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► **To cite this version:**

Carsten Chong, Marc Hoffmann, Yanghui Liu, Grégoire Szymanski, Mathieu Rosenbaum. STATISTICAL INFERENCE FOR ROUGH VOLATILITY: MINIMAX THEORY. 2023. hal-03949577

HAL Id: hal-03949577

<https://hal.science/hal-03949577>

Preprint submitted on 20 Jan 2023

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STATISTICAL INFERENCE FOR ROUGH VOLATILITY: MINIMAX THEORY

CARSTEN CHONG, MARC HOFFMANN, YANGHUI LIU, MATHIEU ROSENBAUM AND GRÉGOIRE SZYMANSKI

ABSTRACT. Rough volatility models have gained considerable interest in the quantitative finance community in recent years. In this paradigm, the volatility of the asset price is driven by a fractional Brownian motion with a small value for the Hurst parameter H . In this work, we provide a rigorous statistical analysis of these models. To do so, we establish minimax lower bounds for parameter estimation and design procedures based on wavelets attaining them. We notably obtain an optimal speed of convergence of $n^{-1/(4H+2)}$ for estimating H based on n sampled data, extending results known only for the easier case $H > 1/2$ so far. We therefore establish that the parameters of rough volatility models can be inferred with optimal accuracy in all regimes.

Mathematics Subject Classification (2020) : 60G22, 62C20, 62F12, 62M09, 62P20.

Keywords: Rough volatility, fractional Brownian motion, wavelets, scaling, minimax optimality, pre-averaging, iterated estimation procedure

1. INTRODUCTION

Rough volatility models have been introduced in quantitative finance in 2014 in [GJR18]. The core idea of this new paradigm in financial engineering is to consider that the volatility process of financial assets exhibits very irregular sample paths. More precisely the prototypical rough volatility model postulates that the price of a one-dimensional asset S satisfies

$$(1) \quad dS_t = \sigma_t dB_t, \quad \sigma_t = a \exp(\eta W_t^H)$$

with B a Brownian motion, W^H a fractional motion with Hurst parameter $H \in (0, 1)$ and positive constants a and η . The key point of rough volatility modelling is to take H very small, of order 10^{-1} . Other rough volatility models involve more complex functionals of the fractional Brownian motion adapted to various applications, but with the same order of magnitude for H , see for example [EER19, GJR20].

Rough volatility has been uncovered following a data-driven approach. The data sets of interest were time series of daily measurement of historical volatility over many years and thousands of assets, see [BLP22, GJR18]. More precisely, volatility was essentially considered constant during each day¹ and its daily value was estimated from high-frequency price observations. This was done using various up-to-date inference methods for high-frequency data, all of them leading to analogous results. Many natural empirical statistics have been computed on these volatility time series, in a model-agnostic way. Then it was shown in [GJR18] that strikingly similar patterns were obtained by computing the same statistics in the simple Model (1) (actually a version of (1) where one considers a piecewise constant approximation of the volatility).

Date: October 3, 2022.

¹A varying volatility does not make much sense in finance over short time intervals where the diffusive nature of the price is lost due to microstructural effects. Intraday seasonalities of the volatility could be considered though.

For example, among the statistics advocating for rough volatility, empirical means of log-volatility increments play an important role. They consist in measuring on volatility time series $(\sigma_t)_{t=1,2,\dots}$, for given $q > 0$, the empirical means of the $|\log(\sigma_{t+\Delta}) - \log(\sigma_t)|^q$, for Δ going from one day to several hundreds of days. On data, as a function of Δ , this quantity behaves systematically as Δ^A with A of order 10^{-1} , for the whole range of Δ . This is obviously reproduced if the volatility follows Model (1) with H of order 10^{-1} , thanks to the scaling of fractional Brownian motion. In addition, the fact that this property also holds for large Δ somehow discards stationary models where the moments of the increments no longer depend on Δ for large Δ . It also immediately rules out the idea that the scaling of log-volatility increments in practice could be due to measurement error in the volatility process.

At first glance, the relevance of the parameters value in Model (1) may be surprising. It is in strong contrast with the first generation of fractional volatility models where $H > 1/2$ in a stationary environment, see [CR98]. The goal of these models was notably to reproduce long memory properties of the volatility process and we know that fractional Brownian motion increments exhibit long memory when $H > 1/2$. However, it turns out that when H is very small, apparent long memory is also generated in a model such as (1) and that despite its non-stationarity, it remains consistent with the behaviour of financial data even on very long time scales, see [GJR18].

In addition to the very convincing stylized facts obtained from historical volatility, the data analysis obtained from implied volatility surfaces also support the rough volatility paradigm, see [ALV07, BFG16, Fuk21, LMPR18]. In other words, rough volatility models are, in financial terms, compatible with both historical and risk-neutral measures, which paved the way to many new directions in quantitative finance. Furthermore, it turns out that rough volatility models can be micro-founded and that in fact, only a rough nature for the volatility can allow financial markets to operate under no-statistical arbitrage conditions, see [EEFR18, JR20].

Given the multiple empirical evidence from market data towards their direction, rough volatility models have gained great interest in the quantitative finance community and a vast literature has been devoted to this new paradigm, both from a theoretical and a practical perspective. For example, among other contributions risk management of (complex) derivatives is considered in [AJ22, ARS22, EER19, ER18, FHT21, GJR20, HTŽ21, JMP21], numerical issues are addressed in [AJEE19, BLP17, CGP21, Gas22, MP18, RZ21], asymptotic expansions are provided in [CT22, EFGR19, FFGS20, FGP22, FZ17, JP22] and theoretical considerations about the probabilistic structure of these models are studied in [AJLP19, BFG⁺20, CT19, FGR22, Gas19, GGP19, GKR19].

As explained, the popularity of rough volatility models comes from their ability to remarkably mimic the behaviour of data. Now that these models are well admitted, we can take a step back and consider a rigorous statistical analysis taking Model (1) as a postulate. From the point of view of a statistician, the intriguing question is then obviously that of the inference of the parameter H . Can we estimate it and with which accuracy? Said differently, we want to know how well we can distinguish between two values of H and therefore overcome the latent nature of the volatility and the noise in its estimation.

Thorough investigations related to these questions can notably be found in [BCPV22, FT19]. In [FT19], the following approximation is considered

$$\log(\widehat{\sigma}_t^2) \sim \log\left(\int_{(t-1)\delta}^{t\delta} \sigma_s^2 ds\right) + \varepsilon_t,$$

where ε_t is an iid Gaussian noise and $\widehat{\sigma}_t^2$ is the quadratic variation computed from high-frequency observations of the log price over the interval $[(t-1)\delta, t\delta)$. Taking this approximation for granted, the authors obtain a Whittle estimator of the parameter H and provide its asymptotic theory in a high-frequency asymptotic framework (fixed time interval of observation). However, the methodology is tightly related to this approximation, which is unfortunately not accurate enough in our Model (1). Another very interesting study is that of [BCPV22]. Here the authors place themselves in an ergodic framework and make stationary assumptions to build estimating procedures. In our work, we rather consider a high-frequency setting, which makes the nature of the statistical problem quite different. Finally, the complementary paper [CHL⁺22] uses a very similar setting as ours but over a slightly different class of models. With a practitioner perspective in mind, this paper focuses on the most useful rough volatility models in practice and associated central limit theorems for estimating H . We rather stay in the prototypical Model (1) to address our very natural mathematical statistics question: the optimal rate of convergence for an estimator of the Hurst parameter in the rough volatility paradigm.

To work first in the spirit of rough volatility models as introduced for financial engineering, we start our study with a version of Model (1) where the volatility is piecewise constant. We consider n regularly sampled observations of the price S from this modified Model (1) with $0 < H < 1$ over a fixed time interval. In this setting, squared price increments are down to spot variance multiplied by noise. Taking the logarithm reduces our problem to the setup of [GH07] where the authors study the estimation of the Hurst parameter for a fractional Brownian motion observed with additive measurement error for $H > 1/2$. Their approach is based on the scaling properties of wavelet-based energy levels of the fractional Brownian motion Q_j (the sum of the squared wavelet coefficients for a given resolution level j). Indeed, we have that $2^{2jH}Q_j$ converges to some constant. Therefore H can be obtained from the ratio Q_{j+1}/Q_j , provided we have access to a good approximation of it. Furthermore, the multiresolution nature of wavelet coefficients and their adaptivity properties makes this approach quite tractable compared for example to using scaling properties of p -variations (sum of increments to some power p), although both approaches are in essence very similar. To build their estimator, the authors of [GH07] estimate energy levels Q_j from the increments of observations. Unfortunately, their approach cannot be directly applied in our setting. This is due to the roughness of the volatility path that creates a large bias error term in the estimation of the energy levels. We mitigate this phenomenon using a pre-averaging technique similar to that in [Szy22]. More precisely, we show that the energy levels computed from the pre-average spot volatility process still have a nice scaling property and derive an estimator from it. This estimator achieves the rate $n^{-1/(4H+2)}$, as in [GH07] for $H > 1/2$, which we prove to be minimax optimal for any $H \in (0, 1)$. In particular, we can conclude that from a rigorous statistical viewpoint, estimating H in rough volatility models can be done with very satisfactory accuracy.

Then we consider Model (1) without any piecewise constant assumption on the volatility path. In this situation, the law of price increments becomes more intricate. The piecewise constant

volatility is replaced by local average of the spot volatility. Interestingly, the question of the inference of H in this model has already been studied in [Ros08] for $H > 1/2$. In this work, energy levels are computed from price increments and are shown to exhibit a scaling property around a stochastic limit. So one can use again ratios of estimated energy levels to estimate H when $H > 1/2$. However, this approach does not extend to the case $H < 1/2$. The reason is that over a short time interval, given the small value of H , the integrated volatility is very badly approximated by the renormalized spot volatility at the beginning of the interval, which is a crucial element in [Ros08]. We get inspired by the empirical means used in [GJR18] as well as by the piecewise constant case. More precisely, we compute energy levels from the logarithm of the squared price increments and not from price increments. However, as explained above, the logarithm of price increments involves the quantity

$$\log \left(\frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \exp(\eta W_s^H) ds \right)$$

which does not enjoy suitable properties when H is small. This is because the roughness of the trajectories makes this quantity far from $\eta W_{i\delta}^H$. This creates a large bias when computing H from a ratio of energy levels. We analyze the extent of this bias and show that the scaling of order 2^{-2jH} is no longer exact. Additional terms of order 2^{-2ajH} with $a > 1$ appear and must be removed with a suitable bias correction procedure. To do so, we start with a first estimator with no bias correction on the energy levels. Then we re-use this estimator to correct the energy levels and compute a new estimator with a faster rate of convergence. This procedure is iterated many times, of the order of H^{-1} , and reaches the minimax optimal speed $n^{-1/(4H+2)}$.

The paper is organized as follows. The model with piecewise constant volatility is considered in Section 2 while this assumption is removed in Section 3. Discussions are gathered in Section 4 and the next sections are devoted to the proofs of the results.

2. ROUGH PIECEWISE CONSTANT VOLATILITY

2.1. Model and notation. We start with the version of the prototypical rough volatility model where the volatility is piecewise constant. In financial terms, it means the volatility is taken constant over the day, which is quite reasonable (up to intraday seasonalities). To do so, we consider for some $H \in (0, 1)$ and $\eta > 0$ a measurable space $(\Omega, \mathcal{A}, \mathbb{P}_{H,\eta})$ on which is defined a process S such that

$$dS_t = \sigma_t dB_t$$

with B a Brownian motion and the volatility σ given by

$$\sigma_t^2 = \sigma_0^2 \exp(X_t) \text{ with } X_t = \eta W_{\lfloor t\delta^{-1} \rfloor \delta}^H$$

where $\delta = \delta_n > 0$, $\sigma_0 > 0$, and where W^H is a fractional Brownian motion with Hurst index H independent from B . We will write $\mathbb{E}_{H,\eta}$ for the expectation under the probability measure $\mathbb{P}_{H,\eta}$. Without loss of generality, we take $\sigma_0 = 1$. We assume that (H, η) lies in a compact subset \mathcal{D} of $(0, 1) \times (0, \infty)$ and we define

$$\begin{aligned} H_- &= \min_{(H,\eta) \in \mathcal{D}} H, & \eta_- &= \min_{(H,\eta) \in \mathcal{D}} \eta, \\ H_+ &= \max_{(H,\eta) \in \mathcal{D}} H, & \eta_+ &= \min_{(H,\eta) \in \mathcal{D}} \eta. \end{aligned}$$

We suppose that we have access to the trajectory of S via discrete data at times i/n with $i = 0, \dots, n$ and we write \mathcal{A}^n for the σ -algebra generated by observations

$$(S_{i/n})_{0 \leq i \leq n}.$$

For simplicity, we further assume that $n = 2^N$ and that $m := n\delta$ is an integer. We aim at recovering the parameters (H, η) from these observations. We define

$$v_n(H) = n^{-1/(4H+2)} \vee \delta^{1/2}.$$

In Section 2.2, we derive a minimax optimality theory for this model and in Section 2.3 we focus on the construction of an estimator achieving the minimax rate.

2.2. Minimax optimality. The rate v_n is said to be a lower rate of convergence over \mathcal{D} for estimating H if there exists $c > 0$ such that

$$\liminf_{n \rightarrow \infty} \inf_{\hat{H}_n} \sup_{(H, \eta) \in \mathcal{D}} \mathbb{P}_{H, \eta}(v_n^{-1} |\hat{H}_n - H| \geq c) > 0$$

where the infimum is taken over all \mathcal{A}^n -measurable random variables \hat{H}_n (i.e. all estimators).

Theorem 1. *The rate $v_n(H)$ is a lower rate of convergence for estimating H over \mathcal{D} .*

Also, $w_n(H) = n^{-1/(4H+1)} \ln(n) \vee \delta^{1/2}$ is a lower rate of convergence for estimating η over \mathcal{D} (with obvious modification in the definition).

Note that we retrieve here the convergence rate of [GH07] and [Ros08]. Additional comments can be found in Section 4. The proof of Theorem 1 is relegated to Section 5.

2.3. Construction of an estimator. Recall that that $m = n\delta$. Notice first that under $\mathbb{P}_{H, \eta}$, we have for $0 \leq i < 2^N$

$$2^N \sum_{j=1}^m (S_{(im+j)/n} - S_{(im+j-1)/n})^2 = \sigma_{i\delta}^2 \sum_{j=1}^m (B_{(im+j)/n} - B_{(im+j-1)/n})^2$$

so that

$$\hat{X}_{i,n} := \log \left(2^N \sum_{j=1}^m (S_{(im+j)/n} - S_{(im+j-1)/n})^2 \right) = \eta W_{i/n}^H + \varepsilon_{i,n,m}$$

where the $(\varepsilon_{i,n,m})_i$ are i.i.d. random variables with same law as that of $\bar{\varepsilon}_m = \log(m^{-1} \sum_{j=1}^m \bar{\xi}_j^2)$, where the $\bar{\xi}_j$ are i.i.d. standard Gaussian variables. Thus $n\delta_n \exp(\varepsilon_{1,n})$ is a χ^2 random variable with $n\delta_n$ degree of freedom and we get by Lemma 36

$$\text{Var}(\bar{\varepsilon}_m) = \psi^{(1)}\left(\frac{m}{2}\right) \leq Cm^{-1} \quad \text{and} \quad \mathbb{E}[\bar{\varepsilon}_m^4] \leq Cm^{-2}$$

for some constant C and where the function $\psi^{(1)}$ is explicit and defined in Section C.

We now introduce the energy levels of the process X . Let $j \geq 2$, $k < 2^{j-1}$ and $p \geq 0$. We write:

$$d_{j,k,p} = \frac{1}{2^{j/2+p}} \sum_{l=0}^{2^p-1} X_{(k+l2^{-p})2^{-j}} - 2X_{(k+1+l2^{-p})2^{-j}} + X_{(k+2+l2^{-p})2^{-j}} \quad \text{and} \quad Q_{j,p} = \sum_{k=0}^{2^{j-1}-1} d_{j,k,p}^2.$$

Note that this sum stops at index $k = 2^{j-1} - 1$ and is defined only for $j \geq 2$ because our observations stop at time 1. In view of these definitions, we introduce their empirical counterparts. We start with d . We define

$$\tilde{d}_{j,k,p,n} = \frac{1}{2^{j/2+p}} \sum_{l=0}^{2^p-1} \widehat{X}_{n2^{-j}(k+l2^{-p}),n} - 2\widehat{X}_{n2^{-j}(k+1+l2^{-p}),n} + \widehat{X}_{n2^{-j}(k+2+l2^{-p}),n}$$

which is \mathcal{A}^n -measurable provided $p + j \leq N$. Note that $\tilde{d}_{j,k,p,n} = d_{j,k,p} + e_{j,k,p,n}$ where

$$(2) \quad e_{j,k,p,n} = \frac{1}{2^{j/2+p}} \sum_{l=0}^{2^p-1} \varepsilon_{n((k+l2^{-p})2^{-j}),n,m} - 2\varepsilon_{n((k+1+l2^{-p})2^{-j}),n,m} + \varepsilon_{n((k+2+l2^{-p})2^{-j}),n,m}.$$

However we cannot estimate d^2 by \tilde{d}^2 because of the noise term e . Following [GH07], we offset these effects by removing $\mathbb{E}[e_{j,k,p,n}^2] = 6 \text{Var}(\bar{\varepsilon}_m)2^{-j-p}$. Thus, we estimate the energy levels by

$$(3) \quad \widehat{Q}_{j,p,n} = \sum_{k=0}^{2^{j-1}-1} \widehat{d}_{j,k,p,n}^2 \quad \text{where} \quad \widehat{d}_{j,k,p,n}^2 = \tilde{d}_{j,k,p,n}^2 - 6 \text{Var}(\bar{\varepsilon}_m)2^{-j-p}.$$

We fix $\nu_0 > 0$. The estimator of H is given by

$$\widehat{H}_n = -\frac{1}{2} \log \left[\frac{\widehat{Q}_{J_n^*+1, N-J_n^*-1, n}}{\widehat{Q}_{J_n^*, N-J_n^*-1, n}} \right] \quad \text{where} \quad J_n^* = \max\{2 \leq j \leq N-1 : \widehat{Q}_{j, N-j-1, n} \geq \nu_0 2^j n^{-1}\}$$

The consistency and convergence rate of this estimator is provided in the following theorem, discussed in Section 4 and proved in Section 6

Theorem 2. *The rate $v_n(H) = n^{-1/(4H+2)} \vee \delta^{1/2}$ is achievable for estimating H over \mathcal{D} .*

More precisely, let $\kappa_0(H) = 4 - 2^{2H}$ and suppose that $\nu_0 < \inf_{(H,\eta) \in \mathcal{D}} \eta^2 \kappa_0(0, H) 2^{2H}$. Then $v_n^{-1}(\widehat{H}_n - H)$ is bounded in $\mathbb{P}_{H,\eta}$ -probability uniformly over \mathcal{D} .

3. GENERAL MODEL

3.1. Model and main results. We now consider the very same model and statistical experiment as in the previous section but with a volatility process σ_t that is no longer piecewise constant:

$$\sigma_t^2 = \sigma_0^2 \exp(\eta W_t^H)$$

where W^H is a fractional Brownian motion with Hurst index H and independent from B . We will write $\mathbb{E}_{H,\eta}$ the expectation under probability $\mathbb{P}_{H,\eta}$. Without loss of generality, we take $\sigma_0 = 1$. We assume that (H, η) lies in a compact subset \mathcal{D} of $(0, 3/4) \times (0, \infty)$ and we define H_-, H_+, η_- and η_+ as before. The upper bound $H_+ < 3/4$ is artificial here and could be improved to $H_+ < 1$ using second-order increments, see Section 4 for more details.

The rest of this section is divided into three parts. First, we present a minimax theory for this model in Section 3.2. Then we gather in Section 3.3 some results used later in the construction of an estimator. These results are separated from the actual proof of the estimator since these results could be of independent interest to the reader. Finally in Section 3.4, we build an estimator achieving the optimal convergence rate derived in Section 3.2.

3.2. Minimax optimality. Recall from Section 2.2 that the rate v_n is said to be a lower rate of convergence over \mathcal{D} for estimating H if there exists $c > 0$ such that

$$\liminf_{n \rightarrow \infty} \inf_{\hat{H}_n} \sup_{(H, \eta) \in \mathcal{D}} \mathbb{P}_{H, \eta}(v_n^{-1} |\hat{H}_n - H| \geq c) > 0,$$

where the infimum is taken over all estimators \hat{H}_n , and similarly for η . We obtain the following result, as an analog to Theorem 1. Its proof can be found in Section 7.

Theorem 3. *The rates $v_n(H) = n^{-1/(4H+2)}$ and $w_n(H) = n^{-1/(4H+2)} \ln(n)$ are lower rate of convergence for estimating H and η respectively over \mathcal{D} .*

3.3. Energy levels of the log integrated volatility. Suppose that $j \geq 0$ and $p \geq 0$ are fixed. For $0 \leq k < 2^p$, we introduce the pre-averaged local energy levels of the log-integrated-volatility:

$$d_{j,k,p} = 2^{-p-j/2} \sum_{l=0}^{2^p-1} \log \left(\int_{(k+1)2^{-j+l}2^{-j-p}}^{(k+1)2^{-j}+(l+1)2^{-j-p}} \sigma_u^2 du \right) - \log \left(\int_{k2^{-j+l}2^{-j-p}}^{k2^{-j}+(l+1)2^{-j-p}} \sigma_u^2 du \right).$$

Then we define the corresponding energy levels by

$$Q_{j,p} = \sum_{k < 2^{j-1}} d_{j,k,p}^2.$$

These energy levels differ from those of [Ros08] since they are not defined on the integrated realise variance of the price but on the logarithm of these quantities. They also scale as 2^{-2jH} as in [Ros08], but the logarithm creates two major differences in this scaling. On the one hand, we get rid of the stochastic limit appearing in [Ros08] but on the other hand, this scaling is not exact and additional terms appear. More precisely, the following concentration property is proved within Section 8.3.

Proposition 4. *There exist explicit functions of H denoted $\kappa_{p,a}$ given in Equation (33) such that if $S \geq 1/(4H_-) + 1/2$ and $S > H_+/(2H_-) - 1/2$, we have*

$$(4) \quad \mathbb{E}_{H, \eta} \left[\left(Q_{j,p} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{p,a}(H) \right)^2 \right] \leq C 2^{-j(1+4H)}$$

for some constant C depending only on S .

Though these functions κ are explicit, their numerical implementation is still to be studied and will require the use of Isserlis' Theorem (see Theorem 34).

The following two lemmas control the functions κ . They will be useful in the construction of the estimator and are proved in Section 8.4.

Lemma 5. *There exist $c_{-,1}$ and $c_{+,1}$ some positive constants such that for any $p \geq 1$ and any $H_- \leq H \leq H_+$*

$$(5) \quad c_{-,1} \leq \kappa_{p,1}(H) \leq c_{+,1}.$$

Moreover, there exists $c_{.,S}$ some positive constant such that for any $p \geq 1$, any $2 \leq a \leq S$ and any $H_- \leq H \leq H_+$,

$$(6) \quad |2^{(2a-1)Hp} \kappa_{p,a}(H)| < c_{.,S}.$$

Lemma 6. *The functions $\kappa_{p,a}$ are differentiable on $[H_-, H_+]$ and for any $a > 0$ there exists c_a such that*

$$\sup_{p \geq 0} |\kappa'_{p,a}(H)| \leq c_a.$$

Next, we prove that the rescaled energy levels $2^{2jH}Q_{j,N-j-1}$ are essentially bounded above and below in probability. This next result is proved in Section 8.5.

Proposition 7. *Let $\varepsilon > 0$. Then there exists $0 < r_-(\varepsilon) < r_+(\varepsilon)$, $J_0(\varepsilon) > 0$ and $N_0(\varepsilon)$ such that for $N \geq N_0(\varepsilon)$, we have*

$$\sup_{H,\eta} \mathbb{P}_{H,\eta} \left(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1} \leq r_-(\varepsilon) \right) \leq \varepsilon$$

and

$$\sup_{H,\eta} \mathbb{P}_{H,\eta} \left(\sup_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1} \geq r_+(\varepsilon) \right) \leq \varepsilon.$$

Because of the additional terms appearing in Equation 4, we want to add a bias correction to the energy levels. Therefore, we define for $S > 0$, $\nu > 0$ and $I > 0$

$$(7) \quad Q_{j,p}^{(S)}(I, \nu) = Q_{j,p} - B_{j,p}^{(S)}(I, \nu) \quad \text{where} \quad B_{j,p}^{(S)}(I, \nu) = \sum_{a=2}^S \nu^{2a} 2^{-2aIj} \kappa_{p,a}(I).$$

Note that unlike in Proposition 4, this sum starts at $a = 2$ so that we have

$$\mathbb{E} \left[(Q_{j,p}^{(S)}(\eta, H) - \eta^2 2^{-2Hj} \kappa_{p,1}(H))^2 \right] \leq C 2^{-j(1+4H)}$$

which now has the same behaviour as the term Q in Proposition 3 of [GH07]. Therefore, we can derive bounds over $2^{2jH}Q_{j,N-j-1}^{(S)}(\eta, H)$ similar to that of adapt Proposition 7. The proof, being identical to that of Proposition 7 in Section 8.5 is skipped.

Proposition 8. *Let $\varepsilon > 0$. Then there exists $0 < r_-^{(S)}(\varepsilon) < r_+^{(S)}(\varepsilon)$, $J_0^c(\varepsilon) > 0$ and $N_0^c(\varepsilon)$ such that for $N \geq N_0^c(\varepsilon)$, we have:*

$$\sup_{H,\eta} \mathbb{P}_{H,\eta} \left(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1}^{(S)}(\eta, H) \leq r_-^{(S)}(\varepsilon) \right) \leq \varepsilon$$

and

$$\sup_{H,\eta} \mathbb{P}_{H,\eta} \left(\sup_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1}^{(S)}(\eta, H) \geq r_+^{(S)}(\varepsilon) \right) \leq \varepsilon.$$

We eventually conclude this section by a short lemma giving explicit Lipschitz bounds for functions $B_{j,p}^{(S)}$.

Lemma 9. *There exists $c_B > 0$ such that for any $\eta_1, \eta_2 \in [\eta_-, \eta_+]$ and $H_1, H_2 \in [H_-, H_+]$, we have*

$$|B_{j,p}^{(S)}(\eta_1, H_1) - B_{j,p}^{(S)}(\eta_2, H_2)| \leq c_B 2^{-4(H_1 \wedge H_2)j} (j|H_1 - H_2| + |\eta_1 - \eta_2|).$$

This lemma is proved in Section 8.6.

3.4. Construction of the estimator. We now aim at building an appropriate estimator for $Q_{j,p}$ based on the price increments. In this section, we fix an integer S such that $S \geq 1/(4H_-) + 1/2$ and $S > H_+/(2H_-) - 1/2$ so that the conclusion of Proposition 4 holds.

First, notice that for a fixed (j, p) such that $j + p \leq N$, the price increment $S_{k2^{-j+(l+1)2^{-j-p}}} - S_{k2^{-j+l}2^{-j-p}}$ is \mathcal{A}^n -measurable and we have

$$(8) \quad \left((S_{(l+1)2^{-j-p}} - S_{l2^{-j-p}})^2 \right)_l = \left(\int_{k2^{-j+l}2^{-j-p}}^{k2^{-j+(l+1)2^{-j-p}}} \sigma_u^2 du \xi_{j,p,l}^2 \right)_l$$

in distribution, where $(\xi_{j,p,l})_l$ are i.i.d. standard Gaussian variables, independent of σ . Therefore, we can estimate $(d_{j,p,k})_k$ by $(\tilde{d}_{j,p,k})_k$ where

$$\begin{aligned} \tilde{d}_{j,p,k} = & 2^{-p-j/2} \sum_{l=0}^{2^p-1} \log \left((S_{(k+1)2^{-j}+(l+1)2^{-j-p}} - S_{(k+1)2^{-j}+l2^{-j-p}})^2 \right) \\ & - 2^{-p-j/2} \sum_{l=0}^{2^p-1} \log \left((S_{k2^{-j}+(l+1)2^{-j-p}} - S_{k2^{-j}+l2^{-j-p}})^2 \right). \end{aligned}$$

Using (8), we can see that $\tilde{d}_{j,p,k} = d_{j,p,k} + e_{j,k,p}$ where

$$e_{j,k,p} = 2^{-p-j/2} \sum_{l=0}^{2^p-1} \log \left(\xi_{j,p,(k+1)2^p+l}^2 \right) - \log \left(\xi_{j,p,k2^p+l}^2 \right).$$

In view of [GH07] and [Ros08], we need to correct the estimation $\hat{d}_{j,p,k}^2$ from the bias introduced by the term $e_{j,k,p}^2$. Therefore we write $\hat{d}_{j,p,k}^2 = \tilde{d}_{j,p,k}^2 - 2^{-j-p+1} \text{Var}(\log \xi^2)$ and we estimate Q by $\hat{Q}_{j,p} = \sum_k \hat{d}_{j,p,k}^2$.

A first estimator of (H, η) is defined by $(\hat{H}_n^{(0)}, \hat{\eta}_n^{(0)})$ with

$$\hat{H}_n^{(0)} = \left(\left(-\frac{1}{2} \log_2 \left[\frac{\hat{Q}_{J_n^*+1, N-J_n^*-1, n}}{\hat{Q}_{J_n^*, N-J_n^*-1, n}} \right] \right) \vee H_- \right) \wedge H_+ \quad \text{with } J_n^* = \max(j : \hat{Q}_{j, N-j-1, n} \geq 2^j n^{-1}).$$

and then

$$\hat{\eta}_n^{(0)} = \left(\left(\frac{\hat{Q}_{\hat{j}_n, N-\hat{j}_n, n} 2^{2\hat{j}_n} \hat{H}_n^{(0)}}{\kappa_{N-\hat{j}_n, 1}(\hat{H}_n^{(0)})} \right)^{1/2} \vee \eta_- \right) \wedge \eta_+ \quad \text{with } \hat{j}_n = \left\lfloor \frac{1}{2\hat{H}_n^{(0)} + 1} \log_2(n) \right\rfloor.$$

Then we define a refinement procedure for this estimator to correct the bias introduced by the functions $\kappa_{p,a}$. Let $\nu > 0$ and $0 < I < 1$. First, let $\hat{Q}_{j,p,n}^{(S)}(I, \nu) = \hat{Q}_{j,p,n} - B_{j,p}^{(S)}(I, \nu)$ where $B_{j,p}^{(S)}$ is defined in Equation (7). Then we define

$$(9) \quad \hat{H}_n^c(I, \nu) = \left(\left(-\frac{1}{2} \log_2 \left[\frac{\hat{Q}_{J_n^{*c}(I, \nu)+1, N-J_n^{*c}(I, \nu)-1, n}(I, \nu)}{\hat{Q}_{J_n^{*c}(I, \nu), N-J_n^{*c}(I, \nu)-1, n}(I, \nu)} \right] \right) \vee H_- \right) \wedge H_+$$

where

$$(10) \quad J_n^{*c}(I, \nu) = \max(j : \hat{Q}_{j, N-j-1, n}^{(S)}(I, \nu) \geq 2^j n^{-1})$$

and then

$$(11) \quad \hat{\eta}_n^c(I, \nu) = \left(\left(\frac{\hat{Q}_{\hat{j}_n, N-\hat{j}_n, n}^{(S)}(I, \nu) 2^{2\hat{j}_n I}}{\kappa_{N-\hat{j}_n, 1}(I)} \right)^{1/2} \vee \eta_- \right) \wedge \eta_+.$$

We use these functions to build iteratively a sequence $(\hat{H}_n^{(m)}, \hat{\eta}_n^{(m)})$ of estimators of (H, η) . $(\hat{H}_n^{(0)}, \hat{\eta}_n^{(0)})$ is already built and for $m > 0$, we write

$$\begin{aligned} \hat{H}_n^{(m)} &= \hat{H}_n^c(\hat{H}_n^{(m-1)}, \hat{\eta}_n^{(m-1)}), \\ \hat{\eta}_n^{(m)} &= \hat{\eta}_n^c(\hat{H}_n^{(m)}, \hat{\eta}_n^{(m-1)}). \end{aligned}$$

Choose now m_{opt} such that $m_{opt} > 1/(4H) - 2H - 1$ for any $H_- < H < H_+$.

Theorem 10. *The rates $v_n(H) = n^{-1/(4H+2)}$ and $w_n(H) = n^{-1/(4H+2)} \ln(n)$ are achievable for estimating respectively H and η over the parameter set \mathcal{D} .*

More precisely, suppose that m_{opt} satisfies $m_{opt} > m > 1/(4H) - 2H - 1$ for any $H_- < H < H_+$. Then $v_n^{-1} |\widehat{H}_n^{(m_{opt})} - H|$ and $w_n^{-1} |\widehat{\eta}_n^{(m_{opt})} - \eta|$ are bounded in probability uniformly over \mathcal{D} .

The choice of m_{opt} and other comments are discussed in Section 4; while the proof is delayed until Section 9.

4. DISCUSSION

We now provide some comments about the results given in Section 2 and Section 3.

- **About the convergence rates.** We obtain in Sections 2 and 3 the same convergence rate $n^{-1/(4H+2)}$ as in [GH07] and [Ros08]. This unusual rate ensures that retrieving the Hurst exponent becomes easier when the trajectory is rougher. Therefore estimates of the roughness of the volatility should be quite accurate in rough volatility models. This can seem counter-intuitive at first glance as one knows that the optimal rate for estimating a β -Hölder continuous function (say for instance in the context of estimating a density from a sample of n independent random variables) is roughly $n^{-\beta/(2\beta+1)}$. Here we can get much better rates because we do not try to reconstruct the signal itself but only its regularity. Though volatility remains hidden behind the multiplicative noise and the realised variance, we can retrieve its roughness with fast convergence rate.
- **About the use of wavelets.** Our estimation strategy relies on wavelets and quadratic functionals of the underlying volatility, as in [GH07] and [Ros08]. The multiresolution nature of wavelets is particularly convenient in our setting from a technical viewpoint, notably when computing the dependence structure of the coefficients. Also, selecting optimal resolution levels can be done in a natural way in this framework. That is why we use this technique instead of p -variations of increments. However, the spirit of both approaches would be very close given the strong links between the two objects *via* Besov spaces, see for instance [Ros09, CKR93].
- **About second-order increments.** The fact that second-order increments may be needed to estimate with accuracy parameters of fractional signals when $H \geq 3/4$ is well known, see for example [IL97, Coe01]. That is why we consider such increments in Section 2. For technical convenience, we restrict ourselves in Section 3 to the case $H > 3/4$. This enables us to avoid additional issues coming from the asymptotic expansion of

$$\log \left(\frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \exp(\eta W_s^H) ds \right) - \log \left(\frac{1}{\delta} \int_{(i-1)\delta}^{i\delta} \exp(\eta W_s^H) ds \right)$$

as developed in Proposition 33.

- **About η .** Although these models have two parameters η and H , the parameter of real interest is obviously H . That is why, in the piecewise constant volatility model of Section 2, we do not provide an estimator for η or any minimax theory. However, in a similar way

as in Section 3, one could show that the estimator

$$\hat{\eta}_n = \left(\frac{\hat{Q}_{\hat{j}_n, N-\hat{j}_n, n} 2^{2\hat{j}_n} \hat{H}_n^{(0)}}{\kappa_{N-\hat{j}_n}(\hat{H}_n^{(0)})} \right)^{1/2} \quad \text{with} \quad \hat{j}_n = \left\lfloor \frac{1}{2\hat{H}_n + 1} \log_2(n) \right\rfloor$$

where κ_p is an explicit function defined within Lemma 17, is consistent and achieves convergence rate $\log(n)n^{-1/(4H+2)}$. Minor modification of the proof of Theorem 1 enables us to also show that this rate is minimax-optimal.

- **Implementation and feasibility.** Optimal estimation rates for estimating H do not depend on the model, however, the construction of the estimators does. In the piecewise constant volatility model, the estimator is easy to implement and fast to compute, the only tuning parameter being ν_0 . From Theorem 2, a suitable choice would be $\nu_0 = \frac{1}{2}\eta_-^2 \min(3, (4 - 2^{2H_+})2^{2H_+})$ where $H_+ = \sup_{(H,\eta) \in \mathcal{D}} H$ and $\eta_- = \inf_{(H,\eta) \in \mathcal{D}} \eta$.

The estimators in the general model are more complex. This is first due to the presence of the function $\kappa_{p,a}$ in the debiasing procedure. A function like for example $\kappa_{p,a}$ (see (33)) involves the computation of (an order of) 2^p expectation of $2a$ correlated Gaussian variables. Explicit formulas for such products are given by Isserlis' theorem (see Theorem 34) but there is a slight computational cost. The second issue is the stopping time of the iterated debiasing procedure. The quantity m_{opt} must satisfy $m_{opt} > m > 1/(4H) - 2H - 1$ for any $H_- < H < H_+$. Since it needs to be an integer, a quick study of the function $x \mapsto m > 1/(4x) - 2x - 1$ ensures that one can always take $m_{opt} = \max(\lfloor 1/(4H_-) - 2H_- \rfloor, 0)$. This choice impacts strongly the computation time when H is small. However, in most cases of interest, this iteration cost remains very reasonable with 5 iterations for $H_- = 0.05$ and 24 iterations for $H_- = 0.01$ for instance.

- **Model choice.** We place ourselves in this paper in the prototypical rough volatility model (1). This is a probably reasonable choice when studying a fundamental inference question such as minimax optimality as it enables us to understand the core statistical structure of rough volatility. However, although rough volatility models were initially presented under such a simple form, see [GJR18], they have since then been extended and many models involving various transforms of the fractional Brownian motion or related rough Gaussian processes have emerged. These extensions were notably driven by practice, where beyond optimal rates having access to confidence bounds is crucial. Taking this into account, we build in the companion paper [CHL⁺22] non-parametric estimators of the roughness of the volatility path for these extensions of Model (1), together with a full central limit theory.

5. PROOF OF THEOREM 1

5.1. Outline of the proof. Suppose first that we directly observe $\sigma_{i\delta}$ for $0 \leq i \leq \delta^{-1}$. This would be equivalent to the observation of $(\eta W_{i\delta}^H)_{i \leq \delta^{-1}}$. Optimal estimation in this model is studied in [Szy22] where the LAN property holds (See Theorems 4 and 5 in [Szy22] with $\tau_n = 0$) and optimal rates for the estimation of H and η are respectively $\delta^{1/2}$ and $\delta^{1/2} \log(n)$. Since the model presented in Section 2.1 carries less statistical information than the direct observation of $(\sigma_{i\delta})_{i \leq \delta^{-1}}$, $\delta^{1/2}$ is a lower rate of convergence for H and $\delta^{1/2} \log(n)$ is a lower rate of convergence for η in the model

of Section 2.1. This proves Theorem 1 whenever $\delta \geq n^{-1/(2H+1)}$.

For the remaining of this Section, we will suppose that $\delta \leq n^{-1/(2H+1)}$ so that $v_n(H) = n^{-1/(4H+2)}$. We aim at proving Theorem 1 by a similar strategy to Theorems 2 and 3 of [GH07]. However, two major differences that require delicate handling here. First, we must include the case $H < 1/2$, which was done in [Szy22]. Secondly, we need to show how our model, somewhat different to the additive noise model of [GH07] and [Szy22], can fit in this proof. For completeness, we will go through the main ideas of the proof and emphasize when delicate changes need to be undertaken.

We need some notation. Write $\|\mu\|_{TV} = \sup_{\|f\|_\infty \leq 1} |\int f d\mu|$ for the the total variation of a signed measure μ . For two probability measures μ and ν , set

$$d_{\text{test}}(\mu, \nu) = \|\mu\|_{TV} = \sup_{0 \leq f \leq 1} \left| \int f d\mu - \int f d\nu \right|$$

so that $d_{\text{test}}(\mu, \nu) = \frac{1}{2} \|\mu - \nu\|_{TV}$. We denote by \mathbb{P}_f^n the law of the observations $(S_{i/n})_i$ given $\eta W_t^H = f(t)$.

The following two results are key to the proof of the lower bounds. They replace respectively Propositions 4 and 5 of [GH07]. First, we show that the law of the observation is somewhat close whenever the underlying volatilities are close.

Proposition 11. *Let f and g be two bounded functions. Then there exists $c_0 > 0$ such that*

$$(12) \quad \|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV} \leq c_0 \sqrt{n} \|f - g\|_\infty.$$

Moreover, there exists $c_1 > 0$ and a universal nonincreasing positive function R such that

$$(13) \quad 1 - \frac{1}{2} \|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV} \geq R(n e^{\|f-g\|_\infty} \|f - g\|_\infty^2)$$

Consider now (H_0, ν_0) in the interior of the domain \mathcal{D} . We pick $I > 0$ large enough and we set

$$H_1 = H_0 + \varepsilon_n \quad \text{and} \quad \sigma_1 = \sigma_0 2^{j_0 \varepsilon_n}$$

where

$$\varepsilon_n = I^{-1} n^{-1/(4H_0+2)} \quad \text{and} \quad j_0 = \lfloor \log_2(n^{1/(2H_0+1)}) \rfloor.$$

The next proposition shows that we can build two processes $\xi^{0,n}$ and $\xi^{1,n}$ that act as approximations of $\eta_0 W^{H_0}$ and $\eta_1 W^{H_1}$.

Proposition 12. *For I large enough, there exists a sequence of probability $(X^n, \mathcal{X}_n, \mathbf{P}^n)$ on which can be defined two sequences of stochastic processes, $(\xi_t^{i,n})_{t \in [0,1]}$ and a measurable transformation $T^n : X^n \rightarrow X^n$ such that the following hold:*

- (i) *If $P^{i,n}(\cdot) = \int_{X^n} \mathbb{P}_{\xi^{i,n}(\omega)}^n(\cdot) \mathbf{P}^n(d\omega)$, then $\|P^{i,n} - \mathbb{P}_{H,\sigma}^n\|_{TV} \rightarrow 0$*
- (ii) *The sequence $n \|\xi^{1,n}(\omega) - \xi^{0,n}(T^n \omega)\|_\infty^2$ is tight under \mathbf{P}^n .*
- (iii) *If n is large enough, the probability measure \mathbf{P}^n and its image $T^n \mathbf{P}^n$ are equivalent on (X^n, \mathcal{X}_n) and there exists $0 < c^* < 2$ such that $\|\mathbf{P}^n - T^n \mathbf{P}^n\|_{TV} \leq 2 - c^* < 2$ for n big enough.*

This proposition replaces Proposition 5 of [GH07]. Part (i) shows that we can asymptotically replace the fractional Brownian motions $\eta_i W^{H_i}$ by the processes $\xi^{i,n}(\omega)$ in the model presented in Section 2.1. However, the processes $\xi^{i,n}(\omega)$ are defined in such a way that we can pathwise transform a process into another in the probability space $(X^n, \mathcal{X}_n, \mathbf{P}^n)$. This property is essential

to the lower bound proof and shows how one statistical experiment can be transformed into the other. This is the main goal of points (ii) and (iii). For sake of completeness, we will cover the main idea of the proof of Proposition 12 in Section 5.3.

We can now complete the proof of Theorem 1. We follow again the proof of [GH07]. The same procedure applies for H and σ so we focus on the efficient rate for H . We start with an arbitrary estimator \widehat{H}_n of H and we choose $I > 0$ large enough so that Proposition 12 holds. Let $M < 1/(2I)$. Then we have, using Proposition 12 and notations therein

$$\begin{aligned} & \sup_{(H,\eta)} \mathbb{P}_{H,\eta}^n(v_n(H)^{-1}|\widehat{H}_n - H| \geq M) \\ & \geq \frac{1}{2}(\mathbb{P}_{H_0,\eta_0}^n(v_n(H_0)^{-1}|\widehat{H}_n - H_0| \geq M) + \mathbb{P}_{H_1,\eta_1}^n(v_n(H_1)^{-1}|\widehat{H}_n - H_1| \geq M)) \\ & \geq \frac{1}{2}(P^{0,n}(v_n(H_0)^{-1}|\widehat{H}_n - H_0| \geq M) + P^{1,n}(v_n(H_1)^{-1}|\widehat{H}_n - H_1| \geq M)) + o(1) \\ & = \frac{1}{2} \int_{X^n} \mathbb{P}_{\xi^{0,n}(\omega)}^n(A^0) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^1) \mathbf{P}^n(d\omega) + o(1) \end{aligned}$$

where $A^i = \{v_n(H_i)^{-1}|\widehat{H}_n - H_i| \geq M\}$. Taking n large enough, it suffices to bound from below the integral appearing here. But since \mathbf{P}^n and $T^n\mathbf{P}^n$ are equivalent, see Proposition (iii), we have

$$\int_{X^n} \mathbb{P}_{\xi^{0,n}(\omega)}^n(A^0) \mathbf{P}^n(d\omega) = \int_{X^n} \mathbb{P}_{\xi^{0,n}(T^n\omega)}^n(A^0) \frac{d\mathbf{P}^n}{dT^n\mathbf{P}^n}(T^n\omega) \mathbf{P}^n(d\omega).$$

For $r > 0$, we denote X_r^n is the set of $\omega \in X^n$ such that

$$n\|\xi^{0,n}(T^n\omega) - \xi^{1,n}(\omega)\|_\infty^2 \leq r.$$

Notice that this definition is slightly different from that of [GH07] due to the differences in Proposition 11. We obtain then, for $\lambda > 0$

$$\begin{aligned} & \int_{X^n} \mathbb{P}_{\xi^{0,n}(\omega)}^n(A^0) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^1) \mathbf{P}^n(d\omega) \\ & = \int_{X^n} (\mathbb{P}_{\xi^{0,n}(T^n\omega)}^n(A^0) \frac{d\mathbf{P}^n}{dT^n\mathbf{P}^n}(T^n\omega) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^1)) \mathbf{P}^n(d\omega) \\ & \geq e^{-\lambda} \int_{X_r^n} (\mathbb{P}_{\xi^{0,n}(T^n\omega)}^n(A^0) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^1)) \mathbb{1}_{\frac{d\mathbf{P}^n}{dT^n\mathbf{P}^n}(T^n\omega) \geq e^{-\lambda}} \mathbf{P}^n(d\omega). \end{aligned}$$

Notice also that for $\omega \in X_r^n$, we have

$$\begin{aligned} & \mathbb{P}_{\xi^{0,n}(T^n\omega)}^n(A^0) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^0) \\ & \geq \mathbb{P}_{\xi^{0,n}(T^n\omega)}^n(n^{1/(4H_0+2)}|\widehat{H}_n - H_0| \geq M) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(n^{1/(4H_0+2)}|\widehat{H}_n - H_1| \geq M) \\ & \geq \mathbb{P}_{\xi^{1,n}(\omega)}^n(n^{1/(4H_0+2)}|\widehat{H}_n - H_0| \geq M) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(n^{1/(4H_0+2)}|\widehat{H}_n - H_1| \geq M) \\ & \quad - d_{test}(\mathbb{P}_{\xi^{0,n}(T^n\omega)}^n, \mathbb{P}_{\xi^{1,n}(\omega)}^n) \\ & \geq 1 - \frac{1}{2}\|\mathbb{P}_{\xi^{0,n}(T^n\omega)}^n - \mathbb{P}_{\xi^{1,n}(\omega)}^n\|_{TV} \end{aligned}$$

since $n^{1/(4H_0+2)}|H_0 - H_1| \geq 2M$ by definition of I and M . We apply now Proposition 11 to get

$$\begin{aligned} \mathbb{P}_{\xi^{0,n}(T^n\omega)}^n(A^0) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^0) & \geq R(ne^{\|\xi^{0,n}(T^n\omega) - \xi^{1,n}(\omega)\|_\infty} \|\xi^{0,n}(T^n\omega) - \xi^{1,n}(\omega)\|_\infty^2) \\ & \geq R(e^{r/\sqrt{n}}r^2) \geq R(e^r r^2) \end{aligned}$$

since R is non-increasing. Thus, we get

$$\begin{aligned}
& \int_{X^n} \mathbb{P}_{\xi^{0,n}(\omega)}^n(A^0) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^1) \mathbf{P}^n(d\omega) \\
& \geq e^{-\lambda} R(e^r r^2) \int_{X^n} \mathbb{1}_{\frac{d\mathbf{P}^n}{dT^n \mathbf{P}^n}(T^n \omega) \geq e^{-\lambda}} \mathbf{P}^n(d\omega) \\
& \geq e^{-\lambda} R(e^r r^2) \mathbf{P}^n \left(X_r^n \cap \left\{ \frac{d\mathbf{P}^n(T^n \omega)}{dT^n \mathbf{P}^n} \geq e^{-\lambda} \right\} \right) \\
& \geq e^{-\lambda} R(e^r r^2) \left(\mathbf{P}^n(X_r^n) - T^n \mathbf{P}^n \left(\frac{dT^n \mathbf{P}^n}{d\mathbf{P}^n} \leq e^\lambda \right) \right)
\end{aligned}$$

But Markov inequality and Proposition 12 yield

$$\begin{aligned}
T^n \mathbf{P}^n \left(\frac{dT^n \mathbf{P}^n}{d\mathbf{P}^n} \leq e^\lambda \right) & \leq \mathbf{P}^n \left(\frac{dT^n \mathbf{P}^n}{d\mathbf{P}^n} \leq e^\lambda \right) + d_{test}(T^n \mathbf{P}^n, \mathbf{P}^n) \\
& \leq e^{-\lambda} + \frac{1}{2} \|T^n \mathbf{P}^n - \mathbf{P}^n\|_{TV} \\
& \leq e^{-\lambda} + 1 - c^*
\end{aligned}$$

so that we infer

$$\int_{X^n} \mathbb{P}_{\xi^{0,n}(\omega)}^n(A^0) + \mathbb{P}_{\xi^{1,n}(\omega)}^n(A^1) \mathbf{P}^n(d\omega) \geq e^{-\lambda} R(e^r r^2) \left(\mathbf{P}^n(X_r^n) - e^{-\lambda} - 1 + \frac{c^*}{2} \right).$$

Moreover,

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{P}^n(X_r^n) = \lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{P}^n(n \|\xi^{0,n}(T^n \omega) - \xi^{1,n}(\omega)\|_\infty^2 \leq r) = 1$$

since $n \|\xi^{0,n}(T^n \omega) - \xi^{1,n}(\omega)\|_\infty^2$ is tight by Proposition 12. We conclude by taking λ and r large enough.

5.2. Proof of Proposition 11. Let $K(\mu, \nu) = \int \log(\frac{d\mu}{d\nu}) d\mu$ denote the Kullback-Leibler divergence between two probability measures μ and ν . Recall also the Pinsker's inequality $\|\mu - \nu\|_{TV}^2 \leq 2K(\mu, \nu)$. Under \mathbb{P}_f^n , the increments of the observations $S_{(j+1)/n} - S_{j/n}$ are independent Gaussian variables whose variance is given by

$$n^{-1} \exp(f(\lfloor jn^{-1} \delta^{-1} \rfloor \delta))$$

so that the Kullback-Leibler divergence between the two measures \mathbb{P}_f^n and \mathbb{P}_g^n is given by

$$K(\mathbb{P}_f^n, \mathbb{P}_g^n) = n\delta \sum_{i=0}^{2^N-1} A((f-g)(i\delta))$$

where $A(x) = e^x - x - 1$. A is increasing on $[0, \infty)$ and $A(x) \leq A(|x|)$ for $x \geq 0$ so

$$(14) \quad K(\mathbb{P}_f^n, \mathbb{P}_g^n) \leq nA(\|f-g\|_\infty).$$

Note in addition that $A(x) \leq x^2$ for $0 \leq x \leq 1$ so $K(\mathbb{P}_f^n, \mathbb{P}_g^n) \leq n\|f-g\|_\infty^2$ whenever $\|f-g\|_\infty \leq 1$. Pinsker's inequality proves estimate (12) in that case. When $\|f-g\|_\infty \geq 1$, we write $K = \lceil \|f-g\|_\infty \rceil$ and $f_k = (kg + (K-k)f)/K$ for all $k = 0, \dots, K$ so that $\|f_k - f_{k+1}\|_\infty \leq 1$ for any k . Using (12) for functions (f_k, f_{k+1}) , we get

$$\|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV} \leq \sum_k \|\mathbb{P}_{f_k}^n - \mathbb{P}_{f_{k+1}}^n\|_{TV} \leq \sum_k Cn^{1/2} \|f_k - f_{k+1}\|_\infty \leq Cn^{1/2} \|f-g\|_\infty$$

which proves (12). We now prove (13). First, notice that for $x \geq 0$, we also have $A(x) \leq x^2 e^x$ so that (14) yields

$$K(\mathbb{P}_f^n, \mathbb{P}_g^n) \leq n e^{\|f-g\|_\infty} \|f-g\|_\infty^2$$

and we can conclude since $\|\mu - \nu\|_{TV}$ remains bounded away from 2 when the divergence $K(\mu, \nu)$ and $K(\nu, \mu)$ are bounded away from $+\infty$.

5.3. Proof of Proposition 12.

Introduction and notation. Proposition 12 is an adaptation of Proposition 5 in [GH07]. Indeed, points (i) and (iii) are the same as in [GH07] and the construction of the approximation processes $\xi^{i,n}$ in our proof is exactly the same as in [GH07]. Therefore, we will quickly recall the main arguments concerning the construction of the $\xi^{i,n}$ while skipping the tedious computations already done therein. We will also skip the proof of points (i) and (iii) and focus instead on (ii).

The main idea of the approximation is to decompose the fractional Brownian motions ηW^H over a wavelet basis and to keep only low frequencies. Indeed, high frequencies entail local information of the fractional Brownian motion, such that its Hlder regularity. Thus it should vary a lot even for small changes of H even if the absolute value of a single coefficient is generally small. On the other side, low-frequency coefficients should be high to determine the global trends of the fractional Brownian motions, but their behaviour should be continuous in H . Thus we will cut our signal corresponding to fractional Brownian motions and keep only low frequencies. Thus we will get a process close enough to the original fractional Brownian motion. Moreover, a slight technical modification in the low-frequency process approximating $\eta_1 W^{H_1}$ ensures this remains close to $\eta_1 W^{H_1}$ while being closer to the low-frequency process approximating $\eta_0 W^{H_0}$. The main choice remaining is the cut-off level in the frequencies and, unsurprisingly, we will see that $\log_2 n^{1/(1+2H_0)}$ is perfectly suitable.

We will now present in more detail the approximation procedure. First, recall from [GH07] that for any H , fractional Brownian motions admit the following series representation

$$(15) \quad 2^{-j_0(H+1/2)} \sum_{k=-\infty}^{\infty} \Theta_{j_0,k}^H(t) \varepsilon_k^H + \sum_{j=j_0}^{\infty} 2^{-j(H+1/2)} \sum_{|k| \leq 2^{j_0+1}} (\psi_{j,k}^H(t) - \psi_{j,k}^H(0)) \varepsilon_{j,k}$$

where $\varepsilon_{j,k}$ are independent standard Gaussian variables, $(\varepsilon_k^H)_k$ is a Gaussian stationary family whose spectral density is given by $|2 \sin(v/2)|^{1-2H_0}$ and where the $\Theta_{j,k}^H$ and $\psi_{j,k}^H$ are defined as fractional derivatives of wavelet functions. For conciseness, we do not detail their construction which can be found in [GH07], Section 7.1. In the following, we will need the following property.

Lemma 13 (Lemma 5.(i) in [GH07]). *For any $M > 0$ there exists $c = c(M)$ such that for all j_0 and all $H \in [H_-, H_+]$*

$$\sum_{j \geq j_0} \sum_{|k| \leq 2^{j+1}} \|\psi_{j,k}^H\|_\infty (1+j)^{1/2} (1+|k|)^{1/2} \leq c(M) 2^{-Mj_0}.$$

A very important feature of the representation (15) is that the random variables appearing in the high-frequency part are independent of the low-frequency terms.

Construction of the space X^n . As in [GH07], we take

$$X^n = \left(\otimes_{k=-2^{j_0+1}}^{2^{j_0+1}} \mathbb{R} \right) \otimes \left(\otimes_{j=j_0}^{\infty} \otimes_{k=-2^{j+1}}^{2^{j+1}} \mathbb{R} \right) =: X_e^n \otimes X_d^n.$$

We write \mathcal{X}^n the Borel product sigma-algebra of X^n , $\omega = (\omega^e, \omega^d)$ the elements of X^n and

$$\varepsilon_k(\omega) = \omega_k^e \quad \text{and} \quad \varepsilon_{j,k}(\omega) = \omega_{j,k}^d$$

the projections on the coordinates of ω .

In view of (15), we then define

$$(16) \quad \xi_t^{0,n} = \eta_0 2^{-j_0(H_0+1/2)} \sum_{|k| \leq 2^{j_0+1}} \Theta_{j_0,k}^{H_0}(t) \varepsilon_k + \eta_0 \sum_{j=j_0}^{\infty} 2^{-j(H_0+1/2)} \sum_{|k| \leq 2^{j+1}} (\psi_{j,k}^{H_0}(t) - \psi_{j,k}^{H_0}(0)) \varepsilon_{j,k}.$$

To ensure this is a correct approximation of $\sigma_0 W^{H_0}$, we define the probability measure

$$\mathbf{P}^n := \mathbf{P}_e^n \otimes \mathbf{P}_d^n$$

such that under \mathbf{P}_e^n , $(\varepsilon_k)_k$ is a centred Gaussian stationary sequence with spectral density

$$|2 \sin(v/2)|^{1-2H_0}$$

and under \mathbf{P}_d^n , $(\varepsilon_{j,k})_{j,k}$ are independent standard Gaussian variables. Therefore, under \mathbf{P}^n , the law of $\xi_t^{0,n}$ is close to the law of $\eta_0 W^{H_0}$. We want now to define an approximation of $\eta_1 W^{H_1}$. As explained in v, replacing ε_k by a stationary sequence $(\varepsilon'_k)_k$ with spectral density $|2 \sin(v/2)|^{1-2H_1}$ is not enough and one should incorporate corrective terms corresponding to the development of $\Theta^{H_1} = \Theta^{H_0+\varepsilon_n}$. Following [GH07], a suitable approximation for $\eta_1 W^{H_1}$ is given by

$$\begin{aligned} \xi_t^{1,n} &= \eta_1 2^{-j_0(H_1+1/2)} \sum_{|k| \leq 2^{j_0+1}} \Theta_{j_0,k}^{H_0}(t) \varepsilon'_k + \eta_1 \sum_{j=j_0}^{\infty} 2^{-j(H_1+1/2)} \sum_{|k| \leq 2^{j+1}} (\psi_{j,k}^{H_1}(t) - \psi_{j,k}^{H_1}(0)) \varepsilon_{j,k} \\ &\quad + \eta_1 2^{j_0(H_1+1/2)} \sum_{|l| \leq 2^{j_0+1}} \sum_{|k| \leq 2^{j_0+1}} (\Theta_{j_0,l}^{H_0}(t) a_{l-k} \varepsilon'_k + (\psi_{j_0,l}^{H_0}(t) - \psi_{j_0,l}^{H_0}(0)) b_{l-k} \varepsilon'_k) \end{aligned}$$

where the sequences a and b are defined in Lemma 5 of [GH07].

[GH07] also provides the mapping T^n of Proposition 12. This transformation is divided into two parts. The first one acts on X_d^n and transforms the stationary sequence $(\varepsilon_k)_k$ with spectral density $|2 \sin(v/2)|^{1-2H_0}$ into a stationary sequence $(\varepsilon'_k)_k$ with spectral density $|2 \sin(v/2)|^{1-2H_1}$ which can be done since these measures are absolutely continuous. The second one is more tricky and uses the sequences a and b to transform linearly the $\varepsilon_{j,k}$. We refer once more to [GH07] for the details. The only important result regarding this construction for the proof of point (ii) of 12 is the following development

$$(17) \quad \begin{aligned} \xi_t^{1,n}(\omega) - \xi_t^{0,n}(T^n \omega) &= \sum_{j \geq j_0} \sum_{|k| \leq 2^{j+1}} \eta_1 2^{j(H_1+1/2)} (\psi_{j,k}^{H_1}(t) - \psi_{j,k}^{H_1}(0)) \varepsilon_{j,k}(\omega) \\ &\quad + \sum_{j \geq j_0} \sum_{|k| \leq 2^{j+1}} \eta_0 2^{j(H_0+1/2)} (\psi_{j,k}^{H_0}(t) - \psi_{j,k}^{H_0}(0)) \varepsilon_{j,k}(\omega). \end{aligned}$$

Proof of Proposition 12.(ii). Notice first that it is enough to prove that $\sqrt{n}\|\xi^{1,n}(\omega) - \xi^{0,n}(T^n\omega)\|_\infty$ is bounded in L^1 since Markov inequality for positive random variables will then ensure tightness. By (17), it is enough to bound the two terms:

$$\sum_{j \geq j_0} \sum_{|k| \leq 2^{j+1}} \eta_1 2^{-j(H_1+1/2)} \|\psi_{j,k}^{H_1}\|_\infty |\varepsilon_{j,k}(\omega)| \text{ and } \sum_{j \geq j_0} \sum_{|k| \leq 2^{j+1}} \sigma_0 2^{-j(H_0+1/2)} \|\psi_{j,k}^{H_0}\|_\infty |\varepsilon_{j,k}(\omega)|.$$

Both these terms are handled similarly so we will focus on the first one here. We now need to use Lemma 3 of [MST99].

Lemma 14. *Let $(\varepsilon_{j,k})_{j \geq 0, k \in \mathbb{Z}}$ be independent standard Gaussian variables. Then there exists $C(\omega)$ a random variable having finite moments of all orders such that*

$$|\varepsilon_{j,k}| \leq C(\log(2+j) \log(2+|k|))^{1/2}.$$

Thus there exists a positive random variable $C = C(\omega)$ with finite moments for any order such that for any $j \geq 0$ and $|k| \leq 2^{j+1}$, we have:

$$|\varepsilon_{j,k}(\omega)| \leq C(\omega)(1+j)^{1/2}(1+|k|)^{1/2}$$

and therefore

$$\begin{aligned} & \sum_{j \geq j_0} \sum_{|k| \leq 2^{j+1}} \eta_1 2^{-j(H_1+1/2)} \|\psi_{j,k}^{H_1}\|_\infty |\varepsilon_{j,k}(\omega)| \\ & \leq C(\omega) \eta_1 2^{-j_0(H_1+1/2)} \sum_{j \geq j_0} \sum_{|k| \leq 2^{j+1}} \|\psi_{j,k}^{H_1}\|_\infty (1+j)^{1/2}(1+|k|)^{1/2} \\ & \leq C(\omega) c(M) \eta_1 2^{-j_0(H_1+1/2)-Mj_0} \end{aligned}$$

by Lemma 13, where $M > 0$ is arbitrary. But $C(\omega)$ is bounded in L^1 , so for any $M > 0$ there exists a constant $c' = c'(M)$ such that

$$\mathbb{E}[\|\xi^{1,n}(\omega) - \xi^{0,n}(T^n\omega)\|_\infty] \leq c' \eta_1 2^{-j_0(M+H_1+\frac{1}{2})} \leq c'' n^{-1/2}$$

where the last inequality is obtained taking M big enough.

6. PROOF OF THEOREM 2

6.1. Preparation for the proof. We start by giving a crucial result on the behaviour of the pre-averaged energy levels and their empirical counterparts. It relies on a similar strategy as in [GH07] and is based on the two following Propositions.

Proposition 15. *Let $\varepsilon > 0$. Then there exists $0 < r_-(\varepsilon) < \exp(-(2H_+ + 1))$, $J_0(\varepsilon) > 0$ and $N_0(\varepsilon)$ depending on \mathcal{D} only such that for $N \geq N_0(\varepsilon)$, we have:*

$$(18) \quad \sup_{(H,\eta) \in \mathcal{D}} \mathbb{P}_{H,\eta} \left(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1} \leq r_-(\varepsilon) \right) \leq \varepsilon.$$

Proposition 16.

$$(19) \quad \sup_{(H,\eta) \in \mathcal{D}} \mathbb{E}_{H,\eta} \left[(\widehat{Q}_{j,p,n} - Q_{j,p})^2 \right] \leq C((n\delta)^{-2} 2^{-j-2p} + (n\delta)^{-1} 2^{-j(1+2H)-p}).$$

Their proofs are presented in Sections 6.2 and 6.3. Then notice that

$$\left| \frac{\widehat{Q}_{J_n^*+1, N-J_n^*-1, n}}{\widehat{Q}_{J_n^*, N-J_n^*-1, n}} - 2^{-2H} \right| \leq B_n + V_n^{(1)} + V_n^{(2)}$$

where

$$\begin{aligned} B_n &= \left| \frac{Q_{J_n^*+1, N-J_n^*-1}}{Q_{J_n^*, N-J_n^*-1}} - 2^{-2H} \right|, \\ V_n^{(1)} &= \left| \frac{\widehat{Q}_{J_n^*+1, N-J_n^*-1, n} - Q_{J_n^*+1, N-J_n^*-1}}{\widehat{Q}_{J_n^*, N-J_n^*-1, n}} \right|, \\ V_n^{(2)} &= \left| \frac{Q_{J_n^*+1, N-J_n^*-1}(\widehat{Q}_{J_n^*, N-J_n^*-1, n} - Q_{J_n^*, N-J_n^*-1})}{\widehat{Q}_{J_n^*, N-J_n^*-1, n} Q_{J_n^*, N-J_n^*-1}} \right|. \end{aligned}$$

Therefore the proof is completed once we prove that $v_n^{-1}B_n$, $v_n^{-1}V_n^{(1)}$ and $v_n^{-1}V_n^{(2)}$ are tight. We deal separately with the cases $\delta \leq n^{-1/(2H+1)}$ and $\delta \geq n^{-1/(2H+1)}$ in Sections 6.4 and 6.5.

For $\varepsilon > 0$, we introduce

$$J_n^-(\varepsilon) = \max \{j \geq 1 : r_-(\varepsilon)2^{-2jH} \geq 2^j n^{-1}\} \wedge (N-1)$$

where $r_-(\varepsilon)$ is defined in Proposition 15.

6.2. Proof of Proposition 15. Our proof is an adaptation of Proposition 1 in [GH07].

Lemma 17. *For any p , there exists a function $\kappa_p : (0, 1) \rightarrow (0, \infty)$ such that for any $(H, \eta) \in \mathcal{D}$ and any $j \geq 0$, we have*

$$(20) \quad \mathbb{E}_{H, \eta}[d_{j, k, p}^2] = \kappa_p(H)\eta^2 2^{-j(1+2H)}.$$

Moreover, we have

$$0 < \inf_{(H, \eta) \in \mathcal{D}, p \geq 0} \kappa_p(H) \leq \sup_{(H, \eta) \in \mathcal{D}, p \geq 0} \kappa_p(H) < \infty.$$

Proof. We start with the explicit computation of $\mathbb{E}_{H, \eta}[d_{j, k, p}^2]$. By self-similarity and the fact that W^H has stationary increments, this expectation equals

$$\begin{aligned} & \frac{\eta^2}{2^{j(1+2H)+2p}} \sum_{\ell_1, \ell_2=0}^{2^p-1} \mathbb{E}_{H, \eta}[(W_{(\ell_1-\ell_2)2^{-p}}^H - 2W_{1+(\ell_1-\ell_2)2^{-p}}^H + W_{2+(\ell_1-\ell_2)2^{-p}}^H)(W_2^H - 2W_1^H)] \\ &= \frac{\eta^2}{2^{j(1+2H)}} 2^{-p} \sum_{\ell \in \mathbb{Z}, |\ell| < 2^p} (1 - |\ell|2^{-p}) \mathbb{E}_{H, \eta}[(W_{\ell 2^{-p}}^H - 2W_{1+\ell 2^{-p}}^H + W_{2+\ell 2^{-p}}^H)(W_2^H - 2W_1^H)]. \end{aligned}$$

It follows that $\kappa_p(H)$ is well defined for all values of H and p . It is also positive since $\mathbb{E}_{H, \eta}[d_{j, k, p}^2] > 0$. Moreover, a direct computation of the expectation above yields

$$\kappa_p(H) = 2^{-p} \sum_{\ell \in \mathbb{Z}, |\ell| < 2^p} (1 - |\ell|2^{-p}) \phi_H(\ell 2^{-p}),$$

with $\phi_H(x) = \frac{1}{2} \sum_{k=0}^4 (-1)^{k+1} \binom{4}{k} |x+k-2|^{2H}$. Since $H \mapsto \phi_H(x)$ is continuous for any x , we deduce that $H \mapsto \kappa_p(H)$ is bounded below and above for fixed p . Moreover, $x \mapsto \phi_H(x)$ is continuous and there exists $C > 0$ independent of H such that $|\phi_H(x) - \phi_H(y)| \leq C|x-y|^{2H \wedge 1}$ for any $-1 \leq x, y \leq 1$. Thus $\kappa_p(H)$ converges to $\kappa_\infty(H) = \int_{-1}^1 (1-|x|)\phi_H(x)dx$ as $p \rightarrow \infty$. More precisely, we have

$$|\kappa_p(H) - \kappa_\infty(H)| \leq \sum_{\ell \in \mathbb{Z}, |\ell| < 2^p} \int_{(\ell-1)2^{-p}}^{\ell 2^{-p}} \left| (1 - |\ell|2^{-p})\phi_H(\ell 2^{-p}) - (1 - |x|)\phi_H(x) \right| dx$$

$$+ \int_{1-2^{-p}}^1 (1-|x|)|\phi_H(x)|dx.$$

The integral in the sum is bounded by $C2^{-2p} + C2^{-(2H \wedge 1)p-p}$ and the last integral is bounded by $C2^{-p}$ so that

$$|\kappa_p(H) - \kappa_\infty(H)| \leq C2^{-p(2H \wedge 1)}$$

and the convergence is uniform. Also, notice that

$$\kappa_\infty(H) = \mathbb{E}_{H,\eta} \left[\left(\int_0^1 (W_u^H - 2W_{u+1}^H + W_{u+2}^H) du \right)^2 \right] > 0,$$

hence Lemma 17 follows from the continuity of $H \mapsto \kappa_\infty(H)$. \square

Lemma 18 (Decorrelation of the Wavelet Coefficients). *For k_1, k_2 such that $|k_1 - k_2| \geq 3$, we have:*

$$|\mathbb{E}_{H,\eta}[d_{j,k_1,p}d_{j,k_2,p}]| \leq C\eta^2 2^{-j(1+2H)}(1+|k_1 - k_2|)^{2H-4}.$$

Proof. Suppose $k_1 \geq k_2 + 3$. We need to show

$$|\mathbb{E}_{H,\eta}[d_{j,k_1,p}d_{j,k_2,p}]| \leq C\eta^2 2^{-j(1+2H)}(1+|k_1 - k_2|)^{2H-4}.$$

Since W^H is self-similar and has stationary increments, we have

$$(21) \quad \mathbb{E}_{H,\eta}[d_{j,k_1,p}d_{j,k_2,p}] = \frac{\eta^2}{2^{j(1+2H)}} 2^{-p} \sum_{\ell \in \mathbb{Z}, |\ell| < 2^p} (1 - \ell 2^{-p}) \phi_H(k_1 - k_2 + \ell 2^{-p})$$

where $\phi_H(x) = \frac{1}{2} \sum_{k=0}^4 (-1)^{k+1} \binom{4}{k} |x+k-2|^{2H}$. Notice that for $x > 2$, the absolute values appearing in the expression of $\phi_H(x)$ can be removed. If $F(x,t) = |x+t|^{2H}$, Taylor's formula yields

$$\begin{aligned} \phi_H(x) &= \frac{1}{2 \cdot 4!} \int_0^1 (1-t)^3 (16\partial_t^4 F(x, -2t) - 4\partial_t^4 F(x, -t) - 4\partial_t^4 F(x, t) + 16\partial_t^4 F(x, 2t)) dt \\ &= \frac{1}{2 \cdot 4!} \int_{-1}^1 (1-|t|)^3 (-4\partial_t^4 F(x, t) + 16\partial_t^4 F(x, 2t)) dt. \end{aligned}$$

We infer $|\phi_H(x)| \leq C|x-2|^{2H-4}$ for some constant C independent of $H \in [H_-, H_+]$. Summing over ℓ in (21) yields the result. \square

Using both Lemmas 17 and 18, we have

$$(22) \quad \mathbb{P}_{H,\eta} [(Q_{j,p} - \kappa_p(H)\eta^2 2^{-2jH-1})^2] \leq C2^{-j(1+4H)}$$

for some constant C independent of H and η . The proof of (22) is obtained exactly in the same way as in Proposition 3 of [GH07] and is thus omitted.

We are ready to prove the estimate (18) of Proposition 15. Let J_0 and N be two arbitrary integers and $r > 0$. We have

$$\begin{aligned} \mathbb{P}_{H,\eta} \left(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1} \leq r \right) &\leq \sum_{j=J_0}^{N-1} \mathbb{P}_{H,\eta} (2^{2jH} Q_{j,N-j-1} \leq r) \\ &\leq \sum_{j=J_0}^{N-1} \mathbb{P}_{H,\eta} (Q_{j,N-j-1} - \kappa_p(H)\eta^2 2^{-2jH-1} \leq (r - \kappa_p(H)\eta^2/2)2^{-2jH}). \end{aligned}$$

By Lemma 17, we can pick r small enough so that $r - \kappa_p(H)\eta^2/2 \leq -c$ for some $c > 0$ fixed, uniformly in $p \geq 0$ and $(H, \eta) \in \mathcal{D}$. The estimate (22) yields

$$\mathbb{P}_{H,\eta} \left(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1} \leq r \right) \leq \sum_{j=J_0}^{N-1} C 2^{-j(1+4H)} c^{-2} 2^{4jH} \leq C' 2^{-J_0}$$

and (18) follows.

6.3. Proof of Proposition 16. Recall from Equation (3) that by $\widehat{Q}_{j,p,n} = \sum_{k=0}^{2^j-1} \widehat{d}_{j,k,p,n}^2$ where $\widehat{d}_{j,k,p,n}^2 = \widetilde{d}_{j,k,p,n}^2 - \lambda_{j,p,n}$ and $\lambda_{j,p,n} = 6 \text{Var}(\varepsilon_{1,m}) 2^{-j-p}$. Moreover, recall that we have the decomposition $\widetilde{d}_{j,k,p,n} = d_{j,k,p} + e_{j,k,p,n}$ where e is defined in Equation (2). Therefore, we get:

$$\widehat{Q}_{j,p,n} = Q_{j,p} + \sum_{k=0}^{2^j-1} (e_{j,k,p,n}^2 - \lambda_{j,p,n}) + 2 \sum_{k=0}^{2^j-1} e_{j,k,p,n} d_{j,k,p}$$

and the estimates

$$(23) \quad \mathbb{E}_{H,\eta} \left[\left(\sum_{k=0}^{2^j-1} (e_{j,k,p,n}^2 - \lambda_{j,p,n}) \right)^2 \right] \leq C(n\delta)^{-2} 2^{-j-2p},$$

$$(24) \quad \text{and } \mathbb{E}_{H,\eta} \left[\left(\sum_{k=0}^{2^j-1} e_{j,k,p,n} d_{j,k,p} \right)^2 \right] \leq C(n\delta)^{-1} 2^{-j(1+2H)-p}.$$

prove Proposition 16. First, notice that the random variables $e_{j,k,p,n}^2 - \lambda_{j,p,n}$ are centred and moreover, $e_{j,k_1,p,n}^2 - \lambda_{j,p,n}$ and $e_{j,k_2,p,n}^2 - \lambda_{j,p,n}$ are independent whenever $|k_1 - k_2| \geq 3$. Thus

$$\begin{aligned} \mathbb{E}_{H,\eta} \left[\left(\sum_k (e_{j,k,p,n}^2 - \lambda_{j,p,n}) \right)^2 \right] &= \sum_{k_1, k_2} \mathbb{E}_{H,\eta} [(e_{j,k_1,p,n}^2 - \lambda_{j,p,n})(e_{j,k_2,p,n}^2 - \lambda_{j,p,n})] \\ &\leq C \sum_k \mathbb{E}_{H,\eta} (e_{j,k,p,n}^2 - \lambda_{j,p,n})^2 \\ &\leq C \sum_k \mathbb{E}_{H,\eta} e_{j,k,p,n}^4. \end{aligned}$$

The noise variable $e_{j,k,p,n}$ is a sum of 2^p independent centred random variables with the same law as $\varepsilon_{1,n} - 2\varepsilon_{2,n} + \varepsilon_{3,n}$. Therefore, its 4-th moment is of order 2^{2p} up to a multiplicative factor bounded above by $\mathbb{E}_{H,\eta} [\varepsilon_{1,n}^4]$, as follows for instance by Rosenthal's inequality. We derive

$$\mathbb{E}_{H,\eta} [e_{j,k,p,n}^4] \leq C 2^{-2j-2p} \mathbb{E}_{H,\eta} [\varepsilon_{1,n}^4].$$

Moreover, $n\delta \exp(\varepsilon_{1,n})$ has a χ^2 distribution with $n\delta$ degrees of freedom so that $\mathbb{E}_{H,\eta} [\varepsilon_{1,n}^4] \leq C(n\delta)^{-2}$ by Lemma 36. Thus

$$\mathbb{E}_{H,\eta} [e_{j,k,p,n}^4] \leq C 2^{-2j-2p} (n\delta)^{-2}$$

so that we have (23).

We now focus on the estimate (24). Notice that the random variables $(e_{j,k,p,n})_k$ are centred and independent of the variables $(d_{j,k,p})_k$. Moreover, $e_{j,k_1,p,n}$ and $e_{j,k_2,p,n}$ are independent if

$|k_1 - k_2| \geq 3$. Therefore we get

$$\begin{aligned} \mathbb{E}_{H,\eta} \left[\left(\sum_{k=0}^{2^{-j-1}-1} e_{j,k,p,n} d_{j,k,p} \right)^2 \right] &\leq C \sum_k \mathbb{E}_{H,\eta} \left[e_{j,k,p,n}^2 d_{j,k,p}^2 \right] \\ &\leq C \sum_k \mathbb{E}_{H,\eta} \left[e_{j,k,p,n}^2 \right] \mathbb{E}_{H,\eta} \left[d_{j,k,p}^2 \right]. \end{aligned}$$

We conclude using Lemma 17 and the estimate $\mathbb{E}_{H,\eta} \left[e_{j,k,p,n}^2 \right] = \lambda_{j,p,n} = 6 \text{Var}(\varepsilon_{1,m}) 2^{-j-p} \leq C 2^{-j-p} (n\delta)^{-1}$.

6.4. Completion of proof when $\delta \leq n^{-1/(2H+1)}$. Suppose first that $\delta \leq n^{-1/(2H+1)}$ so that $n \leq \delta^{-(2H+1)} \leq 2^{(2H+1)N}$. Then we define for any $\varepsilon > 0$

$$\begin{aligned} J_n^-(\varepsilon) &= \max \{ j \geq 1 : r_-(\varepsilon) 2^{-2jH} \geq 2^j n^{-1} \} \wedge (N-1) \\ &= \left\lfloor \frac{\log(r_-(\varepsilon)n)}{2H+1} \right\rfloor \wedge (N-1) \\ &\leq \left\lfloor N + \frac{\log r_-(\varepsilon)}{2H+1} \right\rfloor \wedge (N-1). \end{aligned}$$

Since $r_-(\varepsilon) < \exp(-(2H+1))$ by Proposition 15, $\log r_-(\varepsilon) < -(2H+1)$ and $J_n^-(\varepsilon) = \lfloor \frac{\log(r_-(\varepsilon)n)}{2H+1} \rfloor$ so that

$$(25) \quad \frac{1}{2} (r_-(\varepsilon)n)^{1/(2H+1)} \leq 2^{J_n^-(\varepsilon)} \leq (r_-(\varepsilon)n)^{1/(2H+1)}.$$

The following estimate ensures that with overwhelming probability, J_n^* can be controlled by $J_n^-(\varepsilon)$.

Lemma 19. *For any $\varepsilon > 0$, there exists $L(\varepsilon) > 0$ such that*

$$\sup_{(H,\eta) \in \mathcal{D}} \mathbb{P}_{H,\eta}(J_n^* < J_n^-(\varepsilon) - L(\varepsilon)) \leq \varepsilon + \varphi_n(\varepsilon).$$

Proof. Let $L > 0$. For notational simplicity, we set $\bar{J}_n = J_n^-(\varepsilon) - L$. We have:

$$\begin{aligned} \mathbb{P}_{H,\eta}(J_n^* \geq \bar{J}_n) &\geq \mathbb{P}_{H,\eta}(\widehat{Q}_{\bar{J}_n, N-\bar{J}_n-1, n} \geq \nu_0 2^{\bar{J}_n} n^{-1}) \\ &\geq \mathbb{P}_{H,\eta}(\widehat{Q}_{\bar{J}_n, N-\bar{J}_n-1} - Q_{\bar{J}_n, N-\bar{J}_n-1, n} \geq \nu_0 2^{\bar{J}_n} n^{-1} - 2^{-2H\bar{J}_n} r_-(\varepsilon)) \\ &\quad - \mathbb{P}_{H,\eta}(Q_{\bar{J}_n, N-\bar{J}_n-1} \leq 2^{-2H\bar{J}_n} r_-(\varepsilon)). \end{aligned}$$

Since $J_n^-(\varepsilon) \rightarrow \infty$ as $n \rightarrow \infty$, we have $J_0(\varepsilon) \leq \bar{J}_n \leq N-1$ for large enough n ($J_0(\varepsilon)$ is defined in Proposition 15). This entails by Proposition 15

$$\mathbb{P}_{H,\eta}(Q_{\bar{J}_n, N-\bar{J}_n-1} \leq 2^{-2H\bar{J}_n} r_-(\varepsilon)) \leq \mathbb{P}_{H,\eta} \left(\inf_{J_0(\varepsilon) \leq j \leq N-1} 2^{2Hj} Q_{j, N-j-1} \leq r_-(\varepsilon) \right) \leq \varepsilon.$$

By definition of $J_n^-(\varepsilon)$, we also have $r_-(\varepsilon) 2^{-2J_n^-(\varepsilon)H} \geq 2^{J_n^-(\varepsilon)} n^{-1}$ so that

$$\begin{aligned} \mathbb{P}_{H,\eta}(J_n^* \geq \bar{J}_n) &\geq \mathbb{P}_{H,\eta}(\widehat{Q}_{\bar{J}_n, N-\bar{J}_n-1} - Q_{\bar{J}_n, N-\bar{J}_n-1, n} \geq 2^{J_n^-(\varepsilon)} n^{-1} (\nu_0 2^{-L} - 2^{2HL})) - \varepsilon \\ &\geq 1 - \mathbb{P}_{H,\eta}(\widehat{Q}_{\bar{J}_n, N-\bar{J}_n-1} - Q_{\bar{J}_n, N-\bar{J}_n-1, n} \leq -2^{J_n^-(\varepsilon)} n^{-1}) - \varepsilon \end{aligned}$$

as soon as L is taken sufficiently large so that $\nu_0 2^{-L} - 2^{2HL} \leq -1$. Using Proposition 16, we derive then

$$\mathbb{P}_{H,\eta}(J_n^* < \bar{J}_n) \leq \varepsilon + \mathbb{P}_{H,\eta}(|\widehat{Q}_{\bar{J}_n, N-\bar{J}_n-1} - Q_{\bar{J}_n, N-\bar{J}_n-1, n}| \geq 2^{J_n^-(\varepsilon)} n^{-1})$$

$$\begin{aligned} &\leq \varepsilon + C2^{-2J_n^-(\varepsilon)}n^2(2^{-2N+\bar{J}_n}n^{-2}\delta^{-2} + n^{-1}\delta^{-1}2^{-N-2H\bar{J}_n}) \\ &\leq \varepsilon + C'(2^{-J_n^-(\varepsilon)} + 2^{-2(H+1)J_n^-(\varepsilon)}n) \end{aligned}$$

where the constant C' can depend on ε but is independent of the parameters of the model. The estimate (25) ends the proof. \square

We now prove that $v_n^{-1}B_n$ is tight under the condition $\delta_n \leq n^{-1/(2H+1)}$. Let $n \geq 1$ be sufficiently large so that

$$J_n^-(\varepsilon) - L(\varepsilon) \geq J_0(\varepsilon) \quad \text{and} \quad N \geq N_0(\varepsilon)$$

simultaneously hold. Let $M > 0$. We first write

$$\mathbb{P}_{H,\eta}(v_n^{-1}B_n \geq M) \leq I + II + III,$$

with

$$\begin{aligned} I &= \mathbb{P}_{H,\eta}(v_n^{-1}B_n \geq M, J_n^* \geq J_n^-(\varepsilon) - L(\varepsilon), \inf_{J_0 \leq j \leq N-1} 2^{2jH}Q_{j,N-j-1} \geq r_-(\varepsilon)), \\ (26) \quad II &= \mathbb{P}_{H,\eta}(J_n^* < J_n^-(\varepsilon) - L(\varepsilon)), \\ III &= \mathbb{P}_{H,\eta}(\inf_{J_0 \leq j \leq N-1} 2^{2jH}Q_{j,N-j-1} \leq r_-(\varepsilon)). \end{aligned}$$

The term II is smaller than $\varepsilon + \varphi_n(\varepsilon)$ by Lemma 19, while III is smaller than ε by Proposition 15. For the term I , we apply estimate (25) to obtain

$$\begin{aligned} I &\leq \sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1} \mathbb{P}_{H,\eta}(v_n^{-1}B_n \geq M, J_n^* = j, Q_{j,N-j-1} \geq 2^{-2Hj}r_-(\varepsilon)) \\ &\leq \sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1} \mathbb{P}_{H,\eta}(|Q_{j+1,N-j-1} - 2^{-2H}Q_{j,N-j-1}| \geq \kappa 2^{-2Hj}r_-(\varepsilon)v_n) \\ &\leq \sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1} 2^{-j}CM^{-2}r_-(\varepsilon)^{-2}v_n^{-2} \\ &\leq C2^{-J_n^-(\varepsilon)+L(\varepsilon)}M^{-2}r_-(\varepsilon)^{-2}v_n(H)^{-2}. \end{aligned}$$

and we obtain the tightness of $v_n^{-1}B_n$ thanks to the fact that $2^{-J_n^-(\varepsilon)}v_n^{-2}$ is bounded, see (22).

We now consider the tightness $v_n^{-1}V_n^{(1)}$. Let $M > 0$. We have

$$\mathbb{P}_{H,\eta}(v_n^{-1}V_n^{(1)} \geq M) \leq I' + II$$

where

$$I' = \mathbb{P}_{H,\eta}(v_n^{-1}V_n^{(1)} \geq M, J_n^* \geq J_n^-(\varepsilon) - L(\varepsilon))$$

and where II is defined in (26). Recall that $II \leq \varepsilon + \varphi_n(\varepsilon)$ by Lemma 19. By definition, we also have

$$\widehat{Q}_{J_n^*, N-J_n^*-1, n} \geq \nu_0 2^{J_n^*} n^{-1} \quad \text{on} \quad \{J_n^* \geq J_n^-(\varepsilon) - L(\varepsilon)\}.$$

so that we get using also Proposition 16

$$\begin{aligned}
 I' &\leq \sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1} \mathbb{P}_{H,\eta} \left(v_n^{-1} |\widehat{Q}_{j+1,N-j-1,n} - Q_{j+1,N-j-1}| \geq M\nu_0 2^j n^{-1} \right) \\
 &\leq C \sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1} M^{-2} \nu_0^{-2} 2^{-2j} n^2 v_n^{-2} \left(n^{-2} \delta^{-2} 2^{j-2N} + n^{-1} \delta^{-1} 2^{-2Hj-N} \right) \\
 &\leq C M^{-2} \nu_0^{-2} v_n^{-2} \sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1} 2^{-j} + n 2^{-2(H+1)j} \\
 &\leq C' M^{-2} \nu_0^{-2} v_n^{-2} (2^{-J_n^-(\varepsilon)} + n 2^{-2(H+1)J_n^-(\varepsilon)})
 \end{aligned}$$

where C' can depend on ε . We conclude noticing that Equation (25) implies that $v_n^{-2} 2^{-J_n^-(\varepsilon)}$ and $v_n^{-2} n 2^{-2(H+1)J_n^-(\varepsilon)}$ are bounded.

We eventually prove the tightness of $V_n^{(2)}$. First, notice that when $v_n^{-1} B_n \leq M'$ for some $M' > 0$, necessarily

$$2^{-2H} - M' v_n \leq \frac{Q_{J_n^*+1, N_n - J_n^* - 1}}{Q_{J_n^*, N_n - J_n^* - 1}} \leq 2^{-2H} + M' v_n.$$

For $M' > 0$, we have $2^{-2H} - M' v_n > 0$, at least when n is big enough since $v_n \rightarrow 0$ as $n \rightarrow \infty$. In that case

$$\frac{Q_{J_n^*, N - J_n^* - 1}}{Q_{J_n^*+1, N - J_n^* - 1}} \geq (2^{-2H} + M' v_n)^{-1}.$$

Let $M > 0$. It follows that

$$\begin{aligned}
 \mathbb{P}_{H,\eta} \left(v_n^{-1} V_n^{(2)} \geq M \right) &\leq \mathbb{P}_{H,\eta} \left(v_n^{-1} |\widehat{Q}_{J_n^*, N - J_n^* - 1, n} - Q_{J_n^*, N - J_n^* - 1}| \geq \frac{M \widehat{Q}_{J_n^*, N - J_n^* - 1, n} Q_{J_n^*, N - J_n^* - 1}}{Q_{J_n^*+1, N - J_n^* - 1}} \right) \\
 &\leq \mathbb{P}_{H,\eta} \left(|\widehat{Q}_{J_n^*, N - J_n^* - 1, n} - Q_{J_n^*, N - J_n^* - 1}| \geq \frac{M \widehat{Q}_{J_n^*, N - J_n^* - 1, n}}{2^{-2H} v_n^{-1} + M'} \right) + \mathbb{P}_{H,\eta} \left(v_n^{-1} B_n \geq M' \right)
 \end{aligned}$$

We can then repeat the proof for the tightness of $v_n^{-1} V_n^{(1)}$ like in Step 3 and we conclude by noticing that $2^{-2H} v_n^{-1} + M'$ is of the same order as v_n^{-1} .

6.5. Completion of proof when $\delta \geq n^{-1/(2H+1)}$. Suppose now that $\delta \geq n^{-1/(2H+1)}$. We quickly cover this case using the same arguments as in the first case. Note that here $v_n = \delta^{1/2}$.

Recall that J_n^* is defined by $J_n^* = \max\{2 \leq j \leq N-1 : \widehat{Q}_{j, N-j-1, n} \geq \nu_0 2^j n^{-1}\}$. The following estimate replaces Lemma 19 from the previous case.

Lemma 20. *We have*

$$\sup_{(H,\eta) \in \mathcal{D}} \mathbb{P}_{H,\eta} (J_n^* < N-1) \rightarrow 0$$

Proof. Recall that $\nu_0 < \inf_{H,\eta} \eta^2 \kappa_0(H) 2^{2H}$ so there exist a constant $\iota > 0$ such that for any H, η , $\nu_0 - \eta^2 \kappa_0(H) 2^{2H} < -2\iota$. We have

$$\mathbb{P}_{H,\eta} (J_n^* < N-1) = \mathbb{P}_{H,\eta} (\widehat{Q}_{N-1,0,n} \leq \nu_0 \delta^{2H} / 2) \leq I + II$$

where

$$I = \mathbb{P}_{H,\eta}(\widehat{Q}_{N-1,0,n} - Q_{N-1,0} \leq (\nu_0 - \kappa_0(H)\eta^2 2^{2H} + \iota)\delta^{2H}/2)$$

and $II = \mathbb{P}_{H,\eta}(Q_{N-1,0} \leq (\kappa_0(H)\eta^2 2^{2H} - \iota)\delta^{2H}/2).$

Then we have by Proposition 16

$$\begin{aligned} I &\leq \mathbb{P}_{H,\eta}(\widehat{Q}_{N-1,0,n} - Q_{N-1,0} \leq -\iota\delta^{2H}/2) \\ &\leq C(n^{-2}\delta^{-2}2^{-N} + n^{-1}\delta^{-1}2^{-N(1+2H)})\iota^{-2}\delta^{-4H} \\ &\leq C(n^{-2}\delta^{-(1+4H)} + n^{-1}\delta^{-2H})\iota^{-2} \end{aligned}$$

since $\delta = 2^{-N}$. We show then that $I \rightarrow 0$ uniformly over \mathcal{D} as $n \rightarrow \infty$ using that $\delta \geq n^{-1/(2H+1)}$.

Then we have

$$\begin{aligned} II &= \mathbb{P}_{H,\eta}(Q_{N-1,0} - \kappa_0(H)\eta^2 2^{2H} \delta^{2H}/2 \leq -\iota\delta^{2H}/2) \\ &\leq C\delta\iota^{-2} \end{aligned}$$

by Equation (22) and we can conclude since $\delta \rightarrow 0$ when $n \rightarrow \infty$. \square

We can now prove tightness. We start with $v_n^{-1}B_n$. We have

$$\mathbb{P}_{H,\eta}(v_n^{-1}B_n \geq M) \leq \mathbb{P}_{H,\eta}\left(\left|\frac{Q_{N,0}}{Q_{N-1,0}} - 2^{-2H}\right| \geq M\delta^{1/2}\right) + \mathbb{P}(J_n^* < N-1)$$

so that we can focus on the first probability by Lemma 20. Then by Proposition 15

$$\mathbb{P}_{H,\eta}\left(\left|\frac{Q_{N,0}}{Q_{N-1,0}} - 2^{-2H}\right| \geq M\delta^{1/2}\right) \leq \mathbb{P}_{H,\eta}(|Q_{N,0} - 2^{-2H}Q_{N-1,0}| \geq M\delta^{1/2+2H}r_-(\varepsilon)2^{2H}) + \varepsilon$$

since $2^N = \delta^{-1}$ by definition. Using also (22) we eventually obtain

$$\mathbb{P}_{H,\eta}\left(\left|\frac{Q_{N,0}}{Q_{N-1,0}} - 2^{-2H}\right| \geq M\delta^{1/2}\right) \leq CM^{-2}r_-(\varepsilon)^{-2} + \varepsilon$$

and therefore we can show that $v_n^{-1}B_n$ is tight.

Concerning the term $V_n^{(1)}$, for $M > 0$ we have:

$$\mathbb{P}_{H,\eta}(v_n^{-1}V_n^{(1)} \geq M) \leq \mathbb{P}_{H,\eta}(|\widehat{Q}_{N,0,n} - Q_{N,0}| \geq \frac{1}{2}M\nu_0\delta^{1/2+2H}) + \mathbb{P}_{H,\eta}(J_n^* < N-1)$$

since $\widehat{Q}_{N-1,0,n} \geq \frac{1}{2}\nu_0\delta^{2H}$ on $\{J_n^* = N-1\}$. By Proposition 15 and using $\delta_n \geq n^{-1/(2H+1)}$, we obtain

$$\begin{aligned} \mathbb{P}_{H,\eta}(v_n^{-1}V_n^{(1)} \geq M) &\leq C\kappa^{-2}\delta^{-4H-2}n^{-2} + \mathbb{P}_{H,\eta}(J_n^* < N-1) \\ &\leq CM^{-2} + \mathbb{P}_{H,\eta}(J_n^* < N-1) \end{aligned}$$

where and the tightness of $v_n^{-1}V_n^{(1)}$ follows.

We eventually deal with the tightness of $n_n^{-1}V_n^{(2)}$. Proceeding as for the case $\delta \leq n^{-1/(2H+1)}$, we have for $M, M' > 0$ and n big enough that $\mathbb{P}_{H,\eta}(v_n^{-1}V_n^{(2)} \geq M)$ is bounded by

$$\mathbb{P}_{H,\eta}\left(\left|\widehat{Q}_{J_n^*, N-J_n^*-1, n} - Q_{J_n^*, N-J_n^*-1}\right| \geq \frac{M\widehat{Q}_{J_n^*, N-J_n^*-1, n}}{2^{-2H}v_n^{-1} + M'}\right) + \mathbb{P}_{H,\eta}(v_n^{-1}B_n \geq M').$$

The first term is similar to $V_n^{(1)}$ since $2^{-2H}v_n^{-1} + M'$ is of the same order as v_n^{-1} . The second term can be made as small as we want since $v_n^{-1}B_n$ is tight.

7. PROOF OF THEOREM 3

The proof of Theorem 3 is similar to that of Theorem 1. For conciseness, we only provide here a result similar to Proposition 11. The same development as in Section 5 would then conclude the proof of Theorem 3.

Recall that $\|\mu\|_{TV} = \sup_{\|f\|_\infty \leq 1} |\int f d\mu|$ is the total variation of a signed measure μ . We also denote \mathbb{P}_f^n the law of $(S_{i/n})_i$ given $\eta W_t^H = f(t)$.

Proposition 21. *Let f and g be two bounded functions. Then there exists $c_0 > 0$ such that*

$$(27) \quad \|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV} \leq c_0 \sqrt{n} e^{c_0(\|f\|_\infty + \|g\|_\infty)} \|f - g\|_\infty$$

Moreover, there exists $c_1 > 0$ and a universal nonincreasing positive function R such that

$$(28) \quad 1 - \frac{1}{2} \|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV} \geq R(n e^{3\|f\|_\infty + 3\|g\|_\infty} \|f - g\|_\infty^2)$$

Proof. As in the proof of Proposition 11, let $K(\mu, \nu) = \int \log(\frac{d\mu}{d\nu}) d\mu$ denote the Kullback-Leibler divergence between two probability measures μ and ν . We also recall the Pinsker's inequality $\|\mu - \nu\|_{TV}^2 \leq 2K(\mu, \nu)$.

Notice that $(S_{i/n} - S_{(i-1)/n})_i$ has the same law as $\left(\int_{(i-1)/n}^{i/n} \sigma_t^2 dt\right)^{1/2} \xi_{i,n}$ where the random variables $(\xi_{i,n})_i$ are independent standard Gaussian variables since W^H and B are independent. Therefore we have

$$K(\mathbb{P}_f^n, \mathbb{P}_g^n) \leq \sum_{i=1}^n B\left(\left(\int_{(i-1)/n}^{i/n} e^{f(t)} dt\right)\left(\int_{(i-1)/n}^{i/n} e^{g(t)} dt\right)^{-1}\right).$$

where $B(x) = x - \ln(x) - 1$. and Pinsker's inequality yields

$$\|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV}^2 \leq 2 \sum_{i=1}^n B\left(\left(\int_{(i-1)/n}^{i/n} e^{f(t)} dt\right)\left(\int_{(i-1)/n}^{i/n} e^{g(t)} dt\right)^{-1}\right).$$

Notice then that for $x > 0$, $B(x) \leq (x-1)^2 + (1/x-1)^2$ so that we get

$$\|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV}^2 \leq \sum_{i=1}^n \left(\frac{\int_{(i-1)/n}^{i/n} e^{f(t)} - e^{g(t)} dt}{\int_{(i-1)/n}^{i/n} e^{g(t)} dt}\right)^2 + \left(\frac{\int_{(i-1)/n}^{i/n} e^{g(t)} - e^{f(t)} dt}{\int_{(i-1)/n}^{i/n} e^{f(t)} dt}\right)^2.$$

Moreover,

$$\int_{(i-1)/n}^{i/n} e^{g(t)} dt \geq n^{-1} e^{-\|g\|_\infty} \quad \text{and} \quad \int_{(i-1)/n}^{i/n} e^{f(t)} dt \geq n^{-1} e^{-\|f\|_\infty}$$

so

$$\|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV}^2 \leq 2n^2 (e^{\|f\|_\infty} + e^{\|g\|_\infty}) \sum_{i=1}^n \left(\int_{(i-1)/n}^{i/n} e^{f(t)} - e^{g(t)} dt\right)^2$$

By Jensen inequality, we obtain

$$\begin{aligned} \|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV}^2 &\leq 2n(e^{\|f\|_\infty} + e^{\|g\|_\infty}) \sum_{i=1}^n \int_{(i-1)/n}^{i/n} (e^{f(t)} - e^{g(t)})^2 dt \\ &\leq 2n(e^{\|f\|_\infty} + e^{\|g\|_\infty}) \|e^f - e^g\|_\infty^2. \end{aligned}$$

Notice finally that $\|e^f - e^g\|_\infty \leq e^{\max(\|f\|_\infty, \|g\|_\infty)} \|f - g\|_\infty$ so that we obtain

$$\begin{aligned} \|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV}^2 &\leq 2n(e^{\|f\|_\infty} + e^{\|g\|_\infty})e^{2\max(\|f\|_\infty, \|g\|_\infty)} \|f - g\|_\infty^2 \\ &\leq 2ne^{3\|f\|_\infty + 3\|g\|_\infty} \|f - g\|_\infty^2 \end{aligned}$$

which proves (27). For (28), we proceed exactly as for (13) and the proof is omitted here. \square

8. PROOF OF THE RESULTS OF SECTION 3.3

8.1. Notation and organisation of the proofs. We start by introducing some useful notations. Let $b \geq 1$ and $s \geq 1$. Consider $\mathbf{r} \in \{1, \dots, 2S\}^s$ such that $\sum_i \mathbf{r}_i = b$. If $b = 1$, we write

$$\mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} = 2^{j+p} \int_{k2^{-j+l2^{-j-p}}}^{k2^{-j}+(l+1)2^{-j-p}} W_u^H du = \int_0^1 W_{k2^{-j}+(l+u)2^{-j-p}}^H du$$

and if $b \geq 2$, we write

$$\begin{aligned} \mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} &= \prod_{i=1}^s \frac{2^{j+p}}{\mathbf{r}_i!} \int_{k2^{-j+l2^{-j-p}}}^{k2^{-j}+(l+1)2^{-j-p}} (W_u^H - W_{k2^{-j}+l2^{-j-p}}^H)^{\mathbf{r}_i} du. \\ &= \frac{1}{\mathbf{r}!} \int_{[0,1]^s} \prod_{i=1}^s (W_{k2^{-j}+(l+u_i)2^{-j-p}}^H - W_{k2^{-j}+l2^{-j-p}}^H)^{\mathbf{r}_i} du. \end{aligned}$$

where $\mathbf{r}! = \prod_{i=1}^s \mathbf{r}_i!$. Define also $\mathfrak{W}_{p,l}^{H,\mathbf{r}} = \mathfrak{W}_{0,p,0,l}^{H,\mathbf{r}}$. Since $\mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} = \mathfrak{W}_{j,p,0,2^p k+l}^{H,\mathbf{r}}$ we get by self-similarity of the fractional Brownian motion

$$(29) \quad \left(\mathfrak{W}_{j,p,0,l}^{H,\mathbf{r}} \right)_{\mathbf{r},l} = \left(2^{-jH \sum_i \mathbf{r}_i} \mathfrak{W}_{p,l}^{H,\mathbf{r}} \right)_{\mathbf{r},l} \text{ in distribution.}$$

We fix $0 < H^* < H_-$ such that $(2S+1)H^* \geq H_+$. Then using Proposition 33 and notations introduced within this proposition, we can decompose $d_{j,k,p} = g_{j,k,p} + z_{j,p,k}$ where

$$(30) \quad g_{j,k,p} = 2^{-p-j/2} \sum_{l=0}^{2^p-1} \sum_{b=1}^{2S} \eta^b \sum_{s=1}^{2S} \frac{(-1)^{s-1}}{s} \sum_{\substack{\mathbf{r} \in \{1, \dots, 2S\}^s \\ \sum_j \mathbf{r}_j = b}} \mathfrak{W}_{j,p,k+1,l}^{H,\mathbf{r}} - \mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}}$$

$$(31) \quad z_{j,p,k} = 2^{-p-j/2} 2^{-(j+p)(2S+1)H^*} \sum_{l=0}^{2^p-1} Z((k+1)2^p + l, 2^{-j-p}) - Z(k2^p + l, 2^{-j-p}).$$

Then we write $G_{j,p} = \sum_k g_{j,p,k}^2$. The proofs of the results of Section 3.3 follow from similar results on G together with an appropriate control of the error terms arising from z . Therefore the proofs will be split as follows. First, in Section 8.2, will be gathered useful computational lemmas used throughout the proofs. Then in Section 8.3, we present the proof of Proposition 4. In Section 8.4 are gathered the proofs of the results concerning the functions κ and finally, in Section 8.5 we prove Proposition 7.

In the following, we will also use the notation $\overline{\sum}$ to indicate a sum over all indexes $2 \leq b_1, b_2 \leq 2S$, $1 \leq s_1, s_2 \leq 2S$ and multi-indexes $\mathbf{r}_1 \in \{1, \dots, S\}^{s_1}$ and $\mathbf{r}_2 \in \{1, \dots, S\}^{s_2}$ such that $\sum_j \mathbf{r}_{1j} = b_1$ and $\sum_j \mathbf{r}_{2j} = b_2$. Additional subscripts to $\overline{\sum}$ will denote additional constraints over the indexes.

8.2. Computational lemmas.

Lemma 22.

$$(32) \quad \mathbb{E}\left[g_{j,p,k}^2\right] = 2^{-j} \sum_{a=1}^S \eta^{2a} 2^{-2jaH} \kappa_{p,a}(H) + O(2^{-j(2S+1)H-j})$$

where the O is uniform over H , j, k and p and where $\kappa_{p,a}$ are explicit functions of H given by

$$(33) \quad \kappa_{p,a}(H) = 2^{-2p} \sum_{|l| < 2^p} \sum_{b_1+b_2=2a} \frac{(-1)^{s_1+s_2}}{s_1 s_2} (2^p - |l|) \mathbb{E}\left[(\mathfrak{W}_{p,2^p}^{H,\mathbf{r}_1} - \mathfrak{W}_{p,0}^{H,\mathbf{r}_1})(\mathfrak{W}_{p,2^p+l}^{H,\mathbf{r}_2} - \mathfrak{W}_{p,l}^{H,\mathbf{r}_2})\right].$$

Proof. Using (29), we have

$$\begin{aligned} \mathbb{E}\left[g_{j,p,k}^2\right] &= \frac{1}{2^{2p+j}} \sum_{l_1, l_2=0}^{2^p-1} \sum_{b_1+b_2} \eta^{b_1+b_2} \frac{(-1)^{s_1+s_2}}{s_1 s_2} \mathbb{E}\left[(\mathfrak{W}_{j,p,k+1,l_1}^{H,\mathbf{r}_1} - \mathfrak{W}_{j,p,k,l_1}^{H,\mathbf{r}_1})(\mathfrak{W}_{j,p,k+1,l_2}^{H,\mathbf{r}_2} - \mathfrak{W}_{j,p,k,l_2}^{H,\mathbf{r}_2})\right] \\ &= \frac{1}{2^{2p+j}} \sum_{l_1, l_2=0}^{2^p-1} \sum_{b_1+b_2} \frac{(-1)^{s_1+s_2} \eta^{b_1+b_2}}{s_1 s_2 2^{j(b_1+b_2)H}} \mathbb{E}\left[(\mathfrak{W}_{p,2^p+l_1}^{H,\mathbf{r}_1} - \mathfrak{W}_{p,l_1}^{H,\mathbf{r}_1})(\mathfrak{W}_{p,2^p+l_2}^{H,\mathbf{r}_2} - \mathfrak{W}_{p,l_2}^{H,\mathbf{r}_2})\right]. \end{aligned}$$

By stationarity of the fractional Brownian motion, we also have

$$\mathbb{E}\left[(\mathfrak{W}_{p,2^p+l_1}^{H,\mathbf{r}_1} - \mathfrak{W}_{p,l_1}^{H,\mathbf{r}_1})(\mathfrak{W}_{p,2^p+l_2}^{H,\mathbf{r}_2} - \mathfrak{W}_{p,l_2}^{H,\mathbf{r}_2})\right] = \mathbb{E}\left[(\mathfrak{W}_{p,2^p}^{H,\mathbf{r}_1} - \mathfrak{W}_{p,0}^{H,\mathbf{r}_1})(\mathfrak{W}_{p,2^p+l_2-l_1}^{H,\mathbf{r}_2} - \mathfrak{W}_{p,l_2-l_1}^{H,\mathbf{r}_2})\right].$$

Thus writing $l = l_2 - l_1$, we get

$$\mathbb{E}\left[g_{j,p,k}^2\right] = \frac{1}{2^{2p+j}} \sum_{|l| < 2^p} \sum_{b_1+b_2} \frac{(-1)^{s_1+s_2} \eta^{b_1+b_2}}{s_1 s_2 2^{j(b_1+b_2)H}} (2^p - |l|) \mathbb{E}\left[(\mathfrak{W}_{p,2^p}^{H,\mathbf{r}_1} - \mathfrak{W}_{p,0}^{H,\mathbf{r}_1})(\mathfrak{W}_{p,2^p+l}^{H,\mathbf{r}_2} - \mathfrak{W}_{p,l}^{H,\mathbf{r}_2})\right].$$

We can expand the expectation $\mathbb{E}\left[(\mathfrak{W}_{p,2^p}^{H,\mathbf{r}_1} - \mathfrak{W}_{p,0}^{H,\mathbf{r}_1})(\mathfrak{W}_{p,2^p+l}^{H,\mathbf{r}_2} - \mathfrak{W}_{p,l}^{H,\mathbf{r}_2})\right]$ by linearity. Since the expectation of the product of an odd number of (centred) Gaussian variables is null, we have

$$\mathbb{E}\left[(\mathfrak{W}_{p,2^p}^{H,\mathbf{r}_1} - \mathfrak{W}_{p,0}^{H,\mathbf{r}_1})(\mathfrak{W}_{p,2^p+l}^{H,\mathbf{r}_2} - \mathfrak{W}_{p,l}^{H,\mathbf{r}_2})\right] = 0$$

if $b_1 + b_2$ is odd. Hlder's inequality also ensures that when $b_1 + b_2$ is even, this expectation is bounded uniformly over p, H, l, \mathbf{r}_1 and \mathbf{r}_2 . \square

Lemma 23. For any $j, p \geq 0$ and $k \leq 2^j$, we have

$$\mathbb{E}\left[g_{j,p,k}^4\right] \leq C 2^{-j(4H+2)}.$$

Proof. Recall that g is defined in Equation 32 so that

$$g_{j,k,p}^4 \leq C 2^{-p-2j} \sum_{l=0}^{2^p-1} \sum_{b=1}^{2S} \eta^{4b} \sum_{s=1}^{2S} \frac{1}{s^4} \sum_{\substack{\mathbf{r} \in \{1, \dots, 2S\}^s \\ \sum_j \mathbf{r}_j = b}} (\mathfrak{W}_{j,p,k+1,l}^{H,\mathbf{r}} - \mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}})^4.$$

We conclude using Lemma 24. \square

Lemma 24. For any $j, p \geq 0$, $k \leq 2^j$, $l \leq 2^p$, $1 \leq s \leq 2S$, $1 \leq b \leq 2S$ and $\mathbf{r} \in \{1, \dots, S\}^s$ such that $\sum_j \mathbf{r}_j = b$, we have

$$\mathbb{E}(\mathfrak{W}_{j,p,k+1,l}^{H,\mathbf{r}} - \mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}})^4 \leq \begin{cases} C 2^{-4jH} & \text{if } b = 1 \\ C 2^{-4(j+p)Hb} & \text{otherwise} \end{cases}$$

Proof. Remark that since there are only finitely many indexes s, b and multi-index \mathbf{r} satisfying the conditions of Lemma 24, we can suppose these index fixed and show the result for some constant C uniform in j, p and k . Suppose first that $b = 1$, then

$$\begin{aligned} \mathbb{E} \left[\left(\mathfrak{W}_{j,p,k+1,l}^{H,\mathbf{r}} - \mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} \right)^4 \right] &= \mathbb{E} \left[\left(\int_0^1 W_{(k+1)2^{-j}+(l+u)2^{-j-p}}^H - W_{k2^{-j}+(l+u)2^{-j-p}}^H du \right)^4 \right] \\ &\leq \left[\int_0^1 \mathbb{E} \left(W_{(k+1)2^{-j}+(l+u)2^{-j-p}}^H - W_{k2^{-j}+(l+u)2^{-j-p}}^H \right)^4 du \right] \\ &= 2^{-4jH}. \end{aligned}$$

Suppose now that $b > 1$. Then we have $\mathbb{E} \left(\mathfrak{W}_{j,p,k+1,l}^{H,\mathbf{r}} - \mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} \right)^4 \leq \mathbb{E} \left[\left(\mathfrak{W}_{j,p,k+1,l}^{H,\mathbf{r}} \right)^4 + \left(\mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} \right)^4 \right]$. We only show the bound for $\mathbb{E} \left[\left(\mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} \right)^4 \right]$, the other term is treated similarly. Then by Hlder and Jensen's inequalities

$$\begin{aligned} \mathbb{E} \left[\left(\mathfrak{W}_{j,p,k,l}^{H,\mathbf{r}} \right)^4 \right] &= \mathbb{E} \left[\left(\frac{1}{\mathbf{r}!} \int_{[0,1]^s} \prod_{i=1}^s \left(W_{k2^{-j}+(l+u_i)2^{-j-p}}^H - W_{k2^{-j}+l2^{-j-p}}^H \right)^{\mathbf{r}_i} du \right)^4 \right] \\ &\leq \frac{1}{\mathbf{r}!^4} \int_{[0,1]^s} \mathbb{E} \left[\prod_{i=1}^s \left(W_{k2^{-j}+(l+u_i)2^{-j-p}}^H - W_{k2^{-j}+l2^{-j-p}}^H \right)^{4\mathbf{r}_i} \right] du \\ &\leq \frac{1}{\mathbf{r}!^4} \int_{[0,1]^s} \prod_{i=1}^s \mathbb{E} \left[\left(W_{k2^{-j}+(l+u_i)2^{-j-p}}^H - W_{k2^{-j}+l2^{-j-p}}^H \right)^{4\mathbf{r}_i} \right]^{1/s} du \\ &\leq C \int_{[0,1]^s} \prod_{i=1}^s (u_i 2^{-j-p})^{4\mathbf{r}_i} du \\ &\leq C 2^{-(j+p)4Hb}. \end{aligned}$$

□

Lemma 25.

$$\left| \text{Cov}(g_{j,k_1,p}^2, g_{j,k_2,p}^2) \right| \leq C 2^{-j(2+4H)} \left((1 + |k_1 - k_2|)^{4(H-1)} + 2^{-2jH} (1 + |k_1 - k_2|)^{2(H-1)} \right)$$

Proof. By Lemma 23, we can suppose without loss of generality that $|k_1 - k_2| \geq 4$. By symmetry, we also assume $k_2 > k_1$. Then recall that we have

$$g_{j,k,p}^2 = 2^{-2p-j} \sum_{l_1, l_2=0}^{2^p-1} \eta^{b_1+b_2} \frac{(-1)^{s_1+s_2}}{s_1 s_2} \left(\mathfrak{W}_{j,p,k+1,l_1}^{H,\mathbf{r}_1} - \mathfrak{W}_{j,p,k,l_1}^{H,\mathbf{r}_1} \right) \left(\mathfrak{W}_{j,p,k+1,l_2}^{H,\mathbf{r}_2} - \mathfrak{W}_{j,p,k,l_2}^{H,\mathbf{r}_2} \right)$$

so that we can develop $\text{Cov} \left[g_{j,p,k_1}^2, g_{j,p,k_2}^2 \right]$ as

$$2^{-4p-2j} \sum \eta^{b_1+b_2+b_3+b_4} \frac{(-1)^{s_1+s_2+s_3+s_4}}{s_1 s_2 s_3 s_4} \text{Cov} \left[\left(\mathfrak{W}_{j,p,k_1+1,l_1}^{H,\mathbf{r}_1} - \mathfrak{W}_{j,p,k_1,l_1}^{H,\mathbf{r}_1} \right) \left(\mathfrak{W}_{j,p,k_1+1,l_2}^{H,\mathbf{r}_2} - \mathfrak{W}_{j,p,k_1,l_2}^{H,\mathbf{r}_2} \right), \right. \\ \left. \left(\mathfrak{W}_{j,p,k_2+1,l_3}^{H,\mathbf{r}_3} - \mathfrak{W}_{j,p,k_2,l_3}^{H,\mathbf{r}_3} \right) \left(\mathfrak{W}_{j,p,k_2+1,l_4}^{H,\mathbf{r}_4} - \mathfrak{W}_{j,p,k_2,l_4}^{H,\mathbf{r}_4} \right) \right]$$

where the sum is taken over all indexes $l_1, l_2, l_3, l_4, b_1, b_2, b_3, b_4$, and multi-indexes $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ such that $\sum_i \mathbf{r}_i = b_1, \mathbf{r}_i = b_2, \sum_i \mathbf{r}_i = b_3, \sum_i \mathbf{r}_i = b_4$. Note that the proof is completed once we can show that the covariance appearing inside this sum is dominated by $C 2^{-4jH} (1 + |k_1 - k_2|)^{2(H-1)}$, with C uniform in all indexes. We consider separately the cases $b = 1$ and $b > 1$. For conciseness, we will only deal with the following two cases: $b_1 = b_2 = b_3 = b_4 = 1$ and

$b_1, b_2, b_3, b_4 > 1$. The other cases should be treated similarly. Let's start with the first case. Then by definition we have

$$\mathfrak{W}_{j,p,k+1,l}^{H,r} - \mathfrak{W}_{j,p,k,l}^{H,r} = \int_0^1 W_{(k+1)2^{-j}+(l+u)2^{-j-p}}^H - W_{k2^{-j}+(l+u)2^{-j-p}}^H du.$$

Autosimilarity and stationarity of the fractional Brownian motion (increments), we get

$$\begin{aligned} & \text{Cov} \left[(\mathfrak{W}_{j,p,k_1+1,l_1}^{H,r_1} - \mathfrak{W}_{j,p,k_1,l_1}^{H,r_1}) (\mathfrak{W}_{j,p,k_1+1,l_2}^{H,r_2} - \mathfrak{W}_{j,p,k_1,l_2}^{H,r_2}), \right. \\ & \quad \left. (\mathfrak{W}_{j,p,k_2+1,l_3}^{H,r_3} - \mathfrak{W}_{j,p,k_2,l_3}^{H,r_3}) (\mathfrak{W}_{j,p,k_2+1,l_4}^{H,r_4} - \mathfrak{W}_{j,p,k_2,l_4}^{H,r_4}) \right] \\ &= 2^{-4jH} \int_{[0,1]^4} \text{Cov} \left[(W_{1+(l_1+u_1)2^{-p}}^H - W_{(l_1+u_1)2^{-p}}^H) (W_{1+(l_2+u_2)2^{-p}}^H - W_{(l_2+u_2)2^{-p}}^H), \right. \\ & \quad \left. (W_{(\tau+1)+(l_3+u_3)2^{-p}}^H - W_{\tau+(l_3+u_3)2^{-p}}^H) (W_{(\tau+1)+(l_4+u_4)2^{-p}}^H - W_{\tau+(l_4+u_4)2^{-p}}^H) \right] du \end{aligned}$$

where $\tau = k_2 - k_1$. We use then (44) so that the covariance in the integral reduces to

$$E(l_1, l_3, u_1, u_3)E(l_2, l_4, u_2, u_4) + E(l_1, l_4, u_1, u_4)E(l_2, l_3, u_2, u_3)$$

where

$$\begin{aligned} E(l, m, u, v) &= \mathbb{E} \left[(W_{1+(l+u)2^{-p}}^H - W_{(l+u)2^{-p}}^H) (W_{(\tau+1)+(m+v)2^{-p}}^H - W_{\tau+(m+v)2^{-p}}^H) \right] \\ &= \mathbb{E} \left[W_1^H (W_{(\tau+1)+(m-l+v-u)2^{-p}}^H - W_{\tau+(m-l+v-u)2^{-p}}^H) \right] \\ &= D_H(\tau + (m - l + v - u)2^{-p}) \end{aligned}$$

and $D_H(x) = \frac{1}{2}(|x+1|^{2H} - 2|x|^{2H} + |x-1|^{2H})$. By Taylor's formula, we get: $|D_H(x)| \leq C|x-1|^{2H-2}$ provided $x \geq 1$. For $x \geq 3$, we have even $|D_H(x)| \leq C|x+1|^{2H-2}$, for another constant C , independent of H . Since $(m - l + v - u)2^{-p} \geq -1$ and since $\tau \geq 4$, we have

$$E(l, m, u, v) \leq C|\tau|^{2H-2}$$

Therefore,

$$|E(l_1, l_3, u_1, u_3)E(l_2, l_4, u_2, u_4) + E(l_1, l_4, u_1, u_4)E(l_2, l_3, u_2, u_3)| \leq C|\tau|^{4H-4}$$

which yields to

$$\begin{aligned} & \left| \text{Cov} \left[(\mathfrak{W}_{j,p,k_1+1,l_1}^{H,r_1} - \mathfrak{W}_{j,p,k_1,l_1}^{H,r_1}) (\mathfrak{W}_{j,p,k_1+1,l_2}^{H,r_2} - \mathfrak{W}_{j,p,k_1,l_2}^{H,r_2}), \right. \right. \\ & \quad \left. \left. (\mathfrak{W}_{j,p,k_2+1,l_3}^{H,r_3} - \mathfrak{W}_{j,p,k_2,l_3}^{H,r_3}) (\mathfrak{W}_{j,p,k_2+1,l_4}^{H,r_4} - \mathfrak{W}_{j,p,k_2,l_4}^{H,r_4}) \right] \right| \leq C|\tau|^{4H-4}. \end{aligned}$$

We now suppose that $b_1, b_2, b_3, b_4 \geq 2$. In that case, we develop linearly each difference $\mathfrak{W}_{j,p,k+1,l}^{H,r} - \mathfrak{W}_{j,p,k,l}^{H,r}$ in the covariance. This let us with 16 covariances of the form

$$\text{Cov} \left[\mathfrak{W}_{j,p,k'_1,l_1}^{H,r_1} \mathfrak{W}_{j,p,k'_1,l_2}^{H,r_2}, \mathfrak{W}_{j,p,k'_2,l_3}^{H,r_3} \mathfrak{W}_{j,p,k'_2,l_4}^{H,r_4} \right]$$

with $k'_q, k''_q \in \{k_q, k_q + 1\}$. By definition, this covariance equals

$$\begin{aligned} & \frac{1}{\mathbf{r}_1! \mathbf{r}_2! \mathbf{r}_3! \mathbf{r}_4!} \text{Cov} \left[\int_{[0,1]^{s_1}} \prod_{i=1}^{s_1} (W_{k'_1 2^{-j}+(l_1+u_i)2^{-j-p}}^H - W_{k'_1 2^{-j}+l_1 2^{-j-p}}^H)^{\mathbf{r}_1 i} du, \right. \\ & \quad \left. \int_{[0,1]^{s_2}} \prod_{i=1}^{s_2} (W_{k'_2 2^{-j}+(l_2+u_i)2^{-j-p}}^H - W_{k'_2 2^{-j}+l_2 2^{-j-p}}^H)^{\mathbf{r}_2 i} du, \right. \end{aligned}$$

$$\int_{[0,1]^{s_3}} \prod_{i=1}^{s_3} (W_{k'_2 2^{-j+(l_3+u_i)} 2^{-j-p}}^H - W_{k'_2 2^{-j+l_3} 2^{-j-p}}^H)^{\mathbf{r}_{3i}} du$$

$$\int_{[0,1]^{s_4}} \prod_{i=1}^{s_4} (W_{k''_2 2^{-j+(l_4+u_i)} 2^{-j-p}}^H - W_{k''_2 2^{-j+l_4} 2^{-j-p}}^H)^{\mathbf{r}_{4i}} du \Big].$$

Therefore, it is enough to show that the covariance between

$$\prod_{i=1}^{s_1} (W_{k'_1 2^{-j+(l_1+u_{1,i})} 2^{-j-p}}^H - W_{k'_1 2^{-j+l_1} 2^{-j-p}}^H)^{\mathbf{r}_{1i}} \prod_{i=1}^{s_2} (W_{k'_1 2^{-j+(l_2+u_{2,i})} 2^{-j-p}}^H - W_{k'_1 2^{-j+l_2} 2^{-j-p}}^H)^{\mathbf{r}_{2i}}$$

and

$$\prod_{i=1}^{s_3} (W_{k'_2 2^{-j+(l_3+u_{3,i})} 2^{-j-p}}^H - W_{k'_2 2^{-j+l_3} 2^{-j-p}}^H)^{\mathbf{r}_{3i}} \prod_{i=1}^{s_4} (W_{k''_2 2^{-j+(l_4+u_{4,i})} 2^{-j-p}}^H - W_{k''_2 2^{-j+l_4} 2^{-j-p}}^H)^{\mathbf{r}_{4i}}$$

is bounded by $C2^{-4jH}(1 + |k_1 - k_2|)^{4(H-1)}$ uniformly for $(u_1, \dots, u_4) \in [0, 1]^{s_1+\dots+s_4}$. We aim at applying Proposition 35 to prove this result. Notice that

$$\mathbb{E}(W_{k'_1 2^{-j+(l_1+u_{1,i})} 2^{-j-p}}^H - W_{k'_1 2^{-j+l_1} 2^{-j-p}}^H)^2 \leq 2^{-2(j+p)H}$$

and

$$\begin{aligned} & \mathbb{E}(W_{k'_1 2^{-j+(l_1+u_{1,i})} 2^{-j-p}}^H - W_{k'_1 2^{-j+l_1} 2^{-j-p}}^H)(W_{k'_2 2^{-j+(l_3+u_{3,i})} 2^{-j-p}}^H - W_{k'_2 2^{-j+l_3} 2^{-j-p}}^H) \\ &= 2^{-2jH-1} (|k'_2 - k'_1 + 2^{-p}(l_3 - l_1 + u_3)|^{2H} - |k'_2 - k'_1 + 2^{-p}(l_3 - l_1)|^{2H} \\ & \quad + |k'_2 - k'_1 + 2^{-p}(l_3 - l_1 - u_1)|^{2H} - |k'_2 - k'_1 + 2^{-p}(l_3 - l_1 + u_3 - u_1)|^{2H}). \end{aligned}$$

In addition, we have $k'_2 - k'_1 \geq 3$ and also $l_3 - l_1 + u_3$, $l_3 - l_1$, $l_3 - l_1 - u_1$ and $l_3 - l_1 + u_3 - u_1 \geq 2^p$. Thus we can apply Taylor's formula and we develop $|\cdot|^{2H}$ around $k'_2 - k'_1 + 2^{-p}(l_3 - l_1)$. The last expression reduces to

$$\begin{aligned} & 2^{-2jH-2p} H(2H-1) \left(u_3^2 \int_0^1 (1-t) |k'_2 - k'_1 + 2^{-p}(l_3 - l_1 + tu_3)|^{2H-2} dt \right. \\ & \quad + u_1^2 \int_0^1 (1-t) |k'_2 - k'_1 + 2^{-p}(l_3 - l_1 - tu_1)|^{2H-2} dt \\ & \quad \left. - (u_3 - u_1)^2 \int_0^1 (1-t) |k'_2 - k'_1 + 2^{-p}(l_3 - l_1 + t(u_3 - u_1))|^{2H-2} dt \right). \end{aligned}$$

Since $k'_2 - k'_1 \geq 3$, this is bounded in absolute value by $C2^{-2jH-2p}|k'_2 - k'_1|^{2H-2}$. All other variances and covariances of interest to apply Proposition 35 are controled similarly and we can apply this Proposition. Therefore, the covariance between

$$\prod_{i=1}^{s_1} (W_{k'_1 2^{-j+(l_1+u_{1,i})} 2^{-j-p}}^H - W_{k'_1 2^{-j+l_1} 2^{-j-p}}^H)^{\mathbf{r}_{1i}} \prod_{i=1}^{s_2} (W_{k'_1 2^{-j+(l_2+u_{2,i})} 2^{-j-p}}^H - W_{k'_1 2^{-j+l_2} 2^{-j-p}}^H)^{\mathbf{r}_{2i}}$$

and

$$\prod_{i=1}^{s_3} (W_{k'_2 2^{-j+(l_3+u_{3,i})} 2^{-j-p}}^H - W_{k'_2 2^{-j+l_3} 2^{-j-p}}^H)^{\mathbf{r}_{3i}} \prod_{i=1}^{s_4} (W_{k''_2 2^{-j+(l_4+u_{4,i})} 2^{-j-p}}^H - W_{k''_2 2^{-j+l_4} 2^{-j-p}}^H)^{\mathbf{r}_{4i}} \Big]$$

is bounded by $\leq C2^{-2(1-H)p}|k_2 - k_1|^{2H-2} 2^{-(b_1+b_2+b_3+b_4)(j+p)H}$, so we can conclude this proof. \square

Lemma 26. For any $j, p \geq 0$ and $k \leq 2^j$, we have

$$\mathbb{E}[d_{j,p,k}^2] \leq C2^{-j(2H+1)}.$$

Proof. Recall z is defined in Equation (31). We have

$$\mathbb{E}[z_{j,p,k}^2] \leq C2^{-2p-j}2^{-2(j+p)(2S+1)H^*} \sum_{l=0}^{2^{p+1}-1} \mathbb{E}[Z(k2^p + l, 2^{-j-p})^2] \leq C2^{-j-2j(2S+1)H^*}$$

by Proposition 33. Moreover, $\mathbb{E}[g_{j,p,k}^2] \leq C2^{-j(2H+1)}$ by Lemma 22 so that we get

$$\begin{aligned} \mathbb{E}[d_{j,p,k}^2] &\leq \mathbb{E}[g_{j,p,k}^2] + \mathbb{E}[z_{j,p,k}^2] \\ &\leq C(2^{-j(2H+1)} + 2^{-j-2j(2S+1)H^*}) \leq C2^{-j(2H+1)} \end{aligned}$$

since $(2S+1)H^* \geq H_+ \geq H$. \square

8.3. Asymptotic behaviour of Q . We now prove Proposition 4. The proof is split in two parts. In the first one, we will show a result similar to Proposition 4 directly on G . In the second, we will show how this result extends to Q .

First, summing Equation (32) over k gives

$$\mathbb{E}\left[G_{j,p}\right] = \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{p,a}(H) + O(2^{-j(2S+1)H})$$

where the O is uniform over H, j, k and p . Moreover, by Lemma 25, we have

$$\begin{aligned} \text{Var}\left[G_{j,p}\right] &= \text{Var}\left[\sum_k g_{j,p,k}^2\right] = \sum_{k_1, k_2} \text{Cov}\left[g_{j,p,k_1}^2, g_{j,p,k_2}^2\right] \\ &\leq C2^{-j(2+4H)} \sum_{k_1, k_2} (1 + |k_1 - k_2|)^{4(H-1)} + 2^{-2jH} (1 + |k_1 - k_2|)^{2(H-1)} \\ &\leq C2^{-j(2+4H)} \sum_{|\tau| < 2^j} (2^j - |\tau|)((1 + |\tau|)^{4(H-1)} + 2^{-2jH} (1 + |\tau|)^{2(H-1)}) \\ &\leq C2^{-j(1+4H)} \sum_{\tau=1}^{2^j} \tau^{4(H-1)} + 2^{-2jH} \tau^{2(H-1)} \\ &\leq C2^{-j(1+4H)} (1 + 2^{-2jH} 2^{j(2H-1)}) \leq C2^{-j(1+4H)}. \end{aligned}$$

Thus, bias-variance decomposition ensures that

$$\begin{aligned} \mathbb{E}\left[\left(G_{j,p} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{p,a}(H)\right)^2\right] &= \text{Var}\left[G_{j,p}\right] + \left(\mathbb{E}[G_{j,p}] - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{p,a}(H)\right)^2 \\ &\leq C2^{-j(1+4H)} + O(2^{-2j(2S+1)H}) \end{aligned}$$

since $2(2S+1)H \geq 1 + 4H$ (as $S \geq 1/(4H_-) + 1/2$), we obtain

$$\mathbb{E}\left[\left(G_{j,p} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{p,a}(H)\right)^2\right] \leq C2^{-j(1+4H)}.$$

We focus now on the difference $\mathbb{E}\left[\left(Q_{j,p} - G_{j,p}\right)^2\right]$. By Hlder and Jensen's inequalities, we have

$$\begin{aligned}\mathbb{E}\left[\left(Q_{j,p} - G_{j,p}\right)^2\right] &= \mathbb{E}\left[\left(\sum_k z_{j,p,k}(z_{j,p,k} + 2g_{j,p,k})\right)^2\right] \\ &\leq C2^j \left(\mathbb{E}\left[\sum_k z_{j,p,k}^4\right]\mathbb{E}\left[\sum_k (z_{j,p,k} + 2g_{j,p,k})^4\right]\right)^{1/2}.\end{aligned}$$

Recall z is defined in Equation (31). By Hlder's inequality and by the bound obtained on the variables Z in Proposition 33, we have

$$\mathbb{E}[z_{j,p,k}^4] \leq C2^{-2j}2^{-4j(2S+1)H^*}.$$

Moreover, $\mathbb{E}[g_{j,p,k}^4] \leq C2^{-j(4H+2)}$ by Lemma 23 so that we get

$$\begin{aligned}\mathbb{E}\left[\left(Q_{j,p} - G_{j,p}\right)^2\right] &\leq C2^j \cdot 2^{-j}2^{-2j(2S+1)H^*} \cdot \left(2^{-2j}2^{-4j(2S+1)H^*} + 2^{-j(4H+2)}\right)^{1/2} \\ &\leq C2^{-j(1+4H)}.\end{aligned}$$

since $(2S+1)H^* \geq H_+ \geq H$.

8.4. Behaviour of the function $\kappa_{p,a}$.

Bounds on $\kappa_{p,1}$. We first deal with the functions $\kappa_{p,1}$. From (33), we get that

$$\begin{aligned}\kappa_{p,1}(H) &= 2^{-2p} \sum_{|l| < 2^p} (2^p - |l|) \mathbb{E}\left[\left(\mathfrak{W}_{p,2^p}^{H,1} - \mathfrak{W}_{p,0}^{H,1}\right)\left(\mathfrak{W}_{p,2^p+l}^{H,1} - \mathfrak{W}_{p,l}^{H,1}\right)\right] \\ &= 2^{-2p} \sum_{|l| < 2^p} (2^p - |l|) \int_0^1 \int_0^1 \mathbb{E}\left[\left(W_{1+u2^{-p}}^H - W_{u2^{-p}}^H\right)\left(W_{1+(l+v)2^{-p}}^H - W_{(l+v)2^{-p}}^H\right)\right] dudv \\ &= 2^{-2p} \sum_{|l| < 2^p} (2^p - |l|) \int_{-1}^1 (1 - |w|) \varphi_H((l+w)2^{-p}) dw\end{aligned}$$

where $\varphi_H(x) = \frac{1}{2}(|x+1|^{2H} - 2|x|^{2H} + |x-1|^{2H})$. Notice that φ_H is continuous and there exists C independent of H such that $|\varphi_H(x) - \varphi_H(y)| \leq C2^{2H \wedge 1}$ and $|\varphi_H(x)| \leq C$ for any $-1 \leq x, y \leq 1$. We define $\kappa_{p,1}(H) = \int_{-1}^1 (1 - |x|) \varphi_H(x) dx$ and we will show that $\kappa_{p,1}$ converges towards $\kappa_{\infty,1}$ as $p \rightarrow \infty$ uniformly on H . First, we rewrite $\kappa_{\infty,1}$ as

$$\begin{aligned}\kappa_{p,1}(H) &= 2^{-p} \sum_{|l| < 2^p} (1 - |l2^{-p}|) \varphi_H(l2^{-p}) \\ &\quad + 2^{-2p} \sum_{|l| < 2^p} (2^p - |l|) \int_{-1}^1 (1 - |w|) (\varphi_H((l+w)2^{-p}) - \varphi_H(l2^{-p})) dw\end{aligned}$$

and we study both sums separately. For the first one, we have

$$\begin{aligned}&\left|\kappa_{\infty,1}(H) - 2^{-p} \sum_{|l| < 2^p} (1 - |l2^{-p}|) \varphi_H(l2^{-p})\right| \\ &= \left|2^{-p} \sum_{l=-2^p}^{2^p-1} \int_{l2^{-p}}^{(l+1)2^{-p}} (1 - |x|) \varphi_H(x) - (1 - |l2^{-p}|) \varphi_H(l2^{-p}) dx\right|\end{aligned}$$

$$\begin{aligned}
 &\leq 2^{-2p} \sum_{l=-2^p}^{2^p-1} \int_{l2^{-p}}^{(l+1)2^{-p}} (1-|x|) \left| \varphi_H(x) - \varphi_H(l2^{-p}) \right| + \varphi_H(l2^{-p}) \left| (1-|x|) - (1-|l2^{-p}|) \right| dx \\
 &\leq C 2^{-2p} \sum_{l=-2^p}^{2^p-1} \int_{l2^{-p}}^{(l+1)2^{-p}} (1-|x|) \left| x - l2^{-p} \right|^{2H \wedge 1} + \varphi_H(l2^{-p}) \left| x - l2^{-p} \right| dx \\
 &\leq C 2^{-p(2H \wedge 1)}.
 \end{aligned}$$

For the second sum,

$$\begin{aligned}
 &2^{-2p} \left| \sum_{|l| < 2^p} (2^p - |l|) \int_{-1}^1 (1-|w|) (\varphi_H((l+w)2^{-p}) - \varphi_H(l2^{-p})) dw \right| \\
 &\leq 2^{-2p} \sum_{|l| < 2^p} (2^p - |l|) \int_{-1}^1 (1-|w|) \left| \varphi_H((l+w)2^{-p}) - \varphi_H(l2^{-p}) \right| dw \\
 &\leq C 2^{-2p} \sum_{|l| < 2^p} (2^p - |l|) \int_{-1}^1 (1-|w|) |w 2^{-p}|^{2H \wedge 1} dw \\
 &\leq C 2^{-p(2H \wedge 1)}.
 \end{aligned}$$

Thus, $\kappa_{p,1}(H) \rightarrow \kappa_{p,\infty}(H)$ uniformly. But $\kappa_{p,\infty}$ is continuous from its definition and we easily check that $\kappa_{\infty,1}(H) = \mathbb{E}[(\int_0^1 W_{u+1}^H - W_u^H du)^2] > 0$ so $\kappa_{\infty,1}(H)$ is bounded below and above by positive constants. Since each $\kappa_{p,1}$ is also a positive continuous function, it is also bounded below and above by positive constants. The uniform convergence ensures we can conclude.

Bounds on $\kappa_{p,a}$. Suppose $a \geq 2$. By (33), we have

$$\left| \kappa_{p,a}(H) \right| \leq C \sup_{\substack{|l| < 2^p, b_1+b_2=2a \\ s_1, s_2, \mathbf{r}_1, \mathbf{r}_2, \\ \sum_i \mathbf{r}_{1i}=b_1, \sum_i \mathbf{r}_{2i}=b_2}} \left| \mathbb{E} \left[(\mathfrak{W}_{p,2^p}^{H, \mathbf{r}_1} - \mathfrak{W}_{p,0}^{H, \mathbf{r}_1}) (\mathfrak{W}_{p,2^p+l}^{H, \mathbf{r}_2} - \mathfrak{W}_{p,l}^{H, \mathbf{r}_2}) \right] \right|$$

for some constant C independent of a, p and H . We also have

$$\begin{aligned}
 \left| \mathbb{E} \left[(\mathfrak{W}_{p,2^p}^{H, \mathbf{r}_1} - \mathfrak{W}_{p,0}^{H, \mathbf{r}_1}) (\mathfrak{W}_{p,2^p+l}^{H, \mathbf{r}_2} - \mathfrak{W}_{p,l}^{H, \mathbf{r}_2}) \right] \right| &\leq \mathbb{E} \left[(\mathfrak{W}_{p,2^p}^{H, \mathbf{r}_1} - \mathfrak{W}_{p,0}^{H, \mathbf{r}_1})^2 \right]^{1/2} \mathbb{E} \left[(\mathfrak{W}_{p,2^p+l}^{H, \mathbf{r}_2} - \mathfrak{W}_{p,l}^{H, \mathbf{r}_2})^2 \right]^{1/2} \\
 &\leq C(2^{-pHb_1} + \mathbb{1}_{b_1=1})(2^{-pHb_2} + \mathbb{1}_{b_2=1})
 \end{aligned}$$

by Lemma 24. But since $b_1 + b_2 = 2a \geq 4$ and $b_1, b_2 \geq 1$, we obtain the result.

Bounds on $\kappa'_{p,a}$. Recall that is defined in Equation (33) by

$$\kappa'_{p,a}(H) = 2^{-2p} \sum_{\substack{|l| < 2^p, b_1+b_2=2a, s_1, s_2, \mathbf{r}_1, \mathbf{r}_2, \\ \sum_i \mathbf{r}_{1i}=b_1, \sum_i \mathbf{r}_{2i}=b_2}} \frac{(-1)^{s_1+s_2}}{s_1 s_2} (2^p - |l|) E_{r_1, r_2, p, l}(H)$$

where

$$E_{r_1, r_2, p, l}(H) = \mathbb{E} \left[(\mathfrak{W}_{p,2^p}^{H, \mathbf{r}_1} - \mathfrak{W}_{p,0}^{H, \mathbf{r}_1}) (\mathfrak{W}_{p,2^p+l}^{H, \mathbf{r}_2} - \mathfrak{W}_{p,l}^{H, \mathbf{r}_2}) \right].$$

Therefore Lemma 6 is proven once we can prove that the functions $E'_{r_1, r_2, p, l}$ exist and are bounded on $[H_-, H_+]$ by a constant independent of p and l . (The dependence is r_1 and r_2 does not matter since there are finitely many possible indexes r_1 and r_2 appearing in the sum).

We consider separately the cases $b_1 = 1$ and $b_1 > 1$ and the cases $b_2 = 1$ and $b_2 > 1$. In the following, we only deal with $b_1 = b_2 = 1$ and $b_1, b_2 > 1$. The two others cases, $b_1 = 1, b_2 > 1$ and $b_1 > 1, b_2 = 1$ are treated with the same methods.

Suppose first that $b_1 = b_2 = 1$. Then $r_1 = r_2 = 1$ and we get

$$\begin{aligned} E_{r_1, r_2, p, l}(H) &= \mathbb{E} \left[\int_0^1 W_{1+u}^{H, 2^{-p}} - W_{u, 2^{-p}}^H du \int_0^1 W_{1+(l+v)}^{H, 2^{-p}} - W_{(l+v), 2^{-p}}^H dv \right] \\ &= \int_0^1 \int_0^1 \mathbb{E} \left[(W_{1+u}^{H, 2^{-p}} - W_{u, 2^{-p}}^H)(W_{1+(l+v)}^{H, 2^{-p}} - W_{(l+v), 2^{-p}}^H) \right] dudv \\ &= \frac{1}{2} \int_0^1 \int_0^1 2 \left| \frac{l+v-u}{2^p} \right|^{2H} - \left| \frac{l+v-u}{2^p} - 1 \right|^{2H} - \left| \frac{l+v-u}{2^p} + 1 \right|^{2H} dudv \\ &= \int_{-1}^1 F(w, H) dw \end{aligned}$$

where $F(w, H) = (1 - |w|)(|(l+w)2^{-p}|^{2H} - |1+(l+w)2^{-p}|^{2H})$. We now prove that $E'_{r_1, r_2, p, l}(H) = \int_{-1}^1 \partial_H F(w, H) dw$. Note that

- For any $H, w \mapsto F(w, H)$ is integrable since $|F(w, H)|$ is bounded uniformly on $[-1, 1] \times [H_-, H_+]$.
- For any $w, H \mapsto F(w, H)$ is differentiable since $x \mapsto |a|^x$ is always differentiable, whenever $a = 0$ (this function is constant) or $a \neq 0$ (this function is $e^{x \ln |a|}$). In both cases, the derivative of $x \mapsto |a|^x$ is $x \mapsto |a|^x \ln |a|$, with the convention that $0 \times \ln(0) = 0$. Therefore we get

$$\partial_H F(w, H) = (1 - |w|) \left(\left| \frac{l+w}{2^p} \right|^{2H} \ln \left| \frac{l+w}{2^p} \right| - \left| 1 + \frac{l+w}{2^p} \right|^{2H} \ln \left| 1 + \frac{l+w}{2^p} \right| \right)$$

- $\left| \frac{l+w}{2^p} \right| \leq 2$ and $\left| 1 + \frac{l+w}{2^p} \right| \leq 2$ so $\left| \frac{l+w}{2^p} \right|^{2H} \leq 4$ and $\left| 1 + \frac{l+w}{2^p} \right|^{2H} \leq 4$. Moreover, $x \mapsto x \ln(x)$ is bounded by a constant C on $[0, 4]$ so that

$$(34) \quad |\partial_H F(w, H)| \leq 2C(1 - |w|)/(2H) \leq C/H_-$$

which is integrable on $[-1, 1]$.

Therefore we can differentiate under the integral sign and we obtain

$$\begin{aligned} E'_{r_1, r_2, p, l}(H) &= \int_{-1}^1 \partial_H F(w, H) dw \\ &= \int_{-1}^1 (1 - |w|) \left(\left| \frac{l+w}{2^p} \right|^{2H} \ln \left| \frac{l+w}{2^p} \right| - \left| 1 + \frac{l+w}{2^p} \right|^{2H} \ln \left| 1 + \frac{l+w}{2^p} \right| \right) dw. \end{aligned}$$

Using (34), $|E'_{r_1, r_2, p, l}(H)| \leq 2C/H_-$ where C does not depend on p or l so we can conclude.

We now consider the case $b_1 > 1$ and $b_2 > 1$. In that case, we decompose $E_{r_1, r_2, p, l}(H)$ as $E_{r_1, r_2, p, l}^{(1)}(H) - E_{r_1, r_2, p, l}^{(2)}(H) - E_{r_1, r_2, p, l}^{(3)}(H) + E_{r_1, r_2, p, l}^{(4)}(H)$ with

$$\begin{aligned} E_{r_1, r_2, p, l}^{(1)}(H) &= \mathbb{E} \left[\mathfrak{W}_{p,0}^{H, r_1} \mathfrak{W}_{p,l}^{H, r_2} \right], & E_{r_1, r_2, p, l}^{(2)}(H) &= \mathbb{E} \left[\mathfrak{W}_{p,0}^{H, r_1} \mathfrak{W}_{p, 2^p+l}^{H, r_2} \right], \\ E_{r_1, r_2, p, l}^{(3)}(H) &= \mathbb{E} \left[\mathfrak{W}_{p, 2^p}^{H, r_1} \mathfrak{W}_{p,l}^{H, r_2} \right], & E_{r_1, r_2, p, l}^{(4)}(H) &= \mathbb{E} \left[\mathfrak{W}_{p, 2^p}^{H, r_1} \mathfrak{W}_{p, 2^p+l}^{H, r_2} \right]. \end{aligned}$$

Each of these four functions should be studied separately. For conciseness, we only detail the proof for $E_{r_1, r_2, p, l}^{(1)}(H)$. By definition, we have

$$\begin{aligned} E_{r_1, r_2, p, l}^{(1)}(H) &= \mathbb{E} \left[\mathfrak{W}_{p,0}^{H, r_1} \mathfrak{W}_{p,l}^{H, r_2} \right] \\ &= \mathbb{E} \left[\int_{[0,1]^{s_1}} \prod_{i=1}^{s_1} (W_{u_i}^H)^{r_{1i}} du \int_{[0,1]^{s_2}} \prod_{j=1}^{s_2} (W_{(l+v_j)2^{-p}}^H - W_{l2^{-p}}^H)^{r_{2j}} dv \right] \\ &= \int_{[0,1]^{s_1}} \int_{[0,1]^{s_2}} \mathbb{E} \left[\prod_{i=1}^{s_1} (W_{u_i}^H)^{r_{1i}} \prod_{j=1}^{s_2} (W_{(l+v_j)2^{-p}}^H - W_{l2^{-p}}^H)^{r_{2j}} \right] dudv. \end{aligned}$$

We aim at applying Theorem 34 to compute $\mathbb{E} \left[\prod_{i=1}^{s_1} (W_{u_i}^H)^{r_{1i}} \prod_{j=1}^{s_2} (W_{(l+v_j)2^{-p}}^H - W_{l2^{-p}}^H)^{r_{2j}} \right]$. Note that this expectation can be rewritten as $\mathbb{E} \left[\prod_{k=1}^{b_1+b_2} X_k \right]$ with either $X_k = X_k^H(w) = W_{w_k}^H$ if $k \leq b_1$ and $X_k = X_k^H(w) = W_{(l+w_k)2^{-p}}^H - W_{l2^{-p}}^H$ otherwise, where w is the vector obtained concatenating

$$\underbrace{(u_1, \dots, u_1)}_{r_{11} \text{ times}}, \underbrace{(u_2, \dots, u_2)}_{r_{12} \text{ times}}, \dots, \dots, \underbrace{(u_{s_1}, \dots, u_{s_1})}_{r_{1s_1} \text{ times}}$$

and the corresponding vector with v . Thus Theorem 34 yields

$$\mathbb{E} \left[\prod_{i=1}^{s_1} (W_{u_i}^H)^{r_{1i}} \prod_{j=1}^{s_2} (W_{(l+v_j)2^{-p}}^H - W_{l2^{-p}}^H)^{r_{2j}} \right] = \sum_P \prod_{(i,j) \in P} \mathbb{E}(X_i(w)X_j(w)).$$

Since there are finitely many 2-partitions P , it is enough to prove that for a given partition of $\{1, \dots, b_1 + b_2\}$, the application \tilde{E}_P defined by

$$\tilde{E}_P(H) = \int_{[0,1]^{s_1+s_2}} \prod_{(i,j) \in P} \mathbb{E}(X_i^H(w)X_j^H(w)) dw$$

is differentiable and its derivative is bounded uniformly over p and l . Notice that each term $\mathbb{E}(X_i^H(w)X_j^H(w))$ is bounded and has a derivative with respect to H uniformly bounded over $w \in [0, 1]^{b_1+b_2}$ (using explicit computations similar to the case $b_1 = b_2 = 1$). Therefore $(H, w) \mapsto \prod_{(i,j) \in P} \mathbb{E}(X_i^H(w)X_j^H(w))$ is differentiable with respect to H and its derivative is given by

$$\prod_{(i_0, j_0) \in P} \partial_H (\mathbb{E}(X_{i_0}^H(w)X_{j_0}^H(w))) \prod_{(i,j) \in P \setminus \{(i_0, j_0)\}} \mathbb{E}(X_i^H(w)X_j^H(w))$$

which is also uniformly bounded over $w \in [0, 1]^{b_1+b_2}$ and we can conclude.

8.5. Proof of Proposition 7. Let J_0 and N be two arbitrary integers and let $r > 0$. We have

$$\begin{aligned} \mathbb{P}_{H, \eta} \left(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j, N-j-1} \leq r \right) &\leq \sum_{j=J_0}^{N-1} \mathbb{P}_{H, \eta} (2^{2jH} Q_{j, N-j-1} \leq r) \\ &\leq \sum_{j=J_0}^{N-1} \mathbb{P}_{H, \eta} \left(Q_{j, N-j-1} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{N-j-1, a}(H) \leq r 2^{-2jH} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{N-j-1, a}(H) \right). \end{aligned}$$

Notice then that

$$r2^{-2jH} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{N-j-1,a}(H) \leq r2^{-2jH} - (\eta_-^2 \wedge 1)2^{-2Hj} c_{-,1} + \frac{c_{,S} 2^S}{2} 2^{-4Hj}$$

by Equations (5) and (6). Since $N - j - 1 \geq p_0$, we can take $r < r_0$ small enough and $J_0 > \text{big}$ enough so that $r2^{-2jH} - (\eta_-^2 \wedge 1)2^{-2Hj} c_{-,1} + c_{,S} 2^{-3H(N-j-1)} < -c_0 2^{-2jH}$ for some absolute constant $c_0 > 0$. Proposition 4 and Markov inequality finally gives

$$\mathbb{P}_{H,\eta} \left(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1} \leq r \right) \leq C \sum_{j=J_0}^{N-1} 2^{-j} \leq C 2^{-J_0} \leq \varepsilon$$

provided J_0 is big enough.

Similarly, we have

$$r2^{-2jH} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{N-j-1,a}(H) \geq r2^{-2jH} - (\eta_+^2 \vee 1)2^{-2Hj} c_{+,1} - \frac{c_{,S} 2^S}{2} 2^{-4Hj}$$

so we can conclude following the same reasoning that

$$\mathbb{P}_{H,\eta} \left(\sup_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j,N-j-1} \geq r \right) \leq \varepsilon$$

provided r is large enough and J_0 is large enough.

8.6. Proof of Lemma 9. By definition, $|B_{j,p}^{(S)}(\eta_1, H_1) - B_{j,p}^{(S)}(\eta_2, H_2)|$ is bounded by

$$\sum_{a=2}^S \eta_1^{2a} |2^{-2aH_1 j} \kappa_{p,a}(H_1) - 2^{-2aH_2 j} \kappa_{p,a}(H_2)| + 2^{-2aH_2 j} \kappa_{p,a}(H_2) |\eta_1^{2a} - \eta_2^{2a}|.$$

Moreover, $t \mapsto t^{2a}$ is differentiable with derivative $t \mapsto 2at^{2a-1}$ which is uniformly bounded on $[\eta_-, \eta_+]$. Similarly, $t \mapsto 2^{-2atj} \kappa_{p,a}(t)$ is differentiable with derivative

$$t \mapsto 2^{-2atj} (-2aj \log(2) \kappa_{p,a}(t) + \kappa'_{p,a}(t)).$$

Its absolute value is bounded by $Cj2^{-2atj}$ by Lemma 6. Since $2 \leq a \leq S$, we get

$$|B_{j,p}^{(S)}(H_1, \eta_1) - B_{j,p}^{(S)}(H_2, \eta_2)| \leq C(j2^{-4(H_1 \wedge H_2)j} |H_1 - H_2| + 2^{-4H_2 j} |\eta_1 - \eta_2|).$$

9. PROOF OF THEOREM 10

9.1. Outline and completion of proof. Our estimator is heavily based on the estimation of the energy levels through the quantities $\widehat{Q}_{j,p}$. We quantify the error $\widehat{Q}_{j,p} - Q_{j,p}$ in the following Proposition

Proposition 27. *We have*

$$\mathbb{E}_{H,\eta} \left[(\widehat{Q}_{j,p} - Q_{j,p})^2 \right] \leq C(2^{-2p-j} + 2^{-j(2H+1)-p})$$

The proof of Theorem 10 is then an application of the following four propositions, proved in dedicated Sections.

Proposition 28. *Define $v_n^{(0)} = v_n^{(0)}(H) = n^{-2H} \vee n^{-1/(4H+2)}$. Then $(v_n^{(0)})^{-1} |\widehat{H}_n^{(0)} - H|$ is bounded in probability uniformly over \mathcal{D} .*

Recall that this first estimator is used to derive a new adaptive level choice \widehat{j}_n . Indeed, we have

Corollary 29.

$$\mathbb{P}_{H,\eta}\left(\widehat{j}_n \in \left\{\left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor - 1, \left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor\right\}\right) \rightarrow 1$$

uniformly over \mathcal{D} .

Proof. Since $(v_n^{(0)})^{-1}|\widehat{H}_n^{(0)} - H|$ is bounded in probability uniformly over \mathcal{D} , $(v_n^{(0)})^{-1}\left|\left\lfloor \frac{1}{2\widehat{H}_n^{(0)}+1} \right\rfloor - \left\lfloor \frac{1}{2H+1} \right\rfloor\right|$ is also bounded in probability uniformly over \mathcal{D} . Therefore, we have

$$\begin{aligned} & \mathbb{P}_{H,\eta}\left(\widehat{j}_n \in \left\{\left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor - 1, \left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor\right\}\right) \\ & \geq \mathbb{P}_{H,\eta}\left(\frac{1}{2H+1} \log_2(n) - 1 \leq \frac{1}{2\widehat{H}_n^{(0)}+1} \log_2(n) \leq \frac{1}{2H+1} \log_2(n) + 1\right) \\ & \geq \mathbb{P}_{H,\eta}\left(v_n^{(0)-1}\left|\frac{1}{2\widehat{H}_n^{(0)}+1} - \frac{1}{2H+1}\right| \leq (v_n^{(0)} \log_2(n))^{-1}\right) \rightarrow 1 \end{aligned}$$

since $v_n^{(0)} \log_2(n) \rightarrow 0$ and the convergence is uniform on \mathcal{D} . \square

Proposition 30. Define $w_n^{(0)} = w_n^{(0)}(H) = v_n^{(0)} \log(n)$. Then $(w_n^{(0)})^{-1}|\widehat{\eta}_n - \eta|$ is bounded in probability uniformly over \mathcal{D} .

Proposition 31. Suppose that we have estimators \widehat{H}_n and $\widehat{\eta}_n$ such that $v_n^{-1}|\widehat{H}_n - H|$ and $w_n^{-1}|\widehat{\eta}_n - \eta|$ are bounded in probability uniformly over \mathcal{D} , with $w_n = v_n \log(n) \rightarrow 0$. Suppose also that $\widehat{\eta}_n \in [\eta_-, \eta_+]$ and $\widehat{H}_n \in [H_-, H_+]$. We write $\widehat{H}_n^c = \widehat{H}_n^c(\widehat{H}_n, \widehat{\eta}_n)$ as defined in Equation (9). Define also $v_n^c = v_n^c(H) = (v_n \log(n) n^{-2H/(2H+1)}) \vee n^{-1/(4H+2)}$. Then $(v_n^c)^{-1}|\widehat{H}_n - H|$ is bounded in probability uniformly over \mathcal{D} .

Proposition 32. Suppose that we have estimators \widehat{H}_n and $\widehat{\eta}_n$ such that $v_n^{-1}|\widehat{H}_n - H|$ and $w_n^{-1}|\widehat{\eta}_n - \eta|$ are bounded in probability uniformly over \mathcal{D} . Suppose also that $\widehat{\eta}_n \in [\eta_-, \eta_+]$ and $\widehat{H}_n \in [H_-, H_+]$. We write $\widehat{\eta}_n^c = \widehat{\eta}_n^c(\widehat{H}_n, \widehat{\eta}_n)$ and $w_n^c = \max(\log(n)v_n, n^{-1/(4H+2)} \log(n), w_n n^{-2H/(2H+1)})$. Then $w_n^c^{-1}|\widehat{\eta}_n - \eta|$ is bounded in probability uniformly over \mathcal{D} .

We are now ready to conclude the proof of Theorem 10. We define by induction the sequences $v_n^{(m)}$ and $w_n^{(m)}$ by

$$v_n^{(0)} = n^{-2H} \vee n^{-1/(4H+2)} \quad \text{and} \quad w_n^{(0)} = v_n^{(0)} \log(n)$$

exactly as in Propositions 28 and 30, and then for $m > 0$

$$v_n^{(m)} = (v_n \log(n) n^{-2H/(2H+1)}) \vee n^{-1/(4H+2)} \quad \text{and} \quad w_n^{(m)} = v_n^{(m)} \log(n).$$

By induction, we can see that

$$v_n^{(m)} = (\log^m(n) n^{-2H(1+m/(2H+1))}) \vee n^{-1/(4H+2)}$$

and provided $m > 1/(4H) - 2H - 1$, we can see that $v_n^{(m)} = n^{-1/(4H+2)}$ for n large enough.

Propositions 28 and 30 show that $(v_n^{(0)})^{-1}|\widehat{H}_n^{(0)} - H|$ and $(w_n^{(0)})^{-1}|\widehat{\eta}_n^{(0)} - \eta|$ are bounded in probability uniformly over \mathcal{D} ; while Propositions 31 and 32 ensure that for any $m > 0$, $(v_n^{(m)})^{-1}|\widehat{H}_n^{(m)} - H|$ and $(w_n^{(m)})^{-1}|\widehat{\eta}_n^{(m)} - \eta|$ are bounded in probability uniformly over \mathcal{D} .

Since $m_{opt} > m > 1/(4H) - 2H - 1$ for any $H_- < H < H_+$, we can conclude the proof of Theorem 10.

9.2. **Proof of Proposition 27.** By definition,

$$\widehat{Q}_{j,p} = Q_{j,p} + 2 \sum_k d_{j,p,k} e_{j,k,p} + \sum_k (e_{j,k,p}^2 - 2^{-j-p+1} \text{Var}(\log \xi^2)).$$

Moreover, the random variables $e_{j,k,p}$ are centred and $e_{j,k_1,p}$ is independent of $e_{j,k_2,p}$ if $|k_1 - k_2| \geq 2$. We deduce that

$$\begin{aligned} \mathbb{E}_{H,\eta} \left[\left(\sum_k e_{j,k,p}^2 - 2^{-j-p+1} \text{Var}_{H,\eta}(\log \xi^2) \right)^2 \right] &\leq C \mathbb{E}_{H,\eta} \left[\sum_k \left(e_{j,k,p}^2 - 2^{-j-p+1} \text{Var}_{H,\eta}(\log \xi^2) \right)^2 \right] \\ &\leq C \sum_k \mathbb{E}_{H,\eta} [e_{j,k,p}^4] + 2^{-2j-2p+2} \text{Var}_{H,\eta}(\log \xi^2). \end{aligned}$$

But $e_{j,k,p} = 2^{-p-j/2} \sum_{l=0}^{2 \cdot 2^p - 1} \pm \log \left(\xi_{j,p,k2^p+l}^2 \right)$ and the $\xi_{j,p,k2^p+l}^2$ are independent. Thus BDG inequality yields

$$\mathbb{E}_{H,\eta} [e_{j,k,p}^4] \leq 2^{-4p-2j} \mathbb{E}_{H,\eta} \left[\left(\sum_{l=0}^{2 \cdot 2^p - 1} \log \left(\xi_{j,p,k2^p+l}^2 \right) \right)^2 \right] \leq C 2^{-2p-2j}.$$

We obtain finally

$$\mathbb{E}_{H,\eta} \left[\left(\sum_k e_{j,k,p}^2 - 2^{-j-p+1} \text{Var}_{H,\eta}(\log \xi^2) \right)^2 \right] \leq 2^{-2p-j}$$

We now focus on the other term $\mathbb{E}_{H,\eta} \left[\left(\sum_k d_{j,p,k} e_{j,k,p} \right)^2 \right]$. Note that d and e are independent. Moreover, the random variables $e_{j,k,p}$ are centred and $e_{j,k_1,p}$ is independent of $e_{j,k_2,p}$ if $|k_1 - k_2| \geq 2$. Therefore, working first conditionally on d , we get that

$$\begin{aligned} \mathbb{E}_{H,\eta} \left[\left(\sum_k d_{j,p,k} e_{j,k,p} \right)^2 \right] &\leq C \sum_k \mathbb{E}_{H,\eta} [d_{j,p,k}^2 e_{j,k,p}^2] \\ &\leq C \sum_k \mathbb{E}_{H,\eta} [d_{j,p,k}^2] \mathbb{E}_{H,\eta} [e_{j,k,p}^2] \\ &\leq C 2^j 2^{-j(2H+1)} 2^{-j-p} \leq C 2^{-j(2H+1)-p} \end{aligned}$$

using Lemma 26

9.3. **Proof of Proposition 28.** Since $H \in [H_-, H_+]$ and $t \mapsto 2^{-2t}$ is invertible on $(0, 1)$ with inverse uniformly Lipschitz on the compact sets of $(0, 1)$, it is enough to prove that $\frac{Q_{J_n^*+1, N-J_n^*-1, n}}{Q_{J_n^*, N-J_n^*-1, n}} - 2^{-2H}$ is bounded in probability uniformly over \mathcal{D} .

First, notice that

$$\left| \frac{\widehat{Q}_{J_n^*+1, N-J_n^*-1, n}}{\widehat{Q}_{J_n^*, N-J_n^*-1, n}} - 2^{-2H} \right| \leq B_n^{(1)} + V_n^{(1)} + V_n^{(2)}$$

where

$$\begin{aligned} B_n^{(1)} &= \left| \frac{Q_{J_n^*+1, N-J_n^*-1}}{Q_{J_n^*, N-J_n^*-1}} - 2^{-2H} \right|, \\ V_n^{(1)} &= \left| \frac{\widehat{Q}_{J_n^*+1, N-J_n^*-1, n} - Q_{J_n^*+1, N-J_n^*-1}}{\widehat{Q}_{J_n^*, N-J_n^*-1, n}} \right|, \end{aligned}$$

$$V_n^{(2)} = \left| \frac{Q_{J_n^*+1, N-J_n^*-1}(\widehat{Q}_{J_n^*, N-J_n^*-1, n} - Q_{J_n^*, N-J_n^*-1})}{\widehat{Q}_{J_n^*, N-J_n^*-1, n} Q_{J_n^*, N-J_n^*-1}} \right|.$$

We now aim at proving that $(v_n^{(0)})^{-1} B_n^{(1)}$, $(v_n^{(0)})^{-1} V_n^{(1)}$ and $(v_n^{(0)})^{-1} V_n^{(2)}$ are bounded in probability uniformly over \mathcal{D} .

Preliminary: Behaviour of J_n^ .* For any $\varepsilon > 0$, let

$$J_n^-(\varepsilon) = \max(j : r_-(\varepsilon) 2^{-2jH} \geq 2^j n^{-1}).$$

Notice that $J_n^-(\varepsilon) = \max(j : r_-(\varepsilon) n \geq 2^{j(2H+1)})$ so that

$$(35) \quad \frac{1}{2} (r_-(\varepsilon) n)^{1/(2H+1)} \leq 2^{J_n^-(\varepsilon)} \leq (r_-(\varepsilon) n)^{1/(2H+1)}.$$

We will show that for $\varepsilon > 0$ fixed, there exists $L(\varepsilon) > 0$ and $\varphi_n(\varepsilon) \rightarrow 0$ such that

$$(36) \quad \sup_{H, \eta} \mathbb{P}_{H, \eta}(J_n^* < J_n^-(\varepsilon) - L(\varepsilon)) \leq \varepsilon + \varphi_n(\varepsilon)$$

Let L to be chosen later. We write $r = r_-(\varepsilon)$ and $p = N - j - 1$ when the context is clear. We also write $J_n^- = J_n^-(\varepsilon)$ for conciseness.

$$\begin{aligned} \mathbb{P}_{H, \eta}(J_n^* < J_n^- - L) &\leq \mathbb{P}_{H, \eta}(\widehat{Q}_{J_n^- - L} < 2^{J_n^- - L} n^{-1}) \\ &\leq \mathbb{P}_{H, \eta}(\widehat{Q}_{J_n^- - L} - Q_{J_n^- - L} < 2^{J_n^- - L} n^{-1} - Q_{J_n^- - L}) \\ &\leq \mathbb{P}_{H, \eta}(\widehat{Q}_{J_n^- - L} - Q_{J_n^- - L} < 2^{J_n^- - L} n^{-1} - r 2^{-2(J_n^- - L)H}) + \varepsilon \end{aligned}$$

since $\mathbb{P}_{H, \eta}(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j, N-j-1} \leq r) \leq \varepsilon$ by Proposition 7. Moreover, note that (35) yields

$$\begin{aligned} 2^{J_n^- - L} n^{-1} - r 2^{-2(J_n^- - L)H} &\leq (r_-(\varepsilon) n)^{1/(2H+1)} 2^{-L} n^{-1} - r_-(\varepsilon) 2^{2(L+1)H} (r_-(\varepsilon) n)^{-2H/(2H+1)} \\ &= r_-(\varepsilon)^{1/(2H+1)} n^{-2H/(2H+1)} (2^{-L} - 2^{2(L+1)H}). \end{aligned}$$

For L large enough, $2^{-L} - 2^{2(L+1)H} \leq -1$ so that we get by Proposition 27, Markov inequality and Equation (35)

$$\begin{aligned} &\mathbb{P}_{H, \eta}(\widehat{Q}_{J_n^- - L} - Q_{J_n^- - L} < 2^{J_n^- - L} n^{-1} - r 2^{-2(J_n^- - L)H}) \\ &\leq C(n^{-2} 2^{J_n^-} + 2^{-2H J_n^-} n^{-1}) r_-(\varepsilon)^{-2/(2H+1)} n^{4H/(2H+1)} (2^{-L} - 2^{2(L+1)H})^{-2} \\ &\leq C(n^{-2/(2H+1)} 2^{J_n^-} + 2^{-2H J_n^-} n^{(2H-1)/(2H+1)}) \\ &\leq C(n^{-1/(2H+1)} + n^{-1/(2H+1)}) \end{aligned}$$

which proves (36).

Term $B_n^{(1)}$: By (36) and Proposition 7, $\mathbb{P}_{H, \eta}((v_n^{(0)})^{-1} B_n^{(1)} \geq M)$ is bounded by

$$\begin{aligned} &\mathbb{P}_{H, \eta}((v_n^{(0)})^{-1} B_n^{(1)} \geq M, J_n^* \geq J_n^-(\varepsilon) - L(\varepsilon)) + \mathbb{P}_{H, \eta}(J_n^* < J_n^-(\varepsilon) - L(\varepsilon)) \\ &\leq \sum_{j=J_n^-(\varepsilon) - L(\varepsilon)}^{N-1} \mathbb{P}_{H, \eta}((v_n^{(0)})^{-1} B_n^{(1)} \geq M, J_n^* = j, Q_{j, N-j-1} \geq 2^{-2Hj} r_-(\varepsilon)) + 2\varepsilon + \varphi_n(\varepsilon). \end{aligned}$$

Using the definition of B , the probability appearing in the sum is bounded by

$$\mathbb{P}\left(|Q_{j+1, N-j-1} - 2^{-2H} Q_{j, N-j-1}| \geq M 2^{-2Hj} r_-(\varepsilon) v_n^{(0)}\right)$$

$$\leq M^{-2}2^{4Hj}r_-(\varepsilon)^{-2}(v_n^{(0)})^{-2}\mathbb{E}\left|Q_{j+1,N-j-1} - 2^{-2H}Q_{j,N-j-1}\right|^2.$$

But $\mathbb{E}_{H,\eta}\left|Q_{j+1,N-j-1} - 2^{-2H}Q_{j,N-j-1}\right|^2$ is bounded by a constant times

$$\begin{aligned} & \mathbb{E}_{H,\eta}\left[\left(Q_{j+1,N-j-1} - \sum_{a=1}^S \eta^{2a}2^{-2aH(j+1)}\kappa_{N-j-1,a}(H)\right)^2\right] \\ & + 2^{-4H}\mathbb{E}_{H,\eta}\left[\left(Q_{j,N-j-1} - \sum_{a=1}^S \eta^{2a}2^{-2aHj}\kappa_{N-j-1,a}(H)\right)^2\right] \\ & + \left(\sum_{a=1}^S \eta^{2a}2^{-2aH(j+1)}\kappa_{N-j-1,a}(H) - 2^{-2H}\sum_{a=1}^S \eta^{2a}2^{-2aHj}\kappa_{N-j-1,a}(H)\right)^2. \end{aligned}$$

The two first terms are bounded by $C2^{-j(1+4H)}$ by Proposition 4, and the last term equals

$$\left(\sum_{a=2}^S \eta^{2a}\kappa_{N-j-1,a}(H)2^{-2aHj}(2^{-2aH} - 2^{-2H})\right)^2$$

which is bounded by $C2^{-6HN-2jH}$ by Equation (6). Thus $\mathbb{P}_{H,\eta}\left((v_n^{(0)})^{-1}B_n^{(1)} \geq M\right)$ is bounded by

$$\begin{aligned} & CM^{-2}r_-(\varepsilon)^{-2}(v_n^{(0)})^{-2}\sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1}2^{4Hj}(2^{-j(1+4H)} + 2^{-6HN-2jH}) + 2\varepsilon + \varphi_n(\varepsilon) \\ & \leq C(\varepsilon)M^{-2}(v_n^{(0)})^{-2}\left(2^{-J_n^-(\varepsilon)} + 2^{-4HN}\right) + 2\varepsilon + \varphi_n(\varepsilon) \end{aligned}$$

where the notation $C(\varepsilon)$ is used to precise explicitly that the constant C may depend on ε . But $2^{-J_n^-(\varepsilon)} \leq C(\varepsilon)n^{-1/(2H+1)}$ so we can conclude since $n^{-1/(2H+1)}(v_n^{(0)})^{-2}$ and $n^{-4H}(v_n^{(0)})^{-2}$ are bounded sequences by definition of $v_n^{(0)}$.

Term $V_n^{(1)}$: We now deal with $V_n^{(1)}$. By definition, recall that $\widehat{Q}_{J_n^*,N-J_n^*-1,n} \geq 2^{J_n^*}n^{-1}$, at least when $J_n^* \geq J_n^-(\varepsilon) - L(\varepsilon)$. Therefore $\mathbb{P}_{H,\eta}\left((v_n^{(0)})^{-1}V_n^{(1)} \geq M\right)$ is bounded by

$$\sum_{j=J_n^-(\varepsilon)-L(\varepsilon)}^{N-1}\mathbb{P}_{H,\eta}\left((v_n^{(0)})^{-1}|\widehat{Q}_{j+1,N-j-1,n} - Q_{j+1,N-j-1}| \geq M2^j n^{-1}\right) + \varepsilon + \varphi_n(\varepsilon)$$

by Equation (36) By Proposition 27, the probability in the sum is bounded from beyond by a constant times

$$M^{-2}2^{-2j}n^2v_n^{(0)-2}\left(n^{-2}2^j + n^{-1}2^{-2jH}\right) \leq M^{-2}v_n^{(0)-2}\left(2^{-j} + n2^{-2j(H+1)}\right).$$

Since $L(\varepsilon)$ is fixed, we deduce that

$$\mathbb{P}_{H,\eta}\left((v_n^{(0)})^{-1}V_n^{(1)} \geq M\right) \leq M^{-2}v_n^{(0)-2}\left(2^{-J_n^-(\varepsilon)} + n2^{-2J_n^-(\varepsilon)(H+1)}\right) + \varepsilon + \varphi_n(\varepsilon).$$

But recall from Equation (35) that $2^{-J_n^-(\varepsilon)} \leq Cn^{-1/(2H+1)}$ so that

$$\mathbb{P}_{H,\eta}\left((v_n^{(0)})^{-1}V_n^{(1)} \geq M\right) \leq CM^{-2}v_n^{(0)-2}n^{-1/(2H+1)} + \varepsilon + \varphi_n(\varepsilon)$$

and we conclude with tightness using that $v_n^{(0)-2}n^{-1/(2H+1)}$ is bounded.

Term $V_n^{(2)}$: By definition, we have

$$V_n^{(2)} = \frac{Q_{J_n^*+1, N-J_n^*-1}}{Q_{J_n^*, N-J_n^*-1}} \times \left| \frac{(\widehat{Q}_{J_n^*, N-J_n^*-1, n} - Q_{J_n^*, N-J_n^*-1})}{\widehat{Q}_{J_n^*, N-J_n^*-1, n}} \right|.$$

We show that $v_n^{(0)-1} \left| \frac{(\widehat{Q}_{J_n^*, N-J_n^*-1, n} - Q_{J_n^*, N-J_n^*-1})}{\widehat{Q}_{J_n^*, N-J_n^*-1, n}} \right|$ is bounded in probability uniformly over \mathcal{D} by applying readily the same proof as for $V_n^{(1)}$. Therefore, it is enough to show that $\frac{Q_{J_n^*+1, N-J_n^*-1}}{Q_{J_n^*, N-J_n^*-1}}$ is bounded in probability uniformly over \mathcal{D} . Indeed, we have

$$\begin{aligned} \mathbb{P}_{H, \eta} \left(\frac{Q_{J_n^*+1, N-J_n^*-1}}{Q_{J_n^*, N-J_n^*-1}} \geq M \right) &= \mathbb{P}_{H, \eta} \left(Q_{J_n^*+1, N-J_n^*-1} \geq M Q_{J_n^*, N-J_n^*-1} \right) \\ &\leq \mathbb{P}_{H, \eta} \left(\sup_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j+1, N-j-1} \geq M r_-(\varepsilon) \right) + 2\varepsilon + \varphi_n(\varepsilon) \end{aligned}$$

by Proposition 7 and Equation (36). We can conclude since whenever $M r_-(\varepsilon) \geq r_+(\varepsilon)$, we have $\mathbb{P}_{H, \eta} \left(\sup_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j+1, N-j-1} \geq M r_-(\varepsilon) \right) \leq \varepsilon$ by Proposition 7.

9.4. Proof of Proposition 30. Since $\eta \in [\eta_-, \eta_+]$ and $t \mapsto t^2$ is invertible on $(0, \infty)$ with inverse uniformly Lipschitz on the compact sets of $(0, \infty)$, it is enough to prove that $\widehat{\eta}_n^2 - \eta^2$ is bounded in probability uniformly over \mathcal{D} .

First, notice that

$$|\widehat{\eta}_n^2 - \eta^2| \leq \kappa_{N-\widehat{j}_n, 1}^{-1}(\widehat{H}_n) (B_n^{(1)} + B_n^{(2)} + V_n^{(1)} + V_n^{(2)}) 2^{2\widehat{j}_n(\widehat{H}_n - H)}$$

where

$$\begin{aligned} B_n^{(1)} &= 2^{2\widehat{j}_n H} \left| \sum_{a=2}^S \eta^{2a} 2^{-2aH\widehat{j}_n} \kappa_{N-\widehat{j}_n, a}(H) \right|, \\ B_n^{(2)} &= 2^{2\widehat{j}_n H} \left| Q_{\widehat{j}_n, N-\widehat{j}_n} - \sum_{a=1}^S \eta^{2a} 2^{-2aH\widehat{j}_n} \kappa_{N-\widehat{j}_n, a}(H) \right|, \\ V_n^{(1)} &= 2^{2\widehat{j}_n H} \left| \eta^2 2^{-2H\widehat{j}_n} \kappa_{N-\widehat{j}_n, 1}(H) - \eta^2 2^{-2\widehat{j}_n \widehat{H}_n} \kappa_{N-\widehat{j}_n, 1}(\widehat{H}_n) \right|, \\ V_n^{(2)} &= 2^{2\widehat{j}_n H} \left| \widehat{Q}_{\widehat{j}_n, N-\widehat{j}_n, n} - Q_{\widehat{j}_n, N-\widehat{j}_n} \right| \end{aligned}$$

The term $\kappa_{N-\widehat{j}_n, 1}^{-1}(\widehat{H}_n)$ disappears because $\kappa_{p, 1}$ is a continuous function bounded away from 0 on $[H_-, H_+]$, see Equation (5). We now prove that $2^{2\widehat{j}_n(\widehat{H}_n - H)}$, $(w_n^{(0)})^{-1} B_n^{(1)}$, $(w_n^{(0)})^{-1} B_n^{(2)}$, $(w_n^{(0)})^{-1} V_n^{(1)}$ and $(w_n^{(0)})^{-1} V_n^{(2)}$ are bounded in $\mathbb{P}_{H, \eta}$ -probability uniformly over \mathcal{D} .

Term $2^{2\widehat{j}_n(\widehat{H}_n - H)}$: We want to prove that $2^{2\widehat{j}_n(\widehat{H}_n - H)}$ is bounded in probability uniformly over \mathcal{D} . It is enough to prove this for $\widehat{j}_n(\widehat{H}_n - H)$. But with probability converging to 1, $\widehat{j}_n \geq \lfloor \frac{1}{2H+1} \log_2(n) \rfloor - 1$ by Equation (36) so that

$$\mathbb{P}_{H, \eta} (\widehat{j}_n |\widehat{H}_n - H| \geq M) \leq \mathbb{P}_{H, \eta} (\lfloor \frac{1}{2H+1} \log_2(n) \rfloor - 1 |\widehat{H}_n - H| \geq M) + \varepsilon.$$

But $v_n^{(0)-1} |\widehat{H}_n - H|$ is bounded in probability and $\lfloor \frac{1}{2H+1} \log_2(n) \rfloor v_n^{(0)} \rightarrow 0$ (and $v_n^{(0)} \rightarrow 0$) deterministically so we can conclude.

Term $B_n^{(1)}$: By Equation (6), we know that

$$2^{2\hat{j}_n H} \left| \sum_{a=2}^S \eta^{2a} 2^{-2aH\hat{j}_n} \kappa_{N-\hat{j}_n, a}(H) \right| \leq C 2^{-2H\hat{j}_n - 3H(N-\hat{j}_n)}$$

so we need to prove that $(w_n^{(0)})^{-1} 2^{H\hat{j}_n} n^{-3H}$ is uniformly tight. Recall from Corollary 29 that

$$\mathbb{P}_{H, \eta} \left(\hat{j}_n \in \left\{ \left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor - 1, \left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor \right\} \right) \rightarrow 1$$

uniformly on \mathcal{D} . Moreover

$$(w_n^{(0)})^{-1} 2^{H \left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor} n^{-3H} \leq C (w_n^{(0)})^{-1} 2^{\frac{H}{2H+1} \log_2(n)} n^{-3H} = C (w_n^{(0)})^{-1} n^{3H - \frac{H}{2H+1}}^{-1}$$

which is bounded since $w_n^{(0)} n^{3H - \frac{H}{2H+1}} = v_n^{(0)} n^{\frac{6H^2+2H}{2H+1}} \log(n)$ is bounded below (since $v_n^{(0)} \geq n^{-2H}$), and thus it proves that $(w_n^{(0)})^{-1} 2^{H\hat{j}_n} n^{-3H}$ is uniformly tight.

Term $B_n^{(2)}$: By Corollary 29, $\mathbb{P}_{H, \eta}((w_n^{(0)})^{-1} B_n^{(2)} \geq M)$ is bounded by

$$\begin{aligned} & \mathbb{P}_{H, \eta} \left(\left| Q_{\hat{j}_n, N-\hat{j}_n} - \sum_{a=1}^S \eta^{2a} 2^{-2aH\hat{j}_n} \kappa_{N-\hat{j}_n, a}(H) \right| \geq M w_n^{(0)} 2^{-2\hat{j}_n H} \right) \\ & \leq \sum_{\substack{j = \left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor - r, \\ r \in \{0, 1\}}} \mathbb{P}_{H, \eta} \left(\left| Q_{j, N-j} - \sum_{a=1}^S \eta^{2a} 2^{-2aHj} \kappa_{p, a}(H) \right| \geq M w_n^{(0)} 2^{-2jH} \right) + o(1) \end{aligned}$$

where the o is uniform over \mathcal{D} . Moreover, by Proposition 4, it is bounded by

$$C \sum_j M^{-2} (w_n^{(0)})^{-2} 2^{-j} + o(1) \leq C M^{-2} (w_n^{(0)})^{-2} n^{-1/(2H+1)} + o(1)$$

and we conclude with tightness using that $(w_n^{(0)})^{-2} n^{-1/(2H+1)}$ is bounded.

Term $V_n^{(1)}$: For each $j, p, t \mapsto 2^{-2tj} \kappa_{p, 1}(t)$ is differentiable with derivative

$$t \mapsto 2^{-2tj} (-2j \log(2) \kappa_{p, 1}(t) + \kappa'_{p, 1}(t)).$$

By Lemma 6, its absolute value is bounded by $C(j+1)2^{-2tj}$ for some constant C . Therefore, Taylor's formula yields that

$$V_n^{(1)} \leq C(\hat{j}_n + 1) |\hat{H}_n - H| \leq C \log(n) |\hat{H}_n - H|.$$

Moreover, $v_n^{(0)^{-1}} |\hat{H}_n - H|$ is bounded in probability uniformly over \mathcal{D} so $\log(n)^{-1} v_n^{(0)^{-1}} V_n^{(1)}$ is also bounded in probability uniformly over \mathcal{D} .

Term $V_n^{(2)}$: By Corollary 29,

$$\mathbb{P}_{H, \eta} \left((w_n^{(0)})^{-1} V_n^{(2)} \geq M \right) \leq \sum_{\substack{j = \left\lfloor \frac{1}{2H+1} \log_2(n) \right\rfloor - r, \\ r \in \{0, 1\}}} \mathbb{P}_{H, \eta} \left(\left| \hat{Q}_{j, N-j, n} - Q_{j, N-j} \right| \geq M 2^{-2jH} w_n^{(0)} \right) + o(1)$$

where the o is uniform over \mathcal{D} and by Proposition 27, the probability in the sum is bounded from beyond by a constant times

$$M^{-2} w_n^{(0)^{-2}} (n^{-2} 2^{j(1+4H)} + n^{-1} 2^{2jH}) \leq C M^{-2} w_n^{(0)^{-2}} (n^{-2} n^{(1+4H)/(2H+1)} + n^{-1} n^{2H/(2H+1)})$$

$$= CM^{-2}w_n^{(0)-2}n^{-1/(2H+1)} \rightarrow 0.$$

9.5. **Proof of Proposition 31.** Proceeding as for Proposition 28, we only need to show that

$$(v_n^c(H))^{-1} \left| \frac{\widehat{Q}_{J_n^{*c}+1, N-J_n^{*c}-1, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n)}{\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n)} - 2^{-2H} \right|$$

is bounded in $\mathbb{P}_{H, \eta}$ -probability uniformly over \mathcal{D} . We will use the same scheme of proof though additional care needs to be taken since we cannot use the same Markov estimates because of the use of the first-hand estimator \widehat{H}_n to finetune the procedure. In the following, we will write $J_n^{*c} = J_n^{*c}(\widehat{H}_n, \widehat{\eta}_n)$. Note that we have the following decomposition

$$\left| \frac{\widehat{Q}_{J_n^{*c}+1, N-J_n^{*c}-1, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n)}{\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n)} - 2^{-2H} \right| \leq B_n + V_n^{(1)} + V_n^{(2)}$$

where

$$\begin{aligned} B_n &= \left| \frac{Q_{J_n^{*c}+1, N-J_n^{*c}-1}^c(H, \eta)}{Q_{J_n^{*c}, N-J_n^{*c}-1}^c(H, \eta)} - 2^{-2H} \right| \\ V_n^{(1)} &= \left| \frac{\widehat{Q}_{J_n^{*c}+1, N-J_n^{*c}-1, n}^c(\widehat{H}_n, \widehat{\eta}_n) - Q_{J_n^{*c}+1, N-J_n^{*c}-1}^c(H, \eta)}{\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^c(\widehat{H}_n, \widehat{\eta}_n)} \right| \\ V_n^{(2)} &= \left| \frac{Q_{J_n^{*c}+1, N-J_n^{*c}-1}^c(H, \eta) (\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^c(\widehat{H}_n, \widehat{\eta}_n) - Q_{J_n^{*c}, N-J_n^{*c}-1}^c(H, \eta))}{\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^c(\widehat{H}_n, \widehat{\eta}_n) Q_{J_n^{*c}, N-J_n^{*c}-1}^c(H, \eta)} \right| \end{aligned}$$

and we want to prove that $v_n^{c-1}B_n$, $v_n^{c-1}V_n^{(1)}$ and $v_n^{c-1}V_n^{(2)}$ are bounded in probability uniformly over \mathcal{D} .

*Preliminary: Behaviour of J_n^{*c} .* We fix $\varepsilon > 0$ and we define

$$J_n^{-c}(\varepsilon) = J_n^{-c}(\varepsilon) = \max(j : r_-^{(S)}(\varepsilon) 2^{-2jH} \geq 2^j n^{-1})$$

where $r_-^{(S)}(\varepsilon)$ is defined in Proposition 8. Notice that $J_n^{-c}(\varepsilon)$ is independent of η and H since in Proposition 8, $r_-^{(S)}(\varepsilon)$ is defined uniformly for all η and H . As for Equation (35),

$$(37) \quad \frac{1}{2} (r_-^{(S)}(\varepsilon) n)^{1/(2H+1)} \leq 2^{J_n^{-c}(\varepsilon)} \leq (r_-^{(S)}(\varepsilon) n)^{1/(2H+1)}.$$

We will show that there exists $L^c(\varepsilon) > 0$ and $\varphi_n^c(\varepsilon) \rightarrow 0$ such that

$$(38) \quad \sup_{H, \eta} \mathbb{P}_{H, \eta}(J_n^{*c} < J_n^{-c}(\varepsilon) - L^c(\varepsilon)) \leq \varepsilon + \varphi_n^c(\varepsilon)$$

Let L to be chosen later. We write $r = r_-^{(S)}(\varepsilon)$ and $p = N - j - 1$ when the context is clear. We also write $J_n^{-c} = J_n^{-c}(\varepsilon)$ for conciseness. By definition, $\mathbb{P}_{H, \eta}(J_n^{*c} < J_n^{-c} - L)$ is bounded by

$$\mathbb{P}_{H, \eta}(\widehat{Q}_{J_n^{-c}-L, N-J_n^{-c}+L, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n) - Q_{J_n^{-c}-L, N-J_n^{-c}+L}^c(H, \eta) < \frac{2^{J_n^{-c}-L}}{n} - Q_{J_n^{-c}-L, N-J_n^{-c}+L}^{(S)}(H, \eta)).$$

But $\mathbb{P}_{H, \eta}(\inf_{J_0 \leq j \leq N-1} 2^{2jH} Q_{j, N-j-1}^{(S)}(H, \eta) \leq r) \leq \varepsilon$ by Proposition 8, so it is also bounded by

$$\mathbb{P}_{H, \eta}(\widehat{Q}_{J_n^{-c}-L, N-J_n^{-c}+L, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n) - Q_{J_n^{-c}-L, N-J_n^{-c}+L}^c(H, \eta) < 2^{J_n^{-c}-L}/n - r 2^{-2(J_n^{-c}-L)H}) + \varepsilon.$$

Moreover, note that (37) yields

$$\begin{aligned} 2^{J_n^{-c}-L}n^{-1} - r2^{-2(J_n^{-c}-L)H} &\leq (r_-^{(S)}(\varepsilon)n)^{1/(2H+1)}2^{-L}n^{-1} - r_-^{(S)}(\varepsilon)2^{2(L+1)H}(r_-^{(S)}(\varepsilon)n)^{-2H/(2H+1)} \\ &= r_-^{(S)}(\varepsilon)^{1/(2H+1)}n^{-2H/(2H+1)}(2^{-L} - 2^{2(L+1)H}). \end{aligned}$$

For L large enough, $2^{-L} - 2^{2(L+1)H} \leq -2$ so that we only need to prove that the probabilities

$$\begin{aligned} &\mathbb{P}_{H,\eta}(\widehat{Q}_{J_n^{-c}-L, N-J_n^{-c}+L, n} - \widehat{Q}_{J_n^{-c}-L, N-J_n^{-c}+L, n} < -r_-^{(S)}(\varepsilon)^{1/(2H+1)}n^{-2H/(2H+1)}) \\ &\text{and } \mathbb{P}_{H,\eta}(B_{J_n^{-c}-L, N-J_n^{-c}+L}^{(S)}(\widehat{H}_n, \widehat{\eta}_n) - B_{J_n^{-c}-L, N-J_n^{-c}+L}^{(S)}(H, \eta) < -r_-^{(S)}(\varepsilon)^{1/(2H+1)}n^{-2H/(2H+1)}) \end{aligned}$$

converge to 0 uniformly on \mathcal{D} . The first convergence is proven as in the preliminary of the proof of Proposition 28. We deal with the second convergence using Lemma 9. The probability considered is bounded by

$$\begin{aligned} &\mathbb{P}_{H,\eta}(c_B 2^{-4(\widehat{H} \wedge H)(J_n^{-c}-L)}((J_n^{-c}-L)|\widehat{H}-H| + |\widehat{\eta}-\eta|) \geq r_-^{(S)}(\varepsilon)^{1/(2H+1)}n^{-2H/(2H+1)}) \\ &\leq \mathbb{P}_{H,\eta}(2^{-4(\widehat{H} \wedge H)J_n^{-c}}(\log(n)|\widehat{H}-H| + |\widehat{\eta}-\eta|) \geq n^{-2H/(2H+1)}\widetilde{c}_B^{-1}) \end{aligned}$$

where $\widetilde{c}_B = r_-^{(S)}(\varepsilon)^{-1/(2H+1)}c_B 2^{4H+L}$. But for any $\widetilde{\varepsilon} > 0$, we have $\mathbb{P}_{H,\eta}(v_n^{-1}|\widehat{H}_n - H| \geq \widetilde{M}) \leq \widetilde{\varepsilon}$ for \widetilde{M} large enough, so the last probability is bounded by

$$\mathbb{P}_{H,\eta}(2^{-4(H+\widetilde{M}v_n)J_n^{-c}}(\log(n)|\widehat{H}-H| + |\widehat{\eta}-\eta|) \geq n^{-2H/(2H+1)}\widetilde{c}_B^{-1}) + \widetilde{\varepsilon}.$$

We conclude here using first that $v_n J_n^{-c} \leq v_n \log(n) \rightarrow 0$, then $2^{-4H J_n^{-c}}$ is of the same order as $n^{-4H/(2H+1)}$ and finally that $\log(n)n^{-2H/(2H+1)}|\widehat{H}-H|$ and $n^{-2H/(2H+1)}|\widehat{\eta}-\eta|$ converge to 0.

Term B_n : We rewrite B_n as

$$\frac{\left| Q_{J_n^{*c}+1, N-J_n^{*c}-1} - 2^{-2H} Q_{J_n^{*c}, N-J_n^{*c}-1} - B_{J_n^{*c}+1, N-J_n^{*c}-1}^{(S)}(H, \eta) + 2^{-2H} B_{J_n^{*c}, N-J_n^{*c}-1}^{(S)}(H, \eta) \right|}{Q_{J_n^{*c}, N-J_n^{*c}-1}^c(H, \eta)}.$$

Notice that for $a = 1$,

$$\eta^{2a} 2^{-2aH(J_n^{*c}+1)} \kappa_{N-J_n^{*c}-1, a}(H) = 2^{-2H} \eta^{2a} 2^{-2aH J_n^{*c}} \kappa_{N-J_n^{*c}-1, a}(H)$$

so by definition of $B_{j,p}^{(S)}$, $B_n \leq B_n^{(1)} + 2^{-2H} B_n^{(2)}$ where

$$B_n^{(1)} := \frac{\left| Q_{J_n^{*c}+1, N-J_n^{*c}-1} - \sum_{a=1}^S \eta^{2a} 2^{-2aH(J_n^{*c}+1)} \kappa_{N-J_n^{*c}-1, a}(H) \right|}{Q_{J_n^{*c}, N-J_n^{*c}-1}^c(H, \eta)}$$

$$\text{and } B_n^{(2)} := \frac{\left| Q_{J_n^{*c}, N-J_n^{*c}-1} - \sum_{a=1}^S \eta^{2a} 2^{-2aH J_n^{*c}} \kappa_{N-J_n^{*c}-1, a}(H) \right|}{Q_{J_n^{*c}, N-J_n^{*c}-1}^c(H, \eta)}.$$

Both terms are controlled identically so we only prove here that $v_n^{c-1} B_n^{(1)}$ is bounded in probability uniformly over \mathcal{D} .

By Equation (38) and Proposition 8, $\mathbb{P}_{H,\eta}(v_n^{c-1} B_n^{(1)} \geq M)$ is bounded by

$$\begin{aligned} &\mathbb{P}_{H,\eta}(v_n^{c-1} B_n^{(1)} \geq M, J_n^{*c} \geq J_n^{-c}(\varepsilon) - L^c(\varepsilon)) + \mathbb{P}_{H,\eta}^n(J_n^{*c} < J_n^{-c}(\varepsilon) - L^c(\varepsilon)) \\ &\leq \sum_{j=J_n^{-c}(\varepsilon)-L^c(\varepsilon)}^{N-1} \mathbb{P}_{H,\eta}(v_n^{c-1} B_n^{(1)} \geq M, J_n^* = j, Q_{j, N-j-1}^c(H, \eta) \geq 2^{-2Hj} r_-^{(S)}(\varepsilon)) + 2\varepsilon + \varphi_n(\varepsilon). \end{aligned}$$

Using the definition of $B_n^{(1)}$, the probability in the sum is bounded by

$$\mathbb{P}_{H,\eta} \left(\left| Q_{j+1,N-j-1} - \sum_{a=1}^S \eta^{2a} 2^{-2aH(j+1)} \kappa_{N-j-1,a}(H) \right| \geq M 2^{-2Hj} r_-^{(S)}(\varepsilon) v_n^c \right).$$

By Markov inequality and Proposition 4, this is bounded by $CM^{-2} 2^{-j} r_-^{(S)}(\varepsilon)^{-2} v_n^{c-2}$ and summing over j yields

$$\mathbb{P}_{H,\eta} \left(v_n^{c-1} B_n^{(1)} \geq M \right) \leq CM^{-2} 2^{-J_n^{-c}(\varepsilon) + L^c(\varepsilon)} r_-^{(S)}(\varepsilon)^{-2} v_n^{c-2} + 2\varepsilon + \varphi_n(\varepsilon).$$

We conclude using that $v_n^{c-2} 2^{-J_n^{-c}(\varepsilon)}$ is bounded (by Equation (37)).

Term $V_n^{(1)}$. By definition of $V_n^{(1)}$, we have

$$\begin{aligned} V_n^{(1)} &= \left| \frac{\widehat{Q}_{J_n^{*c}+1, N-J_n^{*c}-1, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n) - Q_{J_n^{*c}+1, N-J_n^{*c}-1}^c(H, \eta)}{\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n)} \right| \\ &\leq \frac{|\widehat{Q}_{J_n^{*c}+1, N-J_n^{*c}-1, n}^{(S)}(\widehat{H}_n, \widehat{\eta}_n) - Q_{J_n^{*c}+1, N-J_n^{*c}-1}^c(H, \eta)|}{\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^c(\widehat{H}_n, \widehat{\eta}_n)} \\ &\quad \frac{|B_{J_n^{*c}+1, N-J_n^{*c}-1}^{(S)}(\widehat{H}_n, \widehat{\eta}_n) - B_{J_n^{*c}+1, N-J_n^{*c}-1}^c(H, \eta)|}{\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^c(\widehat{H}_n, \widehat{\eta}_n)}. \end{aligned}$$

Moreover, $\widehat{Q}_{J_n^{*c}, N-J_n^{*c}-1, n}^c(\widehat{H}_n, \widehat{\eta}_n) \geq 2^{J_n^*} n^{-1}$, at least when $J_n^{*c} \geq J_n^{-c}(\varepsilon) - L^c(\varepsilon)$. It happens at least with probability $1 - \varepsilon - \varphi_n^c(\varepsilon)$ by Equation (38) so it is enough to prove that

$$(39) \quad nv_n^{c-1} 2^{-J_n^{*c}} |\widehat{Q}_{J_n^{*c}+1, N-J_n^{*c}-1, n} - Q_{J_n^{*c}+1, N-J_n^{*c}-1}^c|$$

and

$$(40) \quad nv_n^{c-1} 2^{-J_n^{*c}} |B_{J_n^{*c}+1, N-J_n^{*c}-1}(\widehat{H}_n, \widehat{\eta}_n) - B_{J_n^{*c}+1, N-J_n^{*c}-1}^c(H, \eta)|$$

are bounded in probability uniformly over \mathcal{D} , conditionally to $J_n^{*c} \geq J_n^{-c}(\varepsilon) - L^c(\varepsilon)$.

The term (39) is similar to $V_n^{(1)}$ appearing in the proof of Proposition 28. Indeed, by Proposition 27, we have

$$\begin{aligned} &\mathbb{P}_{H,\eta} \left(nv_n^{c-1} 2^{-J_n^{*c}} |\widehat{Q}_{J_n^{*c}+1, N-J_n^{*c}-1, n} - Q_{J_n^{*c}+1, N-J_n^{*c}-1}^c| \geq M, J_n^{*c} \geq J_n^{-c}(\varepsilon) - L^c(\varepsilon) \right) \\ &\leq \sum_{j=J_n^{-c}(\varepsilon) - L^c(\varepsilon)}^{N-1} \mathbb{P}_{H,\eta} \left(|\widehat{Q}_{j+1, N-j-1, n} - Q_{j+1, N-j-1, n}| \geq Mn^{-1} v_n^c 2^j \right) \\ &\leq C \sum_{j=J_n^{-c}(\varepsilon) - L^c(\varepsilon)}^{N-1} (2^j n^{-2} + 2^{-2Hj} n^{-1}) n^2 (Mv_n^c)^{-2} 2^{-2j} \\ &\leq C(\varepsilon) M^{-2} v_n^{c-2} (2^{-J_n^{-c}(\varepsilon)} + n 2^{-2(H+1)J_n^{-c}(\varepsilon)}) \end{aligned}$$

and we conclude using Equation (37) and the definition of v_n^c .

We now focus on the term (40). By Lemma 9 and using $J_n^{*c} \leq \log(n)$, it is bounded by

$$c_B 2^{-4((\widehat{H}_n - H) \wedge 0) \log(n)} \times nv_n^{c-1} 2^{-(4H+1)J_n^{*c}} (\log(n)) |\widehat{H}_n - H| + |\widehat{\eta}_n - \eta|.$$

First, notice that $2^{-4((\widehat{H}_n - H) \wedge 0) \log(n)}$ is bounded in probability uniformly over \mathcal{D} because $v_n^{-1}|\widehat{H}_n - H|$ is uniformly tight and $\log(n)v_n \rightarrow 0$. Therefore we can focus on $nv_n^{c-1}2^{-(4H+1)J_n^{*c}}(\log(n)|\widehat{H}_n - H| + |\widehat{\eta}_n - \eta|)$. Conditionally to $J_n^{*c} \geq J_n^{-c}(\varepsilon) - L^c(\varepsilon)$, it is bounded by

$$Cn \log(n) v_n v_n^{c-1} 2^{-(4H+1)J_n^{-c}(\varepsilon)} (v_n^{-1}|\widehat{H}_n - H| + (\log(n)v_n)^{-1}|\widehat{\eta}_n - \eta|)$$

and we conclude using Equation (37), hypotheses on v_n , \widehat{H}_n and $\widehat{\eta}_n$ and the definition of v_n^c .

Term $V_n^{(2)}$: We deal with this term using the same method as the corresponding term in the proof of Proposition 28, using Proposition 8 and Equation (38) instead of Proposition 7 and Equation (36).

9.6. Proof of Proposition 32. As for Proposition 30, it is enough to prove that $w_n^{c-1}|(\widehat{\eta}_n^c)^2 - \eta^2|$ is bounded in probability uniformly over \mathcal{D} .

First, notice that

$$|(\widehat{\eta}_n^c)^2 - \eta^2| \leq \kappa_{N-\widehat{j}_n,1}^{-1}(\widehat{H}_n)(B_n^{(1)} + B_n^{(2)} + V_n^{(1)} + V_n^{(2)})2^{2\widehat{j}_n(\widehat{H}_n - H)}$$

where

$$B_n^{(1)} = 2^{2\widehat{j}_n H} \left| \sum_{a=2}^S \eta^{2a} 2^{-2aH\widehat{j}_n} \kappa_{N-\widehat{j}_n,a}(H) - \sum_{a=2}^S (\widehat{\eta}_n)^{2a} 2^{-2a\widehat{H}_n\widehat{j}_n} \kappa_{N-\widehat{j}_n,a}(\widehat{H}_n) \right|,$$

$$B_n^{(2)} = 2^{2\widehat{j}_n H} \left| Q_{\widehat{j}_n, N-\widehat{j}_n} - \sum_{a=1}^S \eta^{2a} 2^{-2aH\widehat{j}_n} \kappa_{N-\widehat{j}_n,a}(H) \right|,$$

$$V_n^{(1)} = 2^{2\widehat{j}_n H} \left| \eta^2 2^{-2H\widehat{j}_n} \kappa_{N-\widehat{j}_n,1}(H) - \eta^2 2^{-2\widehat{j}_n\widehat{H}_n} \kappa_{N-\widehat{j}_n,1}(\widehat{H}_n) \right|,$$

$$V_n^{(2)} = 2^{2\widehat{j}_n H} \left| \widehat{Q}_{\widehat{j}_n, N-\widehat{j}_n, n} - Q_{\widehat{j}_n, N-\widehat{j}_n} \right|$$

The term $\kappa_{N-\widehat{j}_n,1}^{-1}(\widehat{H}_n)$ disappears because $\kappa_{p,1}$ is a continuous function bounded away from 0 on $[H_-, H_+]$, see Equation (5). We prove that $2^{2\widehat{j}_n(\widehat{H}_n - H)}$, $(w_n^c)^{-1}B_n^{(1)}$, $(w_n^c)^{-1}B_n^{(2)}$, $(w_n^c)^{-1}V_n^{(1)}$ and $(w_n^c)^{-1}V_n^{(2)}$ are bounded in probability uniformly over \mathcal{D} . Indeed, all these terms except $B_n^{(1)}$ are the same as in the proof of Proposition 30. The same proofs readily apply here for these terms and we conclude using that $w_n^c \geq \log(n)v_n$ and $w_n^{c-2}n^{-1/(2H+1)} \rightarrow 0$. Therefore, we only need to focus on $B_n^{(1)}$.

By Lemma 9, we have

$$\begin{aligned} B_n^{(1)} &= 2^{2\widehat{j}_n H} \left| B_{\widehat{j}_n, N-\widehat{j}_n}^{(S)}(H, \eta) - B_{\widehat{j}_n, N-\widehat{j}_n}^{(S)}(\widehat{\eta}_n, \widehat{H}_n) \right| \\ &\leq c_B 2^{2\widehat{j}_n H} 2^{-4(H \wedge \widehat{H}_n)\widehat{j}_n} (\widehat{j}_n |\widehat{H}_n - H| + |\widehat{\eta}_n - \eta|) \\ &\leq c_B 2^{-4(0 \wedge (\widehat{H}_n - H))\widehat{j}_n} 2^{-2H\widehat{j}_n} (\widehat{j}_n |\widehat{H}_n - H| + |\widehat{\eta}_n - \eta|). \end{aligned}$$

But $v_n^{-1}|\widehat{H}_n - H|$ is bounded in probability uniformly on \mathcal{D} and $v_n \widehat{j}_n \leq v_n \log(n) \rightarrow 0$, so $2^{-4(0 \wedge (\widehat{H}_n - H))\widehat{j}_n}$ is uniformly tight. Thus it is enough to show that $(w_n^c)^{-1}2^{-2H\widehat{j}_n}(\widehat{j}_n |\widehat{H}_n - H| + |\widehat{\eta}_n - \eta|)$ is bounded in probability uniformly on \mathcal{D} . Recall that

$$\mathbb{P}_{H,\eta}(\widehat{j}_n \in \{ \lfloor \frac{1}{2H+1} \log_2(n) \rfloor - 1, \lfloor \frac{1}{2H+1} \log_2(n) \rfloor \}) \rightarrow 1$$

so it is enough to prove that $(w_n^c)^{-1}n^{-2H/(2H+1)}\log(n)|\widehat{H}_n - H|$ and $(w_n^c)^{-1}n^{-2H/(2H+1)}|\widehat{\eta}_n - \eta|$ are uniformly tight. It is the case since $v_n(w_n^c)^{-1}n^{-2H/(2H+1)}\log(n)$ and $w_n(w_n^c)^{-1}n^{-2H/(2H+1)}$ are bounded and $v_n^{-1}|\widehat{H}_n - H|$ and $w_n^{-1}|\widehat{\eta}_n - \eta|$ are uniformly tight.

ACKNOWLEDGMENTS

Mathieu Rosenbaum and Grégoire Szymansky gratefully acknowledge the financial support of the École Polytechnique chairs *Deep Finance and Statistics* and *Machine Learning and Systematic Methods*. Yanghui Liu is supported by the PSC-CUNY Award 64353-00 52.

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APPENDIX A. ASYMPTOTIC EXPANSION OF THE INTEGRATED VOLATILITY

This Section is of independent interest from the rest of this paper and therefore we shall re-define a few notations defined in Section 3.1.

Let \mathcal{D} be a compact subset of $(0, 1) \times (0, \infty)$ and let $T > 0$. We also fix some arbitrary integer constant $S > 0$ and some constant $0 < H^* < \min_{(H, \eta) \in \mathcal{D}} H$. We consider a measurable space (Ω, \mathcal{A}) on which is defined a process $(\sigma_t)_{t \leq T}$ such that under probability $\mathbb{P}_{H, \eta}$ with $(H, \eta) \in \mathcal{D}$, σ is given by

$$\sigma_t^2 = \exp(\eta W_t^H)$$

where W^H is a fractional Brownian motion with Hurst index H . We will write $\mathbb{E}_{H, \eta}$ the expectation under probability $\mathbb{P}_{H, \eta}$.

For any $\alpha > 0$ we also define the best α -Hlder constant of function $f : [0, T] \rightarrow \mathbb{R}$ by

$$\mathcal{H}_\alpha(f) := \sup_{0 \leq s \neq t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\alpha}$$

and we will write $\mathcal{H}_\alpha^H := \mathcal{H}_\alpha(W^H)$ to shorten expressions.

Proposition 33. *There exists random variable Z_0 bounded in $L^2(\mathbb{P}_{H, \eta})$ uniformly on \mathcal{D} such that for any $\delta > 0$ and i such that $(i + 1)\delta \leq T$, then*

$$\begin{aligned} \log \left(\delta^{-1} \int_{i\delta}^{(i+1)\delta} \sigma_u^2 du \right) &= \sum_{b=2}^{2S} \sum_{s=1}^{2S} \frac{(-1)^{s-1}}{s} \sum_{\substack{\mathbf{r} \in \{1, \dots, S\}^s \\ \sum_j \mathbf{r}_j = b}} \prod_{j=1}^s \frac{\eta^{\mathbf{r}_j}}{\mathbf{r}_j!} \frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} (W_u^H - W_{i\delta}^H)^{\mathbf{r}_j} du \\ &\quad + \frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \eta W_u^H du + Z(i, \delta) \cdot \delta^{H^*(S+1)} \end{aligned}$$

where the random variables $Z(i, \delta)$ satisfy $|Z(i, \delta)| \leq Z_0$.

Proof. Recall that for any reals x and a , we have the Taylor expansion

$$\begin{aligned} e^x &= \sum_{r=0}^{2S} \frac{(x-a)^r e^a}{r!} + \frac{(x-a)^{S+1}}{S!} \int_0^1 (1-z)^{2S} e^{a+z(x-a)} dz \\ &= e^a \left(1 + \sum_{r=1}^{2S} \frac{(x-a)^r}{r!} + \frac{(x-a)^{S+1}}{S!} \int_0^1 (1-z)^{2S} e^{z(x-a)} dz \right). \end{aligned}$$

Applying this equality with $x = \eta W_u^H$ and $a = \eta W_{i\delta}^H$, we get

$$e^{\eta W_u^H} = e^{\eta W_{i\delta}^H} \left(1 + \sum_{r=1}^{2S} \frac{\eta^r (W_u^H - W_{i\delta}^H)^r}{r!} + \frac{\eta^{S+1} (W_u^H - W_{i\delta}^H)^{S+1}}{(S+1)!} \int_0^1 (1-z)^{2S} e^{\eta z (W_u^H - W_{i\delta}^H)} dz \right).$$

Notice that since $H > H^*$, $\mathcal{H}_{H^*}^H$ is almost surely finite. Then

$$\left| \eta^{S+1} (W_u^H - W_{i\delta}^H)^{S+1} \int_0^1 (1-z)^{2S} e^{\eta z (W_u^H - W_{i\delta}^H)} dz \right| \leq \eta^{S+1} (\mathcal{H}_{H^*}^H)^{S+1} |u - i\delta|^{H^*(S+1)} \frac{e^{2\eta \|W^H\|_\infty}}{S+1}.$$

Therefore, for $i\delta \leq u \leq (i+1)\delta$, we have

$$(41) \quad e^{\eta W_u^H} = e^{\eta W_{i\delta}^H} \left(1 + \sum_{r=1}^{2S} \frac{\eta^r (W_u^H - W_{i\delta}^H)^r}{r!} + R^{H,S}(u) \delta^{H^*(S+1)} \right)$$

where $|R^{H,S}(u)| \leq \eta^{S+1} (\mathcal{H}^H)^{S+1} |u/\delta - i|^{H^*(S+1)} \frac{e^{2\eta \|W^H\|_\infty}}{(S+1)!} \leq (\eta \mathcal{H}_{H^*}^H)^{S+1} \frac{e^{2\eta \|W^H\|_\infty}}{(S+1)!} = R_0^{H,S}$ is a random variable independent of δ and u .

Then we integrate both sides of (41) and we take the logarithm. This yields to

$$(42) \quad \log \left(\frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \sigma_u^2 du \right) = \eta W_{i\delta}^H + \log \left(1 + \sum_{r=1}^{2S} \frac{\eta^r}{r!} \mathfrak{W}_{i,\delta}^{H,r} + \tilde{R}^{H,S}(i, \delta) \delta^{H^*(S+1)} \right)$$

where

$$\mathfrak{W}_{i,\delta}^{H,r} = \frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} (W_u^H - W_{i\delta}^H)^r du$$

and where $\tilde{R}^{H,S}(i, \delta) = \frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} R^{H,S}(u) du$ is still dominated by R_0^H . Note that this notation differs from \mathfrak{W} introduced in Section 8. We will now expand the logarithm on the right-hand side of (42). Taylor's expansion of the logarithm writes

$$\log(1+x) = \sum_{s=1}^{2S} \frac{(-1)^{s-1}}{s} x^s + (-1)^S \int_0^x \frac{(x-t)^S}{(1+t)^{S+1}} dt.$$

We want to apply this with $x = \sum_{r=1}^{2S} \frac{\eta^r}{r!} \mathfrak{W}_{i,\delta}^{H,r} + \tilde{R}^{H,S}(i, \delta) \delta^{H^*(S+1)}$. Notice that this quantity is indeed independent of S and we also have $x = \sum_{r=1}^{S'} \frac{\eta^r}{r!} \mathfrak{W}_{i,\delta}^{H,r} + \tilde{R}^{H,S'}(i, \delta) \delta^{S'+1}$ for any S' . In particular, with $S' = 0$, $x = \tilde{R}^{H,0}(i, \delta) \delta$. Moreover, we have $1+x = \frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \frac{\sigma_u^2}{\sigma_{i\delta}^2} du$. We obtain

$$(43) \quad \log \left(\frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \sigma_u^2 du \right) = \eta W_{i\delta}^H + \sum_{s=1}^{2S} \frac{(-1)^{s-1}}{s} \left(\sum_{r=1}^{2S} \frac{\eta^r}{r!} \mathfrak{W}_{i,\delta}^{H,r} + \tilde{R}^{H,S}(i, \delta) \delta^{H^*(S+1)} \right)^s + T^H(i, \delta)$$

where $T^H(i, \delta) = (-1)^S \int_0^{\tilde{R}^{H,0}(i, \delta) \delta} \frac{(\tilde{R}^{H,0}(i, \delta) \delta - t)^S}{(1+t)^{S+1}} dt$. Notice in addition that

$$\left| (-1)^S \int_0^x \frac{(x-t)^k}{(1+t)^{k+1}} dt \right| \leq \begin{cases} \frac{|x|^{S+1}}{S+1} & \text{for } x \geq 0, \\ \frac{|x|^{S+1}}{(1+x)^{S+1}(S+1)} & \text{for } x \leq 0 \end{cases}$$

which translates here in

$$\left| T^H(i, \delta) \right| \leq \begin{cases} \frac{|\tilde{R}^{H,0}(i, \delta) \delta|^{S+1}}{S+1} & \text{for } 1+x \geq 0, \\ \frac{|\tilde{R}^{H,0}(i, \delta) \delta|^{S+1}}{\left(\frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \sigma_u^2 du \right)^{S+1} (S+1)} \sigma_{i\delta}^{2(S+1)} & \text{for } 1+x \leq 0. \end{cases}$$

Since $\exp(-\eta \|W^H\|_\infty) \leq \sigma_u^2 \leq \exp(\eta \|W^H\|_\infty)$ and $|\tilde{R}^{H,0}(i, \delta)| \leq R_0^{H,0} = \eta \mathcal{H}_{H^*}^H e^{2\eta \|W^H\|_\infty}$, we deduce that

$$\left| T^H(i, \delta) \right| \leq \frac{|\tilde{R}^{H,0}(i, \delta) \delta|^{S+1}}{S+1} e^{2(S+1)\eta \|W^H\|_\infty} \leq \frac{1}{S+1} \left(\eta \mathcal{H}_{H^*}^H e^{3\eta \|W^H\|_\infty} \delta \right)^{S+1}$$

Recall that $\mathcal{H}_{H^*}^H$ has moments of all orders bounded independently of H and $\|W^H\|_\infty$ has exponential moments of all orders bounded independently of H so $T^H(i, \delta)$ satisfies the condition

required for Z in Proposition 33.

We now focus on the expression $\left(\sum_{r=1}^{2S} \frac{\eta^r}{r!} \mathfrak{W}_{i,\delta}^{H,r} + \tilde{R}^{H,S}(i,\delta) \delta^{H^*(S+1)}\right)^s$. We expand the power s and remove all the terms of order smaller than $\delta^{H^*(S+1)}$ or smaller. Remark that $|\mathfrak{W}_{i,\delta}^{H,r}| \leq \delta^{H^*r} (\mathcal{H}_{H^*}^H)^r$. Thus

$$\left(\sum_{r=1}^{2S} \frac{\eta^r}{r!} \mathfrak{W}_{i,\delta}^{H,r} + \tilde{R}^{H,S}(i,\delta) \delta^{H^*(S+1)}\right)^s = \sum_{\mathbf{r}} \prod_{j=1}^s X_{\mathbf{r}_j}$$

where the sum is taken over all $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_s)$ with $1 \leq \mathbf{r}_j \leq S+1$ and where we write $X_{\mathbf{r}} = \frac{\eta^{\mathbf{r}}}{\mathbf{r}!} \mathfrak{W}_{i,\delta}^{H,\mathbf{r}}$ for $r \leq S$ and $X_{S+1} = \tilde{R}^{H,S}(i,\delta) \delta^{H^*(S+1)}$. By the preceding remark, we have

$$\prod_{j=1}^s X_{\mathbf{r}_j} \leq \delta^{H^* \sum_j \mathbf{r}_j} (\mathcal{H}_{H^*}^H)^{\sum_j \mathbf{r}_j}.$$

Proceeding as for $T^H(i,\delta)$, we can show that $\prod_{j=1}^s X_{\mathbf{r}_j}$ can be incorporated in the rest of Proposition 33 whenever $\sum_j \mathbf{r}_j \geq S+1$. Therefore we can restrict the sum $\sum_{\mathbf{r}} \prod_{j=1}^s X_{\mathbf{r}_j}$ to indexes \mathbf{r} satisfying $\sum_j \mathbf{r}_j \leq S$. In that case, $\mathbf{r}_j \leq S$ and we get:

$$\left(\sum_{r=1}^{2S} \frac{\eta^r}{r!} \mathfrak{W}_{i,\delta}^{H,r} + \tilde{R}^{H,S}(i,\delta) \delta^{H^*(S+1)}\right)^s = \sum_{\mathbf{r}} \prod_{j=1}^s \frac{\eta^{\mathbf{r}_j}}{\mathbf{r}_j!} \mathfrak{W}_{i,\delta}^{H,\mathbf{r}_j} + \text{remainder of order } \delta^{H^*(S+1)}.$$

Plugging this into (43) and using the symbol \approx to indicate implicitly the rest of order $\delta^{H^*(S+1)}$ incorporated in the term Z of Proposition 33, we obtain:

$$\begin{aligned} \log\left(\frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \sigma_u^2 du\right) &\approx \eta W_{i\delta}^H + \sum_{s=1}^{2S} \frac{(-1)^{s-1}}{s} \sum_{\mathbf{r}} \prod_{j=1}^s \frac{\eta^{\mathbf{r}_j}}{\mathbf{r}_j!} \mathfrak{W}_{i,\delta}^{H,\mathbf{r}_j} \\ &\approx \eta W_{i\delta}^H + \sum_{b=1}^{2S} \sum_{s=1}^{2S} \frac{(-1)^{s-1}}{s} \sum_{\mathbf{r}} \prod_{j=1}^s \frac{\eta^{\mathbf{r}_j}}{\mathbf{r}_j!} \mathfrak{W}_{i,\delta}^{H,\mathbf{r}_j} \\ &\approx \frac{1}{\delta} \int_{i\delta}^{(i+1)\delta} \eta W_u^H du + \sum_{b=2}^{2S} \sum_{s=1}^{2S} \frac{(-1)^{s-1}}{s} \sum_{\mathbf{r}} \prod_{j=1}^s \frac{\eta^{\mathbf{r}_j}}{\mathbf{r}_j!} \mathfrak{W}_{i,\delta}^{H,\mathbf{r}_j}. \end{aligned}$$

□

APPENDIX B. SOME CORRELATION ESTIMATES FOR GAUSSIAN VECTORS

First, we recall the Isserlis' theorem (see [Iss18]) allowing us to compute the expectation of a product of zero-mean correlated normal random variables.

Theorem 34. *Suppose that (X_1, \dots, X_{2n}) is a centred Gaussian vector. Then we have*

$$\mathbb{E}\left(\prod_i X_i\right) = \sum_P \prod_{(i,j) \in P} \mathbb{E}(X_i X_j)$$

where the sum is over all the partitions P of $\{1, \dots, 2n\}$ into subsets of exactly two elements.

In particular, we have:

$$(44) \quad \text{Cov}(X_1X_2, X_3X_4) = \mathbb{E}(X_1X_3)\mathbb{E}(X_2X_4) + \mathbb{E}(X_1X_4)\mathbb{E}(X_2X_3).$$

Proposition 35. *Suppose that (X_1, \dots, X_{n+m}) is a centred Gaussian vector, where n and m are two integers such that $n + m$ is even. Suppose in addition that for any $i \leq n$ and $j \geq n + 1$, we have*

$$|\mathbb{E}(X_iX_j)| \leq \rho\sigma^2,$$

for some $0 \leq \rho \leq 1$ and $\sigma \geq 0$ and suppose that $\mathbb{E}X_i^2 \leq \sigma^2$ for any $i \geq 1$. Then

$$\left| \text{Cov}\left(\prod_{i=1}^n X_i, \prod_{j=n+1}^{n+m} X_j\right) \right| \leq C\rho^\alpha\sigma^{n+m}$$

where $\alpha = 1$ if n is odd and $\alpha = 2$ if n is even, and C is a constant depending only on n and m .

Proof. Denote $\mathcal{P}_2(E)$ the set of all partitions of the set E in subsets of exactly 2 elements. Then we have by Theorem 34

$$\begin{aligned} \text{Cov}\left(\prod_{i=1}^n X_i, \prod_{j=n+1}^{n+m} X_j\right) &= \sum_{P \in \mathcal{P}_2(\{1, \dots, n+m\})} \prod_{(i,j) \in P} \mathbb{E}(X_iX_j) \\ &\quad - \sum_{P \in \mathcal{P}_2(\{1, \dots, n\})} \prod_{(i,j) \in P} \mathbb{E}(X_iX_j) - \sum_{P \in \mathcal{P}_2(\{n+1, \dots, n+m\})} \prod_{(i,j) \in P} \mathbb{E}(X_iX_j). \end{aligned}$$

Moreover, the application

$$\begin{aligned} \mathcal{P}_2(\{1, \dots, n+1\}) \times \mathcal{P}_2(\{n+1, \dots, n+m\}) &\rightarrow \mathcal{P}_2(\{1, \dots, n+m\}) \\ (P, Q) &\mapsto P \cup Q \end{aligned}$$

is injective and its image $\mathcal{Q}(n, m)$ is exactly the set of the partitions P of $\{1, \dots, n+m\}$ such that if $(i, j) \in P$ with $i \leq n$, then $j \leq n$ as well. Thus

$$\text{Cov}\left(\prod_{i=1}^n X_i, \prod_{j=n+1}^{n+m} X_j\right) = \sum_{P \in \mathcal{P}_2(\{1, \dots, n+m\}) \setminus \mathcal{Q}(n, m)} \prod_{(i,j) \in P} \mathbb{E}(X_iX_j).$$

Since there are finitely many partitions of $\{1, \dots, n+m\}$, the proof is completed once we can prove that for any $P \in \mathcal{P}_2(\{1, \dots, n+m\}) \setminus \mathcal{Q}(n, m)$, there exists a constant C depending only on n and m such that

$$\left| \prod_{(i,j) \in P} \mathbb{E}(X_iX_j) \right| \leq \rho^\alpha \sigma^{n+m}.$$

Consider such a partition P . Then there is at least one pair $(i_0, j_0) \in P$ such that $i_0 \leq n$ and $j_0 \geq n+1$. thus

$$\left| \prod_{(i,j) \in P} \mathbb{E}(X_iX_j) \right| \leq \left| \mathbb{E}(X_{i_0}X_{j_0}) \right| \prod_{(i,j) \in P \setminus \{(i_0, j_0)\}} \mathbb{E}(X_iX_j) \leq \rho\sigma^2 \prod_{(i,j) \in P \setminus \{(i_0, j_0)\}} \sigma^2$$

which conclude the case n odd since $\#(P \setminus \{(i_0, j_0)\}) = (n+m-2)/2$.

Suppose in addition that n is even. Then there must be another pair $(i_1, j_1) \in P$ such that $i_1 \leq n$ and $j_1 \geq n+1$, $(i_1, j_1) \neq (i_0, j_0)$ because there is no partition of $\{1, \dots, n\} \setminus \{i_0\}$ into subsets

of 2 elements. Then

$$\left| \prod_{(i,j) \in P} \mathbb{E}(X_i X_j) \right| \leq \left| \mathbb{E}(X_{i_0} X_{j_0}) \mathbb{E}(X_{i_1} X_{j_1}) \prod_{(i,j) \in P \setminus \{(i_0, j_0), (i_1, j_1)\}} \mathbb{E}(X_i X_j) \right| \leq \rho^2 \sigma^{n+m}.$$

□

APPENDIX C. LOG MOMENTS OF χ^2 VARIABLES

We first define a few notations. We write Γ the usual Gamma function, defined by

$$(45) \quad \Gamma(t) = \int_0^\infty t^{x-1} e^{-t} dt.$$

We also introduce the polygamma function $\psi^{(k)}$, which is the k -th logarithmic derivative of the Gamma function. Thus, $\psi^{(0)} = \frac{\Gamma'}{\Gamma}$ and explicit computations also give

$$\begin{aligned} \psi^{(1)} &= \frac{\Gamma''}{\Gamma} - (\psi^{(0)})^2, \\ \psi^{(2)} &= \frac{\Gamma^{(3)}}{\Gamma} - (\psi^{(0)})^3 - 3\psi^{(0)}\psi^{(1)}, \\ \psi^{(3)} &= \frac{\Gamma^{(4)}}{\Gamma} - (\psi^{(0)})^4 - 6(\psi^{(0)})^2\psi^{(1)} - 4\psi^{(0)}\psi^{(2)} - 3(\psi^{(1)})^2. \end{aligned}$$

Note that we can express explicitly ratios $\frac{\Gamma^{(k)}}{\Gamma}$ with $k \leq 4$ in terms of poly-gammas functions from these equations.

Lemma 36. *Suppose that for $m \geq 1$, X_m is a random variable following a χ^2 distribution with m degree of freedom. We write $Y_m = \log(m^{-1} X_m)$. Then there exists $C > 0$ such that for any $m \geq 1$,*

$$\begin{aligned} \text{Var}(Y_m) &= \psi^{(1)}\left(\frac{m}{2}\right) \leq C m^{-1}, \\ \mathbb{E}[Y_m^4] &\leq C m^{-2}. \end{aligned}$$

Proof. We know that

$$\mathbb{E}[\exp(tY_m)] = \left(\frac{2}{m}\right)^t \frac{\Gamma\left(\frac{m}{2} + t\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

Moments of Y_m can be derived through the classical formula

$$\mathbb{E}[Y_m^k] = \frac{d^k}{dt^k} \Big|_{t=0} \left(\left(\frac{2}{m}\right)^t \frac{\Gamma\left(\frac{m}{2} + t\right)}{\Gamma\left(\frac{m}{2}\right)} \right).$$

Thus we get

$$\begin{aligned} \mathbb{E}[Y_m] &= -\ln\left(\frac{m}{2}\right) + \frac{\Gamma'\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}, \\ \mathbb{E}[Y_m^2] &= \ln^2\left(\frac{m}{2}\right) - 2\ln\left(\frac{m}{2}\right) \frac{\Gamma'\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} + \frac{\Gamma''\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}, \\ \mathbb{E}[Y_m^4] &= \ln^4\left(\frac{m}{2}\right) - 4\ln^3\left(\frac{m}{2}\right) \frac{\Gamma'\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} + 6\ln^2\left(\frac{m}{2}\right) \frac{\Gamma''\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} - 4\ln\left(\frac{m}{2}\right) \frac{\Gamma^{(3)}\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} + \frac{\Gamma^{(4)}\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}. \end{aligned}$$

We then rewrite $\text{Var}(Y_m)$ and $\mathbb{E}[Y_m^4]$ in terms of these polygamma functions

$$\begin{aligned} \text{Var}(Y_m) &= \psi^{(1)}\left(\frac{m}{2}\right), \\ \mathbb{E}[Y_m^4] &= \ln^4\left(\frac{m}{2}\right) - 4\ln^3\left(\frac{m}{2}\right)\psi^{(0)}\left(\frac{m}{2}\right) + 6\ln^2\left(\frac{m}{2}\right)(\psi^{(1)}\left(\frac{m}{2}\right) + \psi^{(0)}\left(\frac{m}{2}\right)^2) \end{aligned}$$

$$\begin{aligned}
& -4 \ln\left(\frac{m}{2}\right) \left(\psi^{(2)}\left(\frac{m}{2}\right) + \psi^{(0)}\left(\frac{m}{2}\right)^3 + 3\psi^{(0)}\left(\frac{m}{2}\right)\psi^{(1)}\left(\frac{m}{2}\right)\right) \\
& + \psi^{(3)}\left(\frac{m}{2}\right) + \psi^{(0)}\left(\frac{m}{2}\right)^4 + 6\psi^{(0)}\left(\frac{m}{2}\right)^2\psi^{(1)}\left(\frac{m}{2}\right) + 4\psi^{(0)}\left(\frac{m}{2}\right)\psi^{(2)}\left(\frac{m}{2}\right) + 3\psi^{(1)}\left(\frac{m}{2}\right)^2.
\end{aligned}$$

Asymptotic expansion of the polygamma functions $\psi^{(k)}(x)$ is given when $x \rightarrow \infty$ by Equation 5.15.9 in the internet appendix of [OLBC10]

$$\psi^{(0)}(x) = \ln(x) - \frac{1}{2}x^{-1} - \frac{1}{12}x^{-2} + O(x^{-3}),$$

$$\psi^{(1)}(x) = x^{-1} + \frac{1}{2}x^{-2} + O(x^{-3}),$$

$$\psi^{(2)}(x) = -x^{-2} + O(x^{-3}),$$

$$\psi^{(3)}(x) = O(x^{-3}).$$

Plugging these asymptotic development into the explicit expression of $\text{Var}(Y_m)$ and $\mathbb{E}[Y_m^4]$, we get

$$\text{Var}(Y_m) = \psi^{(1)}\left(\frac{m}{2}\right) = \frac{2}{m} + O(m^{-2}),$$

$$\mathbb{E}[Y_m^4] = 12m^{-2} + o(m^{-2})$$

which concludes the proof. □

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