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Finitely presented groups and geometric simple connectivity, A re-written paper (Trilogy II) – B

– Second, revised version 2020 –

by Valentin POÉNARU*

4 A toy model

The present toy-model, developed in this section, is essential for understanding the next two sections V, VI, and we will often refer to it.

The situation we will present now is highly simplified from several respects: there are no immortal singularities (à la $\text{Sing } \widetilde{M}(\Gamma)$), there are no triple points and no $p_{\infty\infty}$'s either (hence no difference between S_u and S'_u). But then there will not be any group action present now either. We replace (2.1) by the following map which **is** our toy model

$$(4.1) \quad X^2 \xrightarrow{f} R^3,$$

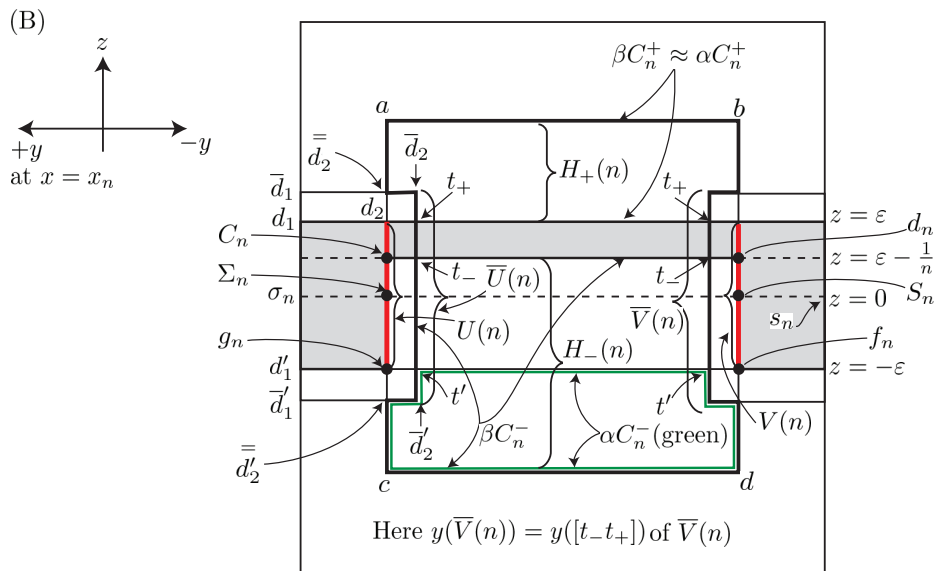
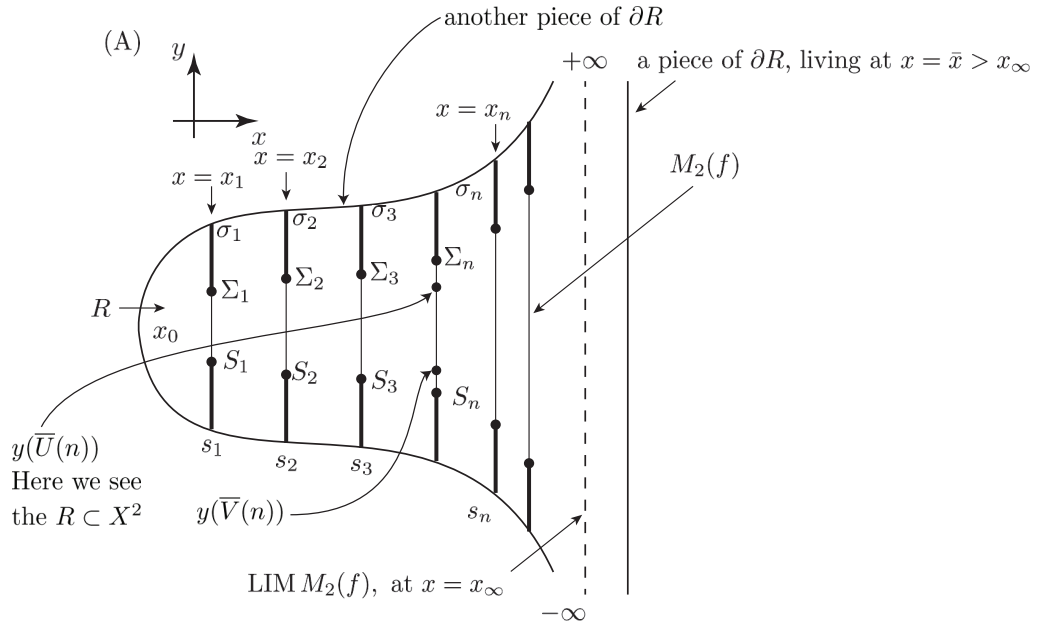
where

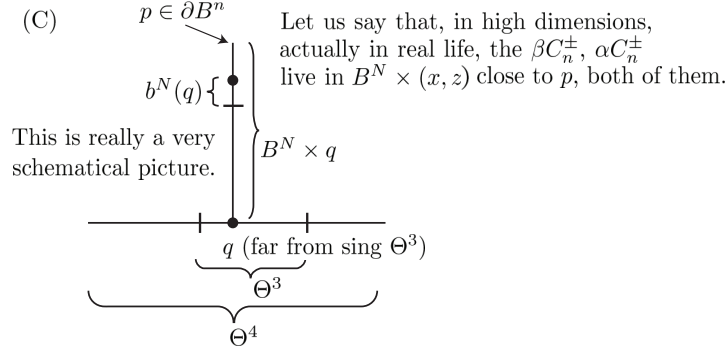
(4.2) $X^2 \equiv \{\text{the } 2^{\text{d}} \text{ region called } R, \text{ homeomorphic to } I \times \text{int } I \text{ and PROPERLY embedded inside } (z = 0) \subset R^3 = \{x, y, z\}, \text{ which is displayed in the figure 4.1.(A)}\} \cup \sum_{n=1}^{\infty} D_n^2$, with R and each of the D_n^2 's glued along $[\sigma_n, \Sigma_n] + [s_n, S_n]$. Here $D_n^2 = \{\text{the disk of diameter } [\sigma_n, s_n] = R \cap (x = x_n), \text{ living in the plane } (x = x_n) \subset R^3 \text{ and displayed in the figure 4.1.(B)}\}$.

Figure 4.1 should explain how the spare parts of X^2 , listed in (4.2) are glued together. Restricted to

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each of these spare parts, the map f is the natural embedding into R^3 .





The disc $D_n^2 \subset X^2$, drawn here as a square, for typographical purposes. Actually, the $[a, b, c, d]$ is meant to be a circle, but it is much easier to draw it in straight lines (which are thick here). When it comes to the map β , then what we see in (B) is a projection via $\Theta_1^{N+4}(X^2 - H) \rightarrow X^2 - H$, while for α the projection is in $\Theta_1^{N+4}(fX^2 - H) \rightarrow fX^2 - H$. That is how figure 4.1-(B) should be read.

Legend:

- (red) = 3^d undrawable singularities of $\Theta^3(X^2)$, containing the points Σ_n, S_n which are 2^d undrawable singularities of X^2 and/or of f .
- (green) = αC_n^- , — (black) = central loop Γ_n .

Figure 4.1.

We display here the spare parts R and D_n^2 of X^2 . The Σ_n, S_n are mortal undrawable singularities of the f (4.1), and the $U(n) \ni \Sigma_n, V(n) \ni S_n$ stand for immortal undrawable singularities of $\Theta^3(X^2)$. The $H_\pm(n)$ are Holes, open sets, drilled out of D_n^2 . The (C) is a very schematical drawing, the meaning of which will become clear a bit later. **VERY IMPORTANTLY**

$$\text{dist}(U(n), \bar{U}(n)) \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \text{dist}(V(n), \bar{V}(n)) \xrightarrow{n \rightarrow \infty} 0.$$

Notice that, with the notation in (B) we have $\Sigma_n \in U(n) = [d_2, g_n], \bar{U}(n) = [\bar{d}'_2, \bar{d}'_2]$. Very importantly too, the $U(n) \ni \Sigma_n, V(n) \ni S_n$, live **outside** of the closed contour in fat lines called “central loop”, namely

$$\Gamma_n \equiv [a, \bar{d}_2, \bar{d}'_2, \bar{d}'_2, \bar{d}_2, c, d, b, a] \supset [t_+ t_-](\bar{U}(n)) + [t_+ t_-](\bar{V}(n)).$$

We will not need any fins now, so proceeding essentially like in (3.5), and/or in figure 3.2, we will define the noncompact smooth 3-manifold, with large boundary

$$(4.3) \quad \Theta^3(fX^2) \equiv (R \times [-\varepsilon \leq z \leq \varepsilon]) \cup \sum_1^\infty D_n^2 \times [-\varepsilon_n \leq x \leq \varepsilon_n] \underbrace{-\text{LIM } M_2(f) \times \{z = \pm \varepsilon\}}_{x=x_\infty, -\infty < y < \infty, z=\pm\varepsilon}.$$

Next, by analogy to (3.17) and to (3.19), we introduce the smooth $(N + 4)$ -manifolds

$$(4.4) \quad \Theta^{N+4}(X^2) = \Theta^4(\Theta^3(X^2), \mathcal{R}) \times B^N$$

and

$$\Theta^{N+4}(fX^2) = \{\text{the already smooth } \Theta^3(fX^2)\} \times I \times B^N.$$

It will be convenient to think that for the singular part of X^2 we work with a thickening $\Theta^4(\Theta^3(X^2), \mathcal{R}) \times B^N$ which, for the non-singular part of X^2 melts smoothly into $\Theta^3(X^2)_{\text{smooth}} \times B^{N+1}$.

In anticipation for what we will do in real life, we will try now to suggest a transformation leading from $\Theta^{N+4}(X^2)$ to $\Theta^{N+4}(fX^2)$ via a succession of steps of the following forms

(4.5.1) Inclusion maps mimicking the infinitely many elementary zipping operations leading from X^2 to fX^2 . There is no question of proving any GSC property now, but we will still insist that all our steps should be, abstractly speaking, i.e. just by themselves, GSC preserving. In real life, the proof that via this we eventually get something which *is*, itself, GSC, will require a compactification argument. But no such infinitistick gimmicks will really occur in our toy-model (just a little whiff of them). [Actually the fX^2 is obviously GSC, but that is irrelevant now.]

(4.5.2) Deletion of $\text{LIM } M_2(f) \times \{z = \pm\varepsilon\} \equiv \partial\mathcal{R}_\infty$, where \mathcal{R}_∞ is the so-called *critical rectangle* from (4.28) below. This is a much simplified version of the $\partial\Sigma(\infty)_*$ from (3.21). This deletion can be realized at the level $\Theta^3(fX^2)$ (or $\Theta^3(X^2)$, or $\Theta^{N+4}(X^2)$, or $\Theta^{N+4}(fX^2)$) either by simple deletion or as an infinite Whitehead dilatation. Anyway, this sending to infinity of $\text{LIM } M_2(f) \times \{\pm\varepsilon\}$ is a GSC preserving step.

Realizing the more serious step (4.5.1) will need the introduction of some basic tools to be used later on in real life too, namely the following mantra, or trinity, basic for this whole paper

$$(4.6) \quad \boxed{\text{The Holes, the Ditches and the (Partial) Ditch-Filling.}}$$

The reason which makes that, even in the context of our toy-model, in trying to perform successfully the change

$$\boxed{\{\text{infinite sequence of space projections}\} \implies \{\text{infinite sequence of GSC-preserving inclusion maps (which we hope converges)}\}}$$

requires heavy technology (like the mantra above), is that even here we have our AVATAR (1.11.1-B), namely

$$M_2(f) \cup \text{Sing}(f) \subset X^2 \text{ is NOT A CLOSED subset.}$$

But above I had used the word “try”, the point being that for our project to succeed, we may be forced to throw in other, still GSC preserving ingredients, into the definition of $\Theta^{N+4}(fX^2)$ too. In anticipation for that we denote, from now on, the $\Theta^{N+4}(fX^2)$ from (4.4) by $\Theta^{N+4}(fX^2)_I$ and we will refer to its context as VARIANT I. The ingredients (4.7) to (4.9) to which we turn next, are there in both variants I and in the real life stuff, in both they will have to be thrown in. Variant I is just a step towards the real life version, like (3.25).

In figure 4.1.(B) the shaded areas which are outside of the central loop

$\{\Gamma_n$ defined in the text coming with figure 4.1, and occurring in fat lines in figure 4.1-(B), and containing inside it the $H_\pm(n)$, and called the *central loop*},

stand for identifications to be performed at level $\Theta^3(X^2)$ leaving us with the singularities $U(n)$, $V(n)$ at 3^d level.

[EXPLANATIONS. The shaded area $[d_1, t_+, t', d'_1]$ is $(N^3(4.14.1)) \mid (N^2 \mid R)$, with N^2 like in (4.14), while $[\bar{d}_1, \bar{d}_2, \bar{d}'_2, \bar{d}'_1]$ is $N^2 \mid D_n^2$.

We refer here to the proof of Lemma 4.1 below. The superposition of the shaded $[d_1, t_+, t', d'_1]$ and of $[\bar{d}_1, \bar{d}_2, \bar{d}'_2, \bar{d}'_1]$ in figure 4.1-(B) is optical illusion. We have two 2^d branches of $X^2 - H$ glued exactly along $[\sigma_n, \Sigma_n]$. See, also, figure 4.2-ter.]

Next, inside each individual $D_n^2 \approx D_n^2 \times [-\varepsilon_n, \varepsilon_n]$ we will drill two Holes $H_1(n)$ like in figure 4.1.(B), deleting open sets (unlike the deletion of the closed $p_{\infty\infty} \times (-\varepsilon, \varepsilon)$ in (3.14)). Notice that our $H_{\pm}(n)$'s do not touch the undrawable singularities $U(n), V(n)$.

This leaves us with the boundary curves

$$(4.7) \quad C_n^{\pm} \equiv \partial \{\text{Hole } H_{\pm}(n)\},$$

coming with, and see here figure 4.1-(C) and 4.2 too, the

$$(4.8) \quad \left\{ \text{canonically framed link } \sum_n C_n^{\pm} \right\} \xrightarrow{\beta} \partial \Theta_1^{N+4}(X^2 - H);$$

here $\Theta_1^{N+4}(X^2 - H) \equiv \{\Theta^{N+4}(X^2) \text{ with the holes } H_{\pm}(n) \text{ deleted}\}$ and, when projected to $\Theta^3(X^2)$, then β is like in figure 4.1-(B). Inside the disc D_n^2 from figure 4.1-(B), we see our central loop Γ_n , not touching the singularities $U(n), V(n)$. It bounds a big "hole" $\bar{H} \supset H_{\pm}(n)$. The shaded area in figure 4.1-(B), contained between the two arcs $[t_-, t]$, and contained in \bar{H} has the following features. It is contained in $\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$, corresponds to $(D_n^2 - H) \cap f^{-1}(R \times [\varepsilon - \frac{1}{n}, \varepsilon])$, and it stays far from the identifications performed at level $\Theta_1^{N+4}(X^2 - H)$. We use here the notation $X^2 - H \equiv X^2 - \sum_n H_{\pm}(n)$.

We also introduce the smooth manifold $\Theta_1^{N+4}(fX^2 - H) \equiv \{\Theta_1^{N+4}(fX^2) \text{ from which we delete the holes } H_+(n) \text{ and } H_-(n) \mid (z < -\varepsilon)\}$ (see here the figure 4.1.(B)).

This comes with its own

$$(4.9) \quad \left\{ \text{canonically framed link } \sum_n C_n^{\pm} \right\} \xrightarrow{\alpha} \partial \Theta_1^{N+4}(fX^2 - H).$$

We find here that $\alpha C_n^+ = f \beta C_n^+$, **BUT** the αC_n^- and $f \beta C_n^-$ are far from each other; the reason for this is that while $\beta C_n^- \subset \Theta_1^{N+4}(X^2 - H)$ goes all the way up to $z = \varepsilon - \frac{1}{n}$, the $\alpha C_n^- \subset \Theta_1^{N+4}(fX^2 - H)$ only reaches up to $z = -\varepsilon$. Both αC_n^- and βC_n^- can be reconstructed from figure 4.2.

[Keep in mind that

$$fX^2 - H \neq f(X^2 - H) \subsetneq fX^2 - H.$$

This important fact hangs together with the PARTIAL DITCH FILLING, see figure 4.2.]

In figure 4.2 we also see the DITCHES, when we restrict ourselves to some fixed value of $y \in [y(\bar{V}(n)), y(\bar{U}(n))]$. The piece $T(n) \subset \Theta_1^{N+4}(X^2 - H)$ -DITCHES, corresponding, in 3^d , to the shaded region in figure 4.1-(B), between the $z = \varepsilon$, $z = \varepsilon - \frac{1}{n}$ and the two arcs $[t_-, t]$ and as defined by figure 4.2 (and then with more detail in the context of (4.11.2) and (4.9.1), (4.12)), lives in the DITCH, without touching the lateral surface δ ditch (n). Here

$$\text{ditch}(n) \subset \Theta_1^{N+4}(X^2 - H) \mid R = \Theta_1^{N+4}(X^2) \mid R.$$

When the gap between $T(n)$ and the lateral surface of the ditch will be filled with material, during the process DIL (4.22) (which is our present version of the geometrical realization of the zipping), then the curves αC_n^+ and $\eta \beta C_n^+$ (as defined by (4.26) below) will be connected by a short simple-minded isotopy, for each n . The filling material mentioned above only occurs inside $[\varepsilon - \frac{1}{n} \leq z \leq \varepsilon] \subset [-\varepsilon \leq z \leq \varepsilon]$, and the same will be true for $T(n)$. We speak hence of a **PARTIAL DITCH FILLING**.

Here are some EXPLANATIONS CONCERNING THE FIGURE 4.2. The ditch (n) $\mid y$ is a region which is concentrated around the following arc

$$A(n, y) \equiv (x = x_n, z - \varepsilon \leq z \leq \varepsilon, u, \{\text{some point } p \in \partial B^N, \text{ here } t = 1\}) \subset R \times [-\varepsilon, \varepsilon] \times I(\ni u) \times B^N,$$

$$\text{for } y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))$$

and which is contained in $(\Theta^{N+4}(fX^2 - H)) | (R | y)$. Here $\Theta^{N+4}(fX^2 - H) | (R, y) = \Theta^{N+4}(X^2 - H) | (R, y)$, and the moving parameter along the arc $A(n, y)$ is $-\varepsilon \leq z \leq +\varepsilon$. Let us say that

$$\text{ditch}(n) | y = A(n, y) \times b^N(n),$$

with $b^N(n) \subset B^N$ concentrated around $p \in \partial B^N$. With this, for a fixed, generic value y , and with fixed generic $u \in I$ too, the (A) presents an infinite family of ditches, occurring as *indentations* inside

$$\{\text{the } R \text{ in figure 4.1}\} \times [-\varepsilon \leq z \leq \varepsilon] \times B^N.$$

They make use of the *additional* dimensions, i.e. of the additional factor B^N , with respect to some smooth $\{\Theta^4$, which has just tamed the undrawable singularities of some $\Theta^3\}$. We insist on this distinction between “additional dimensions” as opposed to the mere “high dimensions” of Smale and Stallings.

In (B) we present

$$T(n) | y \equiv \text{Tube}(n) | y \subsetneq \text{ditch}(n) | y,$$

and this inclusion will be part of the forthcoming map j in (4.13). One should imagine here a complete y -movie coming with the drawing (B). This movie creates then the *1-handle* of dimension $N + 4$, namely

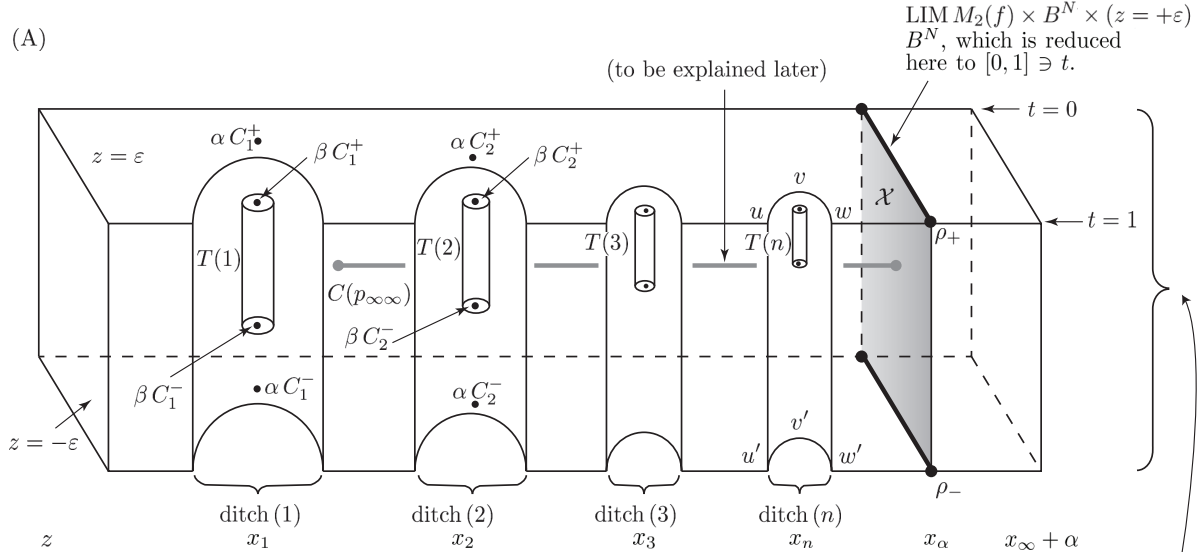
$$(4.9.1) \quad T(n) = \underbrace{\bigcup_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))} T(n) | y}_{\text{The shaded area of } D^2 - H},$$

living inside the central loop Γ_n in the figure 4.1.B, viewed now at the level of $\Theta^{N+4}(X^2 - H)$.

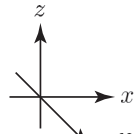
To be more precise, we actually have

$$(4.10) \quad T(n) | y = \Theta_1^{N+4}(X^2 - H) | \left\{ ((D_n^2 - H) | y) \cap \left[\varepsilon - \frac{1}{n} \leq z \leq \varepsilon \right] \right\},$$

for $\{y([t_- t_+] \text{ of } \bar{V}(n)) \leq y \leq y([t_- t_+] \text{ of } \bar{U}(n))\}$, coming with $\lim_{n \rightarrow \infty} T(n) = \infty$. End of (4.10)

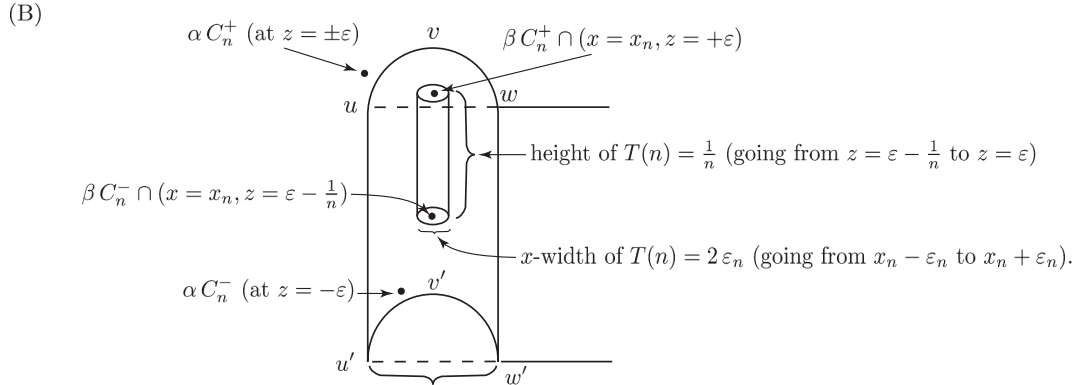


The shaded \mathcal{X} is here $\mathcal{X} = (x = x_\infty) \times [-\varepsilon \leq z \leq \varepsilon] \times B^N$, with the fat lines $(x = x_\infty, z = \pm\varepsilon, B^N)$ living at infinity, i.e. they are punctures, and there are no others.



B^N , which is reduced here to $[0, 1] \ni t$. Here $t = 1$ is the special $p \in \partial B^N$, around which the $b^N(n)$ is concentrated. There are also the dimensions y, u , to be added into $\Theta^4(x, y, z, u)$.

We see here, at a fixed value $y \in (y(\overline{V}(n)), \overline{U}(n))$ the $[x_0 \leq x \leq x_\infty + \alpha] \times [-\varepsilon \leq z \leq \varepsilon] \times B^N$, and we ignore the factor $I \ni u$ of $\Theta^4(\dots, \mathcal{R})$.



The ditch (n) | y = the region $[u, v, w, w', v', u']$
 The x-width of the ditch (n) is $4\varepsilon_n$, going from $x - 2\varepsilon_n$ to $x_n + 2\varepsilon_n$

Figure 4.2.

We concentrate here on the generic ditch (n). The ditch (n) concerns the D_n^2 from figure 4.1. In this figure we are at some fixed value of y , actually at some $y \in (y(\overline{V}(n)), y(\overline{U}(n)))$. Everything we see both in (A) and in (B), lives at some generic value $y \in [y(\overline{V}(n)), y(\overline{U}(n))]$, with $\overline{V}(n), \overline{U}(n)$ like in figure 4.1.(B). In the present (B) we see the ditch (n) | y and, with this ditch (n) $\equiv \bigcup_y$ ditch (n) | y ,

with $\{y([t_-, t_+] \text{ of } \overline{V}(n)) < y < y([t_-, t_+] \text{ of } \overline{U}(n))\}$. Here $[t_-, t_+]$ is like in figure 4.1-(B) so that $y([t_-, t_+] \text{ of } \overline{V}) = y(\overline{V}(n))$ and the same for $\overline{U}(n)$. The coordinates here are like in figure 4.1-(B).

Our figure refers to $j(\Theta_1^{N+4}(X^2 - H) - \{\text{DITCHES}\}) \subset \Theta_1^{N+4}(fX^2 - H)$ (see (4.13) below). In (A) we suggest $\{R \text{ (figure 4.1-(A))}\} \times [-\varepsilon \leq z \leq \varepsilon] \times B^N$. (The $I \ni u$ is not represented.)

From the (A) one should read an easy diffeomorphism isotopic to the identity $\Theta^{N+4}(fX^2 - H)_I | R - \{\text{DITCHES}\} \cong \Theta_1^{N+4}(fX^2 - H) | R$, and here, clearly, we have

$$(4.11) \quad \Theta_1^{N+4}(fX^2 - H) | R = \Theta_1^{N+4}(fX^2) | R = \Theta_1^{N+4}(X^2) | R = \Theta_1^{N+4}(X^2 - H) | R,$$

and in these formulae we may as well replace R by $R | y$. This ends our explanations for figure 4.2.

But more will be said in the legend which comes with the figure in question.

In the context of the figure 4.2 we have, as the legend explains,

$$(4.11.0) \quad \text{ditch}(n) \equiv \underbrace{\bigcup_{y(\overline{V}(n)) < y < y(\overline{U}(n))} \text{ditch}(n) | y}_{y(\overline{V}(n)) < y < y(\overline{U}(n))}$$

In particular, retain that there are **no ditches** at $y = y(\overline{U}(n))$, and $y = y(\overline{V}(n))$.

We have here, $b^N(n) \subset B^N$, which is concentrated around the $p \in \partial B^n$, like in the figure 4.2-bis, below. With this, and with fixed values $y \in (-\infty, \infty)$, $u \in I$, we have

$$(4.11.1) \quad \text{ditch}(n) | (y, u) \equiv \{b^N(n) \times [x_n - 2\varepsilon_n \leq x \leq x_n + 2\varepsilon_n] \times [-\varepsilon \leq z \leq \varepsilon]\} \subset B_0^{N+2} \equiv [x_0 \leq x \leq x_\infty + \alpha] \times [-\varepsilon \leq z \leq \varepsilon] \times B^N, \text{ for fixed } (y, u).$$

There are two concerns in connection with figure 4.4, namely the following.

Concern I. Our construction of the toy-model high-dimensional geometric realization of the zipping process, which is the core of the present section and which ends up with the space $S_b(fX^2 - H)$ in the Lemma 4.3 below, should produce a **metrizable** space $S_b(fX^2 - H)$. After that, it will be even possible for it to be a C^∞ manifold of dimension $N + 4$. The key here is that the DITCH should only be filled up to $\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$ for ditch (n) , when $n \rightarrow \infty$.

-) So, the PARTIAL DITCH FILLING is our key tool at this point, for Concern I.

Concern II. In the context of formula (4.11.1), we want to find a DIFFEOMORPHISM, isotopic to the identity

$$(DIFF) \quad R \times [-\varepsilon \leq z \leq \varepsilon] \times (u \in I) \times B^N - \sum_n \text{ditch}(n) \underset{\text{DIFF}}{\approx} R \times [-\varepsilon \leq z \leq \varepsilon] \times (u \in I) \times B^N.$$

This will be essential for connecting $S_b(fX^2 - H)$ (see below) and $\Theta_1^{N+4}(fX^2 - H)$ in (4.26). In order to take care of this concern, a second very precise **size-control** will be necessary. What we will need here is the following item, which is illustrated in the figure 4.3:

-) A very strong control of the following convergence to zero

$$\lim_{n \rightarrow \infty} \underbrace{\|(\text{ditch}(n) | (y, u)) | z\|}_{\substack{\text{The } B^N\text{-height in } B_0^{N+2}, \text{ for } (y, u, z) \text{ fixed.} \\ \text{This will allow for the (DIFF) above.}}} = 0.$$

The ••) also implies that inside the $[0 \leq t \leq 1]$ from figure 4.2, the ditch corresponds, $[0, 1]$ -wise, to $[1 - b_n, 1]$ with $\lim_{n \rightarrow \infty} b_n = 0$, i.e. living very close to $t = 1$ (the $p \in \partial B^N$).

In all this story, in terms of the figure 4.1 the $T(n)$ goes along the y -axis. And here is how our various coordinates axes go:

$$\Theta^3(x, y, z), \quad \Theta^4(x, y, z, u), \quad B^{N+1} \equiv (I \ni u) \times B^N,$$

with B^{N+1} to be used, as such, when Θ^3 is smooth.

Next, in the figure 4.2-bis we will present a y -movie for the 1-handle $T(n)$ and for the DITCH which contains it. We have here a $p \in \partial B^N$ and then a much smaller $b^N(n) \subset B^N$, concentrated around p . (Both p and $b^N(n)$ could possibly be (y, z, u) -dependent in real life, but not in the present toy-model.

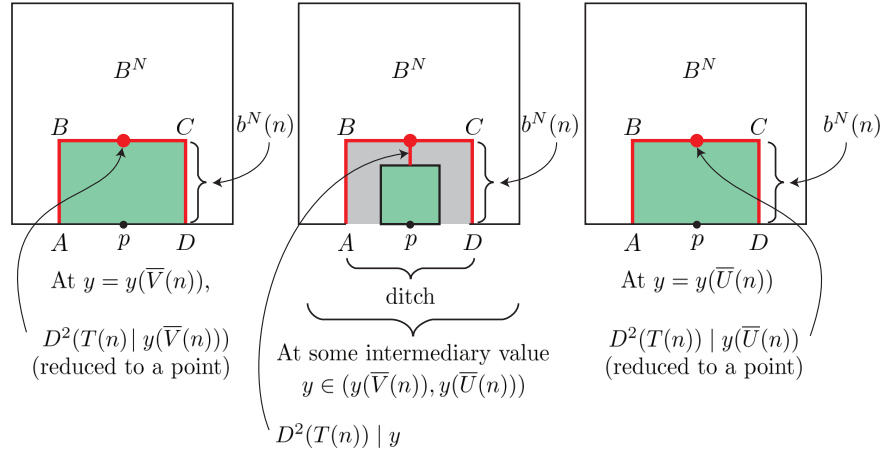


Figure 4.2-bis.

The y -MOVIE. We see here, with the values x, z, u fixed for good, the various $(T(n) \cap b^N(n) = T(n) \cap B^N) | y (= // // // //)$, with which we have:

$$T(n) \cong T(n) \cap b^N(n) = T(n) \cap B^N = \underbrace{\bigcup_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))} T(n) | y}_{\text{ditch}} \cup \underbrace{\bigcup_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))} b^N(n) | y}_{\text{point}}$$

and, at the same time, at the same B^N -level, and with variable y we have

$$\text{ditch}(n) \cap B^N = \underbrace{\bigcup_{y(\bar{V}(n)) < y < y(\bar{U}(n))} b^N(n) | y}_{\text{ditch}}$$

The notation “ $\cap B^N$ ” means that we forget the x -width and the z -height. So far, (x, z, u) were frozen. When we free them, then $T(n)$ goes through $(x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n) \times (\varepsilon - \frac{1}{n} \leq z \leq \varepsilon)$ while $b^N(n) | y$ goes through $(x_n - 2\varepsilon_n \leq x \leq x_n + 2\varepsilon_n) \times (-\varepsilon \leq z \leq \varepsilon)$, like in (4.11.1), while u goes through its factor I , the fourth dimension of Θ^4 .

So, our figure 4.2-bis, with its y -movie, tells us the story of the ditch (n) and of $T(n)$ from the viewpoint of B^N . When we project this story on $\Theta_1^{N+4}(fX^2 - H)$, it is far from the undrawable singularities inherited from $\Theta^3(X^2)$, because in figure 4.1-(B), the $U(n), V(n)$ are outside of the area enclosed by Γ_n . Rephrasing (4.11.1), the complete story is given by the formulae below

(4.11.2) $\text{ditch}(n) = b^N(n) \times [x_n - 2\varepsilon_n \leq x \leq x_n + 2\varepsilon_n] \times [y(\bar{V}(n)) < y < \bar{y}(\bar{U}(n))] \times [-\varepsilon \leq z \leq \varepsilon] \times I(\ni u)$
and, together with this, comes now

$$T(n) = \underbrace{\bigcup_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n)), \text{ see above}} (T(n) \cap b^N(n))}_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n)), \text{ see above}} \mid y \times ([x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n] \times [\varepsilon - \frac{1}{n} \leq z \leq \varepsilon] \times (I \ni n)),$$

living in the ditch. End of (4.11.2)

We will denote the red line $[ABCD]$ (without the (AD)), in figure 4.2-bis by $\delta b^N(n) \equiv \overline{\partial b^N(n) - \partial B^N}$
and, with this we introduce the following objects:

(4.11.3) $D^2([t_-, t_+]) \equiv \{[t_+(\varepsilon), t_-(\varepsilon - \frac{1}{n})] \times [x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n], \text{ at } y = y(\bar{U}(n)) \text{ AND similarly at } y = y(\bar{V}(n)) \text{ (two discs see figure 4.1-(B))}\}.$

In this formula, we have to read $\{\text{arc}[t_+(\varepsilon), t_-(\varepsilon - \frac{1}{n})]\} \equiv [z(t_-(\varepsilon - \frac{1}{n})) \leq z \leq z(t_+(\varepsilon))].$ And, of course,
 $-\varepsilon < \underbrace{z(t_-(\varepsilon - \frac{1}{n}))}_{= \varepsilon - \frac{1}{n}} < z(t_+(\varepsilon)) = \varepsilon.$ For the extreme values $y = y(\bar{V}(n))$ and $y = y(\bar{U}(n))$, we have

$$T(n) \mid y = B^{N+3}([t_-, t_+]) \equiv \underbrace{[x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n] \times [\varepsilon - \frac{1}{n} \leq z \leq \varepsilon]}_{D^2([t_-, t_+])} \times I(\ni u) \times b^N(n),$$

at $y = y(\bar{U}(n))$ or $y = y(\bar{V}(n))$. End of (4.11.3)

Going back now to figure 4.2-bis, I will describe the

(4.11.4) $\{\text{THE DITCH FILLING MATERIAL, (which will occur big in (4.22), in the context of Lemma 4.3 below)}\} = \underbrace{\bigcup_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))} \{\text{the piece shaded in blue in figure 4.2-bis (= // // // //), of } b^N(n) \times y, \text{ and which gets squeezed into } \delta b^N(n) \text{ at the extreme values } y(\bar{V}(n)), y(\bar{U}(n))\}}_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))} \times [x_n - 2\varepsilon_n, x_n + 2\varepsilon_n] \times [\varepsilon - \frac{1}{n} \leq z \leq \varepsilon].$ End of (4.11.4)

With this, we also have a compensating disc $D^2(T(n))$, for our 1-handle, defined as follows:

$$(4.11.5) \quad D^2(T(n)) = \underbrace{\bigcup_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))} \{D^2(T(n)) \mid y \text{ (all in RED in the } y\text{-MOVIE from figure 4.2-bis)}\}}_{y(\bar{V}(n)) \leq y \leq y(\bar{U}(n))}.$$

The $D^2(T(n)) \subset \{\text{DITCH FILLING MATERIAL}\}$, is in **exact** CANCELLING POSITION with the 1-handle $T(n)$. The good which this will do for us is that at the level of the S_b (to be introduced later) we will get something like GSC, even at this toy-model level. And then, keep in mind that the rational for our toy-model, is to present in a simplified form various items which will certainly be there in our REAL-LIFE constructions too, when GSC will become a life or death issue.

The GSC property of S_b , in the real-life context, will be explained in later sections, and it is of paramount importance for this paper. Here it is just a toy gadget. But, then, in real life, getting from $S_b \in \text{GSC}$ to $S_u \in \text{GSC}$ (as the GSC THEOREM demands), we will also need the COMPACTIFICATION PROCESS. No question of it, in the toy model context. Notice, in all this story, that at the extreme values $y = y(\bar{U}(n))$ and $y = y(\bar{V}(n))$, the $T(n) \mid y$ gets glued to $\Theta_1^{N+4}(X^2 - H)$ – DITCH, while at the other, intermediary values, it is floating free in the DITCH, free as long as we ignore the ditch filling material, of course.

Both $\Theta^{N+4}(X^2)$ and $\Theta^{N+4}(fX^2)$ can be reconstructed, up to diffeomorphism, starting from the corresponding objects with Holes and then adding 2-handles along (4.8), respectively along (4.9).

Also, in the same context as in the figure 4.2, we have

$$\text{DITCHES} \equiv \sum_{n=1}^{\infty} \text{ditch}(n) \subset \Theta_1^{N+4}(X^2 - H) \mid R = \Theta_1^{N+4}(X^2) \mid R.$$

In the same vein, when it comes to the curves $\alpha C_n^\pm, \beta C_n^\pm$, we only see there, in figure 4.2, the $C \mid y$'s.

So, what we learn from figure 4.2 is that, by making use of the additional dimensions coming with B^N , we define first the restrictions $\text{ditch}(n) \mid y$ for all the

$$(*) \quad y([t_- t_+] \text{ of } \bar{V}(n)) < y < y([t_- t_+] \text{ of } \bar{U}(n)),$$

then the $\text{DITCH}(n)$ by summing $(*)$ over y , and finally the complete system $\{\text{DITCHES}\} \subset \Theta_1^{N+4}(X^2 - H)$, by summing over n . See here figure 4.2-bis for an illustration, and then also the formulae (4.11.2) to (4.11.5) for a complete description.

Lemma 4.1. – *There is a splitting, to be explicitly defined in the proof below*

$$(4.12) \quad \Theta_1^{N+4}(X^2 - H) - \{\text{DITCHES}\} = \{\text{MAIN } \Theta_1^{N+4}\} + \sum_n \{(N+4)^d \text{ 1-handles } T(n)\}.$$

It will turn out that in this formula, the $\{\text{MAIN } \Theta_1^{N+4}\}$ comes from R (figure 4.1-(A)), while $T(n)$ comes from the formula (4.11.2) which is a by-product of the y -movie from figure 4.2-bis. There is also a PROPER embedding j , such that j sends the $T(n) \subset \Theta_1^{N+4}(X^2 - H) - \{\text{DITCHES}\}$ in the DITCH

$$(4.13) \quad \begin{array}{ccc} \Theta_1^{N+4}(X^2 - H) - \{\text{DITCHES}\} & \xrightarrow{j} & \Theta_1^{N+4}(fX^2 - H) \\ \downarrow p & & \downarrow q \\ X^2 - H & \xrightarrow{f} & fX^2 - H. \end{array}$$

Proof. At the 2^d level, we have $\text{Sing}(f) = \{\text{the points } \Sigma_n \in U(n), S_n \in V(n) \text{ from the figure 4.1-(B)}\} \subset X^2 - H$ and then their isomorphic, now non-singular images $f\text{Sing}(f) \subset fX^2 - H$. Continuing in the same 2^d vein, we go now to the canonical neighbourhoods

$$N^2 = N^{bd}(\text{Sing}(f)) \xrightarrow{\text{PROPER embedding}} X^2 - H.$$

Here is the precise structure of N^2 for the singularity Σ_n and something similar can be done for S_n , and follow here the figure 4.1,

$$(4.14) \quad N^2 = N^2(\Sigma_n) = \underbrace{\{[y(\sigma_n) \leq y \leq y(\bar{U}(n))] \times [x_n - 2\varepsilon_n \leq x \leq x + 2\varepsilon_n]\}}_{\text{call this } R^2(y,x), \text{ part of } R(y,x)=R} \cup \underbrace{[\sigma_n, \Sigma_n] = (x = x_n, y(\sigma_n) \leq y \leq y(\Sigma_n), z = 0)}$$

$\cup \{\text{the 2-cell } \Delta^2 = [\bar{d}_1, \bar{d}_2, \bar{d}'_1, \bar{d}'_2] \text{ (figure 4.1-(B))} \subset D_n^2(y, z) \text{ (figure 4.1-(B))}\}$. Here $\Delta^2 = [y(\sigma_n) \leq y \leq y(\bar{U}(n))] \times [-\varepsilon - \alpha \leq z \leq +\varepsilon + \alpha]$. Also, $y(\Sigma_n) < y(U(n)) < y(\bar{U}(n))$ and N^2 has an undrawable singularity at Σ_n . End of (4.14)

This is, of course, our standard model for a 2^d undrawable singularity, here the Σ_n , living at $y = y(U(n))$ and this follows the cannons from [8], [18], [19], which are also re-explained in the Section 1 above. This way the reader does not have to bother about those older references.

Next, we move to dimension $n = 3$ and go to

$$N^3 = \Theta^3(N^2) = \Theta^3(X^2) \mid (\text{the } U(n) \text{ in figure 4.1-(B)})$$

and now this N^3 is the object below, for which one should also look at the figure 4.2-TER,

$$(4.14.1) \quad N^3 = \{[y(\sigma_n) \leq y \leq y(\bar{U}(n))] \times [x_n - 2\varepsilon_n \leq x \leq x_n + 2\varepsilon_n] \times [-\varepsilon \leq z \leq \varepsilon] \cup \overbrace{[\sigma_n, \Sigma_n] \times [x_n - \varepsilon_n, x_n + \varepsilon_n] \times [-\varepsilon \leq z \leq \varepsilon]} \cup \{[y(\sigma_n) \leq y \leq y(\bar{U}(n))] \times [x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n] \times [-\varepsilon - \alpha \leq z \leq \varepsilon + \alpha]\}.$$

Here $[\sigma_n, \Sigma_n] = [y(\sigma_n) \leq y \leq y(U(n)) (< y(\bar{U}(n)))]$ and we have an undrawable singularity in 3^d

$$\text{singularity } (U(n)) = y(\Sigma_n) \times [x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n] \times [-\varepsilon \leq z \leq \varepsilon].$$

See here the figure 4.2-TER, where the singularity is shaded in RED. End of (4.14.1)

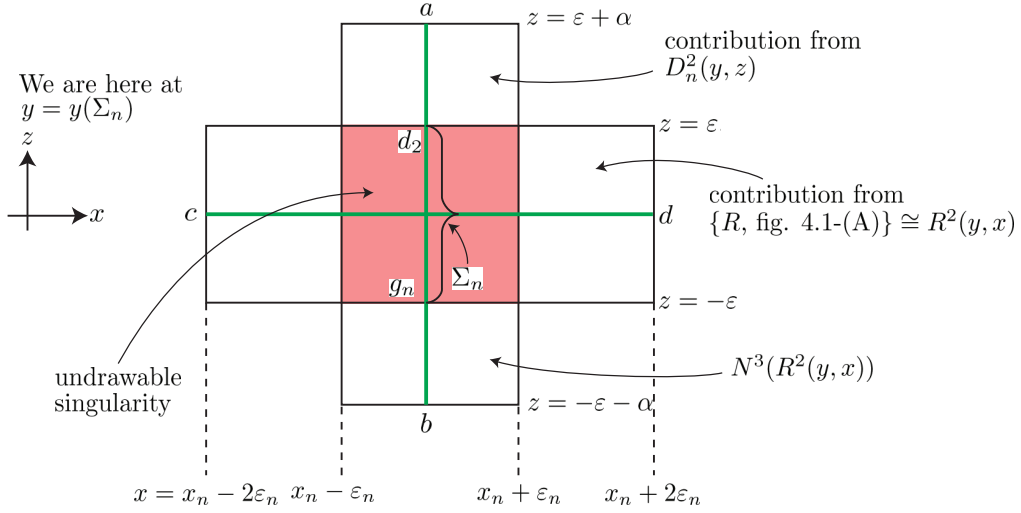


Figure 4.2-ter. $N^3 \mid (y = y(\Sigma_n))$

Legend: $[a, b] = N^2 \mid D_n^2$, $[c, d] = N^2 \mid R$. The d_2, g_n are like in figure 4.1-(B).

Our N^3 is split from the rest of $\Theta^3(X^2)$ by a surface δN^3 homeomorphic to $S^2 \times S^2 - \text{int}D^2$, which is the regular neighbourhood of ∂N^2 (4.14), i.e. it has as spine the following bouquet of circles

$$(4.15) \quad \partial\{\text{the piece } R^2(y, x) \subset N^2\} \cup_{\sigma_n} \partial\Delta^2.$$

Moving to the \mathcal{R} -dependent $\Theta^4(\Theta^3(X^2), \mathcal{R})$, where \mathcal{R} is a desingularization we get like in Definition 1.11 from Section 1, and like in [8], [18], [19] (but you do not really need to worry about those references), an \mathcal{R} -dependent embedding

$$\delta N^3 \subset \partial\Theta^4(\Theta^3(X^2) \mid U(n), \mathcal{R}) = \partial B^4.$$

Notice that the $\{\text{local model (4.14)}\} \supset \Delta^2$ stays essentially untouched by the two holes $H_{\pm}(n)$, figure 4.1-(B).

Remark. The shaded area $[d_1, t_+, t', d'_1]$ in figure 4.1-(B) is NOT a piece of D_n^2 but it corresponds to $N^3 \mid R$. It is the x -projection of $\{[\sigma_n, \Sigma_n] \times [x_0 \leq x \leq x_{\infty}] \times [-\varepsilon \leq z \leq \varepsilon]\}$ (thickened R), in D_n^2 .

Now we finally throw in the Holes $H_{\pm}(n)$. These only touch the N^3 along some small areas of δN^3 , which correspond to the broken arcs $[\overline{d^2}, \overline{d^2}, t_+]$ and $[t_-, \overline{d'_2}, \overline{d'_2}]$, figure 4.1-(B), leaving the discs $D^2([t_+, t_-])$ intact. So, very conveniently as it will turn out, $D^2([t_-, t_+]) \subset \delta N^3$. Also, deleting $H_{\pm}(n)$ does not affect the \mathcal{R} , since we are far from the singularities, like $U(n) \times [x_n - \varepsilon_n \leq x \leq x_n + \varepsilon_n]$, which themselves stay intact.

Under the deletion of the $H_{\pm}(n)$'s, the discs $\underbrace{\sum_{\text{all } \Sigma_n, S_n} D^2([t_- t_+])}$, induce now a splitting, the one in (4.12),

$$\Theta_1^{N+4}(X^2 - H) - \{\text{DITCHES}\} = \{\text{MAIN } \Theta^4\} + \sum_n \{4^{\text{d}} \text{ 1-handles } T(n)\}.$$

When we go from here to $\Theta_1^{N+4}(fX^2 - H) - \{\text{DITCHES}\} \supset \{\text{MAIN } \Theta^4\} \times B^N \supset \{\text{the DITCHES, now deleted}\}$, then the $(N+4)$ -dimensional 1-handles $T(n)$ are attached to $\{\text{MAIN } \Theta^4\} \times B^N$ along the $(N+3)$ -balls $B^{N+3}([t_-, t_+])$ from (4.11.3), at $y = y(\overline{U}(n))$ and at $y = y(\overline{V}(n))$. And, in a nutshell, here is how one proceeds from here on. With a $T(n)$ for which the factor B^N has been reduced to $b^N(n)$, the map j in (4.13) **sends in the DITCH** the 1-handles $T(n)$. More explicitly, here is, how the map j occurring in (4.13) is defined, using (4.12). The $j \mid \{\text{MAIN } \Theta^{N+4}\}$ is the obvious canonical embedding. Next, like in the figure 4.2-bis, we introduce now for each n the small $b^N(n) \subset B^N$ from (4.11.1), corresponding let us say to the indentations in figure 4.2.(A), such that $\{\text{diam } b^N(n)\} \rightarrow 0$ when $n \rightarrow \infty$. With this, $j \mid (T(n) \mid y)$ sends $T(n) \mid y$ inside

$$\text{ditch}(n) \mid y - \partial \text{ditch}(n) \mid y \subset \text{DITCHES} \subset \Theta_1^{N+4}(fX^2 - H),$$

essentially inside $b^N(n) \times [\varepsilon - \frac{1}{n} \leq z \leq \varepsilon]$ as the figure 4.2 suggests to do it and with more details in (4.11.2). This ends the description of (4.13). It is the restriction to $\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$ occurring above which makes that j is PROPER. To be more precise concerning this issue, in the context of the figure 4.2 we have

$$\lim_{n \rightarrow \infty} \{\text{the } z\text{-height of } T(n) \mid y\} = 0,$$

and hence

$$(4.17) \quad \lim_{n \rightarrow \infty} j(T(n) \mid y) = \{\text{the point } (x = x_{\infty}, y, z = \varepsilon, (t = 1) \in \partial B^N)\} \times (I \ni u),$$

which lives at infinity. This ends the proof of our lemma. \square

Very importantly, on the lines of (4.17) we also have here (4.17.1)

$$\text{ditch}(n) \mid [\varepsilon - \frac{1}{n} \leq z \leq \varepsilon] \xrightarrow{n \rightarrow \infty} (\text{LIM } M_2(f) \times (z = \varepsilon)) \times (y(\overline{V}(n)) < y < y(\overline{U}(n)) \times I \times (t = 1),$$

where the LHS contains $jT(n)$, and where $(t = 1) \in \partial \ell^N(n) \cap \partial B^N$.

Moreover, $\text{LIM } M_2(f) \subset (x = x_{\infty})$, and since (4.3) tells us that $\text{LIM } M_2(f) \times (z = \varepsilon)$ lives at infinity, the whole RHS of (4.17.1) **lives at infinity** too.

It is this (4.17.1) which makes sure that in our toy-model construction we stay on the good side of the non-metrizability barrier; see here the CONCERN I after (4.11.1).

In the formulae (4.18), (4.19) below, we work with some $y \in (-\infty, \infty)$ which is fixed. We introduce now the **critical space** χ . Here, the area which is shaded in figure 4.2-(A) is

$$(4.18) \quad \chi|_{y,u} = \{x = x_{\infty}, -\varepsilon < z < \varepsilon, (B^N \ni t)\},$$

which has exactly its two fat sides

$$(4.19) \quad f \text{ LIM } M_2(f) \times \{z = \pm \varepsilon\} \times B^N,$$

living at infinity. Notice that, in the present very simplified context we have $f \text{LIM } M_2(f) = \text{LIM } M_2(f)$, see figure 4.1-(A). In this same context, $\{[\rho_-, \rho_+]$ in figure 4.2-(A) $\} = (x = x_\infty, -\varepsilon < z < \varepsilon, t = 1)$.

Also $\chi \equiv \bigcup_{y \in (-\infty, \infty), u \in I} \chi \mid (y, u)$, which is our critical space.

Lemma 4.2. – *When in the formulae like (4.4), the operation of sending the punctures $\text{LIM } M_2(f) \times \{z = \pm \varepsilon\}$ to infinity is already incorporated into the definition of Θ^3 , then all the four manifolds below are transversally compact*

$$\Theta_1^{N+4}(X^2), \Theta_1^{N+4}(X^2 - H), \Theta_1^{N+4}(fX^2), \Theta_1^{N+4}(fX^2 - H).$$

This means that, when Θ^3 is smooth they are locally of the form $\Theta^3 \times B^{N+1}$ and, when $\Theta^4(\Theta^3, \mathcal{R})$ (= smooth 4^d thickening of Θ^3) is required, then of the form $\Theta^4 \times B^N$.

See here, for instance, what is represented in figure 3.2-(B), which IS transversally compact.

In the context of the figure 4.2 we have, inside the LHS of (4.13), and forgetting about the factor I of Θ^4 ,

$$(4.20) \quad \lim_{n=\infty} \text{ditch}(n) \mid y = \{\text{the whole line } (x = x_\infty, y, -\varepsilon \leq z \leq \varepsilon, t = 1)\},$$

and this formula should be compared to (4.17). [Incidentally too, in both formulae the generic y is fixed.] When we go from $\Theta_1^{N+4}(X^2 - H)$ to $\Theta_1^{N+4}(X^2)$, respectively from $\Theta_1^{N+4}(fX^2 - H)$ to $\Theta_1^{N+4}(fX^2)$ then the 2-handles $D^2(\beta C_n^\pm)$, respectively $D^2(\alpha C_n^\pm)$, are supposed to shoot out of the curves

$$(4.21) \quad \begin{aligned} \beta C_n^\pm &\subset \partial(\Theta_1^{N+4}(X^2 - H) - \text{DITCHES}) \cap \partial\Theta_1^{N+4}(X^2 - H) \subset \\ &\subset \Theta_1^{N+4}(X^2 - H) - \text{DITCHES} \xrightarrow{j} \Theta_1^{N+4}(fX^2 - H) \subset R^3 \times B^{N+1}, \end{aligned}$$

respectively out of

$$\alpha C_n^\pm \subset \partial\Theta_1^{N+4}(fX^2 - H);$$

both $D^2(\beta C_n^\pm)$ and $D^2(\alpha C_n^\pm)$ are without touching the rest of $\Theta_1^{N+4}(fX^2 - H)$ (see figure 4.2).

The very first inclusion in 4.21 means that

$$\beta C_n^\pm \cap \{\text{the contribution of } \delta \text{ DITCH to } \partial\Theta_1^{N+4}(X^2 - H)\} = \emptyset,$$

and we may read this in figure 4.2 too.

When we come to figure 4.2-bis, then projecting things down on $B^N \times (-\infty < y < \infty)$, we have

$$\beta C_n^\pm \cap T(n) = \{\text{the } y\text{-movie of } p\} \times (\text{at } z = \varepsilon \text{ for } C_n^+ \text{ and at } z = \varepsilon - \frac{1}{n} \text{ for } C_n^-).$$

Very importantly, in terms of the smooth manifold $S_b(fX^2 - H)$, which will be introduced below, we have

$$(4.21.0) \quad \beta C_n^\pm \subset \partial S_b(fX^2 - H).$$

Lemma 4.3. – *1) Inside $\Theta_1^{N+4}(fX^2 - H)$ there is an infinite system of smooth dilatations and possible additions of handles of index $\lambda > 1$, call it the transformation*

$$(4.22) \quad j(\Theta_1^{N+4}(X^2 - H) - \text{DITCHES}) \xrightarrow{\text{DIL}} \{j(\Theta_1^{N+4}(X^2 - H) - \text{DITCHES}),$$

with all that part of $\bigcup_y \sum_n \text{ditch}(n) \mid y$ not yet occupied by $j \bigcup_y \sum_n T(n) \mid y$,

filled in with DITCH-FILLING material, but only **PARTIALLY**, namely only for $\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$.

THE DITCH FILLING MATERIAL, defined in (4.11.4) and shaded in BLUE (= //) in figure 4.2-bis, goes from the lateral surface of the DITCH, i.e. from

$$\{\delta \text{DITCH}(n) = \delta b^N(n) \times [x_n - 2\varepsilon_n \leq x \leq x_n + 2\varepsilon_n] \times [y(\bar{V}(n)) < y < y(\bar{U}(n))] \times [-\varepsilon \leq z \leq \varepsilon] \times I,$$

truncated to $[\varepsilon - \frac{1}{n} \leq z \leq \varepsilon]$, to the 1-handle $T(n)$ (4.11.2).

Inside $\Theta_1^{N+4}(fX^2 - H) \supset j(\Theta_1^{N+4}(X^2 - H) - \text{DITCHES})$, the step DIL is PROPER. In anticipation of things to come later, we will introduce the notation

$$(4.23) \quad S_b(fX^2 - H)_I \equiv \{\text{the long } \{\dots\dots\} \text{ occurring as the RHS of (4.22)}.\}$$

This $S_b(fX^2 - H)_I$ is a smooth $(N + 4)$ -manifold.

What makes our ditch filling step (4.13) + (4.22) be only **partial** is that $j \mid T(n)$ is restricted to $\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$ and that the same restriction applies, afterwards, to the ditch filling material in DIL, defined above.

2) The (4.22) induces a PROPER embedding (and for the ‘‘PROPER’’ see the $\lim_{n=\infty} T(n) = \infty$ in (4.10))

$$(4.24) \quad S_b(fX^2 - H)_I \xrightarrow{\mathcal{J}} \Theta_1^{N+4}(fX^2 - H).$$

Putting together (4.13), (4.22) and (4.24), we get the following commutative diagram of PROPER embeddings

$$(4.25) \quad \begin{array}{ccc} \Theta^{N+4}(X^2 - H)\text{-DITCHES} & \xrightarrow{j} & \Theta_1^{N+4}(fX^2 - H) \\ \downarrow \text{DIL} & \nearrow \mathcal{J} & \\ S_b(fX^2 - H)_I & & \end{array}$$

3) The embedding \mathcal{J} is connected by a not boundary respecting isotopy to a simple-minded diffeomorphism η which enters the diagram below, and which continues to be PROPER

$$(4.26) \quad \begin{array}{ccc} & \Theta^{N+3}(\partial X^2) & \\ & \swarrow b \quad \searrow a & \\ S_b(fX^2 - H)_I & \xrightarrow{\eta, \text{simple-minded}} & \Theta_1^{N+4}(fX^2 - H) \\ & \nwarrow \beta \quad \nearrow \alpha & \\ & \Sigma C_n^\pm & \end{array}$$

DIFFEOMORPHISM

Here the $\Theta^{N+3}(\partial X^2) = \{\text{the regular neighbourhood of } \partial X^2 \text{ (figure 4.1.(A))}\}$ in any of our $\partial \Theta_1^{N+4}$ from Lemma 4.1. The embeddings b, β, a, α are the obvious ones and $\text{Im } b \cap \text{Im } \beta = \emptyset = \text{Im } a \cap \text{Im } \alpha$. The upper triangle in (4.26) commutes. The η is suggested in the figure 4.3.

4) When restricted to $\sum_n C_n^+$, the lower triangle in (4.26) also commutes. But when restricted to $\sum_n C_n^-$, then the lower triangle in (4.26) **only commutes up to homotopy**.

COMMENTS. A) The DIL in (4.22) is **the geometric realization of the zipping** in high dimensions, in the present toy model context. It should help understanding the similar process, occurring later on in real life.

B) Concerning the existence of the diffeomorphism η in (4.26) I will restrict the discussion now to the following paradigmatic fact, and come back later to this issue, in the end this section.

Sub-lemma 4.3.1. – 1) Inside the rectangle $K^2 \equiv (x_0 \leq x \leq x_\infty + \alpha, 0 \leq t \leq 1)$ we consider, on the upper side ($x_0 \leq x \leq x_\infty + \alpha, t = 1$), concentrated at $x_1 < x_2 < \dots < x_n < \dots$, with $x_0 < x_1, \lim_{n \rightarrow \infty} x_n = x_\infty$, the **INDENTATIONS** suggested in figure 4.3-(A). If the diameter of these indentations goes **SUFFICIENTLY FAST TO ZERO**, when $n \rightarrow \infty$, then we have a **diffeomorphism**, isotopic to the standard embedding,

$$(4.26.0) \quad \{K^2 \text{ with indentations}\}_{\text{DIF}} = K^2.$$

One can visualize this on figure 4.3.

2) The same is true if one replace K^2 by $K_2 - (x = x_\infty, t = 1)$.

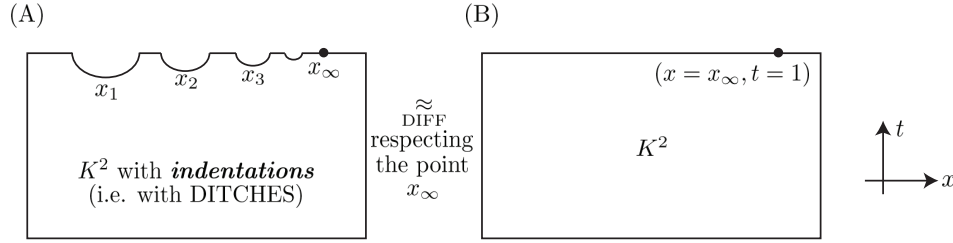


Figure 4.3.

C) The next Lemma 4.4 should explain the “only” in 4) from the Lemma 4.3. It implies that there is no **PROPER** homotopy in the context of 4), Lemma 4.3. \square

We move now to the following two smooth $(N + 4)$ -manifolds

$$(4.26.1) \quad \Theta_I^{N+4}(fX^2) = \Theta_I^{N+4}(fX^2 - H) + \{\text{the 2-handles } D^2(\alpha C_n^\pm)\},$$

$$S_b(fX^2)_I \equiv S_b(fX^2 - H)_I + \{\text{the 2-handles } D^2(\beta C_n^\pm)\}.$$

The $\Theta^{N+4}(fX^2)$ is the substitute for the $S_u^{(l)}$, in the toy-model context.

Lemma 4.4. – 1) There is **NO** homeomorphism taking the form

$$(4.27) \quad (S_b(fX^2)_I, \Theta^{N+3}(\partial X^2)) \longrightarrow (\Theta_I^{N+4}(fX^2), \Theta^{N+3}(\partial X^2)),$$

and inducing the identity on Θ^{N+3} .

2) Hence there is no such diffeomorphism either, and hence diagram (4.26) does **NOT** commute up to a **PROPER** homotopy.

The claim is now that, any conceivable system of singular disks $d_n^2 \subset \Theta_1^{N+4}(fX^2)$ cobounding Γ_n in $\Theta^{N+4}(fX^2)_I$, have to have accumulation points at finite distance. The point here is that

$$\{\text{the bulk of } R \times [-\varepsilon \leq z \leq \varepsilon]\} \subset \Theta^{N+4}(fX^2 - H)_I,$$

and any singular disc $d_n^2 \subset \Theta_1^{N+4}(fX^2)$ cobounding the Γ_n bites through this bulk of $R \times [-\varepsilon \leq z \leq \varepsilon]$. And that piece of ∂R which lives at $x = \bar{x} > x_\infty$, prevents these intersections to be pushed, asymptotically, to infinity. They stay confined at finite distance. End of the Proof of Lemma 4.4.

We did not always bother to add subscripts I to our various $\Theta^3, \Theta^4, \Theta^{N+4}$, but everything discussed so far was in the context of the so-called VARIANT I (for our toy-model) and we move now to the superior level of THE REAL LIFE VARIANT, where the nagging issues of the lemma 4.4 will be superseded. [We mean here, of course, real life as far as the toy model is concerned.]

At any of the 3^d levels $\Theta^3(X^2)$, or $\Theta^3(X^2 - H)$, or $\Theta^3(fX^2)$, or $\Theta^3(fX^2 - H)$, we will introduce the following **critical rectangle** (which we also call the **bad rectangle**)

$$(4.28) \quad \mathcal{R}_\infty \equiv (x = x_\infty, -\infty < y < +\infty, -\varepsilon < z < \varepsilon) \subset \Theta^3.$$

One should notice that, when one takes the projection $\Theta_1^{N+4}(X^2 - H) \xrightarrow{\pi} \Theta^3(X^2 - H)$ and one restricts it to the critical space χ from (4.18), i.e. to

$$\chi = (x = x_\infty, -\infty < y < \infty, -\varepsilon < z < \varepsilon) \times I \times B^N \subset \Theta_1^{N+4}(X_2 - H),$$

then we find that

$$\pi \chi \equiv \mathcal{R}_\infty \quad (\text{see (4.28)}).$$

Let us denote by $\Theta^3(X^2)_I$, the Θ^3 used so far.

We have $\mathcal{R}_\infty \cap \{\text{the undrawable singularities of } \Theta^3(X^2)_I\} = \emptyset$. With this, we go now from our VARIANT I to the REAL LIFE, as that is in our toy-model context, and then we define

$$(4.29) \quad \Theta^3(X^2) \equiv \Theta^3(X^2)_I \underbrace{\cup}_{\mathcal{R}_\infty = \mathcal{R}_\infty \times \{0\}} (\mathcal{R}_\infty \times [0, \infty)),$$

and similarly in the other 3^d contexts. Starting from here, according to the case we will begin building our real-life objects, beyond the variant I of our toy-models.

(4.30) We start with

$$\Theta_1^{N+4}(fX^2 - H) \cup (\mathcal{R}_\infty \times [0, \infty)),$$

with the two pieces glued together along

$$\mathcal{R}_\infty = \mathcal{R}_\infty \times \{0\} \subset \Theta^3(fX^2 - H) \subset \Theta^4(\Theta^3(fX^2 - H), R) \subset \partial(\Theta^4(\Theta^3(fX^2 - H), \mathcal{R}) \times B^N).$$

This object being singular at \mathcal{R}_∞ , we take care of that by adding an $(N + 4)$ -dimensional, transversally compact neighbourhood around $\mathcal{R}_\infty \times [0, \infty)$, going to a space $\Theta_{II}^{N+4}(fX^2 - H)$ (still a provisional object on the road to REAL LIFE) $\equiv \Theta_1^{N+4}(fX^2 - H) \cup \text{Nbd}^{N+4}(\mathcal{R}_\infty \times [0, \infty))$, union made along $\text{Nbd}^{N+3}(\mathcal{R}_\infty \times \{0\}) \subset \partial\Theta_1^{N+4}(fX^2 - H)$. End of (4.30)

But now the $\mathcal{R}_\infty \times [0, \infty)$ has been moved to the interior, and we will need it to be on the boundary. So, we move to the actual, final real life object, which is a smooth $(N + 5)$ -dimensional manifold (in the toy-model case);

(4.30.0) $\Theta^{N+5}(fX^2 - H) \equiv \Theta_{II}^{N+4}(fX^2 - H) \times [0, 1]$, coming with

$$\mathcal{R}_\infty \times [0, \infty) \subset \Theta_{II}^{N+4}(fX^2 - H) \times \{0\} \subset \partial\Theta^{N+5}(fX^2 - H).$$

We do then the same for $S_b(fX^2 - H)$. First we get

$$S_b(fX^2 - H)_{\text{II}} \equiv S_b(fX^2 - H)_{\text{I}} \cup \text{Nbd}^{N+4}(\mathcal{R}_\infty \times [0, \infty)),$$

using $\mathcal{R}_\infty = \mathcal{R}_\infty \times \{0\} \subset \Theta_{\text{I}}^3(X^2 - H) \subset S_b(fX^2 - H)_{\text{I}}$ (see (4.22)). Then the real-life object is now

$$S_b(fX^2 - H) \equiv S_b(fX^2 - H)_{\text{II}} \times [0, 1].$$

End of (4.30.0)

The $\Theta^{N+5}(fX^2 - H)$, $S_b(fX^2 - H)$ which are smooth $(N + 5)$ -dimensional manifolds, contain (like in figure 4.1-(C))

$$\sum_n \alpha C_n^\pm \subset \partial \Theta^{N+5}(fX^2 - H), \quad \sum_n \beta C_n^\pm \subset \partial S_b(fX^2 - H),$$

allowing us to reconstruct the smooth, final *real-life objects*

$$\Theta^{N+5}(fX^2) \quad \text{and} \quad S_b(fX^2).$$

Lemma 4.5. – *When we go from the variant I of Lemma 4.1 to the REAL LIFE case, then 1) in the Lemma 4.3 works just as before, while the higher analogue of the diagram (4.26) from the point 2) commutes now up to PROPER homotopy, not only for $\sum_n C_n^+$ but for $\sum_n C_n^-$ too. As a consequence of this, the real life version of (4.27) is now a diffeomorphism*

$$S_b(fX^2) \underset{\text{DIF}}{=} \Theta^{N+4}(fX^2).$$

End of Lemma 4.5

At the risk of being pedantic, we will give an explicit proof of this lemma. We consider, in the context of the figure 4.1, the following closed curve, at R -level rather than at the D_n^2 level, and which can be moved from $x = x_n$ to $x = x_\infty$

$$(4.31) \quad \gamma_n \equiv \{[c_n, d_n, f_n, g_n], \text{ living at } x = x_n\} \subset R \times [-\varepsilon \leq z \leq \varepsilon].$$

This is drawn in the figure 4.1.(B), except that strictly speaking, what we see there is an embedding $\gamma_n \subset D_n^2$. But then the γ_n clearly also makes sense at level R (figure 4.1-(A)) as written in (4.31). The curve γ_n clearly makes sense at levels $\Theta^{N+4}(fX^2 - H)_{\text{I}}$ and $\Theta^{N+5}(fX^2 - H)$ REAL LIFE too, and in both cases we find that $\lim_{n \rightarrow \infty} \gamma_n = \infty$. One uses here the basic fact (4.27.3) and, of course, one also uses the obvious $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ which allows us to invoke the position at ∞^{ty} of $(x = x_\infty, -\infty < y < \infty, z = \pm\varepsilon)$. And, keep here in mind that one deletes $\text{LIM } M_2(f) \times \{\pm\varepsilon\}$, thereby sending it to infinity.

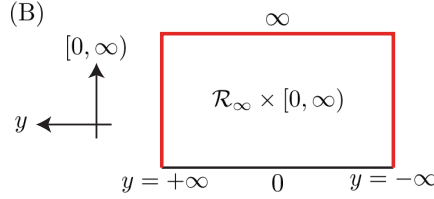
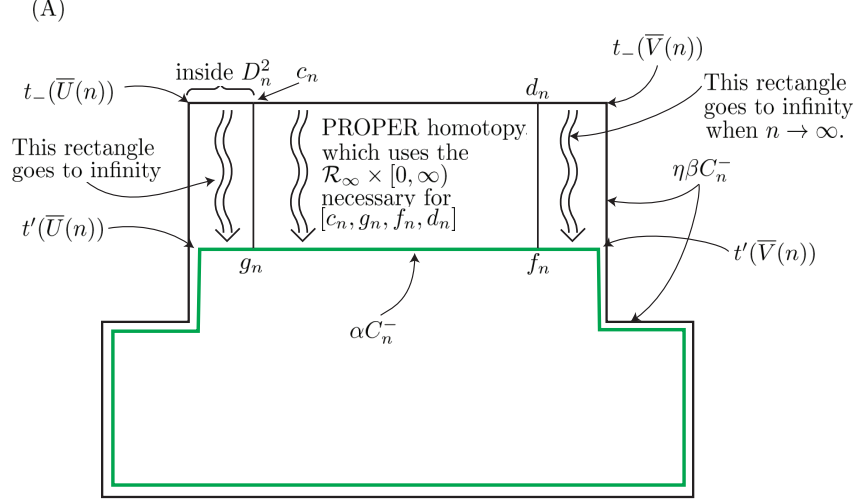


Figure 4.4.

In wiggly lines, we have suggested the PROPER homotopy between $\eta\beta C_n^-$ and the αC_n^- . This happens in the boundary of the smooth manifold $\Theta^{N+5}(fX^2 - H)$. When we consider \mathcal{R}_∞ (4.28), we have $t_\pm(n) \xrightarrow[n \rightarrow \infty]{} y = \pm\infty$ and hence $[c_n, g_n] \xrightarrow[n \rightarrow \infty]{} y = +\infty$, $[d_n, f_n] \xrightarrow[n \rightarrow \infty]{} y = -\infty$. The rectangular box $\mathcal{R}_\infty \times [0, \infty)$ is schematically represented in (B), in a profile view with $[-\varepsilon \leq z \leq \varepsilon)$ omitted. The red contour is at infinity. Our homotopy follows it and, hence it is PROPER. With $y(c_n$ and $g_n) \rightarrow \infty$ and $y(f_n$ and $d_n) \rightarrow -\infty$, think of the 3^d box which (B) suggests, in profile, sitting vertically on top, asymptotically speaking, on top of $[c_n, g_n, f_n, d_n]$ in (A).

In the REAL LIFE CASE, making use of the $\mathcal{R}_\infty \times [0, \infty) \subset \partial\Theta^{N+5}(fX^2 - H)$, we can find now a family of (singular) discs $\Delta_n^2 \subset \partial\Theta^{N+5}(fX^2 - H)$ such that $\partial\Delta_n^2 = \gamma_n$ and $\lim_{n \rightarrow \infty} \Delta_n^2 = \infty$. In the context of the REAL LIFE variant of diagram (4.20), of which the lower triangle is now written as

$$(4.32) \quad \begin{array}{ccc} \partial S_b(fX^2 - H) & \xrightarrow[\approx]{\eta} & \partial\Theta^{N+5}(fX^2 - H) \\ & \swarrow \beta & \searrow \alpha \\ & \sum C_n^\pm & \end{array}$$

the $\sum_n \Delta_n^2$ provide us with the PROPER homotopy modulo which the $\sum_n C_n^-$ part of this triangle commutes too, (while, for the $\sum_n C_n^+$ the strict commutativity is essentially automatic). To make this little argument work, it was necessary to send the bad rectangle \mathcal{R}_∞ to infinity, the way we have just done it above.

Our PROPER homotopy is suggested in figure 4.4 and, as a consequence of it, when one goes to REAL LIFE, contrary to 1) in Lemma 4.3, there is now a diffeomorphism

$$(4.33) \quad S_b(fX^2) \underset{\text{DIF}}{=} \Theta^{N+5}(fX^2).$$

Here are some EXPLANATIONS for that figure. In the figure 4.1-(B) we can see some very thin rectangles, $[c_n, t_-(\bar{U}(n)), t'(\bar{U}(n)), g_n]$, squeezed between $U(n)$ and $\bar{U}(n)$, respectively $[d_n, t_-(\bar{V}(n)), t'(\bar{V}(n)), f_n]$, squeezed between $V(n)$ and $\bar{V}(n)$. When $n \rightarrow \infty$, our thin rectangles go to infinity. Using them and the Δ_n^2 , cobounding γ_n , introduced above (via $\mathcal{R}_\infty \times [0, \infty)$), we can put together a larger rectangle

$$\begin{aligned} & [t_-(\bar{U}(n)), t'(\bar{U}(n)), t_-(\bar{V}(n)), t'(\bar{V}(n))] \text{ (see figure 4.4)} \equiv \\ & \equiv [t_-(\bar{U}(n)), t'(\bar{U}(n)), c_n, g_n] \cup \Delta_n^2 \cup [d_n, f_n, t_-(\bar{V}(n)), t'(\bar{V}(n))] \subset \partial\Theta^{N+5}(fX^2 - H). \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} [t_-(\bar{U}(n)), t'(\bar{U}(n)), t_-(\bar{V}(n)), t'(\bar{V}(n))] = \infty.$$

Moreover, in our high dimensions, $[t_-(\bar{U}(n)), t'(\bar{U}(n)), t_-(\bar{V}(n)), t'(\bar{V}(n))]$ lives embedded inside $\partial\Theta^{N+5}(fX^2 - H)$. It provides us with a PROPER homotopy between $\eta\beta C_n^-$ and αC_n^- in the context of (4.32). Of course

$$\eta\beta C_n^-, \alpha C_n^- \xrightarrow{n \rightarrow \infty} \infty.$$

Figure 4.5 should suggest the PROPER homotopy in the diagram 4.32.

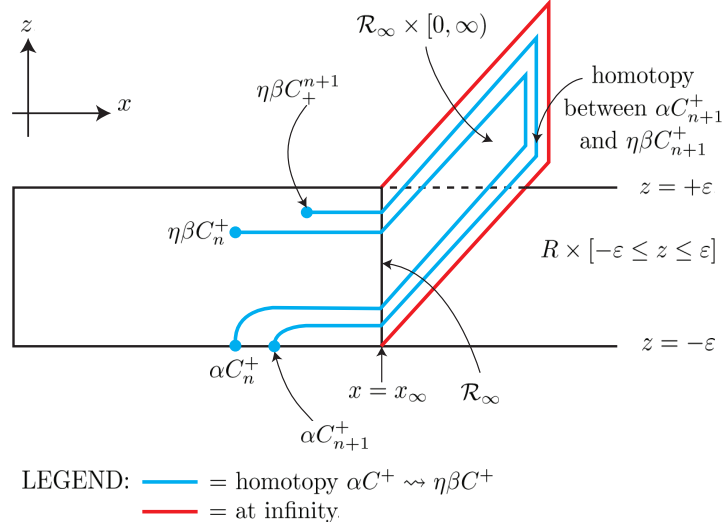


Figure 4.5.

And, finally, in (4.33) we use the fact that in high dimensions (PROPER) **homotopy** of curves means (PROPER) **isotopy**. Living on the boundary is essential here. By now our Lemma 4.5 is proved. \square

FINAL COMMENTS. In Lemma 4.5, the action for the PROPER homotopy was localized in the simple-minded critical rectangle \mathcal{R}_∞ . When one moves from the toy-model to $\bar{M}(\Gamma)$, establishing the analogue of Lemma 4.5 (which will be a very necessary step), will be a much trickier affair, needing a COMPACTNESS LEMMA. There is no question of proving GSC in the context of the toy model. The $\Theta^3(fX^2)$ is, obviously, GSC, so there is no real GSC issue now.

But, as we shall see, the analogue of the Lemma 4.5 is an essential step in the proof of *the* GSC theorem since, there it connects $S_b\widetilde{M}(\Gamma)$ which is GSC by construction and $S_u\widetilde{M}(\Gamma)$ which is shown to be GSC, **by the step in question**. Otherwise, it would just stay a mysterious space.

In the real life case, there is no longer anything so simple-minded as the rectangle \mathcal{R}_∞ to localize our obstruction. So now the key ingredient for getting the desired conclusion, in the likes of the Lemma 4.5, will be the group action of Γ itself, on the representation space, and its equivariance, i.e. we will rely now upon

discrete symmetry with compact fundamental domain

This ends our discussion of the toy-model and, for purely pedagogical reasons we present now a MOCK-PROOF of our main result, i.e. $\forall \Gamma \in \text{QSF}$. For this purpose, let us start with a smooth closed manifold M^p with $\pi_1 M^p = \Gamma$. For large enough m , we have $\pi_1^\infty(\widetilde{M}^p \times R^m) = 0$; here our Γ which we consider is infinite and hence $\widetilde{M}^p \times R^m$ is open. It follows from these things, i.e. that our manifold $\widetilde{M}^p \times R^m$ under investigation is open, simply-connected, with $\pi_1^\infty = 0$ and high-dimensional, that we also have $\widetilde{M}^p \times R^m \in \text{GSC}$. An explicit proof for this more or less well known fact, is to be found in [30], for instance. There is a natural Γ -action on $\widetilde{M}^p \times R^m$, trivial on the factor R^m . We can find an invariant subset $\widetilde{M}^p \times B^m$ on which the action of Γ is co-compact and this comes with the obvious retraction

$$\widetilde{M}^p \times R^m \xrightarrow{r} \widetilde{M}^p \times B^m.$$

Since $\widetilde{M}^p \times R^m \in \text{GSC}$ it is certainly WGSC (in the terminology of L. Funar and D. Otera [7], [14]), meaning that it has an exhaustion by compact, simply connected complexes

$$K_1 \subset K_2 \subset K_3 \dots \subset \widetilde{M}^p \times R^m = \bigcup_i K_i.$$

For any compact $k \overset{j}{\subset} \widetilde{M}^p \times B^m$ there is a $K_N \supset k$. With this in the natural diagram below

$$\begin{array}{ccc} k \subset & \xrightarrow{i} & K_N, \\ & \searrow j & \swarrow r \\ & & \widetilde{M}^p \times B^m \end{array}$$

$r = \text{restriction of the obvious retraction } r \text{ above, to } K_N.$

we have that $i(k) \cap M_2(r) \subset k \cap (\widetilde{M}^p \times \partial B^m)$, since $i(k)$ does not touch $\widetilde{M}^p \times (R^m - B^m)$, $M_2(r) = \widetilde{M}^p \times (R^m - \overset{\circ}{B}^m)$, and $rM_2(r) \subset \widetilde{M}^p \times \partial B^m$. Deceptively, this may look very close to the desired $\widetilde{M}^p \times B^m \in \text{QSF}$ (which would actually ask for the stronger conclusion $i(k) \cap M_2(r) = \emptyset$). But, as long as k touches $\partial(\widetilde{M}^p \times B^m)$, there is no permissible way to disrectangle it from $M_2(r)$. This is as far as this naive argument can go. \square

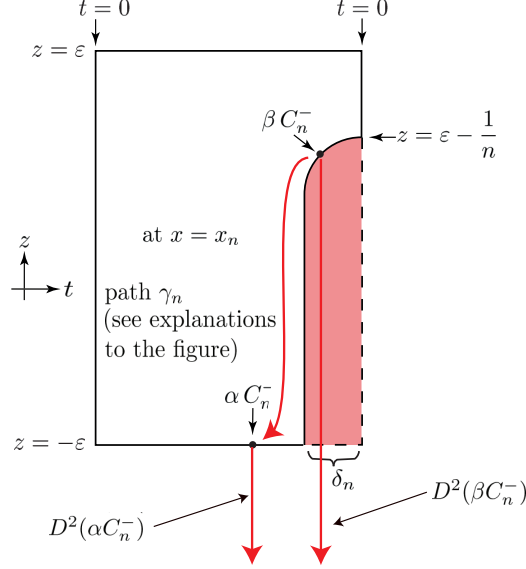


Figure 4.6.

Illustration for the commutativity up to homotopy of diagram (4.26). We consider now two 3^d objects. To begin with the $A_u = (x_1 - k^1 \leq x \leq x_\infty + k^2, z \in [-\varepsilon, \varepsilon], t \in [0, 1]) - \{x = x_\infty, z = \pm \varepsilon, t \in [0, 1]\}$, for some small positive quantities k^1, k^2 . Next, let $A_b = A_u - \sum \{\text{the shaded region from the drawing, for } \forall x \in [x_n - \varepsilon_n, x_n + \varepsilon_n], \forall n\}$. The depth δ_n is $\delta_n > \varepsilon_n$, with $\lim_{n \rightarrow \infty} \delta_n = 0$. The C_n^- 's are now single points. There is an easy diffeomorphism $A_b \xrightarrow{\eta} A_u$. At level A_u , we find that for the points marked $\alpha C_n^- \in A_u \mid (x = x_n)$, $\beta C_n^- \in A_b \mid (x = x_n)$, we have $\lim_{n \rightarrow \infty} \alpha C_n^- = \infty$, $\lim_{n \rightarrow \infty} \eta \beta C_n^- = \infty$. Moreover at the level of ∂A_u , one can join αC_n^- to $\eta \beta C_n^-$ by a path γ_n . But there is no way to prevent the γ_n 's from accumulation at finite distance (far from $z = \pm \varepsilon$). This spells the doom for Variant I, in the context of (4.26).

COMPLEMENTS FOR THE TOY-MODEL STORY.

On the metrizable barrier. The construction of the space $S_b(fX^2 - H)_I$ in (4.24), has to proceed by an infinite sequence of additions, starting from $j(\Theta_1^{N+4}(X^2 - H) - \text{DITCHES})$ in (4.13) + (4.22). The topology normally coming with this is the *weak topology*. [If we are given a sequence of continuous inclusions $X_1 \subset X_2 \subset X_3 \subset \dots$, then in the weak topology of $\varinjlim X_n$ a set $F \subset \varinjlim X_n$ is closed iff every $F \cap X_n$ is closed.]

The (4.17.1), which is a consequence of the fact that our DITCH-FILLING is restricted to $\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$, makes that the weak topology of $S_b(fX^2 - H)_I$ is the one inherited from $\Theta_1^{N+4}(fX^2 - H)$ via the embedding \mathcal{J} in (4.25). This makes the $S_b(fX^2 - H)_I$ a manifold, actually even a smooth one.

Would the definition of $S_b(fX^2 - H)_I$ proceeded by filling the ditches completely, from $z = -\varepsilon$ to $z = +\varepsilon$ (like it is done in the context of Lemma 7 of [50], and the full details are to be found in [56]) then we would have *lost* the basic property (which our construction of S_b has), namely the

$$(4.34) \quad \{\text{The } jT(n) \text{ and the DITCH FILLING MATERIAL which we pile up in the ditch } (n)\} \xrightarrow{n \rightarrow \infty} \infty \text{ (of } \Theta_1^{N+4}(fX^2 - H))$$

and, instead of getting the honest space $S_b(fX^2 - H)_I$ we would have gotten a totally unusable NON-METRIZABLE space.

5 The four functors S_u, S'_u, S_b, S'_b and the geometric high-dimensional realization of the zipping

The object of this (and the next) section is the geometric realization of the zipping in high dimensions. This proceeds in the context of VARIANT I so, most of these two sections will be unrolled in that context. When the geometric realization of the zipping will have been completely developed, then we will be able to perform the big transformation

$$(*) \quad \text{VARIANT I} \implies \text{REAL LIFE VARIANT},$$

already explained in Section III (enough so that we should be able to state correctly the GSC Theorem).

Then, in Section VII we will revisit (*) and **then**, it is in the context of the REAL LIFE VARIANT that the crucial COMPACTNESS LEMMA 5.7 (see below) will be finally proved.

We turn now to the list of ingredients (4.6) and talk about the **Holes** in real life. As a preparation, we start by listing the individual walls (with H^λ meaning λ -bicolored handle)

$$(5.1) \quad W(\text{BLUE}), W(\text{RED}) = W(\text{RED} \cap H^0) \cup W(\text{RED} - H^0)$$

(a splitting to be soon explained below, but the notation should be self-explanatory), and $W_{(\infty)}(\text{BLACK})$.

In a first approximation, it is $\bigcup S_\infty^2 \subset \sum_1(\infty)$ which splits each $W(\text{RED})$ into $W(\text{RED} \cap H^0) \subset \bigcup g(\infty)H_i^0$ and $W(\text{RED} - H^0) \subset \bigcup g(\infty)H_j^1$. More accurately, if H^0 corresponds to $x < x_\infty$ then the border separating the $W(\text{RED} \cap H^0)_n$ from $W(\text{RED} - H^0)_n$ should be placed at $x = x_\infty + \mathcal{J}_n$ with $\mathcal{J}_n > 0$ converging very fast to zero. Also, each of the $W(\text{BLUE}), W(\text{RED} - H^0), W(\text{BLACK} - H^0 - H^1)$ is parallel to a corresponding limit wall in $\sum(\infty)$. We have

$$(5.2) \quad f \text{ LIM } M_2(f) = f \left[\sum(W(\text{RED}) + W_{(\infty)}(\text{BLACK})) \cap \left[\sum S_\infty^2(\text{BLUE}) \cup \sum(S^1 \times I)_\infty(\text{RED}) \right] \right].$$

(5.3) In this formula we have $W(\text{RED}) \cap S_\infty^2(\text{BLUE}) \subset W(\text{RED} \cap H^0)$, in agreement with our separation at $x = x_\infty + \mathcal{J}_n$ above.

Turning now to the global $\sum_1(\infty)$ (2.14), we have

$$(5.4) \quad \Sigma_1(\infty) \supset \overline{\Sigma(\infty)}_*, \text{ with the } \Sigma(\infty)_* \text{ from (3.21), and then}$$

$$(5.5) \quad \Sigma_1(\infty) - \overline{\Sigma(\infty)}_* = \{\text{an infinite disjoint union 2-cells, which will be called the } \mathbf{ideal\ Holes}\}.$$

We have here a homotopy equivalence $\Sigma_1(\infty) - \overline{\Sigma(\infty)}_* \subset \Sigma_1(\infty) - fX^2$.

With this, we introduce now an equivariant system of open 2-cells

$$(5.6) \quad \{\text{normal Holes}\} \subset X^2 - M_2(f),$$

to be pushed afterwards into fX^2 . Here is schematically our complete list of all the Holes of $\Theta^3(fX^2)$ (3.13) and/or of $\Theta^3(fX^2)'$ (3.14)

$$(5.7) \quad \begin{aligned} \text{Holes} &= \{\text{normal Holes, which are open subsets}\} + \{H(p_{\infty\infty}) = \\ &= p_{\infty\infty} \times (-\varepsilon, \varepsilon)\text{'s which are closed subsets (PROPERLY embedded)}\}, \end{aligned}$$

with the further subdivisions $\{\text{normal Holes}\} = \{\text{completely normal Holes}\} + \{\text{BLACK Holes}\}$ and also $\{H(p_{\infty\infty})\} = \{\{H(p_{\infty\infty}(\text{proper}))\}$, which survive in $\Theta^3(fX^2)$ but which are to be deleted when we make

the change $\Theta^3 \Rightarrow (\Theta^3)'$, and not otherwise} + \{\{H(p_{\infty\infty}(S))\}\}, to be **always** deleted, both for $(\Theta^3)'$ and for Θ^3 . Deleted means here deleted from $\Theta^3(fX^2)'$.

Here are some additional explanations. To begin with, the inclusion in (5.6) also means

$$\{\text{normal holes}\} \subset \{\text{smooth part of } fX^2\}.$$

The smooth part in question consists of open 2-cells and, when in BLUE or (RED- H^0) it receives exactly one completely normal hole, exhausting them. BLACK Holes will be discussed later. The Black Holes stay away from the universal curves $x = Z(y)$, as one can see by comparing the figures 2.4-(A) and (B).

At the level of the $H_i^2(\gamma_1), H_i^2(\gamma_2), \dots$ which correspond to a given h_i^2 , each $H_i^2(\gamma_j)$ coming with its unique BLACK Hole, we have

$$\lim_{n=\infty} \partial \text{BLACK Hole}(n) = \infty.$$

It is the completely normal Holes which will go deep into their dual DITCHES, reaching the level $\varepsilon = \frac{1}{n}$. (See here the toy-model.)

The normal Holes are subjected, globally, to the following metric condition

$$(5.8) \quad \lim \{\text{normal Holes}\} = \{\text{a subset of the Ideal Holes}\}.$$

In detail, here is the list of normal Holes.

(5.9) Every maximal smooth 2-cell in $fW(\text{BLUE})$ or in $fW(\text{RED} - H^0)$ contains its unique **completely normal** Hole. Next, any $h_i^2 \subset \widetilde{M^3}(\Gamma)$ generates the infinite system $\sum_{\gamma} H_i^2(\gamma) \subset Y(\infty)$ and each $X^2 \mid H_i^2(\gamma)$, with i given, contains a **unique** $W(\text{BLACK, complete})(\gamma)$, see (2.13). For each i and for all γ 's except for a finite non-void i -dependent subset, the $W(\text{BLACK, complete})(\gamma)$ carries a central BLACK Hole. These are among the normal Holes in (5.6). For given i and for $\gamma \rightarrow \infty$ these BLACK Holes become larger and larger, so that the metrically correct condition (5.8) should be satisfied for them too. End of (5.9).

We move now our discussion to $\Theta_1^{N+4}(X^2)$, which we think of, from now on, on the lines of (3.17), expanded into the splitting

$$(5.10) \quad \Theta_1^{N+4}(X^2) = \Theta_1^{N+4}(X^2 - H) + \sum D^2(C(H))$$

which should mean $\{\Theta_1^{N+4}(X^2)$ (3.44), with **all** the Holes (5.7) **deleted** and then **compensated** for by adding 2-handles (of dimension $N + 4$), $D^2(C(H))\}$.

ADDITIONAL EXPLANATIONS CONCERNING (5.10). We proceed first with $\Theta^4(\Theta^3(X^2), \mathcal{R})$, on the line of (3.19). Then we move to $\Theta_1^{N+4}(X^2)$ proceeding like in (3.25).

This means that we will have a decomposition, on par with the other similar decompositions in Section III (5.10.1)

$$\Theta_1^{N+4}(X^2) = (\Theta^4(\Theta^3(X^2)'_{\text{smooth}}, \mathcal{R}) \times B^N) + \sum_{p_{\infty\infty}(\text{all})} \{\text{the 2-handles } \left(D^2(p_{\infty\infty}) \times \left[\frac{\varepsilon}{4}, \frac{\varepsilon}{4} \right] \right) \times I \times \frac{1}{2} B^N\}.$$

It should be understood that, at this initial time, at the level of $\Theta^3(X^2)'$, the fins with their rims in place are there and nothing has been sent to infinity, **yet**. The $M_2(f)$ is not there, except as a passive spectator, a drawing on X^2 .

Our $\Theta_1^{N+4}(X^2)$ will be the initial object from which the GEOMETRIC REALIZATION OF THE ZIP-PING, main topic of this section and the next will proceed. This process which will include the sending to infinity of $\partial\Sigma(\infty)_*$ will ALL HAPPEN IN THE $(N + 4)$ -dimensional world of VARIANT I as already said. We will move to real life afterwards, climbing to $\dim = N + 5$.

And, since there is no PROPER inclusion $\text{int}(\overset{\circ}{\Sigma}(\infty)_* \cup \text{fins}) \subset \Theta^3(X^2)'_{\text{smooth}}$ (we need fX^2 for that) there is no “real-life” $(N+5)$ -dimensional $\Theta^{N+5}(X^2)$ either, superseding the (5.10.1). End of Explanations.

For the completely normal holes the $D^2(C(H))$'s are the obvious smooth 2-handles corresponding to the framed $\partial(\text{Holes}) \equiv C(H)$. For the $H(p_{\infty\infty})$'s the $D^2(C(H))$'s are like in (3.19) and in the figure 3.3, and now we will describe the $D^2(C(H))$'s, compensating 2-handles for the BLACK Holes. These are displayed in the figure 5.1.

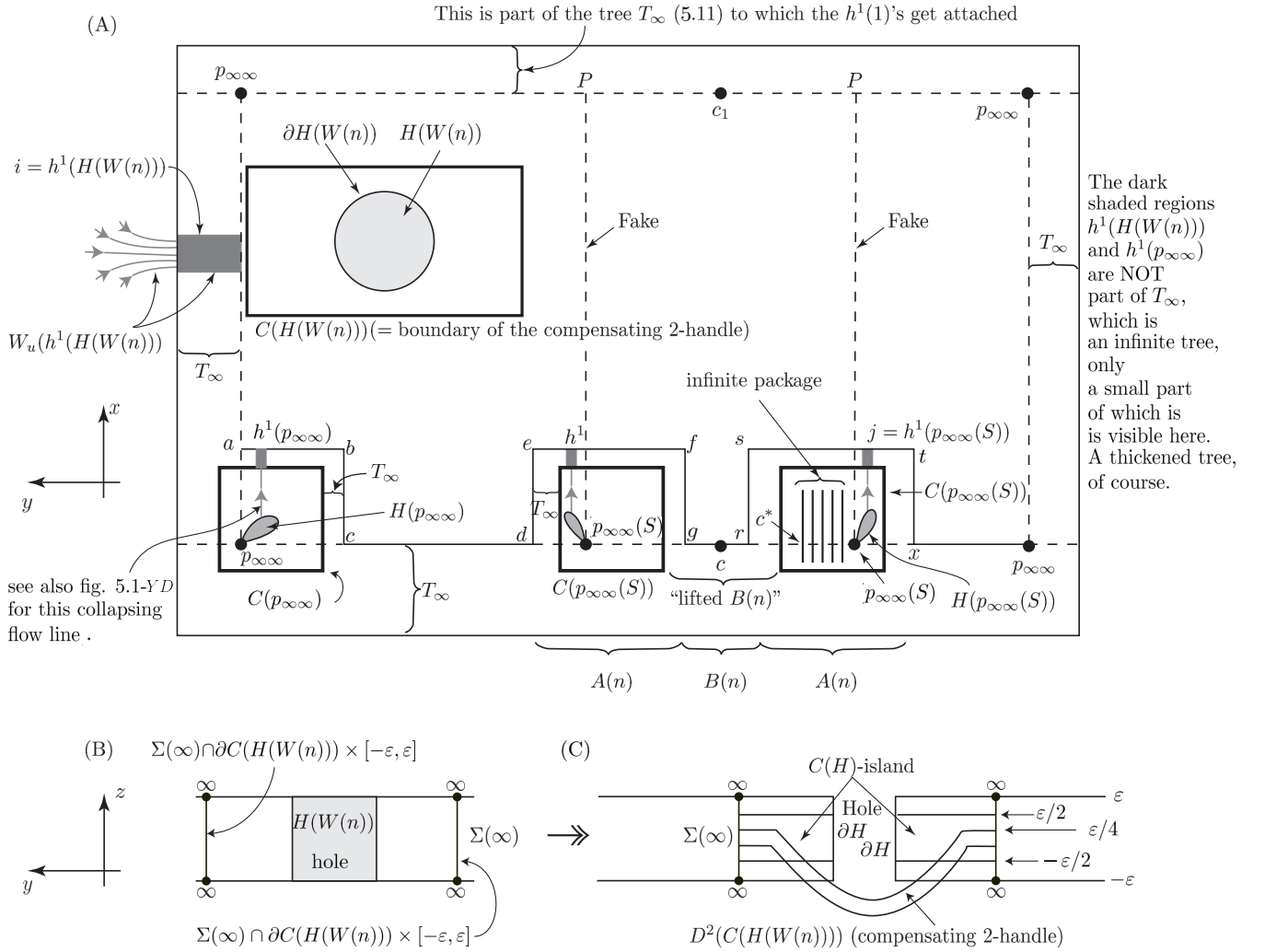


Figure 5.1.

We see here a $W(n) = W(\text{BLACKcomplete})$, and its BLACK Hole $H(W(n))$. At c our $W(n)$ is subdued, while at c^* it overflows and $W^*(n)$ is subdued. Just for better visualization, the holes $p_{\infty\infty}, p_{\infty\infty}(S)$, which in real life are point-like, have been drawn as **closed** teardrops denoted $H(p_{\infty\infty})$. The various h^1 's in the picture are not part of (but rather attached to) $T_\infty =$ the tree in (5.11), while all the **rest** of $\{\text{the region contained between the } \partial W(n)(= \text{the outer rectangle}) \text{ and the closed curve } [p_{\infty\infty}, a, b, c, d, e, f, g, c, r, s, t, x, p_{\infty\infty}, p_{\infty\infty}]\} - \{\text{the interiors of the curves}$

$C(p_{\infty\infty}), C(p_{\infty\infty}(S)), C(p_{\infty\infty}(S))\}$, is part of T_{∞} . LEGEND: - - - - - = future $\sum(\infty)$, real or fake (in the real case the $\partial\sum(\infty)$ is eventually deleted as puncture, but not in the fake case), and we use the word “future” because the present figure is drawn at level X^2 or (5.10), prior to any map for zipping thereof; ————— = curves $C(H), C(H(p_{\infty\infty}))$, to which the 2-handles $D^2(C(H)), D^2(C(H(p_{\infty\infty})))$ are attached, like in the drawing (C), respectively like in figure 3.3-(A); “infinite package” = an infinite package of $\{\text{germs of } \{\text{lifted } B(m \geq n)\}'\text{s}\} \subset C(H(W(m)^*))$'s. When $W(m)^*$ is subdued and the present $W(n)$ is overflowing, their intersection with the dotted lines (= $\sum(\infty)$) are points of type c^* , duals of our present point c , but belonging now to $W(m)^*$. For each S there is an infinite package of W^* 's. As a matter of notation, $C(p_{\infty\infty}) = C(H(p_{\infty\infty}))$. Also, compare this figure to 2.4-(B) and 28.

ADDITIONAL EXPLANATIONS CONCERNING figure 5.1. All the points where $M_2(f) \cap W(\text{BLACK})$ is not, locally, a smooth line, are located inside the islands of $p_{\infty\infty}$ and $p_{\infty\infty}(S)$ and in the S -regions from figure 2.4-(A). Without loss of generality, the $C(H)$'s are inside the easy region (see figure 2.4-(A) too). One may also assume the $C(p_{\infty\infty})$'s far from triple points. The immortal singularities in figure 5.1 are the finitely many c 's and the infinitely many c^* 's, all localized at $x = x_{\infty}$. They never touch the $C(p_{\infty\infty}) \times [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$. These annuli are far from the DITCHES too. So, they are not touched by the geometric realization of the zipping. The immortal singularities created by the partial zipping of $W(n), W(m)^*$, are not yet there, in figure 5.1.

For typographical simplicity's sake, the hexagon in figure 2.4 has become here a square, and only three of the $p_{\infty\infty}$'s are drawn. Notice that both $D(H(p_{\infty\infty}))$ and $D^2(C(H(\text{BLACK})))$ have their attaching curves $C(p_{\infty\infty}), C(H)$ surrounding **from far away** the respective Holes. In the figure 5.1, we are at pre-zipping level, and the dotted (non-fake) lines are here only future punctures. No double points are drawn in figure 5.1. The point marked $c \in \sum(\infty)$ will become one of the finitely many immortal singularities, involving the present $W(n)$ and where this $W(n)$ is subdued, while the infinitely many where it overflows, occur inside the two $p_{\infty\infty}(S)$ -islands, with the “infinite package” signalized in figure 5.1 cutting the dotted line at points c^* . These will also become immortal singularities. We wanted the $h^1(H(W(n)))$ (= the 1-handle dual to the $D^2(C(H(W(n))))$) in the geometric intersection matrix of $\Theta_1^{N+4}(X^2)$ to be in the region contained between $\partial W(n)$ and the concentric dotted polygon, region called “ T_{∞} ” in the figure, and we also want that the region bounded by $C(H(W(n)))$ should not enter in the int T_{∞} , reason which explains the location of $C(H(W(n)))$ (= the attaching curve of $D^2(C(H(W(n))))$, see the drawing (A) and (C). So, now the $D^2(C(H(W(n))))$ is treated very much like the $D^2(H(p_{\infty\infty}))$'s in the figure 3.3, for a while, and hence the different locations for $C(H(W(n)))$ and for $\partial H(W(n))$ in our figure 5.1. Notice, finally, that in Morse-theory terms, every cocore (h^1) comes with a much larger unstable manifold $W_u(h^1)$, which our figure also tries to suggest. In terms of the oriented tree which comes with the geometric intersection matrix, see here the long Definition 1.5, the unstable manifold $W_n(h^1)$ corresponds to the trajectories (or flow lines) coming into h^1 .

What the (5.11) should be, in the neighbourhood of the $W(\text{BLACK})$'s and their $p_{\infty\infty}(\text{proper}), p_{\infty\infty}(S)$, is suggested in figure 5.1. Then, the analogue for $p_{\infty\infty}(W_{\infty}(\text{BLACK}))$ is suggested in figure 5.1-bis. The T_{∞} is the thickening of an infinite tree with its dilatation flow oriented towards the endpoints at infinity AND towards the h^1 's, which induce endpoints at finite distance, with of course $h^1 \not\subset T_{\infty}$. Hence the arrows in figure 5.1-bis; we have omitted to draw them in figure 5.1. End of explanations for figure 5.1. \square

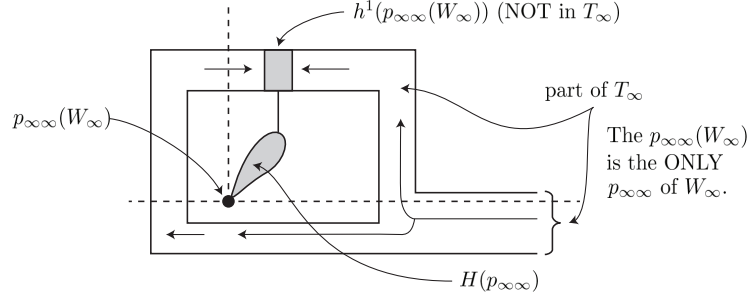


Figure 5.1-bis.

We are here in one of the W_{∞} (BLACK) from the figures 2.5, 2.6. Compare this with a detail of figure 5.1.

The $\Theta_I^{N+4}(X^2)$ in (5.10) will be endowed with a handlebody decomposition flexible enough to allow for our future manipulations. This includes already all the fins (with their rims). For the time being nothing at all is sent to infinity. The handles will be denoted generically $h_i^\lambda(1)$. With this, we will have

$$(5.11) \quad \Theta_I^{N+4}(X^2) = T_{\infty} \text{ (an infinite tree, thickened in dimension } N + 4) + \sum_i h_i^1(1) + \sum_j h_j^2(1) + \{ \text{other handles of index } \lambda \geq 2 \},$$

where:

$$(5.11.1) \quad \text{All the } D^2(C(H)) \text{ in (5.10) are among the } h_j^2(1)_1, \text{ and there is a canonical isomorphism } \{i\} \approx \{j\}.$$

$$(5.11.2) \quad \text{In the geometric intersection matrix we have } \partial h_j^2(1) \cdot \delta h_i^1(1) = \delta_{ji} + \text{nilpotent (easy type).}$$

Now, the (2.1) is a 2^d REPRESENTATION, so X^2 has to be GSC. And, as a matter of fact, the (5.11), (5.11.1), (5.11.2) express explicitly the fact that $\Theta_I^{N+4}(X^2) \in \text{GSC}$. And it is from this basic fact that $X^2 \in \text{GSC}$, that one gets the decomposition (5.11), with a T_{∞} which, in the neighbourhood of the W (BLACK)'s and their $p_{\infty\infty}$'s as well as in the neighbourhood of $p_{\infty\infty}(W_{\infty}) \in W_{\infty}$ (BLACK) (figures 2.5, 2.6), should take the forms prescribed by the figures 5.1, 5.1-bis.

$$(5.12) \quad \text{As a consequence of all this, } \Theta^{N+4}(X^2)_I \text{ certainly is GSC, as already said.}$$

The more schematic figure 5.1-ter, which accompanies figure 5.1, should help understand the matrix $h_j^2(1) \cdot h_i^1(1)$, in the neighbourhood of figure 5.1.

Here come two important facts.

$$(5.13) \quad \text{We have the natural inclusions, when in } \Theta^3(fX^2)' - H \text{ the holes are ALL deleted, but the compensating discs are present,}$$

$$\bigcup_{R_0} \text{int } R_0 = \text{int} \left(\sum^{\circ} (\infty)_* \cup \text{fins} \right) \subset \Theta^3(fX^2)' - H \subset \Theta^3(fX^2)'.$$

So, the $\bigcup_{R_0} \text{int } R_0$ is far from ALL HOLES (5.7).

(5.13.0) The attachment of $\bigcup_{R_0} \text{int } R_0 \times [0, \infty)$ or of its N^{4+N} can be done so that it should not bring any

(new) 1-handles, when we move from $S'_u \widetilde{M}(\Gamma)_1$ to $S'_u \widetilde{M}(\Gamma)$, see below. Here $S'_u \widetilde{M}(\Gamma)$ is the REAL LIFE guy. [And do not mix up the $\Theta^3(fX^2 - H)'$ in (5.13.1) below with the $\Theta^3(fX^2) - H$ from (5.13), where all the compensating discs are removed. In $\Theta^3(fX^2 - H)'$ they are present, see above and below, too.]

We will also introduce the following singular object

(5.13.1) $\Theta^3(fX^2 - H)' \equiv \{\Theta^3(fX^2)$ with all the holes, including $p_{\infty\infty}$ (all) and also all the compensating discs present $\}$. When Holes will be deleted, this will normally happen in the context $(\Theta^3)'$, or S'_u with prime. It is in that context that the Holes are an important useful tool. With this, we have

$$(5.13.2) \quad S'_u(\widetilde{M}(\Gamma) - H)_I = \Theta^4(\Theta^3(fX^2 - H)', \mathcal{R})_I \times B^N \text{ (with } \partial\Sigma(\infty)_*^{\wedge} \text{ sent to infinity).}$$

The smooth manifold (5.13.2) comes endowed with an obvious framed link

$$(5.14) \quad \underbrace{\sum_{H \in (4.7)} C(H)} \xrightarrow[\text{PROPER embedding}]{\alpha} S'_u(\widetilde{M}(\Gamma) - H)_I.$$

In this formula, for $H \in \{\text{completely normal Holes}\}$, we have $\text{Im } \alpha \subset \partial S'_u(\widetilde{M}(\Gamma) - H)_I$, while for $H \in \{\text{BLACK Holes}\} + \{H(p_{\infty\infty})\}$ we have $\text{Im } \alpha \subset \text{int } S'_u(\widetilde{M}(\Gamma) - H)$. Notice the obvious reconstruction formula

$$(5.14.1) \quad S'_u \widetilde{M}(\Gamma)_I = S'_u(\widetilde{M}(\Gamma) - H)_I + \underbrace{\sum_{\alpha C(H), H \in (5.7)} D^2(C(H))}.$$

Here $S'_u \widetilde{M}(\Gamma)_I$ is like in (3.25). And, with fewer details written explicitly in, the next (5.14.2) is the same as in (3.30).

With this, we have the reconstruction formula, in REAL LIFE context

$$(5.14.2) \quad S'_u \widetilde{M}(\Gamma) = \left\{ S'_u(\widetilde{M}(\Gamma) - H) + \sum_{\alpha C(H), H \in (5.7)} D^2(C(H)) \right\}, \text{ to which we add the ingredients } \sum_{R_0} \text{int } R_0 \times$$

$[0, \infty)$, N^{4+N} and “ $\times [0, 1]$ ”, going thus from dimension $N + 4$ to $N + 5$. Enough has been said concerning the Holes, for the time being, and referring to the same list (4.6) we move now next to the DITCHES. These ditches will be *indentations*, inside some of the thickened walls $W \times [-\varepsilon, \varepsilon] \times I \times B^N$, making use of additional N dimensions of B^N . We want our ditches not to interfere with our various mechanisms which concern the $p_{\infty\infty}$'s and so, at least when it comes to the regions $A(p_{\infty\infty}) \subset \Theta^3(fX^2)'$ (figure 3.3) then our indentations will be contained inside smaller N -balls b^N , resting on ∂B^N , in such a way that

$$(5.15) \quad b^N \subset B^N - \left(\frac{1}{2} + \varepsilon \right) B^N.$$

See here the $\frac{1}{2}B^N$ which occurs in Section III for the 2-handles of $S'_u \widetilde{M}(\Gamma)$.

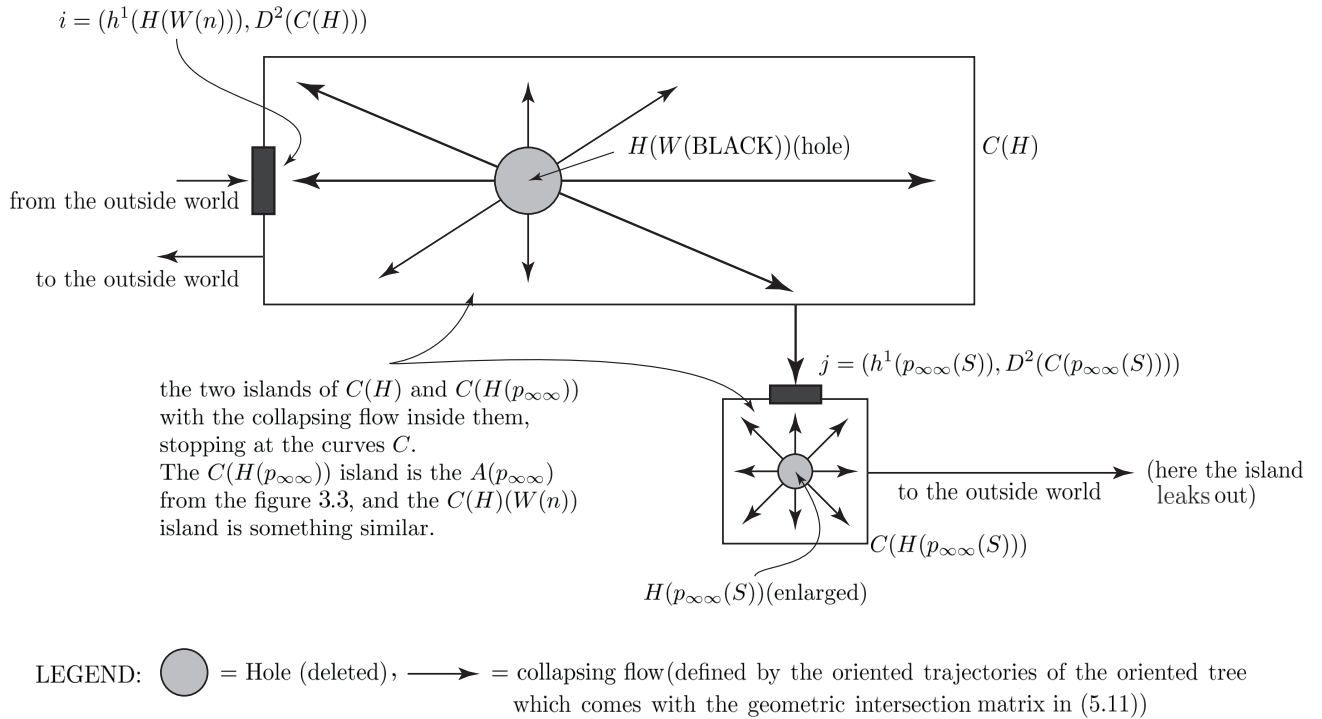


Figure 5.1.ter

This figure should explain the piece of the geometric intersection matrix $h_j^2(1) \cdot h_j^1(1) = (\text{easy}) \text{id} + \text{nil}$, coming with the figure 5.1. In view of the form of this matrix, it defines an oriented graph, and in a somehow metaphorical language, the reader is invited to think of it as the, not everywhere well-defined “collapsing flow” of X^2 . The vertices of our oriented graph are called **states**, and they correspond to couples like for instance

$$i = (h^1(H(W(n))), D^2(C(H(W(n))))), \text{ or } j = (h^1(p_{\infty}(S)), D^2(H(p_{\infty}(S)))) .$$

So, the off-diagonal part of the geometric intersection matrix of X^2 (5.10) and (5.11.2) defines an abstrat flow, our “**collapsing flow**”.

Here is a detail of our flow, coming with the present figure

$$\text{outer world} \left\{ \begin{array}{l} \longrightarrow \\ \longleftarrow \end{array} \right. i \xrightarrow{C(H)} j \longrightarrow \text{outer world} .$$

There is also an internal flow inside the islands, radiating from the corresponding Hole. Exactly one flow line hits j coming from i , and this makes it then to the outer world. These smooth outgoing lines are the ways for the islands to leak out. We define here the islands as being whatever is inside the curves $C(H)$, $C(H(p_{\infty}))$, with the **exclusion** of i, j themselves. Very importantly, nothing coming from the outside world ever hits these islands in their interior. Finally, when it comes to their leaking out, see also the outer boundary of $A(p_{\infty})$, in the figure 3.3, too.

The ditches will be carved inside the

$$\{\text{complementary walls}\} \times [-\varepsilon, \varepsilon] \times I \times B^N,$$

where we define

$$(5.16) \quad \{\text{complementary walls}\} \equiv \{W_{(\infty)}(\text{BLACK}) \text{ and } W(\text{RED} \cap H^0)\}.$$

These walls never carry completely normal holes, only Black Holes, and $H(p_{\infty\infty})$'s. So all the walls are partitioned now into $\{\text{walls}\} = \{\text{complementary walls, which carry DITCHES}\} + \{\text{the non-complementary walls } W(\text{BLUE}), W(\text{RED} - H^0), \text{ which carry completely normal holes}\}$. The complementary walls are the ones cut by the limit walls $S_{\infty}^2 \cup (S^1 \times I)_{\infty}$, which create $\text{LIM } M_2(f)$, reason why they need DITCHES. Without DITCHES we would be in trouble with the non-metrizability barrier. Retain that the $\Sigma(\infty)_* = f \text{ LIM } M_2(f) \times [-\varepsilon, \varepsilon]$ is carried entirely by the (thickened) complementary walls.

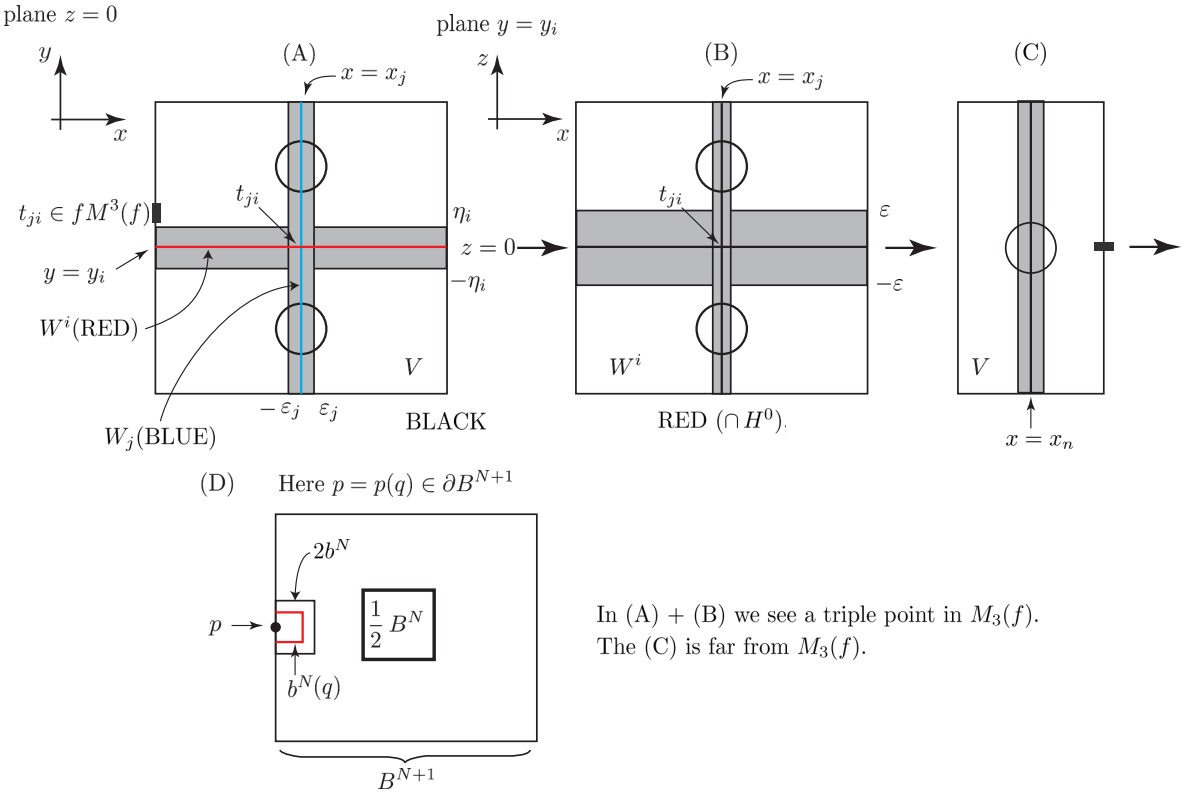


Figure 5.2.

In (A), (B), (C) we see small representative pieces of complementary walls of X^2 , in the neighbourhood of pieces of $M_2(f) \subset X^2$. The (A) and (B) go together, and $x = x_j$ corresponds to a $W_j(\text{BLUE})$ which cuts transversally through them. The $y = y_i$ in (A) corresponds to the W^i (displayed in (B)), and, at least in the generic case we have here, $V = W_{(\infty)}(\text{BLACK})$, $W^i = W(\text{RED} \cap H^0)$. The non-complementary walls, which cut transversally through the displayed complementary walls, carry Holes, signaled by the circles riding on them. But these Holes, two for each circle, live outside of the plane of our drawing. In (A) + (B) we have displayed a triple point t_{ij} . The non smooth ramification point is left for figure 5.3.

The (D) concerns the supplementary N dimensions and it shows the relative positions of

$$1/2 B^N \subset B^N \supset 2b^N \supset b^N.$$

We will need the $2b^N$ much later, in Section VI, when we will construct the (zipping)⁻¹ process.

In the formula (5.16), for any given $W(\text{RED})$ we have a triple splitting $W(\text{RED}) = W(\text{RED} \cap H^0) \cup W(\text{RED} - H^0) \cup W(\text{RED} \cap H^0)$ with the splitting lines corresponding to $x = x_\infty + \zeta$, with $\zeta > 0$ i.e., with $x = x_\infty$, in the “ H^0 ” side, and with

$$\zeta(W(\text{RED})_n) \rightarrow 0 \quad \text{when} \quad W(\text{RED})_n \rightarrow (S^1 \times I)_\infty.$$

With all this, in the figures 5.2 (A, B and C) we have represented small pieces of complementary walls be they V or W^i . The arrow drawn on the right side of each drawing, points towards $\text{LIM } M_2(f)$.

The double lines $fM_2(f)$ are surrounded by **shaded zones** of controlled widths, namely the

$$(5.17.1) \quad [x_j - \varepsilon_j, x_j + \varepsilon_j], [y_i - \eta_i, y_i + \eta_i], [-\varepsilon \leq z \leq \varepsilon].$$

For reasons of compatibility to be developed later, we will insist on the following kind of metric conditions too

$$(5.17.2) \quad \varepsilon_j \ll \{\varepsilon \text{ and } \eta_i\}, \quad \text{and} \quad \lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad \lim_{n \rightarrow \infty} \eta_n = 0 \quad \text{when} \quad \lim x_n = x_\infty.$$

Every wall will be endowed with a canonical **transversal orientation**.

The transversal orientations are chosen individually for each W . In the case $W(\text{RED})$ or $W(\text{BLACK})$ they look towards the corresponding handle-core. In the case of V they go from $z = +\varepsilon$ to $z = -\varepsilon$ and this will determine the **partial** filling of DITCHES, happening close to $z = +\varepsilon$, like in the figure 4.2, and going from ε to $\varepsilon - \frac{1}{n}$.

We consider now a context like in the smooth figure 5.2, but with $V + W^i$ replaced by $V \cup W^i$. This may be already so at level X^2 OR after some preliminary zipping. As it will be explained later, when it comes to any individual triple point

$$V(\text{complementary}) \cap W^i(\text{complementary}) \cap W_j(\text{non-complementary}),$$

then our zipping strategy will be such that any zipping $V + W^i \Rightarrow V \cup W^i$ will be performed **before** the zipping of the BLUE W_j . With this, figure 5.2 is replaced by 5.3 and our shaded zones from (5.17.1) become

$$(5.18) \quad (V \cup W^i) \cap (x_j - \varepsilon_j \leq x \leq x_j + \varepsilon_j) \text{ with the transversal arcs } [-\varepsilon \leq z \leq +\varepsilon] \text{ and } [y_i - \eta_i \leq y \leq y_i + \eta_i] \text{ melted now into the } U(x_0) \text{ from figure 5.3.(B).}$$

We will denote by q the generic point in a shaded zone along a double line

$$(5.19) \quad q \in \{\text{shaded zones}\} \subset \text{complementary wall}.$$

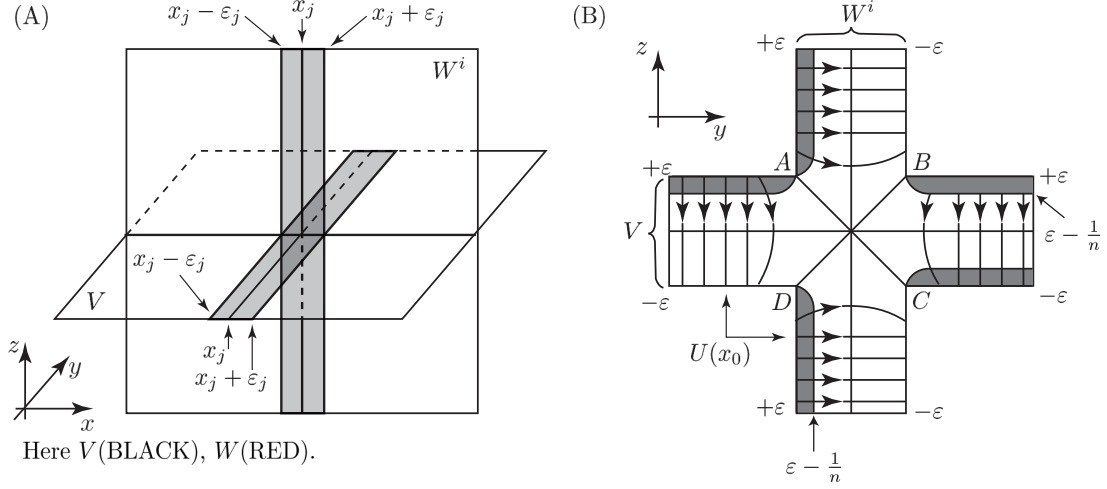
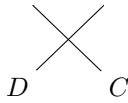


Figure 5.3.

In (A) we show what becomes of the shaded zones from figures 5.2.(A, B), when V, W^i are glued together. This leaves us with an “ X ” $\subset M_2(f)$. In (B) we see a section ($x = x_0$) $\cap (V \cup W^i$ thickened). Here x_0 is a generic point in $[x_j - \epsilon_j, x_j + \epsilon_j]$. The double shading in (B) corresponds to the PARTIAL ditch filling.

Legend: \longrightarrow = The DITCH and its orientation, transversal to W (complementary). Double shading = PARTIAL DITCH-FILLING. The \rightarrow can also stand for the whole DITCH “ $[-\epsilon, \epsilon]$ ”. There is NO DITCH at the cross below: $A \quad B$, which one can see in (B).



The ditches, subject of our present discussion, are far from the fins and from the singularities and, assuming now also that our generic point q (5.27.4) is a smooth point of X^2 , then the local contribution of the $W = W$ (complementary) to $\Theta_I^{N+4}(X^2)$ (and here, as long as we ignore $p_{\infty\infty}$ we might as well write $\Theta_I^{N+4}(X^2 - H)$) is

$$(5.20) \quad \underbrace{W^2 \times [-\epsilon, \epsilon] \times I \times B^N}_{\Theta^3} \times \underbrace{}_{\Theta^4} \times \underbrace{}_{B^{N+1}}.$$

For each $q \in (5.19)$ we will attach, like in the figure 5.2.(D) a smaller q -dependent $b^N = b^N(q)$ verifying (5.15), such that when $\lim q \in \text{LIM } M_2(f)$ then $\lim(\text{diam } b^N(q)) = 0$. The $b^N(q)$ is concentrated around $p(q) \in \partial B^N \cap \partial b^N(q)$, with the p replacing the $t = 1$ from the toy-model figure 4.2. When it comes to the DITCHES, do not mix up the “ $t = 1$ ” $\in \partial B^N$ and the $z = +\epsilon$, end of $-\epsilon \leq z \leq \epsilon$, in terms of which the **PARTIAL DITCH FILLING** only concerns the portion $\epsilon - \frac{1}{n} \leq z \leq \epsilon$ of the complete ditch. The precise definition of the ditches will be given a bit later below.

But this is a good place for describing, in very broad terms, *the main idea of the present paper*, so we will open now a longish prentice. Completely ideally, what we would like to do, would be to replace the

gigantic *quotient*-space projection

$$X^2 \xrightarrow{\text{ZIPPING}} fX^2 = X^2/\Psi(f),$$

connected to the 2^d REPRESENTATION of Γ from (2.1) by a process, indicated for the time being only with a question mark, in the following completely impressionistic formula

$$(5.21) \quad \Theta^{N+4}(X^2)_I \xrightarrow{?} S_u \widetilde{M}(\Gamma)_I$$

consisting of a sequence of *inclusion* maps of GSC-preserving type. It is shown in [36] and also sketched in Section I how this plan (5.21) is actually possible, under the very restrictive condition of *easyness* for the 2^d representation, i.e. under the following explicit condition (i.e. when we are AVATAR-free, see Section I)

$$(5.22) \quad M_2(f) \cup \text{Sing}(f) \subset X^2 \quad \text{and} \quad fX^2 \subset \widetilde{M}(\Gamma) \text{ are } \mathbf{closed} \text{ subsets.}$$

Now when the condition (5.22) is with us, then transforming the ZIPPING $X^2 \rightarrow fX^2$, which is anyway a big quotient space projection, into a high-dimensional inclusion map which is GSC (actually an infinite collection of GSC-preserving inclusion which, in an appropriate sense, converges, is an easy affair, very easy actually. No point of making any fuss about it.

BUT, when the AVATARS from Section I are with us, and not only the two subsets in (5.22) are NOT CLOSED, but on top of it fX^2 is not even locally finite, then our mimicking of a big quotient space projection by an inclusion map process is an affair which is very far from being trivial. Anyway, this is the game which we play in our TRILOGY. And part of this game, the aim of the present section and of the next one too (which will come with more details of proofs), is to present the HIGH-DIMENSIONAL GEOMETRIC REALIZATION OF THE ZIPPING and, very importantly, this will constantly take place in the context of the $(N+4)$ -dimensional VARIANT I. The (5.21) above is a symbol of this geometric realization of zipping. Only much later, after the ZIPPING will have been realized geometrically in the framework of the VARIANT I, will we go to the $(N+5)$ -dimensional REAL WORLD SCENARIO with the

$$\text{Nbd}^{N+4} \left(\sum_{R_0} \text{int } R_0 \times [0, \infty) \right) \quad \text{and the factor } \times [0, 1],$$

thrown in too. All this having been said, in the general case which we have to face now, avoiding the assumption (5.22) is hopeless. We have to live with the AVATARS.

So, we are still left to try to do something on the lines of (5.21), as best as we can. We replace $\Theta_I^{N+4}(X^2)$ by $\Theta_I^{N+4}(X^2)$ -DITCHES and then try to develop a ditch filling process, imitating the zipping, so as to get to $S_u \widetilde{M}(\Gamma)_I$, or to something like it. But we want to stay PROPER and, as we have learned from our toy-model, this requires Holes which make *a partial* ditch-filling possible. The point here is that a ditch filling which is only PARTIAL, is something which can be arranged to be PROPER; see here, for instance, how in the context of the figure 4.2 we have

$$(5.22.0) \quad \lim_{n \rightarrow \infty} T(n) \in \text{LIM } M_2(f) \times B^{N+1} \times \{z = +\varepsilon\}, \quad \text{with } B^{N+1} = I \times B^N,$$

where the RHS of the formula above lives at infinity. But then, in order to get the (5.22.0), the hole $H_-(n)$, reaching all the way to $z = \varepsilon - \frac{1}{n}$, from the figure 4.1.(B) is necessary.

With things like the DIL in (4.22) in mind, we will eventually build, on the lines of (5.21), a process

$$(5.23) \quad \Theta_I^{N+4}(X^2 - H) - \text{DITCHES} \xrightarrow{Z} S'_b(\widetilde{M}(\Gamma) - H)_I,$$

where $S'_b(\widetilde{M}(\Gamma) - H)_I$ is a smooth $(N+4)$ -manifold which is built by the infinite process Z itself, and which is related to $S'_u(\widetilde{M}(\Gamma) - H)_I$; and, as it turns out a posteriori, it is actually diffeomorphic to it (this is an

easy diffeomorphism, to be compared to the η from (4.26), and see figure 4.3 too). The Z mimicks the (2.23) process of zipping, hence the “ Z ”.

But, unlike what happened in the toy-model, in real life we are faced with the following two technical items closely related to the failure of (5.22) and which vastly complicate things.

(5.24.1) The f LIM $M_2(f)$ contains now the obvious complications of the points $p_{\infty\infty}$. In particular the $p_{\infty\infty}(S)$ ’s make havoc of any normal procedure for getting a locally finite $\Theta^3(fX^2)$ or $\Theta^n(\dots)$.

(5.24.2) We also find now a set of triple points $M^3(f) \subset X^2 \times X^2 \times X^2$.

The presence of the (5.24.1), (5.24.2) make that, when we go to (5.23), then the relatively simple-minded step of (partial) ditch filling, which are pure GSC-preserving inclusions (like in the case of the TOY MODEL) will have to be supplemented with other trykier steps, like for instance the **Ditch-jumping**, and others. All this will be developed in great detail in the ZIPPING-LEMMA below. Again, like in the context of the TOY MODEL, there will be now a VARIANT I (à la lemmas 4.1, 4.3) and then also a REAL-LIFE VARIANT, corresponding to Lemma 4.5 of the TOY-MODEL becomes necessary. But then, for our present REAL-LIFE VARIANT, nothing like the simple-minded \mathcal{R}_∞ from (4.28) is available, and a whole new machinery will have to be put up, the big ZIPPING LEMMA 5.1, the **compactness Lemma** 5.7. and everything in the Sections V, VI, VII of the present paper.

But, as already said, the geometric realization of the zipping is done in the context of the VARIANT I.

And then, it is only in the last Section VII that the REAL-LIFE VARIANT can be solidly implemented.

And only in that REAL LIFE variant case we safely prove the GSC theorem.

Coming back to (5.23), it will be shown that the process Z is such that the cell-complex

$$(5.25) \quad S'_b \widetilde{M}(\Gamma)_I \equiv S'_b(\widetilde{M}(\Gamma)_I - H) + \underbrace{\sum_{H \in (4.7)} D^2(C(H))},$$

is GSC and so is also the REAL LIFE $S_b \widetilde{M}(\Gamma)$. Here, the hard step is to get a GSC object $S'_b \widetilde{M}(\Gamma)$ and then, going from that to the GSC $S_b \widetilde{M}(\Gamma)$ is a much easier step. The passage from the VARIANT I to REAL LIFE requires also a change in dimensions, $N + 4 \implies N + 5$.

And so, we would like to show that $S'_u \widetilde{M}(\Gamma) \stackrel{\text{DIF}}{=} S'_b \widetilde{M}(\Gamma)$, so as to be able to conclude the GSC THEOREM, and here the group action will be necessary. In terms of the diagram from (2.24) we move to the lower level of the **compact** $M(\Gamma)$, the fundamental domain of $\Gamma \times \widetilde{M}(\Gamma) \rightarrow \widetilde{M}(\Gamma)$. We also have to move from Variant I to the REAL LIFE Variant, make use of the uniformly zipping length (see [29]), and compactify the $S'_u M(\Gamma)$. Using these kind of ingredients, it will be shown in Section VII below that, now in the real-life context we have, downstairs to begin with

$$(5.26) \quad S'_u M(\Gamma) \stackrel{\text{DIF}}{=} S'_b M(\Gamma).$$

The S'_u, S'_b are functorial and so, by taking the universal cover of (5.26) what we finally get is, now upstairs

$$(5.26.1) \quad S'_u \widetilde{M}(\Gamma) \stackrel{\text{DIF}}{=} S'_b \widetilde{M}(\Gamma),$$

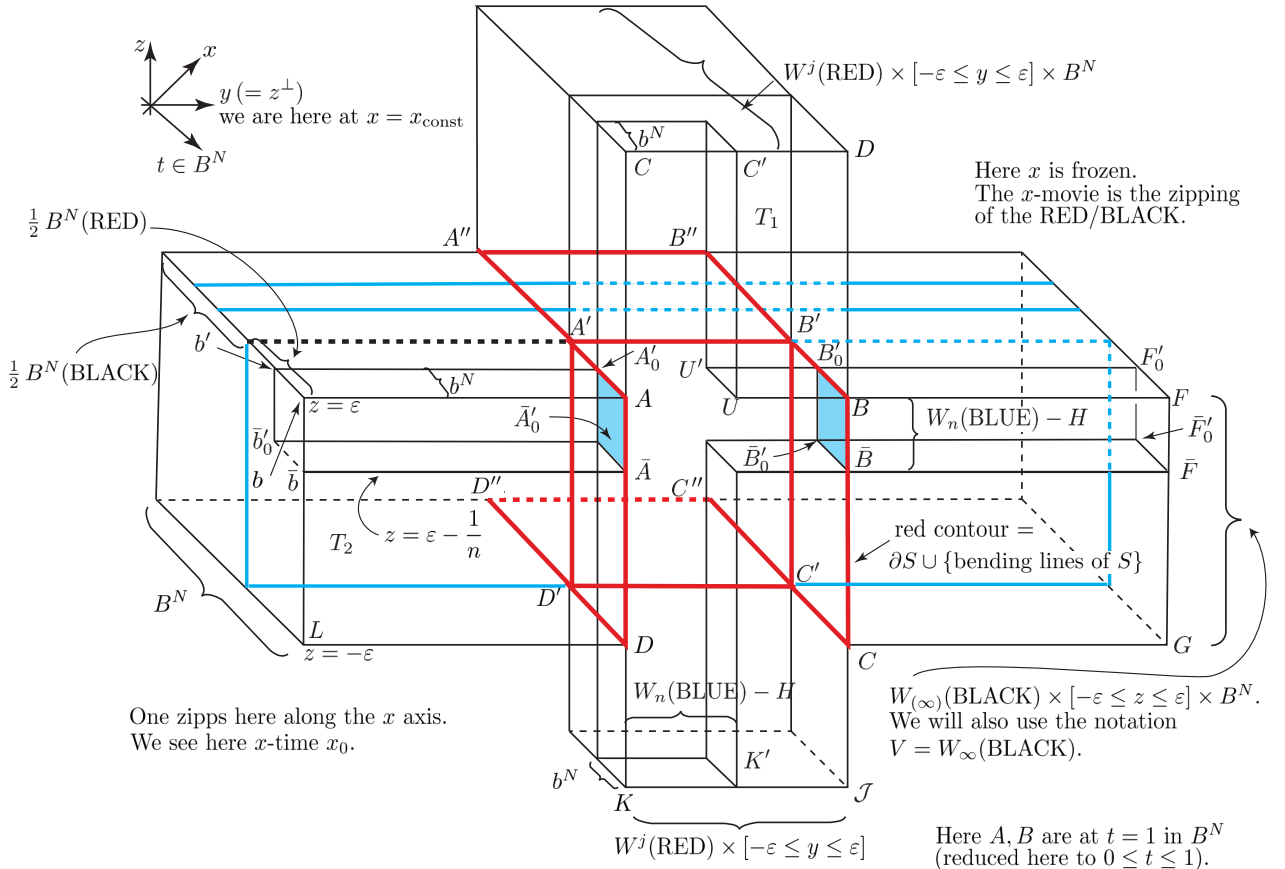
proving that the LHS of this formula is GSC too. The fact that the S'_u, S'_b have functorial properties is essential for our whole approach. They are actually functors, of sorts. And from that view point it is important that we can construct the $S'_u M(\Gamma), S'_b M(\Gamma)$ directly downstairs, not just as quotients of the $S'_u \widetilde{M}(\Gamma), S'_b \widetilde{M}(\Gamma)$, via Γ . Important insights would get lost otherwise, and the whole functoriality issue might become opaque.

Now, from (5.26.1) there is a short argument leading to the conclusion of our GSC THEOREM, in its full glory, as it is stated in Section III. Now, the $S_u \widetilde{M}(\Gamma)$ comes naturally endowed with a free action

$$\Gamma \times S_u \widetilde{M}(\Gamma) \rightarrow S_u \widetilde{M}(\Gamma)$$

and, *if* this action would be co-compact, we could already conclude that $\Gamma \in \text{QSF}$, and this would be the end of our story. But life is not that simple, the action above is not co-compact, the $S_u M(\Gamma) = S_u \widetilde{M}(\Gamma)/\Gamma$ is NOT compact, the only compact space which comes in for free in our story is the very initial $M(\Gamma)$, the singular 3-manifold presenting our Γ as its $\pi_1 M(\Gamma)$. So, in order to get from $S_u \widetilde{M}(\Gamma)_{\text{II}} \in \text{GSC}$ to $\Gamma \in \text{QSF}$, a third paper in this series will still be necessary, the TRILOGY III. And then it will be $S_u \widetilde{M}(\Gamma) \in \text{GSC}$, rather than $S'_u \widetilde{M}(\Gamma) \in \text{GSC}$ which will have been used. [Remember that the first paper in this trilogy is [29], while the present one is the second.]

END of the long prentice. □



LEGEND: The RED contour $[A, A', A'', B'', B', B, C, C', C'', D'', D', D, A]$ is the boundary of a hyper-surface S splitting, in high dimensions, $W(\text{BLACK})$ from $W^j(\text{RED})$. For typographical reasons, here it is 2-dimensional.

Figure 5.4.

This is an illustration for the formulae (5.27.1) to (5.27.3). The *splitting surface* S , splits the $\{\text{thickened } W(\text{BLACK}) \cup W^j(\text{RED})\}$ into $T_1 \cup T_2$. We forget here about the I of $\Theta^4(\dots, \mathcal{R})$.

We make now the definition of the DITCHES completely precise, using the toy-model as a guide, when it is suitable. So we start this now with (5.27.1) below.

(5.27.1) To begin with, once DITCHES \subset {complementary walls}, there are two distinct cases to be considered, namely

•) DITCHES CONCERNING two complementary walls, typically $(W_{(\infty)}(\text{BLACK})(\text{call it } V)) \pitchfork W^i(\text{RED} \cap H^0)$.

••) DITCHES CONCERNING $W(\text{complementary}) \pitchfork W(\text{non-complementary})$.

Here are some additional explanations concerning the (5.27.1)-•) which is illustrated in figure 5.4. What we see explicitly displayed is (with $u \in I$ and $x \in R$ (part of (x, y, z)) frozen), the following SPLITTING

$$(5.27.2) \quad \underbrace{[-\varepsilon \leq y \leq \varepsilon] \cup [-\varepsilon \leq z \leq \varepsilon]}_{\substack{\text{there is a small piece of} \\ \{\text{thickened } V(\text{BLACK}) \cup W(\text{RED})\} \cap R^2(u, z)}} \times B^N \equiv \{[-\varepsilon \leq y \leq \varepsilon] \times B^N - ([-\varepsilon \leq y \leq \varepsilon] \times \\ \times [-\varepsilon \leq z \leq \varepsilon]) \times \frac{1}{2} B^N(\text{BLACK})\} \cup_S \{[-\varepsilon \leq z \leq \varepsilon] \times B^N - ([-\varepsilon \leq y \leq \varepsilon] \times [-\varepsilon \leq z \leq \varepsilon]) \times \frac{1}{2} B^N(\text{RED})\},$$

where S (The SPLITTING SURFACE, also called S -wound) $= (([-\varepsilon \leq y \leq \varepsilon] \cap [-\varepsilon \leq z \leq \varepsilon]) \times B^{N-1}) \cup ((z = \pm\varepsilon) \times [-\varepsilon \leq y \leq \varepsilon] \times \frac{1}{2} B^N(\text{BLACK})) \cup ((y = \pm\varepsilon) \times (-\varepsilon \leq z \leq \varepsilon) \times \frac{1}{2} B^N(\text{RED}))$, and here $B^N (= I \ni t$ in our drawing) $= \frac{1}{2} B^N(\text{BLACK}) \cup \frac{1}{2} B^N(\text{RED})$ and $\frac{1}{2} B^N(\text{RED}) \cap \frac{1}{2} B^N(\text{BLACK}) = B^{N-1}$. End of (5.27.2)

When we go to REAL LIFE, then this splitting surface S should, to begin with, be thickened to dimension $N + 1$, the factor I still ignored. Next, both to the object in (5.27.1) and to S one should add the factor $R \ni x$; so, the dimensions will climb to $N + 3$ for the LHS of (5.27.2), respectively to $N + 2$ for S . Finally, the I is added back too, and then our dimensions become $N + 4$ and $N + 3$.

With these things, we look now into the •) of (5.27.1). Here the ditches are $1/2 B^N \subset B^N$, and the zipping $V + W^i \Rightarrow V \cup W^i$ takes place as already said, before any (BLUE W_n)-action. Forgetting about the dimensions ($x, u \in I$), at the level of the zipped $\Theta^3(V \cup W^i) \times B^N$ the two pieces $\Theta^3(V) \times B^{N+1}$ and $\Theta^3(W^i) \times B^{N+1}$ are separated by a splitting hypersurface S , like in the figure 5.4. Each is supposed to dwell inside the DITCH of the other, in a completely symmetrical manner, topologically speaking, at least; metrically, it may be convenient sometimes to break this symmetry. Our formula (5.27.2) should express how the zipped $V \cup W^i$ looks like, before any zipping of $W(\text{BLUE})$.

One should also keep in mind that in (5.27.2) the two halves are not glued along S . The S -space is there to be filled with the DITCH-FILLING material.

Since figure 5.4, as it stands, has also a $W_n(\text{BLUE})$ contribution, necessary for our further purposes, we have also added, for the convenience of the reader, the figure 5.4-bis where the details concerning $W(\text{BLUE})$, are dropped for the purpose of clarity.

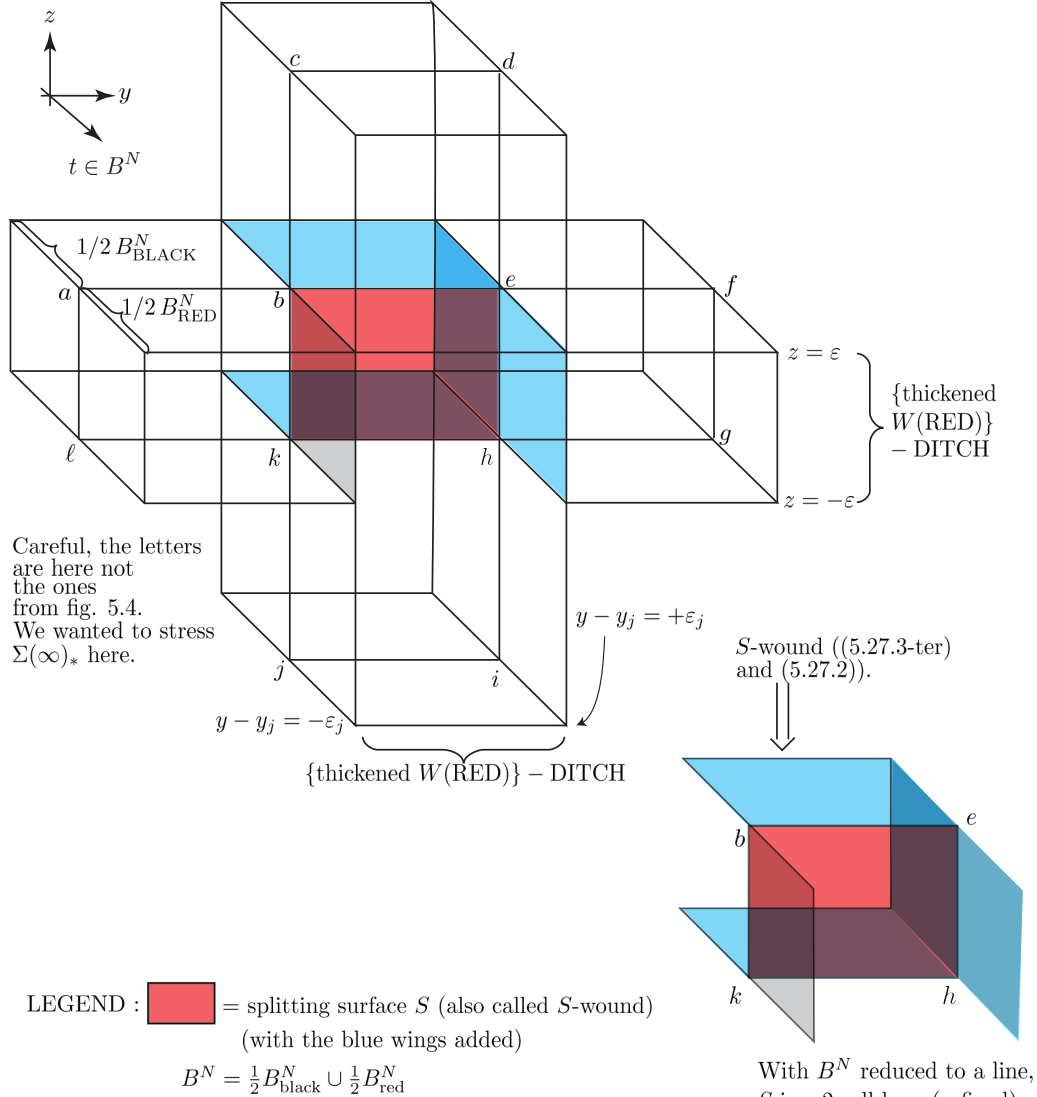


Figure 5.4-bis

We illustrate here the geometric realization, in dimension $N + 4$ of the zipping together of two complementary walls. This figure should be compared of course with 5.4. The big cross $[a, b, c, \dots, j, k, \ell]$ is a piece of $\Sigma(\infty)$, reconstructed now at the level of $S_b^{(l)}(\widetilde{M}(\Gamma)(-H))$. It is made out of pieces $h(1)$ and $h(3)$. For instance, when the shaded S -wound gets a bit thickened, then it consists of $h(3)$ (BLACK/RED). The successive systems of handles $h(\varepsilon)$ are explained in the main text in (5.11), (5.11.1), (5.11.2), (5.27.4), in connection with the GEOMETRIC REALIZATION OF THE ZIPPING.

(5.27.3) We move now to the item $\bullet\bullet$ in (5.27.1), and this time the DITCHES will use the $b^N(q)$ in figure 5.2-(D).

So, to begin with, with a generic $q \in \{\text{shaded areas of } V + W^i \text{ in the smooth figure 5.2}\}$, we take now, for zipping a complementary wall with a non-complementary one, when outside of the triple points, with $\text{DITCH} = \bigcup_{q \in V \cap (W^i \times [-\varepsilon_i, \varepsilon_i])} q \times [-\varepsilon, \varepsilon] \times b^N(q) \subset (V \times [-\varepsilon, \varepsilon] + W^i \times [-\varepsilon_i, \varepsilon_i]) \times B^N \subset \Theta^3(X^2 - H) \times B^N$ (forgetting

here about the $\times I$ of $\Theta^4(\dots, \mathcal{R})$). [Of course, the ditches really only concern the complementary walls V and they are drilled in the supplementary dimensions B^N .]

This formula is valid before the zipping $V + W^i \Rightarrow V \cup W^i$, OR then where this zipping is not concerned. Then, it becomes

(5.27.3-bis) After the zipping $V + W^i \Rightarrow V \cup W^i$ of two complementary walls which changes figure 5.2, into 5.3, we consider again

$$q \in \{\text{shaded area}\} = [V \cup W^i] \cap [x_i - \varepsilon_i \leq x_0 \leq x_i + \varepsilon_i],$$

and the retraction $U(x_0) \xrightarrow{r} (V \cup W^i) | x_0$ of possibly singular fiber, suggested in figure 5.3.(B) and which defines the transversal orientation, from $+\varepsilon$ to $-\varepsilon$. With this, in the present situation, we will have

$$\text{DITCH} = \underbrace{\bigcup_{x_0 \in [x_j - \varepsilon_j, x_j + \varepsilon_j]} U(x_0) \times b^N}_{q \in \{\text{shaded area, fig. (5.3-(A))}\}} = \bigcup_{q \in \{\text{shaded area, fig. (5.3-(A))}\}} q \times "[-\varepsilon, \varepsilon]" \times b^N(q),$$

where the triple point is not concerned here (see figure 5.3-(B)).

When, after $V + W^j \Rightarrow V \cup W^j$, which as already said, for any individual triple point takes place BEFORE the BLUE W_n -action, we move to the next zipping

$$V \cup W^j + (W_n(\text{BLUE}) - H) \Longrightarrow V \cup W^j \cup W_n,$$

where the b^N is used now when room for $W_n - H$ is to be carved out of $V \cup W^j$. The first, initial zipping, heals the

(5.27.3-ter)

$$\{S\text{-wound}\} \equiv \{\text{the splitting surface } S\}, \text{ (see the little shaded drawing accompanying figure 5.4-bis)}$$

leaving, for the purpose of the second zipping the

$$(5.27.3-IV) \quad \{S\text{-scar}\} \equiv S \cap (W_n(\text{BLUE}) - H) \times b^N \text{ (see figure 5.4), shaded blue in the drawing 5.4.}$$

[There are NO indentations at the level of the S -wound only filling of empty space. So, at the time of the second zipping, with $W(\text{BLUE})$, the S -scar is filled with ditch filling material, and only its outer border is brushed by the DITCH-JUMPING STEP of $W(\text{BLUE})$ (to be described). Figure 5.11-bis explains in detail the action, at that point. We will come to it, in due time.]

We use the generic notation $h(3)$ for the system of handles corresponding to the ditch-filling material. With this, at the end of the (geometric realization of the) zipping, we will find, inside the grand geometric intersection matrix, the incidence relation

(5.27.4) $\partial h^{\lambda+1}(3)$ (COMING LATER IN THE TEMPORAL ORDER OF THE ZIPPING PROCESS, (see (5.30) below)) $\cdot \delta h^\lambda(3)$ (COMING EARLIER IN THE ZIPPING) $\neq 0$, while certainly we also have

$$\partial h^{\lambda+1}(3)(\text{EARLY}) \cdot \delta h^\lambda(3)(\text{LATE}) = 0.$$

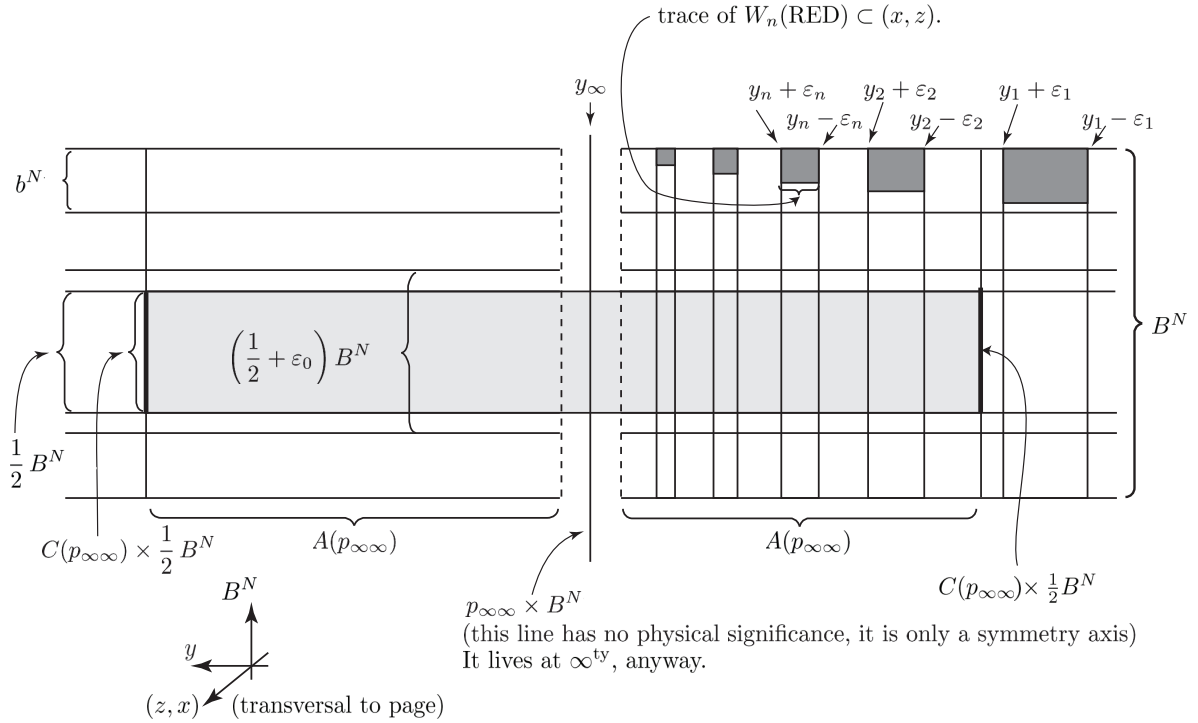
[In the context of figure 5.6-(C), we count the RED/BLACK zippings at A and B as simultaneous.] All this is compatible with GSC. End of (5.27.4).

The figure 5.5 to which we turn next, should help vizualizing the high dimensional geometry of the ditches, in the neighbourhood of the points $p_{\infty\infty}$. Here are EXPLANATIONS for this figure. Both figures 5.5-(A)

and (B) refer to the contribution of a $W_{(\infty)}$ (BLACK) to the $S'_u \widetilde{M}(\Gamma)_I$, $S'_u (\widetilde{M}(\Gamma) - H)_I$, $S'_b (\widetilde{M}(\Gamma) - H)_I$ (to be formally introduced in Lemma 5.1). The coordinate system (x, y, z) is here the same as in the figures 2.4-(B), (C). In 5.5-(A) we see

$$(5.28) \quad \left\{ A(p_{\infty\infty}) \times \left(\frac{1}{2} + \varepsilon_0 \right) B^N, \text{ site of the action (3.34)} \right\} \subset \{ W_{\infty}(\text{BLACK}) \times B^N \\ - \{ \text{the indentation } b^N \subset B^N, \text{ site of the zipping action} \} \},$$

making the SUPER-HEALING action $S'_u \Rightarrow S_u$ in (3.34) and the zipping, independent of each other. We forget here, like we often do, the factor I coming with $\Theta^4(\dots, \mathcal{R})$.



LEGEND:

- = $D^2(H(p_{\infty\infty})) \times \frac{1}{2} B^N$, it is glued to the rest only along $C(p_{\infty\infty}) \times \frac{1}{2} B^N$,
- = $b^N(\ell) \times [y_\ell - \varepsilon_\ell, y_\ell + \varepsilon_\ell]$, where $b^N(\ell) \subset b^N$, $\lim_{\ell \rightarrow \infty} \text{diam } b^N(\ell) = 0$, $\lim_{\ell \rightarrow \infty} \varepsilon_\ell = 0$.
indentation

Our is essentially the trace of DITCH (ℓ) , for $W_{\infty}(\text{BLACK}) = V$ and $W(\text{RED})_n$, in the context of the \bullet from (5.27.1). So $W(\text{RED})$ is here $W(\text{RED} \cap H^0)$.

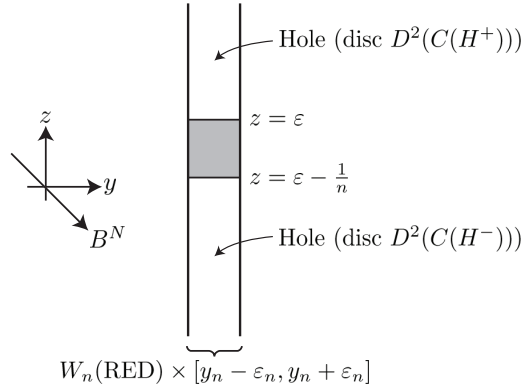
Figure 5.5-(A)

This figure should be compared to figure 3.3-(A), but now in high dimensions and with z deleted, for typographical purposes. This figure is a slice through $W_{\infty}(\text{BLACK}) \times [-\varepsilon \leq z \leq \varepsilon] \times B^N$ at $x = x_{\infty}$, $z = 0$. It may concern a $p_{\infty\infty}$ (proper) or a $p_{\infty\infty}(S)$. We are here in the same context as in figure 3.3, but now in higher dimensions. Both this figure and 5.5-(B) are in

the context of the VARIANT I. Then, compare this figure with 4.2, from the TOY MODEL, with our present y_∞ playing the role of x_∞ in figure 4.2. Then, unlike what had happened in figure 4.2, the ditches are here small in their y -width $[y_n - \varepsilon_n, y_n + \varepsilon_n]$ replacing now the x -width from the figure 4.2. But then figure 3.3-(A) and the present one should also be compared to figure 2.4-(B). The y is common, and from the perspective of figure 2.4-(B) our y_∞ is actually (x_∞, y_∞) , with $p_{\infty\infty}(x_\infty, y_\infty)$ and with $p_{n,\infty}, p_{n+1,\infty}, \dots$ along y converging to $p_{\infty\infty}$. So the $[y_n - \varepsilon_n \leq y \leq y_n + \varepsilon_n]$ are thickened $W^n(\text{RED})$'s (actually $(W^N(\text{RED}) - H^0)$'s. Of course, the ditches start at $x_\infty + \varphi_n$, $\varphi_n \rightarrow 0$. And y_n corresponds to $p_{n\infty}$. When we are in the case $p_{\infty\infty}(\text{proper})$, then, except for $C(p_{\infty\infty})$ we are here far from singularities, and so we can happily forget about the factor I coming from $\Theta^4(\dots, \mathcal{R})$. In the case $p_{\infty\infty}(S)$ then the $p_{n\infty}$'s are $p_{n\infty}(S)$'s which are now immortal singularities. They are placed at (x_∞, y_n) , $n \rightarrow \infty$. The $C(p_{\infty\infty}(S))$ avoids them, this can be easily arranged. We still ignore the factor I , for typographical convenience. This figure 5.5, concerns the zipping of

$$W_\infty(\text{BLACK})(\text{complementary}) / (W(\text{RED} - H^0) - H)(\text{NON-complementary}),$$

and here is how the details $W_n(\text{RED})$ should be read:



In terms of B^N , in our figure 5.5, the vertical spaces of $W_n(\text{RED})$ which are non shaded (not b^N) are purely fictitious.

In figure 5.5-(B), since we are at the level $S'_\varepsilon(\widetilde{M}(\Gamma) - H)_I$ the square $[A_-, A_+, B_-, B_+] = [-\varepsilon \leq z \leq \varepsilon] \times B^N$ (in the coordinates from the figure) has actually been deleted, but our (B) is supposed to continue, with less details beyond $y = y_\infty$ to $y > y_\infty$; so does also the 2-handle $D^2(H(p_{\infty\infty})) \times [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}] \times \frac{1}{2} B^N$. Notice that, at the end of the collapse + dilatation in (3.34), when the dust has settled then, in the case when $p_{\infty\infty} = p_{\infty\infty}(\text{proper})$, what we get locally is a diffeomorphic model for $S_u \widetilde{M}(\Gamma)_I$, where instead of putting back the whole of $[A_-, A_+, B_-, B_+] = p_{\infty\infty} \times [-\varepsilon \leq z \leq \varepsilon] \times B^N$ we only put back $p_{\infty\infty} \times [-\varepsilon < z(c) \leq z \leq z(b) < \varepsilon] \times (\frac{1}{2} + \varepsilon) B^N$, which leaves out the $p_{\infty\infty} \times [-\varepsilon, \varepsilon] \times b^N$. But this alone is still not of much help when dealing with the sting of $p_{\infty\infty}$. To see how we will actually proceed, watch the zipping Lemma 5.1 and its various sequels in this section, up to and including the “Final arguments”. [The sting mentioned means unwanted accumulations at finite distance.]

For $S'_u \widetilde{M}(\Gamma)_I$, the contribution of each $W^i(\text{RED})$ adds to what we can see already in (B), the full contribution

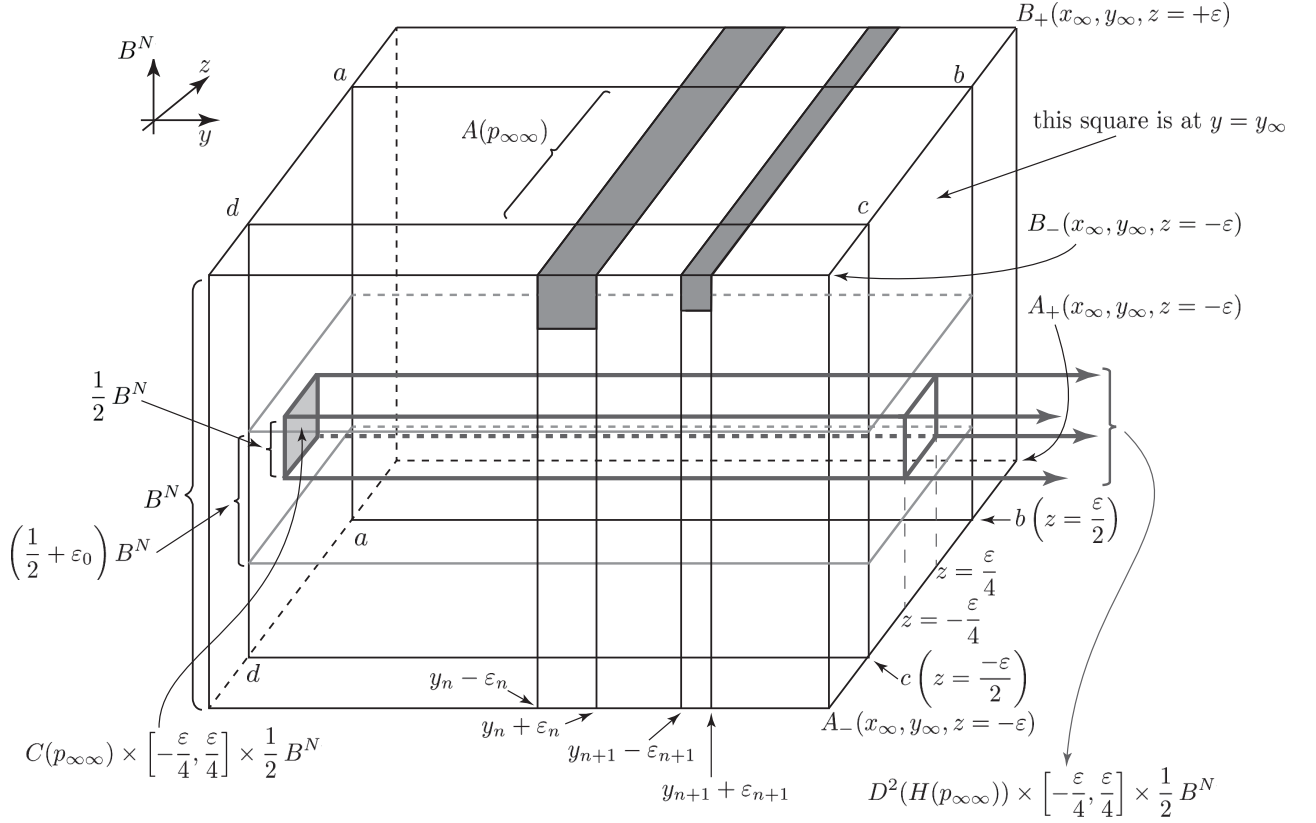


Figure 5.5-(B)

Another view of the RHS of figure 5.5-(A), with the axis z added. The $[a, b, c, d]$ is here like in the figure 3.3. The $[A_-, A_+, B_-, B_+]$ lives at infinity. It is a legacy of $p_{\infty\infty}$.

$$(*_1) \quad [y_i - \varepsilon_i \leq y \leq y_i + \varepsilon_i] \cap \{|z| \geq \varepsilon\} \times B^N.$$

But then, finally, both (A) and (B) are at $x = x_{\infty}$, hence the contribution of $\partial \sum(\infty)$ is to be deleted. Forgetting about the $(*_1)$ above, what this deletion means for (B) is that the whole lateral surface $(z = \pm \varepsilon) \times B^N$ is to go, at $x = x_{\infty}$.

What our figure 5.5 displays is the following fact. At the level of $\Theta^{N+4}(X^2 - H)_I$ – DITCHES or $S'_b(\widetilde{M}(\Gamma) - H)_I$, when localized at $p_{\infty\infty}$ (proper or S) $\subset W_{(\infty)}(\text{BLACK}) = \{\text{let us say the } V \text{ in model (3.8.i)}\}$, then the thickened V contains the region

$$(5.28.1) \quad [A(p_{\infty\infty}) \times B^N - \{\text{indentations}\}] \supset \left\{ C(H(p_{\infty\infty})) \times \frac{1}{2} B^N, \right.$$

which is the attaching zone of the 2-handle $D^2(H(p_{\infty\infty})) \times \frac{1}{2} B^N \}$,

where the **indentations** (= DITCHES) make room for the infinite set of zippings of V . In the geometric realization of the zipping, this means ditch-filling operations, involving first $V = W_{\infty}(\text{BLACK})$ and the

infinite family $W(\text{RED})^\ell$, this is what we see in figure 5.5-(A), which is very much like an infinite sequence of figures 5.4-bis (or 5.4 WITHOUT the BLUE contributions). Infinitely many WOUNDS, smaller and smaller when $\ell \rightarrow \infty$ get created. The $W(\text{RED})^\ell$ lives here in the DITCH (ℓ), figure 5.5-(A).

Then next, outside the world of figures 5.5-(A), (B) comes the zipping of the infinite family $W(\text{BLUE})_j$ with the $V \cup \sum_{\ell} W(\text{RED})^\ell$. This is now like adding the BLUE walls in figure 5.4. We have now S -SCARS, and an infinite family of much smaller ditches NOT touching the previous ones (AND REMEMBER: NO DITCHES ON TOP OF TRIPLE POINTS!).

But the point is that all $b^N(\ell)$'s and $b^N(j)$'s have diameters going very fast to zero, when $\ell \rightarrow \infty, j \rightarrow \infty$. NONE of our zippings, or to be precise their geometric realization in high d , gets entangled with the 2-handle $D^2(H(p_{\infty\infty})) \times \frac{1}{2} B^N$. We have, also, a diffeomorphism

$$A(p_{\infty\infty}) \times B^N - \{\text{indentations}\} \underset{\text{DIF}}{=} A(p_{\infty\infty}) \times B^N .$$

And the indentations protect our $C(H(p_{\infty\infty})) \times \frac{1}{2} B^{N+1}$ from contacts with the zipping process Z .

Now, a similar protection is **not** necessary for $C(H)$ (figure 5.1), since as we can see in the figure 5.1, $M_2(f) \cap C(H) = \emptyset$ (to see this, compare actually the figures 5.1 and 2.4-(A)). Details like $[g, c, r]$ in figure 5.1, fit now into the

{doubly shaded indentations of a figure like 5.5, but for $p_{\infty\infty}(S) \in W(m)^*, m < n$,

where $W(m)^*$ overflows, not represented in figure 5.1}.

Also, figure 2.4-(B) shows that, contrary to the $M_2(f) \cap C(H) = \emptyset$ above, we have

$$C(H(p_{\infty\infty})) \cap M_2(f) \neq \emptyset .$$

Important Remark. Notice that, while in figure 5.4 the $W(\text{RED})/W_{(\infty)}(\text{BLACK})$ played a symmetrical role, in 5.5 this is no longer the case, although both refer to (5.27.1).

But in figure 5.5 the $p_{\infty\infty}$ is there, and only the $W_{(\infty)}(\text{BLACK})$ carry $p_{\infty\infty}$'s. This forces on us some **metrical adjustments** when it comes to the ZIPPING of complementary walls (5.27.1). We will come back to it.

After these preliminaries, we move now to the

GEOMETRIC REALIZATION OF THE ZIPPING, in high dimensions. This discussion will occupy all the rest of this section, of which it is the core. There will be actually two levels for the discussion in question, **upstairs**, at level $\bar{M}(\Gamma)$ and **downstairs**, at level $M(\Gamma)$. For the time being we will function upstairs and it will be signaled when we will move downstairs.

The central piece of the present discussion will be the ZIPPING LEMMA 5.1, for which we give now some PRELIMINARIES. But, before anything else, here is the **abstract zipping** which we will strive to imitate geometrically, by steps which all have to be GSC-preserving, in dimension $N + 4$, making use of the factor B^N in formulae (5.10.1), (5.13.1), ... Our GEOMETRIC REALIZATION OF THE ZIPPING will proceed, first, in the context of VARIANT I, and only afterwards will we be able to present it in the REAL LIFE context. We will start with the equivariant zipping (2.23), which we will restrict now to a version with Holes deleted

$$(5.28.1) \quad X^2 - H \rightarrow \text{---} \rightarrow fX^2 - H \quad (\text{infinitely many } 0(i) \text{ steps}),$$

and “ $-H$ ” means here with ALL Holes (5.7) deleted and WITHOUT ANY compensating discs at all, either. In an equivariant manner, the (5.28.1) will be broken into two successive parts

$$(5.28.2) \quad X^2 - H \xrightarrow{\text{ACYCLIC PART}} f(Q)(X^2 - H) \xrightarrow{O(3)\text{-part}} fX^2 - H$$

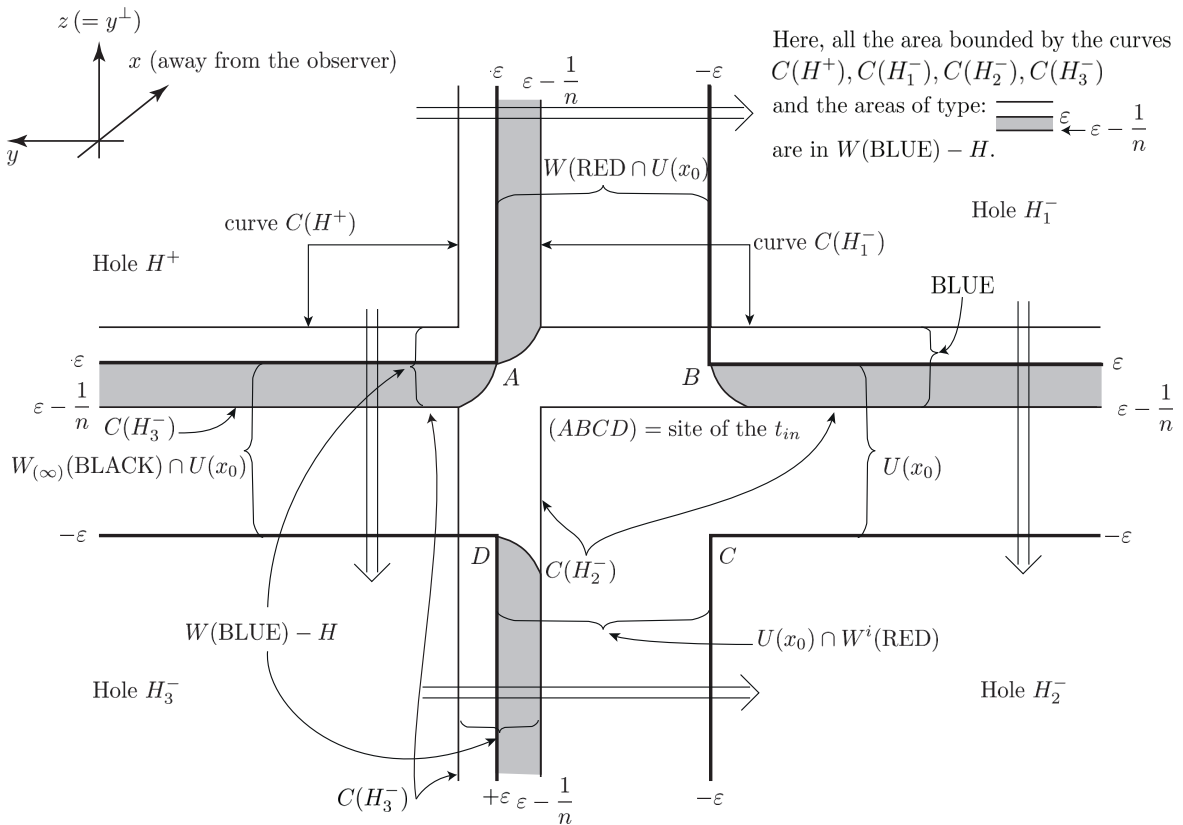
where:

•) The acyclic part consist of $0(0), 0(1), 0(2)$ -moves (see figure 1.2), so that the map $f(Q)$ below is a homotopy equivalence

$$X^2 - H \xrightarrow{f(Q)} f(Q)(X^2 - H).$$

The zipping paths $\lambda(x, y)$, like in Theorem 2.1 are only concerned with $f(Q)$, and it is here that the ZIPPING LENGTH gets measured too.

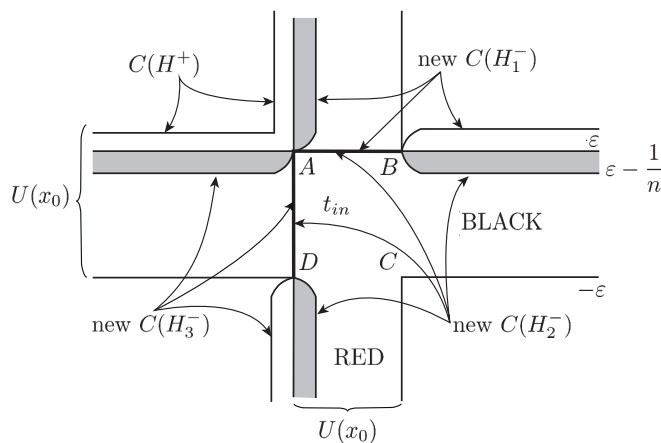
••) The $O(3)$ -part in (5.28.2) consists of infinitely many $0(3)$'s, each homotopy equivalent to the addition of a 2-cell. They are far from each other, and also far from immortal singularities, from $p_{\infty\infty}$'s, from $M_3(f)$ or from any other special spots. And they do not accumulate at finite distance either. Remember, also, that at level $S'_u(\widetilde{M}(\Gamma) - H)$ the $O(3)$ -moves become smooth additions of 2-handles, since we are in dimensions > 4 . The B^N , $N \geq 1$ producing this, also removes the R -dependence, making our construction equivariant. All this, is essential.



LEGEND: = ditch filling material, contribution of $W_n(\text{BLUE}) - H$ seen here as projected by the $\{\text{extension of the } f \text{ from (2.1) to } \Theta^3(X^2) \xrightarrow{f} \widetilde{M}(\Gamma)\}$ to the plane $x = x_0$ of this figure. This is, also, the projection of the partially filled DITCH. For the fate of the unshaded part of $U(x_0) \cap (W(\text{BLUE}) - H)$, see what will be coming next; \Rightarrow = canonical transversal orientation for $U(x_0)$. The source of the projection above is $S'_u(\widetilde{M}(\Gamma) - H)$.

Figure 5.6-(A).

We are here at some level $x = x_0$, inside $\widetilde{M}(\Gamma)$, and we can see the relative situations of $U(x_0) \subset V \cup W^i$ and $W_n(\text{BLUE}) - H$, in the neighbourhood of the triple point t_{in} , before any modulations concerning the special BLUE 1-handles are in effect. The present 2^d image is a projection by the map f , which superposes the two contributions involved. The $U \approx V(\text{BLACK}) \cup W^i(\text{RED})$ and the $W_n(\text{BLUE})$ have not yet been zipped together and, as a preparation for the modulations which are to come, we will start by collapsing the $W_n(\text{BLUE}) - H$ into the thinner version presented in figure 5.6.(B). But the important modulation is the one described schematically by the transformation Fig. 5.6.(B) \Rightarrow Fig. 5.8. This is a transformation of the special BLUE handles $[AB] + [AD]$, which also drags along the four curves $C = \partial(\text{Hole})$, without changing their contribution to the geometric intersection matrix which comes with (5.11), (5.11.1), (5.11.2).



Here t_{in} = triple point BLACK \cap RED \cap BLUE. The $[AB] + [AD]$ are the *special BLUE ARCS* (or *1-handles*).

Figure 5.6-(B).

This is a more physically manageable version of figure 5.6-(A), gotten by some collapsing. The 5.6-(C) is a version of 5.6-(B), where more details have been thrown in.

On the lines of (5.10.1), we start now with the following object, which is a cell-complex, and which certainly is GSC

$$(5.29) \quad [\Theta^{N+4}(X^2 - H)_I - \text{DITCHES}] + \sum_{H \in (5.7)} D^2(C(H)).$$

For the time being we will unroll things upstairs.

The $S_0 \equiv [\Theta^{N+4}(X^2 - H)_I - \text{DITCHES}]$, which occurs in (5.29) is a smooth $(N + 4)$ -manifold related to $\Theta^{N+4}(X^2 - H)$ by a simple-minded diffeomorphism. Remember that “ $-H$ ” means with all Holes, including the $H(p_{\infty\infty})$ ’s deleted, and with the compensating $D^2(C(H))$ ’s deleted too.

The (5.29) comes with an initial cell-decomposition $h(1)$ like in (5.11), which, as said, is GSC (see (5.12)). Starting from S_0 the Lemma 5.1 below will develop an infinite process Z of successive steps and here “ Z ” stands for “zipping”, our infinite process *is* actually the geometric realization of zipping, in high dimensions. During the process Z (see (5.30)) some of the steps involved are inclusions $S_n \subset S_{n+1}$. These steps will add new handles, coming in addition to the system $h(1)$ from (5.11), and here one should remember that $h(1)$

contains already all the fins, with their rims, truncated at finite distance. And here is the list of handles to be added on top of $h(1)$, in order to progress from $\Theta^{N+4}(X^2 - H)_I$ -DITCHES through the forthcoming infinite system of transformations from (5.30) below, leading eventually to the all-important object $S'_b(\widehat{M}(\Gamma) - H)_I$ from (5.32):

(5.29.1) There will be a system of handles $h(2)$, which are added during an infinite subsequence $S_{n_1} \subset S_{n_1+1}, S_{n_2} \subset S_{n_2+1}, \dots$ of the infinite sequence of inclusions/and transformations

$$\Theta^{n+4}(X^2 - H)_I\text{-DITCHES} \equiv S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow \dots,$$

building up the process Z from (5.30). Our steps $S_{n_i} \subset S_{n_i+1}$ send to infinity, via PROPER Whitehead dilatations, the

$$(\partial\Sigma(\infty)_*^\wedge - \{\text{ALL the } p_{\infty\infty} \times [-\varepsilon, \varepsilon] \text{ (which get already deleted in } S_0)\}) \supset \text{the rims of fins (see (3.22))}.$$

[IMPORTANT REMARK. Consider now a paradigmatic figure like 2.4-(B) with the corresponding figure 3.2-(A, B), corresponding to $\Theta^3(fX^2)^{(\cdot)}$. Then with the infinitely many BLUE lines accumulating on $\partial\Sigma(\infty)_*^\wedge$, especially when $p_{\infty\infty} \times [-\varepsilon, \varepsilon]$ is deleted it seems very awkward to use our infinite system of Whitehead dilatations $S_{n_i} \subset S_{n_i+1}$. But at the level of Z (5.30), the accumulation process is potential at the time when the $S_{n_i} \subset S_{n_i+1}$ are performed. So, there is NO INFINITE TRAP at level $S'_b(\widehat{M}(\Gamma))$ which will really interest us.]

(5.29.2) Then there will be the main bulk of inclusions $S_n \subset S_{n+1}$ of the process Z (5.30). Each of these inclusions consists of two parts. First, analogously to some of the things which happen during (4.13) in the context of the toy-model, pieces of our S_0 are being send “in the ditch”, meaning here, of course, in the interiors of the DITCH.

This part is, essentially, an isotopy which bends S_0 , so as to bring it to enter the ditch, via parts of it. Then, comes a real inclusion map which adds more stuff, binding now S_0 , inside the ditch, to $\partial\text{DITCH} \cap \partial S_0$. This means throwing in a new system of handles (mainly of indices $\lambda \geq 1$) and/or cells. We call this system $h(3)$. We will call this the “ditch-filling material”.

VERY IMPORTANTLY, like in the context of figure 4.2 from the TOY MODEL, as soon as we are in the dangerous zone \overline{N}_∞ , the **ditch filling is only partial**. And here, for the dangerous zone \overline{N}_∞^2 , see figure 2.4-(B). This partial filling means exactly the following, remember: in terms of formulae (5.27.1), (5.27.2) (which is to be supplemented by the figure 5.3), when a piece of S_0 (+ the corresponding ditch-filling material $h(3)$) goes into the ditch, this occupies only the part $[\varepsilon - \frac{1}{n} \leq z \leq \varepsilon]$, when one is in the context $\bullet\bullet$) from (5.27.1), i.e. ii) below. Here is one way of conceiving our operations of sending parts of S_0 into the DITCH. To begin with think of the standard zipping operation $X^2 \rightarrow fX^2$, actually restricted to $X^2 - H \rightarrow fX^2 - H$ like in (5.28.1). Locally speaking, there are here two possible cases, like in (5.27.1),

i) zipping together two complementary walls $W_{(\infty)}$ (BLACK) and $W(\text{RED} \cap H^0)$, (and these contain the DITCHES).

ii) zipping together the non-complementary $W_1 = W(\text{BLUE})$ or $W(\text{RED} - H^0)$ (or rather the $W_1 - H$) to W_2 (complementary). (The non-complementary walls contain the normal HOLES.)

Going now from dimension 2 to $N + 4$, *ideally* (a word to be explained later on), in case i) each of the two complementary walls involved, goes inside the ditch of the other like in (5.27.2) and/or like in figure 5.4 and, for i), this is true without further qualifications, i.e. the word “ideally” may be dropped here. In the case ii), the W_1 (non complementary) $- H$ goes now into the ditch of the W_2 (complementary). All this mimicks the zipping in high d , in the sense that, in high d the result should look, topologically, like $W_1 \cup W_2$ thickened in dimension $N + 4$. Put it also another way, what we just said means, in the context of $\bullet\bullet$) from the (5.27.1), an inclusion map

$$\begin{aligned} \{\text{parts of } S_0, \text{ going in the DITCH}\} \subset \{ \text{a shaded region of the complementary } W \} \times \\ \times [-\varepsilon \leq z \leq \varepsilon] \times (b^N \subset B^N) \} = \text{DITCH}, \end{aligned}$$

and the last piece, between the $\{\dots\}$, makes sense both inside S_0 and inside the $S'_u(\widetilde{M}(\Gamma) - H)_I$.

This should also serve for defining the map j from S_0 to $S'_u(\widetilde{M}(\Gamma) - H)_I$ occurring in (5.33) below. We will come back to the word “ideally” later on.

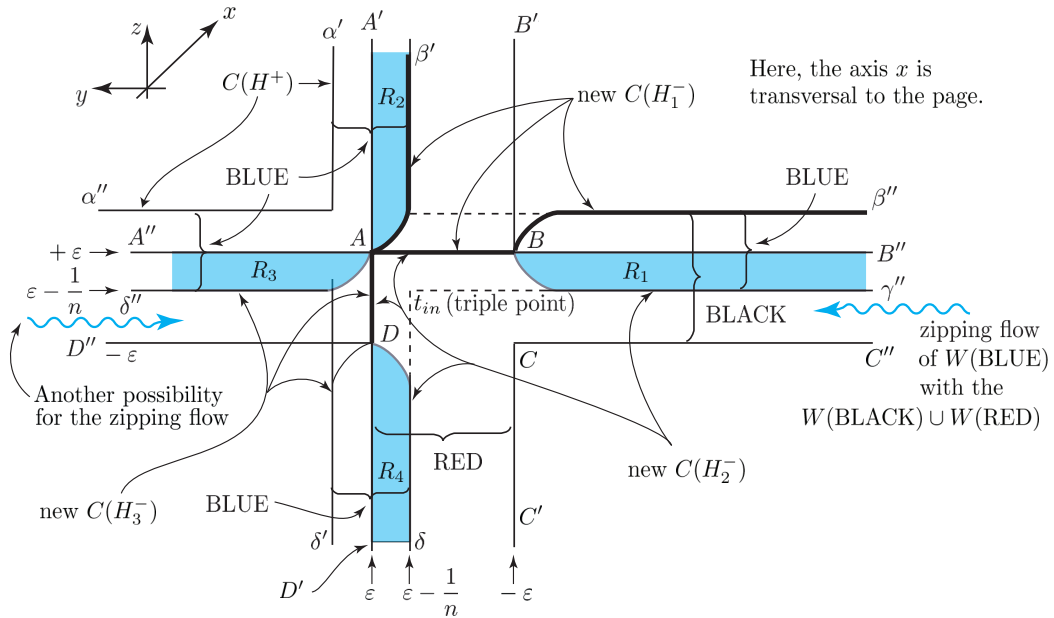
Coming back to our successive handle-system $h(\varepsilon)$, with $1 \leq \varepsilon \leq 3$, for each individual ε we will have that $\partial h^2(\varepsilon) \cdot \delta h^1(\varepsilon) = \text{easy id} + \text{nil}$. Moreover, we will also find that when $\varepsilon' < \varepsilon''$, then $\partial h^2(\varepsilon') \cdot \delta h^1(\varepsilon'') = 0$, while generally speaking we will find that $\partial h^2(\varepsilon'') \cdot \delta h^1(\varepsilon') \neq 0$. With these things, there cannot be any violation of GSC coming from our $h(\varepsilon)$'s. But all this will require proof, particularly so in the case of the $\partial h^2(1) = C(H)$.

Here are some more explanations concerning the conflict with metrizable, mentioned in connection with figure 5.6-(C). To begin with, it is the PARTIAL ditch-filling ($\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$) which is an essential ingredient of our cure of the metrizable problem. And normally this asks that, in terms of a projection on $\Theta^3(fX^2 - H)$, the $W(\text{BLUE}) - H$ should run along $\partial(W(\text{BLACK}) \cup W(\text{RED}))$. But, when we come to the triple points, then things like the special BLUE handles $[BA]$ or $[AD]$ have to plunge through

$$\text{int}((W(\text{BLACK}) \times [-\varepsilon, \varepsilon]) \cap (W(\text{RED}) \times [-\varepsilon, \varepsilon])),$$

because in the figures 5.6-(A, B) there is an x -axis too, running along $W(\text{BLACK}) \cap W(\text{RED})$ and the special BLUE arcs run deep in the interior, far from the boundary. And this precludes any kind of partial DITCH filling. And, for that matter, there are anyway NO DITCHES at the triple points.

So, another gimmick will become our tool now, the DITCH-JUMPING steps



LEGEND: = BLUE, in the DITCH (in dimension $N + 4$), actually in $b^N \subsetneq B^N$.

Figure 5.6-(C)

This is figure 5.6-(B), with more details thrown in. The $[AB]$, $[AD]$ are so-called **special BLUE 1-handles**. The greek cross stands for $W_n(\text{BLUE}) - H$ while the latin cross stands for $W(\text{BLACK})(\text{horizontal}) \cup W(\text{RED})(\text{vertical})$, already zipped together. The shaded areas of the greek cross live in the $\{\text{DITCHES}\} \subset (W(\text{BLACK}) \cup W(\text{RED})) \times [-\varepsilon, \varepsilon] \times b^N$ (see figure 5.4).

There are **no DITCHES** at the level of the square $[ABCD]$, where the triple point $t \in M^3(f)$ lives. Such DITCHES, occupied by the $[AB] + [AD]$, if they would exist, would come into conflict with the **metrizability** of the space S'_b introduced below. In our figure we are at $x = x_n$, with a coordinate axis x going transversally through the plane of our drawing, towards $x = x_\infty$, where $\sum(\infty)_*$ lives.

The COLOURS marked in the figure refer to the local models from (3.8). Specifically, the “RED” and “BLACK” are like in the paradigmatic case (3.8.i). Other combinations are possible for them.

Notice that the whole BLUE cross $[\delta'', \alpha'', \alpha', \beta', \beta'', \gamma'', \delta, \delta']$ (but to have the cross, imagine the dotted lines present too . . . and beyond the cross, in the plane of $W_n(\text{BLUE})$, where it lives, are Holes), collapses into $\{\text{the two special BLUE 1-handles } [AB] + [CD]\} \cup \{\text{the four shaded BLUE areas, which during } Z \text{ will go into the DITCH, namely the } R_{\varepsilon \leq 4} \text{'s}\}$. In the collapse, the dotted lines vanish.

The whole point here is that, as explained above, the special BLUE 1-handles **cannot** be sent into the DITCH. Compare this with our term “ideally” from above. In this figure, things like new $C(H_3^-)$, new $C(H_2^-)$ are part of the $\beta(\partial C(H^-))$'s, while things like $[B', B, B'']$, $[C', C, C'']$ are part of the corresponding $\alpha(\partial C(H^-))$'s. We use here the α, β from (5.36) below, but see the similar partitions from the TOY MODEL too. The $C(H_1^+)$ is both α and β while the new $C(H_1^-)$ which is β is just a variation of the corresponding α which is $[B', B, B'']$.

We move now to the figure 5.6-(C). With respect to the RED ditch inside which the R_2 lives, the present z -axis plays exactly the same role as the axis y plays in figure 4.2; this y is not explicitly drawn in figure 4.2, but the DITCHES stretch along it, like in the formula (4.11.0). Keep this in mind, it is important. Keep also in mind that the present figure 5.6-(C) is gotten from 5.6-(A) by collapsing the BLUE cross on $([BA] \cup [AD]) \cup (R_1 + R_2 + R_3 + R_4)$.

VERY IMPORTANT REMARK. In the jargon of section IV, in all the present and also the next section, we will be constantly in the context of the VARIANT I. Later on **after** all the movements of the ZIPPING LEMMA below will have been performed, then we will move to the context of the REAL-WORLD VARIANT. This will only effectively happen in the section VII. But we do have to unfold the GEOMETRIC REALIZATION OF THE ZIPPING in the framework of VARIANT I, first, **before** we can move on to REAL LIFE.

Before going on, here are some additional EXPLANATIONS CONCERNING the NON-METRIZABILITY issue mentioned above, for instance in connection with figure 5.6-(C). Consider a figure like 4.2, but now, with the $T(n)$'s extending all the way from $-\varepsilon$ to $+\varepsilon$, for all n 's. They would then accumulate on the segments $[\rho_-, \rho_+]$ from figure 4.2.(A), of which only the extreme points ρ_\pm live at infinity, and the open arc (ρ_-, ρ_+) at finite distance. Something similar would happen, if we would use the ditches for housing the special BLUE 1-handles. Imagine now that the Z -process (see (5.30)) would make use of this. If we would still insist that $\lim_{n \rightarrow \infty} S_n + \sum D^2(C(H))$ be GSC, as we shall certainly do, this would oblige us to use for $\lim S_n$ a weak topology (consistent with the topology of the individual S_n) which would no longer be metrizable, the all-important $S'_u(M(\Gamma) - H)_I$ (and/or the $S'_u(\widetilde{M}(\Gamma) - H)_I$) would no longer be a DIFF, smooth manifold and our technology for proving anything like the GSC Theorem 1.4, stated in its full glory in Section III, would be in shambles. The cure for this serious disease consists in:

- α) Use the **partial** ditch filling when outside the triple points (inside the difficult regions \overline{N}_∞^2).
- β) Use the not yet defined ditch-jumping steps, when one is at triple points (inside the difficult regions \overline{N}_∞^2).
- γ) Last, but not least is also, our treatment of the p_{∞} 's, i.e. delete them at the price of introducing compensating 2-handles. Concerning the whole issue of the **non-metrizability barrier**, see also [56], where

the non-metrizable spaces fill the whole scene. In a world of metrizable spaces the rather strange result of [56] would not stand a chance. END OF EXPLANATIONS.

When we go to the REAL-LIFE variant, then the bad rectangles (3.24.1) (actually their interiors) will be sent to infinity, in the manner of (3.17), using PROPER Whitehead dilatations. This comes with still another system of handles $h(4)$ to be added, **after** the scenario from the lemma below will be played through. So, we will have $\partial h^2(\varepsilon) \cdot \delta h^1(4) = 0$, for $1 \leq \varepsilon \leq 3$. Moreover, internally to $h(4)$, $\partial h^4(2) \cdot \delta h^4(1) = \text{easy id} + \text{nil}$. So no violation of GSC comes with the transformation

Variant I \implies REAL LIFE Variant, either.

The next “lemma” is one of the main steps in all this paper. And I have put here “lemma” between quotation marks because the statement which follows now is a description of a long process, the unrolling of the GEOMETRIC REALIZATION OF THE ZIPPING IN HIGH DIMENSIONS. So it is not a standard kind of lemma at all.

The Zipping Lemma 5.1. – 1) *Starting from the $S_0 = \Theta^{N+4}(X^2 - H)_1$ -DITCHES in (5.29), there is an infinite system of transformation, which we will call the Process Z (like zipping)*

$$(5.30) \quad (\Theta^{N+4}(X^2 - H)_1\text{-DITCHES} \equiv S_0) \implies S_1 \implies S_2 \implies S_3 \implies \dots$$

*This infinite chain (5.30) will essentially follow the (5.28.1) and, hence, there will be two successive parts: A first, **acyclic part**, where the zipping length is controlled (it is uniformly bounded) and then, an 0(3)- part (which Lemma 7.1-bis should make completely explicit).*

For a while, in this long statement, we will concentrate on that acyclic part, and we will come to the 0(3)-part only in the end. Anyway, the (5.30) is the GEOMETRIC HIGH-DIMENSIONAL REALIZATION OF THE ZIPPING.

Each S_i is a smooth $(N + 4)$ -manifold, coming with an inclusion $S_i \subset S_{i+1} + \underbrace{\sum_{H \in (5.7)} D^2(C(H))}$ (which is a cell-complex). This last inclusion is not part of (5.30), but we have commutative diagrams, at each index i in (5.30),

$$(5.30.0) \quad \begin{array}{ccc} S_i & \xrightarrow{\quad\quad\quad} & S_{i+1} \\ \downarrow & & \downarrow \\ S_i + \sum_H D^2(C(H)) & \xrightarrow{\quad\quad\quad} & S_{i+1} + \sum_H D^2(C(H)), \end{array}$$

*for the corresponding H 's. Each step in (5.30), amplified to (5.30.0), is followed by an appropriate **admissible subdivision** of the handle-body and/or cell decomposition $h(\varepsilon \leq 3)$, which never subdivide the $D^2(C(H))$'s. The admissible subdivisions are stellar, barycentric or, more generally, Siebenmann's bisections, never violating the GSC property; remember that more general linear subdivisions may do so. Hence, we will keep $\partial h^2 \cdot \delta h^1 = \text{id} + \text{nil}$, under the admissible subdivisions, and these will also provide us with the necessary flexibility for performing the step 4) below, and then the next ones too.*

*2) Each individual step in (5.30) is either a normal inclusion $S_n \subset S_{n+1}$ during which a finite number of new cells $h(2)$ or $h(3)$ are added, like in (5.29.1), (5.29.2), in a GSC-preserving manner, actually a J.H.C. Whitehead dilatation, followed possibly by adding handles of index $\lambda > 1$, OR a so-called **DITCH-jumping steps** $S_m \implies S_{m+1}$, which is also GSC preserving, as we shall see, and which is also always attached to triple points $t \in M_3(f)$. Globally speaking, the whole process Z in (5.30) is a PROPER sequence of Whitehead dilatation, higher ($\lambda \geq 2$) handle additions and diffeomorphisms (which may be disguised as folding maps, but which are diffeomorphisms).*

So, no danger for GSC can come from Z itself, (5.30), as such. But then, the $\partial h^2(1) = C(H)$ are dragged along by the ditch-jumping steps. Now, in terms of (5.11), (5.11.1), (5.11.2), the $D^2(C(H))$ are (part of) the $\{h_j^2(1)\}$, and actually they are the only part of $\{h_j^2(1)\}$ we will look explicitly into. They come with a dual family of 1-handles $h_i^1(1)$ and the following non-trivial fact will be proved.

(5.30.0-bis) Under the ditch-jumping steps, the geometric intersection matrix stays of the form

$$\partial h_j^2(1) \cdot \delta h_i^1(1) = \text{easy id} + \text{nilpotent}.$$

Also, more generally, ditch-jumping stays compatible with GSC. This will have to be proved, of course.
End of (5.30.0-bis)

Anyway, each complementary wall comes with a canonical transversal orientation, going conventionally from $z = +\varepsilon$ to $z = -\varepsilon$. When the ditches were explicitly defined, in (5.27.1) to (5.27.3), they had carried, corresponding to these transversal orientations, the canonical factors $[-\varepsilon \leq z \leq \varepsilon]$. There are two possibilities for the DITCH FILLING, namely

(5.30.0-ter) Either the complete filling, for all the $[-\varepsilon \leq z \leq \varepsilon]$, OR the **partial** ditch-filling restricted to $\varepsilon - \frac{1}{n} \leq z \leq \varepsilon$, with an $n \rightarrow \infty$ to be made explicit later.

The partial ditch filling is to be used in the situation of a zipping $W(\text{complementary}) \cap W(\text{non-complementary}) \cap \overline{N}_\infty^2$. The full ditch filling means the full

$$\left\{ \left[V(\text{BLACK}) \times [-\varepsilon, \varepsilon] \times \frac{1}{2} B^N(\text{BLACK}) \right] + \left[W^j(\text{RED}) \times [-\varepsilon, \varepsilon] \times \frac{1}{2} B^N(\text{RED}) \right] \right\} \Big| (V(\text{BLACK}) \cap W^j(\text{RED})),$$

in the manner of (5.27.2), possibly modified by metric adjustments; which we forget about right now. When two complementary walls, let's say $W(\text{BLACK})$ and $W^j(\text{RED} \cap H^0)$ are to be zipped together, then in $(N+4)$ -dimensions their respective contributions

$$\begin{aligned} & \{W(\text{BLACK})(\subset (x, y))\} \times [-\varepsilon \leq z \leq \varepsilon] \times B^N - \\ & - \left\{ \text{the DITCH} = \{ \text{the shaded area}(-\varepsilon_j \leq y - y_j \leq \varepsilon_j, x) \} \times [-\varepsilon \leq z \leq \varepsilon] \times \frac{1}{2} B_{\text{red}}^N \text{ (see figure 5.4.bis)} \right\} \end{aligned}$$

and

$$\begin{aligned} & \{W^j(\text{RED})(\subset (x, z))\} \times [-\varepsilon_j \leq y - y_j \leq \varepsilon_j] \times B^N - \\ & - \left\{ \text{the DITCH} = [\text{the shaded area}(-\varepsilon \leq z \leq \varepsilon, x)] \times [-\varepsilon_j \leq y - y_j \leq \varepsilon_j] \times \frac{1}{2} B_{\text{black}}^N \right\} \end{aligned}$$

can be clamped together into a perfectly symmetrical fit, like in figure 5.4.bis. This figure is a more symmetrical version of the figure 5.4, with the $W_n(\text{BLUE})$ ignored now. Notice that richer symmetry acquired now by the red splitting surface S . What we see in figure 5.4.bis corresponds, of course, to (5.27.1), and it is valid both in N_∞^2 and \overline{N}_∞^2 . In \overline{N}_∞^2 , when it comes to zipping $W(\text{BLUE})$ or $W(\text{RED} - H^0)$ with $W(\text{BLACK})$ (and now we are in the situation (5.27.1) ...), then we proceed like in the TOY MODEL, in figure 4.2. (See here, also, figure 2.4-(B).)

Inside N_∞^2 there is no prescribed a priori order for the elementary zipping steps, let us say that they all "commute", in particular, when we are at a triple point we are free to proceed arbitrarily. Inside \overline{N}_∞^2 they do not commute any longer, and their order is fixed by the Lemma 6.3 in the next section. Moreover, when we are inside \overline{N}_∞^2 , then for double lines which involve (in $(N+4)$ dimension)

$$W(\text{complementary}) \times [-\varepsilon \leq z \leq \varepsilon] \times B^N - \{\text{DITCH}\}$$

and the non-complementary

$$W_n = W(\text{BLUE OR } (\text{RED} - H^0)) \times B^N - \{\text{Hole } H\},$$

the ditch gets filled only up to $q \times [\varepsilon - \frac{1}{n} \leq z \leq \frac{1}{n}] \times b^N(q)$ and, in the formula above, one should not mix up the “H” for Holes and “H $^\lambda$ ” for λ -handles. (We have ignored here the factor $I \ni u$.)

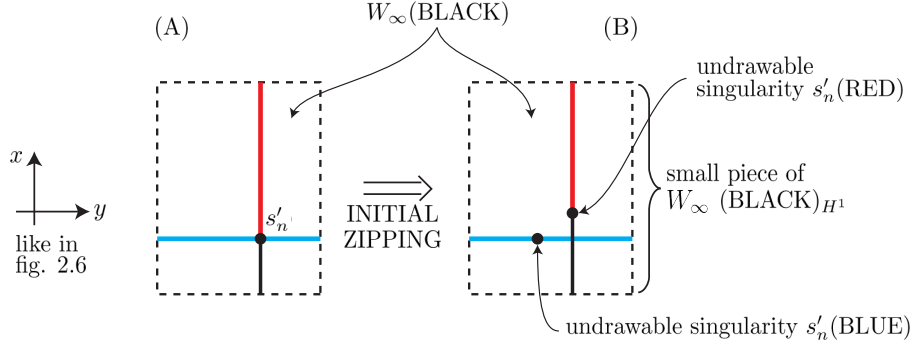


Figure 5.7.

An initial zipping, which we include in the definition of X^2 , and which changes the messy singularity s'_n into two bona fide undrawable singularities (like in (1.16.1)), $s'_n(\text{RED})$ and $s'_n(\text{BLUE})$. All singularities here are of the mortal type.

LEGEND : ——— (thick black) = here things are glued at the source in $X^2 - H \subset X^2$,
 ——— = RED double line, ——— = BLUE double line

Coming back to the special step $S_m \implies S_{m+1}$, to any $t \in M^3(f)$ like above, two **special BLUE** 1-handles are attached. This is suggested in figure 5.6-(C). The 5.6-(C) is supplemented by the additional figures 5.6-(A) and 5.6-(B). So, to our $t \in M^3(f)$, the special BLUE 1-handles $[XY] \equiv \{[BA] \text{ and/or } [AD], \text{ figure 5.6-(C)}\} \subset W_n(\text{BLUE}) - H$, are getting attached. We will assume, in the figure 5.6-(C), that the zipping flow comes to the triple point t , via the road $B'' \rightarrow B$, going from right to left in our figure, at stage m in the process Z . We will denote by $[0, m]$ the initial piece of Z (5.30), up to time m included, i.e. the piece

$$(5.30.1) \quad S_0 \implies S_1 \implies S_2 \implies \dots \implies S_m.$$

The next step, $m \Rightarrow m+1$ will be the **DITCH jumping step**, which we will describe soon, actually in the 4) of this lemma, below. But still at time m comes the piece $R_1 = [(X \equiv B), (B'', \gamma'')] \subset W_n(\text{BLUE}) - H$, see the figure 5.6-(C), which is inductively, already in the **DITCH**, glued to the rest by the ditch filling, material $h(3) \upharpoonright [0, m]$.

But, before coming to the triple point and to the corresponding **DITCH-jumping step** (s), we will discuss now what Z does, just before that. This should help understand the geometric realization of the **ZIPPING**, when far from triple points, or other complications.

3) We explain now how the **Process Z** proceeds, at its very beginning, **before** the moment when in terms of the (5.28.1) which it tries to mimick, it encounters three 2-sheets which meet in a triple point $M_3(f)$. (The third sheet is to be zipped later.)

We have to start from an initial singularity of $X^2 - H$ (or $\Theta^3(X^2 - H)$). This singularity will involve $W(\text{RED or BLUE}) \cap W_\infty(\text{BLACK})$, in the figures 2.5, 2.6, and, according to things already said, we have to start with $W(\text{RED})$, let us say from an s' in figure 2.5 or 2.6 and, to make things a bit more interesting, we

will consider $W(\text{RED}) \cap W_\infty(\text{BLACK})_{H^1}$ (figure 2.6). [First a little preliminary step, which we will constantly do. Let us say we consider s'_n in figure 2.6. To demess things we start with a little initial zipping, like in the figure 5.7, which we include in the definition of $X^2 - H$ (5.28.2).] From $s'_n(\text{RED})$ in the figure 5.7, the acyclic part $f(Q)$ in (5.28.2) continues until it meets and $0(3)$ move involving $W(\text{RED})$ and $W_\infty(\text{BLACK})_{H^1}$ which in terms of the x -axis in figure 2.6, is located at the same level as the Holes $\subset W(\text{RED})$, which are suggested in that figure. At the $0(3)$, our (5.30) temporarily stops. We discuss here the situation when that $0(3)$ is effectively there and, how and when this comes about is discussed in Lemma 7.1-bis, Section VII. The present discussion is rather pedagogical and we do not seek complete generality.

This is the 2^d story which the Z has to mimick. And, to be very precise, let us say that in the context of figure 5.7, we have, **with the same** (y, z) as in figure 5.4-bis, $W(\text{RED}) \subset (x, z)$ and $W_\infty(\text{BLACK}) \subset (x, y)$. In order to explain the piece of Z which interests us now, I will use an x -axis along the RED double line in figure 5.7-(B) with x -times going from $x(s'_n(\text{RED}))$ through all the values

$$(*) \quad x(s'_n(\text{RED})) \leq x < x(0(3) \text{ move}).$$

For further purposes notice, in the context of $(*)$, that the x_∞ is contained in the open interval $(x(s'_n(\text{red})), x(0(3)\text{-move}))$.

At time $x = x(s'_n(\text{RED}))$ we see figure 5.4-bis with nothing glued along S which is now in its full $N + 2$ dimensional glory. [Here is the account of dimensions. We retain N and the y, z of $\Theta^4(\dots, \mathcal{R})$, but we forget $I \ni u$ and freeze the x , so the x -axis is suppressed.] Nothing is glued to S , when we are at $x = x(s'_n(\text{RED}))$. But S is thickened into a **thin** codimension zero S_{thick} . The dimensions here were schematized, for the needs of figure 5.4-bis. But now we will get more realistic.

In the x -movie, at each fixed x on the x -axis, the movie is **frozen** and that x -value is frozen too. So, we pass from dimensions $N + 4$ to $N + 3$. Indulging in a bit of pedantry, here are some more details. In $R^2(y, z)$ (transversal to the x -axis), with the $W_\infty(\text{BLACK})$ going along the y -axis and the $W(\text{RED})$ along the z -axis, we have the 2^d cross (see figure 5.4)

$$(**) \quad W(\text{RED}) \times [-\varepsilon \leq y \leq \varepsilon] \cup (W_\infty(\text{BLACK}) \times [-\varepsilon \leq z \leq \varepsilon]) \supset [-\varepsilon \leq y \leq \varepsilon] \cap [-\varepsilon \leq z \leq \varepsilon].$$

Our $N + 3$ dimensions, at time x (fixed), are the two dimensions (y, z) of the $(**)$ cross, the $I \ni u$ of Θ^4 added back now, and B^N , which we break into (see figure 5.4-bis)

(5.30.1-bis)

$$B^N = [-1, 1] \times B^{N-1} = ([-1, 0] \times B^{N-1}) \underbrace{\cup}_{\{0\} \times B^{N-1}} ([0, 1] \times B^{N-1}) \equiv \frac{1}{2} B^N(\text{BLACK}) \cup \frac{1}{2} B^N(\text{RED}).$$

We assume that in terms of figure 5.4, the $[-1, +1]$ runs along $+t$, like $A' \rightarrow A' \rightarrow A$. With this, the high-dimensional analogue of the 2^d object

$$S = ([-\varepsilon \leq y \leq \varepsilon] \cap [-\varepsilon \leq z \leq \varepsilon]) \times \{0\} \cup ((z = \pm\varepsilon) \times [-\varepsilon \leq y \leq \varepsilon] \times [-1, 0]) \cup \\ \cup ((y = \pm\varepsilon) \times (-\varepsilon \leq z \leq \varepsilon) \times [0, 1])$$

(with, here, $t \in [-1, +1] \ni 0$) from the figures 5.4, 5.4-bis, is now

$$(5.30.1-ter) \quad ([-\varepsilon \leq y \leq \varepsilon] \cap [-\varepsilon \leq z \leq \varepsilon]) \times I \times (\{0\} \times B^{N-1}) \cup \left(((z = \pm\varepsilon) \times [-\varepsilon \leq y \leq \varepsilon]) \times I \times \right. \\ \left. \times \underbrace{([-1, 0] \times B^{N-1})}_{\frac{1}{2} B^N(\text{BLACK})} \right) \cup \left(((y = \pm\varepsilon) \times (-\varepsilon \leq z \leq \varepsilon)) \times I \times \underbrace{([0, 1] \times B^{N-1})}_{\frac{1}{2} B^N(\text{RED})} \right) = S \times I \times B^{N-1} \underset{\text{TOP}}{\cong} B_0^{N+2}$$

and compare this with (5.27.2).

Please, do not mix up this 2^d S (figure 5.4) and the high-dimensional S_ℓ 's occurring in (5.30).

The piece of Z (5.30) which corresponds to the zipping of $W(\text{RED})$ and $W_\infty(\text{BLACK})$ along $(*)$ consists of a sequence of inclusions $S_0 \subset S_1 \subset \dots$ which we will describe now.

Locally, what one sees at $x(s'_n(\text{RED}))$ in $(*)$ is
(5.30.2)

$$\begin{array}{ccc}
 B_0^{N+3} \equiv (**) \times I \times B^N & \xleftarrow{\text{a proper, SPLITTING-inducing, inclusion}} & B_0^{N+2} \\
 & & \text{(= The high-dimensional splitting surface } S) \\
 & \swarrow & \searrow \\
 & B_0^{N+2} \times [-\eta, \eta] & \\
 & \text{(tubular neighbourhood of } B_0^{N+2} \text{ in } B_0^{N+3}) &
 \end{array}$$

[Remember that “proper” means interior to interior and boundary to boundary, while “PROPER” in capital letters means that $f^{-1}(\text{compact}) = \text{compact}$. So, if $A \xrightarrow{f} B$ and in A the sequence $x_1, x_2, \dots \in A$ is such that $\lim_{n \rightarrow \infty} x_n = \infty$, then also, in B , we have $\lim_{n \rightarrow \infty} f(x_n) = \infty$. Of course, this comes with $\{\text{PROPER}\} \Rightarrow \{\text{infinity} \rightarrow \text{infinity}\}$.]

Now, in the context of (5.30.2), the B_0^{N+2} splits B_0^{N+3} into two halves $B_0^{N+3}(\text{RED})$ and $B_0^{N+3}(\text{BLACK})$, glued along $B_0^{N+2} = B_0^{N+2} \times \{0\}$. With this, here is what the x -movie $(*)$ does, for all $x(s'_n(\text{RED})) < x < x(0(3)\text{-move})$:

•) At any fixed x (see below), $B_0^{N+3}(\text{RED})$ and $B_0^{N+3}(\text{BLACK})$ are separated from each other. This is very much like in (5.27.2), with each half living inside the DITCH of the other half, symmetrically. The DITCH $| x$ corresponds here to $B_0^{N+2} \times [-\eta, \eta]$ (at x). To be more precise, and referring here to the figure 5.4,

$$\begin{aligned}
 \text{DITCH} | W(\text{RED}) &= (-\infty < x < \infty) \times ([-\varepsilon \leq y \leq \varepsilon] \cap [-\varepsilon \leq z \leq \varepsilon]) \times \frac{1}{2} B^N(\text{BLACK}) \subset \\
 &\subset W(\text{RED}) \times [-\varepsilon \leq y \leq \varepsilon] \times B^N,
 \end{aligned}$$

and

$$\begin{aligned}
 \text{DITCH} | W_\infty(\text{BLACK}) &= (-\infty < x < \infty) \times ([-\varepsilon \leq y \leq \varepsilon] \cap [-\varepsilon \leq z \leq \varepsilon]) \times \frac{1}{2} B^N(\text{RED}) \subset \\
 &\subset W_\infty(\text{BLACK}) \times [-\varepsilon \leq z \leq \varepsilon] \times B^N.
 \end{aligned}$$

So, at all values $x > x(s'_n(\text{RED}))$, the two BLACK and RED $(N + 4)$ -dimensional branches, which at $x = x(s'_n(\text{RED}))$ were glued together along

$$B_0^{N+3}(\text{RED}) \underbrace{\cup}_{B_0^{N+2} = B_0^{N+2} \times \{0\}} B_0^{N+3}(\text{BLACK})$$

get unglued, each $W(\text{COLOUR})$ part losing the corresponding DITCH $| W(\text{COLOUR})$ piece. This becomes now a singular situation (singularity at $x = x(s'_n(\text{RED}))$), with a gap $B_0^{N+2} \times [-\eta, \eta]$ at the values $x > x(s'_n(\text{RED}))$. See here what happens with the toy-model in the context of figure 4.2-bis.

••) We then add the $B_0^{N+2} \times [-\eta, \eta]$ back, as DITCH-FILLING MATERIAL, and when we look at the whole movie, from $x(s'_n(\text{RED}))$ to some higher $x < x(0(3)\text{-move})$, then what we see in terms of (5.30) is an inclusion

$$S_0 = \Theta^{N+4}(X^2 - H)_I \subset \text{some higher } S_m \text{ (corresponding to the high } x).$$

•••) When we add to this the DITCH-FILLING MOVIE, then what we finally get is a smooth J.H.C. Whitehead dilatation, which also fixes the geometric intersection matrix internal to our ditch-filling material,

$$(5.30.3) \quad S_m \supset S_0 \nearrow S_m.$$

When it comes to that (part of the) handlebody decomposition of $h(3)$ which comes with (5.30.3), then we have the following important fact too

(5.30.4) When it comes to a handle $h^\lambda(3)$, then

$$\delta h^\lambda(3) \cap S_0 = \emptyset \text{ (for } S_0 \subset S_m \text{), see (5.30),}$$

and we will never have $\partial h^{\lambda+1}(1) \cdot \delta h^\lambda(3) \neq \emptyset$, but only contacts $\partial h^{\lambda+1}(3) \cdot \delta h^\lambda(1) \neq \emptyset$. These kind of things will continue to be with us, as long as the zipping process Z mimics the acyclic part of (5.28.1), see (5.28.2). End of (5.30.4).

We have just discussed an initial zipping of two complementary walls. And here, by “initial” we mean from some mortal singularity of S_0 until we meet either the next $0(3)$ OR the next triple point where two of the three involved branches are already glued together. And this will certainly happen when, in the context of figure 5.7, instead of starting at s'_n (RED) as we just did, we start at s'_n (BLUE). Concerning this Process Z , which mimicks the zipping (5.28.2), and which we have just discussed, let us denote, in the context of (5.30) and of the (*) above, by

$$\{S_0 \implies S(\infty)\} \text{ (supposedly part of (5.30)),}$$

the {zipping (*), which is part of (5.30), TRUNCATED AT $x = x_\infty$ }.

The B_0^{N+3} from (5.30.2) occurs as part of $S(\infty)$ and here we will be a bit more precise. And keep in mind here that $S(\infty)$ is a smooth manifold.

(5.30.5) Actually, with (**) the cross above, the whole space below is in $S(\infty)$:

$$\mathcal{X} \equiv \underbrace{\bigcup_{x(s'_n(\text{red})) \leq x \leq x_\infty} (**)} \times I \times \partial B^N \subset \partial S(\infty); \text{ (think here of } \partial B^N = (t=1) \text{ in the figure 5.4).}$$

For x fixed, in the context of figure 5.4, $\mathcal{X} | x$ is exactly

$$[(W(\text{RED}) \times [-\varepsilon \leq y \leq \varepsilon]) \cup (W_\infty(\text{BLACK}) \times [-\varepsilon \leq z \leq \varepsilon])] \times I \times (t=1)$$

looking towards the observer, in the figure 5.4 in question.

Very importantly for what will be following soon, we also have

$$(***) \quad \underbrace{\left([(W(\text{RED}) \times [-\varepsilon \leq y \leq \varepsilon]) | (z = \varepsilon) \times [x(s'_n(\text{red})) \leq x \leq x_\infty]] \cup_{x_\infty} (\text{FIN}) \right)}_{\text{this is in } \Theta^4(\Theta^3(fX^2)', \mathcal{R}) \text{ which comprises both the additional } I \ni u \text{ and the FINS}}$$

What is understood here is that, in the context of (5.30.5), we add all the fins to \mathcal{X} too.

We describe now the same things for the zipping of a non-complementary wall with a complementary one, let us say $W(\text{BLUE})$ and $W_1(\text{BLACK})$ starting from s'_n (BLUE) in figure 5.7-(B) or from s_n in figure 2.5. One novelty now is that, in this situation the $W(\text{BLUE})$ carries now Holes. The description is now very much like in the TOY MODEL, with a y -movie going along

$$\underbrace{\bigcup_{y(\bar{U}(n)) \leq y \leq y(\bar{V}(n))} T(n) | y, \text{ in (4.91) with } T(n) | y \subset b^N.}$$

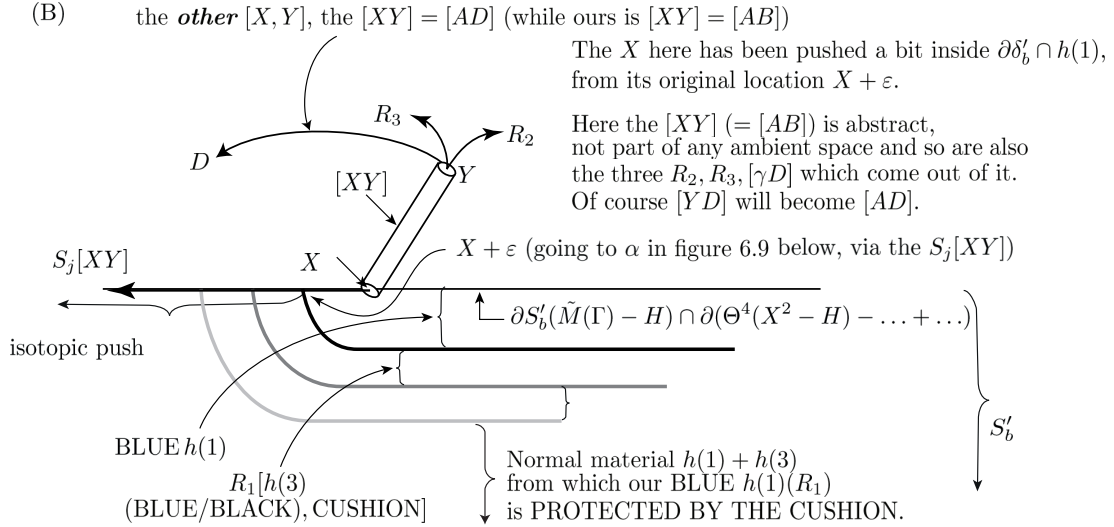


Figure 5.7-bis

This figure corresponds to the S -scar at $[B\bar{B}B'_0\bar{B}'_0]$ in figure 5.4. The X here is the B in figure 5.6-(B), and the isotopic push has not yet started. This push will drag along the strata drawn here, without disturbing the layered structure. Saying that $[XY], \dots$ are “abstract” means that they are floating freely, for the time being not embedded anywhere, and just attached to the rest of the world at X . It is $[XY]$ which is there attached and then, internally to the whole structure growing out of X , the three $R_2, R_3, [YD]$ are attached at Y . And when it comes to those three guys, they follow the fate of X (and afterwards of Y). The two upper layers in the drawing represent the DITCH (R_1), inside which they live.

And figure 4.2-bis is to be used here again. As long as we have left the singularity behind, the action takes place in (see figure 5.4) $[F, F'_0, \bar{F}, \bar{F}'_0; B, B'_0, \bar{B}, \bar{B}'_0]$ or the similar region $[A, A', \bar{A}, \bar{A}'_0; \dots]$ in the same figure 5.4. The reader will not find it hard to fill in the details.

4) We come next to the (first) triple point which the Z meets, let us say the one in figure 5.4 and/or 5.6. The b^N -thin cross in figure 5.4, to which $W_n(\text{BLUE}) - H$ is reduced, this is now further simplified like in figure 5.6 to the four regions R_1, R_2, R_3, R_4 (figure 5.6-(C)) going into the DITCH $b^N \times [\varepsilon - \frac{1}{n} \leq z \leq \varepsilon]$, bound together by the two very thin **special BLUE 1-handles** $[X, Y] = \{[AB] \text{ and } [AD]\}$. And, for these $[XY]$'s, the b^N (non-complementary) factor, already reduced to b^N for $W(\text{BLUE})$ in the DITCH should be even further reduced to an even much smaller b^N (**special small**).

To every special BLUE 1-handle $[XY]$ there is attached, an embedded shadow arc, $S_j[XY]$, present at the beginning of $m \Rightarrow m + 1$, coming with

$$(5.30.5) \quad S_j[XY] \subset \partial [\{\Theta^{N+4}(X^2 - H) - \text{DITCH} - \{\text{special BLUE 1-handles,} \\ \text{for all times } [0, \infty)\} \cup \{(h(2) + h(3)) \mid [0, m]\}\}],$$

and this arc is liftable to $\partial S'_b(\tilde{M}(\Gamma) - H)$, see here (5.33) too.

[NOTATIONS: For $[XY] \subset W(\text{BLUE}) - H$, the end-points are X, Y . When it comes to $S_j[XY]$, and when we want to stress it, rather than the $[XY]$, then the endpoints may be denoted by \bar{X}, \bar{Y} . With this we have $\bar{X} \equiv X \equiv \{\text{that endpoint of } [XY] \text{ already reached by the geometric realization of the zipping at time } m\}$. The other end of $S_j[XY]$ is at $\bar{Y} \equiv \{\text{the corresponding other endpoint, already reached by the previous zipping BLACK/RED}\}$. In our specific case of $[AB]$ figure 5.6, with the indicated zipping flow, we have $X = \bar{X} = B$ (with BLACK/BLUE identification), $\bar{Y} = A$ (with BLACK/RED identification).] So, in our specific example, $\bar{X} = X = B \in \{\text{the shaded region } R_1 \equiv (B, (B'', \gamma''))\}$ (figure 5.6-(C)) consisting entirely of BLUE $h(1)$'s}. One can also see our A, B above in figure 5.4, where the BLUE cross is only a ghost. The four regions R_i will eventually go into the DITCH of $W(\text{BLACK}) \cup W(\text{RED})$, actually in $W \times [-\varepsilon, \varepsilon] \times b^N$, making use of the supplementary dimensions, while, as we shall see, $[AB] + [AD]$ will eventually go very far from the site of figure 5.6, smeared along their corresponding shadow arcs, without ever going into the DITCH, which they never enter.

It is very convenient for us to speak in terms of 1^d BLUE 1-handles $[XY]$. But keep in mind that, in real life, these $[XY]$'s should be thickened, so that, in a figure like 5.6-(C)

$$(*) \quad [AB] \cup [AD] \cup R_1 \cup R_2 \cup R_3 \cup R_4 \cup \Sigma D^2(C(H)) \quad (\text{figure 5.6-(C)})$$

should exactly catch all the $h(1)$'s corresponding to that situation. So, all of the presently relevant part of (5.29) is caught by the (*) above.

We will describe right now the geometry of the shadow arc $S_j[AB]$, which the easiest, leaving the much trickier $S_j[AD]$ for the next section.

We will have,

$$(5.31) \quad S_j[AB] \subset [((W(\text{RED}) \times [-\varepsilon \leq y \leq \varepsilon]) \mid ((z = \varepsilon) \times [x(s'_n(\text{red}) \leq x \leq x_\infty])) \underbrace{\cup}_{x_\infty} \text{FIN}) \times \partial B^N \subset \partial S(\infty).$$

(Compare this with the (***) which follows after (5.30.5).) This stretches, like $[AB]$ itself, from B to A . Figure 5.8 should give an idea of what it looks like. This figure is supposed to live in $\partial S(\infty) \subset \partial S'_b(\widetilde{M}(\Gamma) - H)$. Very importantly the whole action involving the arcs $S_j[AB]$ (i.e. the pushing + smearing to be soon described) happens on $\delta S'_b(\widetilde{M}(\Gamma) - H)$ far from the DITCHES (except occasional scar-jumping) and has nothing to do with the b^N 's.

[We discuss here, with a bit of details, what happens at a triple point

$$(**) \quad W(\text{BLUE}) \cap (W(\text{BLACK}) \cap W(\text{RED})).$$

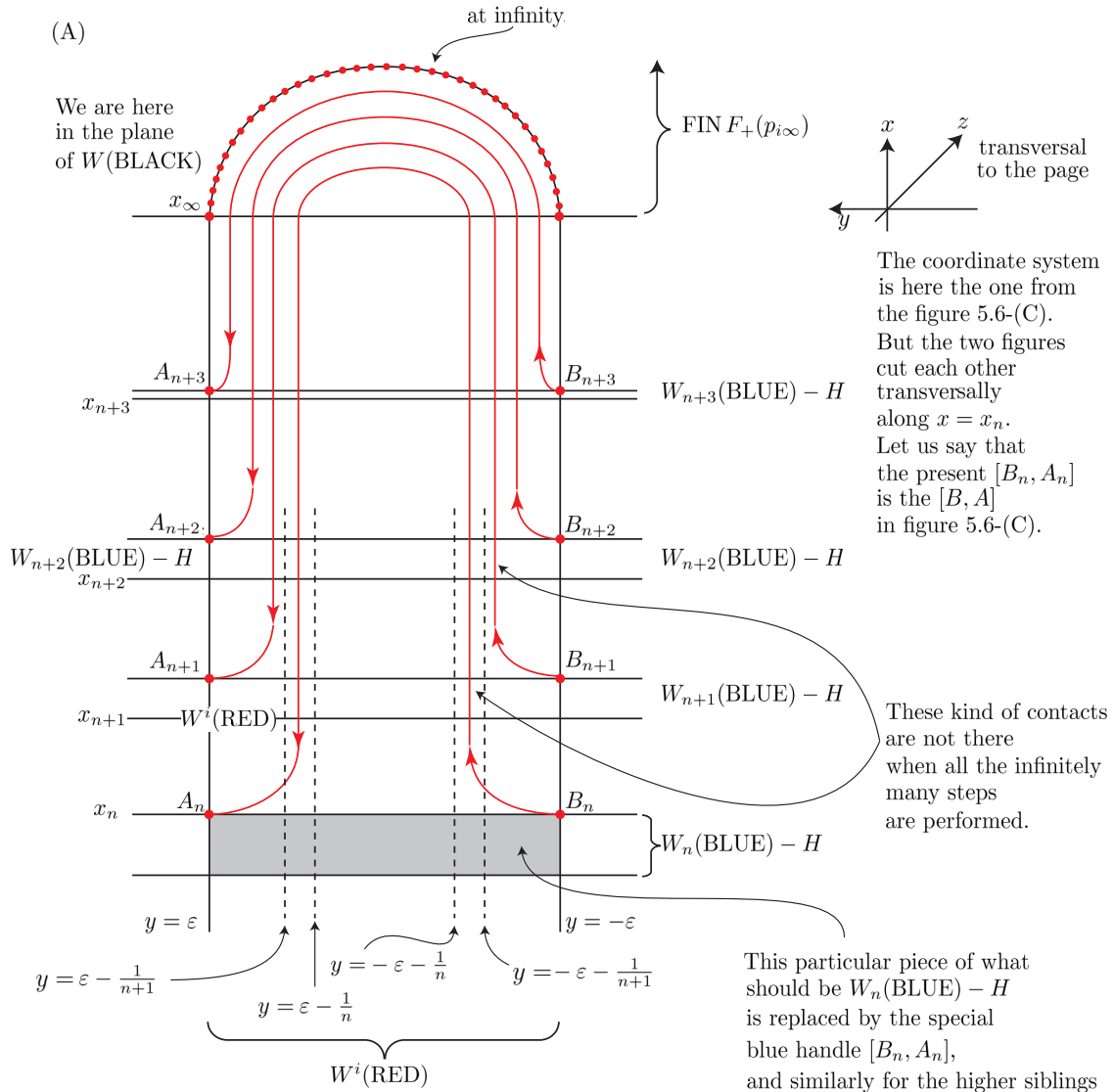
But, then, when it comes to the regions \bar{S} (figure 2.4) which are certainly there in the generic case

$$\Gamma \neq \pi_1 M^3(\text{smooth 3-manifold}),$$

there we have triple points

$$(***) \quad W(\text{BLUE}) \cap \underbrace{(W(\text{BLACK}))}_{\text{overflowing}} \cap \underbrace{W^*(\text{BLACK})}_{\text{subdued}},$$

which will get exactly the same treatment as the (**) above. To be more specific, let us say that we are zipping, in figure 6.2 (Section VI) $W(\text{BLUE})_\ell$ and the horizontal $W(\text{BLACK})$ at $(p'''_\infty(S))'$ (which is overflowing) from $(p''_\infty(S))'$ to the triple point $t_{\ell n}$ where the vertical $W^*(\text{BLACK})$ (which is subdued), plays the role of $W(\text{RED})$. We use now the fins at $t_{\ell n}$, which figure 6.2 presents, exactly as we use the fins in figure 5.8-(B) below. See here also the proof of the Lemma 6.3, in the next section.]



each other. One should also notice that each line $[x(s'_n(\text{red})) \leq x \leq x_{\text{fixed}} < x_\infty]$ sees only FINITELY many RED S_j 's. Their full infinity gets corralled on the RIMS OF THE FINS.

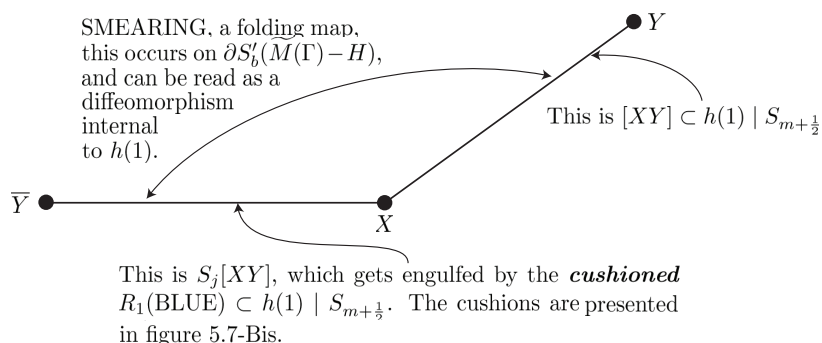


Figure 5.9

A schematical view of **smearing**. This smearing is a folding map, confined inside the $h(1) | S_{m+\frac{1}{2}}$. So, the smearing is an internal affair of $h(1) | S_{m+\frac{1}{2}} \subset h(1)$, readable as a **DIFFEOMORPHISM**.

5) So, for $[XY] \equiv [BA]$ (with the attention fixed on our $[B_n A_n]$) we have explained the geometry of $S_j[XY]$, which one thinks of as going from X to Y , like the arrows on the red arcs in figure 5.8. Notice that, on its road to $A = Y$, the shadow arc will have to cross of lot of handles $\{h(1)$ (of various COLOURS), $h(2), h(3)\}$.

Just before we meet the triple point t_{in} , we have three distinct systems of handle systems $h(1), h(3)$ (RED/BLACK) and $h(3)$ (BLUE/BLACK); then there is, of course, the $h(2)$ too, but it will play no role in the present discussion. Each of them is GSC, there are no relations (yet) between the two $h(3)$'s, and if we think of the geometric intersection matrices as oriented graphs, we have oriented edges, independent of each other, right now,

(5.31.1)

$$\begin{array}{ccc} h(3)(\text{RED/BLACK}) & & h(3)(\text{BLUE/BLACK}) \\ (\text{COMPL./COMPLEMENTARY}) \rightarrow h(1) \leftarrow & & (\text{NON-COMPLEMENTARY/COMPLEMENTARY}) \end{array}$$

Next, in terms of figure 5.6-(C) and the other related ones we leave R_1 (for the moment) in peace. Keeping also the attachment of $[XY] = [BA]$ to it, but moving the $X \in [XY]$ a bit to the interior of R_1 , as the figure 5.7-bis suggests. The old $X \in R_1$ is called now $X + \varepsilon$ (= the new position of X).

For any individual $\varepsilon = 1, 2, 3$, the next mini-step, which is essentially isotopic, will not change the incidence relations $\{\partial h^{\lambda+1}(\varepsilon) \cdot \delta h^\lambda(\varepsilon), \varepsilon \leq 3\}$ at all. Making use of the flexibility which is provided by the admissible subdivision from the end of point 1) above, we will **push isotopically** the BLUE shaded region R_1 along the portion $[X + \varepsilon, Y \equiv A] \subset S_j[XY]$, until it engulfs completely the whole $S_j[XY]$, starting from B to the end A , and also a bit beyond A .

The isotopic push, **PUSHES** in front of the $h(1)(\text{BLUE})(\subset R_1)$ a **CUSHION** consisting entirely of $h(3)(\text{BLUE/BLACK})$ and our $h(1)$ BLUE, and once pushed, it will only touch the rest of the world via the **protective CUSHIONS**. The **LAYERED SYSTEM OF CUSHIONING** is suggested in the figure 5.7-bis. This has the following **IMPORTANT CONSEQUENCE**, which will be a constant, universally true throughout the whole Z -process, namely the (5.31.1-bis) below.

And here is what goes on, once the special BLUE arcs $[AB]$ have been changed to the positions $S_j[AB]$ and then engulfed by the **CUSHION**, via our isotopic push

(5.31.1-bis) To begin with, the incidence relations internal to the $h(1)$'s or among the individual $h(3)$ (COLOUR/COLOUR)'s stay unchanged. But then, the (5.31.1) changes as follows

$$\begin{array}{ccc}
 h(3)(\text{RED/BLACK, both complementary}) & \longrightarrow & h(1) \longleftarrow \underbrace{h(3)(\text{BLUE/BLACK})}_{\text{CUSHION}} \\
 \uparrow & & \downarrow \\
 & \xrightarrow{\text{(see figure 5.11-bis)}} &
 \end{array}$$

which is OK, GSC-wise, as it will turn out. End of (5.31.1-bis)

We will denote by $S_m \Rightarrow S_{m+\frac{1}{2}}$ this isotopic push of the BLUE region R_1 , until it engulfs the whole arc $S_j[BA]$, and a bit beyond A . During this pushing operation, the region R_1 (or the active part of it) is in contact with $h(3)$, which **cushions** it like in the figure 5.7-bis, and it is only indirectly through the $h(3)$ -CUSHION, that it comes into contact with the outer world. We will introduce the notation $h(1) | S_m \equiv \{\text{That part } R_1 \cup (\text{the } [XY] \text{ which is attached to it}) \subset S_m, \text{ consisting entirely of BLUE } h(1) \text{'s, like in the formula (5.31.1), and which is actively involved in covering the shadow arc}\}$.

The next mini-step after this $m \Rightarrow m+\frac{1}{2}$, our isotopic push, which we will denote by $m+\frac{1}{2} \Rightarrow m+\frac{3}{4}$, and which we will call the **smearing**, is a folding map strictly confined inside the $h(1) | S_{m+\frac{1}{2}}$, see the (5.31.2) below, and which realizes the identification $\{[XY] = S_j[XY] \text{ (for } [XY] = [BA])\}$, leaving B fixed. The Y here is like in the figure 5.7-bis. This creates the identification $\{Y = A(\text{BLUE}) = (A(\text{RED/BLACK})) = \bar{Y}\}$, see here the figures 5.4, 5.6-(B, C). This smearing appears now in the diagram below

$$\begin{array}{ccccccc}
 (5.31.2) & & S_m & \xrightarrow{\text{pushing}} & S_{m+\frac{1}{2}} & \xrightarrow[\text{(folding map)}]{\text{smearing}} & S_{m+\frac{3}{4}} & \xrightarrow{\quad} & S_{m+1} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & h(1) | S_m & \xrightarrow{\quad} & h(1) | S_{m+\frac{1}{2}} & \xrightarrow{\quad} & h(1) | S_{m+\frac{3}{4}} & &
 \end{array}$$

The smearing is illustrated in the figure 5.9. There is a lot of action coming with the diagram (5.31.2) and we will describe that now. The whole action $\{\text{pushing}\} + \{\text{smearing}\}$ is a purely internal affair to $\{S_0 + \sum_H D^2(C(H))\}$, actively involving that part of this object which is in figure 5.6-(A, B, C) and 5.8-(B) + $\{\text{the CUSHION } h(3) \text{ (BLUE/BLACK)}\}$, with the rest being just passive spectators.

In this diagram (5.31.2), the double arrows are transformations while the vertical ones are canonical inclusions. The $h(1) | S_{m+\frac{1}{2}} \equiv \{\text{what becomes of } h(1) | S_m \text{ via the pushing}\}$ and, since the pushing in question is an isotopy, we have a diffeomorphism

$$(5.31.3) \quad h(1) | S_m \xrightarrow[\text{DIFF}]{\quad} h(1) | S_{m+\frac{1}{2}}.$$

(5.31.3) The smearing step induces a second DIFFEOMORPHISM

$$h(1) | S_{m+\frac{1}{2}} \xrightarrow[\text{DIFF}]{\quad} h(1) | S_{m+\frac{3}{4}},$$

and this carries along with it any $D^2(C(H))$ which is such that $[XY] \subset \partial C(H) = \partial D^2(C(H))$. This gets naturally transported with the rest, making sure that the internal geometric intersection matrix of $h(1)$ stays unchanged, hence of the GSC type. With this, for any m in (5.30) we find that

$$(*) \quad \left\{ S_0 + \sum_H D^2(C(H)) = (\Theta^{N+4}(X^2 - H))_I - \text{DITCHES} \right\} + \sum_H D^2(C(H)),$$

with its admissible subdivision NOT touching the $D^2(C(H))$, and which is GSC $\left. \right\} \subset S_m + \sum_H D^2(C(H))$ (which hence contains a complete copy of $S_0 + \sum_H D^2(C(H))$), possibly submitted to some admissible subdivisions and,

we may add, in some grandly distorted position with respect to the original one, but still diffeomorphic to it). Moreover, the LHS in (*) above contains all the $h(1)$'s and nothing else. The $S_m + \sum_H D^2(C(H))$ is also GSC, $\forall m$. End of (5.31.3)

Notice that our (5.31.3) takes care of (5.30.0-bis).

So, retain that in the context of (5.31.2) we have

$$h(1) | S_m \xrightarrow{\text{push}} h(1) | S_{m+\frac{1}{2}} \xrightarrow{\text{smear}} h(1) | S_{m+\frac{3}{4}},$$

which is a purely internal $h(1)$ -affair, dragging along the $D^2(C(H))$, inducing a *DIFFEOMORPHISM* and transporting the GSC cell-structure, possibly subjected to a GSC-preserving subdivision. We forget here the *CUSHIONING*, which induces an internal $h(3)$ transformation too. Actually, via pushing, additional contacts $h(3)$ (BLUE/BLACK) $\rightarrow h(1)$ may occur in the context (5.31.1.1), which gets thus enriched, but that is OK for GSC. Let us say that these additional arrows are of the form

$$\{\text{cushion } h(3) \text{ (BLUE/BLACK) of } R_1\} \longrightarrow h^1 \text{ (RED/BLACK)}.$$

But, very importantly, the pushing, which comes with its cushions (seable in figure 5.7-bis) *NEVER* brings arrows $h(1) \rightarrow h(3)$ in the geometric intersection matrix.

Inductively, the step $[0, m]$ from (5.30.0) is GSC preserving and $h(1) | S_m$ has its handlebody decomposition inherited from the GSC handlebody decomposition of S_m . We will use the diffeomorphism from (5.31.3) for simple-mindedly **transporting** the handlebody decomposition of $h(1) | S_m$ into a handlebody decomposition of the manifold $h(1) | S_{m+\frac{1}{2}}$. Here the transport only concerns those pushed items, actively speaking.

Outside of $h(1) | S_{m+\frac{1}{2}}$, the $S_{m+\frac{1}{2}}$ inherits quite naturally a handlebody decomposition from S_m . By the (diffeomorphism (5.31.3)) + {the transport of handlebody decomposition}, the $h(1) | S_{m+\frac{1}{2}}$ inherits the features from (5.31.1) and it follows from all these things that $S_{m+\frac{1}{2}} + \sum D^2(C(H)) \in \text{GSC}$. Keep in mind the following fact too.

(5.31.3-bis) The transport does not modify the internal incidence relations $\partial h^2(1) \cdot \delta h^1(1)$, while it certainly may change the $\partial h^2(\varepsilon) \cdot \delta h^2(1)$, for $\varepsilon > 1$. The smearing step which follows next in (5.31.2), bends $[XY]$ modulo X and glues it to $S_j[XY] \subset h(1) | S_{m+\frac{1}{2}}$, blending $[XY]$ into it. In our high dimensions, this can be read as another diffeomorphism, which continues the one in (5.31.2) and which we have already noticed in (5.31.3), call it

$$(5.31.4) \quad h(1) | S_{m+\frac{1}{2}} \xrightarrow[\text{DIFF}]{\text{SMEARING}} h(1) | S_{m+\frac{3}{4}}.$$

By the same argument of transporting and/or reshuffling handlebody decompositions as above, we get that $S_{m+\frac{3}{4}} + \sum D^2(C(H)) \in \text{GSC}$, just like $S_m + \sum D^2(C(H))$ or $S_{m+\frac{1}{2}} + \sum D(C(H))$.

This second transport is a more serious business than the first one. It concerns now

$$R_1 \cup \{\text{the special BLUE 1-handle } [XY]\},$$

and the effect of transporting, which continues to abide to (5.31.3.bis), takes care of the potential nontriviality of the folding map.

Here is now how the ditch-jumping continues from $m + \frac{3}{4}$ to $m + 1$ and it will turn out that this piece is also GSC-preserving.

But before that, notice that starting at A there are two *DITCHES* which, for the time being are empty, a *RED DITCH* R_2 and a *BLUE DITCH* R_3 . At $Y = A$, the R_2, R_3 (and the $[AD]$ too) stick out of

$\{h(1) \mid S_{m+\frac{3}{4}}, \text{ sitting on its } h(3) \text{ (BLUE/BLACK) CUSHION}\}$. [These $R_2, R_3, [AD]$ sticking out, are seable in the figure 6.9, in the next Section VI too.] This having been said, here is how one proceeds with R_2 . The {cushion} + {the $h(1)$ which has already been pushed and smeared in (5.31.2)}, will descend a bit in the DITCH, enough so that R_2 can be sent in the DITCH too. Very schematically, this is suggested in the figures 5.10-(A), (B). Then, one adds new DITCH-FILLING material, like in 5.10-(C).

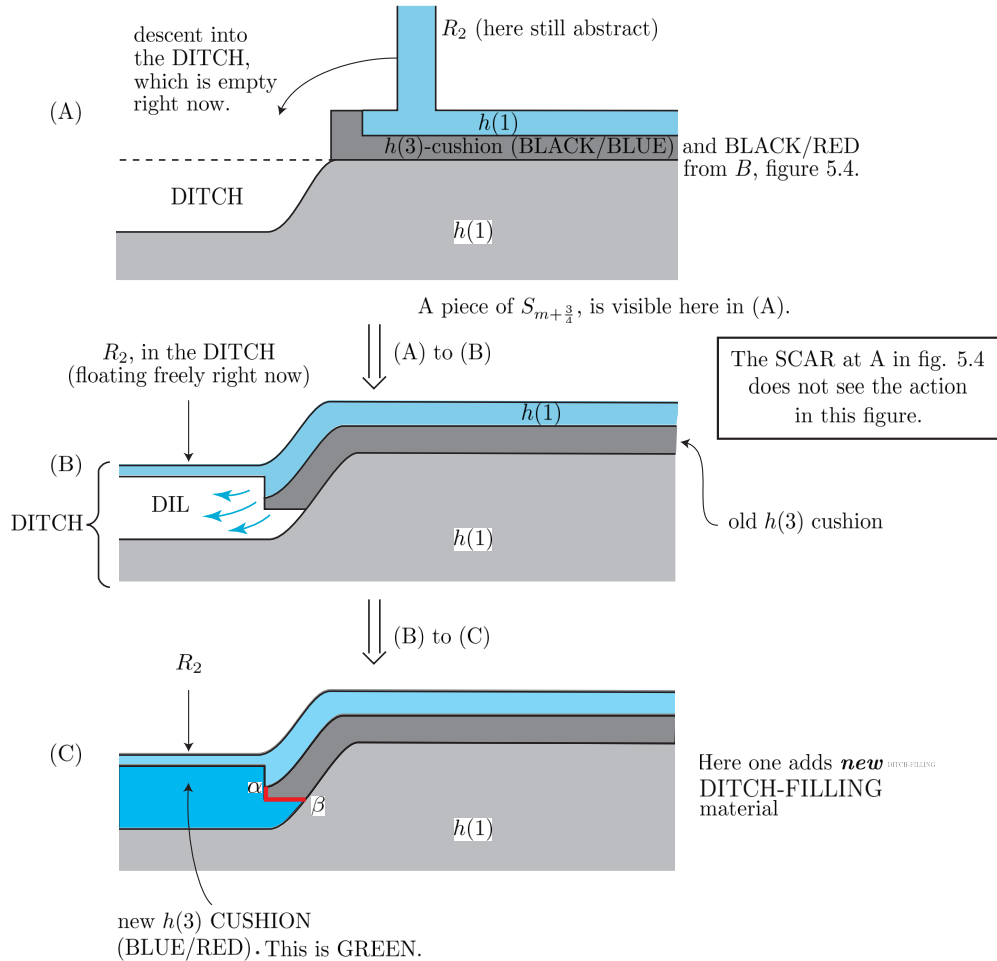


Figure 5.10.

Fate of R_2 (RED), after $S_{m+\frac{3}{4}}$ from (5.31.2). The important fact here is that no DITCH FILLING $h(3)$ (BLACK/RED) is being met in the present situation.

At $[\alpha\beta]$ appear new incidence relations

$$h(3)(\text{BLUE/BLACK})_{\text{old}} \leftarrow h(3)(\text{BLUE/RED})_{\text{new}},$$

which are OK GSC-wise.

The triecquier fate of R_3 (BLUE) is presented in figure 5.11-bis, to be compared to the present one.

And, in this figure 5.10-(C) one should notice that, along the red $[\alpha, \beta]$ one has **new** incidence relations of the type

$$\partial h^{\lambda+1}(3)(\text{NEW CUSHION}) \cdot \delta h^\lambda(3)(\text{old CUSHION}) \neq 0,$$

and this will turn out to be OK, GSC-wise. Of course, we also find $\partial h^{\lambda+1}(3)(\text{NEW CUSHION}) \cdot \delta h^\lambda(1) \neq 0$, which certainly is OK too. Now, something similar can be done for R_3 , but there is here a little additional complication. This time one also has to cross the S -scar at $[A, A'_0, \bar{A}, \bar{A}'_0]$, in figure 5.4, and like it is suggested in the figure 5.11-bis, this comes now with additional arrows in the context (5.31.1), namely

$$h(3)(\text{BLUE/BLACK AND \{RED/BLACK from B\}}, \text{fig. 5.4}) \rightarrow h(3)(\text{RED/BLACK, at the scar above, from A}).$$

Once one has pushed from B to A , here is how things continue.

i) The two BLUE zones R_2, R_3 which start at A (figure 5.6-(B)) are now sent into their respective DITCHES, together with the necessary $h(3)$ ditch-filling material. We have here a **partial** ditch-filling, again, of course. (See, for instance, figure 5.10.)

ii) Next, we have to handle the $[XY] = [AD]$ (figure 5.6) too. This will start with a pushing + smearing like in $m \Rightarrow m + \frac{3}{4}$ above. But the $S_j[AD]$, which is needed for this new pushing + smearing, is now a much trickier thing than before. Notice that we have not said anything, yet, about the explicit location of the shadow arcs other than $S_j[BA]$. It turns out that for $S_j[AB]$ this was a relatively simple affair. We just made use of the closest fin, remembering that our concern is to keep the whole Z PROPER; see figure 5.8. But for $S_j[AD]$ we will have to go quite far from $[AD]$. Out of the zipping flow

$$\{\text{singularities} \rightarrow B\},$$

essentially played in reverse, one will be able to extract the continuous arc $S_j[XY]$ for $[XY] \equiv [AD]$. This will be the so-called (zipping)⁻¹ process, to be explained with great detail in the next section. Then, like before, X now at A , will be pushed slightly, into the interior of one of the BLUE regions R_2, R_3 by now already in the ditch, via i) above, then we push $[X, X + \varepsilon] \subset R_{2 \text{ or } 3}$ along $S_j[XY]$ and then we finally perform the smearing $[XY] = S_j[XY]$, now for $[XY] = [AD]$.

iii) We end with a partial ditch filling starting at D (figure 5.6-(C)), concerning now R_4 and going in the $-z$ direction.

iv) Very importantly, at this point, once the zipping is to be invoked albeit in reverse, the uniformly bounded zipping length from TRILOGY I [29] will be one of the main actors.

For the needs of the long statement of our ZIPPING LEMMA, enough has been said concerning the ditch-jumping steps $S_m \Rightarrow S_{m+1}$.

Only the acyclic part of the Process Z has been discussed, so far, and now we turn to the $0(3)$ -part. Of course, this happens far from triple points and the only novelty is that, the $h(3)$ DITCH-FILLING is now no longer just a Whitehead dilatation. We have now the addition of 2-handles internal to $h(3)$, not compensated by any 3-handle. But, otherwise, things are like before and we are GSC-safe:

6) For what will come next, it will be necessary to invoke the following list of ingredients: The fins, the fact that $M_2(f) \subset X^2$ can only accumulate on $\text{LIM } M_2(f) \approx \sum(\infty)$, and then finally the process (zipping)⁻¹ which will turn out to be PROPER. But it will only be explicitly defined in the next Section VI. Making use of these ingredients, we can **choose** the location of the shadow arcs, which are anyway demanded by the steps $m \Rightarrow m + 1$ which are DITCH-jumpings, so that they should not accumulate at finite distance, but on $\partial \sum(\infty)_*$, which lives at infinity. Symbolically, we may write this as

$$\lim_{n=\infty} (S_j[XY]_n) = \infty.$$

7) Each of the individual $h_i^\lambda(\varepsilon)$'s, when $\varepsilon \leq 3$, has a first moment $p(h_i^\lambda(\varepsilon)) \in Z_+$ (5.30) when it occurs, meaning that $p = p(h_i^\lambda(\varepsilon))$ is the smallest index such that, inside the process Z from (5.30) we should have that

$$h_i^\lambda(\varepsilon) \subset S_p.$$

Then, during the process Z in question, $h_i^\lambda(\varepsilon)$ gets finitely many time subdivided, the subdivisions being repositioned during the pushes and smearings of the ditch-jumping steps. During all the ditch-jumping steps, for every given $(\lambda, \varepsilon, i)$ we will find that geometric incidence is bounded, in the sense that

$$\left[\sum_{\text{all } [XY]} \{\text{shadow arcs } S_j[XY]\} \right] \cdot h_i^\lambda(\varepsilon) < \infty.$$

After the finitely many subdivisions and repositionings, there is a final transformation, defined for every triplet $(\lambda, i, \varepsilon)$ and stabilizing things forever after, as far as $(\lambda, i, \varepsilon)$ is concerned,

$$(5.31.5) \quad h_i^\lambda(\varepsilon)_{\text{initial}} \implies \sum_{j=1}^N h_{ij}^\lambda(\varepsilon), \quad N = N(\lambda, \varepsilon, i).$$

For any given $h_i^\lambda(\varepsilon)$ all this happens during a finite span of time, after which the $h_{ij}^\lambda(\varepsilon)$'s are no longer subdivided nor touched by the pushes + smearings, during the infinite process (5.30). Let us say that any finite piece of $S_0 + \sum_H D^2(C(H))$ can undergo only a finitely bounded action, via the contorsions of pushing + smearing or the subdivisions coming with them. And the same for any given higher $S_m + \sum D^2(C(H))$. With this, we have a grand "geometric incidence matrix", involving all the stable $h_{ij}^\lambda(\varepsilon)$'s. These can be glued now together into a well-defined limit-object for the infinite process Z from (5.30) which creates it, call it

$$(5.32) \quad S'_b(\widetilde{M}(\Gamma) - H)_I \equiv \lim_{n=\infty} S_n.$$

Here the $\lim_{n=\infty}$ includes not only inclusions but also ditch-jumps. But the point is that, if we consider all those infinitely many indices quadruples occurring in the RHS of (5.31.5), and which by now are stabilized, it makes perfectly good sense to write, along with (5.32) the

$$(5.32.1) \quad S'_b(\widetilde{M}(S) - H) = \left\{ \sum_{(\lambda, \varepsilon, i, j)} h_{ij}^\lambda(\varepsilon) \text{ glued naturally together} \right\} = \bigcup_{(\lambda, \varepsilon, i, j)} h_{ij}^\lambda(\varepsilon).$$

The ditch jumping consists of (2-by-2) disjoint elementary steps, corresponding to the triple points, and these are forming an infinite sequence, without accumulation at finite distance. Also, the infinite process Z (5.30), induces a smooth PROPER embedding

$$S_0 \xrightarrow{Z} S'_b(\widetilde{M}(\Gamma) - H)_I.$$

We claim that this $S'_b(\widetilde{M}(\Gamma) - H)_I$ has a natural structure of smooth $(N+4)$ -manifold, with large boundary. With all this, one may also interpret now our Z as an embedding map, occurring in the following commutative diagram of PROPER embeddings, to be compared to (4.25),

$$(5.33) \quad \begin{array}{ccc} S_0 \equiv \Theta^{N+4}(X^2 - H)_I - \text{DITCH} & \xrightarrow{j} & S'_u(\widetilde{M}(\Gamma) - H)_I \\ \downarrow Z & \nearrow \mathcal{J} & \\ S'_b(\widetilde{M}(\Gamma) - H)_I & & \end{array}$$

Here, the map j essentially, SENDS THINGS IN THE DITCH. The process Z builds up the $S'_b(\widetilde{M}(\Gamma) - H)_I$, defined in (5.32), and as an embedding of $S_0 \rightarrow S'_b$, the “ Z ” should be clear too. Then, every newly created piece has its natural location inside $S'_u(\widetilde{M}(\Gamma) - H)_I$, once things go into the DITCH. This is what defines the map \mathcal{J} .

8) Just like in the toy-model, the embedding \mathcal{J} is actually isotopic, via a simple-minded isotopy of non-boundary respecting injections, to a simple-minded **diffeomorphism** like in (4.26), the η below

$$(5.34) \quad S'_b(\widetilde{M}(\Gamma) - H)_I \xrightarrow[\approx]{\eta} S'_u(\widetilde{M}(\Gamma) - H)_I.$$

[But we prefer to keep \mathcal{J} and η distinct, since it is \mathcal{J} which makes the diagram (5.33) strictly commutative, and not the η .]

9) Completely analogous to the α from (5.14), there is also second embedded framed link

$$(5.35) \quad \underbrace{\sum_{H \in (5.7)} C(H)}_{\beta} \xrightarrow{\beta} S'_b(\widetilde{M}(\Gamma) - H)_I,$$

and we have the following diagram, connecting together these various objects

$$(5.36) \quad \begin{array}{ccc} S'_b(\widetilde{M}(\Gamma) - H)_I & \xrightarrow[\text{DIFFEOMORPHISM}]{\eta} & S'_u(\widetilde{M}(\Gamma) - H)_I \\ & \swarrow \beta & \searrow \alpha \\ & \underbrace{\sum_{H \in (5.7)} C(H)} & \end{array}$$

In (5.36), for the H (normal) we have

$$\partial C(H) \underset{\beta}{\subset} \partial S'_b, \quad \partial C(H) \underset{\alpha}{\subset} \partial S'_u.$$

This diagram has the following features:

(5.36.1) When it comes to the $H \in \{\text{BLACK Holes}\} + \{H(p_{\infty\infty})\}$, for which the corresponding curves are such that $C(H) \subset \text{int } S'_z$, then our diagram commutes strictly (up to isotopy at least).

(5.36.2) But when it comes to the other $H \in \{\text{completely normal Holes}\}$, which are such that for the corresponding curves $C(H) \subset \partial S'_z$, then the (5.36) only **commutes up to homotopy** (at the level of $\partial S'_z$).

10) Let us go back now to the region R_2 . After the step $S_{m+\frac{3}{4}}$, as presented in figure 5.10, has been performed, we are far from the S -wound from (5.27.3-ter).

This is so because, as figure 5.4 may show, one can proceed from $Y = A$ to the DITCH of R_2 , without crossing the wound S . But this is NOT SO for R_3 , which will be discussed next. To begin with, in figure 5.11 we suggest how the dith filling material sits inside the S -wound.

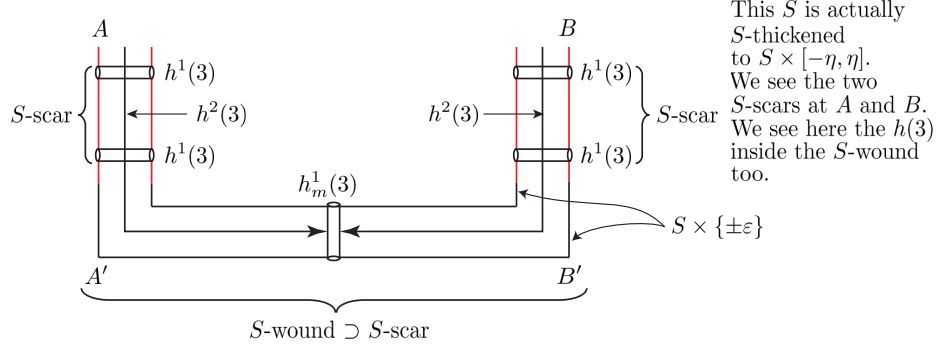


Figure 5.11.

In this figure, which lives at $x = x_n$ we have suggested, schematically, the S -wound from the figure 5.4 (to which the “A”, “B” refer), figure 5.4-bis and formula (5.27.3-ter). The S -scar is the small shaded (pair of) region(s) in figure 5.4 and in (5.27.3-IV).

Here are further EXPLANATIONS FOR FIGURE 5.11. The two arrows suggest the collapsing flow, at $x = x_n$, in the arch S . This has to continue then from $h_m^1(3)$ on, towards $x < x_n$, in the style of the description from 3) above, so as to make sure that, just by itself, the $h(3)$ -material is GSC, actually a J.H.C. Whitehead dilatation. [We are NOT GSC just at $x = x_n$ above, for $h(3)$, actually, the whole zipping trajectory, from the singularity up to $x = x_n$, is necessary so as to have the GSC-feature for $h(3)$ fully displayed. The zipping comes with its own internal collapsing flow.]

Remember that this collapsing flow means here $\xrightarrow{\partial h^2(3) \cdot \delta h^1(3)}$. When going into the action

$$W(\text{BLACK}) + W^j(\text{RED}) \implies W(\text{BLACK}) \cup W^j(\text{RED}),$$

then the two W (complementary) which are involved come with associated $\frac{1}{2} B^N(\text{RED})$ and $\frac{1}{2} B^N(\text{BLACK})$ (see figure 5.4-bis) and then, for the purposes of the next zipping with $W_n(\text{BLUE}) - H$, we also have a much smaller $b^N \subset \frac{1}{2} B^N(\text{RED}) \subset B^N$, see figure 5.4. It is understood that the ditch-filling $h(3)$ (BLACK/RED), which heals the wound S (figure 5.4-bis), is added in such a way that b^N , which corresponds to the S -scar, should interact with it like the figure 5.4 suggests. For R_2 the scar is just brushed without interacting, while for R_3 we have the non-trivial interaction from figure 5.11-bis. For R_2, R_3 the DITCHES are indentations in $b^N \subset B^N$. And, as a side remark, the splitting $B^N = \frac{1}{2} B^N(\text{RED}) \cup \frac{1}{2} \text{BLACK}$ **only** concerns $[-\varepsilon \leq y \leq \varepsilon] \times [-\varepsilon \leq z \leq \varepsilon]$, which is outside the world of the present DITCHES.

We come now to the analogues of figure 5.10, when we move from R_2 to R_3 and when the ditch-filling by R_3 interacts with the S -scar. When R_3 is still an abstract stick growing out of $h(1)$ (see figure 5.7-bis), we **chose**, with the corresponding DITCH for the time being completely empty, to keep the whole S -scar from figure 5.11 alive. With this, we have now the scenario from figure 5.11-bis, to be compared to figure 5.10.

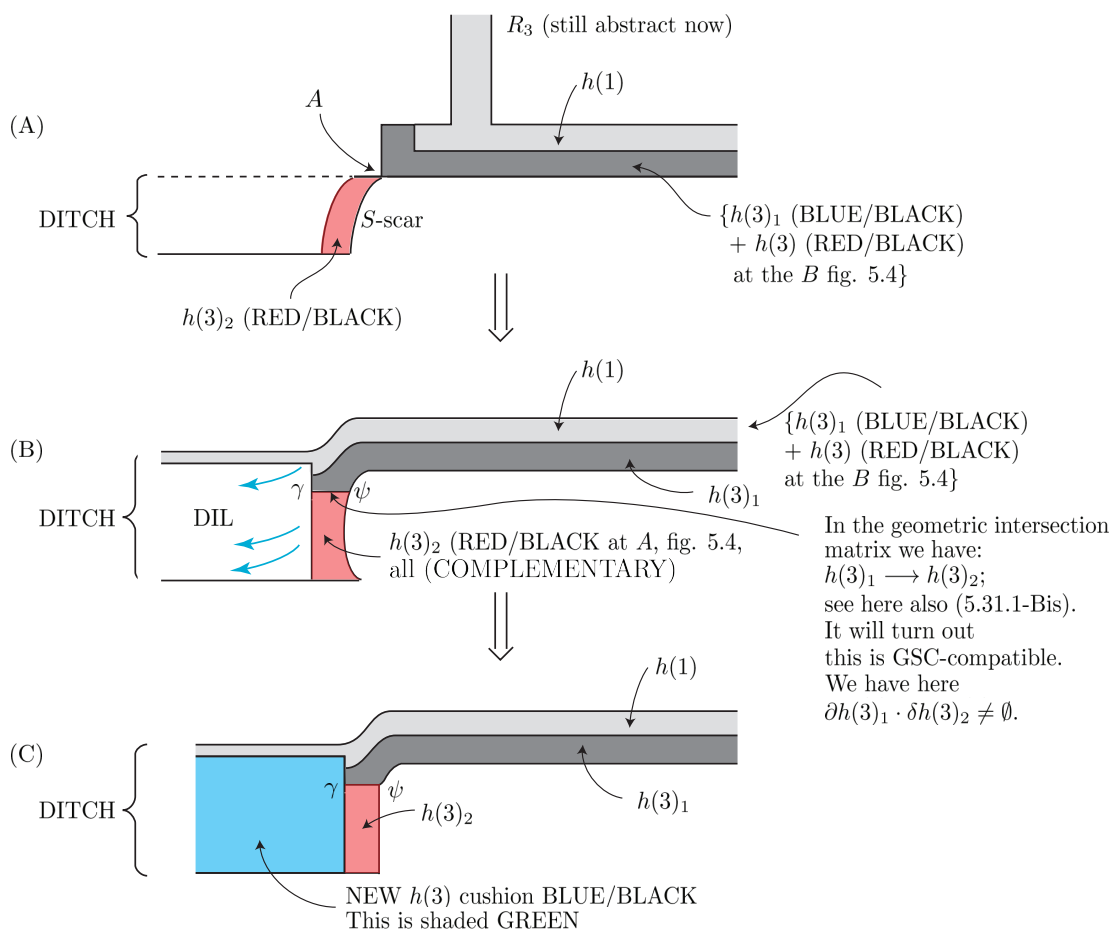
11) Making use of (5.35) we can define the following cell-complex

$$(5.37) \quad S'_b \widetilde{M}(\Gamma)_I \equiv S'_b(\widetilde{M}(\Gamma) - H)_I \cup \underbrace{\sum_{\beta C(H), \text{ for } H \in (5.7)} D^2(C(H))}_{\beta C(H), \text{ for } H \in (5.7)}.$$

This cell-complex comes with a decomposition into spare parts $S'_b(\widetilde{M}(\Gamma))_I = [\{S_0 + \sum_H D^2(C(H))\}]$, which is GSC, submitted to the infinite sequence of normal GSC-preserving steps, plus an infinite sequence of pushing

+ smearing steps each of them GSC-preserving too} + {the dilatation $\Sigma h(2)$ } + {the ditch filling addition $\Sigma h(3)$ by itself GSC preserving and also affecting GSC-wise the previous items}, all of this put nicely together like in (5.32), (5.32.1)]. So, essentially as a formal consequence of things already said in this very long lemma, it follows that $S'_b M(\Gamma)_I \in \text{GSC}$. This immediately also implies, for the REAL LIFE context, that $S'_b \widetilde{M}(\Gamma) \in \text{GSC}$. Then, with the help of an all-important COMPACTNESS LEMMA, we will deduce that $S'_u \widetilde{M}(\Gamma)_I \in \text{GSC}$, which implies $S'_u M(\Gamma) \in \text{GSC}$. Remember here that the whole idea of going from S'_u to S'_b is to replace quotient space projections by inclusion maps (of the GSC preserving type). This ends the statement of the very long ZIPPING Lemma.

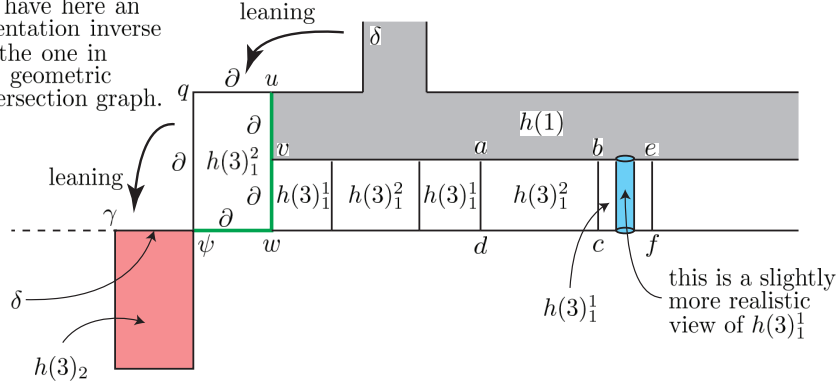
Remarks. I) The easy special BLUE 1-handles $[AB]$ go across the $W(\text{RED})$; we have looked at them, with a certain amount of detail. But there are also the difficult special BLUE 1-handles $[AD]$ going across $W_{(\infty)}(\text{BLACK})$. We will look at them with more details, in the next section.



Here is, in time, $(A) \Rightarrow (B) \Rightarrow (C)$; we see how R_3 gets into the DITCH, after the pushing and smearing from (5.31.2). In (B) the $h(1)$ leans down in the DITCH, where it lives freely. But in (C) the DITCH FILLING material is added too. In (D) we see a detail of (A) stressing the cell-decomposition of $h(3)_1$. The inverse of the upper arrow and the lower arrow suggests how the (B), (C) influence the geometric incidence matrix and how $h(3)_1 \rightarrow h(3)_2$ occurs.

(D)

For the upper leaning arrow, we have here an orientation inverse to the one in the geometric intersection graph.



The leaning, bringing δ and ∂ into contact creates more entries in the geometric intersection matrix.

Figure 5.11-bis.

For $h(3)_1^1$ (with the upper 1 meaning $\lambda = 1$), things like $[b, e] + [c, f]$ are attaching zone (∂) while $[b, c] + [e, f]$ are lateral surface (δ). For $h(3)_1^2$ (with 2 meaning now $\lambda = 2$) it is $[a, b, c, d]$ (or in the extreme left $h(3)_2^1$) the $[q, u, v, w, \psi]$, which are ∂ in the context (A). In (A), the non-green part of the extreme left $\partial h(3)_2^1$ is free, before bending.

II) When it comes to the DITCH-jumping step which makes use of the easy RED $S_j[BA]$ then, as it was explained in 5) of the ZIPPING LEMMA, the *isotopic* push of the $R_1 \subset h(1)$ along our $S_j[XY]$, with $[XY] = [BA]$ is permanently CUSHIONED by $h(3)$.

This makes that the isotopic pushing in question never brings additional contacts $C(H)_i(\text{BLUE}) \subset h_2^1(1) \cdot h_j^1(1)$. Moreover, the same isotopy + cushioning does not change the (5.31.1), at all, as far as $h(1)$ itself is concerned.

III) Concerning the impact of the DITCH-jumping step on $\partial h^2(3) \cdot \delta h^1(3)$, more will be said in the next section.

IV) So far, we have mostly concentrated on the non-complementary wall $W(\text{BLUE})$, but there are also the non-complementary walls $W(\text{RED} - H^0)$'s. We have

$$(W(\text{RED} - H^0)) \cap M_3(f) = \emptyset$$

and so there are no special (RED) 1-handles for $W(\text{RED}) - H^0$. End of Remarks.

The proof of the zipping lemma will occupy all of the next Section VI.

COMMENTS. A) Consider, in the neighbourhood of $p_{\infty\infty}$, a

$$\{W^i(\text{RED}) \mid (x \leq x_\infty + \zeta_i, \text{ with } \zeta_i > 0, \zeta_i \rightarrow 0 \text{ when } i \rightarrow \infty)\} \approx W^i(\text{RED} \cap H^0);$$

this occupies its corresponding DITCH $\subset V = W(\text{BLACK})$, all the way from $z = -\varepsilon$, to $z = +\varepsilon$. The *partial* ditch filling, inside $V - \text{DITCH}$ is to be used here *only* for a piece of $W(\text{RED})$ namely the $\{W^i(\text{RED} - H^0) \mid$

$x \geq x_\infty + \zeta_i\} - H$, which is a non-complementary wall, see here, also the figure 2.4.(B). Without all these various things, the very important object $S'_b(\widetilde{M}(\Gamma) - H)$ would have been unmanageably on the wrong side of the non-metrizability barrier, **if** we would not have drilled out the $p_{\infty\infty} \times [-\varepsilon, \varepsilon]$'s. But, as has been seen, once this is done, a whole can of worms is opened; and the present paper will certainly have to deal with them. The $\xi_i > 0$ above is forced on us by the accumulation of $W_n(\text{BLUE})$, $n \rightarrow \infty$, on S_∞^2 . Retain here that inside N_∞^2 there is no partial ditch-filling at all. And then, inside \overline{N}_∞^2 we have partial DITCH filling for $\{W(\text{BLUE}) - H\}$ and for the $\{W(\text{RED} - H^0) - H\}$. Here, $N_\infty^2/\overline{N}_\infty^2$ are like in figure 2.4-(B).

B) Generically, our special BLUE 1-handles occur essentially as

$$[XY] \subset C(H(\text{completely normal})) \subset W(\text{BLUE}) - H \quad (\text{see figure 5.6-(C)}),$$

and, in terms of (5.33) + (5.35) + (5.36) we will find that $j[XY] = \eta \circ \beta[XY] \subset \partial S'_u(\widetilde{M}(\Gamma) - H)$. Moreover, in terms of what has been said at 4) in the Lemma 5.1, and according to (5.35) in 9), we get $\beta[XY] = S_j[XY] \subset \partial S'_b(\widetilde{M}(\Gamma) - H)$. In view of these things, we may sometimes use the loose notation “[XY] = $\beta[XY]$ ”, a reminder of the smearing step in (5.31.2). In this same vein, the “ β ” in (5.35) is here a reminder of the fact that this arrow is “bizarre”, since it uses the possibly complicated repositionning of $C(H) \supset [XY]$, along the shadow arc $S_j[XY]$.

So, we have the commutative diagram

$$\begin{array}{ccccc} S_0 \supset [XY] \subset C(H) & \xrightarrow{\beta} & S'_b(\widetilde{M}(\Gamma) - H)_I & \xrightarrow{\eta} & S'_u(\widetilde{M}(\Gamma) - H)_I \\ \downarrow & & & & \uparrow \\ & & \xrightarrow{j(5.33) = \eta \circ \beta} & & \end{array}$$

C) While the reconstruction formula (5.14.1) is a posteriori, in the sense that it concerns two already defined objects, the superficially analogous formula (5.37) is a priori, it **defines** the $S'_b \widetilde{M}(\Gamma)$.

In (5.14.1), the $\alpha C(H)$'s (see (5.14)), play exactly the same role as the $\beta C(H)$ in (5.37). End of Comments.

The next lemmas 5.2 to 5.7 are various complements to the ZIPPING LEMMA.

Lemma 5.2. – 1) The process Z from (5.30) is modelled after a precise zipping strategy for our $X^2 \xrightarrow{f} fX^2$ from (2.1), which is **equivariant**

$$(5.38) \quad X^2 \equiv X_0^2 \longrightarrow X_1^2 \longrightarrow X_2^2 \longrightarrow X_3^2 \longrightarrow \dots$$

This strategy will be made explicit in Section VI.

2) So, the whole process Z which mimicks (5.38) is equivariant too. It follows that there is a free Γ -action on $S'_b \widetilde{M}(\Gamma)_I$ entering into the following commutative diagram, the vertical arrows of which are the natural inclusions

$$(5.38.1) \quad \begin{array}{ccc} (\Gamma \times S'_b(\widetilde{M}(\Gamma)))_I & \longrightarrow & S'_b(\widetilde{M}(\Gamma))_I \\ \uparrow \text{id} & & \uparrow \\ (\Gamma \times S'_b(\widetilde{M}(\Gamma) - H))_I & \longrightarrow & S'_b(\widetilde{M}(\Gamma) - H)_I \end{array}$$

Whenever it makes sense, the zipping lemma is equivariant.

3) There are actually homotopy equivalences

$$(5.39) \quad S_i + \sum D^2(C(H)) \longrightarrow X_i^2$$

which, together with (5.30) + (5.38) can be put into an infinite, homotopy-commutative ladder. \square

The next lemma expresses a very important property of the objects of type S'_ε (where ε can mean u or b) and where the prime ($'$) means that $p_{\infty\infty}$ (all) are excised, as opposed to the objects of type S_ε , where only the $p_{\infty\infty}(S)$ are excised. This property, expressed here at level $\widetilde{M}(\Gamma)$, survives at level $M(\Gamma)$ too, and then at this last level, it will play a crucial role in the proof of the COMPACTIFICATION LEMMA 5.6, in Section VII. But before we can really implement our next lemma we have to come back with some **metrical adjustments** concerning the zipping together of two COMPLEMENTARY WALLS. When I say “metrical adjustments” what I mean is that, TOPOLOGICALLY speaking we will not contradict anything already said but, in order to keep the zipping of two complementary walls together, away from the compensatory discs $D^2(H(p_{\infty\infty})) \times \frac{1}{2} B^N$, and to be able to cope with the accumulation of double lines in $fM_2(f)$ on $\text{LIM } M_2(f)$, we need to make some metrical changes.

At this point, to begin with, there are two cases of interest now, namely:

I) ZIPPING OF $W_{(\infty)}(\text{BLACK})$ and $W(\text{RED} \cap H^0)$.

II) In the presence of the immortal singularities \bar{S} , ZIPPING of type $W(\text{BLACK}) \cap W(\text{BLACK})^*$.

We will start with I). Formula (5.30.1-bis) will be changed into the following, less symmetric version,

$$(5.39.1) \quad B^N = b^N \cup \overline{(B^N - b^N)}, \text{ which is a SPLITTING along } B^{N-1} \text{ (the new } S) = b^N \cap \overline{(B^N - b^N)}, \text{ a new meaning for } B^{N-1}. \text{ Also } \|b^N\| \ll \|\frac{1}{2} B^N\|.$$

Then, formula (5.30.1-ter) is changed into the following, when the (z, y) are like in figure 5.4,

$$(5.39.2) \quad B_0^{N+2} \text{ (with new meaning now)} = (([-\varepsilon \leq z \leq \varepsilon] \cap [-\eta_n \leq y \leq \eta_n]) \times I \times (\{0\} \times B^{N-1} \text{ (like above)}) \cup [(z = \pm\varepsilon) \times (-\eta_n \leq y \leq \eta_n)] \times I \times (b^N \equiv \frac{1}{2} B^N(\text{RED})) \cup [(y = \pm\eta_n) \times (-\varepsilon \leq z \leq \varepsilon)] \times I \times (\overline{(B^N - b^N)} \equiv \frac{1}{2} B^N(\text{BLACK})), \text{ and here } \lim_{n=\infty} \eta_n = 0. \text{ End of (5.39.2)}$$

These new formulae also come with a new vision of the S -wound (see figure 5.4 and (5.27.2)), namely

$$(5.39.3) \quad S(\text{wound}) = [[-\eta_n \leq y \leq \eta_n - \varepsilon \leq z \leq \varepsilon]] \cup \left(\underbrace{[-\eta_n \leq y \leq \eta_n] \times (z = \pm\varepsilon) \times \frac{1}{2} B^N(\text{BLACK})}_{\text{the new vision}} \right) \cup \left(\underbrace{[-\varepsilon \leq z \leq \varepsilon] \times (y = \pm\eta_n) \times \frac{1}{2} B^N(\text{RED})}_{\text{the new vision}} \right).$$

End of (5.39.3)

In (5.39.1) a new b^N occurs, different from the b^N we have already used in our zipping C/NC . We have

$$\left\| \frac{1}{2} B^N(\text{BLACK}) \right\| \gg \left\| \text{new } b^N (= \frac{1}{2} B^N(\text{RED})) \right\| > \left\| \text{old } b^N(\text{non complementary}) \right\|,$$

of course. Also old $b^N \subsetneq$ new b^N . All this should fit well with what we see in figure 5.4.

We know already that figure 5.5 deals with the zipping

COMPLEMENTARY / COMPLEMENTARY .

But then, because of the presence of $D^2(p_{\infty\infty}) \times \frac{1}{2} B^N$ the $W_{(\infty)}(\text{BLACK})$ and the $W(\text{RED} \cap H^0)$ *cannot* play a symmetrical role. And this forces our metrical adjustments.

With this, the ZIPPING I) is like before and now we move to II). At this point we should look at the figures 2.8, 6.2, 7.2.4 and we are concerned now with the zipping together of a $W(\text{BLACK})$ (overflowing) with an infinite sequence $W_n(\text{BLACK})^*$ (subdued), with

$$\lim_{n=\infty} W \cap W_n^* = p_{\infty\infty}(S),$$

and of course we may also want to study $W(\text{BLACK})^*$ (overflowing)/ $W(\text{BLACK})$ (subdued). We proceed here exactly like for I), with the changes

$$W_{(\infty)}(\text{BLACK}) \Rightarrow W(\text{BLACK}) \text{ (overflowing)}, W(\text{RED} \cap H^0) \Rightarrow W(\text{BLACK}) \text{ (subdued)}.$$

It is only with these changes that the next lemma holds in whole generality.

Lemma 5.3. – *We consider some arbitrary sequence of points*

$$P_n = (x_n, y_n, z_n, t_n) \in S'_\varepsilon(\widetilde{M}(\Gamma) - H) \mid W_{(\infty)}(\text{BLACK})$$

which is such that $\lim_{n=\infty} (x_n, y_n) = (x_{\infty}, y_{\infty}) = p_{\infty\infty}$, $-\varepsilon \leq z_n \leq +\varepsilon$, i.e. which is **without restrictions** on the values z_n , and with $t_n \in b^{N+1} \subset B^{N+1}$ (see (5.15)). With this, we have the following items.

[Here $B^{N+1} = B^N \times I$, with a mute I (which we will sometimes just omit), and the same for the b^{N+1} .]

1) Both at the level $S'_\varepsilon(\widetilde{M}(\Gamma) - H)$ and at the levels

$$(5.40) \quad S'_\varepsilon(\widetilde{M}(\Gamma) - H) \underbrace{\cup}_{\sum C(p_{\infty\infty}(\text{all}))} \sum \left(D^2(H(p_{\infty\infty})) \times \frac{1}{2} B^N \right) \subset S'_\varepsilon \widetilde{M}(\Gamma),$$

always with an S'_ε , and it is very important that we work here with S'_ε AND NOT with S_ε , we have that

$$(5.41) \quad \lim_{n=\infty} P_n = \infty.$$

2) Our sequence $\{P_n\}$ stays far both from the $[(A(p_{\infty\infty}) \subset [-\frac{\varepsilon}{2} \leq z \leq \frac{\varepsilon}{2}]) \times \frac{1}{2} B^N - \{\text{the indentations}\}]$, and also the $\{P_n\}$ is undisturbed by the SUPERHEALING process. End of Lemma 5.3.

Proof. This is an easy consequence of the fact that, at level S'_ε all the $p_{\infty\infty} \times [-\varepsilon \leq z \leq \varepsilon]$ are completely deleted, and that

$$b^N \cap \frac{1}{2} B^N = \emptyset \subset B^N.$$

□

IMPORTANT COMPLEMENT TO Lemma 5.3. Our Lemma 5.3 remains valid if one replaces b^N with the just slightly larger (but still very small) $2b^N$, from figure 5.2-(D). This fact will be useful later on. □

In the zipping Lemma 5.1, at (5.32) we have defined the $S'_b(\widetilde{M}(\Gamma) - H)_I$ and at (5.37) the $S'_b \widetilde{M}(\Gamma)_I$. Now, we introduce a slightly smoother version of the cell-complex $S_b \widetilde{M}(\Gamma)_I$, as follows

$$(5.42) \quad S_b \widetilde{M}(\Gamma)_I \equiv \left\{ S'_b(\widetilde{M}(\Gamma) - H)_I \cup \sum_{H \in (5.7)} D^2(C(H)) \right\},$$

so far just like in (5.37), but then to this the following two operations are, afterwards applied:

a) first a simple-minded change $\{C(H(\text{BLACK hole}))\} \Rightarrow \partial H(\text{BLACK Hole})$ i.e., with the notations from the figure 5.1

$$\partial C(H(W(n))) \implies \partial H(W(n));$$

b) Next, the SUPER-HEALING step (3.34) is applied to the $p_{\infty\infty}(\text{proper})$, leaving only the $C(H(p_{\infty\infty}(S))) = \partial D^2(H(p_{\infty\infty}(S)))$ singular}. End of (5.42).

It is understood that a), b) are to be incorporated in the definition of $S_b \widetilde{M}(\Gamma)$.

Here the operation a) makes our S'_b be smooth at the BLACK Holes (now filled), and it is already smooth at the usual Holes (filled too). The (3.34) was defined, a priori, in the context S'_u , but it makes perfectly sense here too. It leaves us with an $S_b \widetilde{M}(\Gamma)$ which is a smooth $(N+4)$ -manifold, except for mild singularities at the $C(H(p_{\infty\infty}(S)))$'s.

The next lemma, puts together some technicalities which we shall need later on.

Lemma 5.4. – 1) *Without any loss of generality, the following inclusions take place in the neighbourhood of any $p_{\infty\infty}$ (and see here figures 3.3, 5.5 too).*

(5.43.1) $A(p_{\infty\infty}) \times (\frac{1}{2} + \varepsilon_0)B^N \mid \{S'_b \text{ or } \Theta^{N+4}\} \subset [-\frac{\varepsilon}{2} \leq z \leq \frac{\varepsilon}{2}] \times B^N - \{\text{the indentation in (5.28.1), i.e. the DITCHES in (5.29)}\}$; here and also in the next items, we ignore those dimensions of the S'_b and/or Θ^{N+4} , when not explicitly mentioned; we have also ignored here the $I \ni u$.

(5.43.2) $\{\text{Ditch filling and } h(3)\} \mid p_{\infty\infty} \subset b^N \subset B^N - (\frac{1}{2} + \varepsilon_0)B^N$;

(5.43.3) $\{\text{adding fins and sending } \partial \sum^\wedge(\infty) \text{ to infinity}\} \subset \{|z| \geq \varepsilon\}$;

(5.43.4) $\{\text{Ditch-jumping, pushing and smearing}\} \subset \{|z| \geq \frac{3\varepsilon}{4}\}$ (let's say close to $z = \varepsilon$, or just on the surface of S'_ε (and $\partial S'_\varepsilon$)).

The point, for items (5.43.3), (5.43.4) is that they are far from $A(p_{\infty\infty}(\text{all}))$.

2) As a consequence of the 1) above, the Z-action (5.30) is disjointed from $A(p_{\infty\infty}) \times (\frac{1}{2} + \varepsilon_0)B^N$ which is the site of the action in Lemma 3.2, see in particular (3.34). This last action concerns the $p_{\infty\infty}$ (proper)'s, and never the $p_{\infty\infty}(S)$'s, remember. No question of HEALING the $p_{\infty\infty}(S)$ as long as we are working on the GSC-Theorem, $S_u \widetilde{M}(S) \in \text{GSC}$, i.e. in the present Trilogy II. We need here all the immortal singularities S to be at $x = x_\infty$, which keeps the $p_{\infty\infty}(S)$ very pathological.

3) Starting from any of the two smooth manifolds $S'_\varepsilon(\widetilde{M}(\Gamma) - H)_I$, where $\varepsilon = u$ or b , we have the following chain of transformations

$$(5.44) \quad S'_\varepsilon(\widetilde{M}(\Gamma) - H)_I \xrightarrow[\text{add } D^2(H(\text{normal}))]{\textcircled{1}} S'_\varepsilon(\widetilde{M}(\Gamma) - H(p_{\infty\infty}(\text{all})))_I \xrightarrow[\text{add } D^2(H(p_{\infty\infty}(S)))]{\textcircled{2}} \\ S'_\varepsilon(\widetilde{M}(\Gamma) - H(p_{\infty\infty}(\text{proper})))_I \xrightarrow[\text{add } D^2(H(p_{\infty\infty}(\text{proper})))]{\textcircled{3}} S'_\varepsilon \widetilde{M}(\Gamma)_I \xrightarrow[\text{(3.34)}]{\textcircled{4}} S_\varepsilon \widetilde{M}(\Gamma)_I.$$

The notations $(S'_\varepsilon(\widetilde{M}(\Gamma) - H(p_{\infty\infty}(\eta))))_I$'s mean here that only the $H(p_{\infty\infty}(\eta))$'s and their compensating disks are being removed. Here only the first two objects are smooth $(N+4)$ -manifolds, the other are cell-complexes with mild singularities. Moreover, concerning this chain of transformations (5.44), we have the following facts too

(5.44.1) A lot of pairs of steps in (5.44) commute with each other; we have $[\textcircled{1}, \textcircled{2}] = [\textcircled{2}, \textcircled{3}] = [\textcircled{3}, \textcircled{1}] = 0$ and then, also $[\text{adding the } D^2(H(\text{normal}))\text{'s}, \{\text{adding } D^2(H(p_{\infty\infty}(\text{all})))\} + \{\textcircled{4}, \text{ which only concerns the } p_{\infty\infty}(\text{proper})\}] = 0$. But then, when it comes to $\textcircled{4}$, this requires the previous addition of the $D^2(H(p_{\infty\infty}(\text{proper})))$'s. Remember that this $\textcircled{4}$ is our process of SUPERHEALING. And, importantly, this is NOT the brutal process of putting back the deleted $p_{\infty\infty}(\text{proper}) \times (-\varepsilon, \varepsilon)$'s, but the more gentle and manageable

process of performing first an infinite collapse, and then an infinite dilatation (not the inverse of the previous process).

(5.44.2) The whole (5.44), written explicitly upstairs at level $\widetilde{M}(S)$ functions also downstairs, at level $M(\Gamma)$.

(5.44.3) In the context of (5.44) consider the two transformations, for $\varepsilon = u$ and $\varepsilon = b$.

$$(*) \quad S'_\varepsilon(\widetilde{M}(\Gamma) - H)_I \xrightarrow{\text{SUPERHEALING}} S_\varepsilon(\widetilde{M}(\Gamma))_I.$$

These two transformations are **completely isomorphic**.

EXPLANATION AND PROOF. For both transformations we can find canonical pieces which exist on both LHS and RHS sides, $S'_u(\widetilde{M}(\Gamma) - H)_I \supset A \subset S'_b(\widetilde{M}(\Gamma) - H)_I$ and $S_u(\widetilde{M}(\Gamma))_I \supset B \subset S_b(\widetilde{M}(\Gamma))_I$, such that the active part of (*) should be exactly $A \Rightarrow B$, for both ε 's. For the ① in (5.44), these A, B are reduced to the various $\partial D^2(H(\text{normal})) = C(H)$'s while for ②, ③, and certainly the SUPERHEALING ④ they are canonically isomorphic, i.e. $A \approx B$. \square

All the properties above have been claimed TRUE for VARIANT I, but the same holds for the REAL LIFE too. The active action they involve takes place far from the additions $\sum_{R_0} \text{int } R_0 \times [0, \infty]$ of the REAL LIFE version.

From the fact that $S'_b \widetilde{M}(\Gamma)_I$ is GSC, see here the 11) in the ZIPPING LEMMA, and making use of the step ④ in (5.44), for S'_b , we will be able to deduce the following

Corollary 5.5. – We have $S_b \widetilde{M}(\Gamma)_I \in \text{GSC}$. (Here $S_b \widetilde{M}(\Gamma)_I$ is like in (5.42).)

Proof. What one needs to prove is that the transformation (THE SUPERHEALING)

$$(5.44.4) \quad S'_b \widetilde{M}(\Gamma)_I \xrightarrow{(3.34)} S_b \widetilde{M}(\Gamma)_I$$

does not change the GSC property which $S'_b \widetilde{M}(\Gamma)_I$ already has, see 11) in the ZIPPING LEMMA. One should remember that the transformation ④ in (5.44) is localized inside (see figure 5.1.ter too, but now with only $p_{\infty\infty}(\text{proper})$ in focus) the following region

$$(5.44.5) \quad \sum_{p_{\infty\infty}(\text{proper})} \left(A(p_{\infty\infty}) \times \left(\frac{1}{2} + \varepsilon \right) B^N \right) \cup \{j = (h^1(p_{\infty\infty}), D^2(H(p_{\infty\infty})))\} \times \frac{1}{2} B^N,$$

confined inside $-\frac{\varepsilon}{2} \leq z \leq \frac{\varepsilon}{2}$, far from anything interesting, like the zipping action with its (possibly partial) DITCH-filling, the pushing + smearing coming with the special BLUE 1-handles or the $C(H(\text{normal})) = \partial D^2(H(\text{normal}))$.

From (5.33) and (5.44) we can derive the following diagram which also makes use of the commutativities from (5.44), and we are now in the context of the VARIANT I, remember (so there is a subscript “I” **all over** in (5.44.6) below)

$$(5.44.6) \quad \begin{array}{ccc} S'_b \widetilde{M}(\Gamma) \supset S'_b(\widetilde{M}(\Gamma) - H) & \xrightarrow{\text{add } D^2(H(p_{\infty\infty}))} & S'_b(\widetilde{M}(\Gamma) - H(\text{normal})) & \xrightarrow{(3.34)} & S_b(\widetilde{M}(\Gamma) - H(\text{normal})). \\ \uparrow \parallel Z & & & & \text{add } D^2(H(\text{normal})) \parallel \downarrow \\ S_0 \text{ (see (5.33))} & & & & S_b \widetilde{M}(\Gamma) \end{array}$$

The (3.34) is here again our SUPERHEALING, of course. The Z is confined inside $b^N \times \{|z| \geq \frac{3\varepsilon}{4}\}$, more precisely to values of z close to $\pm\varepsilon$, far from the (5.44.5).

When one considers the transformation of geometric intersection matrix where, in the context of (5.44.6), we go from $S'_b \widetilde{M}(\Gamma)$ to $S_b \widetilde{M}(\Gamma)$ then one can see the following fact.

The only place where the change $S'_b \Rightarrow S_b$ is really active at the level of the GEOMETRIC INTERSECTION MATRIX is the following:

(*) Inside each island $C(p_{\infty\infty}(\text{proper}))$ the internal arrows irradiating inside it (starting from $H(P_{\infty\infty})$, like in the figure 5.1-ter) get erased. Moreover, the $j(p_{\infty\infty}(\text{proper}))$ from the same figure disappears and the composite arrow

$$i \longrightarrow j(p_{\infty\infty}(\text{proper})) \longrightarrow \text{outer world}$$

gets replaced by $i \longrightarrow \text{outer world}$.

The (*) above concerns exactly the region of (5.44.5). All this is without impact on the GSC which stays intact under our transformation (5.44.4), as claimed.

Concerning (*), see again the figure 5.1.ter, but for the $p_{\infty\infty}(\text{proper})$ -island, and not the $p_{\infty\infty}(S)$ -island, which is not touched by the (5.44.4). \square

Once the Corollary 5.5 has been proved, we turn now our attention back from the $S_b \widetilde{M}(\Gamma)$ (from (5.42)), to the actually more inclusive $S_\varepsilon^{(\prime)} \widetilde{M}(\Gamma)$. The following lemma is completely analogous to Lemma 3.3.

Lemma 5.6. – 1) (Reminder) *There is a natural free action*

$$\Gamma \times S_b^{(\prime)} \widetilde{M}(\Gamma)_I \longrightarrow S_b^{(\prime)} \widetilde{M}(\Gamma)_I.$$

2) *One can, also, define directly $S_b^{(\prime)} M(\Gamma)$ downstairs and this last object has the properties*

$$(5.45.1) \quad S_b^{(\prime)} M(\Gamma)_I = (S_b^{(\prime)} \widetilde{M}(\Gamma))_I / \Gamma$$

and

$$(5.45.2) \quad S_b^{(\prime)} \widetilde{M}(\Gamma)_I = (S_b^{(\prime)} M(\Gamma))_I^\sim,$$

and, of course, $\pi_1 S_b^{(\prime)} M(\Gamma)_I = \Gamma$.

Now, for a while we will stay in the context of the VARIANT I, but for typographical commodity, we may omit most times the notation “I”.

So far, our discussion of the geometric realization of the zipping in high dimensions has proceeded UPSTAIRS, at the level of $\widetilde{M}(\Gamma)$, but now we will move DOWSTAIRS, at the level of the compact $M(\Gamma)$; this crucial **compactness** reflects, of course, the fact that our Γ is a finitely presented group. It will turn out that, once we move downstairs, then by various COMPACTNESS ARGUMENTS, we can connect $S_u^{(\prime)} M(\Gamma)$ and $S_b^{(\prime)} M(\Gamma)$, actually show that they are diffeomorphic, then via functoriality, this connection can be transported upstairs. With this, we will finally deduce that $S_u^{(\prime)} \widetilde{M}(\Gamma)$ is GSC, from the known fact that $S_b^{(\prime)} \widetilde{M}(\Gamma)$ is GSC.

According to our Lemma 5.2 above, Z was modelled on an **equivariant zipping** $X^2 \rightarrow fX^2$, but once we have this equivariance, the process Z (5.30) makes sense downstairs too, leading like in (5.32) to an object $S'_b(M(\Gamma) - H)$ downstairs where, of course, we have a more obvious $S_u^{(\prime)}(M(\Gamma) - H)$ too. It is important, at this point, that there are here two distinct, but eventually equivalent roads, leading to the same scenario DOWSTAIRS:

A) Everything which we did so far, in this paper, has been Γ -equivariant and so we can take the viewpoint that the story downstairs is just the quotient by Γ of the equivariant story upstairs. This means using the formulae like $S_\varepsilon^{(\prime)} M(\Gamma) = (S_\varepsilon^{(\prime)} \widetilde{M}(\Gamma))/\Gamma$ or the same with Holes, as definitions.

B) But then, the zipping being equivariant, it makes sense directly downstairs, instead of zipping the upper line in (2.24) we zip now the lower line and then built directly the analogue of Z (5.30), downstairs too. Next, one notices that the $S_\varepsilon^{(\prime)}$ have good functorial properties like localization and glueing (going from local to global), and like in Lemma 3.3, under appropriate maps one can push them forward and pull them backwards. So, one constructs now the $S_\varepsilon^{(\prime)} M(\Gamma)$ (possibly with Holes) directly downstairs, one notices that $S_\varepsilon^{(\prime)}(\widetilde{M}(\Gamma)(-H)) = S_\varepsilon^{(\prime)}(M(\Gamma)(-H))^\sim$ and then one deduces the definition from A) as a theorem. In the same vein, the $S_u^{(\prime)}$ certainly has good localization and glueing properties and if one looks at things from the correct viewpoint, so has $S_b^{(\prime)}$ too. So the two viewpoints A) and B) **are** equivalent, and this is important for the self-consistency of our approach. And much more flexibility is gained if one can proceed directly downstairs too.

C) We know from TRILOGY I [29] that, upstairs, we have a uniformly bounded zipping length. The zipping is equivariant, the one downstairs is a quotient image of the upper one, so there is uniformly bounded zipping length downstairs too.

D) Notice also things like

$$\Theta^3(fX^2)/\Gamma = \Theta^3(f/\Gamma(X^2/\Gamma)),$$

and the same with Holes, $X^2 - H$ instead of X^2 .

So, I will start by indicating now, rather briefly, how the zipping Lemma 5.1 and its sequels fare downstairs. The starting point is now

$$(5.45.3) \quad [\Theta^{N+4}((X^2 - H)/\Gamma) - \text{DITCHES}] + \sum D^2(C(H)) = \{\text{the quotient (5.29)/}\Gamma\}.$$

From here on, invoking now Lemma 5.2 which tells us that the process Z in (5.30) is equivariant, we can make sense of it DOWNSTAIRS too, and get then the analogue of the ZIPPING LEMMA 5.1, which holds downstairs too, except for the following two items.

The Γ -action has clearly disappeared, and any reference to GSC loses its meaning, of course, so GSC is no longer with us either. But the following items certainly will survive. To begin with the analogue of the diagram (5.33), namely the following commutative diagram of PROPER embeddings, is with us:

$$(5.46) \quad \begin{array}{ccc} \Theta^{N+4}((X^2 - H)/\Gamma)_I - \text{DITCHES} & \xrightarrow{j/\Gamma} & S'_u(\widetilde{M}(\Gamma) - H)_I \\ \downarrow z/\Gamma & \nearrow \mathcal{J}/\Gamma & \\ S'_b(M(\Gamma) - H)_I & & \end{array}$$

Again, like in the context of the map (5.33) \mathcal{J}/Γ is connected, via a simple-minded isotopy to a diffeomorphism, to be denoted again by

$$S'_b(M(\Gamma) - H)_I \xrightarrow[\approx]{\eta} S'_u(M(\Gamma) - H)_I.$$

The analogue of (5.36) survives too, but right here we will only retain from it the following $C(H(\text{normal}))$

part, which continues to be *homotopy-commutative*

$$(5.47) \quad \begin{array}{ccc} S'_b(M(\Gamma) - H)_I & \xrightarrow[\approx]{\eta} & S'_u(M(\Gamma) - H)_I \\ \uparrow & & \uparrow \\ \partial S'_b(M(\Gamma) - H)_I & \xrightarrow[\approx]{\eta} & \partial S'_u(M(\Gamma) - H)_I \\ & \swarrow \beta & \searrow \alpha \\ & \sum_{H(\text{normal})} \{C(H(\text{normal})) + \text{framing}\} & \end{array}$$

(5.47.1) Of course, just by itself the upper square in (5.47) commutes strictly. Also, the map η respects, strictly, the contribution

$$(*) \quad \sum_{p_{\infty\infty}(\text{all})} \left[A(p_{\infty\infty}) \times \left(\frac{1}{2} + \varepsilon_0 \right) B^{N+1} \right] \supset \sum_{p_{\infty\infty}(\text{all})} C(p_{\infty\infty}) \times \frac{1}{2} B^{N+1}.$$

End of (5.47)

This contribution $(*)$ lives buried deep inside the int $S'_\varepsilon(M(\Gamma) - H)$, safely away from the boundary, which is the site of the homotopy which connects $\eta \circ \beta$ and α in (5.47).

This ends our zipping lemma discussion downstairs, and with this we can move now forward.

We will state below a COMPACTNESS LEMMA, which will be proved later, in Section VII. But I start by listing now the four basic ingredients which are all necessary for this very basic lemma.

- i) The group Γ is finitely presented, hence the $M(\Gamma) = \widetilde{M}(\Gamma)/\Gamma$ which, loosely speaking is the fundamental domain of Γ , is COMPACT; the action $\Gamma \times \widetilde{M}(\Gamma) \rightarrow \widetilde{M}(\Gamma)$ is *cocompact*.
- ii) We have a uniform upper bound for the zipping length, a fact proved in [29]. And, as already noticed once we have it for $\widetilde{M}(\Gamma)$ we have it for $M(\Gamma)$ too.
- iii) We do not work now with S_ε but with S'_ε , which will allow us to take advantage of Lemma 5.3, applied DOWNSTAIRS.
- iv) VERY IMPORTANTLY too, we move from the $(N + 4)$ -dimensional VARIANT I, to the $(N + 5)$ -dimensional REAL-LIFE context. As we will see, this will allow us to render PROPER various homotopy commutativities. And, under our appropriate dimensional conditions, homotopy implies isotopy.

With these things, we get the following basic item

Compactness Lemma 5.7. – *Once we take into account the four ingredients above, the diagram (5.47) considered now in the context of the REAL LIFE VARIANT, and which we rewrite here as*

$$(5.47\text{-REAL LIFE}) \quad \begin{array}{ccc} \partial S'_b(M(\Gamma) - H) & \xrightarrow[\text{DIFFEOMORPHISM}]{\eta} & \partial S'_u(M(\Gamma) - H) \\ & \swarrow \beta & \searrow \alpha \\ & \sum \{C(H(\text{normal})) + \text{framing}\} & \end{array}$$

commutes up to a PROPER homotopy, which in our high dimensions implies ISOTOPY.

This crucial lemma will be proved in the last Section VII.

We call this “compactness lemma” both because $M(\Gamma)$ is compact and because compactification of $S'_\varepsilon(M(\Gamma) - H)$ is used for the proof.

VERY IMPORTANT REMARK. The $\alpha C(H)$, $\eta\beta C(H)$ are contained inside $S'_u(\widetilde{M}(\Gamma) - H)_I$, but the PROPER homotopy which connects them needs the larger, REAL-LIFE object

$$\partial S'_u(\widetilde{M}(\Gamma) - H) \supsetneq S'_u(\widetilde{M}(\Gamma) - H)_I.$$

□

Let us take now another look at the triple points. Once our W 's are endowed with transversal orientations, then at each triple point, we can assign signs \pm to each of the four Holes H adjacent to the triple point, more exactly signs will be associated to the four H -corners at the triple point. The signs are corner-dependent, the same H may come with different signs at its various corners. At the triple point t we have a W (BLUE), carrying four holes, and two complementary walls W_1, W_2 . When the two orientations, transversal to W_1, W_2 look away from H , the corner is “+”, otherwise it is “-”, making that exactly one hole is H^+ at t and three are H^- . This rule is illustrated in figure 5.6-(A). We move now from these corner-dependent signs, to global ones. A hole H , or rather its $C(H)$ will be C^- (respectively C^+) if at least one of its corners is - (respectively if they are all +). See here the figure 5.12 for the corner dependent situation.

We go back now to the diagrams (5.36) + (5.47), in connection with which we make the following

Claim (5.48). - Both at the level $S'_\varepsilon(\widetilde{M}(\Gamma) - H)_I$ (from (5.36)) and at the level $S'_\varepsilon(M(\Gamma) - H)_I$ (from (5.47)) we may assume, without any loss of generality that we have $\alpha = \eta \circ \beta$, when it comes to the curves

$$\{C(p_{\infty\infty}(\text{all})), C(H(\text{BLACK Hole})) \text{ and } C^+(H)\} \in \sum_H C(H),$$

complete COMMUTATIVITY here.

So, when it comes to the concern of being PROPER for the homotopy in the diagram (5.37), it is *only* the $\beta C^-(H)$ which should worry us at all. The next lemma is valid now, both upstairs and downstairs. But, in its form down in $M(\Gamma)$ it will provide an essential ingredient for the proof of the COMPACTNESS Lemma 5.6. So we only state it downstairs here. And we are here in the context of VARIANT I, still.

Lemma 5.8. - (DOWNSTAIRS) *We consider now all the various $C^-(H)$'s,*

$$(5.49) \quad C^-(H_1), C^-(H_2), \dots \subset \Theta^3(X^2 - H)/\Gamma = \Theta^3(fX^2 - H),$$

where remember that “ $-H$ ” means not only with all Holes deleted, but with all the $\{D^2(C(H)) \supset D^2(C(H(p_{\infty\infty})))\}$'s removed too.

For each H_i in (5.49) there is an embedded arc $\gamma_i \subset \partial S'_u(M(\Gamma) - H)_I$, which joins $\alpha C^-(H_i)$ to $\eta\beta C^-(H_i)$ and which is such that, when we go to the closed loop

$$(5.50) \quad \Lambda(H_i^-) \equiv \alpha C^-(H_i) \underbrace{\quad}_{\gamma_i} (\eta\beta C^-(H_i))^{-1} \rightarrow \partial S'_u(M(\Gamma) - H)_I,$$

then we have the following items

(5.50.1) *As a consequence of the homotopy-commutativity of the lower triangle in (5.47), the loop $\Lambda(H_i^-)$ as defined above, is null-homotopic in $\partial S'_u(M(\Gamma) - H)_I$. [This is standard algebraic topology text-book, of course. But then, the next two items claim that we can make very interesting choices for locating those arcs γ_i . There are hard, highly non-trivial facts, to be proved later.]*

(5.50.2) *There is a uniform bound s.t. $\|\Lambda(H_i^-)\| < N$, for all i 's.*

(5.50.3) *For every compact $K \subset \partial S'_u(M(\Gamma) - H)$ there are only finitely many j 's s.t. $K \cap \Lambda(H_j^-) \neq \emptyset$. In a more schematical language, we have $\lim_{n=\infty} \Lambda(H_n^-) = \infty$, inside $\partial S'_u(M(\Gamma) - H)_I$.*

Comments and some hints on the proof of the Lemma 5.8. – Concerning (5.50.2), once we work downstairs, it is very easy to control the $\|\alpha C^-(H_i)\|$.

The construction of the $\eta\beta C^-(H_i)$'s and of the γ_i 's will be a major objective of the next section. Since one of the main ingredients of the construction is the process “(zipping)⁻¹”, explained in the next section, the control of $\|\eta\beta C^-(H_i)\|$ and $\|\gamma_i\|$ will need the uniform boundedness of the zipping length, proved in the first paper of this series [29]. It will also need the Lemmas 6.1, 6.3 from the next section (actually the various estimates given in those lemmas). So much about (5.50.2). The (5.50.3) will follow from Lemma 5.3, in conjunction with the (6) in Lemma 6.3, next section. \square

FINAL ARGUMENTS FOR THE PROOF OF THE GSC THEOREM ($\forall \Gamma, S_u \widetilde{M}(\Gamma) \in \text{GSC}$).

We move now to the $(N + 5)$ -dimensional REAL-LIFE CONTEXT, and we start from the COMPACTNESS LEMMA 5.7, which will be proved at the end of the last Section VII, of this paper. In view of the high dimensions $N + 5 \gg 1$, which we are working with, in the context of the diagram (5.47-REAL LIFE), the PROPER homotopy from Lemma 5.7 *implies smooth ambient isotopy inside $\partial S'_u(M(\Gamma) - H)$* . So, from (5.47-REAL LIFE), we can pull now the following commutative diagram, where only smooth objects and smooth maps are involved, with the d below being a DIFFEOMORPHISM, and the two unions being along $C(H(\text{normal}))$, to which β or α has been applied, and of course we are here in the REAL LIFE context

$$(5.51) \quad \begin{array}{ccc} S'_b(M(\Gamma) - H) \cup \bigcup \sum D^2(H(\text{normal})) & \xrightarrow{d} & S'_u(M(\Gamma) - H) \cup \bigcup \sum D^2(H(\text{normal})) \\ & \swarrow \quad \searrow & \\ & \sum_{p_{\infty}(\text{ALL})} A(p_{\infty}) \supset \sum_{p_{\infty}(\text{ALL})} C(p_{\infty}) & \end{array}$$

We lean here heavily on the basic COMPACTNESS LEMMA 5.7, with the diffeomorphism d extending the η from (5.47). Also, in (5.51) the vertical/oblique arrows are here the canonical inclusions. We have denoted here by $\sum_{p_{\infty}(\text{all})} A(p_{\infty})$ the $(*)$ from (5.47.1). In connection with (5.51), both for $\varepsilon = u$ and for $\varepsilon = b$, we have the obvious equality (written here for REAL LIFE, but valid of course for VARIANT I too)

$$S'_\varepsilon(M(\Gamma) - H) \cup \sum D^2(H(\text{normal})) = S'_\varepsilon M(\Gamma) - \underbrace{\sum_{p_{\infty}(\text{all})} D^2(H(p_{\infty}))}_{\text{all}}$$

VERY IMPORTANT REMARK (and reminder). When it comes to the COMPACTNESS LEMMA 5.7, it is only for the curves $C^-(H)$, i.e. $C^-(H_1), C^-(H_2), \dots$, in (5.49), that it is non-trivial (and actually quite highly so). End of Remark.

(5.52) Also, considering the commutativity of (5.51), the

$$\sum_{p_{\infty}(\text{ALL})} A(p_{\infty}) \supset \sum_{p_{\infty}(\text{proper})} A(p_{\infty}) \text{ (of interest for us, now)}$$

concerns, just like in 5.44.3 (let now with I being changed to REAL LIFE and the $\widetilde{M}(\Gamma)$ (upstairs) to $M(\Gamma)$ (downstairs)) some regions which, canonically, are the same in S'_b and in the S'_u . End of (5.52).

Also, since (5.51) commutes, we can put back the $D^2(H(p_{\infty\infty}))$'s and get another commutative diagram, the upper line being now a diffeomorphism among cell-complexes (which are certainly non-manifolds)

$$(5.53) \quad \begin{array}{ccc} S'_b M(\Gamma) & \xrightarrow[\approx]{d} & S'_u M(\Gamma) \\ & \nwarrow \quad \nearrow & \\ & \sum_{p_{\infty\infty}(\text{all})} (A(p_{\infty\infty}) \cup D^2(H(p_{\infty\infty}))) & \end{array}$$

Now, both $S'_b M(\Gamma)$ and $S'_u M(\Gamma)$ contain $\left(\sum_{R_0} \text{int } R_0 \times [0, \infty)\right)/\Gamma \subset \{\text{complementary walls}\}$ and d in (5.53) send one onto the other. For simplicity's sake we will omit to write the $"/\Gamma$ " from now on.

We concentrate now on some $p_{\infty\infty} = p_{\infty\infty}(\text{proper}) \in W_{(\infty)}(\text{BLACK})$, and inside $W_{(\infty)}(\text{BLACK})$ we consider a disc $\overline{D}^2(p_{\infty\infty})$, centered at $p_{\infty\infty}$ such that the concentric $D^2(H(p_{\infty\infty}))$ is strictly smaller. Both $S'_u M(\Gamma)$ and $S'_b M(\Gamma)$ contain, for every $p_{\infty\infty} = p_{\infty\infty}(\text{proper})$ the piece below

$$\mathcal{X}^{N+5}(p_{\infty\infty}) = \left\{ \left((\overline{D}^2(p_{\infty\infty}) - \{p_{\infty\infty}\}) \times [-\varepsilon, \varepsilon] \right) \bigcup_{\left(\sum_{R_0} \text{int } R_0 \right) \cap \overline{D}^2(p_{\infty\infty})} \left(\left(\sum_{R_0} \text{int } R_0 \right) \cap \overline{D}^2(p_{\infty\infty}) \right) \times [0, \infty) \cup \underbrace{\bigcup_{C(p_{\infty\infty}) \times \left[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right]} D^2(H(p_{\infty\infty})) \times \left[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right]}_{C(p_{\infty\infty}) \times \left[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right]}, \text{ to which, afterwards all the steps of the passage } S'_u(\widetilde{M}(\Gamma))_I \implies S'_u \widetilde{M}(\Gamma) \text{ from Chapter III have been applied} \right\}.$$

These steps include $\Theta^4(\dots, \mathcal{R})$, coming in our case just with a factor I , the $\times B^N$ on $(\overline{D}^2(p_{\infty\infty}) - \{p_{\infty\infty}\}) \times [-\varepsilon, \varepsilon] \times I$ and the $\times \frac{1}{2} B^N$ on $D^2(H(p_{\infty\infty})) \times \left[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right] \times I$, the $N^{4+N}(\dots \text{ on } \text{int } R_0 \times [0, \infty) \dots)$ and then the final $\times [0, 1]$, making the passage from dimensions $N + 4$ to $N + 5$. Notice that, anyway $\sum_{R_0} \text{int } R_0$ does NOT touch the $p_{\infty\infty} \times [-\varepsilon, \varepsilon]$.

With this, the diagram (5.53) extends to the next commutative diagram

$$(5.54) \quad \begin{array}{ccc} S'_b M(\Gamma) & \xrightarrow[\approx]{d} & S'_u M(\Gamma) \\ & \nwarrow \quad \nearrow & \\ & \sum_{p_{\infty\infty}(\text{proper})} \mathcal{X}^{N+5}(p_{\infty\infty}) & \end{array}$$

where $A(p_{\infty\infty})$ is replaced by the larger $\overline{D}^2(p_{\infty\infty}) \times [-\varepsilon, \varepsilon]$.

Now, a priori, the passage VARIANT I \implies REAL LIFE is in the way for SUPERHEALING. The reason is that we have

$$A(p_{\infty\infty}) \cap \sum_{R_0} \text{int } R_0 \neq \emptyset.$$

A priori, again, since they do not commute exactly, if we want to combine them, we have to proceed by the following successive steps, when we want to apply SUPERHEALING to $S'_\varepsilon M(\Gamma)$ (REAL LIFE):

$$\text{REAL LIFE} \implies \text{VARIANT I, SUPERHEALING, VARIANT I} \implies \text{REAL LIFE}.$$

But we can actually do better than that.

Claim (5.55). – In the context of the commutative diagram (5.54) the operations (SUPERHEALING of S'_b) | $\mathcal{X}^{N+5}(p_{\infty\infty})$ and (SUPERHEALING of S'_u) | $\mathcal{X}^{N+5}(p_{\infty\infty})$ are completely ISOMORPHIC operations.

Proof. For the SUPERHEALING action on $S'_\varepsilon M(\Gamma)$ everything really active is concentrated inside $\sum_{p_{\infty\infty}(\text{proper})} \mathcal{X}^{N+5}(p_{\infty\infty})$ has the following description, which is the same, up to isomorphism, for $\varepsilon = u$ and $\varepsilon = b$:

-) Erase the term $(\sum_{R_0} \text{int } R_0) \cap \overline{D}^2(p_{\infty\infty}) \times [-\varepsilon, \varepsilon] \times [0, \infty)$.
-) Apply SUPERHEALING inside $\mathcal{X}^{N+5}(p_{\infty\infty})_I$, like in (3.34).
-) Put back the erased term, which anyway avoids the $p_{\infty\infty} \times [-\varepsilon, \varepsilon]$ NOW BACK. End of Proof.

Because the operation described above is exactly the same for $\varepsilon = u$ and $\varepsilon = b$, let us say SUPERHEALING $S'_u M(\Gamma)$ and $S'_b M(\Gamma)$ ARE ISOMORPHIC OPERATIONS, the diffeomorphism d in the upper arrow of the diagram below induces a diffeomorphism for the arrow $d(\text{HEALED})$, below

$$(5.56) \quad \begin{array}{ccc} S'_b M(\Gamma) & \xrightarrow[\text{DIFF}]{d} & S'_u M(\Gamma) \\ \Downarrow & & \Downarrow \\ S_b M(\Gamma) & \xrightarrow[\text{DIFF}]{d(\text{HEALED})} & S_u M(\Gamma) \end{array}$$

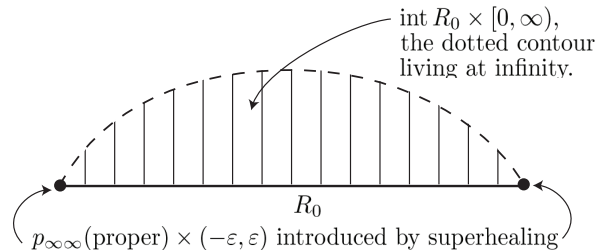
So, in an appropriate sense, our diagram (5.56) commutes. And it induces the diffeomorphism

$$(5.57) \quad S_b M(\Gamma) \underset{\text{DIFF}}{=} S_u M(\Gamma).$$

At this point, we can invoke the functoriality property of the S_ε 's (as stated for $S_u^{(\prime)}$ in the Lemma 3.3 and for $S_b^{(\prime)}$ in Lemma 5.6, and then taking the universal covering spaces of both terms in the formula (5.53), get the following diffeomorphisms **UPSTAIRS** again, in the context of the **REAL LIFE VARIANT**

$$(5.58) \quad S_u \widetilde{M}(\Gamma) = (S_u(M(\Gamma)))^\sim \underset{(5.53)^\sim}{=} (S_b(M(\Gamma)))^\sim = S_b \widetilde{M}(\Gamma).$$

Important remark. The superhealing and the $\sum_{R_0} \text{int } R_0 \times [0, \infty)$ introduced by the REAL LIFE CONTEXT, are not in conflict with each other. The little drawing below may suggest that. The point is that, at the **COMPACTIFIED LEVEL OF** $S_u(M(\Gamma))^\wedge$ (occurring in the proof of the COMPACTNESS LEMMA, the $p_{\infty\infty} \times [0, \infty]$'s are anyway crushed on $p_{\infty\infty} \times \{0\}$. End of Remark. But notice that our remark is INDEPENDENT of that was said in connection with (5.54).



I think it is nice to synthesize all this in the following commutative diagram, when the two vertical arrows stand for universal covering maps, where the lower horizontal one is our diffeomorphism (5.5) and where, HENCE, the upper horizontal arrow is a diffeomorphism too

$$\begin{array}{ccc}
 S_b \widetilde{M}(\Gamma) & \xrightarrow{\text{DIFF!}} & S_u \widetilde{M}(\Gamma) \\
 \downarrow & & \downarrow \\
 S_b M(\Gamma) & \xrightarrow{d(\text{HEALED})(\text{DIFF in (5.56)})} & S_u M(\Gamma) .
 \end{array}$$

And since $S_b \widetilde{M}(\Gamma) \in \text{GSC}$ we also have $S_u \widetilde{M}(\Gamma) \in \text{GSC}$, which CLINCHES THE PROOF OF THE GSC THEOREM, our main result in the present paper.

COMMENTS. But $S_u \widetilde{M}(\Gamma)/\Gamma = S_u M(\Gamma)$ is **NOT** compact, so even with $S_u \widetilde{M}(\Gamma) \in \text{GSC}$, we are not yet done for $\Gamma \in \text{QSF}$. The action $\Gamma \times S_u \widetilde{M}(\Gamma) \rightarrow S_u \widetilde{M}(\Gamma)$ fails to be co-compact.

A third paper in this series will be necessary, for providing us with the ingredient which we are still missing, i.e. the implication

$$S_u \widetilde{M}(\Gamma) \in \text{GSC} \implies \Gamma \in \text{GSF} .$$

In a nutshell, here is the strategy for this implication. We have a little cascade of implications: $\{S_u \widetilde{M}(\Gamma) \in \text{GSC}, \text{ the main result of the present paper}\} \implies \{\text{A certain smooth 4-manifold, on which the } S_u \widetilde{M}(\Gamma) \text{ retracts is } 4^{\text{d}} \text{ Dehn exhaustible}\} \implies \{\text{A certain singular } 3^{\text{d}} \text{ manifold, result of a further retraction, is } 3^{\text{d}} \text{ Dehn exhaustible}\} \implies \{\text{There exists a certain 3-complex } X^3 \text{ with the following features: } \pi_1 X^3 = 0, \text{ there is a free action } \Gamma \times X^3 \rightarrow X^3 \text{ with compact quotient } X^3/\Gamma \text{ such that } \pi_1(X^3/\Gamma) = \Gamma \text{ and then, finally also, } X^3 \in \text{QSF}\}.$

From the last item, it follows by standard results that $\Gamma \in \text{QSF}$. The Dehn-exhaustibility, which will be explicitly defined in the next paper of this series but see [49] and [53] too, is a forefather of the QSF concept, and it was used by the present author in the very early nineties [23], [24] and [25]. In the cascade above, the first two arrows need that the $S_u \widetilde{M}(\Gamma)$ should be transversally compact, i.e. they make use of the fact that we safely stay on the good side of the Stallings barrier. Somehow dually to this, in the present paper we are also striving to stay on the good side of the non metrizable barrier. Concerning these two barriers, which a priori push in opposite directions, making the navigation hazardous, see the introduction to the present paper and also the [27], [49], [50].

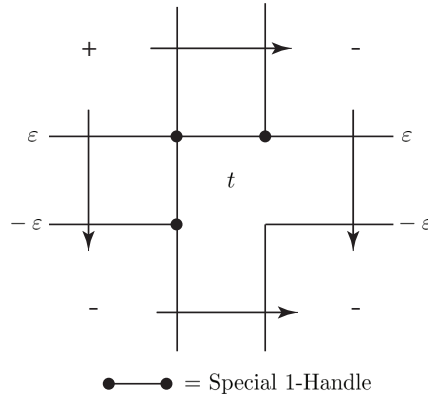


Figure 5.12.

The \pm signs for the four Holes at a triple point t .

A last comment on the zipping.

For purposes of later developments of the theory developed here, let us explain in some detail how a 2^d zipping move inside (5.28.1), let us say a zipping together of two complementary walls $W_{(\infty)}(\text{BLACK})$ and $W(\text{RED} \cap H^0)$, $(X^2 - H)_i \rightarrow (X - H)_{i+1}$ gets realized geometrically as $S_i \implies S_{i+1}$ in (5.30). We will forget the refinements (5.39.1) to (5.39.3), in order to make the exposition lighter, and look now at figure 5.4.

Let $(x = x_0, t = t_0)$ be the section through $[A'B'C'D']$. Then one sees the $\{W_{\infty}(\text{BLACK}) \times [-\varepsilon \leq z \leq \varepsilon], W(\text{RED} \cap H^0)\} | x_0$, glued along $[A'B'C'D']$.

Split now figure 5.4 along the SPLITTING SURFACE S , and also unroll completely the x -movie. One sees then the two *disjoined* contributions of our 2^d complementary wall + the $(N + 4)$ dimensional S_b .

Mounting then back together, with the correct fixing realizes

$$S_i \implies S_{i+1} \supset \Theta^3(fX^2 - H) | x_0,$$

which, when one takes into account the singularity from which this move starts, can be read as a DIFFEOMORPHISM induced by a smooth J.H.C. Whitehead dilatation. End of Story.

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