1.10)$$

One may then expect that the mean velocity \bar{V}_N has magnitude $N^{2/3}$ unless $u = 0$. In this latter case, we must have that ne_3 is a gradient or equivalently that (Hom) holds true. Even when (Hom) holds true, it will appear that the components of \bar{V}_N have different magnitudes. Below, we call sedimentation velocity the projection of \bar{V}_N along e_3 :

$$\bar{V}_N^{sed} = \bar{V}_N \cdot e_3.$$

To end this subsection, we point out that (Hom) together with (H0) entail that, if the axis ξ and the force e_3 are orthogonal, then n is necessarily constant in the direction ξ which contradicts that n is a probability measure. This is not the situation that we are interested in here.

1.2 Main results

For N fixed, the system (1.8) is well posed *via* the following construction. A classical framework is the space of extended velocity-fields:

$$H_0[N] := \{w \in H_0^1(\Omega) \text{ s.t. } \operatorname{div} w = 0 \text{ on } \Omega \text{ and } D(w) = 0 \text{ on } B_i^N \text{ for all } i\} \quad (1.11)$$

We remind that, since the B_i are connected, for arbitrary $w \in H_0[N]$ there exists vectors (W_1, \dots, W_N) and vectors (R_1, \dots, R_N) so that:

$$w(x) = W_i + R_i \times (x - X_i), \quad \forall x \in B_i.$$

In particular, an extended velocity-field $w \in H_0[N]$ encodes u_N but also $(V_i, \Omega_i)_{i=1, \dots, N}$. Classically, we only need to compute these unknowns to solve our system since the pressure p_N is then recovered as the Lagrange multiplier of the divergence-free constraint. Eventually, we have the weak formulation of (1.8):

Find $u_N \in H_0[N]$ such that,

$$\int_{\Omega} \nabla u_N : \nabla w = \sum_{i=1}^N \frac{e_3 \cdot W_i}{N^{1/3}}, \quad \forall w \in H_0[N].$$

Such a weak formulation is obtained by multiplying formally the Stokes equation with w and performing integration by parts to apply (pointwise and integral) boundary conditions on u_N . From this weak formulation, we immediately deduce

$$\|\nabla u_N\|_{L^2(\Omega)}^2 = \sum_{i=1}^N \frac{e_3 \cdot V_i}{N^{1/3}} = N^{2/3} \bar{V}_N^{sed}. \quad (1.12)$$

We see on this energy identity that there is a non-trivial relationship between u_N and \bar{V}_N . One could have expected that the sedimentation velocity \bar{V}_N^{sed} is of the same order (with respect to N) as the fluid velocity u_N itself. The energy identity, however, relates the sedimentation velocity \bar{V}_N^{sed} to the *gradient* of the fluid velocity u_N and reveals a factor $N^{2/3}$ between $\|\nabla u_N\|_{L^2(\Omega)}^2$ and \bar{V}_N^{sed} . Our first main result is then the identification of the magnitude of \bar{V}_N in both cases when (Hom) holds true and does not hold true:

Theorem 1.1. *Assume that (H0)–(H2) are satisfied.*

(i) *Assume that (Hom) is not satisfied. Then, there exists C depending only on Ω , on n and on C_0, c , from (H2) and (H1) such that*

$$\limsup_{N \rightarrow \infty} N^{-\frac{2}{3}} |\bar{V}_N| \leq C, \quad (1.13)$$

$$\liminf_{N \rightarrow \infty} N^{-\frac{2}{3}} \bar{V}_N^{sed} \geq \frac{1}{C}. \quad (1.14)$$

(ii) *If (Hom) is satisfied then there exists C depending on Ω and on C_0, c such that*

$$\limsup_{N \rightarrow \infty} N^{-\frac{1}{3}} |\bar{V}_N| \leq Cr^{-1/2}, \quad (1.15)$$

$$\limsup_{N \rightarrow \infty} |\bar{V}_N^{sed} - V_r^{St}| \leq C. \quad (1.16)$$

We remark that the factor r is related to the volume fraction ϕ through $\phi \sim r^3$. For instance, if n is the indicator of some connected open set $K \Subset \Omega$ with $|K| = 1$ (say a unit cube for instance), we can compute a local volume fraction $\phi = 4\pi r^3/3$. We recall also that $V_r^{St} \sim r^{-1}$. In particular, since r is independent of N we have $N^{2/3} \gg V^{St}$ for $N \gg 1$, and therefore, in case (Hom) is not satisfied, (1.14) is coherent with (1.2). In case (Hom) holds true, (1.16) is a prerequisite in order that an expansion (1.3) can be valid.

If (Hom) holds true, the solution to (1.10) is a pure pressure. With similar arguments as previously, a more relevant approximation to (u_N, p_N) for large N is then $N^{2/3}(\tilde{u}, \tilde{p})$ where (\tilde{u}, \tilde{p}) is the solution to:

$$\left. \begin{aligned} -\Delta \tilde{u} + \nabla \tilde{p} &= 6\pi(\rho_N - n)e_3 && \text{in } \Omega, \\ \operatorname{div} \tilde{u} &= 0 && \text{in } \Omega \\ \tilde{u} &= 0 && \text{on } \partial\Omega. \end{aligned} \right\} \quad (1.17)$$

According to the rate of convergence (H2), one may then expect that the mean velocity \bar{V}_N is of size $N^{1/3}$. The even smaller size of the sedimentation velocity (in powers of N) comes from the remark that:

$$\bar{V}_N^{sed} = \bar{V}_N \cdot e_3 \sim \langle N^{2/3} \tilde{u}, \rho_N e_3 \rangle = N^{2/3} \langle \tilde{u}, (\rho_N - n)e_3 \rangle$$

We used here again that, under assumption (Hom), the term ne_3 is a pressure gradient. The further gain of $N^{1/3}$ then yields from (H2) again. This gain can be generalized to the component of \bar{V}_N along any vector $e \in \mathbb{S}^2$ such that $\nabla n \times e = 0$.

In order to derive and characterize an expansion of the form (1.3), we introduce the two following additional structural assumptions. The first assumption regards a refined convergence of ρ_N to n . To this end, we first smooth out the density ρ_N as follows

$$\sigma_N := \frac{1}{N} \sum_{i=1}^N \frac{1}{|Q_i|} \mathbf{1}_{Q_i}, \quad \bar{\rho}_N = \frac{1}{N} \sum_{i=1}^N \frac{1}{|\partial B_i|} \mathcal{H}_{\partial B_i}^2. \quad (1.18)$$

Here $\mathcal{H}_{\partial B_i}^2$ is the Hausdorff measure on ∂B_i while the Q_i are disjoint cubes centered at X_i of volume

$$\frac{1}{C_1 N} \leq |Q_i| \leq \frac{C_1}{N}. \quad (1.19)$$

with C_1 independent of N . We emphasize that it is always possible to find such cubes thanks to assumption (H1) with $C_1 = c^{-3}$. However, the Q_i are not unique and we might change construction depending on the computations. To characterize defects of ρ_N to n , we impose that for a suitable choice of the cubes Q_i , the following strong convergence holds:

$$N^{\frac{1}{3}}(\sigma_N - n) \rightarrow g \quad \text{in } H^{-1}(\Omega) \quad \text{for some } g \in H^{-1}(\Omega). \quad (\text{Str})$$

We remark that by Proposition 2.1 below $N^{\frac{1}{3}}(\sigma_N - n)$ is already bounded in $\dot{H}^{-1}(\Omega)$ under assumption (H2)–(H1).

The second assumption is an almost periodicity assumption on the particles.:

$\exists d > 0, t_N \in \mathbb{R}^3, E_N \subseteq \Omega, I_N \subseteq \{1, \dots, N\}$ s.t.

$$\left\{ \begin{array}{l} |t_N|_\infty \leq N^{-1/3}, E_N \subseteq E_{N+1}, \frac{|I_N|}{N} \rightarrow 1, \\ \{X_i : i \in I_N\} = E_N \cap (t_N + dN^{-\frac{1}{3}}\mathbb{Z})^3, \quad (\text{Per}) \\ E_N = \bigcup_{i \in I_N} X_i + [-N^{-1/3}d, -N^{-1/3}d]^3 \end{array} \right.$$

Here E_N should be understood as the set on which the configuration is periodic, I_N the set of particles which are periodically distributed and t_N allows those particles to be uniformly translated with respect to a lattice centered at the origin. We will give an example for a particle configuration that satisfies both (Per) and (Str) with a nontrivial g in Section 2.2.

To give a characterization of the mean velocity, we introduce the following velocity fields. We define $v_{N,1} \in \dot{H}^1(\mathbb{R}^3)$ as the solution of the following Stokes equations in the whole space \mathbb{R}^3 :

$$\left. \begin{array}{l} -\Delta v_{N,1} + \nabla p_{N,1} = \sum_{i=1}^N \left(\frac{1}{|\partial B_R(X_i)|} \mathcal{H}^2|_{\partial B_R(X_i)} - \frac{1}{|Q_i|} \mathbf{1}_{Q_i} \right) e_3, \\ \operatorname{div} v_{N,1} = 0. \end{array} \right\} \quad (1.20)$$

Moreover, we consider the solution $v_{\infty,3} \in H_0^1(\Omega)$ to the Stokes equations in Ω

$$\left. \begin{array}{l} -\Delta v_{\infty,3} + \nabla p_{\infty,3} = g, \quad \operatorname{div} v_{\infty,3} = 0 \quad \text{in } \Omega, \\ v_{\infty,3} = 0 \quad \text{on } \partial\Omega. \end{array} \right\} \quad (1.21)$$

We keep the index 2 for a further velocity-fields that we require for technical convenience below. With these definitions, our expansion is the content of the following result:

Theorem 1.2. *Assume that assumption (H0)–(H2) and (Hom) are satisfied.*

(i) *If in addition (Str) is satisfied, then, for all $\delta > 0$, there exists $C > 0$, depending only on C_0, C_1, c, δ from (H2), (H1) and (H1) respectively such that*

$$\limsup_{N \rightarrow \infty} \left| \bar{V}_N^{sed} - \left(N^{-2/3} \|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla v_{\infty,3}\|_{L^2(\Omega)}^2 \right) \right| \leq Cr^{1-\delta}. \quad (1.22)$$

Moreover,

$$\|\nabla v_{\infty,3}\|_{L^2(\Omega)}^2 = \lim_{N \rightarrow \infty} N^{1/3} \langle v_{\infty,3}, \bar{\rho}_N e_3 \rangle. \quad (1.23)$$

and there exists a sequence $w_N \in H_0^1(\Omega)$ with $\|\nabla w_N\|_{L^2(\Omega)} \leq CN^{1/3}r^{3/2-\delta}$ such that

$$N^{-1/3}(u_N - w_N) \rightharpoonup v_{\infty,3} \quad \text{weakly in } H_0^1(\Omega). \quad (1.24)$$

(ii) *If in addition (Per) is satisfied and $Q_i = X_i + [-N^{-1/3}d, N^{-1/3}d]^3$ for all $i \in I_N$, then*

$$\lim_{N \rightarrow \infty} N^{-2/3} \|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2 = \|\nabla v_{per}\|_{L^2(\mathbb{T}_d^3)}^2 = V_r^{\text{St}} \left(1 - a_{per} \frac{r}{d} + o(r) \right), \quad (1.25)$$

for some constant $a_{per} > 0$ and where v_{per} is the unique solution to

$$\left. \begin{aligned} -\Delta v_{per} + \nabla p_{per} &= \left(\frac{1}{|\partial B_r|} \mathcal{H}^2|_{\partial B_r(0)} - \mathbf{1}_{\mathbb{T}_d^3} \right) F && \text{in } \mathbb{T}_d^3, \\ \operatorname{div} v_{per} &= 0 && \text{in } \mathbb{T}_d^3, \\ \int_{\mathbb{T}_d^3} v_{per} \, dx &= 0, \end{aligned} \right\} \quad (1.26)$$

where $\mathbb{T}_d^3 = \mathbb{R}^3 / (d\mathbb{Z})^3$.

A few remarks are in order. We first recall, in order to compare with the expansions for \bar{V}_N^{sed} discussed at the beginning of the introduction, that $r \sim \phi^{1/3}$. The estimate (1.22) characterizes the sedimentation velocity \bar{V}_N^{sed} up to an $O(r^{1-\delta})$ error as the sum of two contributions. The first contribution, encoded in $v_{N,1}$, only depends on the particle configuration. It is completely independent of the container Ω . Under the periodicity assumption (Per), we characterize this contribution in (1.25) as the sum of the Stokes velocity V_r^{St} and a correction of order rV_r^{St} that can be computed from the problem on the torus (1.26). Recall from (1.7) that $r|V_r^{St}| = O(1)$.

The second contribution to \bar{V}_N^{sed} in (1.22) is encoded in $v_{\infty,3}$. Note that $v_{\infty,3}$ is independent of r and therefore the contribution $\|\nabla v_{\infty,3}\|_{L^2(\Omega)}$ is of order 1 if $\operatorname{curl} g \neq 0$. Moreover, the characterization (1.24) means that $v_{\infty,3}$ is the leading order normalized macroscopic fluid flow and by (1.23) the contribution $\|\nabla v_{\infty,3}\|_{L^2(\Omega)}$ equals the average of this leading order macroscopic fluid flow in the particles. Note that even though the macroscopic fluid flow is of order $N^{1/3}$, (1.23) implies that its average at the particles is of order 1.

We also remark that the constant a_{per} corresponds to the one from (1.3) analyzed in [Has59]. We do not investigate further the computation of $\|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2$ for particle configurations other than those satisfying (Per). One could expect though that the energy $\|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2$ can be generally expressed in terms of the 2-point correlation, similar as for the second order correction of the effective viscosity of a suspension obtained in [GH20; DG20].

1.3 Organization of the remainder of the paper and notations

The remainder of the paper is devoted to the proofs of Theorem 1.1 and 1.2.

Section 2, contains preliminary investigations on the probability densities involved in the analysis, namely the empirical measure of the particles smeared out to ∂B_i , the measure σ_N and the limit density n . Section 2.1 contains estimates between these densities which will be crucial for the subsequent analysis. In Section 2.2, we provide an example for assumption (Str) with a nontrivial function g .

In Section 3 we prove Theorem 1.1 (ii) as well as Theorem 1.2 (i). The proof is based on the splitting of u_N into $v_{N,1}, v_{N,2}, v_{N,3}$ and w_N that account for a whole space solution, boundary corrections, the defect between the measures σ_N and n as well as higher order hydrodynamical interactions between the particles.

In Section 4 we show Theorem 1.2 (ii) by analyzing periodic particle configurations.

Finally, in Section 5, we give the proof of Theorem 1.1 (i) that concerns the mean particle velocity in the ill-prepared case, when (Hom) is not satisfied. We complement the proof by additional structural information, namely the strong convergence $v_N \rightarrow v_*$ in $H_0^1(\Omega)$ of the leading part v_N of u_N in terms of the particle volume fraction r^3 as well as a characterization of the leading order of the limiting behavior of the mean velocity \bar{V}_N in terms of v_* .

In what follows, we use classical notations for function spaces. We do not specify whether we handle vector or scalar functions. This shall be clear in the context. If $U \subseteq \mathbb{R}^3$ is bounded, we denote

$$\int_U f(x)dx = \frac{1}{|U|} \int_U f(x)dx \quad \forall f \in L^p(U),$$

and $L_0^p(U)$ the subset of $L^p(U)$ containing mean-free functions. Such definitions may be generalized to functions defined on hypersurface of \mathbb{R}^3 . Finally, for arbitrary $U \subseteq \mathbb{R}^3$, we denote

$$\dot{H}^1(U) = \{u \in L^6(U) \text{ s.t. } \nabla u \in L^2(U)\}.$$

If U is bounded we have $\dot{H}^1(U) = H^1(U)$ that we endow with the classical norm. If U is unbounded we endow $\dot{H}^1(U)$ with the norm

$$\|u\|_{\dot{H}^1(U)} = \|\nabla u\|_{L^2(U)}$$

for which it is also a Hilbert space.

Below we use also constantly the symbol \lesssim for an inequality involving a harmless (multiplying) constant.

2 Properties of (smoothened-)empirical measures

In our problem, particle distributions are encoded:

- *via* the associated empirical measures ρ_N at the discrete level,
- *via* the density n in the continuous model.

For technical convenience, we need in the sequel smoothened versions of ρ_N . Namely, we will use:

$$\sigma_N = \frac{1}{N} \sum_{i=1}^N \frac{1}{|Q_i|} \mathbb{1}_{Q_i} \quad \bar{\rho}_N = \frac{1}{N} \sum_{i=1}^N \frac{1}{|\partial B_i|} \mathcal{H}_{\partial B_i}^2 \quad (2.1)$$

where we recall that Q_i are cubes centered in the X_i of volume scaling like $1/N$ (see assumption (1.19)) while $\mathcal{H}_{\partial B_i}^2$ is the Hausdorff measure on ∂B_i . In this section we prove at first some preliminary Poincaré type estimates that are crucial for the later analysis. These inequalities enable to control distances between smoothened empirical measures and between empirical measures and their continuous counterparts. We provide then examples of particle distributions for which assumption (Str) holds true with an explicit g .

2.1 Poincaré type inequalities

The first purpose of this section is the following estimates regarding particle distributions:

Proposition 2.1. *Let $p \in (1, \infty)$ and assume that $B_i \subseteq Q_i$ for all i .*

- (i) *If $p \neq 3$, there exists a constant C that depends only on p and the constant C_1 from (1.19) such that*

$$\|\bar{\rho}_N - \sigma_N\|_{(W^{1,p}(\Omega))^*} \leq C r^{-(3/p-1)_+} N^{-\frac{1}{3}}, \quad (2.2)$$

and, if $p > 3/2$,

$$\|\bar{\rho}_N - \sigma_N\|_{(W^{2,p}(\Omega))^*} \leq CN^{-2/3}, \quad (2.3)$$

where $(\cdot)_+$ stands for the positive part of real numbers.

(ii) If $p \neq 3$ there exists a constant C that depends only on p and the constants C_0 , c and C_1 from (H2), (H1) and (1.19) such that

$$\|\sigma_N - n\|_{(W^{1,p}(\Omega))^*} \leq CN^{-1/3}, \quad (2.4)$$

(iii) If $p \neq 3$ there exists a constant C that depends only on p , n and the constants C_0 and c from (H2) and (H1) such that

$$\|\bar{\rho}_N - n\|_{(W^{1,p}(\Omega))^*} \leq Cr^{-(3/p-1)_+} N^{-1/3}, \quad (2.5)$$

We note that item (i) entails in particular that for all $p \in (1, \infty)$

$$N^{1/3}(\bar{\rho}_N - \sigma_N) \rightharpoonup 0 \quad \text{in } (W^{1,p}(\Omega))^*. \quad (2.6)$$

It might be surprising that the scale in N changes between (2.2) and (2.3) making (2.2) seem far from optimal. It must be noted, though, that by symmetry, all affine functions tested on $\sigma_N - \bar{\rho}_N$ vanish. We will then obtain our result by comparing expansions of test-functions around each center X_i . The discrepancy between both estimates is due to the fact that only zero-order expansions are available in $W^{1,p}$ while first-order expansions are available in $W^{2,p}$. Finally, inequality (2.3) in case $p > 3/2$ could be complemented with a similar inequality in case $p < 3/2$. This will be however useless to our purpose.

For the proof, we furthermore introduce

$$\tilde{\rho}_N := \frac{1}{N} \sum_{i=1}^N \frac{1}{|B_i|} \mathbb{1}_{B_i}, \quad (2.7)$$

and we first show the following estimates involving $\tilde{\rho}_N$:

Lemma 2.2. *Let $p \in [1, \infty] \setminus \{3\}$ and assume that $B_i \subseteq Q_i$ for all i .*

$$\|\tilde{\rho}_N - \sigma_N\|_{(W^{1,p}(\Omega))^*} \leq Cr^{-(3/p-1)_+} N^{-1/3}, \quad (2.8)$$

where C depends only on p and the constant C_1 from (1.19).

Proof. We start with $p < 3$. Then, we may use the continuous embedding $W^{1,p}(Q_i) \subseteq L^{p^*}(Q_i)$, where $1/p^* = 1/p - 1/3$, which implies here that for any $\varphi \in W^{1,p}(Q_i) \cap L_0^p(Q_i)$ we have:

$$\|\varphi\|_{L^{p^*}(Q_i)} \leq C(C_1, q) \|\nabla \varphi\|_{L^q(Q_i)}$$

with a constant $C(C_1, p)$ independent of φ by a straightforward homogeneity argument. Consequently, we have for all $v \in W^{1,p}(\Omega)$, via a sequence of discrete and continuous Hölder

inequalities:

$$\begin{aligned}
|\langle \tilde{\rho}_N - \sigma_N, v \rangle| &= \frac{3}{N4\pi R^3} \sum_{i=1}^N \int_{B_i} \left(v(x) - \int_{Q_i} v(z) dz \right) dx \\
&\lesssim \frac{1}{r^3} \sum_{i=1}^N |B_i|^{1-\frac{1}{p^*}} \|\nabla v\|_{L^p(Q_i)} \lesssim \frac{1}{r^3} r^{3(1-\frac{1}{p^*})} N^{1-\frac{1}{p}-(1-\frac{1}{p^*})} \|\nabla v\|_{L^p(\Omega)} \\
&\lesssim \frac{1}{r^{\frac{3}{p^*}}} N^{\frac{1}{p^*}-\frac{1}{p}} \|\nabla v\|_{L^p(\Omega)}.
\end{aligned}$$

We conclude by recalling that $1/p^* = 1/p - 1/3$.

In the case $p > 3$ we have the embedding $W^{1,p}(Q_i) \subseteq C^{0,\theta}(Q_i)$ with $\theta = 1/3 - 1/p$. This implies here that, for arbitrary $\varphi \in W^{1,p}(Q_i) \cap L_0^p(Q_i)$ we have, by standard homogeneity arguments:

$$\|\varphi\|_{L^\infty(Q_i)} \leq \frac{C(p, C_1)}{N^{\frac{1}{3}-\frac{1}{p}}} \|\nabla \varphi\|_{L^p(Q_i)}$$

with a constant $C(p, C_1)$ depending only on p and C_1 . By standard arguments, we have then that:

$$|\langle \tilde{\rho}_N - \sigma_N, v \rangle| \leq \frac{1}{N} N^{-(\frac{1}{3}-\frac{1}{p})} \sum_{i=1}^N \|\nabla v\|_{L^p(Q_i)} \leq \frac{1}{N} N^{-(\frac{1}{3}-\frac{1}{p})} N^{1-\frac{1}{p}} \|v\|_{W^{1,p}(\Omega)}.$$

This finishes the proof of (2.2). \square

We are then in position to prove our main result.

Proof of Proposition 2.1. The convergence (2.5) follows from (2.2) and (2.4). It remains to show (2.2), (2.3) and (2.4).

Step 1: Proof of (2.2): Our result follows also immediately from Lemma 2.2 and the standard Poincaré-like inequality

$$\|v - \int_{\partial B_i} v\|_{L^p(B_i)} \leq C_p R \|\nabla v\|_{L^p(B_i)}, \quad (2.9)$$

where C_p depends only on $p \in (1, \infty)$. Indeed, we split $\bar{\rho}_N - \sigma_N = \bar{\rho}_N - \tilde{\rho}_N + \tilde{\rho}_N - \sigma_N$. The second part is estimated *via* the previous lemma while for the first part, we have:

$$|\langle \tilde{\rho}_N - \bar{\rho}_N, v \rangle| \leq \frac{1}{N} \sum_{i=1}^N \int_{B_i} \left| v - \int_{\partial B_i} v \right| \quad (2.10)$$

$$\leq \left(\frac{1}{N} \sum_{i=1}^N \int_{B_i} \left| v - \int_{\partial B_i} v \right|^p \right)^{\frac{1}{p}} \leq C_p N^{-\frac{1}{p}} R^{1-\frac{3}{p}} \|\nabla v\|_{L^p(\Omega)} \quad (2.11)$$

$$= C_p N^{-\frac{1}{3}} r^{1-\frac{3}{p}} \|\nabla v\|_{L^p(\Omega)}. \quad (2.12)$$

Step 2: Proof of (2.3): The argument is analogous as the proof of Lemma 2.2 in the case $p > 3$. Indeed, we observe that due to the assumption that Q_i is centered in X_i , we have

$$\langle \bar{\rho}_N - \sigma_N, v \rangle = \frac{1}{N4\pi R^2} \sum_{i=1}^N \int_{\partial B_i} \left(v(x) - \int_{Q_i} v(z) dz - \int_{Q_i} \nabla v(z) dz \cdot (x - X_i) \right) dx. \quad (2.13)$$

Moreover, for all $p > 3/2$ and all $\varphi \in W^{2,p}(Q_i)$ satisfying

$$\int_{Q_i} \varphi = 0 \quad \int_{Q_i} \nabla \varphi = 0$$

we have, by a standard homogeneity argument

$$\|\varphi\|_{C^0(\bar{Q}_i)} \leq \frac{C(p, C_1)}{N^{\frac{2}{3} - \frac{1}{p}}} \|\nabla^2 \varphi\|_{L^p(Q_i)}$$

where $C(p, C_1)$ depends only on p and C_1 from (1.19). This entails that:

$$|\langle \bar{\rho}_N - \sigma_N, v \rangle| \leq \frac{C(p, C_1)}{N^{1-1/p} N^{2/3}} \sum_{i=1}^N \|\nabla^2 v\|_{L^p(Q_i)}.$$

The assertion then follows again from application of the discrete Hölder inequality.

Step 3: Proof of (2.4): We observe that by the triangle inequality, assumption (H2) and the definition of σ_N , we have

$$\mathcal{W}_\infty(\sigma_N, n) \leq \mathcal{W}_\infty(\bar{\sigma}_N, \rho_N) + \mathcal{W}_\infty(\rho_N, n) \leq CN^{-1/3}. \quad (2.14)$$

where \mathcal{W}_∞ is the Wasserstein distance built on the *sup*-norm. By definition, the first term on the right-hand side is bounded by $1/N^{1/3}$. Now the desired estimate follows from the result

$$\|\mu - \nu\|_{(W^{1,p})^*} \leq C(\|\mu\|_\infty + \|\nu\|_\infty)^{1/p} \mathcal{W}_\infty(\mu, \nu). \quad (2.15)$$

see [San15, Exercise 38] and [HS21, Proposition 5.1]. This concludes the proof. \square

2.2 Explicit construction of distributions satisfying (Str)

We focus now on the construction of an example of particle distributions so that (Str) holds true:

$$N^{1/3}(\sigma_N - n) \text{ converges in } H^{-1}(\Omega).$$

To this end, we consider the case $\Omega = (-1, 1) \times (0, 1) \times \mathbb{R}$.

Fix $M \in \mathbb{N}^*$ and $N = 2M^3$. Firstly, we distribute $N/2$ particles covering $(0, 1)^2$. For this, we construct the cubes \tilde{Q}_k ($k \in \{0, \dots, M-1\}^3$) with centers in $\tilde{X}_k = 1/M(k_1 + 1/2, k_2 + 1/2, k_3 + 1/2)$, radius $1/M$ and thus volume $2/N$. We choose then $\lambda \in (0, 1/2)$ and set $X_k = \tilde{X}_k - \lambda/M e_1$.

$$Q_k = \begin{cases} \text{cube with center } X_k \text{ and radius } 1/(2M) \text{ if } k_1 > 1 \\ \text{cube with center } X_k \text{ and radius } 1/(2M) - \lambda/M \text{ if } k_1 = 0 \end{cases}$$

The remaining particles and cubes are obtained by transforming the X_k with the symmetry σ_1 with respect to the plane $\{x_1 = 0\}$. One easily checks that (H2) is satisfied for $n = \frac{1}{2} \mathbf{1}_{(-1,1) \times (0,1)^2}$ by considering the transport map $T(x) = X_k$ for $x \in \tilde{Q}_k$. Note that $n = \frac{1}{2} \mathbf{1}_{(-1,1) \times (0,1]^2}$ satisfies (Hom). Explicit computations then show that, denoting $\hat{k} = (0, k_2, k_3)$ for arbitrary $(k_2, k_3) \in \{0, \dots, M-1\}^2$:

$$N^{1/3}(\sigma_N - n)|_{(0,1)^2 \times \mathbb{R}} = 2^{-\frac{2}{3}} M \left(\left[\frac{2}{N} \sum_{k_2, k_3=0}^{M-1} \frac{1}{|Q_{\hat{k}}|} \mathbf{1}_{Q_{\hat{k}}} - \mathbf{1}_{\{x_1 \in (0, (1-\lambda)/M)\}} \right] - \mathbf{1}_{\{x_1 \in (1-\lambda/M, 1)\}} \right).$$

Classical computations then entail that:

$$M \left[\sum_{k_2, k_3=0}^{M-1} \frac{1}{M^3 |Q_{\hat{k}}|} \mathbb{1}_{Q_{\hat{k}}} - \mathbb{1}_{\{x_1 \in (0, (1-\lambda)/M)\}} \right] \rightarrow \lambda \delta_{\{x_1=0\}} \text{ in } (W^{1,2}((0,1)^2 \times \mathbb{R}))^*,$$

$$M \mathbb{1}_{\{x_1 \in (1-\lambda/M, 1)\}} \rightarrow -\lambda \delta_{\{x_1=1\}} \text{ in } (W^{1,2}((0,1)^2 \times \mathbb{R}))^*.$$

Using symmetry at $x_1 = 0$ and that $\delta_{\{x_1=1\}} = 0$ in $H^{-1}((-1,1) \times (0,1) \times \mathbb{R})$, we deduce that (Str) holds true with $g = 2^{1/3} \lambda \delta_{x_1=0}$ in $H^{-1}((0,1)^2 \times \mathbb{R})$. We see on this example that the term g encodes a finer description of the particle distribution. Indeed, we created artificially a distribution in which particles around $x_1 = 0$ are closer and thus have larger interactions. Particles near $x_1 = 0$ will therefore be slowed down in comparison to the particles near $x_1 = 1$. Such a difference will induce a variation of the velocity distribution in the cloud that is captured by the term $v_{\infty,3}$ solution to (1.21).

3 Computation of \bar{V}_N^{sed} when (Hom) holds true

Throughout this section, we assume that (H0)–(H2) and (Hom) are satisfied. Let (u_N, p_N) be the solution to (1.8). We remind the definition of $\bar{\rho}_N$ from (2.1) introduce v_N as the solution to

$$\left. \begin{aligned} -\Delta v_N + \nabla q_N &= N^{\frac{2}{3}} \bar{\rho}_N e_3 & \text{in } \Omega, \\ \operatorname{div} v_N &= 0 & \text{in } \Omega, \\ v_N &= 0 & \text{on } \partial\Omega. \end{aligned} \right\} \quad (3.1)$$

and the remainder

$$w_N := u_N - v_N. \quad (3.2)$$

We will estimate the contribution of w_N through the variational characterization of Stokes solution that entails $\|\nabla w_N\|_{L^2(\Omega)} \leq C \|D(v_N)\|_{L^2(\cup_i B_i)}$. We therefore first turn to the analysis of v_N itself.

We furthermore remind the definition of σ_N from (2.1) and split v_N further into $v_N = v_{N,1} + v_{N,2} + v_{N,3}$ (resp. $q_N = q_{N,1} + q_{N,2} + q_{N,3}$) where

$$\left. \begin{aligned} -\Delta v_{N,1} + \nabla q_{N,1} &= N^{\frac{2}{3}} (\bar{\rho}_N - \sigma_N) e_3 & \text{in } \mathbb{R}^3, \\ \operatorname{div} v_{N,1} &= 0 & \text{in } \mathbb{R}^3, \end{aligned} \right\} \quad (3.3)$$

$$\left. \begin{aligned} -\Delta v_{N,2} + \nabla q_{N,2} &= 0 & \text{in } \Omega \\ \operatorname{div} v_{N,2} &= 0 & \text{in } \Omega, \\ v_{N,2} &= -v_{N,1} & \text{on } \partial\Omega, \end{aligned} \right\} \quad (3.4)$$

and

$$\left. \begin{aligned} -\Delta v_{N,3} + \nabla q_{N,3} &= N^{\frac{2}{3}} (\sigma_N - n) e_3 & \text{in } \Omega, \\ \operatorname{div} v_{N,3} &= 0 & \text{in } \Omega, \\ v_{N,3} &= 0 & \text{on } \partial\Omega. \end{aligned} \right\} \quad (3.5)$$

The identity $v_N = v_{N,1} + v_{N,2} + v_{N,3}$ holds because, due to assumption [\(Hom\)](#), the term involving n on the right-hand side of [\(3.5\)](#) can be absorbed into the pressure: there exists a function $p_n \in L^2_{loc}(\Omega)$ such that $\nabla p_n = ne_3$.

We will show the following properties of these functions.

Proposition 3.1. *There exists a constant $C > 0$ depending only on Ω , and on C_0 and c from [\(H2\)](#)–[\(H1\)](#) as well as on C_1 from [\(1.19\)](#) such that the following holds.*

(i) $N^{-1/3}v_{N,1} \rightharpoonup 0$ weakly in $\dot{H}^1(\mathbb{R}^3)$ and $N^{-1/3}v_{N,1} \rightarrow 0$ strongly in $W^{1,p}(\mathbb{R}^3 \setminus \bar{\Omega})$ for all $p \in (1, \infty)$. Moreover,

$$\left| V_r^{\text{St}} - N^{-2/3} \|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2 \right| \leq C. \quad (3.6)$$

(ii) For $p = 2$,

$$N^{-1/3} \|\nabla v_{N,2}\|_{L^2(\Omega)} \rightarrow 0. \quad (3.7)$$

(iii) For $p = 2$,

$$N^{-1/3} \|\nabla v_{N,3}\|_{L^2(\Omega)} \leq C. \quad (3.8)$$

For all $p \in (1, \infty)$ and all bounded sets $\Omega' \subseteq \Omega$ there holds:

$$N^{-1/3} \|\nabla v_{N,3}\|_{L^p(\Omega')} \leq C'. \quad (3.9)$$

with C' depending furthermore on Ω' . If in addition [\(Str\)](#) is satisfied, then $N^{-1/3}v_{N,3} \rightarrow v_{\infty,3}$, strongly in $H^1(\Omega)$, where $v_{\infty,3}$ is the solution to [\(1.21\)](#).

(iv) For all $\delta > 0$

$$\limsup_{N \rightarrow \infty} N^{-1/3} \|D(v_N)\|_{L^2(\cup_i B_i)} \leq Cr^{3/2-\delta} \quad (3.10)$$

where the constant C depends in addition on δ .

The proof of this proposition is postponed to Subsection [3.2](#).

3.1 Proof of Theorem [1.1](#) when [\(Hom\)](#) holds true

To treat the error w_N , we note that w_N can be associated to a pressure \bar{q}_N to yield a solution to

$$\left. \begin{aligned} -\Delta \psi + \nabla q &= 0 && \text{in } \Omega \setminus \bigcup_{i=1}^N \bar{B}_i, \\ \operatorname{div} \psi &= 0 && \text{in } \Omega \setminus \bigcup_{i=1}^N \bar{B}_i, \\ \psi &= 0 && \text{on } \partial\Omega, \\ D(\psi) &= D(\varphi) && \text{in } B_i \text{ for all } 1 \leq i \leq N, \\ \int_{\partial B_i} \sigma[\psi, q]n &= 0 = \int_{\partial B_i} \sigma[\psi, q]n \times (x - X_i) && \text{for all } 1 \leq i \leq N. \end{aligned} \right\} \quad (3.11)$$

with $\varphi = -v_N$. The estimate for w_N then follows from the following standard estimate (see e.g. [\[GH21, Equation \(27\)\]](#))

Proposition 3.2. *Let $\varphi \in H^1(\cup_i B_i)$ and let ψ be the solution to (3.11). Then*

$$\|\psi\|_{H^1(\Omega)} \leq C \|D(\varphi)\|_{L^2(\cup_i B_i)} \quad (3.12)$$

for a universal constant C .

We show how Proposition 2.1 and Proposition 3.1 imply Theorem 1.1 (ii) and Theorem 1.2 (i).

Proof of Theorem 1.1 (ii). We fix the choice of the cubes Q_i by $|Q_i| = c^3 N^{-1}$ where c is the constant from (H1). In this way, dependencies on C_1 from (1.19) become dependencies on c .

Using that $V_i = \int_{\partial B_i} u_N$, we first note that, for arbitrary direction $e \in \mathbb{S}^2$, there holds:

$$\bar{V}_N \cdot e = \langle \bar{\rho}_N e, u_N \rangle$$

Writing that $u_N = v_{N,1} + v_{N,2} + v_{N,3} + w_N$ and that $\bar{\rho}_N = \bar{\rho}_N - n + n$, we combine (3.6)-(3.7)-(3.9) together with (2.5) in case $p = 2$ to yield (1.15).

Using assumption (Hom) and the fact that u_N is divergence free and that $V_i = \int_{\partial B_i} u_N$, we rewrite

$$\bar{V}_N^{sed} = \langle (\bar{\rho}_N - n)e_3, u_N \rangle \quad (3.13)$$

We recall the decomposition $u_N = v_N + w_N = v_{N,1} + v_{N,2} + v_{N,3} + w_N$. By (3.10), Proposition 3.2 and (2.5) with $p = 2$

$$\limsup_{N \rightarrow \infty} |\langle (\bar{\rho}_N - n)e_3, w_N \rangle| \leq C_\delta r^{1-\delta}. \quad (3.14)$$

Moreover, using the Stokes equations that v_N solves,

$$\langle (\bar{\rho}_N - n)e_3, v_N \rangle = \langle -\Delta v_N + \nabla q_N, v_N \rangle = N^{-2/3} \|\nabla v_N\|_{L^2(\Omega)}^2$$

From (3.7) we infer that we have a remainder rem_N going to 0 as $N \rightarrow \infty$ such that:

$$\langle (\bar{\rho}_N - n)e_3, v_N \rangle = N^{-2/3} \|\nabla(v_{N,1} + v_{N,3})\|_{L^2(\Omega)}^2 + rem_N.$$

and thus:

$$\limsup_{N \rightarrow \infty} |\bar{V}_N^{sed} - V_r^{St}| \leq \limsup_{N \rightarrow \infty} |N^{-2/3} \|\nabla(v_{N,1} + v_{N,3})\|_{L^2(\Omega)}^2 - V_r^{St}| + C_\delta r^{1-\delta} \quad (3.15)$$

At this point, we realize that, with (2.2) and (3.9) with $p > 3$

$$N^{-\frac{2}{3}} \int_{\Omega} \nabla v_{N,1} : \nabla v_{N,3} = \langle (\bar{\rho}_N - \sigma_N)e_3, v_{N,3} \rangle \leq C.$$

Therefore, expanding the square in (3.15), and using also (3.8) yields

$$\limsup_{N \rightarrow \infty} |\bar{V}_N^{sed} - V_r^{St}| \leq \limsup_{N \rightarrow \infty} |N^{-2/3} \|\nabla v_{N,1}\|_{L^2(\Omega)}^2 - V_r^{St}| + C$$

and we obtain the expected result thanks to (3.6). \square

Proof of Theorem 1.2 (i) . We now turn to the proof of Theorem 1.2 (i). This time, we choose the cubes Q_i such that assumption (Str) is satisfied. We revisit the latter computations, using that Proposition 3.1 provides the weak convergence $N^{-1/3}v_{N,1} \rightharpoonup 0$ in $\dot{H}^1(\Omega)$ and that, thanks to (Str), we have the strong convergence $N^{-1/3}v_{N,3} \rightarrow v_{\infty,3}$. We infer:

$$\begin{aligned} \limsup_{N \rightarrow \infty} \langle (\bar{\rho}_N - n)e_3, v_N \rangle &= \limsup_{N \rightarrow \infty} N^{-2/3} \|\nabla v_N\|_{L^2(\Omega)}^2 \\ &= \limsup_{N \rightarrow \infty} N^{-2/3} \left(\|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla v_{\infty,3}\|_{L^2(\Omega)}^2 \right), \end{aligned} \quad (3.16)$$

where we also used that $N^{-1/3}v_{N,1} \rightarrow 0$ strongly in $\dot{H}^1(\mathbb{R}^3 \setminus \Omega)$ in order to replace Ω by \mathbb{R}^3 . Combining (3.16) with (3.13) and (3.14) yields (1.22). Moreover, by definition of $v_{\infty,3}$ there holds:

$$\begin{aligned} \|\nabla v_{\infty,3}\|_{L^2(\Omega)}^2 &= \lim_{N \rightarrow \infty} N^{1/3} \langle v_{\infty,3}, (\sigma_N - n)e_3 \rangle \\ &= \lim_{N \rightarrow \infty} \left(N^{1/3} \langle v_{\infty,3}, \bar{\rho}_N e_3 \rangle + N^{1/3} \langle v_{\infty,3}, (\sigma_N - \bar{\rho}_N)e_3 \rangle \right), \end{aligned} \quad (3.17)$$

where we used again that $\langle v_{N,3}, ne_3 \rangle = 0$. We conclude (1.23) by observing that $N^{1/3}(\sigma_N - \rho_N) \rightharpoonup 0$ weakly in $\dot{H}^{-1}(\Omega)$ due to (2.2)–(2.3).

Finally, (1.24) is a consequence of the convergence $N^{-1/3}(v_{N,1} + v_{N,2}) \rightharpoonup 0$ in $\dot{H}^1(\Omega)$ from Proposition 3.1 as well as the bound on w_N that follows from Proposition 3.2 and (3.10). \square

3.2 Proof of Proposition 3.1

Item (iii) is independent and proven in a first step. Item (ii) is a consequence to the properties of $v_{N,1}$ outside Ω and is proven in a last step after tackling item (i). Item (iv) will follow from estimates that we show along the proof of items (i)–(iii). All the constants C involved in the following computations are harmless constants. They may depend on the involved exponent p and the constants c , C_0 and C_1 appearing in (H2)–(H1)–(H0).

Step 1: Proof of (iii):

To obtain (3.9) we proceed in two steps showing in passing the other statements in item (iii). Firstly, since Ω is bounded in one direction (orthogonal to ξ) it is standard to adapt the classical construction of solutions to (3.5) (see for instance [Gal11, Section IV.1]) to yield that $v_{N,3} \in H_0^1(\Omega)$ with

$$N^{-1/3} \|v_{N,3}\|_{H_0^1(\Omega)} \leq C$$

and claimed convergence when $N \rightarrow \infty$ thanks to (Str) and the linearity of the Stokes equations.

Then, we introduce a truncation function χ such that $\Omega^x = \text{supp}(\chi) \cap \Omega$ is C^2 and satisfies $\Omega' := \{\chi = 1\} \subseteq \Omega^x \subseteq \Omega$. We set

$$v_{N,3}^x = \chi v_{N,3} - \tilde{v}_\chi, \quad q_{N,3}^x = \chi q_{N,3} \quad (3.18)$$

where \tilde{v}_χ lifts the divergence of $\chi v_{N,3}$ in $H_0^1(\Omega^x \setminus \Omega')$. We have then that $\text{div}(v_{N,3}^x) = 0$ and

$$\int_{\Omega^x} \nabla v_{N,3}^x : \nabla w = \langle N^{2/3}(\sigma_N - n)e_3, \chi w \rangle + \int_{\Omega^x} v \cdot (2\nabla w \nabla \chi + \Delta \chi w) - \int_{\Omega^x} \nabla \tilde{v}_\chi : \nabla w + \int_{\Omega^x} p \nabla \chi \cdot w.$$

for any divergence-free $w \in C_c^\infty(\Omega^\chi)$. Firstly, we use the embedding $H_0^1(\Omega) \subseteq L^6(\Omega^\chi)$ to yield that up to a trivial extension $\tilde{v}_\chi \in W_0^{1,6}(\Omega^\chi)$ (see [Gal11, Theorem III.3.1]). Thus, since $p \in L^2(\Omega^\chi)$ (see [Gal11, Lemma IV.1.1]) we deduce $v_{N,3}^\chi \in W_0^{1,6}(\Omega^\chi)$ (see [Gal11, Theorem IV.6.1]) with bounds that entail

$$N^{-1/3} \|v_{N,3}\|_{W^{1,6}(\{\chi=1\})} \leq C^\chi$$

with C^χ depending furthermore on Ω^χ . This entails that $v_{N,3} \in L^\infty(\{\chi=1\})$. We can then reproduce the same argument with a second χ with support a little smaller to yield:

$$N^{-1/3} \|\nabla v_{N,3}\|_{L^p(\Omega')} \leq C'$$

whatever $p \in (1, \infty)$ with the expected dependencies for C' .

Step 2: Proof of (i): The assertion that $N^{-1/3}v_{N,1} \rightharpoonup 0$ in $\dot{H}^1(\mathbb{R}^3)$ is a consequence of (2.6). We mention here only that the estimate we derived in $(W^{1,2}(\Omega))^*$ extends straightforwardly into an estimate in the dual of $\dot{H}^1(\mathbb{R}^3)$. We write then:

$$v_{N,1} = \sum_{i=1}^N U_i, \tag{3.19}$$

where

$$-\Delta U_i + \nabla P_i = N^{-1/3}(\delta_i^R - \frac{1}{|Q_i|}\mathbf{1}_{Q_i})e_3 \quad \text{div } U_i = 0 \quad \text{in } \mathbb{R}^3 \tag{3.20}$$

and $\delta_i^R = \mathcal{H}^2|_{\partial B_i}/|\partial B_i|$ is the normalized uniform measure on ∂B_i . Then,

$$\|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2 = \sum_{i,j=1}^N \int \nabla U_i : \nabla U_j \, dx. \tag{3.21}$$

For the diagonal terms, we split $U_i = U_{i,1} - U_{i,2}$, $P_i = P_{i,1} - P_{i,2}$ corresponding respectively to the solutions of Stokes equations on \mathbb{R}^3 with source terms $N^{-1/3}\delta_i^R e_3$ and $N^{-1/3}|Q_i|^{-1}\mathbf{1}_{Q_i}e_3$. Thanks, to the theory on Stokes problem on \mathbb{R}^3 (see [Gal11, Section IV.2]), we know that such solutions can be computed by convolution with a fundamental solution. We denote Φ the fundamental solution for the velocity-field. We shall use below extensively that Φ is (-1) -homogeneous (see [Gal11, Eq. IV.2.3] for the exact formula). In case of $U_{1,i}$ the existence theory for Stokes problem in exterior domains yields that we have also an exact solution (see [Gal11, Section V, Eq. (V.0.4)]). This formula entails in particular that $U_{i,1} = V_r^{\text{St}}$ in B_i .

With these remarks at-hand now, we obtain by multiplying the Stokes equations for $U_{i,1}$ with $U_{i,1}$ that:

$$\|\nabla U_{i,1}\|_{L^2(\mathbb{R}^3)}^2 = N^{-1/3}V_r^{\text{St}}. \tag{3.22}$$

Moreover, we have, using first the weak formulation of the Stokes equations and then standard estimates for the convolution with Φ :

$$\begin{aligned} \|\nabla U_{i,2}\|_{L^2}^2 + |(\nabla U_{i,1}, \nabla U_{i,2})_{L^2(\mathbb{R}^3)}| &\leq N^{-1/3}\|U_{i,2}\|_{C^0(\overline{Q_i})} \\ &\leq CN^{-1/3}N^{2/3}\|\mathbf{1}_{Q_i}\|_\infty^{1/3}\|\mathbf{1}_{Q_i}\|_1^{2/3} \leq CN^{-1/3}. \end{aligned} \tag{3.23}$$

Thus, expanding the sum for U_i when computing the L^2 -norm, we obtain:

$$\left| \|\nabla U_i\|_{L^2}^2 - N^{-1/3} V_r^{\text{St}} \right| \leq CN^{-1/3}. \quad (3.24)$$

Finally, since $U_{i,1}$ is constant in B_i , we may reproduce the convolution arguments with $\nabla U_{i,2}$ to yield:

$$\|\nabla U_i\|_{L^\infty(B_i)} \leq N^{2/3} \|\mathbf{1}_{Q_i}\|_\infty^{2/3} \|\mathbf{1}_{Q_i}\|_1^{1/3} \leq CN^{1/3}. \quad (3.25)$$

We are now in position to estimate the off-diagonal terms. For fixed $i \in \{1, \dots, N\}$, we consider two cases for index $j \neq i$. Firstly, we say that Q_j is a neighbor of Q_i if $i \neq j$ and $\text{dist}(Q_i, Q_j) \leq C_1$ with C_1 being the constant from (1.19). We note that for each i there are at most M neighbors Q_j of Q_i where $M \in \mathbb{N}$ depends only on C_1 . We observe now from the explicit formula for $U_{j,1}$

$$|U_{j,1}(x)| \leq C \frac{N^{-1/3}}{|x - X_j|} \quad (3.26)$$

for all $x \in \mathbb{R}^3 \setminus B_j$. Thus, combining this with the bound of $U_{j,2}$ derived in (3.23), we have for all $i \neq j$

$$\|U_j(x)\|_{L^\infty(Q_i)} \leq C. \quad (3.27)$$

This yields after integration by parts:

$$\left| \sum_{i=1}^N \sum_{Q_j \text{ neighb. } Q_i} \int_{\mathbb{R}^3} \nabla U_i : \nabla U_j \right| \leq N^{-1/3} \left| \sum_{i=1}^N \sum_{j | Q_j \text{ neighb. } Q_i} \|U_j\|_{L^\infty(Q_i)} \right| \leq CN^{2/3} \quad (3.28)$$

When Q_j is not a neighbor of Q_i we may use the following estimate for smooth test-functions which is reminiscent of (2.13):

$$\langle \delta_i^R - |Q_i|^{-1} \mathbf{1}_{Q_i}, \varphi \rangle \leq CN^{-2/3} \|\nabla^2 \varphi\|_{L^\infty(Q_i)}. \quad (3.29)$$

Applying this twice, with Φ :

$$\begin{aligned} \left| \int \nabla U_i : \nabla U_j \, dx \right| &= N^{-1/3} \left| \langle \delta_i^R - |Q_i|^{-1} \mathbf{1}_{Q_i}, U_j \cdot e_3 \rangle \right| \\ &\leq N^{-1} \|\nabla^2 U_j\|_{L^\infty(Q_i)} \\ &= N^{-4/3} \|\langle \delta_j^R - |Q_j|^{-1} \mathbf{1}_{Q_j}, \nabla^2 \Phi(x - \cdot) e_3 \rangle\|_{L^\infty(Q_i)} \leq \frac{N^{-2}}{|X_i - X_j|^5} \end{aligned} \quad (3.30)$$

since $|X_i - X_j|$ is comparable to $|x - X_j|$ uniformly in $x \in Q_j$ when Q_i and Q_j are not neighbors. Thus,

$$\left| \sum_{i=1}^N \sum_{\substack{i \neq j \\ Q_i \text{ not neighb. } Q_j}} \int_{\mathbb{R}^3} \nabla U_i : \nabla U_j \, dx \right| \leq C \sum_{i=1}^N \sum_{i \neq j} \frac{N^{-2}}{|X_i - X_j|^5} \leq CN^{2/3}. \quad (3.31)$$

Combining (3.24) and (3.28)–(3.31) yields (3.6).

Moreover, X_j is in the center of Q_j so that B_i is far from the support of the convolution defining U_j (the distance scales like $1/N^{1/3}$ with a constant depending on the parameters c, C_1 involved in (H1)–(1.19)). Arguing as for (3.30), we find then a constant C such that, for $i \neq j$

$$\|\nabla U_j\|_{L^\infty(B_i)} \leq C \frac{N^{-1}}{|X_i - X_j|^4} \quad \left\| \sum_{j \neq i} \nabla U_j \right\|_{L^\infty(B_i)} \leq CN^{1/3}. \quad (3.32)$$

Combining with (3.25) yields

$$\|D(v_{N,1})\|_{L^2(\cup_i B_i)} \leq Cr^{3/2} \sup_i \|D(v_{N,1})\|_{L^\infty(B_i)} \leq Cr^{3/2} N^{1/3}. \quad (3.33)$$

It remains to analyse the convergence of $N^{-1/3}v_{N,1} \rightarrow 0$ outside Ω . For this, we first provide an L^∞ -bound that we formulate in the following lemma for future reference:

Lemma 3.3. *Assume that for all $i = 1, \dots, N$ we have $Q_i \subseteq \Omega'$ for some compact $\Omega' \subseteq \mathbb{R}^3$. Then, for any $x \in \mathbb{R}^3 \setminus \overline{\Omega'}$, there holds:*

$$\left| \sum_{i=1}^N U_i \right| \leq C \left(\mathbf{1}_{\text{dist}(x, \Omega') < 2} + \frac{\mathbf{1}_{\text{dist}(x, \Omega') > 1}}{\text{dist}(x, \Omega')^3} \right)$$

$$\left| \sum_{i=1}^N \nabla U_i \right| \leq C \left(\min(N^{1/3}, \text{dist}(x, \Omega')^{-1}) \mathbf{1}_{\text{dist}(x, \Omega') < 2} + \frac{\mathbf{1}_{\text{dist}(x, \Omega') > 1}}{\text{dist}(x, \Omega')^4} \right).$$

Proof. We provide a computation of the second bound since the first one is obtained similarly. Fix $x \in \mathbb{R}^3 \setminus \Omega'$. We have then:

$$\sum_{i=1}^N \nabla U_i = \sum_{Q_i \text{ neighb. } x} (\nabla U_{i,1} + \nabla U_{i,2}) + \sum_{Q_i \text{ not neighb. } x} \nabla U_i. \quad (3.34)$$

where we define “ Q_i neighboring x ” as $\text{dist}(Q_i, x) < 2(C_1/N)^{1/3}$ (with C_1 given in (1.19)). For the second sum, we proceed similarly to (3.32) to obtain that:

$$\left| \sum_{i|Q_i \text{ not neighb. of } x} \nabla U_i \right| \leq \sum_{i|Q_i \text{ not neighb. of } x} \frac{CN^{-1}}{|x - X_i|^4} \leq C \text{dist}(x, \cup\{Q_i \text{ not neighb. of } x\})^{-1}$$

$$\leq C \min(N^{1/3}, \text{dist}(x, \Omega')^{-1})$$

We note then that we may only have a finite number of indices in the first sum in (3.34) and that, for each i neighbor of x there holds:

$$|\nabla U_{i,1}(x)| \leq \frac{CN^{-1/3}}{|x - X_i|^2} \leq CN^{1/3}.$$

since B_i is $C/N^{1/3}$ far from ∂Q_i . We treat the second term with convolution arguments as in (3.25) and we obtain $|\nabla U_{i,2}(x)| \leq CN^{1/3}$. Eventually, we conclude that:

$$\left| \sum_{i=1}^N \nabla U_i(x) \right| \leq C \left(\min(N^{1/3}, \text{dist}(x, \Omega')^{-1}) + N^{1/3} \mathbf{1}_{\{\text{dist}(x, \Omega') < (2C_1/N)^{1/3}\}} \right).$$

We obtain the first bound when $\text{dist}(x, \Omega') \leq 2$. When $\text{dist}(x, \Omega') > 1$ we remark that there are no neighboring Q_i to x and the above computations yield:

$$\left| \sum_{i=1}^N \nabla U_i(x) \right| \leq \frac{C}{N} \sum_{i=1}^N \frac{1}{|x - X_i|^4}$$

we conclude by noting that $|x - X_i| \geq \text{dist}(x, \Omega')$ for each i in the sum. \square

We continue with the proof of Proposition 3.1 (i). Let $\Omega' \subseteq \Omega$ be chosen independent of N containing all the cubes Q_i which is possible due to assumption (1.6). Then, since $v_{N,1} = \sum_i U_i$, the above lemma implies with dominated convergence that for arbitrary $p \in (1, \infty)$:

$$\lim_{N \rightarrow \infty} N^{-1/3} \|\nabla v_{N,1}\|_{L^p(\mathbb{R}^3 \setminus \Omega')} = 0. \quad (3.35)$$

In particular, we have the same convergence in $W^{1,p}(\mathbb{R}^3 \setminus \overline{\Omega})$.

Proof of (ii): Using that $v_{N,2}$ is solution to the (homogeneous) Stokes solution inside Ω (with boundary condition $-v_{N,1}$ on $\partial\Omega$), we have the variational characterization

$$\|\nabla v_{N,2}\|_{L^2(\Omega)} = \min\{\|\nabla v\|_{L^2(\Omega)} : v \in \dot{H}^1(\Omega), \text{div} v = 0, v|_{\partial\Omega} = -v_{N,1}\}.$$

To construct a suitable competitor, we consider again a bounded (and connected) set Ω' as above and set $v = v_{N,1}$ in $\mathbb{R}^3 \setminus \Omega'$. Inside of Ω' we then take a divergencefree extension of v . It is classical that such an extension can be constructed (e.g. by use of a Bogovkii operator) since the condition $\int \partial\Omega' v \cdot n = 0$ is satisfied because $\text{div} v_{N,1} = 0$, and that the extension satisfies

$$\|\nabla v\|_{L^2(\mathbb{R}^3)} \lesssim C \|\nabla v\|_{L^2(\mathbb{R}^3 \setminus \Omega')} = C \|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3 \setminus \Omega')}$$

In view of (3.35), this concludes the proof of (ii).

Proof of (iv): The statement is an immediate consequence of (3.33), item (ii) and item (iii) applied with Ω' that contains K from assumption (1.6) and with p sufficiently large.

4 Explicit computation of the first order correction for periodic configurations

In this section, we complete the proof of Theorem 1.2 by justifying item (ii). We will thus assume (Per) throughout this section. We will assume without loss of generality that $t_d = 0$ in (Per). Indeed, since we consider the norm of $v_{N,1}$ in the whole space \mathbb{R}^3 , the shift t_d does not have any influence.

We first note that, by classical arguments, there is a unique $v_{per} \in \dot{H}^1(\mathbb{T}_d^3)$ (homogeneous means here that we consider mean-free functions) to which we can associate a pressure $p_{per} \in L^2(\mathbb{T}_d^3)$ such that (1.26) holds true. We consider then in analogy to the cubes Q_i , $1 \leq i \leq N$ the covering of \mathbb{R}^3 by cubes $(Q_\alpha)_{\alpha \in \mathbb{Z}^3}$ where $Q_\alpha = N^{-1/3}d(\alpha + (-1/2, 1/2)^3)$. Similarly, we adapt the notations introduced in Section 3.2: for $\alpha \in \mathbb{Z}^3$, (U_α, P_α) is the solution to

$$-\Delta U_\alpha + \nabla P_\alpha = N^{-1/3}(\delta_\alpha^R - \frac{1}{|Q_\alpha|} \mathbf{1}_{Q_\alpha})e_3 \quad \text{div} U_\alpha = 0 \quad \text{in } \mathbb{R}^3, \quad \lim_{|x| \rightarrow \infty} |U_\alpha(x)| = 0. \quad (4.1)$$

and δ_α^R is the normalized uniform measure on $\partial B_\alpha = \partial B_{r_{N^{-1/3}}}(dN^{-1/3}\alpha)$. We keep for technical convenience the labels $\alpha \in \mathbb{Z}^3$. We then note by a scaling argument that:

$$v_{per}(x) = \sum_{\alpha \in \mathbb{Z}^3} U_\alpha(N^{-1/3}x) - \int_{[0,d]^3} U_\alpha(N^{-1/3}y) dy \quad v_{N,1} = \sum_{i \in I_N} U_i + \sum_{i \notin I_N} U_i,$$

where we recall the set I_N from (Per) and use the convention that the sums over the index i runs over the set $\{1, \dots, N\}$.

The fact that the first sum converges in $\dot{H}^1(\mathbb{T}_d^3)$ follows from the decay of ∇U_α (cf. (3.32)). Let $Z_N \subseteq \mathbb{Z}^3$ be such that $\cup_{\alpha \in Z_N} Q_\alpha = \cup_{i \in I_N} Q_i = E_N$ with E_N as in (Per). We then obtain then:

$$\|\nabla v_{N,1}\|_{L^2(\mathbb{R}^3)}^2 = N^{2/3} \|\nabla v_{per}(N^{1/3}x)\|_{L^2(E_N)}^2 + rem_{1,N} + rem_{2,N}$$

where

$$\begin{aligned} rem_{1,N} &= -\left\| \sum_{\alpha \notin Z_N} \nabla U_\alpha \right\|_{L^2(E_N)}^2 - 2 \int_{E_N} N^{1/3} \nabla v_{per}(N^{1/3}x) : \sum_{\alpha \notin Z_N} \nabla U_\alpha dx \\ rem_{2,N} &= \left\| \sum_{i \in I_N} \nabla U_i \right\|_{L^2(\mathbb{R}^3 \setminus E_N)}^2 + \left\| \sum_{i \notin I_N} \nabla U_i \right\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{\mathbb{R}^3} \nabla v_{N,1} : \sum_{i \notin I_N} \nabla U_i dx \end{aligned}$$

By standard arguments, we have:

$$\lim_{N \rightarrow \infty} \|\nabla v_{per}(N^{1/3}x)\|_{L^2(E_N)}^2 = \|v_{per}\|_{\dot{H}^1(\mathbb{T}_d^3)}^2$$

and the the second identity in (1.25) yields from the analysis of the periodic problem in [Has59]. Our proof thus reduces to obtaining that:

$$\limsup_{N \rightarrow \infty} N^{-2/3} (rem_{1,N} + rem_{2,N}) = 0.$$

Concerning $rem_{1,N}$ we note that we have first the bound:

$$|rem_{1,N}| \leq C \left\| \nabla \sum_{\alpha \notin Z_N} U_\alpha \right\|_{L^2(E_N)} \left(1 + \left\| \nabla \sum_{\alpha \notin Z_N} U_\alpha \right\|_{L^2(E_N)} \right)$$

Then, we apply Lemma 3.3 to yield that, for arbitrary $x \in E_N$ there holds:

$$\sum_{\alpha \notin Z_N} |\nabla U_\alpha(x)| \leq C \min(N^{1/3}, \text{dist}(x, \partial E_N)^{-1}).$$

We note here that, to apply properly Lemma 3.3 we must invoke an ‘‘invading domain’’ argument and firstly approximate the infinite sum by finite sums. The above bound yields from the remark that the right-hand side does not depend on the finite subset of $\mathbb{Z}^3 \setminus Z_N$ that we would choose. Recalling $E_N \subseteq E_{N+1}$ from (Per) and that on the other side the sets E_N are contained in a compact set independently of N due to (1.6), we deduce with the dominated convergence theorem

$$\limsup_{N \rightarrow \infty} N^{-1/3} \left\| \nabla \sum_{\alpha \notin Z_N} U_\alpha \right\|_{L^2(E_N)} = 0, \quad \limsup_{N \rightarrow \infty} N^{-2/3} rem_{1,N} = 0.$$

Concerning $rem_{2,N}$, we can get similarly as above

$$\limsup_{N \rightarrow \infty} N^{-1/3} \left\| \sum_{i \in I_N} \nabla U_i \right\|_{L^2(\mathbb{R}^3 \setminus E_N)}^2 = 0.$$

Moreover, since by assumption (Per) $\#\{i \notin I_N\} \ll N$, we have from the bound on U_i in (3.25) that

$$N^{-1/3} \left\| \sum_{i \notin I_N} \nabla U_i \right\|_{L^2(\mathbb{R}^3)} = 0. \quad (4.2)$$

Combining these estimates yields $\lim_{N \rightarrow \infty} N^{-2/3} rem_{2,N} = 0$ which concludes the proof.

5 Computations in the ill-prepared case

We provide here the computations in the ill-prepared case when (Hom) is not satisfied. Let (u_N, p_N) be the solution to (1.8). We introduce again (v_N, q_N) the solution to

$$\left. \begin{aligned} -\Delta v_N + \nabla q_N &= N^{2/3} \bar{\rho}_N e_3 && \text{in } \Omega \\ \operatorname{div} v_N &= 0 && \text{in } \Omega \\ v_N &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (5.1)$$

and $w_N = u_N - v_N$. We point out that, without assumption (Hom) we may not normalize the pressure to add the $-ne_3$ term to the right-hand side without modifying v_N .

The main goal of this section is a proof of item (i) in Theorem 1.1. We complement the proof with a more refined description of \bar{V}_N at the end of this section. To achieve our main goal we first provide the following proposition:

Proposition 5.1. *The vector-fields v_N and w_N introduced above satisfy the following statements:*

(i) *there exists $(v_*, q_*) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ for which $N^{-2/3} v_N \rightarrow v_*$ in $H_0^1(\Omega)$ and:*

$$\left. \begin{aligned} -\Delta v_* + \nabla q_* &= ne_3 && \text{in } \Omega, \\ \operatorname{div} v_* &= 0 && \text{in } \Omega. \end{aligned} \right\} \quad (5.2)$$

(i) *there exists a constant C which depends only on c from (H1) and Ω such that, for N sufficiently large:*

$$\|w_N\|_{H^1(\Omega)} \leq CN^{2/3} r^{3/2}. \quad (5.3)$$

Proof. Item i) is a direct consequence to (2.5) in Proposition 2.1 by standard arguments on generalized solutions to Stokes system (see [Gal11, Theorem IV.1.1]). We point out that the result holds actually whether data are well-prepared or ill-prepared. The main difference between the ill-prepared and well-prepared setting is that $v_* = 0$ in the latter one.

Since $v_N/N^{2/3}$ converges to v_* in $H_0^1(\Omega)$, we can bound w_N as follows thanks to Proposition 3.2:

$$\|\nabla w_N\|_{L^2(\Omega)} \leq C \|D(v_N)\|_{L^2(\cup B_i)} \leq CN^{2/3} \left(\|D(v_*)\|_{L^2(\cup B_i)} + C \|N^{-2/3} v_N - v_*\|_{H^1(\Omega)} \right)$$

where the second term in the parenthesis can be made arbitrary small for N large. We remark then that $n \in L^\infty(\Omega)$ so that standard elliptic regularity results entail in particular that $v_* \in W^{2,4}(\Omega')$ for arbitrary bounded $\Omega' \subseteq \Omega$ and thus $v_* \in C^1(\bar{\Omega})$. We infer then that the first term in the parenthesis is bounded by $r^{3/2}$. This ends the proof. \square

Proof of Theorem 1.1(i). Assume (Hom) is not satisfied. Then, by combining Proposition 5.1 and (2.5) in Proposition 2.1 (which implies the strong convergence of $\bar{\rho}_N$ to n in $H^{-1}(\Omega)$), we infer:

$$\begin{aligned} N^{-2/3}\bar{V}_N &= N^{-2/3}\langle v_N, \bar{\rho}_N \rangle_{H^1, H^{-1}} + N^{-2/3}\langle w_N, \bar{\rho}_N \rangle_{H^1, H^{-1}} \\ &\rightarrow \langle v_*, n \rangle_{H^1, H^{-1}} + O(r^{3/2}) = \int_{\Omega} v_* n + O(r^{3/2}), \end{aligned}$$

which yields (1.13) since $\|v_*\|_{H^1(\Omega)} \leq C\|n\|_{L^\infty(\Omega)}$. Moreover, we have via a standard energy estimate:

$$\int_{\Omega} v_* \cdot n e_3 = \|\nabla v_*\|_{L^2(\Omega)}^2, \quad (5.4)$$

which yields (1.14) since $v_* \neq 0$ if (Hom) is not satisfied. This ends our proof. \square

To complement the analysis of the ill-prepared case, we provide a sharper description of \bar{V}_N for large values of N . For this, we introduce further notations for solutions to (5.2). Indeed, we remark that this solution is fixed by the vector e_3 so that changing this value to another vector $\tilde{e} \in \mathbb{R}^3$ would yield a different velocity-field. Below, we highlight this possible dependency by writing $v_*[\tilde{e}]$ the solution associated with the vector $\tilde{e} \in \mathbb{R}^3$. We can now state our main proposition:

Proposition 5.2. *Assume that (Hom) does not hold. Then, there exists $V_* \in \mathbb{R}^3$ such that:*

(i) *there exists a constant C independent of $N \in \mathbb{N}$ sufficiently large for which*

$$\bar{V}_N = N^{\frac{2}{3}}(V_* + \text{rem}_N) \quad \text{with} \quad |\text{rem}_N| \leq Cr.$$

(i) *there holds:*

$$V_* \cdot e = \int_{\Omega} \nabla v_*[e_3] : \nabla v_*[e] \quad \forall e \in \mathbb{S}^2$$

Remark 5.3. We first point out that v_* does not depend on r . So, when $\nabla n \times e \neq 0$ we have indeed captured the first order of \bar{V}_N with a remainder smaller than $O(r)$. We note that we can use the system satisfied by v_* to rewrite:

$$V_* \cdot e = \int_{\Omega} v_*[e_3] \cdot n e = \int_{\Omega} v_*[e] \cdot n e_3$$

We also recall that, in the degenerate case $\nabla n \times e = 0$, there holds $v_*[e] = 0$. In this case, the computations of the previous section hold and show that $\bar{V}_N \cdot e \leq CN^{1/3}$ similarly as we obtained (1.15). In particular, the results obtained in the present section are not optimal in this direction.

Proof. We prove that, for arbitrary $e \in \mathbb{S}^2$, there holds:

$$\bar{V}_N \cdot e = N^{\frac{2}{3}} \left(\int_{\Omega} \nabla v_*[e_3] : \nabla v_*[e] + O(r) \right)$$

This shall complete the two items of the proposition simultaneously.

Given $e \in \mathbb{S}^2$, let us denote by $u_N[e], v_N[e], w_N[e]$ the velocity-fields associated to the problem (1.8) replacing e_3 by e (analogously as the notation $v_*[e]$ introduced above). By Proposition 5.1, we have:

$$\|\nabla v_N[e_3]\|_{L^2} + \|\nabla v_N[e]\|_{L^2} \leq CN^{2/3}, \quad \|\nabla w_N[e_3]\|_{L^2} \leq CN^{2/3}r^{\frac{3}{2}}$$

and

$$N^{-\frac{2}{3}}v_N[e_3] \rightarrow v_*[e_3] \quad N^{-\frac{2}{3}}v_N[e] \rightarrow v_*[e] \quad \text{in } H_0^1(\Omega).$$

Furthermore, like in the previous proof, there holds:

$$\begin{aligned} N^{-\frac{2}{3}}\bar{V}_N \cdot e &= N^{-\frac{4}{3}}\langle v_N[e_3], -\Delta v_N[e] + \nabla q_N[e] \rangle_{H_0^1(\mathbb{R}^3), H^{-1}(\mathbb{R}^3)} + N^{-\frac{2}{3}}\langle w_N[e_3], \bar{\rho}_N e \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &= N^{-\frac{4}{3}} \int_{\Omega} \nabla v_N[e_3] : \nabla v_N[e] + N^{-\frac{2}{3}}\langle w_N[e_3], \bar{\rho}_N e \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \end{aligned}$$

We conclude by remarking that by Proposition 5.1

$$\lim_{N \rightarrow \infty} \int_{\Omega} \nabla v_N[e_3] : \nabla v_N[e] = \int_{\Omega} \nabla v_*[e_3] : \nabla v_*[e]$$

and, by (2.2) and (2.4) in Proposition 2.1 and the above bound on w_N , that:

$$N^{-\frac{2}{3}} \left| \langle w_N[e_3], \bar{\rho}_N e \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \right| \leq N^{-\frac{2}{3}} \|\bar{\rho}_N\|_{H^{-1}(\Omega)} \|w_N[e_3]\|_{H^1(\Omega)} \leq Cr.$$

□

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