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Abstract

In this work, we propose an *a posteriori* goal-oriented error estimator for the harmonic $\mathbf{A}$-φ formulation arising in the modeling of eddy current problems, approximated by non-conforming finite element methods. It is based on the resolution of an adjoint problem associated with the initial one. For each of these two problems, a guaranteed equilibrated estimator is developed using some flux reconstructions. These fluxes also allow to obtain a goal-oriented error estimator that is fully computable and can be split in a principal part and a remainder one. Our theoretical results are illustrated by numerical experiments.

Keywords : Maxwell equations, potential formulations, goal-oriented *a posteriori* estimators, finite element method.

1 Introduction

The finite element method is widely used to solve a large variety of electromagnetic problems, and many papers have been devoted to this topic for the last decades. More particularly, in the context of low-frequency electromagnetics allowing a quasi-static approximation, some specific models are usually introduced. Among them, the so-called $\mathbf{A}$-φ formulation consists in computing a magnetic vector potential $\mathbf{A}$ as well as an electric scalar potential φ, allowing to obtain approximated values of the magnetic flux density $\mathbf{B}$ as well as of the electric field $\mathbf{E}$ arising in the Maxwell equations. From there, an important question to address is to ensure the good quality of the numerical solutions obtained. Consequently, some *a posteriori* error estimators have been developed in order to provide a global upper bound of the numerical error, as well as some local lower bounds, very useful to drive a mesh-refinement strategy. We refer to [11, 26, 5] for residual estimators and to [12, 9] for equilibrated ones, allowing in this second case to obtain a sharp upper bound of the error without any unknown multiplicative constant. All these estimators have been tested in several configurations, from academic to more industrial ones (see e.g. [27, 28, 31]). In all the papers quoted above, the error to control is defined globally, and

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corresponds to the value of a global energy, directly linked to the bilinear form arising in the variational formulation of the model.

Nevertheless, in many applications, engineers are interested in some local physical quantities, that are called quantities of interest (QOI). To stay in the field of electromagnetic problems, let us mention for example the computation of the magnetic flux through a coil [19] or the magnetic flux density at a given point of an electromagnetic device [16, 20]. Some specific estimators, called "goal-oriented estimators", have consequently to be developed in order to derive an upper bound for this kind of errors. In the case of magnetostatic problems, we refer e.g. to [29] where a dual formulation of the model is used.

In this paper, we derive a goal-oriented *a posteriori* estimator for the harmonic $\mathbf{A}$-φ formulation, to control an error defined by a linear form operating on the magnetic vector potential $\mathbf{A}$. Using some flux reconstructions of the direct and adjoint problems, the upper bound is obtained without any multiplicative constant. Moreover, this fully computable estimator can be split in a principal part and a remainder one, the last one being in some cases of higher order and can be most of the time disgarded. This result can be seen as an extension of [10], which is devoted to classical diffusion problems. Moreover, this new goal-oriented estimator is based on some equilibrated error estimators developed for the direct and the adjoint problems. They are based on equilibrated flux reconstructions, and have already been derived in a previous work for conforming finite element approximations [9]. Let us note that the present work also allows to generalize them to the case of non-conforming approximations.

The paper is organized as follows. Section 2 gives the continuous formulation of the eddy-current problem in the $\mathbf{A}$-φ formulation. Moreover, some regularity results are derived on the solution (see Lemma 2.2 to 2.4), which have their own interest but will be useful in the interpretation of our numerical results. Section 3 provides a guaranteed equilibrated error estimator for general approximations (conforming or not) of the problem, leading to Theorem 3.2 ensuring the guaranteed reliability of the estimation. Then, Section 4 details the goal-oriented functional as well as the associated adjoint problem and its discrete approximation. In Section 5, the error representation of the error is given in Theorem 5.1, and Theorem 5.2 allows to prove that the remainder term can be controlled by the product of the equilibrated estimators devoted to the direct and adjoint problems. Finally, the obtained theoretical results are illustrated by some numerical experiments in Section 6.

2 The continuous formulation

Let us consider a bounded simply connected polyhedral domain $D \subset \mathbb{R}^3$ with a Lipschitz and connected boundary $\Gamma = \partial D$. D is composed of two subdomains: the conducting domain D_c and the non-conducting domain $D_{nc} = D \setminus \bar{D}_c$. Let us remark that D_c is supposed to be bounded, simply connected with a Lipschitz connected boundary ∂D_c and strictly included into D, in the sense that $\overline{D_c} \subset D$. The source domain D_s where the divergence free current density $\mathbf{J}_s$ is imposed is usually included into D_{nc}, but our mathematical analysis does not require this assumption. The eddy current problem is given by:

Find the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ solution of

$$\begin{aligned} \mathrm{curl}\mathbf{E} &= -j\omega\mathbf{B} \text{ in } D, && \text{(1a)}\\ \mathrm{curl}\mathbf{H} &= \mathbf{J}_s + \mathbf{J}_e \text{ in } D, && \text{(1b)}\\ \mathrm{div}\mathbf{B} &= 0 \text{ in } D, && \text{(1c)} \end{aligned}$$

where $j^2 = -1$ is the unit imaginary number, ω is the pulsation that is supposed to be scalar, namely a positive real number, and the magnetic flux $\mathbf{B}$ and the eddy current $\mathbf{J}_e$ are given by the constitutive laws

$$\mathbf{B} = \mu\mathbf{H} \text{ in } D \text{ and } \mathbf{J}_e = \sigma\mathbf{E} \text{ in } D_c, \tag{2}$$

where μ denotes the magnetic permeability and σ the electric conductivity. Here we suppose that $\mu, \sigma \in L^\infty(D)$ and that there exists positive real numbers μ_0, σ_0 such that

$$\begin{aligned} \mu &\geq \mu_0 && \text{a.e. in } D \\ \sigma &\geq \sigma_0 && \text{a.e. in } D_c, \\ \sigma &= 0 && \text{a.e. in } D_{nc}. \end{aligned}$$

Note that the divergence free property of $\mathbf{J}_s$, the fact that $\partial D_c \subset D$, and the second equation of (1) implies that

$$\mathrm{div}\mathbf{J}_e = 0 \text{ in } D_c, \tag{3}$$

as well as

$$\mathbf{J}_e \cdot \mathbf{n} = 0 \text{ on } \partial D_c, \tag{4}$$

where $\mathbf{n}$ stands for the unit outward normal to D or D_c depending on the context.

System (1)-(2) is completed with the following boundary conditions on Γ

$$\mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \Gamma. \tag{5}$$

Before stating the $\mathbf{A}$-φ formulation of this problem, let us introduce some notations used throughout the paper. On a given domain $\mathcal{D}$, the $L^2(\mathcal{D})$-norm is denoted by $||\cdot||_{\mathcal{D}}$, and the corresponding $L^2(\mathcal{D})$-inner product by $(\cdot,\cdot)_{\mathcal{D}}$. In the case of $\mathcal{D} = D$, the index D is dropped. $H_0^1(\mathcal{D})$ is the subspace of $H^1(\mathcal{D})$ with vanishing trace on $\partial\mathcal{D}$ and

$$H_0(\mathrm{curl}, \mathcal{D}) = \Big\{\mathbf{F} \in L^2(\mathcal{D})^3 : \mathrm{curl}\mathbf{F} \in L^2(\mathcal{D})^3, \mathbf{F} \times \mathbf{n} = 0 \text{ on } \partial\mathcal{D}\Big\}.$$

Finally, in order to ensure later the uniqueness of the fields, let us introduce the gauge spaces:

$$\begin{aligned} \widetilde{X}(\mathcal{D}) &= \Big\{\mathbf{F} \in H_0(\mathrm{curl}, \mathcal{D}) : (\mathbf{F}, \nabla\xi)_{\mathcal{D}} = 0,\ \forall \xi \in H_0^1(\mathcal{D})\Big\}, \\ \widetilde{H^1}(\mathcal{D}) &= \Big\{f \in H^1(\mathcal{D}) : (f, 1)_{\mathcal{D}} = 0\Big\}. \end{aligned}$$

The harmonic $\mathbf{A}$-φ formulation is based on the introduction of a magnetic vector potential $\mathbf{A}$ in D and an electric scalar potential φ in D_c such that:

$$\mathbf{B} = \mathrm{curl}\mathbf{A} \text{ in } D \text{ and } \mathbf{E} = -j\omega\mathbf{A} - \nabla\varphi \text{ in } D_c. \tag{7}$$

From system (1), the harmonic $\mathbf{A}$-φ formulation reads:

$$\mathrm{curl}\left(\mu^{-1}\mathrm{curl}\mathbf{A}\right) + \sigma\Big(j\omega\mathbf{A} + \nabla\varphi\Big) = \mathbf{J}_s \text{ in } D, \tag{8a}$$

$$\mathrm{div}(\sigma(j\omega\mathbf{A} + \nabla\varphi)) = 0 \text{ in } D_c, \tag{8b}$$

with the boundary conditions, derived from (4)-(5), given by

$$\mathbf{A} \times \mathbf{n} = 0 \text{ on } \Gamma \text{ and } \sigma(j\omega\mathbf{A} + \nabla\varphi) \cdot \mathbf{n} = 0 \text{ on } \partial D_c. \tag{9}$$

As mentioned in [11], the great interest of the formulation (8)-(9) lies in its effectiveness in both domains D_c and D_{nc}. The Coulomb gauge on $\mathbf{A}$, namely $\mathrm{div}\mathbf{A} = 0$ in D, and the zero mean of the potential φ in D_c are imposed to ensure the uniqueness of these potentials (see [11], as well as [25, 30] for examples of practical implementations). Since φ does not make sense in

D_{nc}, we fix an arbitrary extension of φ in the whole domain D. This choice does not impact the problem since $\sigma \equiv 0$ in D_{nc}. Let us note that if $D_c = \emptyset$, then the formulation (8)-(9) remains valid and corresponds to the classical magnetic vector potential $\mathbf{A}$ formulation for magnetostatic problems (where φ does not exist any more). If $D_{nc} = \emptyset$ (or equivalently $D = D_c$), then the formulation (8)-(9) remains also valid. In that case, no gauge condition is needed, if the so-called $\mathbf{A}^\star$ formulation is used, where the sole unknown is then $\mathbf{A}^* = j\omega\mathbf{A} + \nabla\varphi$.

The corresponding weak formulation is given by:

Find $(\mathbf{A}, \varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ such that

$$B((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) = (\mathbf{J}_s, \mathbf{A}'), \quad \forall (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c), \tag{10}$$

where

$$\begin{aligned} B((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) &= \left(\mu^{-1}\mathrm{curl}\mathbf{A}, \mathrm{curl}\mathbf{A}'\right)_D \\ &+ j\omega^{-1}\left(\sigma(j\omega\mathbf{A} + \nabla\varphi), (j\omega\mathbf{A}' + \nabla\varphi')\right)_{D_c}, \forall (\mathbf{A}, \varphi), (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c). \end{aligned}$$

Lemma 2.1 of [11] ensures the existence and uniqueness of the weak solution $(\mathbf{A}, \varphi)$ of this problem since it was shown there that

$$\|(\mathbf{A}', \varphi')\|_B := |B((\mathbf{A}', \varphi'), (\mathbf{A}', \varphi'))|^{\frac{1}{2}}, \forall (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c),$$

is a norm on $\widetilde{X}(D) \times \widetilde{H^1}(D_c)$ equivalent to the natural one

$$\|(\mathbf{A}, \varphi)\|_V = \left(\|\mathbf{A}'\|_D^2 + \|\mu^{-1/2}\mathrm{curl}\mathbf{A}'\|_D^2 + |\varphi'|_{1,D_c}^2\right)^{\frac{1}{2}}, \forall (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c).$$

Recall that from the definition of B, we have

$$|B((\mathbf{A}', \varphi'), (\mathbf{A}', \varphi'))|^{\frac{1}{2}} = \left(\left\|\mu^{-1/2}\mathrm{curl}\mathbf{A}'\right\|^2 + \left\|\omega^{-1/2}\,\sigma^{1/2}(j\omega\mathbf{A}' + \nabla\varphi')\right\|_{D_c}^2\right)^{1/2},$$

for all $(\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$.

As $\mathbf{J}_s$ is divergence free, it was further shown in Lemma 2.3 of [11] that (10) remains valid for non divergence free fields $\mathbf{A}'$, namely

$$B((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) = (\mathbf{J}_s, \mathbf{A}'), \quad \forall (\mathbf{A}', \varphi') \in H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c). \tag{11}$$

As usual, the convergence of numerical schemes are related to regularity results of the solution $(\mathbf{A}, \varphi)$ of (10). But up to now, such results are not available in the literature. Let us then give some of them in some particular cases. Before stating them, we introduce the following subspace of $H_0(\mathrm{curl}, D)$ (see [1, 6]) defined that

$$X_N(D) = \{\mathbf{A}' \in H_0(\mathrm{curl}, D) \;:\; \mathrm{div}\mathbf{A}' \in L^2(D)\},$$

that is a Hilbert space with its natural inner product. With this definition, we can formulate the following results.

Lemma 2.1 *If $(\mathbf{A}, \varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ is solution of (10), then*
1. $\mathbf{A}$ belongs to $X_N(D)$ and is solution of the (regularized) Maxwell problem

$$\int_D (\mu^{-1}\mathrm{curl}\mathbf{A} \cdot \mathrm{curl}\overline{\mathbf{A}'} + \mathrm{div}\mathbf{A}\;\mathrm{div}\overline{\mathbf{A}'}) = \int_D \mathbf{f} \cdot \overline{\mathbf{A}'}, \forall \mathbf{A}' \in X_N(D). \tag{12}$$

where for any extension $\tilde{\varphi} \in H^1(D)$ *of* φ *to* D,

$$\mathbf{f} = \mathbf{J}_s + \sigma(j\omega\mathbf{A} + \nabla\tilde{\varphi}) \tag{13}$$

belongs to $L^2(D)^3$ *and is divergence free,*
2. φ *belongs to* $\widetilde{H^1}(D_c)$ *and is solution of the non-homogeneous Neumann problem*

$$\int_{D_c} \sigma\nabla\varphi \cdot \nabla\bar{\varphi}' = -j\,\omega \int_{D_c} \sigma\mathbf{A} \cdot \nabla\bar{\varphi}', \forall\varphi' \in H^1(D_c). \tag{14}$$

Proof. The second property is a direct consequence of (10) by taking $\mathbf{A}' = \mathbf{0}$ (noticing that the obtained identity remains valid for any $\varphi' \in H^1(D_c)$). For the first property, we notice that (11) with $\varphi' = 0$ yields

$$\int_D \mu^{-1}\mathrm{curl}\mathbf{A} \cdot \mathrm{curl}\overline{\mathbf{A}'} - \int_{D_c} \sigma(j\omega\mathbf{A} + \nabla\varphi) \cdot \overline{\mathbf{A}'} = (\mathbf{J}_s, \mathbf{A}'), \quad \forall\mathbf{A}' \in H_0(\mathrm{curl}, D).$$

As $\mathbf{A}$ is divergence free, it clearly belongs to $X_N(D)$, and as $X_N(D)$ is a subset of $H_0(\mathrm{curl}, D)$, the previous identity directly implies

$$\int_D (\mu^{-1}\mathrm{curl}\mathbf{A} \cdot \mathrm{curl}\overline{\mathbf{A}'} + \mathrm{div}\mathbf{A}\ \mathrm{div}\overline{\mathbf{A}'}) - \int_{D_c} \sigma(j\omega\mathbf{A} + \nabla\varphi) \cdot \overline{\mathbf{A}'} = (\mathbf{J}_s, \mathbf{A}'), \forall\mathbf{A}' \in X_N(D).$$

Since $\sigma = 0$ on $D \setminus D_c$, the second term of this left-hand side can be written as

$$\int_{D_c} \sigma(j\omega\mathbf{A} + \nabla\varphi) \cdot \overline{\mathbf{A}'} = \int_D \sigma(j\omega\mathbf{A} + \nabla\tilde{\varphi}) \cdot \overline{\mathbf{A}'}.$$

These two identity directly lead to (12). The divergence free property of $\mathbf{f}$ comes from the divergence free of $\mathbf{J}_s$ and from (14). ■

According to this Lemma, the regularity of $\mathbf{A}$ is related to the regularity of the Maxwell problem (12), while the regularity of φ is related to the regularity of the Neumann problem (14). Let us start with the regularity of $\mathbf{A}$.

Lemma 2.2 *Let* $(\mathbf{A}, \varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ *be the unique solution of* (10) *with a divergence free field* $\mathbf{J}_s \in L^2(D)$. *Then the next results hold:*
1. If D *has a* $C^{1,1}$ *boundary and* $\mu \in C^{0,1}(D)$, *then* $\mathbf{A} \in H^2(D)^3$.
2. If D *is a convex polyhedron and* $\mu = 1$, *then* $\mathbf{A} \in H^{1+\varepsilon}(D)^3$, *for some* $\varepsilon \in (0, 1]$ *that depends on the interior angles along the edges of* D *and of the corner singularities of the Neumann problem in* D.
3. If D *is a parallelepiped and* $\mu = 1$, *then* $\mathbf{A} \in H^2(D)^3$.

Proof. Point 1 follows from the fact that the system associated with (12) is an elliptic system of order 2 (see [8, §4.5]), hence the H^2 regularity of $\mathbf{A}$ follows from a standard shift theorem as $\mathbf{f}$ is in $L^2(D)^3$.

Points 2 and 3 follow from [6, §4.4.2 (b) and (c)]. ■

For μ piecewise smooth and/or D a non convex polyhedron, some regularity results for (12) are also available, see for instance [7].

Let us go on with the regularity of φ.

Lemma 2.3 *Let* $(\mathbf{A}, \varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ *be the unique solution of* (10) *with a divergence free field* $\mathbf{J}_s \in L^2(D)^3$. *Then the next results hold:*
1. If $\mathbf{A} \in H^2(D)^3$, D_c *has a* $C^{2,1}$ *boundary, and* $\sigma \in C^{1,1}(D_c)$, *then* $\varphi \in H^3(D_c)$.
2. If $\mathbf{A} \in H^1(D)^3$, D_c *has a* $C^{1,1}$ *boundary and* $\sigma \in C^{0,1}(D_c)$, *then* $\varphi \in H^2(D_c)$.
3. If $\mathbf{A} \in H^1(D)^3$, D_c *is a convex polyhedron and* $\sigma = 1$, *then* $\varphi \in H^{1+s}(D_c)$, *for all* $s \in (0, 1)$.

Proof. As the strong formulation of (14) is (see (8b) and (9))

$$\begin{cases} \operatorname{div}(\sigma\nabla\varphi) & = -j\,\omega\operatorname{div}(\sigma\mathbf{A}) \text{ in } D_c \\ \sigma\nabla\varphi\cdot\mathbf{n} & = -j\,\omega\sigma\mathbf{A}\cdot\mathbf{n} \text{ on } \partial D_c, \end{cases} \tag{15}$$

both points follow from an appropriate shift theorem for the non-homogeneous Neumann problem in D_c and trace theorems.

For point 1, the regularities on $\mathbf{A}$ and σ guarantee that $\sigma\mathbf{A} \in H^2(D_c)^3$, hence $\operatorname{div}(\sigma\mathbf{A}) \in H^1(D_c)$, and its trace $\sigma\mathbf{A} \in H^{3/2}(\partial D_c)^3$. From the regularity of the boundary of D_c, its normal trace $\mathbf{n}$ belongs to $C^{1,1}(\partial D_c)$ and therefore $\mathbf{n} \in W^{2,p}(\partial D_c)$, for all $p > 1$. By Theorem 1.4.4.2 from [17] (multiplication in Sobolev spaces), we deduce that $\sigma\mathbf{A}\cdot\mathbf{n} \in H^{3/2}(\partial D_c)$. By a standard shift theorem for (15), we deduce that $\varphi \in H^3(D_c)$.

Point 2 is obtained similarly, but here we only get $\sigma\mathbf{A} \in H^1(D_c)^3$, hence $\operatorname{div}(\sigma\mathbf{A}) \in L^2(D_v)$, $\sigma\mathbf{A} \in H^{1/2}(\partial D_c)^3$. and $\mathbf{n}$ belongs to $C^{0,1}(\partial D_c)$ (and therefore $\mathbf{n} \in W^{1,p}(\partial D_c)$, for all $p > 1$). Again by Theorem 1.4.4.2 from [17], we deduce that $\sigma\mathbf{A}\cdot\mathbf{n} \in H^{1/2}(\partial D_c)$. By a standard shift theorem for (15), we deduce that $\varphi \in H^2(D_c)$.

For the third point, the problem comes from the poor regularity of the normal trace along ∂D_c. In that case, we can only say that $\mathbf{n}$ is piecewise smooth, and therefore deduce that $\sigma\mathbf{A}\cdot\mathbf{n} \in H^{1/2}(F)$, for all faces F of ∂D_c. Owing to Corollary 1.4.4.5 of [17], we obtain that $\sigma\mathbf{A}\cdot\mathbf{n} \in H^{s-1/2}(\partial D_c)$, for all $s \in [0,1)$. The conclusion then follows from Theorem 23.3 of [13]. ■

Finally a bootstrapping argument allows to obtain improved regularities for $\mathbf{A}$. For shortness, we restrict ourselves to some particular cases.

Lemma 2.4 *Let $(\mathbf{A},\varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ be the unique solution of* (10) *with a divergence free field $\mathbf{J}_s \in H^{1/2}(D)^3$. Assume that D_c has a $C^{1,1}$ boundary and $\sigma \in C^{0,1}(D_c)$ or D_c is a convex polyhedron and $\sigma = 1$. Then if D has a $C^{2,1}$ boundary and $\mu \in C^{1,1}(D)$ or D is a parallelepiped and $\mu = 1$, $\mathbf{A} \in H^{2+s}(D)^3$, for all $s \in (0,1/2)$.*

Proof. The assumptions on D and μ allow to apply Lemma 2.2, and obtain $\mathbf{A} \in H^2(D)^3$. In a second step, due to our assumption on D_c and σ, we can apply Lemma 2.3 (point 2 or 3) and get $\varphi \in H^{1+s}(D_c)$, for all $s \in (0,1)$. This regularity and the regularity of $\mathbf{A}$ yield

$$\sigma(j\omega\mathbf{A} + \nabla\varphi) \in H^s(D_c), \forall s \in (0,1).$$

And again by Corollary 1.4.4.5 of [17], we deduce that

$$\sigma(j\omega\mathbf{A} + \nabla\tilde{\varphi}) \in H^s(D), \forall s \in (0,1/2).$$

Using the definition (13) of $\mathbf{f}$ and the assumption on $\mathbf{J}_s$, we conclude that $\mathbf{f}$ belongs to $H^s(D)$, for all $s \in (0,1/2)$.

Coming back to problem (12), we find $\mathbf{A} \in H^{2+s}(D)^3$, for all $s \in (0,1/2)$, by a standard shift theorem if D has a $C^{2,1}$ boundary and $\mu \in C^{1,1}(D)$ or by [6, §4.4.2 (c)] if D is a parallelepiped and $\mu = 1$. ■

Remark 2.5 As usual, the regularity of the solution affects the performance of the chosen finite element scheme, Lemmas 2.2 to 2.4 allow to quantify such a regularity and then to obtain *a priori* error estimates, but they are not needed for the construction and the implementation of our *a posteriori* error estimators. Nevertheless they will be useful in the interpretation of our numerical results.

3 A guaranteed equilibrated error estimator for conforming or non conforming approximations of the A-φ formulation

In this Section, we obtain a guaranteed equilibrated error estimator for nonforming approximations of (10), extending the results from [9, §3.2 and §4] to nonconforming and higher order approximations with general equilibrated fluxes.

More precisely, to discretize problems (10), we suppose that we are given a partition $\mathcal{T}$ of D into polyhedral elements T that covers exactly D. Each element T of $\mathcal{T}$ is assumed to belong either to $\bar{D}_c$ or to $\overline{D_{nc}}$, furthermore we denote by h_T its diameter. For simplicity, we set $\mathcal{T}_c = \{T \in \mathcal{T} \,:\, T \subset \overline{D_c}\}$. On such a mesh we introduce the so-called broken Sobolev space

$$\begin{aligned} H^1(\mathcal{T}) &= \{v \in L^2(D) \,|\, v_{|T} \in H^1(T), \forall T \in \mathcal{T}\}, \\ H^1(\mathcal{T}_c) &= \{v \in L^2(D_c) \,|\, v_{|T} \in H^1(T), \forall T \in \mathcal{T}_c\}. \end{aligned}$$

As in [14, 21, 10], in order to analyse simultaneously different approximation schemes, the problem is approximated in a finite dimensional subspace V_h of $H^1(\mathcal{T})^3 \times H^1(\mathcal{T}_c)$. In other words, we suppose given an approximation $(\mathbf{A}_h, \varphi_h) \in V_h$ of the solution $(\mathbf{A}, \varphi)$ of (10).

For the sake of simplicity, for $\mathbf{A}'_h \in H^1(\mathcal{T})^3$ (resp. $\varphi'_h \in H^1(\mathcal{T}_c)$), we denote by $\mathrm{curl}_h \mathbf{A}'_h$ (resp. its $\nabla_h \varphi'_h$) its broken rotation (resp. its broken gradient), namely

$$\begin{aligned} \mathrm{curl}_h \mathbf{A}'_h &= \mathrm{curl} \mathbf{A}'_h \quad \text{on } T, \forall T \in \mathcal{T}, \\ \nabla_h \varphi'_h &= \nabla \varphi'_h \quad \text{on } T, \forall T \in \mathcal{T}_c. \end{aligned}$$

We now introduce the discrete couterparts of (7) by setting

$$\mathbf{B}_h = \mathrm{curl}_h \mathbf{A}_h \text{ and } \mathbf{E}_h = -\,(j\,\omega\,\mathbf{A}_h + \nabla_h \varphi_h). \tag{16}$$

We further assume that a potential reconstruction $(\mathbf{S}_h, \psi_h) \in H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c)$ of $(\mathbf{A}_h, \varphi_h)$, and some flux reconstructions $\mathbf{H}_h$ and $\mathbf{J}_{e,h}$ are available (see Remark 3.1) that belong respectively to $H(\mathrm{curl}, D)$ and $H(\mathrm{div}, D_c)$ and satisfy the following conservation properties (compare with [14, identity (18)] and [9, Lemma 4.1])

$$(\mathrm{curl}\mathbf{H}_h - \tilde{\mathbf{J}}_{e,h} - \mathbf{J}_s, \mathbf{e})_T = 0, \forall T \in \mathcal{T}, \mathbf{e} \in \mathbb{C}^3, \tag{17}$$

$$\mathrm{div}\mathbf{J}_{e,h} = 0 \text{ in } D_c, \tag{18}$$

$$\mathbf{J}_{e,h} \cdot \mathbf{n} = 0 \text{ on } \partial D_c. \tag{19}$$

For further uses, we denote by $\tilde{\mathbf{J}}_{e,h}$ the extension of $\mathbf{J}_{e,h}$ by zero outside D_c, that remains divergence free.

Note that $\mathbf{H}_h$ represents an approximation of $\mu^{-1}\mathrm{curl}\mathbf{A}$, while $\mathbf{J}_{e,h}$ is an approximation of $\sigma\mathbf{E}$.

Notice that (17) is a minimal assumption to guarantee that $\mathbf{H}_h$ is a correct approximation of the continuous flux $\mu^{-1}\mathrm{curl}\mathbf{A}$, while (18) and (19) is the exact counterpart of (3) and (4), therefore $\mathbf{J}_{e,h}$ is a correct approximation of $\mathbf{J}_e = \sigma\mathbf{E}$.

Remark 3.1 1. The reconstruction of these fields is not the main goal of our paper, we refer to Appendix A for some discussions on practical implementations of the reconstructed fluxes.
2. If $(\mathbf{A}_h, \varphi_h)$ is a conforming approximation of $(\mathbf{A}, \varphi)$ in the sense that $(\mathbf{A}_h, \varphi_h)$ belongs to $H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c)$, then there is no need to build up a reconstructed potential $(\mathbf{S}_h, \psi_h)$, since we can simply choose $(\mathbf{S}_h, \psi_h) = (\mathbf{A}_h, \varphi_h)$.

With these objects at hands, we now define the global estimator η as

$$\eta = 2\eta_{NC} + \eta_{\mathrm{flux}} + \eta_{\mathcal{O}}, \tag{20}$$

where the nonconforming estimator η_{NC} is defined by

$$\begin{aligned}\eta_{NC} &= \left(\left\|\mu^{-1/2}\mathrm{curl}_h(\mathbf{A}_h-\mathbf{S}_h)\right\|^2\right.\\ &+ \left.\left\|\omega^{-1/2}\,\sigma^{1/2}\left(j\,\omega(\mathbf{A}_h-\mathbf{S}_h)+\nabla_h(\varphi_h-\psi_h)\right)\right\|_{D_c}^2\right)^{1/2},\end{aligned} \tag{21}$$

the flux estimator η_{flux} is defined by (see [9, (15)])

$$\eta_{\text{flux}} = \left(\eta_{\text{magn}}^2+\eta_{\text{elec}}^2\right)^{1/2}, \tag{22}$$

where η_{magn} and η_{elec} are defined by

$$\eta_{\text{magn}} = \left\|\mu^{1/2}(\mathbf{H}_h-\mu^{-1}\mathbf{B}_h)\right\|_D,\ \eta_{\text{elec}} = \left\|(\omega\sigma)^{-1/2}(\mathbf{J}_{e,h}-\sigma\mathbf{E}_h)\right\|_{D_c}, \tag{23}$$

and finally the oscillation estimator $\eta_{\mathcal{O}}$ is defined by

$$\eta_{\mathcal{O}} = C_L\mu_{\max}^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}} c_{P,T}h_T^2\|\mathbf{J}_s-\mathrm{curl}\mathbf{H}_h+\tilde{\mathbf{J}}_{e,h}\|_T^2\right)^{\frac{1}{2}}, \tag{24}$$

where

$$\mu_{\max} = \sup_{x\in D}\mu(x).$$

For further uses, we also recall the following Poincaré inequality

$$\|u-\mathcal{M}_T u\|_T^2 \le c_{P,T}h_T^2|u|_{1,T}^2, \forall u\in H^1(T), T\in\mathcal{T}, \tag{25}$$

where $\mathcal{M}_T u = |T|^{-1}\int_T u(x)$ is the mean of u on T, and $c_{P,T}$ is a positive constant depending only on T. If T is convex, then $c_{P,T}\le\frac{1}{\pi^2}$ (hence for pratical uses, in (25), we can replace $c_{P,T}$ by $\frac{1}{\pi^2}$), see [2, 24]. For nonconvex T, some estimations of $c_{P,T}$ can be found in [3, Lemma 10.2] or [15, §2].

Besides the above estimate, we also need the following consequence of [7, Thm 3.4] (see also [4, p. 45]). Namely there exists a positive constant C_L such that for all $\mathbf{A}'\in\widetilde{X}(D)$, there exist a unique function $\Phi\in H_0^1(D)$ and a unique vector field $\Psi\in H_0(\mathrm{curl},D)\cap H^1(D)^3$ such that

$$\mathbf{A}' = \nabla\Phi+\Psi, \tag{26}$$

with

$$|\Phi|_{1,D}+|\Psi|_{1,D}\le C_L\|\mathrm{curl}\mathbf{A}'\|_D. \tag{27}$$

Note that $C_L=1$ if D is convex, due to [6, Theorem 1.1] and [7, Theorem 2.1 and Lemma 2.2] since in that case $\Phi=0$ and $\Psi=\mathbf{A}'$.

We are ready to prove the following upper error bound (compare with [9, Theorem 4.2]) of the energy norm of the $\mathbf{A}$-φ error $\epsilon_{A,\varphi}$, given by:

$$\epsilon_{A,\varphi} = \left(\left\|\mu^{-1/2}\mathrm{curl}_h\epsilon_A\right\|^2+\left\|\omega^{-1/2}\,\sigma^{1/2}(j\,\omega\epsilon_A+\nabla_h\epsilon_\varphi)\right\|_{D_c}^2\right)^{1/2}, \tag{28}$$

where

$$\epsilon_A = \mathbf{A}-\mathbf{A}_h \text{ and } \epsilon_\varphi = \varphi-\varphi_h.$$

Theorem 3.2 *Let us suppose that* $\mathbf{J}_s\in(L^2(D))^3$*, that* $\mathbf{H}_h\in H(\mathrm{curl},D)$*, and that* $\mathbf{J}_{e,h}\in H(\mathrm{div},D_c)$ *satisfy* (17)-(19)*. Then the following upper bound holds:*

$$\epsilon_{A,\varphi}\le\eta \tag{29}$$

Proof. Introduce

$$\epsilon_S = \mathbf{A} - \mathbf{S}_h \text{ and } \epsilon_\psi = \varphi - \psi_h,$$

as well as

$$\begin{aligned}\epsilon_{S,\psi} &= |B((\epsilon_S,\epsilon_\psi),(\epsilon_S,\epsilon_\psi))|^{\frac{1}{2}} \\ &= \left(\left\|\mu^{-1/2}\mathrm{curl}\epsilon_S\right\|^2 + \left\|\omega^{-1/2}\,\sigma^{1/2}(j\,\omega\epsilon_S + \nabla\epsilon_\psi)\right\|^2_{D_c}\right)^{1/2}.\end{aligned} \tag{30}$$

By the triangular inequality, we directly deduce that

$$\epsilon_{A,\varphi} \leq \epsilon_{S,\psi} + \eta_{NC}. \tag{31}$$

Hence it remains to estimate $\epsilon_{S,\psi}$.

From the definition of B, we have

$$\begin{aligned}B((\epsilon_S,\epsilon_\psi),(\epsilon_S,\epsilon_\psi)) &= \int_D \mu^{-1}\mathrm{curl}(\mathbf{A}-\mathbf{S}_h)\cdot\mathrm{curl}\overline{\epsilon_S} \\ &+ \int_{D_c}\frac{j\,\sigma}{\omega}\,(j\omega(\mathbf{A}-\mathbf{S}_h)+\nabla(\varphi-\psi_h))\cdot\overline{(j\omega\epsilon_S+\nabla\epsilon_\psi)}.\end{aligned}$$

By defining (compare with (16))

$$\mathbf{B}_h^S = \mathrm{curl}\mathbf{S}_h \text{ and } \mathbf{E}_h^S = -\,(j\,\omega\,\mathbf{S}_h + \nabla_h\psi_h), \tag{32}$$

and adding the quantities $\pm\int_D \mathbf{H}_h\cdot\mathrm{curl}\overline{\epsilon_S} \pm \frac{j}{\omega}\int_{D_c}\mathbf{J}_{e,h}\cdot\overline{(j\omega\epsilon_S+\nabla\epsilon_\psi)}$, the above identity becomes

$$\begin{aligned}B((\epsilon_S,\epsilon_\psi),(\epsilon_S,\epsilon_\psi)) &= B((\mathbf{A},\varphi),(\epsilon_S,\epsilon_\psi)) \\ &+ \int_D(\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S)\cdot\mathrm{curl}\overline{\epsilon_S} \\ &+ \frac{j}{\omega}\int_{D_c}(\sigma\mathbf{E}_h^S - \mathbf{J}_{e,h})\cdot\overline{(j\omega\epsilon_S+\nabla\epsilon_\psi)} + \frac{j}{\omega}\int_{D_c}\mathbf{J}_{e,h}\cdot\overline{(j\omega\epsilon_S+\nabla\epsilon_\psi)} \\ &- \int_D \mathbf{H}_h\cdot\mathrm{curl}\overline{\epsilon_S}.\end{aligned}$$

Using the weak formulation (10) valid for test-functions in $H_0(\mathrm{curl},D)\times\widetilde{H^1}(D_c)$ see [11, Lemma 2.2]) and applying Green's formula to the last term of this right-hand side, we find term

$$\begin{aligned}B((\epsilon_S,\epsilon_\psi),(\epsilon_S,\epsilon_\psi)) &= \int_D \mathbf{J}_s\cdot\overline{\epsilon_S} - \int_D \mathrm{curl}\mathbf{H}_h\cdot\overline{\epsilon_S} \\ &+ \int_D(\mathbf{H}_h-\mu^{-1}\mathbf{B}_h^S)\cdot\mathrm{curl}\overline{\epsilon_S} + \frac{j}{\omega}\int_{D_c}(\sigma\mathbf{E}_h^S-\mathbf{J}_{e,h})\cdot\overline{(j\omega\epsilon_S+\nabla\epsilon_\psi)} \\ &+ \frac{j}{\omega}\int_{D_c}\mathbf{J}_{e,h}\cdot\overline{(j\omega\epsilon_S+\nabla\epsilon_\psi)}.\end{aligned}$$

Keeping unchanged the third and fourth terms of this right-hand side and rearranging the other terms, we find

$$\begin{aligned}B((\epsilon_S,\epsilon_\psi),(\epsilon_S,\epsilon_\psi)) &= \int_D(\mathbf{H}_h-\mu^{-1}\mathbf{B}_h^S)\cdot\mathrm{curl}\overline{\epsilon_S} \\ &+ \frac{j}{\omega}\int_{D_c}(\sigma\mathbf{E}_h^S-\mathbf{J}_{e,h})\cdot\overline{(j\omega\epsilon_S+\nabla\epsilon_\psi)} \\ &+ \int_D(\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot\overline{\epsilon_S} \\ &+ \frac{j}{\omega}\int_{D_c}\mathbf{J}_{e,h}\cdot\nabla\overline{\epsilon_\psi}.\end{aligned}$$

Using the assumptions (18)-(19), the last term vanishes, therefore by the identity (30), and the triangular inequality, we arrive at

$$\begin{aligned}\epsilon_{S,\psi}^2 \leq &\left|\int_D (\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S)\cdot \mathrm{curl}\overline{\epsilon_S}\right| \\ &+ \frac{1}{\omega}\left|\int_{D_c}(\sigma\mathbf{E}_h^S - \mathbf{J}_{e,h})\cdot\overline{(j\omega\epsilon_S + \nabla\epsilon_\psi)}\right| \\ &+ \left|\int_D (\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot\overline{\epsilon_S}\right|.\end{aligned} \tag{33}$$

Let us estimate each term of the right hand-side of this estimate. But we first transform (and then estimate) the third term of this right-hand side. For that purpose, we use the Helmholtz decomposition of [22, Lemma 4.5]

$$H_0(\mathrm{curl}, D) = \nabla H_0^1(D) \overset{\perp}{\oplus} \widetilde{X}(D), \tag{34}$$

so that

$$\epsilon_S = \nabla\phi + \epsilon_\perp, \tag{35}$$

with $\phi \in H_0^1(D)$ and $\epsilon_\perp \in \widetilde{X}(D)$ and

$$\|\epsilon_S\|_D^2 = \|\nabla\phi\|_D^2 + \|\epsilon_\perp\|_D^2. \tag{36}$$

This decomposition, Green's formula and the divergence free property of $\tilde{\mathbf{J}}_{e,h}$ allow to get

$$\int_D (\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot\overline{\epsilon_S} = \int_D (\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot\overline{\epsilon_\perp}. \tag{37}$$

Now as $\epsilon_\perp \in \widetilde{X}(D)$, there exist a unique function $\Phi \in H_0^1(D)$ and a unique vector field $\Psi \in H_0(\mathrm{curl}, D)\cap H^1(D)^3$ such that (see (26) and (27))

$$\epsilon_\perp = \nabla\Phi + \Psi, \tag{38}$$

with

$$|\Phi|_{1,D} + |\Psi|_{1,D} \leq C_L\|\mathrm{curl}\epsilon_\perp\|.$$

Furthermore since $\mathrm{curl}\epsilon_S = \mathrm{curl}\epsilon_\perp$, the previous estimate implies

$$\begin{aligned}|\Phi|_{1,D} + |\Psi|_{1,D} &\leq C_L\,\|\mathrm{curl}\epsilon_S\| \\ &\leq C_L\,\mu_{\max}^{\frac{1}{2}}\|\mu^{-1/2}\mathrm{curl}\epsilon_S\|.\end{aligned}$$

Therefore by the defintion of $\epsilon_{S,\psi}$, we get

$$|\Phi|_{1,D} + |\Psi|_{1,D} \leq C_L\mu_{\max}^{\frac{1}{2}}\epsilon_{S,\psi}. \tag{39}$$

Using (38) into (37) and using Green's formula and the divergence free property of $\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h}$, we obtain

$$\int_D (\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot\overline{\epsilon_S} = \int_D (\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot\overline{\Psi}.$$

By the property (17), we obtain

$$\int_D (\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot\overline{\epsilon_S} = \int_D (\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})\cdot(\overline{\Phi} - \mathcal{M}_h\overline{\Psi}),$$

where $\mathcal{M}_h\Psi \in H^1(\mathcal{T})^3$ is defined by

$$(\mathcal{M}_h\Psi)_{|T} = \mathcal{M}_T(\Psi_{|T}), \forall T \in \mathcal{T}.$$

By (continuous and discrete) Cauchy-Schwarz's inequality and Poincaré's inequality (25), we deduce that

$$\int_D (\mathbf{J}_s - \text{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h}) \cdot \overline{\epsilon_S} \le \left(\sum_{T\in\mathcal{T}} c_{P,T} h_T^2 \|\mathbf{J}_s - \text{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h}\|_T^2\right)^{\frac{1}{2}} |\Psi|_{1,D}.$$

Using the estimate (39), we arrive at

$$\left|\int_D (\mathbf{J}_s - \text{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h}) \cdot \overline{\epsilon_S}\right| \le \eta_{\mathcal{O}}\, \epsilon_{S,\psi}. \tag{40}$$

A simple consequence of (continuous and discrete) Cauchy-Schwarz's inequality allows to estimate the first two terms of the right-hand side of (33) as follows

$$\begin{aligned}
&\left|\int_D (\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S) \cdot \text{curl}\overline{\epsilon_S}\right| + \frac{1}{\omega}\left|\int_{D_c} (\sigma\mathbf{E}_h^S - \mathbf{J}_{e,h}) \cdot \overline{(j\omega\epsilon_S + \nabla\epsilon_\psi)}\right| \qquad (41)\\
&\le \left\|\mu^{1/2}(\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S)\right\|_D \left\|\mu^{-1/2}\text{curl}\epsilon_S\right\|_D \\
&+ \left\|(\omega\,\sigma)^{-1/2}(\mathbf{J}_{e,h} - \sigma\mathbf{E}_h^S)\right\|_{D_c} \left\|\sigma^{1/2}\omega^{-1/2}(j\omega\epsilon_S + \nabla\epsilon_\psi)\right\|_{D_c} \\
&\le \left(\left\|\mu^{1/2}(\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S)\right\|_D^2 + \left\|(\omega\,\sigma)^{-1/2}(\mathbf{J}_{e,h} - \sigma\mathbf{E}_h^S)\right\|_{D_c}^2\right)^{\frac{1}{2}} \epsilon_{S,\psi}.
\end{aligned}$$

Let us now transform the first factor of this right-hand side in order to display the estimator η_{flux}. Indeed inserting $\pm\mathbf{E}_h$ and $\pm\mathbf{B}_h$ and using the discrete Cauchy-Schwarz's inequality, and using the definition (22) of η_{flux}, we have

$$\begin{aligned}
&\left(\left\|\mu^{1/2}(\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S)\right\|_D^2 + \left\|(\omega\,\sigma)^{-1/2}(\mathbf{J}_{e,h} - \sigma\mathbf{E}_h^S)\right\|_{D_c}^2\right)^{\frac{1}{2}} \\
\le\ & \eta_{\text{flux}} + \left(\left\|\mu^{-1/2}(\mathbf{B}_h - \mathbf{B}_h^S)\right\|_D^2 + \left\|\omega^{-1/2}\sigma^{1/2}(\mathbf{E}_h - \mathbf{E}_h^S)\right\|_{D_c}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

The definition of $\mathbf{E}_h$, $\mathbf{E}_h^S$, $\mathbf{B}_h$ and $\mathbf{B}_h^S$ directly give

$$\mathbf{B}_h - \mathbf{B}_h^S = \text{curl}_h(\mathbf{A}_h - \mathbf{S}_h), \text{ and } \mathbf{E}_h - \mathbf{E}_h^S = j\omega(\mathbf{A}_h - \mathbf{S}_h) + \nabla(\varphi_h - \psi_h),$$

which leads to

$$\left(\left\|\mu^{-1/2}(\mathbf{B}_h - \mathbf{B}_h^S)\right\|_D^2 + \left\|\omega^{-1/2}\,\sigma^{1/2}(\mathbf{E}_h - \mathbf{E}_h^S)\right\|_{D_c}^2\right)^{\frac{1}{2}} = \eta_{NC}.$$

Therefore we have found that

$$\left(\left\|\mu^{1/2}(\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S)\right\|_D^2 + \left\|(\omega\,\sigma)^{-1/2}(\mathbf{J}_{e,h} - \sigma\mathbf{E}_h^S)\right\|_{D_c}^2\right)^{\frac{1}{2}} \le \eta_{\text{flux}} + \eta_{NC}.$$

This estimate in (41) leads to

$$\begin{aligned}
&\left|\int_D (\mathbf{H}_h - \mu^{-1}\mathbf{B}_h^S) \cdot \text{curl}\overline{\epsilon_S}\right| + \frac{1}{\omega}\left|\int_{D_c} (\sigma\mathbf{E}_h^S - \mathbf{J}_{e,h}) \cdot \overline{(j\omega\epsilon_S + \nabla\epsilon_\psi)}\right| \\
&\le (\eta_{\text{flux}} + \eta_{NC})\epsilon_{S,\psi}.
\end{aligned}$$

Using this last estimate and (40) in (33), we arrive at

$$\epsilon_{S,\psi} \le \eta_{\text{flux}} + \eta_{NC} + \eta_{\mathcal{O}}.$$

With the help of (31), we conclude that (29) holds (using the definition (20) of η). ■

4 The goal oriented functional and the adjoint problem

We here consider the output functional that represents the physical quantity of interest given by

$$Q(\mathbf{A}) = \int_D \mathbf{q} \cdot \mathrm{curl}\bar{\mathbf{A}}\, dx, \forall \mathbf{A} \in H(\mathrm{curl}, D), \tag{42}$$

where $\mathbf{q} \in L^2(D)^3$ is a given function.

Remark 4.1 *In many engineering applications, engineers are interested in the computation of the flux through a coil. Indeed, in the case where a coil is included in D, in which a given current $\mathbf{J}_s$ of intensity i is imposed, $\mathbf{N}$ being the unit direction of the coil, it can be shown that the magnetic flux through the surface S of a coil is given by*

$$\Phi = \int_S \mathbf{curlA} \cdot \mathbf{n}\, dS,$$

and that it can be evaluated by [19]

$$\bar{\Phi} = \frac{1}{i} Q(\mathbf{A}) = \frac{1}{i} \int_D \mathbf{q} \cdot \mathbf{curl}\bar{\mathbf{A}}\, dx,$$

using $\mathbf{q} = \mathbf{H}_s$ where $\mathbf{curlH}_s = \mathbf{J}_s$, and where as usual $\mathbf{B} = \mathbf{curlA}$. It corresponds to the numerical test proposed in Subsections 6.1 and 6.2. In other applications, engineers can be interested in the computation of the magnetic flux density at a given point of an electromagnetic device, see [16, 20]. It corresponds to the numerical test proposed in Subsection 6.3.

Accordingly by setting

$$\begin{aligned} B^*((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) &= \overline{B((\mathbf{A}', \varphi'), (\mathbf{A}, \varphi))} \\ &= \left(\mu^{-1}\mathrm{curl}\mathbf{A}, \mathrm{curl}\mathbf{A}'\right)_D \\ -j\omega^{-1} &\left(\sigma(j\omega\mathbf{A} + \nabla\varphi), (j\omega\mathbf{A}' + \nabla\varphi')\right)_{D_c}, \forall (\mathbf{A}, \varphi), (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c), \end{aligned}$$

the associated adjoint problem consists in looking for $(\mathbf{A}^*, \varphi^*) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ solution of the adjoint problem

$$B^*((\mathbf{A}^*, \varphi^*), (\mathbf{A}', \varphi')) = Q(\mathbf{A}'), \quad \forall (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c), \tag{43}$$

that also has a unique solution. Note that the Helmholtz decomposition (34) implies that (43) remains valid for any test-functions in $H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c)$, namely the next Lemma holds.

Lemma 4.2 *If $(\mathbf{A}^*, \varphi^*) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ is the unique solution of the adjoint problem (43), then we have*

$$B^*((\mathbf{A}^*, \varphi^*), (\mathbf{A}'', \varphi'')) = Q(\mathbf{A}''), \quad \forall (\mathbf{A}'', \varphi'') \in H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c). \tag{44}$$

Proof. Fix $(\mathbf{A}'', \varphi'') \in H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c)$, then the Helmholtz decomposition (34) implies that there exist $\psi \in H_0^1(D)$ and $\mathbf{A}' \in \widetilde{X}(D)$ such that

$$\mathbf{A}'' = \nabla\psi + \mathbf{A}'.$$

Therefore, by the sesquilinearity of B^* and (43), we have

$$\begin{aligned} B^*((\mathbf{A}^*, \varphi^*), (\mathbf{A}'', \varphi'')) &= B^*((\mathbf{A}^*, \varphi^*), (\nabla\psi, 0)) + B^*((\mathbf{A}^*, \varphi^*), (\mathbf{A}', \varphi'')) \\ &= B^*((\mathbf{A}^*, \varphi^*), (\nabla\psi, 0)) + Q(\mathbf{A}'). \end{aligned}$$

Since $Q(\nabla\psi) = 0$, we then obtain

$$B^*((\mathbf{A}^*, \varphi^*), (\mathbf{A}'', \varphi'')) = B^*((\mathbf{A}^*, \varphi^*), (\nabla\psi, 0)) + Q(\mathbf{A}''). \tag{45}$$

Finally as we readily check that

$$B^*((\mathbf{A}^*, \varphi^*), (\nabla\psi, 0)) = B^*((\mathbf{A}^*, \varphi^*), (0, j\omega\psi)).$$

and since $\nabla\psi = \nabla(\psi - \mathcal{M}_{D_c}\psi)$, where $\mathcal{M}_{D_c}\psi$ is the mean of ψ in D_c, again by (43), we find

$$B^*((\mathbf{A}^*, \varphi^*), (\nabla\psi, 0)) = B^*((\mathbf{A}^*, \varphi^*), (0, j\omega(\psi - \mathcal{M}_{D_c}\psi)))) = 0.$$

This property in (45) yields (44). ■

This Lemma implies that the strong formulation of (43) is

$$\begin{aligned} \operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{A}^*\right) - \sigma\Big(j\omega\mathbf{A}^* + \nabla\varphi^*\Big) &= \operatorname{curl}\mathbf{q} \text{ in } D, \\ \operatorname{div}(\sigma(j\omega\mathbf{A}^* + \nabla\varphi^*)) &= 0 \text{ in } D_c. \end{aligned}$$

Let us notice that Lemmas 2.2, 2.3 and 2.4 remain valid for the adjoint problem.

Similarly to the direct problem (see the beginning of Section 3), the adjoint one is approximated in a finite dimensional subspace V_h^* of $H^1(\mathcal{T})^3 \times H^1(\mathcal{T}_c)$, that may be different from V_h. In other words, we suppose given an approximation $(\mathbf{A}_h, \varphi_h) \in V_h$ of the solution $(\mathbf{A}, \varphi)$ of (10) and $(\mathbf{A}_h^*, \varphi_h^*) \in V_h^*$ of the solution $(\mathbf{A}^*, \varphi^*)$ of (43). As already specified, we assume that some flux reconstructions $\mathbf{H}_h$ and $\mathbf{J}_{e,h}$ are available (using $(\mathbf{A}_h, \varphi_h)$ and the datum $\mathbf{J}_s$) that belong respectively to $H(\operatorname{curl}, D)$ and $H(\operatorname{div}, D_c)$ and satisfy the conservation properties (17)-(19). In the same manner, by assuming that $\mathbf{q} \in H(\operatorname{curl}, D)$, we suppose that some flux reconstructions $\mathbf{H}_h^*$ and $\mathbf{J}_{e,h}^*$ are available (using $(\mathbf{A}_h^*, \varphi_h^*)$ and the datum $\operatorname{curl}\mathbf{q}$) that belong respectively to $H(\operatorname{curl}, D)$ and $H(\operatorname{div}, D_c)$ satisfy the following conservation properties:

$$(\operatorname{curl}\mathbf{H}_h^* + \tilde{\mathbf{J}}_{e,h}^* - \operatorname{curl}\mathbf{q}, \mathbf{e})_T = 0, \forall T \in \mathcal{T}, \mathbf{e} \in \mathbb{C}^3, \tag{46}$$

$$\operatorname{div}\mathbf{J}_{e,h}^* = 0 \text{ in } D_c, \tag{47}$$

$$\mathbf{J}_{e,h}^* \cdot \mathbf{n} = 0 \text{ on } \partial D_c. \tag{48}$$

Recall that $\mathbf{H}_h$ represents an approximation of $\mu^{-1}\operatorname{curl}\mathbf{A}$, $\mathbf{J}_{e,h}$ an approximation of $\sigma\mathbf{E}$. Similarly, note that $\mathbf{H}_h^*$ represents an approximation of $\mu^{-1}\operatorname{curl}\mathbf{A}^*$ and $\mathbf{J}_{e,h}^*$ an approximation of $\sigma\mathbf{E}^* = -\sigma(j\omega\mathbf{A}^* + \nabla\varphi^*)$.

Let us now state a guaranteed error estimate for the adjoint problem. For that purpose, we suppose given a potential reconstruction $(\mathbf{S}_h^*, \psi_h^*) \in H_0(\operatorname{curl}, D) \times \widetilde{H^1}(D_c)$ of $(\mathbf{A}_h^*, \varphi_h^*)$. Then we define the global estimator η^* as follows

$$\eta^* = 2\eta_{NC}^* + \eta_{\text{flux}}^* + \eta_{\mathcal{O}}^*, \tag{49}$$

where the nonconforming estimator η_{NC}^* is defined by

$$\begin{aligned} \eta_{NC}^* &= \Bigg(\left\|\mu^{-1/2}\operatorname{curl}_h(\mathbf{A}_h^* - \mathbf{S}_h^*)\right\|^2 \\ &+ \left\|\omega^{-1/2}\,\sigma^{1/2}\left(j\,\omega(\mathbf{A}_h^* - \mathbf{S}_h^*) + \nabla_h(\varphi_h^* - \psi_h^*)\right)\right\|_{D_c}^2\Bigg)^{1/2}, \end{aligned} \tag{50}$$

the flux estimator η_{flux}^* is defined by

$$\eta_{\text{flux}}^* = \left((\eta_{\text{magn}}^*)^2 + (\eta_{\text{elec}}^*)^2)\right)^{1/2}, \tag{51}$$

where η^*_{magn} and η^*_{elec} are defined by

$$\eta^*_{\text{magn}} = \left\|\mu^{1/2}(\mathbf{H}^*_h - \mu^{-1}\mathbf{B}^*_h)\right\|_D, \ \eta^*_{\text{elec}} = \left\|(\omega\sigma)^{-1/2}(\mathbf{J}^*_{e,h} - \sigma\mathbf{E}^*_h)\right\|_{D_c}, \tag{52}$$

and finally the oscillation estimator $\eta^*_{\mathcal{O}}$ is defined by

$$\eta^*_{\mathcal{O}} = C_L\mu_{\max}^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}} c_{P,T}h_T^2\|\text{curl}\mathbf{H}^*_h + \tilde{\mathbf{J}}^*_{e,h} - \text{curl}\mathbf{q}\|_T^2\right)^{\frac{1}{2}}. \tag{53}$$

The same arguments as the ones used in the proof of Theorem 3.2 allow to prove the following guaranteed error estimate on the error

$$\epsilon_{A^*,\varphi^*} = \left(\left\|\mu^{-1/2}\text{curl}_h\epsilon_{A^*}\right\|^2 + \left\|\omega^{-1/2}\,\sigma^{1/2}(j\,\omega\epsilon_{A^*} + \nabla_h\epsilon_{\varphi^*})\right\|_{D_c}^2\right)^{1/2}, \tag{54}$$

where

$$\epsilon_{A^*} = \mathbf{A}^* - \mathbf{A}^*_h \text{ and } \epsilon_{\varphi^*} = \varphi^* - \varphi^*_h.$$

Corollary 4.3 *Under the assumptions from this Section, we have*

$$\epsilon_{A^*,\varphi^*} \leq \eta^*. \tag{55}$$

5 The error representation

With the help of the equilibrated fluxes of the direct and adjoint problems, in the spirit of [23] (see also [21, 10]), we here show that the error on the quantity of interest can be expressed into a fully computable expression and a remainder that will be estimated by a fully computable quantity (but is usually of higher order and can then be disregarded).

Theorem 5.1 *Let $(\mathbf{S}_h, \psi_h) \in H_0(\text{curl}, D) \times \widetilde{H^1}(D_c)$ be a potential reconstruction of $(\mathbf{A}_h, \varphi_h)$, then the error on the quantity of interest defined by*

$$\mathcal{E} = \sum_{T\in\mathcal{T}}\int_T \mathbf{q}\cdot\text{curl}\overline{(\mathbf{A}-\mathbf{A}_h)}\,dx,$$

admits the splitting

$$\mathcal{E} = \eta_{\text{QOI}} + \mathcal{R}, \tag{56}$$

where the estimator η_{QOI} is given by

$$\begin{aligned}
\eta_{\text{QOI}} &= \sum_{T\in\mathcal{T}}\int_T \mathbf{q}\cdot\text{curl}\overline{(\mathbf{S}_h-\mathbf{A}_h)}\,dx \\
&+ \int_D \mathbf{S}^*_h\cdot\overline{(\mathbf{J}_s - \text{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\,dx \\
&- j\omega^{-1}\int_{D_c}\sigma^{-1}\mathbf{J}^*_{e,h}\cdot\overline{(\sigma(j\omega\mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h})}\,dx \\
&- \int_D \mathbf{H}^*_h\cdot(\text{curl}\overline{\mathbf{S}_h} - \mu\overline{\mathbf{H}_h})\,dx
\end{aligned} \tag{57}$$

while the remainder term $\mathcal{R}$ is defined by

$$\begin{aligned}
\mathcal{R} &= \int_D(\mathbf{A}^* - \mathbf{S}^*_h)\cdot\overline{(\mathbf{J}_s - \text{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\,dx \\
&+ j\omega^{-1}\int_{D_c}(\sigma^{-1}\mathbf{J}^*_{e,h} - \mathbf{E}^*)\cdot\overline{(\sigma(j\omega\mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h})}\,dx \\
&- \int_D(\mu^{-1}\text{curl}\mathbf{A}^* - \mathbf{H}^*_h)\cdot(\text{curl}\overline{\mathbf{S}_h} - \mu\overline{\mathbf{H}_h})\,dx
\end{aligned} \tag{58}$$

Proof. First we notice that

$$(\mathbf{J}_s, \mathbf{A}^*)_D = B((\mathbf{A},\varphi),(\mathbf{A}^*,\varphi^*)) = \overline{B^*((\mathbf{A}^*,\varphi^*),(\mathbf{A},\varphi))} = \overline{Q(\mathbf{A})}.$$

Hence by the definition of the error, we have

$$\mathcal{E} = \overline{(\mathbf{J}_s, \mathbf{A}^*)_D} - \sum_{T\in\mathcal{T}} \int_T \mathbf{q}\cdot \mathrm{curl}\overline{\mathbf{A}_h}\, dx.$$

Introducing artificially $\mathbf{S}_h$ and using (43) with $(\mathbf{A}', \varphi') = (\mathbf{S}_h, \psi_h)$, this is equivalent to

$$\begin{aligned} \mathcal{E} &= \overline{(\mathbf{J}_s, \mathbf{A}^*)_D} - \sum_{T\in\mathcal{T}} \int_T \mathbf{q}\cdot \mathrm{curl}\overline{(\mathbf{A}_h - \mathbf{S}_h)}\, dx \\ &- \left(\mu^{-1}\mathrm{curl}\mathbf{A}^*, \mathrm{curl}\mathbf{S}_h\right)_D \\ &+ j\omega^{-1}\left(\sigma(j\omega\mathbf{A}^* + \nabla\varphi^*), (j\omega\mathbf{S}_h + \nabla\psi_h)\right)_{D_c}. \end{aligned} \tag{59}$$

Using (16), and adding and subtracting the terms

$$\int_D \mathrm{curl}\mathbf{A}^*\cdot\overline{\mathbf{H}_h}\, dx, \text{ and } j\omega^{-1}\int_{D_c} \nabla\varphi^*\cdot\bar{\mathbf{J}}_{e,h}\, dx,$$

we find

$$\begin{aligned} \mathcal{E} &= \overline{(\mathbf{J}_s, \mathbf{A}^*)_D} - \sum_{T\in\mathcal{T}} \int_T \mathbf{q}\cdot \mathrm{curl}\overline{(\mathbf{A}_h - \mathbf{S}_h)}\, dx \\ &- \int_D \mathrm{curl}\mathbf{A}^*\cdot(\mu^{-1}\mathrm{curl}\overline{\mathbf{S}_h} - \overline{\mathbf{H}_h})\, dx \\ &- \int_D \mathrm{curl}\mathbf{A}^*\cdot\overline{\mathbf{H}_h}\, dx \\ &- \int_{D_c} \sigma\mathbf{A}^*\cdot\overline{(j\omega\mathbf{S}_h + \nabla\psi_h)}\, dx \\ &+ j\omega^{-1}\int_{D_c} \nabla\varphi^*\cdot\overline{(\sigma(j\omega\mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h})}\, dx \\ &- j\omega^{-1}\int_{D_c} \nabla\varphi^*\cdot\overline{\mathbf{J}_{e,h}}\, dx. \end{aligned}$$

Using Green's formula in the fourth and seventh terms of this right-hand side and using (18)-(19), we find

$$\begin{aligned} \mathcal{E} &= \overline{(\mathbf{J}_s, \mathbf{A}^*)_D} - \sum_{T\in\mathcal{T}} \int_T \mathbf{q}\cdot \mathrm{curl}\overline{(\mathbf{A}_h - \mathbf{S}_h)}\, dx \\ &- \int_D \mathrm{curl}\mathbf{A}^*\cdot(\mu^{-1}\mathrm{curl}\overline{\mathbf{S}_h} - \overline{\mathbf{H}_h})\, dx \\ &- \int_D \mathbf{A}^*\cdot\mathrm{curl}\overline{\mathbf{H}_h}\, dx \\ &- \int_{D_c} \sigma\mathbf{A}^*\cdot\overline{(j\omega\mathbf{S}_h + \nabla\psi_h)}\, dx \\ &+ j\omega^{-1}\int_{D_c} \nabla\varphi^*\cdot\overline{(\sigma(j\omega\mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h})}\, dx. \end{aligned}$$

Adding and subtracting the term $\int_{D_c} \mathbf{A}^* \cdot \overline{\mathbf{J}_{e,h}}\, dx$ and rearranging the terms of this right-hand side, we find equivalently

$$\begin{aligned}
\mathcal{E} &= -\sum_{T\in\mathcal{T}} \int_T \mathbf{q} \cdot \mathrm{curl}\overline{(\mathbf{A}_h - \mathbf{S}_h)}\, dx \\
&+ \int_D \mathbf{A}^* \cdot \overline{(\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\, dx \\
&- j\omega^{-1} \int_{D_c} \mathbf{E}^* \cdot \overline{(\sigma(j\omega \mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h})}\, dx \\
&- \int_D \mathrm{curl}\mathbf{A}^* \cdot (\mu^{-1}\mathrm{curl}\overline{\mathbf{S}_h} - \overline{\mathbf{H}_h})\, dx,
\end{aligned}$$

where we recall that $\tilde{\mathbf{J}}_{e,h}$ means the extension by zero of $\mathbf{J}_{e,h}$ outside D_c and that $\mathbf{E}^* = -(j\omega\mathbf{A}^* + \nabla\varphi^*)$. Writing

$$\mathbf{A}^* = \mathbf{S}_h^* + \mathbf{A}^* - \mathbf{S}_h^*, \quad \mathbf{E}^* = \sigma^{-1}\mathbf{J}_{e,h}^* + \mathbf{E}^* - \sigma^{-1}\mathbf{J}_{e,h}^*,$$

and

$$\mu^{-1}\mathrm{curl}\mathbf{A}^* = \mathbf{H}_h^* + \mu^{-1}\mathrm{curl}\mathbf{A}^* - \mathbf{H}_h^*,$$

we arrive at (56). ∎

Let us now show that the remainder can be explicitly estimated using the error estimators for $(\mathbf{A}, \varphi)$ and $(\mathbf{A}^*, \varphi^*)$ obtained in Section 3.

Theorem 5.2 *With η (resp. η^*) defined before, we have*

$$|\mathcal{R}| \le 6\eta\eta^*. \tag{60}$$

Proof. We estimate each term of $\mathcal{R}$ separetely. For the first term,

$$\mathcal{R}_1 = \int_D (\mathbf{A}^* - \mathbf{S}_h^*) \cdot \overline{(\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\, dx,$$

as in the proof of Theorem 3.2 we use the Helmholtz decomposition

$$\mathbf{A}^* - \mathbf{S}_h^* \;=\; \nabla\phi^* \;+\; \epsilon_\perp^*\,, \tag{61}$$

with $\phi^* \in H_0^1(D)$ and $\epsilon_\perp^* \in \widetilde{X}(D)$ and

$$\|\mathbf{A}^* - \mathbf{S}_h^*\|_D^2 \;= \|\nabla\phi^*\|_D^2 + \|\epsilon_\perp^*\|_D^2. \tag{62}$$

As $\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h}$ is divergence free, we then get

$$\mathcal{R}_1 = \int_D \epsilon_\perp^* \cdot \overline{(\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\, dx.$$

Again as in the proof of Theorem 3.2, as $\epsilon_\perp^* \in \widetilde{X}(D)$, there exist a unique function $\Phi^* \in H_0^1(D)$ and a unique vector field $\Psi^* \in H_0(\mathrm{curl}, D) \cap H^1(D)^3$ such that (see (26) and (27))

$$\epsilon_\perp^* = \nabla\Phi^* + \Psi^*,$$

with

$$|\Phi^*|_{1,D} + |\Psi^*|_{1,D} \le C_L\|\mathrm{curl}\epsilon_\perp^*\|_D \le C_L\mu_{\max}^{\frac{1}{2}}\|\mu^{-1/2}\mathrm{curl}(\mathbf{A}^* - \mathbf{S}_h^*)\|, \tag{63}$$

since $\mathrm{curl}(\mathbf{A}^* - \mathbf{S}_h^*) = \mathrm{curl}\epsilon_\perp^*$. As $\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h}$ is divergence free and using the property (17), we then get

$$\begin{aligned}
\mathcal{R}_1 &= \int_D \Psi^* \cdot \overline{(\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\, dx \\
&= \int_D (\Psi^* - \mathcal{M}_h\Psi^*) \cdot \overline{(\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\, dx.
\end{aligned}$$

Cauchy-Schwarz's inequality and Poincaré's inequality (25) yield

$$|\mathcal{R}_1| \leq \eta_{\mathcal{O}} \|\mu^{-1/2}\mathrm{curl}(\mathbf{A}^* - \mathbf{S}_h^*)\|, \tag{64}$$

where we have used the estimate (63) and (24). Coming back to $\mathbf{A}^* - \mathbf{A}_h^*$, we get

$$\begin{aligned} |\mathcal{R}_1| &\leq \eta_{\mathcal{O}}(\|\mu^{-1/2}\mathrm{curl}_h(\mathbf{A}^* - \mathbf{A}_h^*)\| + \|\mu^{-1/2}\mathrm{curl}_h(\mathbf{A}_h^* - \mathbf{S}_h^*)\|) \\ &\leq \eta_{\mathcal{O}}(\epsilon_{A^*,\varphi^*} + \eta_{NC}^*), \end{aligned}$$

where we recall that ϵ_{A^*,φ^*} is defined by (54). Owing to the estimate (55), we get finally

$$|\mathcal{R}_1| \leq \eta_{\mathcal{O}}\ (\eta^* + \eta_{NC}^*). \tag{65}$$

For the second term

$$\mathcal{R}_2 = j\omega^{-1}\int_{D_c}(\sigma^{-1}\mathbf{J}_{e,h}^* - \mathbf{E}^*)\cdot\overline{(\sigma(j\omega\mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h})}\,dx,$$

we use Cauchy-Schwarz's inequality, to get

$$|\mathcal{R}_2| \leq \|(\omega\sigma)^{-1/2}(\mathbf{J}_{e,h}^* - \sigma\mathbf{E}^*)\|_{D_c}\|(\omega\sigma)^{-1/2}\left(\sigma(j\omega\mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h}\right)\|_{D_c}.$$

Adding $\pm(\omega\sigma)^{-1/2}\mathbf{E}_h^*$ and $\pm(\omega\sigma)^{-1/2}\mathbf{E}_h$, and using the triangular inequality, we get

$$\begin{aligned} |\mathcal{R}_2| \leq &\left(\|(\omega\sigma)^{-1/2}(\mathbf{J}_{e,h}^* - \sigma\mathbf{E}_h^*)\|_{D_c} + \|\omega^{-1/2}\sigma^{1/2}(\mathbf{E}_h^* - \mathbf{E}^*)\|_{D_c}\right)\times \\ &\left(\|(\omega\sigma)^{-1/2}\left(-\sigma\mathbf{E}_h + \mathbf{J}_{e,h}\right)\|_{D_c} + \|\omega^{-1/2}\sigma^{1/2}\left(j\omega\mathbf{S}_h + \nabla\psi_h + \mathbf{E}_h\right)\|_{D_c}\right). \end{aligned}$$

Reminding (7), (16), and the definitions of η_{flux}, η_{NC}, $\epsilon_{A,\varphi}$, and ϵ_{A^*,φ^*}, we find

$$|\mathcal{R}_2| \leq \left(\eta_{\text{flux}}^* + \epsilon_{A^*,\varphi^*}\right)\left(\eta_{\text{flux}} + \eta_{NC} + \epsilon_{A,\varphi}\right).$$

By (29) and (55), we get finally

$$|\mathcal{R}_2| \leq \left(\eta_{\text{flux}} + \eta_{NC} + \eta\right)\left(\eta_{\text{flux}}^* + \eta^*\right). \tag{66}$$

Again for the third term

$$\mathcal{R}_3 = -\int_D(\mu^{-1}\mathrm{curl}\mathbf{A}^* - \mathbf{H}_h)\cdot(\mathrm{curl}\overline{\mathbf{S}_h} - \mu\overline{\mathbf{H}_h})\,dx,$$

we use Cauchy-Schwarz's inequality to obtain

$$|\mathcal{R}_3| \leq \|\mu^{1/2}(\mu^{-1}\mathrm{curl}\mathbf{A}^* - \mathbf{H}_h)\|_D\|\mu^{1/2}(\mu^{-1}\mathrm{curl}\mathbf{S}_h - \mathbf{H}_h\|_D.$$

Introducing $\mathrm{curl}_h\mathbf{A}_h^*$ and $\mathrm{curl}_h\mathbf{A}_h$, the triangular inequality yields

$$\begin{aligned} |\mathcal{R}_3| \leq &\left(\|\mu^{1/2}(\mu^{-1}\mathrm{curl}\mathbf{A}_h^* - \mathbf{H}_h)\|_D + \|\mu^{-1/2}\mathrm{curl}_h(\mathbf{A}^* - \mathbf{A}_h^*)\|_D\right)\times \\ &\left(\|\mu^{1/2}(\mu^{-1}\mathrm{curl}\mathbf{A}_h - \mathbf{H}_h\|_D + \|\mu^{-1/2}\mathrm{curl}_h(\mathbf{S}_h - \mathbf{A}_h)\|_D\right). \end{aligned}$$

As before, this means that

$$|\mathcal{R}_3| \leq (\eta_{\text{flux}} + \eta_{NC} + \epsilon_{A,\varphi})(\eta_{\text{flux}}^* + \eta_{NC}^*),$$

and again (29) yields

$$|\mathcal{R}_3| \leq (\eta_{\text{flux}} + \eta_{NC} + \eta)(\eta_{\text{flux}}^* + \eta_{NC}^*). \tag{67}$$

The estimates (65) to (67) lead to

$$|\mathcal{R}| \leq \eta_{\mathcal{O}} \, (\eta^* + \eta^*_{NC}) + (\eta_{\text{flux}} + \eta_{NC} + \eta)(2\eta^*_{\text{flux}} + \eta^*_{NC} + \eta^*).$$

Using the respective definitions (20) and (49) of η and η^*, and simple calculations yield

$$|\mathcal{R}| \leq 9\eta_{NC}\eta^*_{NC} + 9\eta_{NC}\eta^*_{\text{flux}} + 3\eta_{NC}\eta^*_{\mathcal{O}} + 6\eta_{\text{flux}}\eta^*_{NC} + 6\eta_{\text{flux}}\eta^*_{\text{flux}} + 2\eta_{\text{flux}}\eta^*_{\mathcal{O}} + 6\eta_{\mathcal{O}}\eta^*_{NC} + 4\eta_{\mathcal{O}}\eta^*_{\text{flux}} + 2\eta_{\mathcal{O}}\eta^*_{\mathcal{O}},$$

as wells as

$$\begin{aligned} \eta\eta^* &= (2\eta_{NC} + \eta_{\text{flux}} + \eta_{\mathcal{O}})(2\eta^*_{NC} + \eta^*_{\text{flux}} + \eta^*_{\mathcal{O}}) \\ &= 4\eta_{NC}\eta^*_{NC} + 2\eta_{NC}\eta^*_{\text{flux}} + 2\eta_{NC}\eta^*_{\mathcal{O}} + 2\eta_{\text{flux}}\eta^*_{NC} + \eta_{\text{flux}}\eta^*_{\text{flux}} + \eta_{\text{flux}}\eta^*_{\mathcal{O}} + 2\eta_{\mathcal{O}}\eta^*_{NC} + \eta_{\mathcal{O}}\eta^*_{\text{flux}} + \eta_{\mathcal{O}}\eta^*_{\mathcal{O}}. \end{aligned}$$

As the smallest constant C such that

$$\begin{aligned} &9\eta_{NC}\eta^*_{NC} + 9\eta_{NC}\eta^*_{\text{flux}} + 3\eta_{NC}\eta^*_{\mathcal{O}} + 6\eta_{\text{flux}}\eta^*_{NC} + 6\eta_{\text{flux}}\eta^*_{\text{flux}} + 2\eta_{\text{flux}}\eta^*_{\mathcal{O}} + 6\eta_{\mathcal{O}}\eta^*_{NC} + 4\eta_{\mathcal{O}}\eta^*_{\text{flux}} + 2\eta_{\mathcal{O}}\eta^*_{\mathcal{O}} \\ &\leq C \left(4\eta_{NC}\eta^*_{NC} + 2\eta_{NC}\eta^*_{\text{flux}} + 2\eta_{NC}\eta^*_{\mathcal{O}} + 2\eta_{\text{flux}}\eta^*_{NC} + \eta_{\text{flux}}\eta^*_{\text{flux}} + \eta_{\text{flux}}\eta^*_{\mathcal{O}} + 2\eta_{\mathcal{O}}\eta^*_{NC} + \eta_{\mathcal{O}}\eta^*_{\text{flux}} + \eta_{\mathcal{O}}\eta^*_{\mathcal{O}}\right), \end{aligned}$$

is $C = 6$, we have proved that (60) holds.

■

Before going on, if $(\mathbf{A}_h, \varphi_h)$ is a conforming approximation of $(\mathbf{A}, \varphi)$, let us give an estimate on the error of quantity of interest that will be used for our numerical examples but has also its own interest.

Lemma 5.3 *Let X_h be a finite dimensional subspace of $H_0(\text{curl}, D) \times H^1(D_c)$ and denote by*

$$K_h = \{(\mathbf{A}'_h, \varphi'_h) \in X_h \, : \, \text{curl}\, \mathbf{A}'_h = 0\} = X_h \cap \left(\nabla H^1_0(D) \times H^1(D_c)\right),$$

its subspace of elements whose first component is curl free. Then denote by

$$\mathbf{V}_h = \{(\mathbf{A}'_h, \varphi'_h) \in X_h \, : \, (\mathbf{A}'_h, \mathbf{A}''_h)_D = 0, \forall\, (\mathbf{A}''_h, \varphi''_h) \in K_h \text{ and } \int_{D_c} \varphi'_h = 0\},$$

the finite dimensional subspace of X_h made of elements whose first component is discrete divergence free and second component is of zero mean. Let $(\mathbf{A}_h, \varphi_h) \in \mathbf{V}_h$ be the Galerkin approximation of the solution $(\mathbf{A}, \varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ of (10), namely the unique solution to

$$B((\mathbf{A}_h, \varphi_h), (\mathbf{A}'_h, \varphi'_h)) = (\mathbf{J}_s, \mathbf{A}'_h), \quad \forall (\mathbf{A}'_h, \varphi'_h) \in \mathbf{V}_h. \tag{68}$$

Then the error on the quantity of interest is equal to

$$\mathcal{E} = B^*((\mathbf{A}^* - \mathbf{A}'_h, \varphi^* - \varphi'_h), (\mathbf{A} - \mathbf{A}_h, \varphi - \varphi_h)), \forall (\mathbf{A}'_h, \varphi'_h) \in X_h. \tag{69}$$

Proof. Let us first notice that by the definition of $\mathbf{V}_h$, problem (68) has indeed a unique solution because if $(\mathbf{A}_h, \varphi_h) \in \mathbf{V}_h$ satisfy

$$B((\mathbf{A}_h, \varphi_h), (\mathbf{A}_h, \varphi_h)) = 0,$$

then recalling that D is simply connected, $(\mathbf{A}_h, \varphi_h)$ belongs to K_h, and therefore $(\mathbf{A}_h, \varphi_h) = (0, 0)$. Now by its definition and our assumption on $\mathbf{V}_h$, we have

$$\mathcal{E} = \int_D q \cdot \text{curl}\overline{(\mathbf{A} - \mathbf{A}_h)} = Q(\mathbf{A} - \mathbf{A}_h).$$

Therefore as $\mathbf{A} - \mathbf{A}_h$ belongs to $H_0(\text{curl}, D)$, by Lemma 4.2, we have

$$\mathcal{E} = B^*((\mathbf{A}^*, \varphi^*), (\mathbf{A} - \mathbf{A}_h, \varphi - \varphi_h)). \tag{70}$$

As Lemma 2.3 of [11] proved that

$$B((\mathbf{A}-\mathbf{A}_h,\varphi-\varphi_h),(\mathbf{A}'_h,\varphi'_h))=0,\forall(\mathbf{A}'_h,\varphi'_h)\in X_h,$$

by the definition of B^*, we get equivalently

$$B^*((\mathbf{A}'_h,\varphi'_h),(\mathbf{A}-\mathbf{A}_h,\varphi-\varphi_h)),\forall(\mathbf{A}'_h,\varphi'_h)\in X_h.$$

Subtracting this identity to (70), we obtain (69). ■

Corollary 5.4 *Suppose that D and D_c are polyhedra and define (see [11])*

$$\mathbf{V}_h=\tilde{X}_h\times\tilde{\Theta}_h, \tag{71}$$

where

$$\begin{aligned}
\tilde{X}_h&=\{\mathbf{A}'_h\in X_h\ :\int_D\mathbf{A}'_h\cdot\nabla\psi_h=0,\forall\psi_h\in\Theta_h^0\},\\
X_h&=\{\mathbf{A}'_h\in H_0(\mathrm{curl},D)\ :\ \mathbf{A}'_{h|T}\in\mathcal{ND}_1(T),\forall T\in\mathcal{T}\},\\
\tilde{\Theta}_h&=\{\varphi'_h\in\widetilde{H^1}(D_c)\ :\varphi'_{h|T}\in\mathbb{P}_1(T),\forall T\in\mathcal{T}\cap\bar{D}_c\},\\
\Theta_h^0&=\{\psi_h\in H_0^1(D)\ :\psi_{h|T}\in\mathbb{P}_1(T),\forall T\in\mathcal{T}\},
\end{aligned}$$

and for $k\in\mathbb{N}^*$, $\mathcal{ND}_k(T)$ is the k^{th}-order Nédélec element defined by

$$\mathcal{ND}_k(T)=[\mathbb{P}_{k-1}(T)]^3\oplus\mathbb{S}_k(T), \tag{72}$$

with $\mathbb{P}_{k-1}(T)$ the space of polynomials of order at most $k-1$ on T and $\mathbb{S}_k(T)$ defined by

$$\mathbb{S}_k(T)=\{\mathbf{p}\in[\tilde{\mathbb{P}}_k(T)]^3\ :\ \mathbf{x}\cdot\mathbf{p}(\mathbf{x})=0,\forall\mathbf{x}\in T\},$$

where $\tilde{\mathbb{P}}_k(T)$ is the space of homogeneous polynomials of order k on T. Let $(\mathbf{A}_h,\varphi_h)\in\mathbf{V}_h$ be the Galerkin approximation of the solution $(\mathbf{A},\varphi)\in\widetilde{X}(D)\times\widetilde{H^1}(D_c)$ of (10). Suppose further that the solution $(\mathbf{A},\varphi)\in\widetilde{X}(D)\times\widetilde{H^1}(D_c)$ of (10) belongs to $H^2(D)^3\times H^2(D_c)$ and that the solution $(\mathbf{A}^*,\varphi^*)\in\widetilde{X}(D)\times\widetilde{H^1}(D_c)$ of (43) belongs to $H^2(D)^3\times H^{1+s}(D_c)$, for some $s\in(0,1]$. If $\mathcal{T}$ is a regular triangulation, then

$$|\mathcal{E}|\le Ch^{1+s}, \tag{73}$$

for a positive constant C that does not depend on h.

Proof. First we notice our choice of $\mathbf{V}_h$ enters in the framework of Lemma 5.3 with X_h defined by

$$X_h=\{(\mathbf{A}'_h,\varphi'_h)\in H_0(\mathrm{curl},D)\times H^1(D_c)\ :A'_{h|T}\in\mathcal{ND}_1(T),\text{ and }\varphi'_{h|T}\in\mathbb{P}_1(T),\forall T\in\mathcal{T}\},$$

since we have (see [22, Lemma 5.38])

$$K_h=\nabla\Theta_h^0.$$

By the identity (69) with $\mathbf{A}'_h=I_{ND}\mathbf{A}^*_h$ and $\varphi'_h=I_L\varphi^*$, where I_{ND} (resp. I_L) is the (low-order) Nédélec (resp. $\mathbb{P}_1$ Lagrange) interpolant, we have

$$|\mathcal{E}|\le C_1(\|\mathbf{A}^*-I_{ND}\mathbf{A}^*\|_{H(\mathrm{curl},D)}+|\varphi^*-I_L\varphi^*|_{H^1(D_c)})(\|\mathbf{A}-\mathbf{A}_h\|_{H(\mathrm{curl},D)}+|\varphi-\varphi_h|_{H^1(D_c)}),$$

for some positive constant C_1 independent of h. As there exists a positive constant C_2 independent of h such that

$$\|\mathbf{A}-\mathbf{A}_h\|_{H(\mathrm{curl},D)}+|\varphi-\varphi_h|_{H^1(D_c)}\le C_2(\|\mathbf{A}-I_{ND}\mathbf{A}\|_{H(\mathrm{curl},D)}+|\varphi-I_L\varphi|_{H^1(D_c)}),$$

we conclude that

$$\begin{aligned}
|\mathcal{E}|\quad\le\quad & C_1\max\{C_2,1\}(\|\mathbf{A}^*-I_{ND}\mathbf{A}^*\|_{H(\mathrm{curl},D)}+|\varphi^*-I_L\varphi^*|_{H^1(D_c)})\\
&(\|\mathbf{A}-I_{ND}\mathbf{A}\|_{H(\mathrm{curl},D)}+|\varphi^*-I_L\varphi|_{H^1(D_c)}).
\end{aligned}$$

We conclude by standard interpolation error estimates for the Nédélec and Lagrange elements.
■

6 Numerical validation

In this Section we propose a numerical test inspired by [11] (see Section 5.1), in order to underline and confirm our theoretical predictions, using the FreeFem++ software [18]. The data are built in order to have in hand an exact solution, allowing to compare the estimator to the exact error. The domains are defined by $D = [-2,5] \times [-2,2] \times [-2,2]$, $D_s = [-1,1]^3$ and $D_c = [2,4] \times [-1,1] \times [-1,1]$ (see Figure 1).

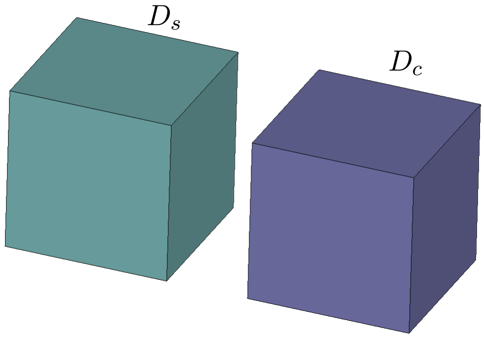


Figure 1: Domains configuration.

We set $\mu \equiv 1$ in D, $\sigma \equiv 1$ in D_c and $\omega = 2\pi$. Then, the exact solution is given by $\varphi \equiv 0$ and

$$\mathbf{A} = \mathrm{curl}\begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} \text{ with } f(x,y,z) = \begin{cases} (x^2-1)^4(y^2-1)^4(z^2-1)^4 & \text{in} \quad D_s, \\ 0 & \text{in} \quad D\backslash D_s. \end{cases}$$

The value of $\mathbf{J}_s$ is computed accordingly using (8a), and is given by

$$\mathbf{J}_s = \mathrm{curl}(\mathrm{curl}\mathbf{A}),$$

as the support of $\mathbf{A}$ is equal to D_s (that is disjoint from D_c).

First of all, we compute $(\mathbf{A}_h, \varphi_h) \in \mathbf{V}_h$, with $\mathbf{V}_h$ defined by (71), as the Galerkin approximation of the solution $(\mathbf{A}, \varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ of (10). The estimator η defined by (20) is computed using $\mathbf{S}_h = \mathbf{A}_h$ and $\psi_h = \varphi_h$ as well as with the reconstructed fluxes $\mathbf{H}_h$ and $\mathbf{J}_{e,h}$ which are computed like explained in Appendix A. As mentioned in Remark 4.1, we choose

$$\mathbf{q} = \mathbf{H}_s = \mathrm{curl}\mathbf{A}, \tag{74}$$

and we are interested in the value of $\mathcal{E}$ given by:

$$\mathcal{E} = \int_D \mathbf{H}_s \cdot \mathrm{curl}\overline{(\mathbf{A} - \mathbf{A}_h)}\, dx$$

Consequently in order to validate the identity (56) and the estimate (60), it remains to compute a numerical approximation $(\mathbf{A}_h^*, \varphi_h^*)$ of the solution $(\mathbf{A}^*, \varphi^*)$ of the adjoint problem (43), as well as the estimator η^*. Let us first notice that for this example, we directly check that $(\mathbf{A}^*, \varphi^*) = (\mathbf{A}, 0)$. Here we take $(\mathbf{A}_h^*, \varphi_h^*) \in \mathbf{V}_h^*$ as the Galerkin approximation of $(\mathbf{A}^*, \varphi^*)$, with $\mathbf{V}_h^*$ defined by

$$\mathbf{V}_h^* = \tilde{X}_h^* \times \tilde{\Theta}_h^*, \tag{75}$$

where

$$\begin{aligned}
\tilde{X}_h^* &= \{\mathbf{A}'_h \in X_h^* \ : \int_D \mathbf{A}'_h \cdot \nabla \psi_h = 0, \forall \psi_h \in \Theta_h^{*,0}\},\\
X_h^* &= \{\mathbf{A}'_h \in H_0(\mathrm{curl}, D) \ : \ \mathbf{A}'_{h|T} \in \mathcal{ND}_2(T), \forall T \in \mathcal{T}\},\\
\tilde{\Theta}_h^* &= \{\varphi'_h \in \widetilde{H^1}(D_c) \ : \varphi'_{h|T} \in \mathbb{P}_2(T), \forall T \in \mathcal{T} \cap \bar{D}_c\},\\
\Theta_h^{*,0} &= \{\psi_h \in H_0^1(D) \ : \psi_{h|T} \in \mathbb{P}_2(T), \forall T \in \mathcal{T}\},
\end{aligned}$$

$\mathcal{ND}_2(T)$ being defined by (72). The estimator η^* defined by (49) is computed using $\mathbf{S}_h^* = \mathbf{A}_h^*$ and $\psi_h^* = \varphi_h^*$, as well as with the reconstructed fluxes $\mathbf{H}_h^*$ and $\mathbf{J}_{e,h}^*$ which are computed like explained in Appendix A. Finally, the estimator η_{QOI} is computed by (57).
Tests are performed using two sets of meshes as described below.

6.1 Results with globally refined meshes

The first set consists in some uniformly refined meshes. Figure 2 displays the fourth mesh of the series (Mesh 4), and Table 1 indicates for each mesh the number of tetrahedra and the number of degrees of freedom associated to each formulation (direct or adjoint one). Recall that since the adjoint problem is computed with finite elements of higher degrees, the number of degrees of freedom is more important on the same mesh.

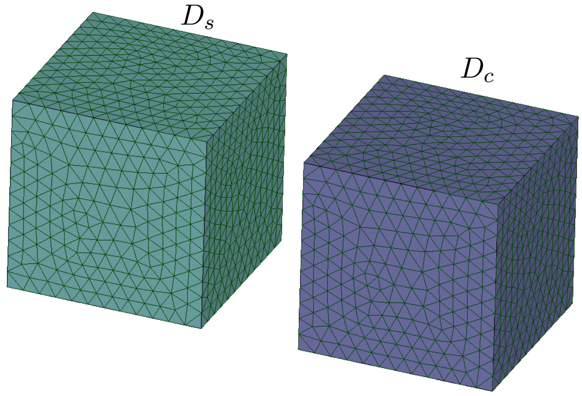


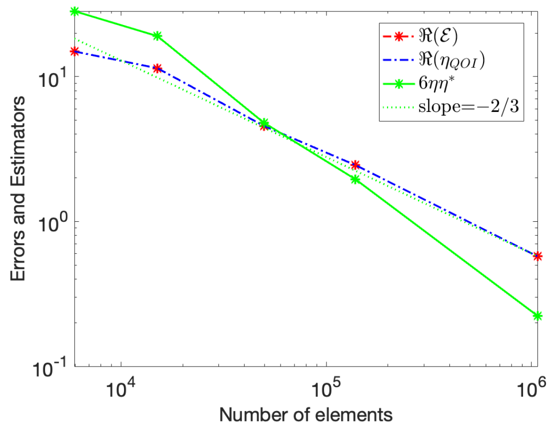
Figure 2: Mesh 4, global refinement,138 404 elements : 9 826 elements in D_s, 9 428 elements in D_c and 119 150 elements in $D\backslash(D_s \cup D_c)$.

Mesh	Nb of elements	Nb dof for $(\mathbf{A}_h, \varphi_h)$	Nb dof for $(\mathbf{A}_h^*, \varphi_h^*)$
1	5 935	8 097	41 843
2	15 052	20 372	105 507
3	49 642	63 031	334 722
4	138 404	173 809	924 858
5	1 068 028	1 302 975	7 013 477

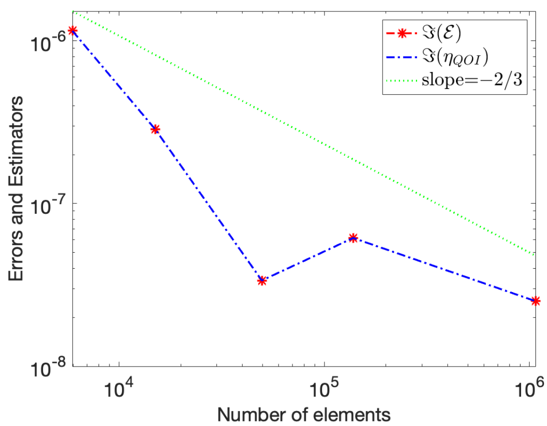
Table 1: The meshes used for the simulations, global refinement.

To begin with, we plot in Figure 3 the values of $\mathcal{E}$, η_{QOI} and $6\eta\eta^*$ as a function of the number of elements N, in a log-log scale. More precisely, the real parts are displayed in Figure 3(a) and the imaginary ones in Figure 3(b). First, it can be seen in Figure 3(a) that $\Re(\mathcal{E})$ goes towards zero as $O(N^{-2/3}) = O(h^2)$. This is a consequence of Corollary 5.4 (with $s = 1$), since in this example $\mathbf{A} = \mathbf{A}^*$ are in $H^3(D)^3$, while $\varphi = \varphi^* = 0$.

Then it can be seen that the value of $6\eta\eta^*$ converges faster towards zero than the real part $\Re(\mathcal{E})$ and becomes smaller than $\Re(\mathcal{E})$ when the mesh is sufficiently refined. Again from the regularity of $\mathbf{A}$ and $\mathbf{A}^*$, the error $\mathbf{A} - \mathbf{A}_h$ (resp. $\mathbf{A}^* - \mathbf{A}_h^*$) will be of order h (resp. h^2), and therefore we may expect a convergence in h for η and in h^2 for η^*. From Theorem 5.2 and estimate (60), it follows that the remainder $\mathcal{R}$ is a superconvergent term. It is illustrated in Figure 4 where the values of $\mathcal{R}$ (computed from (56) with the knowledge of $\mathcal{E}$ and η_{QOI}) and the ones of $6\eta\eta^*$ have been plotted. Moreover, coming back to Figure 3(a), we observe that $\Re(\eta_{\text{QOI}})$ has exactly the same behaviour as $\Re(\mathcal{E})$, that once again from Theorem 5.1 and estimate (56) shows that the remainder $\mathcal{R}$ can be neglected. Concerning the imaginary parts, it can be seen in Figure 3(b) that the values are very small, these values and the bumps are therefore not significant, even if the behaviour of the imaginary part $\Im(\mathcal{E})$ and $\Im(\eta_{\text{QOI}})$ are once again exactly the same. Then we plot in Figure 5 the values of $\Re(\mathcal{E})/\Re(\eta_{\text{QOI}})$ (Figure 5(a)) and $\Im(\mathcal{E})/\Im(\eta_{\text{QOI}})$ (Figure 5(b)), defined as the real effectivity index and the imaginary effectivity index. We can see that they are both exactly equal to one whatever the mesh in consideration. It illustrates the fact that the estimator η_{QOI} gives a very accurate evaluation of the error $\mathcal{E}$.



(a) $\Re(\mathcal{E})$, $\Re(\eta_{\text{QOI}})$ and $6\eta\eta^*$



(b) $\Im(\mathcal{E})$ and $\Im(\eta_{\text{QOI}})$

Figure 3: Error and Estimator, global refinement, $\mathbf{q} = \mathbf{H}_s$.

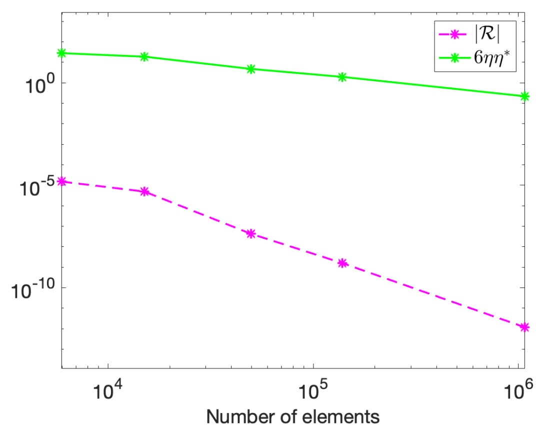
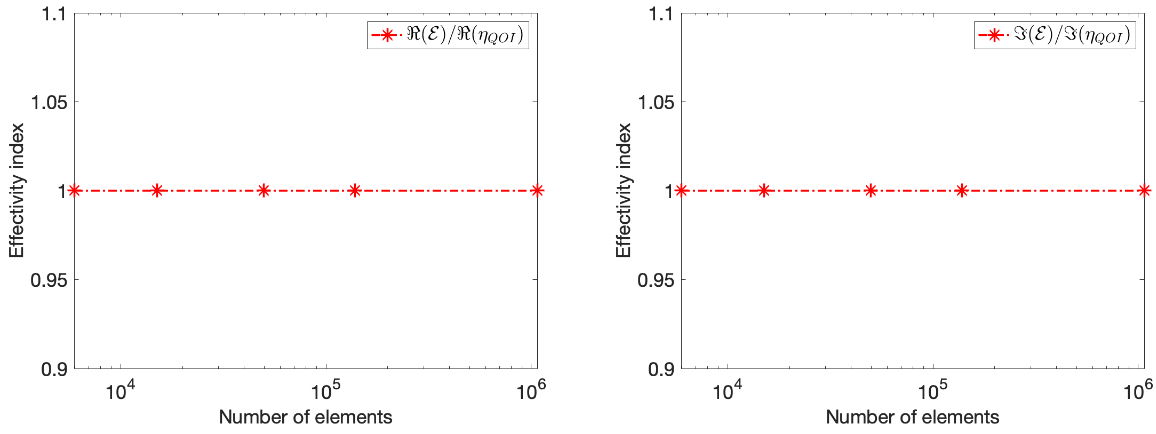


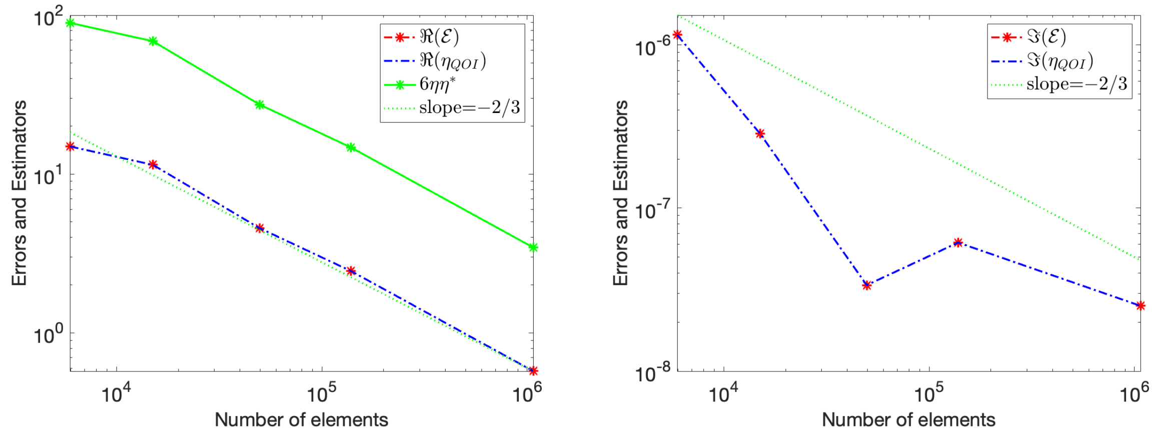
Figure 4: $|\mathcal{R}|$ and $6\eta\eta^*$, global refinement, $\mathbf{q} = \mathbf{H}_s$.



(a) $\Re(\mathcal{E})/\Re(\eta_{\mathrm{QOI}})$ (b) $\Im(\mathcal{E})/\Im(\eta_{\mathrm{QOI}})$

Figure 5: Ratios of error over estimator, real and imaginary parts, global refinement, $\mathbf{q} = \mathbf{H}_s$.

Remark 6.1 *In order to illustrate the consequence to use a higher order approximation of the adjoint solution, we respectively display in Figures 6, 7 and 8 the same results as in Figures 3, 4 and 5, but computing this time* $(\mathbf{A}_h^*, \varphi_h^*) \in \mathbf{V}_h$ *instead of* $(\mathbf{A}_h^*, \varphi_h^*) \in \mathbf{V}_h^*$. *Similarly, we compute* $(\mathbf{T}_h^*, \Omega_h^*) \in \tilde{Y}_h \times \tilde{Z}_h$ *(necessary for the computation of* $\mathbf{H}_h^*$ *and* $\mathbf{J}_{e,h}^*$*, see Appendix A) instead of* $(\mathbf{T}_h^*, \Omega_h^*) \in \tilde{Y}_h^* \times \tilde{Z}_h^*$. *It can be seen that in Figure 6(a) that this time, the value of* $6\eta\eta^*$ *converges in* h^2 *since the convergence remains in* h *for* η*, but becomes also in* h *for* η^*. *Consequently, Theorem 5.2 and estimate* (60) *are not sufficient to prove the superconvergence of the remainder* $\mathcal{R}$. *Nevertheless, we see in Figure 7 that* $\mathcal{R}$ *remains much smaller than* $6\eta\eta^*$*, so that the effectivity index remains equal to one as displayed in Figure 8.*



(a) $\Re(\mathcal{E})$, $\Re(\eta_{\mathrm{QOI}})$ and $6\eta\eta^*$ (b) $\Im(\mathcal{E})$ and $\Im(\eta_{\mathrm{QOI}})$

Figure 6: Error and Estimator, global refinement, $\mathbf{q} = \mathbf{H}_s$, with same finite element spaces for direct and adjoint problems.

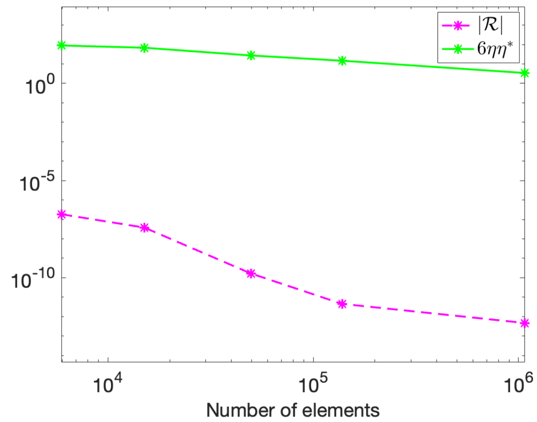
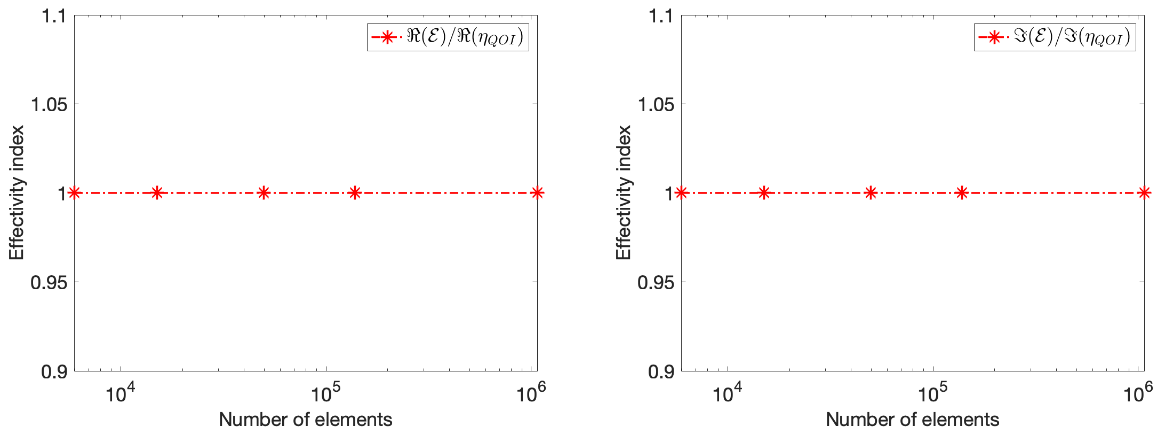


Figure 7: $|\mathcal{R}|$ and $6\eta\eta^*$, global refinement, $\mathbf{q} = \mathbf{H}_s$, with same finite element spaces for direct and adjoint problems.



(a) $\Re(\mathcal{E})/\Re(\eta_{\mathrm{QOI}})$ (b) $\Im(\mathcal{E})/\Im(\eta_{\mathrm{QOI}})$

Figure 8: Ratios of error over estimator, real and imaginary parts, global refinement, $\mathbf{q} = \mathbf{H}_s$, with same finite element spaces for direct and adjoint problems.

6.2 Results with locally refined meshes

The second set of meshes consists of some locally refined meshes. Here, the meshes have been further refined in D_s, which corresponds to the support of $\mathbf{J}_s$. Figure 9 displays the third mesh of the series (Mesh 3), and Table 2 indicates for each mesh the number of tetrahedra and the number of degrees of freedom associated to each formulation (direct or adjoint one). The same tests are performed, and Figures 10, 11 and 12 display the same results as Figures 3, 4 and 5 on this second set of meshes. The conclusions are very similar, and we can see moreover that the value of $6\eta\eta^*$ goes towards zero faster than in the globally refined case, what can be explained by the fact that the support of $\mathbf{q} = \mathbf{H}_s$, namely D_s, is here better refined.

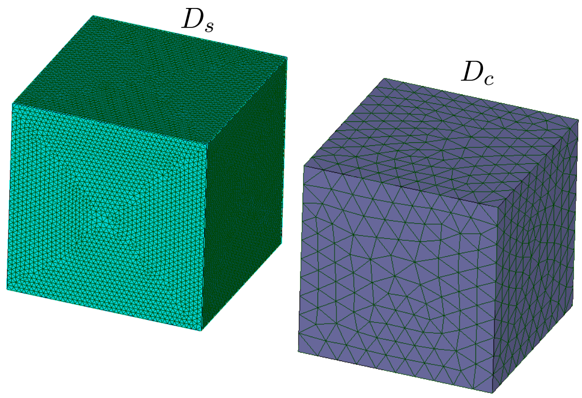
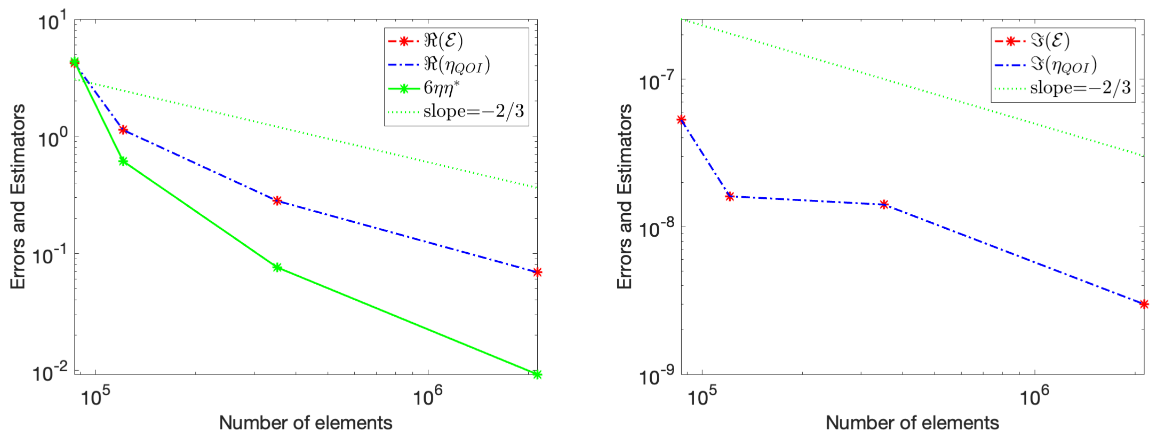


Figure 9: Mesh 3, local refinement, 352 162 elements : 243 232 elements in D_s, 5 794 elements in D_c and 103 136 elements in $D \backslash (D_s \cup D_c)$.

Mesh	Nb of elements	Nb dof for $(\mathbf{A}_h, \varphi_h)$	Nb dof for $(\mathbf{A}_h^*, \varphi_h^*)$
1	86 335	108 740	577 696
2	121 005	149 374	797 644
3	352 162	420 454	2 264 432
4	2 128 618	2 500 710	13 530 768

Table 2: The meshes used for the simulations, local refinement.



(a) $\Re(\mathcal{E})$, $\Re(\eta_{\mathrm{QOI}})$ and $6\eta\eta^*$

(b) $\Im(\mathcal{E})$ and $\Im(\eta_{\mathrm{QOI}})$

Figure 10: Error and Estimator, local refinement, $\mathbf{q} = \mathbf{H}_s$.

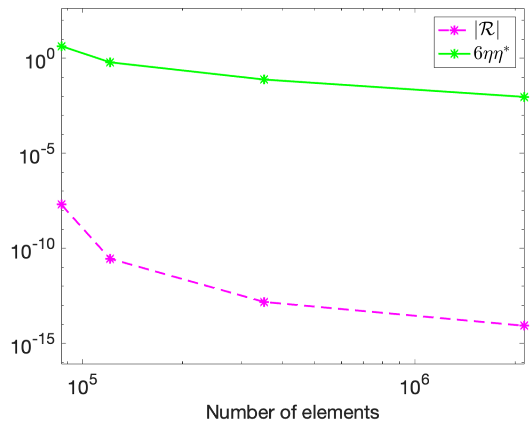
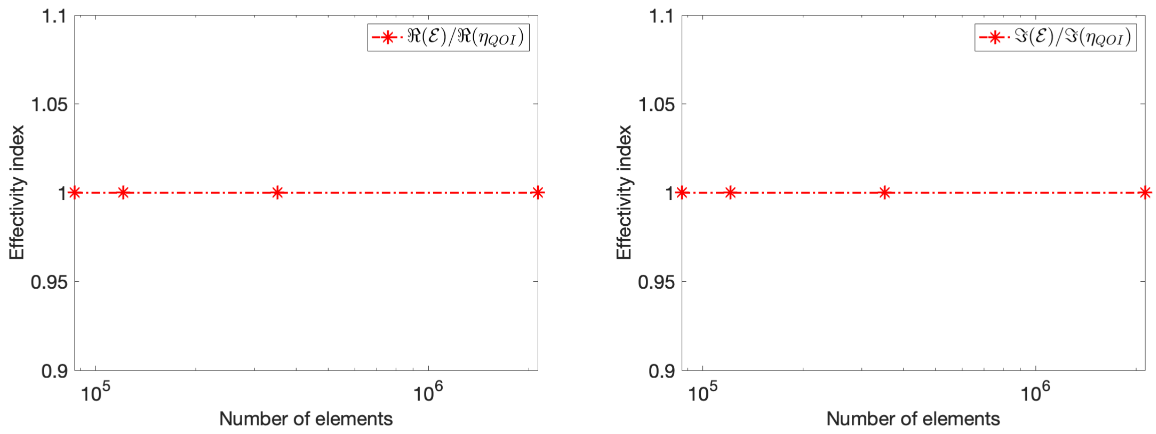


Figure 11: $|\mathcal{R}|$ and $6\eta\eta^*$, local refinement, $\mathbf{q} = \mathbf{H}_s$.



(a) $\Re(\mathcal{E})/\Re(\eta_{\mathrm{QOI}})$ (b) $\Im(\mathcal{E})/\Im(\eta_{\mathrm{QOI}})$

Figure 12: Ratios of error over estimator, real and imaginary parts, local refinement, $\mathbf{q} = \mathbf{H}_s$.

6.3 Results with a singular solution of the adjoint problem

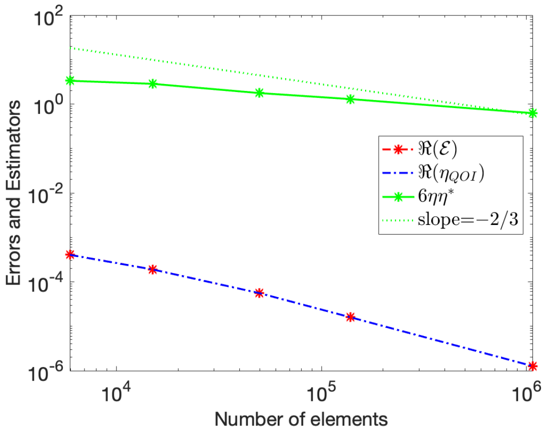
A last test is now proposed. It consists of the same setting as in Subsection 6.1, but instead of defining $\mathbf{q}$ using (74), we choose:

$$\mathbf{q} = \begin{pmatrix} \rho_s \\ 0 \\ 0 \end{pmatrix}, \tag{76}$$

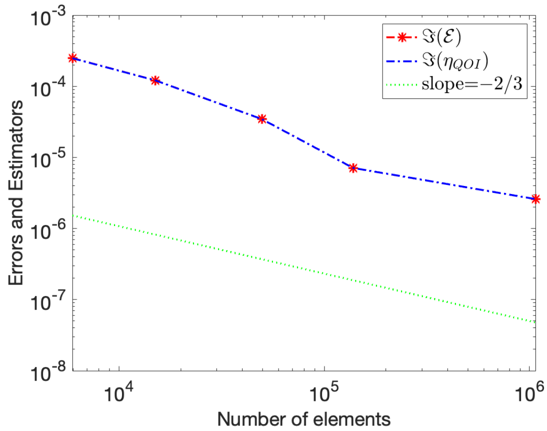
with

$$\rho_s(x, y, z) = e^{-\frac{(x-3)^2+y^2+z^2}{log(10)/4}}, \forall (x, y, z) \in D.$$

The error we are interested in is given by (42). Figures 13 to 15 display the same results as Figures 3 to 5 for this new error definition.



(a) $\Re(\mathcal{E})$, $\Re(\eta_{\mathrm{QOI}})$ and $6\eta\eta^*$



(b) $\Im(\mathcal{E})$ and $\Im(\eta_{\mathrm{QOI}})$

Figure 13: Error and Estimator, global refinement, $\mathbf{q}$ given by (76).

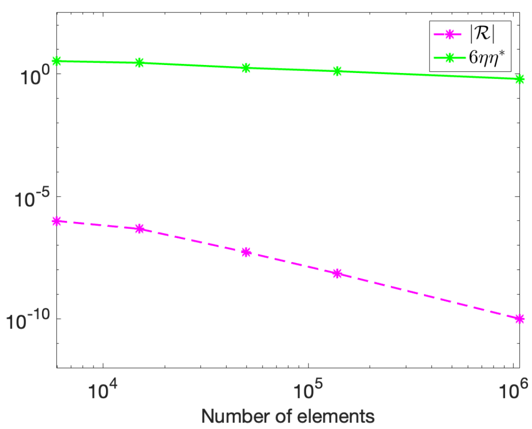
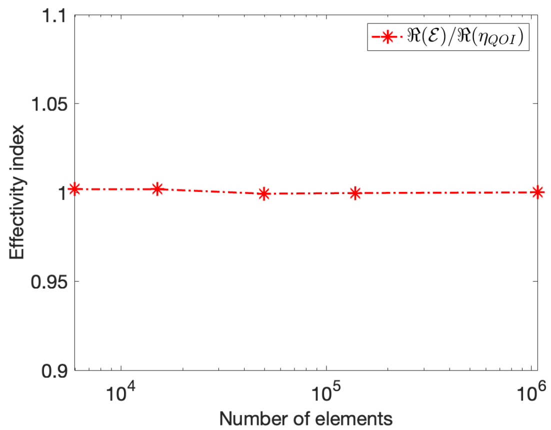
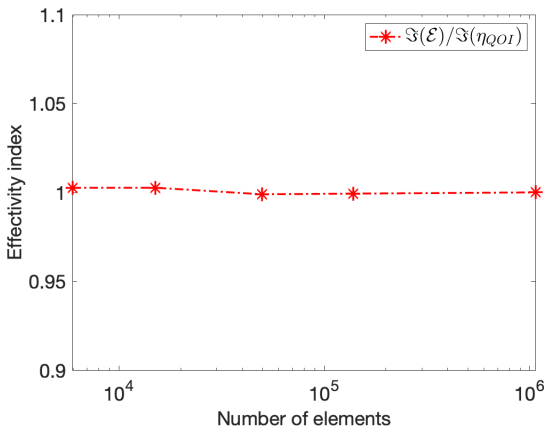


Figure 14: $|\mathcal{R}|$ and $6\eta\eta^*$, global refinement, $\mathbf{q}$ given by (76).



(a) $\Re(\mathcal{E})/\Re(\eta_{\mathrm{QOI}})$



(b) $\Im(\mathcal{E})/\Im(\eta_{\mathrm{QOI}})$

Figure 15: Ratios of error over estimator, real and imaginary parts, global refinement, $\mathbf{q}$ given by (76).

The difference with the previous example relies on the limited regularity of φ^*. Indeed Lemma 2.2 (point 3) guarantees $\mathbf{A}^* \in H^2(D)^3$ and Lemma 2.3 (point 3) guarantees $\varphi^* \in H^{1+s}(D_c)$, for all $s \in (0,1)$. This means that η^* could converge to 0 only in h^s, for all $s \in (0,1)$. This is indeed the case since it can be seen in Figure 13(a) that the quantity $6\eta\eta^*$ does no more converge faster towards zero than the error $\mathcal{E}$, and remains significantly higher. Nevertheless, the error $\mathcal{E}$ and the estimator η_{QOI} remain nearly the same (see the real parts in Figure 13(a) and the imaginary ones in Figure 13(b)). $\Re(\mathcal{E})$ and $\Re(\eta_{\mathrm{QOI}})$ converge towards zero in h^2 (consequence of Corollary 5.4, with s as close as we want to 1). In this case as in previous ones, the estimation (60) given by Theorem 5.2 strongly overestimates the value of $\mathcal{R}$ (see Figure 14). Nevertheless and despite this phenomenon, the estimator η_{QOI} provides once again a very accurate evaluation of the error $\mathcal{E}$, as indicated by the ratios of the error over the estimator displayed in Figure 15, and once again nearly equal to one.

A Practical computation of the estimators

As mentioned in Remark 3.1, the potential reconstruction $(\mathbf{S}_h, \psi_h)$ of $(\mathbf{A}_h, \varphi_h)$ as well as the flux reconstructions $\mathbf{H}_h$ and $\mathbf{J}_{e,h}$ fulfilling (17)-(18)-(19) and corresponding respectively to approximations of $\mu^{-1}\mathrm{curl}\mathbf{A}$ and $\sigma\mathbf{E}$ are supposed to be available. We explain here how these reconstructions have been performed in Section 6 devoted to the numerical validation.

First of all, since the chosen finite element spaces imply that $(\mathbf{A}_h, \varphi_h)$ belongs to $H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c)$ (see (71) for the finite element spaces definition), we then simply take $(\mathbf{S}_h, \psi_h) = (\mathbf{A}_h, \varphi_h)$. Now, let us explain how we have computed $\mathbf{H}_h$ and $\mathbf{J}_{e,h}$.

Like specified in the introduction, we first recall that the $\mathbf{A}$-φ formulation (8) with *ad hoc* boundary conditions is derived from the eddy current system (1). More precisely, we introduce the vector and scalar potentials $\mathbf{A}$ and φ given by (7). Then, using the constitutive laws (2), we only have to reformulate equations (1b) and (3) in terms of the $\mathbf{A}$ and φ variables to obtain the $\mathbf{A}$-φ formulation (8).

Another choice can also be done. Indeed, from (3) we can introduce a vector potential $\mathbf{T}$ such that $\mathbf{J}_e = \mathrm{curl}\,\mathbf{T}$ in D_c. Moreover, introducing $\mathbf{H}_s$ as an arbitrary source magnetic field satisfying $\mathrm{curl}\,\mathbf{H}_s = \mathbf{J}_s$ in D, from (1b) a scalar potential Ω can be introduced such that $\mathbf{H} = \mathbf{H}_s + \mathbf{T} - \nabla\Omega$. Consequently, the $\mathbf{T}$-Ω formulation of the eddy current problem (cf. [28] for more details) reads

$$\mathrm{curl}(\sigma^{-1}\,\mathrm{curl}\mathbf{T}) + j\omega\mu(\mathbf{H}_s + \mathbf{T} - \nabla\Omega) \;=\; \mathbf{0} \text{ in } D_c, \tag{77a}$$
$$\mathrm{div}(\mu(\mathbf{H}_s + \mathbf{T} - \nabla\Omega)) \;=\; 0 \text{ in } D_c, \tag{77b}$$
$$\mathrm{div}(\mu(\mathbf{H}_s - \nabla\Omega)) \;=\; 0 \;\text{ in } D. \tag{77c}$$

The numerical approximation $(\mathbf{T}_h, \Omega_h)$ of $(\mathbf{T}, \Omega)$ is computed with similar finite element spaces than the ones used for the previous $\mathbf{A}_h$-φ_h formulation. More precisely, we look this time for $(\mathbf{T}_h, \Omega_h) \in \tilde{Y}_h \times \tilde{Z}_h$, with

$$\begin{aligned}
\tilde{Y}_h &= \{\mathbf{T}'_h \in Y_h \;:\; \int_{D_c} \mathbf{T}'_h \cdot \nabla\psi_h = 0, \forall \psi_h \in Z_h^0\},\\
Y_h &= \{\mathbf{T}'_h \in H_0(\mathrm{curl}, D_c) \;:\; \mathbf{T}'_{h|T} \in \mathcal{ND}_1(T), \forall T \in \mathcal{T} \cap \bar{D}_c\},\\
\tilde{Z}_h &= \{\Omega'_h \in \widetilde{H^1}(D) \;:\; \Omega'_{h|T} \in \mathbb{P}_1(T), \forall T \in \mathcal{T}\},\\
Z_h^0 &= \{\psi_h \in H_0^1(D_c) \;:\; \psi_{h|T} \in \mathbb{P}_1(T), \forall T \in \mathcal{T} \cap \bar{D}_c\},
\end{aligned}$$

where $\mathcal{ND}_1(T)$ is defined by (72). Now, it remains to respectively define $\mathbf{H}_h$ and $\mathbf{J}_{e,h}$ by :

$$\mathbf{H}_h = \mathbf{H}_s + \mathbf{T}_h - \nabla\Omega_h \text{ and } \mathbf{J}_{e,h} = \mathrm{curl}\mathbf{T}_h,$$

which by construction are good approximations of $\mu^{-1}\text{curl}\mathbf{A}$ and $\sigma\mathbf{E}$ and fulfill (17)-(18)-(19).

Similarly, we evaluate the values of $(\mathbf{A}_h^*, \varphi_h^*) \in \mathbf{V}_h^*$, approximation of the solution $(\mathbf{A}^*, \varphi^*)$ of (43), where $\mathbf{V}_h^*$ is defined by (75). Finally, the flux reconstructions $\mathbf{H}_h^*$ (approximation of $\mu^{-1}\text{curl}\mathbf{A}^*$) and $\mathbf{J}_{e,h}^*$ (approximation of $\sigma\mathbf{E}^* = -\sigma(j\omega\mathbf{A}^* + \nabla\varphi^*)$), that belong respectively to $H(\text{curl}, D)$ and $H(\text{div}, D_c)$ and satisfy the conservation properties (46) are evaluated using the corresponding $\mathbf{T}^*$-Ω^* formulation of the adjoint problem, namely

$$\begin{aligned}
\text{curl}(\sigma^{-1}\,\text{curl}\mathbf{T}^*) - j\omega\mu(\mathbf{q} + \mathbf{T}^* - \nabla\Omega^*) &= \mathbf{0} \text{ in } D_c, && \text{(78a)}\\
\text{div}(\mu(\mathbf{q} + \mathbf{T}^* - \nabla\Omega^*)) &= 0 \text{ in } D_c, && \text{(78b)}\\
\text{div}(\mu(\mathbf{q} - \nabla\Omega^*)) &= 0 \text{ in } D. && \text{(78c)}
\end{aligned}$$

The numerical approximation $(\mathbf{T}_h^*, \Omega_h^*)$ of $(\mathbf{T}^*, \Omega^*)$ is computed with similar finite element spaces than the ones used for the previous $\mathbf{A}_h^*$-φ_h^* formulation. More precisely, we look this time for $(\mathbf{T}_h^*, \Omega_h^*) \in \tilde{Y}_h^* \times \tilde{Z}_h^*$, with

$$\begin{aligned}
\tilde{Y}_h^* &= \{\mathbf{T}_h' \in Y_h^* \ : \int_{D_c} \mathbf{T}_h' \cdot \nabla\psi_h = 0, \forall\psi_h \in Z_h^{*,0}\},\\
Y_h^* &= \{\mathbf{T}_h' \in H_0(\text{curl}, D_c) \ : \ \mathbf{T}_{h|T}' \in \mathcal{ND}_2(T), \forall T \in \mathcal{T} \cap \bar{D}_c\},\\
\tilde{Z}_h^* &= \{\Omega_h' \in \widetilde{H^1}(D) \ : \Omega_{h|T}' \in \mathbb{P}_2(T), \forall T \in \mathcal{T}\},\\
Z_h^{*,0} &= \{\psi_h \in H_0^1(D_c) \ : \psi_{h|T} \in \mathbb{P}_2(T), \forall T \in \mathcal{T} \cap \bar{D}_c\},
\end{aligned}$$

where $\mathcal{ND}_2(T)$ is defined by (72). Finally, it remains to respectively define $\mathbf{H}_h^*$ and $\mathbf{J}_{e,h}^*$ by :

$$\mathbf{H}_h^* = \mathbf{q} + \mathbf{T}_h^* - \nabla\Omega_h^* \text{ and } \mathbf{J}_{e,h}^* = \text{curl}\mathbf{T}_h^*,$$

which by construction are good approximations of $\mu^{-1}\text{curl}\mathbf{A}^*$ and $\sigma\mathbf{E}^*$ and satisfy the conservation properties (46).

Let us note that in order to compute $\mathbf{H}_h$, $\mathbf{J}_{e,h}$, $\mathbf{H}_h^*$ and $\mathbf{J}_{e,h}^*$, we have to solve some global problems, namely the $\mathbf{T}_h$-Ω_h one given by (77) and the $\mathbf{T}_h^*$-Ω_h^* one given by (78). In that case, these reconstructions are not local. Nevertheless, they depend neither on $(\mathbf{A}_h, \varphi_h)$, nor on $(\mathbf{A}_h^*, \varphi_h^*)$, so that the computation of $(\mathbf{A}_h, \varphi_h)$, $(\mathbf{A}_h^*, \varphi_h^*)$, $(\mathbf{T}_h, \Omega_h)$ and $(\mathbf{T}_h^*, \Omega_h^*)$ can be done simultaneously. An alternative should be to compute $\mathbf{H}_h$, $\mathbf{J}_{e,h}$, $\mathbf{H}_h^*$ and $\mathbf{J}_{e,h}^*$ locally by a post-processing of $(\mathbf{A}_h, \varphi_h)$ and $(\mathbf{A}_h^*, \varphi_h^*)$, we refer to [9] section 3.2 for some explicit examples.

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