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# The role of correlation in diffusion control ranking games

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This paper studies Nash equilibria in two player continuous time stochastic differential games with diffusion control, and where the Brownian motions driving the state processes are correlated. We consider zero-sum *ranking* games, in the sense that the criteria to optimize only depends on the difference of the two players' state processes. We explicitly compute the players' equilibrium strategies, depending on the correlation of the Brownian motions driving the two state equations: in particular, if the correlation coefficient is smaller than some explicit threshold, then the equilibrium strategies consist of strong controls, whereas if the correlation exceeds the threshold, then the optimal controls are mixed strategies. To do so, we rely on a relaxed formulation of the game based on solutions to martingale problems, allowing the players to randomize their actions.

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## Introduction

We study a continuous-time dynamic two player zero-sum game with the following features:

- Each player is assigned a continuous martingale. We refer to the martingales as state processes of the players. We denote by  $X$  the state process of player 1 and by  $Y$  the state process of player 2.

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- Both players receive a reward at some finite time horizon  $T \in (0, \infty)$ , depending on the difference of the states at time  $T$ . More precisely, we assume that there exists a real function  $g$  such that the reward of player 1 is  $g(X_T - Y_T)$ . Since we are in a zero-sum setting, the reward of player 2 is then, up to a constant, given by  $-g(X_T - Y_T)$ . We assume that both players aim at maximizing the expectation of their respective reward.
- At any time  $t \in [0, T]$ , both players can choose actions from a set  $A = [\sigma_1, \sigma_2]$ , where  $0 \leq \sigma_1 < \sigma_2$ . If player 1 chooses  $a \in A$  and player 2 chooses  $b \in A$  at time  $t$ , then the quadratic variation of  $X$  increases with rate  $a^2$ , the quadratic variation of  $Y$  with  $b^2$ , and the covariation process of  $X$  and  $Y$  changes at the rate  $\rho ab$ , where  $\rho \in [-1, 1]$  is a given correlation coefficient.
- The players can observe all states and hence they can make their choice of action depend on their current position. The dynamic strategies of the players are modeled as Markovian controls. We distinguish between strict controls and the larger class of relaxed controls. The strict controls are defined as measurable functions  $\alpha : [0, T] \times \mathbb{R}^2 \rightarrow [\sigma_1, \sigma_2]$ . The relaxed controls are defined as measurable functions  $q : [0, T] \times \mathbb{R}^2 \rightarrow \mathcal{P}(A)$ , where  $\mathcal{P}(A)$  denotes the set of probability measures on  $A$ .

As usual, for predicting and explaining the players' behavior, we fall back on the concept of Nash equilibria. Since we are in a zero-sum setting, the set of equilibria coincides with the set of saddle points of the function that maps any pair of controls onto the expected reward of player 1. Moreover, any saddle point provides the same expected reward for player 1, referred to as the value of the game.

The special case of the game with  $\rho = 0$ ,  $\sigma_1 > 0$  and  $g(x) = 1_{[0, \infty)}(x)$  has been studied in Section 6 of [1]. If  $\rho = 0$  and  $\sigma_1 > 0$ , then for any pair of strict controls  $(\alpha, \beta)$  the state processes  $X$  and  $Y$  can be characterized as a weak solution of the system of SDEs

$$\begin{aligned} dX_t &= \alpha(t, X_t, Y_t) dW_t^X, \\ dY_t &= \beta(t, X_t, Y_t) dW_t^Y, \end{aligned} \tag{1}$$

where  $W^X$  and  $W^Y$  represent two independent Brownian motions. A reward of  $g(x) = 1_{[0, \infty)}(x)$  for player 1 means that both players aim at being ahead of the other at time  $T$ . This winner-takes-all reward implies that each player chooses the maximal diffusion rate  $\sigma_2$  whenever her state is smaller than the opponent's state, and the minimal diffusion rate  $\sigma_1$  whenever her state is greater than or equal to the opponent's. To put it differently, the player behind will take maximal risk, and the player ahead minimal risk.

The situation changes if one assumes that the Brownian motions  $W^X$  and  $W^Y$  coincide, or equivalently, if  $\rho = 1$ . In this case, the player ahead wants to mimic the player behind in order to keep the lead constant. The player behind, however, strives to choose a control as different as possible from the leader's control in order to maximize the probability of overtaking the opponent at some time point before  $T$ .

The example with  $\rho = 1$  and  $g(x) = 1_{[0, \infty)}(x)$  indicates two things. First, if there is a saddle point, then it has to be of a mixed type, i.e. the players need to involve some

randomization in their choice of controls. Second, there should be a threshold correlation beyond which there is no saddle point in strict controls.

The purpose of this paper is to confirm these assumptions, and to reveal that they are correct not only for the specific reward function  $g(x) = 1_{[0,\infty)}(x)$ , but for *arbitrary* reward functions  $g$  having at most countably many discontinuities and satisfying an exponential growth condition (see Assumption 1.2 below).

Indeed, our first main result states that if the correlation coefficient does not exceed the threshold  $\sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ , then for all reward functions  $g$  the game has a saddle point in strict controls. We explicitly describe a saddle point and provide an analytic expression of the value of the game as a function of the initial states (see Theorem 1.4 below).

We provide a counterexample showing that, in general, there is no saddle point in strict controls if the correlation exceeds the threshold  $\sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$  (see Proposition 1.8 below). By allowing also for relaxed controls the game has a value for *every* correlation level  $\rho$ . For correlation levels exceeding the threshold we explicitly characterize a saddle point in relaxed controls and compute the value of the game (see Theorem 2.6 below).

Our results reveal that a suitable notion of a mixed strategy for the game at hand is that of a relaxed control. It has long been known that certain deterministic differential games do not have an equilibrium in strict controls, but do have an equilibrium in the extended set of relaxed controls (see [9]). Elliot, Kalton and Markus observe that “the introduction of relaxed controls into differential games is analogous to the introduction by von Neumann of mixed strategies into two person, zero sum games” (see abstract of [9]).

Stochastic differential games with relaxed control of the *drift rate* are considered in [5]. The authors use fixed point and minimax theorems of Fan, and compactness of the set of relaxed controls, to show that the games possess equilibria. We stress that we do not use relaxed controls in order to compactify the set of controls and to fall back on a minimax theorem. For the game at hand we explicitly verify that a given pair of (relaxed) controls is a saddle point.

As our results show, the introduction of relaxed controls is also useful for two-player games where the diffusion rate depends on the control. To the best of our knowledge, the game studied in the present paper is the first two player zero-sum game with diffusion control for which a saddle point in relaxed controls is computed explicitly.

In recent years, relaxed controls have been successfully employed in the analysis of models with McKean-Vlasov dynamics and mean field games. E.g., [7], [17], [18] and [20] prove existence of a relaxed control equilibrium within some mean field games. Here the relaxed control equilibria are mainly used as an intermediate for showing existence of an equilibrium in strict controls. The articles [2], [3] and [19] show existence of an optimal relaxed control for some mean field control problems, and prove that the optimal control can be approximated with strict ones.

In our rigorous description of the game, see Section 1 and 2 below, we fall back on the concept of martingale problems due to Stroock and Varadhan [25] for specifying the law of state processes  $X$  and  $Y$ . The theory of martingale problems bears some

advantages, compared to a pure SDE approach, in striving for a unified treatment of the game with strict and with relaxed controls. To explain this, consider again the example with  $\rho = 0$  and  $\sigma_1 > 0$ . Suppose first that the players choose strict controls  $\alpha$  and  $\beta$ , respectively. Then the state processes can be characterized as a weak solution of (1). Now suppose both players choose relaxed controls  $p$  and  $q$ , respectively. Then, if at time  $t \in [0, T)$  the states are in  $(x, y)$ , the quadratic variation process of  $X$  grows at  $t$  with rate  $\int_A a^2 p(t, x, y)(da)$ , the quadratic variation of  $Y$  with rate  $\int_A b^2 q(t, x, y)(db)$ , and the covariation process changes with rate  $\int_A \int_A \rho ab q(t, x, y)(da)q(t, x, y)(db)$ . The dynamics of the two states can again be characterized in terms of a weak solution of an SDE. In contrast to (1), the representation of the states' dynamics under relaxed controls requires, in general, four Brownian motions (see Corollary 3.7 below).

If we defined the state processes by the SDE

$$\begin{aligned} dX'_s &= \int_A a p(s, X'_s, Y'_s)(da) dW_s^X, \\ dY'_s &= \int_A b q(s, X'_s, Y'_s)(db) dW_s^Y, \end{aligned} \tag{2}$$

we would *not* obtain a state model with randomized covariation rates, although the generalized dynamics (2) seem reasonable from the perspective of SDEs. The diffusion coefficients in (2) are the expected values of  $p$  and  $q$ , respectively, and hence assume values in  $A$ . The relaxed control  $p$  (or  $q$ ) has the same effect on the joint dynamics as the strict control given by the expectation of  $p$  in  $a$  (or  $q$  in  $b$ ). Therefore, a model with state dynamics given by (2) is essentially the same as a model with strict controls only.

The reasoning above shows that the right way of relaxing a diffusion control is to relax the quadratic variation of the state processes instead of the diffusion coefficient itself as in (2). Within the martingale problem framework, a relaxation of the quadratic variation nicely translates into a simple integration of the infinitesimal generator over the action space. Therefore, the martingale problem framework allows us to describe the game in strict and relaxed controls in a concise and uniform manner.

A direct “relaxation” of the diffusion coefficient as in (2) is also not suitable from a compactification point of view. In control theory, the set of relaxed controls is usually introduced in order to compactify the set of controls and hence guarantee the existence of an optimal control (see, e.g., [8]). The extension of the control set must not, however, change the value function. Section 2.2.1 in [3] illustrates with an explicit example of a diffusion control problem that a relaxation as in (2) can entail that value functions for the strict and “relaxed” control sets do not coincide.

Instead of using relaxed controls, one can define mixed strategies within stochastic differential games by assuming that the agents can observe the other players' states only with a (infinitesimal) delay. This approach is used, e.g., in [6] within a model, where the states are modeled in terms of SDEs with diffusion coefficients depending on the controls. We stress that we do not directly model the impact of the controls on diffusion coefficients of SDEs: the impact rather results from using an SDE representation of the solution to a martingale problem. The difference between these two approaches can be seen in our Corollary 3.7, where an SDE representation of the solution to the

martingale problem we are interested in is given by the system (11). Therefore, using relaxed controls via martingale problems leads to a control model formulated on SDEs which is different from the model used in [6], since in particular, it can be seen from (11) that the variances of the relaxed controls chosen by the players appear as diffusion coefficients of stochastic integrals with respect to new Brownian motions.

Among the games already studied in the literature the most related to the game of the present paper can be found in [23] and [1]. Section 6 of the latter considers the zero correlation case and provides an equilibrium in strict controls. Within the game of [23] the players control a single state, driven by a one-dimensional Brownian motion. The game bears the most similarities with our game in the case of zero correlation.

## 1. The game in strict controls

We start by describing the two player diffusion control game in a rigorous manner.

Let  $T > 0$  be a finite time horizon,  $0 \leq \sigma_1 < \sigma_2$  and define the action set  $A = [\sigma_1, \sigma_2]$ . The set of control functions of player 1 and player 2 is denoted by  $\tilde{\mathcal{A}}$  and consists of the set of measurable functions  $\alpha : [0, T] \times \mathbb{R}^2 \rightarrow A$ . We refer to elements as *controls* or *strategies*.

Let  $\rho \in [-1, 1]$  be a correlation coefficient and

$$C(a, b) := \begin{pmatrix} a^2 & \rho ab \\ \rho ab & b^2 \end{pmatrix}, \quad a, b \in A,$$

be the matrix that describes the instantaneous covariation of the players' states in dependence of the chosen actions. We describe the dynamics of the players' states in terms of a *controlled martingale problem*. To this end, we set  $\Omega = \mathcal{C}([0, T]; \mathbb{R}^2)$ , the space of continuous functions  $f : [0, T] \rightarrow \mathbb{R}^2$ . We denote by  $(X, Y)$  the canonical processes on  $\Omega$ , i.e.  $(X_t, Y_t)(\omega) = (\omega_1(t), \omega_2(t))$  for  $\omega \in \Omega$  and  $t \in [0, T]$ . We interpret  $X$  as the state of player 1 and  $Y$  as the state of player 2. We equip  $\Omega$  with the Borel  $\sigma$ -algebra  $\mathcal{F}$  generated by the topology given by the supremum norm. Moreover, we define  $(\mathcal{F}_t)_{t \in [0, T]}$  as the filtration generated by the canonical processes  $(X, Y)$ , i.e.  $\mathcal{F}_t = \sigma((X_s, Y_s) : 0 \leq s \leq t), t \in [0, T]$ . We denote by  $\mathcal{C}^2(\mathbb{R}^2)$  the space of all twice continuously differentiable functions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Moreover, we define the infinitesimal generator  $L$  of the state processes as a map on  $\mathcal{C}^2(\mathbb{R}^2)$  by

$$Lf(x, y, a, b) = \frac{1}{2} \text{tr} (C(a, b) D^2 f(x, y)), \quad f \in \mathcal{C}^2(\mathbb{R}^2), \quad (3)$$

where  $D^2 f$  denotes the Hessian matrix of  $f$ , and  $\text{tr}(\cdot)$  the trace of a matrix.

**Definition 1.1.** Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and  $\alpha, \beta \in \tilde{\mathcal{A}}$ . We say that a probability measure  $P^{\alpha, \beta}$  on  $(\Omega, \mathcal{F})$  is a feasible state distribution if

$$(i) \quad P^{\alpha, \beta} \circ (X_t, Y_t)^{-1} = \delta_{(x, y)},$$

(ii) for any  $f \in \mathcal{C}_c^2(\mathbb{R}^2)$  the process  $(M_s^f)_{s \in [t, T]}$ , where for  $t \leq s \leq T$

$$M_s^f := f(X_s, Y_s) - f(x, y) - \int_t^s Lf(X_u, Y_u, \alpha(u, X_u, Y_u), \beta(u, X_u, Y_u)) du, \quad (4)$$

is a  $P^{\alpha, \beta}$ -martingale.  $\mathcal{C}_c^2(\mathbb{R}^2)$  denotes the set of all  $f \in \mathcal{C}^2(\mathbb{R}^2)$  with compact support.

We denote by  $\mathcal{Q}(t, x, y, \alpha, \beta)$  the set of all measures  $P^{\alpha, \beta}$  satisfying (i)-(ii) above.

It is not clear if the martingale problem (4) has a solution for all choices  $\alpha, \beta \in \tilde{\mathcal{A}}$ . We comment on solvability and give sufficient conditions later in Section 3. To avoid making restrictive assumptions on the controls we introduce, similar to the setting in [24], Section 3, the set

$$\mathcal{A}(t, x, y) := \{\alpha \in \tilde{\mathcal{A}} \mid \exists \beta \in \tilde{\mathcal{A}} : \mathcal{Q}(t, x, y, \alpha, \beta) \neq \emptyset\},$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ .  $\mathcal{A}(t, x, y)$  is the set of controls such that there exists at least one control for the opponent that guarantees existence of a solution to the martingale problem (4). Whenever the initial data  $(t, x, y)$  is understood we write for simplicity  $\mathcal{A}$  for  $\mathcal{A}(t, x, y)$ .

Let now  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function that describes the rank-based reward at time  $T$ . Throughout the paper we assume that the following assumption is in force.

**Assumption 1.2.**  $g$  has only countably many discontinuities and at most exponential growth, i.e. there exist  $C_1, C_2 > 0$  such that  $|g(x)| \leq C_1 e^{C_2|x|}$  for all  $x \in \mathbb{R}$ .

We suppose that at time  $T$  player 1 receives a reward of  $g(X_T - Y_T)$  while player 2 receives  $-g(X_T - Y_T)$ . This means we consider a zero-sum game and hence it is sufficient to consider the payoff of player 1 only. Given  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ ,  $\alpha, \beta \in \mathcal{A}(t, x, y)$  and  $P \in \mathcal{Q}(t, x, y, \alpha, \beta)$  the expected payoff of player 1 under the measure  $P$  is given by

$$J^P(t, x, y, \alpha, \beta) = E^P [g(X_T - Y_T)].$$

Furthermore, we introduce the maximal payoff under solutions to the martingale problem

$$J^+(t, x, y, \alpha, \beta) := \sup_{P \in \mathcal{Q}(t, x, y, \alpha, \beta)} J^P(t, x, y, \alpha, \beta),$$

and the minimal payoff

$$J^-(t, x, y, \alpha, \beta) := \inf_{P \in \mathcal{Q}(t, x, y, \alpha, \beta)} J^P(t, x, y, \alpha, \beta).$$

Here we use the convention  $\inf \emptyset := +\infty$  and  $\sup \emptyset := -\infty$ . Note that if uniqueness holds true for the martingale problem (8), then we have  $J^- = J^+$ . The lower value of the game (in strict controls) is defined by

$$V_{\text{strict}}^-(t, x, y) = \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{A}} J^-(t, x, y, \alpha, \beta),$$

and the upper value (in strict controls) by

$$V_{\text{strict}}^+(t, x, y) = \inf_{\beta \in \mathcal{A}} \sup_{\alpha \in \mathcal{A}} J^+(t, x, y, \alpha, \beta).$$

Notice that the definition implies  $V_{\text{strict}}^- \leq V_{\text{strict}}^+$ . If the upper value coincides with the lower value, i.e.  $V_{\text{strict}}^- = V_{\text{strict}}^+$ , then the game is said to have a *value* in strict controls given by  $V_{\text{strict}} := V_{\text{strict}}^+$ . One can show that the upper and lower value function is finite, see Remark 2.4.

**Definition 1.3.** Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . A tuple  $(\alpha^*, \beta^*) \in \mathcal{A}(t, x, y) \times \mathcal{A}(t, x, y)$  is said to be a *saddle point* of the game if for any  $P \in \mathcal{Q}(t, x, y, \alpha^*, \beta^*)$

$$J^P(t, x, y, \alpha^*, \beta^*) = \sup_{\alpha \in \mathcal{A}} J^+(t, x, y, \alpha, \beta^*) = \inf_{\beta \in \mathcal{A}} J^-(t, x, y, \alpha^*, \beta).$$

Note that any saddle point is also a *Nash equilibrium*. If  $(\alpha^*, \beta^*)$  is a saddle point, then all feasible state distributions  $P^{\alpha^*, \beta^*}$  have the same payoff.

To formulate the main result of this section we introduce the function  $w : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$w(t, z) = \int_{-\infty}^{\infty} g(z + c(\rho)\sqrt{T-t}x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, \quad (t, z) \in [0, T] \times \mathbb{R}. \quad (5)$$

and

$$c(\rho) := \begin{cases} \sqrt{\sigma_2^2 - 2\rho\sigma_2(\sigma_1 \vee \rho\sigma_2) + (\sigma_1 \vee \rho\sigma_2)^2}, & \text{if } \rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}, \\ \sqrt{\left(\frac{\sigma_1 + \sigma_2}{2\rho}\right)^2 - \sigma_1\sigma_2}, & \text{if } \rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}. \end{cases}$$

Here  $x \vee y := \max\{x, y\}$ . Note that  $w \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ , i.e.  $w$  is twice continuously differentiable in the space variable and once in the time variable on the domain  $[0, T] \times \mathbb{R}$ . Moreover,  $w(T, \cdot) = g$  and

$$w(t, z) = \int_{-\infty}^{\infty} g(x) \frac{1}{\sqrt{2\pi c(\rho)^2 (T-t)}} e^{-\frac{(x-z)^2}{2c(\rho)^2 (T-t)}} dx, \quad (t, z) \in [0, T] \times \mathbb{R},$$

i.e.  $w$  is the convolution of the terminal reward  $g$  with the probability density function of a normal distribution. For further properties of  $w$  see Lemma 5.1. We define the convex and concave regions of  $w$  by

$$\begin{aligned} D_+ &= \{(t, z) \in [0, T] \times \mathbb{R} : \partial_{zz} w(t, z) > 0\}, \\ D_- &= \{(t, z) \in [0, T] \times \mathbb{R} : \partial_{zz} w(t, z) < 0\}. \end{aligned}$$

Our first result is the following.

**Theorem 1.4.** *Let either  $\rho > 0$  or  $\sigma_1 > 0$ . The game has a value in strict controls if*

$$\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}. \quad (6)$$



In this case the value function is given by

$$V_{strict}(t, x, y) = w(t, x - y), \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},$$

and the pair of controls  $(\alpha^*, \beta^*)$ , given by

$$\alpha^*(t, x, y) = \begin{cases} \sigma_2, & \text{if } (t, x - y) \in \overline{D_+}, \\ \sigma_1 \vee \rho\sigma_2, & \text{if } (t, x - y) \in D_-, \end{cases}$$

$$\beta^*(t, x, y) = \begin{cases} \sigma_1 \vee \rho\sigma_2, & \text{if } (t, x - y) \in \overline{D_+}, \\ \sigma_2, & \text{if } (t, x - y) \in D_-, \end{cases}$$

is a saddle point of the game, and hence a Nash equilibrium.

**Remark 1.5.** If  $g$  is continuous, we can drop the assumptions  $\sigma_1 > 0$  or  $\rho > 0$  in Theorem 1.4. In this case, the results of Theorem 1.4 hold true for all  $\sigma_1 \geq 0$  and  $\rho \in [-1, 1]$  because the statement of Lemma 5.3 is still valid.

If  $g$  is discontinuous, then the assumption that  $\sigma_1 > 0$  or  $\rho > 0$  can not be omitted as Example 2.8 illustrates. The problem is that players can be incentivized to deviate from  $(\alpha^*, \beta^*)$  if the states' difference is equal to a point of discontinuity of  $g$ . To avoid this incentive one can redefine  $(\alpha^*, \beta^*)$  such that both controls are equal to  $\sigma_2$  whenever the difference of the state processes visits a discontinuity of  $g$ . In this case, if the probability of the difference  $X - Y$  hitting a discontinuity of  $g$  at time  $T$  is equal to zero whenever one player deviates from  $(\alpha^*, \beta^*)$  (similar to Lemma 5.3), the statement of Theorem 1.4 remains true without the assumption  $\rho > 0$  or  $\sigma_1 > 0$ .

**Example 1.6.** Let  $g = \mathbb{1}_{[0, \infty)}$ . Then

$$w(t, z) = \Phi\left(\frac{z}{c(\rho)\sqrt{T-t}}\right), \quad (t, z) \in [0, T] \times \mathbb{R},$$

where  $\Phi$  denotes the cumulative distribution function of a standard normal distribution. We observe that  $D_+ = [0, T] \times (-\infty, 0)$  and  $D_- = [0, T] \times (0, \infty)$ , i.e.

$$\alpha^*(t, x, y) = \alpha^*(x, y) = \begin{cases} \sigma_2, & \text{if } x \leq y, \\ \sigma_1 \vee \rho\sigma_2, & \text{if } x > y, \end{cases}$$

$$\beta^*(t, x, y) = \beta^*(x, y) = \begin{cases} \sigma_1 \vee \rho\sigma_2, & \text{if } x \leq y, \\ \sigma_2, & \text{if } x > y, \end{cases}$$

for  $t \in [0, T]$ . Note that  $g$  satisfies Assumption 1.2. The reward  $g$  is an example why the assumption  $\rho > 0$  or  $\sigma_1 > 0$  can not be omitted: if  $\rho \leq 0$  and  $\sigma_1 = 0$  then  $\alpha^*(x, x) = \sigma_2$  and  $\beta^*(x, x) = 0$  for  $x \in \mathbb{R}$ . This means if  $X_t = Y_t$ , then player 1 has an incentive to deviate from  $\alpha^*$  and to choose action 0, because then  $g(X_T - Y_T) = 1$ . Indeed,

$$\frac{1}{2} = J^+(t, x, x, \alpha^*, \beta^*) < \sup_{\alpha \in \mathcal{A}} J^+(t, x, x, 0, \beta^*) = 1, \quad x \in \mathbb{R},$$

because  $d(X - Y)_t = a(X_t, Y_t)dW_t$  with a Brownian motion  $W$  and  $a := (\alpha^*)^2 + (\beta^*)^2 - 2\rho\alpha^*\beta^*$  (see Lemma 5.3 and (23)). Hence,  $\alpha^*$  is not a best response to  $\beta^*$ , and  $(\alpha^*, \beta^*)$  is not a saddle point.

**Example 1.7.** If the terminal reward  $g$ , in addition to Assumption 1.2, satisfies

- (i)  $g$  is continuous,
- (ii)  $g$  is convex on  $(-\infty, 0]$  and concave on  $[0, \infty)$ ,
- (iii) for any  $x \geq 0$  it holds

$$g(x) + g(-x) = 2g(0),$$

then  $D_+ = [0, T) \times (-\infty, 0]$  and  $D_- = [0, T) \times [0, \infty)$ , and  $(\alpha^*, \beta^*)$  are given as in Example 2.8. This follows from the control problem studied in [21].

For certain terminal rewards  $g$  we can rigorously prove that the condition (6) is not only sufficient, but also necessary. One can show the following.

**Proposition 1.8.** *Let  $g = \mathbf{1}_{[0, \infty)}$ . If  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ , then the game does not have a value in strict controls, i.e.  $V_{strict}^- < V_{strict}^+$ .*

We prove Proposition 1.8 by considering the upper and lower Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations in strict controls that characterize the upper and lower value function, respectively. There exist classical solutions to these equations and we verify that they coincide with the upper and lower value function. See Section 5 for details.

The statement of Proposition 1.8 also holds true for more general functions  $g$  that satisfy, e.g., the assumptions (i)-(iii) in Example 1.7. For such functions the proof of Proposition 1.8 immediately applies. More general terminal rewards  $g$ , that only satisfy Assumptions 1.2, require different proof techniques. In particular, it involves a diffusion control problem with terminal reward  $g$ , for which the optimal control is unknown. For further details see Remark 5.7 below.

The next section extends the game to a class of control for which there always exists a saddle point.

## 2. The game in relaxed controls

Proposition 1.8 shows that for certain terminal conditions there is no saddle point in strict controls. In this section we extend our game model to relaxed controls.

Let  $\mathcal{P}(A)$  denote the set of all probability measures on the action set  $A$  endowed with the topology of weak convergence. Define  $\tilde{\mathcal{V}}$  as the set of all Borel measurable  $q : [0, T] \times \mathbb{R}^2 \rightarrow \mathcal{P}(A)$ . We refer to elements of  $\tilde{\mathcal{V}}$  as *relaxed controls*. We work in this paper with controls of Markovian type, i.e. all controls are only functions of time and state. Note that strict controls  $\alpha \in \tilde{\mathcal{A}}$  can be embedded into  $\tilde{\mathcal{V}}$  by identifying  $\alpha$  with  $\delta_\alpha \in \tilde{\mathcal{V}}$ . Hence, the class of relaxed controls is larger.

**Remark 2.1.** In the literature relaxed controls are usually defined as measures on the product of the time and action set with first marginal equal to the Lebesgue measure.

This set of measures is convex and compact, e.g. w.r.t. the topology of weak convergence or some Wasserstein metric. These properties are useful for existence proofs. A relaxed control  $\nu$  in this sense can be decomposed into a measure  $\nu_t(da)dt$ , with  $\nu_t$  being a measure on the action set arising by disintegration of  $\nu$ . One can then identify a relaxed control with its disintegrated version, which is uniquely determined up to sets of Lebesgue measure zero.

In our game formulation we allow the players to control these marginals and we call them relaxed controls. This means players can choose at each time a randomization on the action set. Note that we equip the set of relaxed controls with a topology only for measurability reasons, but not for showing existence of a saddle point using compactness. Indeed, we determine a saddle point in closed form.

For any given two relaxed controls one can define a feasible state distributions via the corresponding martingale problem on the canonical space. As before we work on the canonical space  $\Omega$  with canonical processes  $(X, Y)$  and we equip this space with the same  $\sigma$ -algebra  $\mathcal{F}$  and filtration  $(\mathcal{F}_t)_{t \in [0, T]}$ , as before. We extend the definition of the infinitesimal generator  $L$  to relaxed controls and therefore set

$$Lf(x, y, \mu_1, \mu_2) = \int_A \int_A Lf(x, y, a, b) \mu_1(da) \mu_2(db), \quad (7)$$

for any  $f \in \mathcal{C}^2(\mathbb{R}^2)$ ,  $(x, y) \in \mathbb{R}^2$  and  $\mu_1, \mu_2 \in \mathcal{P}(A)$ . If  $f$  also depends on time, we write  $Lf(t, x, y, \mu_1, \mu_2)$  for  $(Lf(t, \cdot))(x, y, \mu_1, \mu_2)$ .

**Definition 2.2.** Let  $q_1, q_2 \in \tilde{\mathcal{V}}$  and  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . We say that a probability measure  $P^{q_1, q_2}$  on  $(\Omega, \mathcal{F})$  is a feasible state distribution if

- (i)  $P^{q_1, q_2} \circ (X_t, Y_t)^{-1} = \delta_{(x, y)}$ ,
- (ii) for any  $f \in \mathcal{C}_c^2(\mathbb{R}^2)$  the process  $(M_s^f)_{s \in [t, T]}$ , where for  $t \leq s \leq T$

$$M_s^f := f(X_s, Y_s) - f(x, y) - \int_t^s Lf(X_u, Y_u, q_1(u, X_u, Y_u), q_2(u, X_u, Y_u)) du, \quad (8)$$

is a  $P^{q_1, q_2}$ -martingale.

We denote by  $\mathcal{Q}(t, x, y, q_1, q_2)$  the set of all measures  $P^{q_1, q_2}$  satisfying (i)-(ii) above.

To treat the issue of a non-solvable state equation we introduce the set

$$\mathcal{V}(t, x, y) := \{q_1 \in \tilde{\mathcal{V}} \mid \exists q_2 \in \tilde{\mathcal{V}} : \mathcal{Q}(t, x, y, q_1, q_2) \neq \emptyset\},$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ , similar to the setting in [24], Section 3. For simplicity we write  $\mathcal{V}$  for  $\mathcal{V}(t, x, y)$  whenever the initial data is understood. Under certain assumptions on the model parameters we have  $\mathcal{V} = \tilde{\mathcal{V}}$ , see Proposition 3.1. For other parameters the set  $\mathcal{Q}(t, x, y, q_1, q_2)$  can be empty even if  $q_1, q_2 \in \mathcal{V}$ . If there is, however, a probability measure  $P \in \mathcal{Q}(t, x, y, q_1, q_2)$  we can define the associated reward by

$$J^P(t, x, y, q_1, q_2) := E^P [g(X_T - Y_T)].$$

Given  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and  $q_1, q_2 \in \mathcal{V}(t, x, y)$  we define the maximal payoff by

$$J^+(t, x, y, q_1, q_2) := \sup_{P \in \mathcal{Q}(t, x, y, q_1, q_2)} J^P(t, x, y, q_1, q_2),$$

and the minimal payoff by

$$J^-(t, x, y, q_1, q_2) := \inf_{P \in \mathcal{Q}(t, x, y, q_1, q_2)} J^P(t, x, y, q_1, q_2),$$

with the convention  $\inf \emptyset := +\infty$  and  $\sup \emptyset := -\infty$ . Note that if uniqueness holds true for the martingale problem (8), then we have  $J^- = J^+$ . The upper and lower value of the game in relaxed controls is now defined by

$$\begin{aligned} V^+(t, x, y) &:= \inf_{q_2 \in \tilde{\mathcal{V}}} \sup_{q_1 \in \mathcal{V}} J^+(t, x, y, q_1, q_2), \\ V^-(t, x, y) &:= \sup_{q_1 \in \mathcal{V}} \inf_{q_2 \in \tilde{\mathcal{V}}} J^-(t, x, y, q_1, q_2). \end{aligned}$$

The upper value of the game can be interpreted as the value of a static game with action set  $\mathcal{V}$ , where player 1 chooses her action after observing player 2's action, and the lower value as the value of this game with player 2 choosing an action after observing the action of player 1.

**Remark 2.3.** Under the assumptions of Proposition 3.1 we have  $\tilde{\mathcal{V}} = \mathcal{V}$ . If, however,  $|\rho| = 1$  or  $\sigma_1 = 0$ , we obtain by Remark 3.10 that  $\tilde{\mathcal{V}} \setminus \mathcal{V} \neq \emptyset$ , and it makes a difference if we define the upper and lower value using  $\tilde{\mathcal{V}}$  or  $\mathcal{V}$ . We want to argue why our definition, using the set  $\mathcal{V}$ , makes sense.

If player 2 chooses a strategy  $q_2 \in \mathcal{V}$ , then player 1 is incentivized to choose a strategy  $q_1 \in \tilde{\mathcal{V}}$  such that the martingale problem (8) possesses a solution, because otherwise  $J^+(t, x, y, q_1, q_2) = -\infty$ . This means that player 1 chooses some  $q_1 \in \mathcal{V}$  such that  $\mathcal{Q}(t, x, y, q_1, q_2) \neq \emptyset$ . However, if player 2 chooses a strategy  $q_2 \in \tilde{\mathcal{V}} \setminus \mathcal{V}$  then  $J^+(t, x, y, q_1, q_2) = -\infty$  for all  $q_1 \in \tilde{\mathcal{V}}$ , because  $\mathcal{Q}(t, x, y, q_1, q_2) = \emptyset$ , and hence

$$\inf_{q_2 \in \tilde{\mathcal{V}}} \sup_{q_1 \in \tilde{\mathcal{V}}} J^+(t, x, y, q_1, q_2) = -\infty.$$

Similarly,  $\sup_{q_1 \in \tilde{\mathcal{V}}} \inf_{q_2 \in \tilde{\mathcal{V}}} J^-(t, x, y, q_1, q_2) = +\infty$ . To exclude those cases it makes sense to consider the game only on the strategy set  $\mathcal{V}$ . Then we see that the players always have a motivation to choose strategies for which the martingale problem is solvable.

**Remark 2.4.** One can bound  $|J^P(t, x, y, q_1, q_2)|$  uniformly in  $q_1, q_2 \in \mathcal{V}$  and  $P \in \mathcal{Q}(t, x, y, q_1, q_2)$ , because the process  $X - Y$  can be viewed as a time-changed Brownian motion with time change given by its quadratic variation, that is bounded by a constant only depending on  $\sigma_1, \sigma_2$  and  $T$  (see Lemma 5.4 and the proof of Lemma 5.5 for more details). This means the upper and lower value function is finite.

**Definition 2.5.** Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . A tuple  $(q_1^*, q_2^*) \in \mathcal{V}(t, x, y) \times \mathcal{V}(t, x, y)$  is said to be a *saddle point* of the game (in relaxed controls) if for any  $P \in \mathcal{Q}(t, x, y, q_1^*, q_2^*)$

$$J^P(t, x, y, q_1^*, q_2^*) = \sup_{q_1 \in \mathcal{V}} J^+(t, x, y, q_1, q_2^*) = \inf_{q_2 \in \mathcal{V}} J^-(t, x, y, q_1^*, q_2).$$

From the definition it immediately follows that if there exists a saddle point then all feasible state distributions  $P \in \mathcal{Q}(t, x, y, q_1^*, q_2^*)$  induce the same payoff.

Our main result for relaxed controls is the following:

**Theorem 2.6.** *Let either  $\rho > 0$  or  $\sigma_1 > 0$ . Then the game has a value in relaxed controls, given by*

$$V(t, x, y) = w(t, x - y).$$

Moreover, the tuple  $(q_1^*, q_2^*) \in \mathcal{V} \times \mathcal{V}$ , defined by

$$q_1^*(t, x, y) = \begin{cases} \delta_{\sigma_2}, & \text{if } (t, x - y) \in \overline{D_+}, \\ \delta_{\sigma_1 \vee \rho \sigma_2}, & \text{if } (t, x - y) \in D_-, \end{cases}$$

$$q_2^*(t, x, y) = \begin{cases} \delta_{\sigma_1 \vee \rho \sigma_2}, & \text{if } (t, x - y) \in \overline{D_+}, \\ \delta_{\sigma_2}, & \text{if } (t, x - y) \in D_-, \end{cases}$$

for  $\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ , and

$$q_1^*(t, x, y) = \begin{cases} \frac{1}{\sigma_2 - \sigma_1} \left( \left( \sigma_2 - \frac{\sigma_1 + \sigma_2}{2\rho^2} \right) \delta_{\sigma_1} + \left( \frac{\sigma_1 + \sigma_2}{2\rho^2} - \sigma_1 \right) \delta_{\sigma_2} \right), & \text{if } (t, x - y) \in \overline{D_+}, \\ \delta_{\frac{\sigma_1 + \sigma_2}{2\rho}}, & \text{if } (t, x - y) \in D_-, \end{cases}$$

$$q_2^*(t, x, y) = \begin{cases} \delta_{\frac{\sigma_1 + \sigma_2}{2\rho}}, & \text{if } (t, x - y) \in \overline{D_+}, \\ \frac{1}{\sigma_2 - \sigma_1} \left( \left( \sigma_2 - \frac{\sigma_1 + \sigma_2}{2\rho^2} \right) \delta_{\sigma_1} + \left( \frac{\sigma_1 + \sigma_2}{2\rho^2} - \sigma_1 \right) \delta_{\sigma_2} \right), & \text{if } (t, x - y) \in D_-, \end{cases}$$

for  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ , is a saddle point.

**Example 2.7.** Let  $g$  satisfy either the assumptions in Example 2.8 or in Example 1.7. Then for  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$  we have

$$q_1^*(x, y) = \begin{cases} \frac{1}{\sigma_2 - \sigma_1} \left( \left( \sigma_2 - \frac{\sigma_1 + \sigma_2}{2\rho^2} \right) \delta_{\sigma_1} + \left( \frac{\sigma_1 + \sigma_2}{2\rho^2} - \sigma_1 \right) \delta_{\sigma_2} \right), & \text{if } x \leq y, \\ \delta_{\frac{\sigma_1 + \sigma_2}{2\rho}}, & \text{if } x > y, \end{cases}$$

and  $q_2^*(x, y) = q_2^*(y, x)$ . This means the player ahead chooses the action  $\frac{\sigma_1 + \sigma_2}{2\rho}$ , while the player behind randomizes over the actions  $\sigma_1$  and  $\sigma_2$ .

**Remark 2.8.** In control theory relaxed controls can be an auxiliary tool for showing the existence of  $\varepsilon$ -optimal controls, or even optimal controls under some convexity assumptions. The value functions of a strict control problem and its relaxed version coincide, because relaxed controls can be seen as the closure of the set of strict controls (when equipped with an appropriate topology). We refer the reader to [8] for more details.

The situation in our game is fundamentally different, because the value functions of strict and relaxed control can be different if the correlation is too large. For example,

if  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$  and  $g = \mathbf{1}_{[0, \infty)}$ , then a strict control tuple  $(\alpha, \beta) \in \mathcal{A} \times \mathcal{A}$  can never be an  $\varepsilon$ -Nash equilibrium for small  $\varepsilon > 0$ . Each player has an incentive to deviate from her strategy in the tuple  $(\alpha, \beta)$ . By deviating, player 1 can ensure a reward of at least  $V_{\text{strict}}^+$ , while player 2 can ensure that player 1 does not receive more than  $V_{\text{strict}}^-$ . Proposition 1.8 implies that  $V_{\text{strict}}^+ > V_{\text{strict}}^-$ , and hence, we observe that  $(\alpha, \beta)$  are not  $\varepsilon$ -optimal responses to each other, because the optimal rewards have a difference of at least  $V_{\text{strict}}^+ - V_{\text{strict}}^-$ . Note, however, that one can find strict controls that approximate the relaxed control tuple  $(q_1^*, q_2^*)$  arbitrarily well. For this, one can use results such as the so-called *chattering lemma* (see [8], Lemma 4.7) to approximate the measure  $q_1^*(t, x, y)(da)q_2^*(t, x, y)(db)dt$  on  $A \times A \times [0, T]$ .

### 3. Solutions to the martingale problem and their representation

A priori it is not clear if the martingale problems (4) and (8) possess a solution. In this section we fill this gap and comments on conditions that guarantee existence and uniqueness. Moreover, we connect solutions to the martingale problem to certain classes of stochastic differential equations (SDEs) that give some insight on the dynamics of the players' states. We state most of the results for relaxed controls, i.e. controls contained in the set  $\mathcal{V}$  or  $\tilde{\mathcal{V}}$ , since strict controls are embedded in this class. There are, however, results that only hold true for strict controls.

For certain parameters the martingale problem always possesses a solution and therefore the sets  $\mathcal{V}$  and  $\tilde{\mathcal{V}}$  coincide.

**Proposition 3.1.** *Let  $|\rho| < 1$ ,  $\sigma_1 > 0$  and  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ .*

- (i) *For any  $q_1, q_2 \in \tilde{\mathcal{V}}$  there exists a solution to the martingale problem (8), i.e. the set  $\mathcal{Q}(t, x, y, q_1, q_2)$  is non-empty and  $\mathcal{V}(t, x, y) = \tilde{\mathcal{V}}$ .*
- (ii) *For any  $\alpha, \beta \in \tilde{\mathcal{A}}$  there exists a solution  $P^{\alpha, \beta}$  to the martingale problem (4), i.e. the set  $\mathcal{Q}(t, x, y, \alpha, \beta)$  is non-empty and  $\mathcal{A}(t, x, y) = \tilde{\mathcal{A}}$ . The solution  $P^{\alpha, \beta}$  is unique if  $\alpha$  and  $\beta$  do not depend on time.*

We postpone the proof and details to the Appendix A on page 32. The proof relies on the connection of the martingale problem to a certain auxiliary SDE, for which results on weak existence of solutions are known. These results do *not* cover the cases  $|\rho| = 1$  and  $\sigma_1 = 0$  since the diffusion coefficient becomes degenerate for certain controls. For the equilibrium control tuple one can yet obtain solutions to the martingale problem.

**Proposition 3.2.** *Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ .*

- (i) *For all  $\rho \in [-1, 1]$  and  $\alpha^*, \beta^*$ , defined in Theorem 1.4, there exists a solution to the martingale problem (4), i.e.  $\mathcal{Q}(t, x, y, \alpha^*, \beta^*) \neq \emptyset$  and  $\alpha^*, \beta^* \in \mathcal{A}(t, x, y)$ .*
- (ii) *For all  $\rho \in [-1, 1]$  and  $q_1^*, q_2^*$ , defined in Theorem 2.6, there exists a solution to the martingale problem (8), i.e.  $\mathcal{Q}(t, x, y, q_1^*, q_2^*) \neq \emptyset$  and  $q_1^*, q_2^* \in \mathcal{V}(t, x, y)$ .*

For the proof we refer the reader again to the Appendix A, in particular to page 34, where all cases of Proposition 3.2 are treated. Proposition 3.2 implies that the tuple  $(q_1^*, q_2^*)$  is an admissible tuple of controls for a saddle point, because it belongs to the set  $\mathcal{V} \times \mathcal{V}$ .

The martingale problem (8) is only formulated for functions that do not depend on time. For the purpose of verification arguments in the proof of our main result, Theorem 2.6, we generalize the martingale problem to time-dependent functions that do not have compact support.

**Proposition 3.3.** *Let  $t \in [0, T]$ ,  $(x, y) \in \mathbb{R}^2$ ,  $q_1, q_2 \in \mathcal{V}(t, x, y)$  and  $P \in \mathcal{Q}(t, x, y, q_1, q_2)$ . Then for any  $f \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^2)$  the process  $\tilde{M}^f$ , defined by*

$$\tilde{M}_s^f := f(s, X_s, Y_s) - f(t, x, y) - \int_t^s (\partial_t f(u, X_u, Y_u) + Lf(u, X_u, Y_u, q_1, q_2)) du, \quad (9)$$

for  $s \in [t, T]$ , is a local martingale under  $P$ , where

$$Lf(s, x', y', q_1, q_2) := Lf(s, x', y', q_1(s, x', y'), q_2(s, x', y')), \quad (s, x', y') \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

*Proof of Proposition 3.3.* Corollary 3.7 below implies that the canonical process  $(X, Y)$  solves the SDE (11) on an extension of  $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0, T]}, P)$  with correlated Brownian motions  $W, B$  and an independent Brownian motion  $\tilde{B}$ . Applying Itô's formula to  $f(s, X_s, Y_s)$  implies that  $\tilde{M}^f$  is a local martingale.  $\square$

**Remark 3.4.** The result of Proposition 3.3 also holds true if  $f \in \mathcal{C}^{1,1}([0, T] \times \mathbb{R})$  and the partial derivative  $\partial_x f(t, \cdot)$  is absolutely continuous for all  $t \in [0, T]$ , because in this case Itô's formula can still be applied (see, e.g., [12], Section 3.7).

The connection of martingale problems and weak solutions is well understood, see, e.g., monograph [12], Section 5.4.B, for a thorough treatment. We describe the distribution of the state process via a martingale problem, because it permits a straightforward introduction of the state distribution determined by relaxed controls. One just needs to generalize the covariation structure. Nevertheless, the characterization of the state processes via SDEs can be very helpful, in particular for showing the existence and uniqueness of solutions, but also for an interpretation of the states' dynamics.

We start by generalizing the definition  $C(a, b)$  to the matrix  $C(\mu_1, \mu_2)$ , where

$$C_{i,j}(\mu_1, \mu_2) = \int_A \int_A C_{i,j}(a, b) \mu_1(da) \mu_2(db), \quad \mu_1, \mu_2 \in \mathcal{P}(A).$$

Note that  $C(\mu_1, \mu_2)$  is the matrix where each entry of  $C(a, b)$  is integrated w.r.t. the product measure  $\mu_1 \otimes \mu_2$ . For  $a, b \in A$  one obtains that  $C(\delta_a, \delta_b) = C(a, b)$ . With this notation equation (8) is equivalent to

$$f(X_s, Y_s) - f(x, y) - \int_t^s \frac{1}{2} \text{tr} (C(q_1(u, X_u, Y_u), q_2(u, X_u, Y_u)) D^2 f(X_u, Y_u)) du.$$

To connect the martingale problem (8) with an SDE we need to find a measurable matrix  $\Sigma$  with  $\Sigma\Sigma^T = C$ , that describes the diffusion coefficient of the SDE.  $\Sigma$  is a “square root” of the matrix  $C$ . For certain parameters the matrix  $C$  is degenerate which suggest that we need to enlarge the probability space such that it supports sufficiently many Brownian motions. Moreover, we need to consider matrices that have dimension  $2 \times 4$  in order to construct  $\Sigma$  satisfying  $\Sigma\Sigma^T(\mu_1, \mu_2) = C(\mu_1, \mu_2)$ ,  $\mu_1, \mu_2 \in \mathcal{P}(A)$ , i.e. the solution to an SDE with diffusion coefficient  $\Sigma$  possesses four independent Brownian motions. Note that our probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0, T]}, P)$  can not support four independent Brownian motions and hence we need an extension. Following Remark 3.4.1 in [12], we define an extension of the probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0, T]}, P)$  as follows.

**Definition 3.5.** We say that a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{P})$  is an extension of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$  if for some other probability space  $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \in [0, T]}, P')$  we have

- (i)  $\tilde{\Omega} = \Omega \times \Omega'$ ,
- (ii)  $\tilde{P} = P \otimes P'$ ,
- (iii)  $\tilde{\mathcal{F}} = \sigma(\mathcal{G} \cup \mathcal{N})$ , where  $\mathcal{G} = \mathcal{F} \otimes \mathcal{F}'$  and  $\mathcal{N}$  is the set of all  $\tilde{P}$ -null sets in  $\mathcal{G}$ ,
- (iv)  $\tilde{\mathcal{F}}_t = \bigcap_{s > t} \sigma(\mathcal{F}_s \otimes \mathcal{F}'_s \cup \mathcal{N})$ ,

Any random variable  $Z$ , defined on  $(\Omega, \mathcal{F}, P)$  can be extended onto  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$  by setting  $Z(\omega, \omega') := Z(\omega)$  for any  $(\omega, \omega') \in \tilde{\Omega}$ .

We can now connect a solution to the martingale problem to a class of SDEs, as in Theorem 2.5 of [8].

**Proposition 3.6.** *Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and  $q_1, q_2 \in \mathcal{V}(t, x, y)$ . Let  $\Sigma : \mathcal{P}(A) \times \mathcal{P}(A) \rightarrow \mathbb{R}^{2 \times 4}$  be measurable and such that  $\Sigma\Sigma^T = C$ . The existence of a solution to the martingale problem (8) is equivalent to the existence of a weak solution to the SDE*

$$dZ_t = \Sigma(q_1(s, Z_s), q_2(s, Z_s)) dW_s, \quad Z_t = (x, y). \quad (10)$$

Moreover, for any  $P \in \mathcal{Q}(t, x, y, q_1, q_2)$  there exists a four-dimensional Brownian motion  $W$ , defined on an extended probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{P})$ , such that the canonical process  $(X, Y)$  solves (10).

*Proof.* Follows from Proposition 5.4.11 in [12]. □

For a particular choice for  $\Sigma$  we obtain a more insightful representation.

**Corollary 3.7.** *Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ ,  $q_1, q_2 \in \mathcal{V}(t, x, y)$ ,  $\mathcal{Q}(t, x, y, q_1, q_2) \neq \emptyset$  and  $P \in \mathcal{Q}(t, x, y, q_1, q_2)$ . Then there exist Brownian motions  $W, B$  and  $\tilde{B} = (\tilde{B}^1, \tilde{B}^2)$ , defined on an extended probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{P})$ , such that  $\tilde{P}$ -a.s. for all  $s \in [t, T]$*

$$\begin{aligned} X_s &= x + \int_t^s \int_A a q_1(r, X_r, Y_r)(da) dW_r dr + \int_t^s \sqrt{\text{Var}(q_1(r, X_r, Y_r))} d\tilde{B}_r^1 dr, \\ Y_s &= y + \int_t^s \int_A b q_2(r, X_r, Y_r)(db) dB_r dr + \int_t^s \sqrt{\text{Var}(q_2(r, X_r, Y_r))} d\tilde{B}_r^2 dr, \end{aligned} \quad (11)$$



where both  $W$  and  $B$  are independent of  $\tilde{B}$ , and  $\langle W, B \rangle_s = \rho s$ ,  $s \in [t, T]$ .

**Remark 3.8.** In view of Corollary 3.7 the case  $\rho = 1$  allows for an interesting interpretation: the leading player always wants to increase the correlation, while the opponent tries to decrease the correlation by including an independent noise proportional to the standard deviation.

Note that by comparing (2) and (11) we see that the SDEs describing the dynamics of the state processes require an additional stochastic integral that depends on the variance of the relaxed controls. In the special case that  $q_1 = \delta_\alpha$  and  $q_2 = \delta_\beta$  one can just recover the SDE (1) since the variance of  $q_1$  and  $q_2$  is equal to zero.

Proposition 3.6 entails that the set of solutions to the martingale problem (8) is precisely the set of laws given by SDEs of the form (10). The same characterization holds true for strict controls. For strict controls there are even cases where there is no need to extend the probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0, T]}, P)$ , because one can find  $2 \times 2$ -matrices  $\Sigma$  such that  $\Sigma \Sigma^T = C$ . A sufficient condition is that for fixed  $\alpha, \beta \in \mathcal{A}$  the matrix  $C(\alpha(s, X_s, Y_s), \beta(s, X_s, Y_s))$  has constant rank  $r \in \{1, 2\}$  for a.e.  $s \in [t, T]$ ,  $P$ -a.s. (see Remark 3.4.3 in [12]). Then only  $r$  Brownian motions are required. If, e.g.,  $|\rho| < 1$  and  $\sigma_1 > 0$ , then  $C(\alpha, \beta)$  is non-degenerate and hence has rank 2. Thus no extension is required. In more detail:

**Corollary 3.9.** *Let  $|\rho| < 1$ ,  $\sigma_1 > 0$ ,  $(t, x, y) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}$ ,  $\alpha, \beta \in \mathcal{A}$  and let  $\sigma : A \times A \rightarrow \mathbb{R}^{2 \times 2}$  be measurable and satisfy  $\sigma \sigma^T = C$ . For  $P \in \mathcal{Q}(t, x, y, \alpha, \beta)$  there exists a two-dimensional Brownian motion  $W$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0, T]}, P)$  such that the canonical processes  $(X, Y)$  satisfy*

$$dZ_s = \sigma(\alpha(s, Z_s), \beta(s, Z_s)) dW_s, \quad Z_t = (x, y). \quad (12)$$

for all  $s \in [t, T]$ ,  $P$ -a.s.

*Proof.* Follows from [12], Proposition 5.4.6 together with Theorem 3.4.2 and Remark 3.4.3 in [12], because  $C(\alpha(s, x, y), \beta(s, x, y))$  has constant rank of two for all  $(s, x, y) \in [t, T] \times \mathbb{R} \times \mathbb{R}$ .  $\square$

**Remark 3.10.** We summarize some facts about the SDEs that represent our state processes.

- (i) In general the SDE (10) does not have a strong solution (even if we just consider strict controls). A counterexample is given by Barlow [4], who considers a one-dimensional SDE with a bounded diffusion coefficient that is uniformly bounded away from zero.
- (ii) For  $|\rho| = 1$  or  $\sigma_1 = 0$  there exists no weak solution to the SDE (12) for certain strategies  $\alpha, \beta \in \tilde{\mathcal{A}}$ , and hence also no solution to the martingale problems (4) and (8), because the matrix  $C$  is degenerate. In more detail, consider, e.g., the diffusion matrix

$$\sigma(x, y) = \begin{pmatrix} \alpha(x, y) & 0 \\ \rho\beta(x, y) & \sqrt{1 - \rho^2}\beta(x, y) \end{pmatrix}, \quad \alpha, \beta \in \tilde{\mathcal{A}}.$$

We have  $\sigma\sigma^T = C(\alpha, \beta)$ . If  $\sigma_1 = 0$ , then for  $\alpha(x, y) := \mathbb{1}_{\{0\}}(x)$  there exists no weak solution to (12) (see, e.g., [12], Remark 5.5.5). If  $|\rho| = 1$ , then for  $\alpha(x, y) := \sigma_1 \mathbb{1}_{(0, \infty)}(|x - \text{sgn}(\rho)y|) + \sigma_2 \mathbb{1}_{\{0\}}(x - \text{sgn}(\rho)y)$  and  $\beta(x, y) := \sigma_1$  there is no weak solution to (12): assume on the contrary that there is a solution  $Z$  to (12). Then  $\tilde{Z} := Z^1 - \text{sgn}(\rho)Z^2$  solves

$$d\tilde{Z}_s = (\sigma_2 - \sigma_1) \mathbb{1}_{\{0\}}(\tilde{Z}_s) dW_s.$$

This is a contradiction, because this SDE does not have a solution (as in the case of  $\sigma_1 = 0$ ).

- (iii) Let  $\sigma_1 = 0$ . Then weak uniqueness does not hold true for the SDE (12) for some  $\alpha, \beta \in \tilde{\mathcal{A}}$ . For example, the choice  $\rho = 0$  and  $\alpha(x) = \beta(x) := |x|^\delta \wedge \sigma_2$  with  $0 < \delta < \frac{1}{2}$  leads to multiple weak solutions to (12), see [12], Theorem 5.5.7. To the best of our knowledge there are no results on uniqueness in law for the SDE (12) if  $\alpha, \beta \in \tilde{\mathcal{A}}$  depend on time (even if  $\sigma_1 > 0$ ). Hence, in general we can only deal with weak solutions that are not unique in law. See also the abstract of [16].

## 4. An auxiliary static game

In this section, we consider the upper and lower *Hamilton-Jacobi-Bellman-Isaacs* (HJBI) PDE and prove that the associated *Isaacs condition* is satisfied for a certain class of functions. To this end, we introduce an auxiliary static game, which consists of two players who aim at maximizing or minimizing a quadratic function  $r$  by choosing actions from the set  $A$ . We prove that the game has a value in mixed actions, and even in pure actions if the correlation does not exceed the bound in (6).

In the literature, Fleming and Souganidis [10] were the first that characterized the upper and lower value of stochastic differential games as the unique viscosity solution of the upper and lower Hamilton-Jacobi-Bellman-Isaacs equation. They showed that these solutions coincide under the so-called Isaacs condition. For our game in relaxed controls the upper and lower HJBI equation are given by

$$\begin{aligned} -\partial_t v^+(t, x, y) - H^+(D^2 v^+(t, x, y)) &= 0, \\ v^+(T, x, y) &= g(x - y), \end{aligned} \tag{13}$$

and

$$\begin{aligned} -\partial_t v^-(t, x, y) - H^-(D^2 v^-(t, x, y)) &= 0, \\ v^-(T, x, y) &= g(x - y), \end{aligned} \tag{14}$$

for  $(t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}$  and with Hamiltonians  $H^+$  and  $H^-$  given by

$$\begin{aligned} H^+(\Lambda) &:= \frac{1}{2} \inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} \text{tr}(C(\mu_1, \mu_2)\Lambda), \\ H^-(\Lambda) &:= \frac{1}{2} \sup_{\mu_1 \in \mathcal{P}(A)} \inf_{\mu_2 \in \mathcal{P}(A)} \text{tr}(C(\mu_1, \mu_2)\Lambda), \end{aligned}$$

for any symmetric matrix  $\Lambda \in \mathbb{R}^{2 \times 2}$ . With the infinitesimal generator  $L$  of the state processes we obtain that

$$\begin{aligned} H^+(D^2v^+(t, x, y)) &= \inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} Lv^+(t, x, y, \mu_1, \mu_2), \\ H^-(D^2v^-(t, x, y)) &= \sup_{\mu_1 \in \mathcal{P}(A)} \inf_{\mu_2 \in \mathcal{P}(A)} Lv^-(t, x, y, \mu_1, \mu_2), \end{aligned}$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ , see (3) and (7) for the definition of  $L$ . The upper HJBI equation (13) characterizes the upper value of the game in relaxed controls, and the lower HJBI equation (14) the lower value. We prove later in Lemma 5.6 that the upper value function and the lower value function indeed solve the upper and lower HJBI equation, respectively, and that they coincide. It is reasonable to assume that the upper and lower value function,  $V^+$  and  $V^-$ , only depend on  $(x, y)$  through the difference  $x - y$ , because of the definition of the payoff functions  $J^+$  and  $J^-$ , and of course because we consider rank-based rewards depending only on the difference of the state processes. We prove this assertion rigorously in Section 5. For solving the HJBI equations (13) and (14) it makes thus sense to consider functions contained in

$$\mathcal{D} := \{v : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R} \mid \exists \tilde{v} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} : v(t, x, y) = \tilde{v}(t, x - y)\}. \quad (15)$$

If the Hessian matrix (w.r.t. the state variables) of a function in  $\mathcal{D}$  exists, then it is contained in the set

$$\mathcal{M} := \left\{ \begin{pmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\} \subseteq \mathbb{R}^{2 \times 2},$$

that is a subset of the set of symmetric matrices with real entries. For the proof of our main results, namely Theorem 1.4 and Theorem 2.6, we show that the upper and lower value function coincide, i.e.  $V^+ = V^-$ . For this the so-called *Isaacs condition*, i.e. the condition  $H^+ = H^-$  for all symmetric matrices, is essential, because then the PDEs (13) and (14) coincide. Since we suspect that the value functions are contained in  $\mathcal{D}$ , we only need the Isaacs condition to be satisfied on  $\mathcal{M}$ . Therefore, we consider the weaker Isaacs condition

$$H^+ = H^- \text{ on } \mathcal{M}. \quad (16)$$

We refer to (16) in the following just as Isaacs condition. If a function in  $\mathcal{D}$  solves either (13) or (14), and the Isaacs condition (16) holds true, then it also solves the other. Hence, the Isaacs condition can indicate the existence of the game value. Note that there exist games that satisfy the Isaacs condition, but that do not have a value (see e.g. [24], Example 2.1).

For  $\Lambda \in \mathcal{M}$ , with entries  $\Lambda_{ij}$ ,  $i, j = 1, 2$ , we see that

$$\begin{aligned} H^+(\Lambda) &= \frac{1}{2} \inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} (\Lambda_{11}r(\mu_1, \mu_2)), \\ H^-(\Lambda) &= \frac{1}{2} \sup_{\mu_1 \in \mathcal{P}(A)} \inf_{\mu_2 \in \mathcal{P}(A)} (\Lambda_{11}r(\mu_1, \mu_2)), \end{aligned}$$

where

$$r(a, b) := a^2 - 2\rho ab + b^2, \quad (a, b) \in A^2,$$

and we write  $r(\mu_1, \mu_2) := \int_A \int_A r(a, b) \mu_1(da) \mu_2(db)$  for  $\mu_1, \mu_2 \in \mathcal{P}(A)$  to simplify notation. We observe that the Isaacs condition (16) is satisfied, if and only if the values

$$\inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2) \quad (17)$$

and

$$\sup_{\mu_1 \in \mathcal{P}(A)} \inf_{\mu_2 \in \mathcal{P}(A)} r(\mu_1, \mu_2) \quad (18)$$

coincide. We prove equality of (17) and (18) by studying the static zero-sum game, where the pure action set of both players is  $A = [\sigma_1, \sigma_2]$ , and the reward of player 1 is given by the function  $r$ . Recall that a pair  $(a^*, b^*) \in A^2$  is a saddle point of the static game if and only if it is a saddle point of  $r$ , i.e. if

$$\max_{a \in A} r(a, b^*) = r(a^*, b^*) = \min_{b \in A} r(a^*, b).$$

We identify the set of mixed actions with  $\mathcal{P}(A)$ , the set of probability measures on  $A$ . A saddle point in mixed actions is a tuple  $(\mu_1^*, \mu_2^*) \in \mathcal{P}(A) \times \mathcal{P}(A)$  satisfying

$$\sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2^*) = r(\mu_1^*, \mu_2^*) = \inf_{\mu_2 \in \mathcal{P}(A)} r(\mu_1^*, \mu_2).$$

Note that the value (17) and (18) correspond exactly to the upper and lower value of the static game in mixed actions, respectively. The existence of a value and a saddle point is standard in the game theoretic literature, see e.g. [11]. We can even explicitly calculate the upper value (17) and the lower value (18), and identify the saddle point along the way. We obtain the following result.

**Proposition 4.1.** *The static game always has a value in mixed actions, i.e. we have*

$$\inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2) = \sup_{\mu_1 \in \mathcal{P}(A)} \inf_{\mu_2 \in \mathcal{P}(A)} r(\mu_1, \mu_2) =: r^*.$$

The value of the game is given by

$$r^* = c(\rho)^2 = \begin{cases} \sigma_2^2 - 2\rho\sigma_2(\rho\sigma_2 \vee \sigma_1) + (\rho\sigma_2 \vee \sigma_1)^2, & \text{if } \rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}, \\ \left(\frac{\sigma_1 + \sigma_2}{2\rho}\right)^2 - \sigma_1\sigma_2, & \text{if } \rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}. \end{cases}$$

Moreover, the tuple  $(\mu_1^*, \mu_2^*) \in \mathcal{P}(A) \times \mathcal{P}(A)$ , given by

$$\begin{aligned} \mu_1^* &= \begin{cases} \delta_{\sigma_2}, & \text{if } \rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}, \\ \frac{\sigma_2 - \frac{\sigma_1 + \sigma_2}{2\rho^2}}{\sigma_2 - \sigma_1} \delta_{\sigma_1} + \frac{\frac{\sigma_1 + \sigma_2}{2\rho^2} - \sigma_1}{\sigma_2 - \sigma_1} \delta_{\sigma_2}, & \text{if } \rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}, \end{cases} \\ \mu_2^* &= \begin{cases} \delta_{\sigma_1}, & \text{if } \rho \leq \frac{\sigma_1}{\sigma_2}, \\ \delta_{\rho\sigma_2}, & \text{if } \frac{\sigma_1}{\sigma_2} < \rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}, \\ \delta_{\frac{\sigma_1 + \sigma_2}{2\rho}}, & \text{if } \rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}, \end{cases} \end{aligned} \quad (19)$$

is a saddle point.

Before we prove Proposition 4.1 we consider the following auxiliary result.

**Lemma 4.2.** *Let  $f : A \rightarrow \mathbb{R}$  be continuous. Then*

$$\max_{a \in A} f(a) = \max_{\mu \in \mathcal{P}(A)} \int_A f(a) \mu(da).$$

*Proof.* We have for any  $\mu \in \mathcal{P}(A)$

$$\int_A f(a) \mu(da) \leq \max_{a \in A} f(a),$$

and  $\sup_{\mu \in \mathcal{P}(A)} \int_A f(a) \mu(da) \geq \max_{a \in A} f(a)$ . Hence,  $\max_{\mu \in \mathcal{P}(A)} \int_A f(a) \mu(da)$  exists and is equal to  $\max_{a \in A} f(a)$ .  $\square$

Lemma 4.2 implies that any saddle point in pure actions is also one in mixed actions.

*Proof of Proposition 4.1.* We prove the statement by verifying that  $(\mu_1^*, \mu_2^*)$  are mutually best responses.

*Step 1.* Let  $\rho \leq \frac{\sigma_1}{\sigma_2}$ . Then  $r(\mu_1^*, \mu_2^*) = r(\sigma_2, \sigma_1)$ . Note that  $a \mapsto r(a, \sigma_1)$  is monotonically increasing on  $[\rho\sigma_1, \sigma_2]$ , and thus attains its maximum on  $[\sigma_1, \sigma_2]$  at  $a = \sigma_2$ , i.e.

$$\max_{a \in [\sigma_1, \sigma_2]} r(a, \sigma_1) = r(\sigma_2, \sigma_1).$$

Moreover,  $b \mapsto r(\sigma_2, b)$  is monotonically increasing on  $[\sigma_1, \sigma_2]$  because  $\rho\sigma_2 \leq \sigma_1$ . Hence,

$$\min_{b \in [\sigma_1, \sigma_2]} r(\sigma_2, b) = r(\sigma_2, \sigma_1).$$

This altogether implies that  $(\sigma_2, \sigma_1)$  is a saddle point in pure actions. Lemma 4.2 implies that it is also a saddle point in mixed actions.

*Step 2.* Let  $\frac{\sigma_1}{\sigma_2} < \rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ . Then  $r(\mu_1^*, \mu_2^*) = r(\sigma_2, \rho\sigma_2)$ . The quadratic function  $a \mapsto r(a, \rho\sigma_2)$  is symmetric around its minimum at  $a = \rho^2\sigma_2 \leq \frac{\sigma_1 + \sigma_2}{2}$ . Hence, the maximum is attained at  $a = \sigma_2$ , i.e.

$$\max_{a \in [\sigma_1, \sigma_2]} r(a, \rho\sigma_2) = r(\sigma_2, \rho\sigma_2).$$

In addition,  $b \mapsto r(\sigma_2, b)$  attains its minimum at  $\rho\sigma_2 \in [\sigma_1, \sigma_2]$ , and therefore

$$\min_{b \in [\sigma_1, \sigma_2]} r(\sigma_2, b) = r(\sigma_2, \rho\sigma_2).$$

Again Lemma 4.2 implies that  $(\sigma_2, \rho\sigma_2)$  is also a saddle point in mixed actions.

*Step 3.* Let  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ . Then

$$r(\mu_1^*, \mu_2^*) = p r\left(\sigma_1, \frac{\sigma_1 + \sigma_2}{2\rho}\right) + (1 - p) r\left(\sigma_2, \frac{\sigma_1 + \sigma_2}{2\rho}\right)$$

with  $p := \frac{1}{\sigma_2 - \sigma_1} \left( \sigma_2 - \frac{\sigma_1 + \sigma_2}{2\rho^2} \right)$ . The function  $a \mapsto r \left( a, \frac{\sigma_1 + \sigma_2}{2\rho} \right)$  is symmetric around its minimum at  $a = \frac{\sigma_1 + \sigma_2}{2}$  and hence

$$\begin{aligned} \max_{a \in [\sigma_1, \sigma_2]} r \left( a, \frac{\sigma_1 + \sigma_2}{2\rho} \right) &= r \left( \sigma_1, \frac{\sigma_1 + \sigma_2}{2\rho} \right) = r \left( \sigma_2, \frac{\sigma_1 + \sigma_2}{2\rho} \right) \\ &= r \left( \mu_1^*, \frac{\sigma_1 + \sigma_2}{2\rho} \right). \end{aligned}$$

The function  $b \mapsto r(\mu_1^*, b) = pr(\sigma_1, b) + (1-p)r(\sigma_2, b)$  attains its minimum at  $b = \frac{\sigma_1 + \sigma_2}{2\rho} \in [\sigma_1, \sigma_2]$ , i.e.

$$\min_{b \in [\sigma_1, \sigma_2]} r(\mu_1^*, b) = r \left( \mu_1^*, \frac{\sigma_1 + \sigma_2}{2\rho} \right).$$

Lemma 4.2 implies that  $(\mu_1^*, \mu_2^*)$  is a saddle point.

*Step 4.* The existence of a saddle point implies that the upper and lower value of the game coincide, because

$$\begin{aligned} \inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2) &\leq \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2^*) \\ &= r(\mu_1^*, \mu_2^*) \\ &= \inf_{\mu_2 \in \mathcal{P}(A)} r(\mu_1^*, \mu_2) \\ &\leq \sup_{\mu_1 \in \mathcal{P}(A)} \inf_{\mu_2 \in \mathcal{P}(A)} r(\mu_1, \mu_2) \\ &\leq \inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2). \end{aligned}$$

Finally, one can obtain the explicit value of the game by calculating  $r(\mu_1^*, \mu_2^*)$ .  $\square$

**Remark 4.3.** We are able to solve problems (17) and (18) above explicitly because they consist in optimizing  $r(\mu_1, \mu_2)$ , which is a linear functional of  $(\mu_1, \mu_2)$ . For instance, for a given  $\mu_2$ , the value of  $\sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2)$ , which is solved thanks to Lemma 4.2, only depends on the average  $\bar{\mu}_2 := \int_A a \mu_2(da)$ . As a consequence, the problem  $\inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2)$  can be divided in subproblems consisting of minimizing a linear (and hence concave) functional of  $\mu_2$  under a moment constraint. From Theorem 3.2 in [26], these subproblems are known to be reduced to optimization over measures  $\mu_2$  which take the form of convex combination of at most two Dirac masses: this explains the form of the optimal probability measures given by (19).

**Proposition 4.4.** *The static game has a value in pure actions if and only if  $\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ .*

*Moreover, if  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ , then*

$$\begin{aligned} \inf_{b \in A} \sup_{a \in A} r(a, b) &= r \left( \sigma_i, \frac{\sigma_1 + \sigma_2}{2\rho} \right), \quad i = 1, 2, \\ \sup_{a \in A} \inf_{b \in A} r(a, b) &= r(\sigma_2, \rho\sigma_2), \end{aligned}$$

*and hence  $\inf_{b \in A} \sup_{a \in A} r(a, b) > \sup_{a \in A} \inf_{b \in A} r(a, b)$ .*

*Proof.* If  $\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ , then the existence of a saddle point in pure actions follows from Proposition 4.1. Let  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ . We prove that the upper and lower value do not coincide. The proof follows from minimization and maximization of the quadratic function  $r$ . For some  $b \in A = [\sigma_1, \sigma_2]$  it follows that

$$\sup_{a \in A} r(a, b) = \begin{cases} r(\sigma_1, b), & \text{if } \frac{\sigma_1 + \sigma_2}{2} \leq \rho b, \\ r(\sigma_2, b), & \text{if } \frac{\sigma_1 + \sigma_2}{2} \geq \rho b. \end{cases}$$

Now, minimizing over  $b$  implies that  $\inf_{b \in A} \sup_{a \in A} r(a, b) = r\left(\sigma_i, \frac{\sigma_1 + \sigma_2}{2\rho}\right)$ ,  $i = 1, 2$ . For some  $a \in A$  we obtain

$$\inf_{b \in A} r(a, b) = \begin{cases} r(a, \rho a), & \text{if } \rho a \in A, \\ r(a, \sigma_1), & \text{if } \rho a < \sigma_1. \end{cases}$$

Then maximizing over  $a$  yields  $\sup_{a \in A} \inf_{b \in A} r(a, b) = r(\sigma_2, \rho\sigma_2)$ . It is easy to see that  $r(\sigma_2, \rho\sigma_2) < r\left(\sigma_2, \frac{\sigma_1 + \sigma_2}{2\rho}\right)$ , and hence the static game does not have a value in pure actions.  $\square$

**Remark 4.5.** The upper and lower HJBI equation in strict controls is given as in (13) and (14), but  $H^+$  and  $H^-$  need to be replaced with new Hamiltonians  $H_{\text{strict}}^+$  and  $H_{\text{strict}}^-$ , defined as

$$\begin{aligned} H_{\text{strict}}^+(\Lambda) &:= \frac{1}{2} \inf_{b \in A} \sup_{a \in A} \text{tr}(C(a, b)\Lambda), \\ H_{\text{strict}}^-(\Lambda) &:= \frac{1}{2} \sup_{a \in A} \inf_{b \in A} \text{tr}(C(a, b)\Lambda), \end{aligned} \tag{20}$$

for symmetric  $\Lambda \in \mathbb{R}^{2 \times 2}$  (see equations (27) and (28) below). With similar arguments as in the case of relaxed controls one can argue that the Isaacs condition (16) is fulfilled for  $H_{\text{strict}}^+$  and  $H_{\text{strict}}^-$ , if and only if the auxiliary static game has a value in pure actions. Therefore, Proposition 4.4 entails that the Isaacs condition (16) is satisfied for  $H_{\text{strict}}^+$  and  $H_{\text{strict}}^-$ , if and only if  $\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ . This is the crucial point in the proof of Proposition 1.8.

## 5. Proof of the main results

### 5.1. Proof of Theorem 1.4 and Theorem 2.6

In this section, we prove our main results Theorem 1.4 and Theorem 2.6. To this end, we first show that  $w$  solves the HJBI equations (13) and (14), and then we prove that the upper and lower value are equal to  $w$  by using verification arguments.

**Lemma 5.1.**  $w \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$  and  $w$  solves the PDE

$$-\partial_t w(t, z) - \frac{c(\rho)^2}{2} \partial_{zz} w(t, z) = 0, \quad (t, z) \in [0, T] \times \mathbb{R}, \tag{21}$$

with  $w(T, \cdot) = g$ .

*Proof.* The smoothness of  $w$  follows from its definition in (5), because  $w$  is the convolution of the function  $g$  with the probability density function of a standard normal distribution. The derivatives can be calculated by interchanging integration and differentiation. The function  $w$  can be written as a convolution

$$w(t, z) = (g * \varphi(t, \cdot))(z), \quad \text{with } \varphi(z) := \frac{1}{\sqrt{2\pi c(\rho)^2(T-t)}} e^{-\frac{z^2}{2c(\rho)^2(T-t)}}, \quad z \in \mathbb{R}.$$

The function  $\varphi$  is the probability density function of a normal distribution with mean 0 and variance equal to  $c(\rho)^2(T-t)$ , and it solves the heat equation

$$\partial_t \varphi(t, x) + \frac{c(\rho)^2}{2} \partial_{xx} \varphi(t, x) = 0, \quad (t, x) \in [0, T) \times \mathbb{R},$$

which is straightforward to verify. We obtain that  $w$  solves the PDE

$$-\partial_t w(t, z) - \frac{c(\rho)^2}{2} \partial_{zz} w(t, z) = - \left( g * \left( \partial_t \varphi(t, \cdot) + \frac{c(\rho)^2}{2} \partial_{xx} \varphi(t, \cdot) \right) \right) (z) = 0,$$

on  $[0, T) \times \mathbb{R}$  by the linearity of the convolution and because differentiating  $w$  is the same as the convolution of  $g$  and the derivatives of  $\varphi$ . Finally, note that  $w(T, z) = g(z)$  by the very definition (5).  $\square$

Proposition 4.1 implies that  $w$  also satisfies

$$\begin{aligned} 0 &= -\partial_t w(t, z) - \frac{1}{2} \inf_{\mu_2 \in \mathcal{P}(A)} \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2) \partial_{zz} w(t, z) \\ &= -\partial_t w(t, z) - \frac{1}{2} \sup_{\mu_1 \in \mathcal{P}(A)} \inf_{\mu_2 \in \mathcal{P}(A)} r(\mu_1, \mu_2) \partial_{zz} w(t, z), \quad (t, z) \in [0, T) \times \mathbb{R}. \end{aligned} \tag{22}$$

Moreover, we have:

**Lemma 5.2.** *For all  $(t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}$  it holds*

$$\begin{aligned} 0 &= -\partial_t w(t, x-y) - \frac{1}{2} \inf_{\mu_2 \in \mathcal{P}(A)} r(q_1^*(t, x, y), \mu_2) \partial_{zz} w(t, x-y) \\ &= -\partial_t w(t, x-y) - \frac{1}{2} \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, q_2^*(t, x, y)) \partial_{zz} w(t, x-y). \end{aligned}$$

*Proof.* By definition of  $(q_1^*, q_2^*)$  we have

$$q_1^*(t, x, y) = \begin{cases} \mu_1^*, & \text{if } (t, x-y) \in \overline{D_+}, \\ \mu_2^*, & \text{if } (t, x-y) \in D_-, \end{cases} \quad q_2^*(t, x, y) = \begin{cases} \mu_2^*, & \text{if } (t, x-y) \in \overline{D_+}, \\ \mu_1^*, & \text{if } (t, x-y) \in D_-, \end{cases}$$

for  $(t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}$  and where  $\mu_1^*, \mu_2^*$  are given by (19). Proposition 4.1 implies that  $(\mu_1^*, \mu_2^*)$  is a saddle point, i.e.

$$c(\rho)^2 = r(\mu_1^*, \mu_2^*) = \max_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_2^*) = \min_{\mu_2 \in \mathcal{P}(A)} r(\mu_1^*, \mu_2).$$



If  $(t, x - y) \in \overline{D_+}$ , then

$$\begin{aligned} \sup_{\mu_1 \in \mathcal{P}(A)} \partial_{zz} w(t, x - y) r(\mu_1, q_2^*(t, x, y)) &= \partial_{zz} w(t, x - y) \sup_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_1^*) \\ &= \partial_{zz} w(s, X_s - Y_s) c(\rho)^2, \end{aligned}$$

and if  $(t, x - y) \in D_-$ , then

$$\begin{aligned} \sup_{\mu_1 \in \mathcal{P}(A)} \partial_{zz} w(t, x - y) r(\mu_1, q_2^*(t, x, y)) &= \partial_{zz} w(t, x - y) \inf_{\mu_1 \in \mathcal{P}(A)} r(\mu_1, \mu_1^*) \\ &= \partial_{zz} w(s, X_s - Y_s) c(\rho)^2. \end{aligned}$$

Similarly, we have

$$\inf_{\mu_2 \in \mathcal{P}(A)} \partial_{zz} w(t, x - y) r(q_1^*(t, x, y), \mu_2) = \partial_{zz} w(s, X_s - Y_s) c(\rho)^2, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

□

The next two lemmas present probabilistic properties of the state processes.

**Lemma 5.3.** *Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ ,  $q_1, q_2 \in \mathcal{V}(t, x, y)$  and  $P \in \mathcal{Q}(t, x, y, q_1, q_2)$ . Assume that*

(i) *either  $\sigma_1 > 0$  or  $\rho > 0$ , and*

(ii)  *$q_1 = q_1^*$  or  $q_2 = q_2^*$ ,*

then

$$P(X_T - Y_T = z) = 0, \quad z \in \mathbb{R}.$$

*Proof.* Proposition 3.6 implies that there exists a four-dimensional Brownian motion  $W$  on an extension  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{P})$  of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$  such that  $\tilde{P}$ -a.s.

$$\begin{aligned} dX_s &= \bar{q}_1(s, X_s, Y_s) dW_s^1 + \sqrt{\text{Var}(q_1(s, X_s, Y_s))} dW_s^3, \\ dY_s &= \rho \bar{q}_2(s, X_s, Y_s) dW_s^1 + \sqrt{1 - \rho^2} \bar{q}_2(s, X_s, Y_s) dW_s^2 + \sqrt{\text{Var}(q_2(s, X_s, Y_s))} dW_s^4, \end{aligned}$$

for  $s \in [t, T]$ , where  $\bar{\mu} := \int_A a \mu(da)$ ,  $\bar{\mu}^2 := \int_A a^2 \mu(da)$ , and  $\text{Var}(\mu) := \bar{\mu}^2 - (\bar{\mu})^2$  for  $\mu \in \mathcal{P}(A)$ . Introduce the short-hand notation  $\sigma_s^1 := \bar{q}_1(s, X_s, Y_s)$ ,  $\sigma_s^2 := \sqrt{\text{Var}(q_1(s, X_s, Y_s))}$ ,  $\sigma_s^3 := \rho \bar{q}_2(s, X_s, Y_s)$ ,  $\sigma_s^4 := \sqrt{1 - \rho^2} \bar{q}_2(s, X_s, Y_s)$ ,  $\sigma_s^5 := \sqrt{\text{Var}(q_2(s, X_s, Y_s))}$  for  $s \in [t, T]$ . Then note that  $(\tilde{W}_s)_{s \in [t, T]}$ , defined by

$$\tilde{W}_s := \frac{(\sigma_s^1 - \sigma_s^3) W_s^1 - \sigma_s^4 W_s^2 + \sigma_s^2 W_s^3 - \sigma_s^5 W_s^4}{\sqrt{(\sigma_s^1 - \sigma_s^3)^2 + (\sigma_s^4)^2 + (\sigma_s^2)^2 + (\sigma_s^5)^2}}, \quad s \in [t, T],$$

is a Brownian motion. Moreover,

$$d(X - Y)_s = \sqrt{(\sigma_s^1 - \sigma_s^3)^2 + (\sigma_s^4)^2 + (\sigma_s^2)^2 + (\sigma_s^5)^2} d\tilde{W}_s, \quad s \in [t, T]. \quad (23)$$

The diffusion coefficient of this equation is bounded, and uniformly bounded away from zero if (i) and (ii) hold true. Theorem 1 in [22] implies the result. □

**Lemma 5.4.** For all  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ ,  $q_1, q_2 \in \mathcal{V}(t, x, y)$  and  $P \in \mathcal{Q}(t, x, y, q_1, q_2)$  the process  $X - Y$  is a time-changed Brownian motion (possibly on an extended probability space). In more detail, there exists a Brownian motion  $B$  on an extended probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{P})$  such that

$$X_s - Y_s = x - y + B_{\langle X - Y \rangle_s}, \quad s \in [t, T], \quad \tilde{P}\text{-a.s.}$$

*Proof.* Follows, e.g., from [12], Theorem 3.4.6 (also note Problem 3.4.7).  $\square$

Now, we prove that  $(q_1^*, q_2^*)$  is a saddle point.

**Lemma 5.5.** The pair  $(q_1^*, q_2^*)$  is a saddle point and we have

$$\begin{aligned} w(t, x - y) &= J^+(t, x, y, q_1^*, q_2^*) = J^-(t, x, y, q_1^*, q_2^*) \\ &= \sup_{q_1 \in \mathcal{V}} J^+(t, x, y, q_1, q_2^*) = \inf_{q_2 \in \mathcal{V}} J^-(t, x, y, q_1^*, q_2), \end{aligned}$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ .

*Proof.* Let  $\sigma_1 > 0$  or  $\rho > 0$ . Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and let  $q_1 \in \mathcal{V}(t, x, y)$ . Note that if  $\mathcal{Q}(t, x, y, q_1, q_2^*) = \emptyset$ , then by convention  $J^+(t, x, y, q_1, q_2^*) = -\infty$ . Thus, player 1 gains from choosing a different control  $q_1' \in \mathcal{V}$  (that exists since  $q_2^* \in \mathcal{V}$  by Proposition 3.2) such that  $\mathcal{Q}(t, x, y, q_1', q_2^*) \neq \emptyset$ , because then  $J^+(t, x, y, q_1', q_2^*)$  is finite. Consequently, for an optimal response of player 1 there always exists a solution to the corresponding martingale problem. We obtain  $J^+(t, x, y, q_1, q_2^*) < \sup_{q \in \mathcal{V}} J^+(t, x, y, q, q_2^*)$  if  $\mathcal{Q}(t, x, y, q_1, q_2^*) = \emptyset$ , and we can restrict the following considerations to the case where  $\mathcal{Q}(t, x, y, q_1, q_2^*) \neq \emptyset$ . Let  $P \in \mathcal{Q}(t, x, y, q_1, q_2^*)$ . We first aim at showing that

$$J^+(t, x, y, q_1^*, q_2^*) = \sup_{q_1 \in \mathcal{V}} J^+(t, x, y, q_1, q_2^*) = w(t, x - y).$$

To this end, let  $\delta \in (0, T - t)$ . From Proposition 3.3 we obtain that  $\tilde{M}^w$ , defined in (9), is a local martingale under  $P$ . Taking the expectation of  $\tilde{M}^w$  under  $P$  (plus possibly a localization) implies that

$$\begin{aligned} &E^P [w(T - \delta, X_{T-\delta} - Y_{T-\delta})] - w(t, x - y) \\ &= E^P \int_t^{T-\delta} \partial_t w(s, X_s - Y_s) ds \\ &\quad + E^P \int_t^{T-\delta} \frac{1}{2} \partial_{zz} w(s, X_s - Y_s) r(q_1(s, X_s, Y_s), q_2^*(s, X_s, Y_s)) ds \\ &\leq E^P \int_t^{T-\delta} \partial_t w(s, X_s - Y_s) ds \\ &\quad + E^P \int_t^{T-\delta} \frac{1}{2} \sup_{\mu \in \mathcal{P}(A)} (\partial_{zz} w(s, X_s - Y_s) r(\mu, q_2^*(s, X_s, Y_s))) ds. \end{aligned}$$

Lemma 5.2 implies

$$E^P [w(T - \delta, X_{T-\delta} - Y_{T-\delta})] - w(t, x - y) \leq 0. \quad (24)$$

We now want to apply dominated convergence. Because  $g$  has at most exponential growth (see Assumption 1.2), there exists  $C_1, C_2 > 0$  such that  $|g(z)| \leq C_1 e^{C_2|z|}$ ,  $z \in \mathbb{R}$ , and hence we can estimate  $w$  by

$$\begin{aligned} |w(t, z)| &\leq \int_{-\infty}^{\infty} |g(z + c(\rho)\sqrt{T-t}\xi)| \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi, \\ &\leq C_1 e^{C_2|z|} \int_{-\infty}^{\infty} e^{C_2 c(\rho)\sqrt{T}\xi} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi =: C_3 e^{C_2|z|}, \quad (t, z) \in [0, T] \times \mathbb{R}, \end{aligned} \quad (25)$$

using the definition (5) of  $w$ . By Lemma 5.4, the process  $X - Y$  is a time-changed Brownian motion, i.e. there exists a Brownian motion  $B$  on an extension  $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \in [0, T]}, \hat{P})$  of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$  such that

$$X_s - Y_s = x - y + B_{\langle X - Y \rangle_s}, \quad s \in [t, T].$$

The quadratic variation  $\langle X - Y \rangle$  is bounded on  $[0, T]$ , e.g. by  $T((\sigma_2 - \sigma_1)^2 + 3\sigma_2^2) =: \tilde{T}$  (use representation of  $X - Y$  in (23)). We can therefore estimate

$$\begin{aligned} |X_{T-\delta} - Y_{T-\delta}| &\leq \sup_{s \in [t, T]} |X_s - Y_s| \\ &\leq |x - y| + \sup_{s \in [t, T]} |B_{\langle X - Y \rangle_s}| \\ &\leq |x - y| + \sup_{s \in [0, \tilde{T}]} |B_s| \\ &\leq |x - y| + \sup_{s \in [0, \tilde{T}]} B_s + \sup_{s \in [0, \tilde{T}]} (-B_s). \end{aligned}$$

This implies with (25)

$$|w(T - \delta, X_{T-\delta} - Y_{T-\delta})| \leq C_3 e^{C_2|x-y|} \exp\left(C_2 \sup_{s \in [0, \tilde{T}]} B_s\right) \exp\left(C_2 \sup_{s \in [0, \tilde{T}]} (-B_s)\right) =: Z,$$

and

$$\begin{aligned} E^{\hat{P}} [|Z|] &\leq C_3 e^{C_2|x-y|} \left( E^{\hat{P}} \left[ \exp\left(2C_2 \sup_{s \in [0, \tilde{T}]} B_s\right) \right] E^{\hat{P}} \left[ \exp\left(2C_2 \sup_{s \in [0, \tilde{T}]} (-B_s)\right) \right] \right)^{\frac{1}{2}} \\ &= C_3 e^{C_2|x-y|} E^{\hat{P}} [\exp(2C_2 B_{\tilde{T}})], \end{aligned}$$

using the Cauchy-Schwarz inequality. This means the random variable  $Z$  is dominating  $w(T - \delta, X_{T-\delta} - Y_{T-\delta})$ . Let  $D \subseteq \mathbb{R}$  be the countable set of discontinuities of  $g$ . Note

that  $P(X_T - Y_T \in D) = 0$  by Lemma 5.3. Applying dominated convergence implies

$$\begin{aligned}
& \lim_{\delta \downarrow 0} E^P [w(T - \delta, X_{T-\delta} - Y_{T-\delta})] \\
&= \lim_{\delta \downarrow 0} E^P [w(T - \delta, X_{T-\delta} - Y_{T-\delta})] \\
&= \lim_{\delta \downarrow 0} E^P [\mathbf{1}_{\mathbb{R} \setminus D}(X_T - Y_T) w(T - \delta, X_{T-\delta} - Y_{T-\delta})] \\
&= E^P [g(X_T - Y_T)].
\end{aligned} \tag{26}$$

Together with (24) we arrive at

$$J^P(t, x, y, q_1, q_2^*) = E^P [g(X_T - Y_T)] \leq w(t, x - y),$$

as  $\delta \downarrow 0$  because  $w(T, \cdot) = g$ . Since  $P$  and  $q_1$  are chosen arbitrarily we get that

$$\sup_{q_1 \in \mathcal{V}} J^+(t, x, y, q_1, q_2^*) = \sup_{q_1 \in \mathcal{V}} \sup_{P \in \mathcal{Q}(t, x, y, q_1, q_2^*)} E^P [g(X_T - Y_T)] \leq w(t, x - y).$$

We obtain equality in (24) if we choose  $q_1 = q_1^*$  (see Lemma 5.2). This means

$$J^+(t, x, y, q_1^*, q_2^*) = \sup_{q_1 \in \mathcal{V}} J^+(t, x, y, q_1, q_2^*) = w(t, x - y).$$

Similarly one can show that

$$J^-(t, x, y, q_1^*, q_2^*) = \inf_{q_2 \in \mathcal{V}} J^-(t, x, y, q_1^*, q_2) = w(t, x - y).$$

Hence, the result follows.  $\square$

From Lemma 5.5 we derive:

**Lemma 5.6.** *The upper and lower value function satisfy*

$$V^+(t, x, y) = V^-(t, x, y) = w(t, x - y), \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},$$

*i.e. the game has a value  $V := V^+ = V^-$ , and  $V$  solves the upper and lower HJBI equation (13) and (14).*

*Proof.* Lemma 5.5 implies that the strategies  $q_1^*$  and  $q_2^*$  are mutually best responses. Hence,

$$\begin{aligned}
V^+(t, x, y) &\leq \sup_{q_1 \in \mathcal{V}} J^+(t, x, y, q_1, q_2^*) = J^+(t, x, y, q_1^*, q_2^*) = w(t, x - y) \\
&= J^-(t, x, y, q_1^*, q_2^*) = \inf_{q_2 \in \mathcal{V}} J^-(t, x, y, q_1^*, q_2) \leq V^-(t, x, y) \leq V^+(t, x, y),
\end{aligned}$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . This means that the upper and lower value coincide and the value of the game is given by  $V(t, x, y) := w(t, x - y)$ . Moreover,  $V$  solves (13) and (14), because the Isaacs condition (16) is fulfilled for the Hessian of  $V$ , and  $w$  satisfies (21) and (22).  $\square$

Finally, with all results above we can conclude the proof of Theorem 1.4 and Theorem 2.6

*Proof of Theorem 1.4 and Theorem 2.6.* First note that Theorem 1.4 follows from Theorem 2.6, so it is sufficient to prove the latter. Lemma 5.6 implies that the game has a value and Lemma 5.5 implies that the strategies  $q_1^*$  and  $q_2^*$  are mutually best responses. Thus  $(q_1^*, q_2^*)$  is a saddle point.  $\square$

## 5.2. Proof of Proposition 1.8

In this section we prove Proposition 1.8 by considering the upper and lower HJBI equation in strict controls, which are given by

$$\begin{aligned} -\partial_t w^+(t, x, y) - H_{\text{strict}}^+(D^2 w^+(t, x, y)) &= 0, \\ w^+(T, x, y) &= \mathbb{1}_{[0, \infty)}(x - y), \end{aligned} \tag{27}$$

and

$$\begin{aligned} -\partial_t w^-(t, x, y) - H_{\text{strict}}^-(D^2 w^-(t, x, y)) &= 0, \\ w^-(T, x, y) &= \mathbb{1}_{[0, \infty)}(x - y), \end{aligned} \tag{28}$$

for  $(t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}$  and with Hamiltonians  $H_{\text{strict}}^+$  and  $H_{\text{strict}}^-$  defined in (20). In the first step of the proof we construct solutions to (27) and (28) that are contained in  $\mathcal{D}$ , defined in (15), by considering an auxiliary diffusion control problem. Then in the second and final step we verify that those solutions coincide with the upper and lower value function in strict controls.

*Proof of Proposition 1.8. Step 1.* We derive candidate functions for the upper and lower value by considering an auxiliary diffusion control problem. Let

$$\begin{aligned} \sigma_+ &:= \inf_{b \in A} \sup_{a \in A} r(a, b) = r\left(\sigma_1, \frac{\sigma_1 + \sigma_2}{2\rho}\right), \text{ and} \\ \sigma_- &:= \sup_{a \in A} \inf_{b \in A} r(a, b) = r(\sigma_2, \rho\sigma_2), \end{aligned}$$

see Proposition 4.4. Note that  $0 < \sigma_- < \sigma_+$ , because  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ . Consider the control problem

$$\sup \left\{ E \left[ g(X_T^{t,x,\alpha}) \right] \mid \alpha : \mathbb{R} \rightarrow [\sigma_-, \sigma_+] \text{ measurable} \right\},$$

where  $X^{t,x,\alpha}$  is the weak solution (uniqueness in law holds true) to

$$dX_s^{t,x,\alpha} = \alpha(X_s^{t,x,\alpha}) dW_s, \quad X_t^{t,x,\alpha} = x.$$

By [21], Remark 8, we obtain that an optimal control is given by the feedback function

$$\sigma^*(x) := \begin{cases} \sigma_+, & x \leq 0, \\ \sigma_-, & x > 0. \end{cases}$$

The optimally controlled state process  $X^{t,x,\sigma^*}$  is an oscillating Brownian motion (OBM). We refer the reader to [13] for more details on the OBM. Note that the OBM possesses a probability density function in explicit form, that can be found, e.g., in [13], Theorem 1, or [1], Proposition 2.3. Using this explicit representation one can show that

$$v^+(t, x) := E \left[ g(X_T^{t,x,\sigma^*}) \right] = P(X_T^{t,x,\sigma^*} \geq 0) = \begin{cases} \frac{2\sigma_-}{\sigma_+ + \sigma_-} \Phi \left( \frac{x}{\sigma_- \sqrt{T-t}} \right) + \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-}, & \text{if } x \geq 0, \\ \frac{2\sigma_+}{\sigma_+ + \sigma_-} \Phi \left( \frac{x}{\sigma_+ \sqrt{T-t}} \right), & \text{if } x < 0, \end{cases}$$

for  $(t, x) \in [0, T) \times \mathbb{R}$ , and  $v^+(T, \cdot) := \mathbb{1}_{[0, \infty)}(\cdot)$ . Moreover, one can observe that:

- (i)  $v^+ \in \mathcal{C}^{1,2}([0, T) \times \mathbb{R} \setminus \{0\}) \cap \mathcal{C}^{1,1}([0, T) \times \mathbb{R})$ ,
- (ii) the first derivative  $\partial_x v^+(t, \cdot)$  is absolutely continuous for all  $t \in [0, T)$ ,
- (iii)  $v^+(t, \cdot)$  is convex on  $(-\infty, 0]$  and concave on  $[0, \infty)$ ,  $t \in [0, T)$ ,
- (iv)  $v^+$  satisfies

$$\begin{aligned} 0 &= -\partial_t v^+(t, x) - \frac{1}{2} \sigma^*(x)^2 \partial_{xx} v^+(t, x) \\ &= -\partial_t v^+(t, x) - \frac{1}{2} \sup_{a \in [\sigma_-, \sigma_+]} a^2 \partial_{xx} v^+(t, x) \\ &= -\partial_t v^+(t, x) - \frac{1}{2} \inf_{b \in A} \sup_{a \in A} r(a, b) \partial_{xx} v^+(t, x), \quad (t, x) \in [0, T) \times \mathbb{R} \setminus \{0\}, \end{aligned}$$

with  $v^+(T, \cdot) = g = \mathbb{1}_{[0, \infty)}$ , and thus solves the PDE (27) in a classical sense.

Define the feedback function

$$\tilde{\sigma}(x) := \begin{cases} \sigma_-, & x \leq 0, \\ \sigma_+, & x > 0, \end{cases}$$

and note denote by  $X^{t,x,\tilde{\sigma}}$  the corresponding state process. The process  $X^{t,x,\tilde{\sigma}}$  is also an OBM, and we set

$$v^-(t, x) := P(X_T^{t,x,\tilde{\sigma}} \geq 0) = \begin{cases} \frac{2\sigma_+}{\sigma_+ + \sigma_-} \Phi \left( \frac{x}{\sigma_+ \sqrt{T-t}} \right) - \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-}, & \text{if } x \geq 0, \\ \frac{2\sigma_-}{\sigma_+ + \sigma_-} \Phi \left( \frac{x}{\sigma_- \sqrt{T-t}} \right), & \text{if } x < 0. \end{cases}$$

Because  $\sigma^*$  is optimal, we obtain  $v^- \leq v^+$ . Moreover, we see that for  $t \in [0, T)$

$$v^+(t, x) - v^-(t, x) \geq \frac{2(\sigma_+ - \sigma_-)}{\sigma_+ + \sigma_-} \left( 1 - \Phi \left( \frac{x}{\sigma_+ \sqrt{T-t}} \right) \right) > 0, \quad x \geq 0,$$

and

$$v^+(t, x) - v^-(t, x) \geq \frac{2\sigma_-}{\sigma_+ + \sigma_-} \left( \Phi \left( \frac{x}{\sigma_+ \sqrt{T-t}} \right) - \Phi \left( \frac{x}{\sigma_- \sqrt{T-t}} \right) \right) > 0, \quad x < 0.$$

Therefore,  $v^-(t, x) < v^+(t, x)$  for  $(t, x) \in [0, T] \times \mathbb{R}$ . Note that  $v^-$  satisfies the properties (i)-(iii) above, and it solves the PDE (28), because

$$\begin{aligned} 0 &= -\partial_t v^-(t, x) - \frac{1}{2} \tilde{\sigma}(x)^2 \partial_{xx} v^-(t, x) \\ &= -\partial_t v^-(t, x) - \frac{1}{2} \inf_{a \in [\sigma_-, \sigma_+]} a^2 \partial_{xx} v^-(t, x) \\ &= -\partial_t v^-(t, x) - \frac{1}{2} \sup_{a \in A} \inf_{b \in A} r(a, b) \partial_{xx} v^-(t, x), \quad (t, x) \in [0, T] \times \mathbb{R} \setminus \{0\}, \end{aligned}$$

and  $v^-(T, \cdot) = g = \mathbb{1}_{[0, \infty)}$ .

*Step 2.* Now we show that  $V_{\text{strict}}^+(t, x, y) = v^+(t, x - y)$  and  $V_{\text{strict}}^-(t, x, y) = v^-(t, x - y)$  for  $t \in [0, T]$ ,  $x, y \in \mathbb{R}$ , because then the game does not have a value. This result follows by verification techniques. We only prove  $V_{\text{strict}}^+ = v^+$ . The proof of  $V_{\text{strict}}^- = v^-$  follows along the same lines. Let

$$\hat{\alpha}(x, y) = \begin{cases} \sigma_2, & \text{if } x \leq y, \\ \rho \sigma_2, & \text{if } x > y, \end{cases} \quad \text{and} \quad \hat{\beta}(x, y) = \begin{cases} \frac{\sigma_1 + \sigma_2}{2\rho}, & \text{if } x \leq y, \\ \sigma_2, & \text{if } x > y, \end{cases} \quad (x, y) \in \mathbb{R}^2.$$

Note that

$$\begin{aligned} r(\hat{\alpha}(x, y), \hat{\beta}(x, y)) &= \begin{cases} \inf_{b \in A} \sup_{a \in A} r(a, b), & \text{if } x \leq y, \\ \sup_{b \in A} \inf_{a \in A} r(a, b), & \text{if } x > y, \end{cases} \\ &= \begin{cases} \sup_{a \in A} r(a, \hat{\beta}(x, y)), & \text{if } x \leq y, \\ \inf_{a \in A} r(a, \hat{\beta}(x, y)), & \text{if } x > y, \end{cases} \end{aligned}$$

see Proposition 4.4. As in the proof of Lemma 5.5 one can show that

$$\sup_{\alpha \in \mathcal{A}} J^+(t, x, y, \alpha, \hat{\beta}) = J^+(t, x, y, \hat{\alpha}, \hat{\beta}) = v^+(t, x - y), \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},$$

and hence  $V_{\text{strict}}^+ \leq v^+$ . Note that Remark 3.4 implies that the results of Proposition 3.3 can be still applied, although  $v^+$  is not twice continuously differentiable in  $x = 0$ .

To obtain equality, let  $\beta \in \mathcal{A}$  and choose

$$\alpha^\beta(t, x, y) \in \begin{cases} \arg \max_{a \in A} r(a, \beta(t, x, y)), & \text{if } x \leq y, \\ \arg \min_{a \in A} r(a, \beta(t, x, y)), & \text{if } x > y, \end{cases} \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

Then for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$

$$\begin{aligned} \partial_{zz} v^+(t, x - y) r(\alpha^\beta(t, x, y), \beta(t, x, y)) &= \sup_{a \in A} (\partial_{zz} v^+(t, x - y) r(a, \beta(t, x, y))) \\ &\geq \inf_{b \in A} \sup_{a \in A} (\partial_{zz} v^+(t, x - y) r(a, b)) \\ &= \partial_{zz} v^+(t, x - y) r(\hat{\alpha}(x, y), \hat{\beta}(x, y)). \end{aligned}$$

Again, as in the proof of Lemma 5.5 but with a lower estimate, one obtains that

$$J^+(t, x, y, \alpha^\beta, \beta) \geq v^+(t, x - y),$$

and hence also  $V_{\text{strict}}^+ \geq v^+$ . Finally, it follows that  $V_{\text{strict}}^+ = v^+$ .  $\square$

**Remark 5.7.** Note that the proof of Proposition 1.8 also applies for terminal rewards  $g$  satisfying (i)-(iii) in Example 1.7, in addition to Assumption 1.2. The proof is completely analogous since the results of McNamara [21] apply (see Theorem 6 in [21]).

For a general reward function  $g$ , that, e.g., does not satisfy the symmetry assumptions (iii) in Example 1.7, it is not clear how to prove the statement of Proposition 1.8. In particular, the first step of the proof can not be generalized, because to the best of our knowledge there are no results on optimal controls for the diffusion control problem with terminal reward  $g$ . Without this symmetry condition the value function is not convex on the negative reals, and concave on the positive reals, independently of the time argument. One can conjecture that there is a threshold, where the value function changes from convex to concave, that is time-dependent. In this case, the optimal control is still of bang-bang type with a time-dependent threshold determined by the sign of the second derivative of the value function (see also [21], Remark 9).

## A. Appendix: Proofs of Section 3

The goal of this section is to prove results of Section 3, in particular Proposition 3.1 and Proposition 3.2. For this recall that a matrix  $A \in \mathbb{R}^{d \times d}$  is called *uniformly elliptic* if there exists  $\delta > 0$  such that

$$x^T A x \geq \delta |x|^2, \quad x \in \mathbb{R}^d.$$

Due to Krylov is the following result about weak existence of solutions to a certain class of SDEs.

**Lemma A.1** (cf. [15], Theorem 2.6.1). *Let  $x \in \mathbb{R}^d$  and  $\Sigma : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  be measurable, bounded and uniformly elliptic. Then the SDE*

$$dX_t = \Sigma(t, X_t) dW_t, \quad X_0 = x,$$

*has a weak solution.*

*Proof of Proposition 3.1.* We first prove (i) and show existence of a solution to the martingale problem (8) by considering the auxiliary four-dimensional SDE

$$dZ_s = \Sigma(q_1(s, Z_s^1, Z_s^2), q_2(s, Z_s^1, Z_s^2)) dW_s, \quad Z_t = (x, y, 0, 0), \quad (29)$$

where  $\Sigma$  is the  $4 \times 4$ -block matrix

$$\Sigma(\mu_1, \mu_2) := \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ -\Sigma_2 & \text{Id} \end{pmatrix} (\mu_1, \mu_2), \quad (30)$$



with

$$\begin{aligned}\Sigma_1(\mu_1, \mu_2) &:= \frac{1}{c(\mu_1, \mu_2)} \begin{pmatrix} \overline{\mu_1^2} + \left(\overline{\mu_1^2} \overline{\mu_2^2} - \rho^2 (\overline{\mu_1} \overline{\mu_2})^2\right)^{\frac{1}{2}} & \rho \overline{\mu_1} \overline{\mu_2}, \\ \rho \overline{\mu_1} \overline{\mu_2} & \overline{\mu_2^2} + \left(\overline{\mu_1^2} \overline{\mu_2^2} - \rho^2 (\overline{\mu_1} \overline{\mu_2})^2\right)^{\frac{1}{2}} \end{pmatrix}, \\ c(\mu_1, \mu_2) &:= \left( \overline{\mu_1^2} + \overline{\mu_2^2} + 2 \left(\overline{\mu_1^2} \overline{\mu_2^2} - \rho^2 (\overline{\mu_1} \overline{\mu_2})^2\right)^{\frac{1}{2}} \right)^{\frac{1}{2}}, \\ \Sigma_2(\mu_1, \mu_2) &:= \begin{pmatrix} \sqrt{\text{Var}(\mu_1)} & 0 \\ 0 & \sqrt{\text{Var}(\mu_2)} \end{pmatrix} \\ \text{Id} &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

for  $\mu_1, \mu_2 \in \mathcal{P}(A)$ . Here we use the notation:  $\overline{\mu} := \int_A a \mu(da)$ ,  $\overline{\mu^2} := \int_A a^2 \mu(da)$ , and  $\text{Var}(\mu) := \overline{\mu^2} - (\overline{\mu})^2$  for  $\mu \in \mathcal{P}(A)$ . To show existence of a weak solution to (29) we verify that  $\Sigma$  is bounded and uniformly elliptic. The matrix  $\Sigma_1$  is symmetric and the real eigenvalues  $\lambda_1$  and  $\lambda_2$  are given by

$$\begin{aligned}\lambda_1(\mu_1, \mu_2) &= \frac{1}{2} \left( \left(\overline{\mu_1^2} + \overline{\mu_2^2} + 2s(\mu_1, \mu_2)\right)^{\frac{1}{2}} - \left(\overline{\mu_1^2} + \overline{\mu_2^2} - 2s(\mu_1, \mu_2)\right)^{\frac{1}{2}} \right), \\ \lambda_2(\mu_1, \mu_2) &= \frac{1}{2} \left( \left(\overline{\mu_1^2} + \overline{\mu_2^2} + 2s(\mu_1, \mu_2)\right)^{\frac{1}{2}} + \left(\overline{\mu_1^2} + \overline{\mu_2^2} - 2s(\mu_1, \mu_2)\right)^{\frac{1}{2}} \right),\end{aligned}$$

for  $\mu_1, \mu_2 \in \mathcal{P}(A)$  and  $s(\mu_1, \mu_2) := \sqrt{\overline{\mu_1^2} \overline{\mu_2^2} - \rho^2 (\overline{\mu_1} \overline{\mu_2})^2}$ . We have  $0 \leq \lambda_1 \leq \lambda_2$ . The eigenvalue  $\lambda_1$  is uniformly bounded away from zero, because using the mean value theorem we observe

$$\begin{aligned}2\lambda_1(\mu_1, \mu_2) &\geq \sqrt{\overline{\mu_1^2} + \overline{\mu_2^2} + 2\sqrt{1 - \rho^2} \overline{\mu_1} \overline{\mu_2}} - \sqrt{\overline{\mu_1^2} + \overline{\mu_2^2}} \\ &\geq \sqrt{\overline{\mu_1^2} + \overline{\mu_2^2} + 2\sqrt{1 - \rho^2} \sigma_1^2} - \sqrt{\overline{\mu_1^2} + \overline{\mu_2^2}} \\ &\geq \min_{\xi \in (0, 2\sqrt{1 - \rho^2} \sigma_1^2)} \frac{\sqrt{1 - \rho^2} \sigma_1^2}{\sqrt{\overline{\mu_1^2} + \overline{\mu_2^2} + \xi}} \\ &\geq \frac{\sqrt{1 - \rho^2} \sigma_1^2}{\sqrt{2\sigma_2^2 + 2\sqrt{1 - \rho^2} \sigma_1^2}} =: l > 0,\end{aligned}$$

for all  $\mu_1, \mu_2 \in \mathcal{P}(A)$ , i.e. the matrix  $\Sigma_1$  is positive definite and uniformly elliptic. Note that also the highest eigenvalue is bounded: for any  $\mu_1, \mu_2 \in \mathcal{P}(A)$

$$\lambda_2(\mu_1, \mu_2) \leq \frac{1}{2} \left( \sqrt{2\sigma_2^2 + 2\sqrt{\sigma_2^4 - \rho^2 \sigma_1^4}} + \sqrt{2\sigma_2^2 + 2\sqrt{\sigma_2^4 - \rho^2 \sigma_1^4}} \right) =: u.$$

Hence, for any  $\mu_1, \mu_2 \in \mathcal{P}(A)$

$$l|z|^2 \leq z^T \Sigma_1(\mu_1, \mu_2)z \leq u|z|^2, \quad z \in \mathbb{R}^2.$$

This implies that also  $\Sigma$  is positive definite: for any  $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$  and  $\mu_1, \mu_2 \in \mathcal{P}(A)$  we have

$$z^T \Sigma(\mu_1, \mu_2)z = (z_1, z_2) \Sigma_1(\mu_1, \mu_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + z_3^2 + z_4^2,$$

and thus

$$|z|^2 \min\{l, 1\} \leq z^T \Sigma(\mu_1, \mu_2)z \leq |z|^2 \max\{u, 1\}.$$

Consequently,  $\Sigma$  is positive definite, bounded and uniformly elliptic. Lemma A.1 (Theorem 2.6.1 in [15]) implies that there exists a weak solution  $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_t, P', Z, W)$ . The measure  $P^{q_1, q_2} := P \circ (Z^1, Z^2)^{-1}$  on the canonical space  $(\Omega, \mathcal{F})$  solves the martingale problem (8) due to Itô's formula, and hence belongs to  $\mathcal{Q}(t, x, y, q_1, q_2)$ . Alternatively, we can employ Proposition 5.4.11 in [12] to obtain the solvability of the martingale problem (8). This proves part (i).

Part (ii) follows from (i) if we choose  $q_1 = \delta_\alpha$  and  $q_2 = \delta_\beta$ . Note that in this case it is sufficient to consider a two-dimensional auxiliary SDE with diffusion matrix  $\Sigma_1$  instead of (29). If  $\alpha$  and  $\beta$  do not depend on time, the solution to the auxiliary SDE is unique in law, because of Theorem 3 in [14]. This uniqueness transfers to the martingale problem: let  $P, P' \in \mathcal{Q}(t, x, y, \alpha, \beta)$ . Then Proposition 3.6 implies that there exist two Brownian motions, each defined on an extended probability space, such that the canonical process  $(X, Y)$  solves the SDE

$$dZ_s = \Sigma_1(\alpha(Z_s^1, Z_s^2), \beta(Z_s^1, Z_s^2))dW_s, \quad Z_t = (x, y),$$

on these extended probability spaces and with these Brownian motions. But solutions to this SDE are unique in law, and hence  $P \circ (X, Y)^{-1} = P' \circ (X, Y)^{-1}$ . Therefore,  $P = P'$  since  $(X, Y)$  is the canonical process on  $\Omega$ .  $\square$

*Proof of Proposition 3.2. Part (i): Case 1.* For  $\sigma_1 > 0$ ,  $|\rho| < 1$  the result follows from Proposition 3.1 (ii).

*Case 2.* Let  $\sigma_1 = 0$ ,  $\rho \in (0, 1)$ . Note that

$$\alpha^*(t, x, y) = \begin{cases} \sigma_2, & \text{if } (t, x - y) \in \overline{D_+}, \\ \rho\sigma_2, & \text{if } (t, x - y) \in D_-, \end{cases} \quad \beta^*(t, x, y) = \begin{cases} \rho\sigma_2, & \text{if } (t, x - y) \in \overline{D_+}, \\ \sigma_2, & \text{if } (t, x - y) \in D_-, \end{cases}$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ , and  $\rho\sigma_2 > 0$ . Hence, one can argue as in the proof of Proposition 3.1: for  $q_1 = \delta_{\alpha^*}$  and  $q_2 = \delta_{\beta^*}$  the matrix  $\Sigma(q_1(t, x, y), q_2(t, x, y))$  is bounded and uniformly elliptic for all  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . Therefore, there exists a weak solution to (29) and Itô's formula implies that the law of the solution satisfies the martingale problem (4).

For the remaining cases we need to construct processes  $X', Y'$  on some probability space with quadratic variation and covariation given by

$$\begin{aligned}\langle X', X' \rangle_s &= \int_t^s \alpha^*(r, X'_r, Y'_r)^2 dr, \\ \langle Y', Y' \rangle_s &= \int_t^s \beta^*(r, X'_r, Y'_r)^2 dr, \\ \langle X', Y' \rangle_s &= \int_t^s \rho \alpha^*(r, X'_r, Y'_r) \beta^*(r, X'_r, Y'_r) dr, \quad s \in [t, T].\end{aligned}$$

Then Itô's formula (and the representation as an Itô process) implies that the law of  $(X', Y')$  solves the martingale problem (4).

*Case 3.* Let  $\sigma_1 = 0$ ,  $\rho \in [-1, 0]$ ,  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and let  $B$  be a Brownian motion on some filtered probability space  $(\Omega', \mathcal{F}', (\mathcal{F}'_s)_{s \in [0, T]}, P')$ . We have

$$\alpha^*(t, x, y) = \sigma_2 \mathbf{1}_{\overline{D_+}}(t, x - y) = \sigma_2 - \beta^*(t, x, y), \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

Define the process  $Z$  by

$$Z_s = x - y + \sigma_2(B_s - B_t), \quad s \in [t, T].$$

Moreover, we define

$$\begin{aligned}X'_s &= x + \int_t^s \sigma_2 \mathbf{1}_{\overline{D_+}}(r, Z_r) dB_r, \\ Y'_s &= y - \int_t^s \sigma_2 \mathbf{1}_{D_-}(r, Z_r) dB_r, \quad s \in [t, T].\end{aligned}$$

We have  $X'_s - Y'_s = Z_s$  and

$$\begin{aligned}\langle X', X' \rangle_s &= \int_t^s \sigma_2^2 \mathbf{1}_{\overline{D_+}}(r, Z_r) dr = \int_t^s \alpha^*(r, X'_r, Y'_r)^2 dr, \\ \langle Y', Y' \rangle_s &= \int_t^s \sigma_2^2 \mathbf{1}_{D_-}(r, Z_r) dr = \int_t^s \beta^*(r, X'_r, Y'_r)^2 dr, \\ \langle X', Y' \rangle_s &= 0 = \int_t^s \rho \alpha^*(r, X'_r, Y'_r) \beta^*(r, X'_r, Y'_r) dr,\end{aligned}$$

for  $s \in [t, T]$ . The law  $P' \circ (X', Y')^{-1}$  solves the martingale problem (4) which follows again from Itô's formula.

*Case 4.* Let  $\sigma_1 > 0$ ,  $\rho = -1$ . The construction of a solution to the martingale problem is similar to the 2. case. Let  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and let  $B$  be a Brownian motion on some filtered probability space  $(\Omega', \mathcal{F}', (\mathcal{F}'_s)_{s \in [t, T]}, P')$ . Note that

$$\alpha^*(t, x, y) = \begin{cases} \sigma_2, & \text{if } (t, x - y) \in \overline{D_+}, \\ \sigma_1, & \text{if } (t, x - y) \in D_-, \end{cases} \quad \beta^*(t, x, y) = \begin{cases} \sigma_1, & \text{if } (t, x - y) \in \overline{D_+}, \\ \sigma_2, & \text{if } (t, x - y) \in D_-, \end{cases}$$

for  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . Define the process  $Z$  by

$$Z_s = x - y + (\sigma_1 + \sigma_2)(B_s - B_t), \quad s \in [t, T].$$

Moreover, we define

$$\begin{aligned} X'_s &= x + \int_t^s \left( \sigma_1 \mathbf{1}_{D_-}(r, Z_r) + \sigma_2 \mathbf{1}_{\overline{D_+}}(r, Z_r) \right) dB_r, \\ Y'_s &= y - \int_t^s \left( \sigma_2 \mathbf{1}_{D_-}(r, Z_r) + \sigma_1 \mathbf{1}_{\overline{D_+}}(r, Z_r) \right) dB_r, \quad s \in [t, T]. \end{aligned}$$

We have  $X'_s - Y'_s = Z_s$  and

$$\begin{aligned} \langle X', X' \rangle_s &= \int_t^s \left( \sigma_1^2 \mathbf{1}_{D_-}(r, Z_r) + \sigma_2^2 \mathbf{1}_{\overline{D_+}}(r, Z_r) \right) dr = \int_t^s \alpha^*(r, X'_r, Y'_r)^2 dr, \\ \langle Y', Y' \rangle_s &= \int_t^s \left( \sigma_2^2 \mathbf{1}_{D_-}(r, Z_r) + \sigma_1^2 \mathbf{1}_{\overline{D_+}}(r, Z_r) \right) dr = \int_t^s \beta^*(r, X'_r, Y'_r)^2 dr, \\ \langle X', Y' \rangle_s &= - \int_t^s \sigma_1 \sigma_2 dr = \int_t^s \rho \alpha^*(r, X'_r, Y'_r) \beta^*(r, X'_r, Y'_r) dr, \end{aligned}$$

for  $s \in [t, T]$ . The law  $P' \circ (X', Y')^{-1}$  solves the martingale problem (4) which follows again from Itô's formula.

*Case 5.* Let  $\sigma_1 \geq 0$ ,  $\rho = 1$ . We have  $\alpha^* = \beta^* \equiv \sigma_2$ . Hence, the construction of a solution to the martingale problem is straightforward.

*Part (ii):* For  $\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$  the result follows from part (i). Thus, let  $\rho > \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}$ .

*Case 1.* Let  $\sigma_1 > 0$ ,  $\rho < 1$ . The result follows from Proposition 3.1 (i).

*Case 2.* Let  $\sigma_1 = 0$ ,  $\rho \in \left(\frac{1}{\sqrt{2}}, 1\right)$  and  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . The proof follows along the same lines as the proof of Proposition 3.1. The matrix  $\Sigma(q_1^*(s, x, y), q_2^*(s, x, y))$ , defined in (30), is bounded and uniformly elliptic: we have

$$\overline{(q_1^*)^2} + \overline{(q_2^*)^2} = \frac{3}{4} \frac{\sigma_2^2}{\rho^2}, \quad \overline{(q_1^*)^2} \overline{(q_2^*)^2} = \frac{1}{8} \frac{\sigma_2^4}{\rho^4} \quad \text{and} \quad \overline{q_1^* q_2^*} = \frac{1}{4} \frac{\sigma_2^2}{\rho^3}.$$

Hence, the lowest eigenvalue  $\lambda_1$  of  $\Sigma_1(q_1^*(s, x, y), q_2^*(s, x, y))$  is uniformly bounded away from zero, since

$$\lambda_1(q_1^*(s, x, y), q_2^*(s, x, y)) = \frac{1}{4} \frac{\sigma_2}{\rho} (\sqrt{5} - 1) > 0, \quad (s, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

Moreover, highest eigenvalue of  $\Sigma_1(q_1^*(s, x, y), q_2^*(s, x, y))$  is bounded. This implies that there exists a weak solution to (29) and Itô's formula implies that the law of this solution satisfies the martingale problem (8).

*Case 3.* Let  $\sigma_1 \geq 0$ ,  $\rho = 1$ ,  $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and  $(W, B)$  be a two-dimensional Brownian motion on some filtered probability space  $(\Omega', \mathcal{F}', (\mathcal{F}'_s)_{s \in [t, T]}, P')$ . Note that

$$q_1^*(t, x, y) = \begin{cases} \frac{1}{2}(\delta_{\sigma_1} + \delta_{\sigma_2}), & \text{if } (t, x - y) \in \overline{D}_+, \\ \delta_{\frac{\sigma_1 + \sigma_2}{2}}, & \text{if } (t, x - y) \in D_-, \end{cases}$$

$$q_2^*(t, x, y) = \begin{cases} \delta_{\frac{\sigma_1 + \sigma_2}{2}}, & \text{if } (t, x - y) \in \overline{D}_+, \\ \frac{1}{2}(\delta_{\sigma_1} + \delta_{\sigma_2}), & \text{if } (t, x - y) \in D_-. \end{cases}$$

Define the process  $Z$  by

$$Z_s = x - y + \frac{\sigma_2 - \sigma_1}{2}(B_s - B_t), \quad s \in [t, T],$$

and the processes  $X', Y'$  by

$$X'_s = x + \frac{\sigma_1 + \sigma_2}{2}(W_s - W_t) + \int_t^s \frac{\sigma_2 - \sigma_1}{2} \mathbb{1}_{\overline{D}_+}(Z_r) dB_r,$$

$$Y'_s = y + \frac{\sigma_1 + \sigma_2}{2}(W_s - W_t) - \int_t^s \frac{\sigma_2 - \sigma_1}{2} \mathbb{1}_{D_-}(Z_r) dB_r, \quad s \in [t, T].$$

We have  $Z = X' - Y'$  and

$$\begin{aligned} \langle X', X' \rangle_s &= \left( \frac{\sigma_1 + \sigma_2}{2} \right)^2 (t - s) + \left( \frac{\sigma_2 - \sigma_1}{2} \right)^2 \int_t^s \mathbb{1}_{\overline{D}_+}(Z_r) dr \\ &= \int_t^s \overline{q}_1^*(t, X'_r, Y'_r)^2 dr + \int_t^s \text{Var}(q_1^*(t, X'_r, Y'_r)) dr, \\ \langle Y', Y' \rangle_s &= \left( \frac{\sigma_1 + \sigma_2}{2} \right)^2 (t - s) + \left( \frac{\sigma_2 - \sigma_1}{2} \right)^2 \int_t^s \mathbb{1}_{D_-}(Z_r) dr \\ &= \int_t^s \overline{q}_2^*(t, X'_r, Y'_r)^2 dr + \int_t^s \text{Var}(q_2^*(t, X'_r, Y'_r)) dr, \\ \langle X', Y' \rangle_s &= \left( \frac{\sigma_1 + \sigma_2}{2} \right)^2 (t - s) = \int_t^s \overline{q}_1^*(t, X'_r, Y'_r) \overline{q}_2^*(t, X'_r, Y'_r) dr, \quad s \in [t, T]. \end{aligned}$$

Itô's formula implies that the law of  $(X', Y')$  satisfies the martingale problem (8).  $\square$

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