

Dual natural-norm *a posteriori* error estimators for parametrized equations

P. Edel

*DTIS, ONERA, Université Paris Saclay
F-91123 Palaiseau - France*

Y. Maday

*Sorbonne Université, CNRS, Université Paris Cité
Laboratoire Jacques-Louis Lions
F-75005 Paris - France
Institut Universitaire de France*

Abstract

In this work, we introduce the concept of dual natural-norm for parametrized linear equations and use it to derive residual-based *a posteriori* error bounds characterized by a $\mathcal{O}(1)$ stability constant. We translate these error bounds into very effective practical *a posteriori* error estimators for reduced basis approximations and show how they can be efficiently computed following an offline/online strategy. We prove that our practical dual natural-norm error estimator outperforms the classical inf-sup based error estimators in the self-adjoint case. Our findings are illustrated on anisotropic Helmholtz equations showing resonant behavior. Numerical results suggest that the proposed error estimator is able to successfully catch the correct order of magnitude of the reduced basis approximation error, thus outperforming the classical inf-sup based error estimator even for non self-adjoint problems.

Keywords: *A posteriori* error estimation; parametrized equations; reduced basis methods; inf-sup constant; natural-norm.

1 Introduction

This paper is concerned with the numerical solution of large-scale parametrized linear equations of the form: *find* $u(\mu)$ *such that* $A(\mu)u(\mu) = f(\mu)$, where the operator $A(\mu)$ and right-hand side $f(\mu)$ are given and depend on a parameter μ in some parameter set $\mathcal{D} \subset \mathbb{R}^p$, $p \geq 1$. Such a problem typically occurs in

the "many query" context, where we wish to solve a linear PDE for many different parameter values. Model order reduction techniques, such as the proper orthogonal decomposition (POD) methods [3, 14]; or reduced basis (RB) methods [16, 19, 21] can be used to obtain a low-cost approximation $\widetilde{u}_N(\mu)$ for any $\mu \in \mathcal{D}$ in a low-dimensional approximation space. In this context, the development of rigorous *a posteriori* error estimators for estimating the error $\|u(\mu) - \widetilde{u}_N(\mu)\|$ is crucial for certifying the reduced order model.

The original approach for error estimation is based on the so-called *inf-sup constant* $\alpha(\mu)$. More precisely, it is well-known that the error can be bounded from above by the residual $A(\mu)\widetilde{u}_N(\mu) - f(\mu)$ in the appropriate norm divided by the inf-sup constant [16, 19, 17, 8]. Yet two main issues arise with this error bound: (i) it is difficult to translate into a readily computable error estimator, because the inf-sup constant $\alpha(\mu)$ is the smallest eigenvalue of a large-scale generalized eigenvalue problem which (in practice) cannot be solved at all possible values of $\mu \in \mathcal{D}$ and (ii) this error bound is not sharp in general and may significantly overestimate the error [9]. A major step towards overcoming (i) is the successive constraints method (SCM) proposed in Ref. [12], which permits the construction of low-cost lower and upper bounds for the inf-sup constant over parameter space; while an attractive approach for overcoming (ii) is to be found in the natural-norm approach in Ref. [22], which consists in building an equivalent norm (the so called *natural-norm*), such that the error in this norm can be bounded by the residual norm, divided by a $\mathcal{O}(1)$ stability constant.

In this work, we derive new error bounds, specifically designed to be sharp. To do so, we retain the original natural-norm idea of achieving a $\mathcal{O}(1)$ stability constant. The originality of our approach resides in that we build a natural-norm for measuring the residual rather than the error. Our natural-norm – the *dual natural-norm* – is thus different from the original natural-norm introduced in Ref. [22]. For this reason, we shall speak of *primal natural-norm* when referring to the original natural-norm concept. In this paper, we show that the dual natural-norm approach is equivalent to preconditioning the $A(\mu)u(\mu) = f(\mu)$ equation to the left, which shares some similarities with Refs. [25] and [1], although the present approach considers no dependency in μ in the preconditioner and relies on deterministic rather than probabilistic arguments. Moreover, this leads us to re-interpret the primal natural-norm approach as a right preconditioning approach.

The error estimator resulting from the new error bound is very effective. In this work, we prove that when $A(\mu)$ is self-adjoint our error estimator

is guaranteed to provide less overestimation than the inf-sup based error estimator.

The paper is organized as follows. Section 2 introduces the Hilbert setting and reviews two classical *a posteriori* error bounds. The first is based on the inf-sup condition and the second relies on the original natural-norm concept. In section 3, we introduce the dual natural-norm and use this new tool to derive a new *a posteriori* error estimate. Then, in section 4, we translate the new *a posteriori* error estimate into a practical error estimator, which can be efficiently computed in the RB context. Section 5 addresses the computational aspects in detail. Finally, in section 6, numerical experiments illustrate the properties of the new dual natural-norm error estimator on a challenging Helmholtz problem.

2 Inf-sup based a posteriori error estimators

2.1 Setting

Let $\mathcal{D} \subset \mathbb{R}^p$ be a compact set, with $p \geq 1$ the number of parameters. We consider the following parametrized linear PDE: *find* $u^{\text{ex}}(\mu) \in V^{\text{ex}}$ *such that* $A^{\text{ex}}(\mu)u^{\text{ex}}(\mu) = f^{\text{ex}}(\mu)$, where the operator $A^{\text{ex}}(\mu) \in \mathcal{L}(V^{\text{ex}}, (W^{\text{ex}})')$ and right-hand side $f^{\text{ex}}(\mu) \in (W^{\text{ex}})'$ are given and depend on the parameter $\mu \in \mathcal{D}$. The space V^{ex} denotes the infinite-dimensional complex Hilbert space in which the PDE is well-posed, typically this corresponds to a Sobolev space $H_0^1(\Omega) \subset V^{\text{ex}} \subset H^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$ is the domain in which the PDE is solved. The space W^{ex} denotes the infinite-dimensional complex Hilbert space for the test functions in the weak form associated to the PDE and $(W^{\text{ex}})'$ denotes the topological dual.

Using discretization techniques such as the finite element method, the parametrized PDE is approximated by the following parametrized linear equation: *find* $u(\mu) \in V$ *such that*

$$A(\mu)u(\mu) = f(\mu) \text{ in } W', \quad (2.1)$$

where the operator $A(\mu) \in \mathcal{L}(V, W')$ and right-hand side $f(\mu) \in W'$ are given and depend on the parameter $\mu \in \mathcal{D}$. The spaces V and W are complex Hilbert spaces with finite dimension \mathcal{N} . The space V may be thought of as a conforming approximation space $V \subset V^{\text{ex}}$, in which case the norm on V

coincides with the norm on V^{ex} ; yet the present framework is also fit to non-conforming approximation spaces, in which case V can be equipped with an adequate, usually mesh-dependent energy norm [11]. The dimension \mathcal{N} can be chosen large enough so that the difference between the exact PDE solution $u^{\text{ex}}(\mu)$ and the numerical approximation $u(\mu)$ is adequately small. Thus, our focus is on solution $u(\mu)$ to the parametrized linear equation (2.1) and we forget about the exact PDE solution since it is unobtainable in practice.

We denote ${}_{V'}\langle \ell, v \rangle_V$ (or $\langle \ell, v \rangle$, when there is no ambiguity) the duality bracket between $v \in V$ and a member $\ell \in V'$ from the topological dual of V (denoted V'). We adopt the convention that the duality bracket is linear with respect to the first variable, anti-linear with respect to the second, that is $\forall \lambda, \eta \in \mathbb{C}$, $\langle \lambda \ell, \eta v \rangle = \lambda \bar{\eta} \langle \ell, v \rangle$, where $\bar{\eta}$ designates the complex conjugate of η . We denote $R_V \in \mathcal{L}(V, V')$ the inverse Riesz operator, thus the norm on V denoted $\|\cdot\|_V$ verifies $\|v\|_V^2 = \langle R_V v, v \rangle$ and the norm $\|\cdot\|_{V'}$ on the topological dual V' is given by

$$\|\ell\|_{V'} = \sup_{v \in V} \frac{|\langle \ell, v \rangle|}{\|v\|_V} = (\langle \ell, R_V^{-1} \ell \rangle)^{1/2}. \quad (2.2)$$

We assume that $A(\mu)$ is a weakly coercive operator satisfying the assumptions of the Banach Necas Babuska theorem [7]; *i.e.*, for all $v \in V$ and all $\mu \in \mathcal{D}$,

$$\alpha(\mu) \|v\|_V \leq \|A(\mu)v\|_{W'} \leq \gamma(\mu) \|v\|_V, \quad (2.3)$$

with the so-called (strictly positive) *inf-sup constant*,

$$\alpha(\mu) = \inf_{v \in V} \sup_{w \in W} \frac{|\langle A(\mu)v, w \rangle|}{\|v\|_V \|w\|_W} > 0, \quad (2.4)$$

and with the (bounded) *continuity constant*,

$$\gamma(\mu) = \sup_{v \in V} \sup_{w \in W} \frac{|\langle A(\mu)v, w \rangle|}{\|v\|_V \|w\|_W} < \infty. \quad (2.5)$$

Under these assumptions, for any $\mu \in \mathcal{D}$ the solution $u(\mu)$ to Eq. (2.1) exists and is unique. In this work, we are interested in the manifold $\{u(\mu), \mu \in \mathcal{D}\}$, comprised of all solutions under variation of the parameter $\mu \in \mathcal{D}$.

2.2 Error bound using the inf-sup constant

We recall the classical inf-sup based error estimate, which states that the norm of the error is bounded by the residual norm divided by the inf-sup

constant. The following theorem is a direct consequence of the Banach-Necas-Babuska assumptions.

Theorem 1. *For all $\mu \in \mathcal{D}$, the solution $u(\mu) \in V$ to (2.1) satisfies,*

$$\forall \tilde{v} \in V, \quad \|u(\mu) - \tilde{v}\|_V \leq \frac{1}{\alpha(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{W'}.$$

The question that arises is: how sharp is this error bound? The inequality becomes an equality only if the error $e(\mu) = \tilde{v} - u(\mu)$ satisfies the infimizer property

$$\frac{\|A(\mu)e(\mu)\|_{W'}}{\|e(\mu)\|_V} = \inf_{v \in V} \frac{\|A(\mu)v\|_{W'}}{\|v\|_V} = \alpha(\mu). \quad (2.6)$$

Of course, this situation is not likely to occur in practice. Thus the inf-sup based error bound is typically a strict inequality, which overestimates (possibly significantly) the error. The next proposition, the proof of which is straightforward, establishes the worst possible overestimation scenario.

Proposition 2.1. *For all $\mu \in \mathcal{D}$, the solution $u(\mu) \in V$ to (2.1) satisfies,*

$$\forall \tilde{v} \in V, \quad 1 \leq \frac{\frac{1}{\alpha(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{W'}}{\|u(\mu) - \tilde{v}\|_V} \leq \frac{\gamma(\mu)}{\alpha(\mu)}.$$

Typically, in resonant problems, the inf-sup constant may approach 0 near the resonant values of μ . In this situation, the upper bound $\gamma(\mu)/\alpha(\mu)$ may be very large, therefore the inf-sup based error bound provided by Theorem 1 tends to overestimate the error by possibly many orders of magnitude. This effect is well-known in the reduced basis community and effort is currently being made to circumvent this issue. Among the most recent approaches we cite the hierarchical approach in Ref. [9] and the randomized approach in Ref. [24]. The natural norm approach, originally introduced in the reduced basis context in Ref. [22], has been used with success since then to resolve effectivity issues [6, 15, 5].

2.3 Error bound using the primal natural-norm

The original natural-norm concept has been introduced in the context of parameterized equations in Ref. [22]. In this work, we rename this original approach the *primal natural-norm* approach for reasons that will soon become obvious.

The primal natural-norm approach relies on the (primal) supremizer operator $T(\mu) \in \mathcal{L}(V, W)$, defined by

$$T(\mu) = R_W^{-1} A(\mu). \quad (2.7)$$

We can show that this operator satisfies the following supremizer property

$$\forall v \in V, \quad \frac{|\langle A(\mu)v, T(\mu)v \rangle|}{\|T(\mu)v\|_W} = \sup_{w \in W} \frac{|\langle A(\mu)v, w \rangle|}{\|w\|_W}. \quad (2.8)$$

We can use this supremizer operator to define the so-called *primal natural-norm*, which is a μ -dependent norm on V . We shall denote it $\|\cdot\|_{\mu, V}$. We define it as

$$\forall v \in V, \quad \|v\|_{\mu, V} = \|T(\mu)v\|_W, \quad (2.9)$$

or equivalently, using the property on Riesz maps $\|\cdot\|_{W'} = \|R_W^{-1} \cdot\|_W$,

$$\forall v \in V, \quad \|v\|_{\mu, V} = \|A(\mu)v\|_{W'} = (\langle A(\mu)v, R_W^{-1} A(\mu)v \rangle)^{1/2}. \quad (2.10)$$

Recalling that $A(\mu)$ is a weakly coercive operator satisfying the Banach-Necas-Babuska assumptions, it is clear that the norm $\|\cdot\|_{\mu, V}$ is indeed a norm on V equivalent to the $\|\cdot\|_V$ norm, with equivalence constants independent from the dimension \mathcal{N} .

The original natural-norm concept consists in providing an error estimate not in the $\|\cdot\|_V$ norm, but rather in the $\|\cdot\|_{\mu, V}$ norm. To start with, note that for all $\mu \in \mathcal{D}$, the solution $u(\mu) \in V$ to (2.1) satisfies, for all $\tilde{v} \in V$

$$\|u(\mu) - \tilde{v}\|_{\mu, V} = \|A(\mu)(u(\mu) - \tilde{v})\|_{W'} = \|A(\mu)\tilde{v} - f(\mu)\|_{W'}. \quad (2.11)$$

Thus, the primal natural-norm of the error coincides with the residual norm. In practice however, one is not satisfied with this result, because the natural-norm depends on μ . In order to circumvent this, we fix a value $\bar{\mu} \in \mathcal{D}$ and provide an error estimate in the $\|\cdot\|_{\bar{\mu}, V}$ norm (which is no longer dependent on μ since $\bar{\mu}$ is fixed). In this situation, one can prove the following theorem.

Theorem 2. *Let $\bar{\mu} \in \mathcal{D}$. Define for all $\mu \in \mathcal{D}$ the primal natural-norm constant*

$$\alpha_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\|v\|_{\mu, V}}{\|v\|_{\bar{\mu}, V}}.$$

Then, for all $\mu \in \mathcal{D}$, the solution $u(\mu) \in V$ to (2.1) satisfies

$$\forall \tilde{v} \in V, \quad \|u(\mu) - \tilde{v}\|_{\bar{\mu}, V} \leq \frac{1}{\alpha_{\bar{\mu}}(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{W'},$$

furthermore, the inequality is an equality for $\mu = \bar{\mu}$ and $\alpha_{\bar{\mu}}(\bar{\mu}) = 1$.

Again, the question of the sharpness of this bound is raised. A significant improvement over the inf-sup-based error bound from Theorem 1 is that we now achieve equality at least when $\mu = \bar{\mu}$. The next proposition provides insight on the worst overestimation case scenario.

Proposition 2.2. *Let $\bar{\mu} \in \mathcal{D}$. Define for all $\mu \in \mathcal{D}$,*

$$\gamma_{\bar{\mu}}(\mu) = \sup_{v \in V} \frac{\|v\|_{\mu, V}}{\|v\|_{\bar{\mu}, V}}.$$

Then, for all $\mu \in \mathcal{D}$, the solution $u(\mu) \in V$ to (2.1) satisfies

$$\forall \tilde{v} \in V, \quad 1 \leq \frac{\frac{1}{\alpha_{\bar{\mu}}(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{W'}}{\|u(\mu) - \tilde{v}\|_{\bar{\mu}, V}} \leq \frac{\gamma_{\bar{\mu}}(\mu)}{\alpha_{\bar{\mu}}(\mu)},$$

furthermore, $\gamma_{\bar{\mu}}(\bar{\mu}) = \alpha_{\bar{\mu}}(\bar{\mu}) = 1$.

This proposition reveals the potential benefits of the primal natural-norm approach. Indeed, under basic regularity assumptions, the quantity $\gamma_{\bar{\mu}}(\mu)/\alpha_{\bar{\mu}}(\mu)$, being equal to 1 for $\mu = \bar{\mu}$, will continue to be $\mathcal{O}(1)$ for values of μ in a neighborhood of $\bar{\mu}$. Thus, the amount of overestimation of the primal natural-norm error bound provided by Theorem 1 will be $\mathcal{O}(1)$ for values of μ adequately close to $\bar{\mu}$. Since $\bar{\mu}$ is a fixed value, chosen by the user, one can always consider a family of points $\{\bar{\mu}^k\}_{1 \leq k \leq K}$ and thus be able to estimate the correct order of magnitude of the error for any $\mu \in \mathcal{D}$. Notice however that the primal natural-norm approach still suffers from the problem of resonances.

Remark that the primal natural-norm error bound given by Theorem 2 does not bound the error in the norm of our choice. Indeed, the theorem bounds the error in the primal natural-norm $\|\cdot\|_{\bar{\mu}, V}$ but not in the user-defined $\|\cdot\|_V$ norm. The dual natural-norm error bound (see next section) circumvents this; since it bounds the error in the $\|\cdot\|_V$ norm while maintaining a $\mathcal{O}(1)$ stability constant.

3 A new error bound using the dual natural-norm

3.1 An additional hypothesis

In order to go beyond the primal natural-norm approach, we need to introduce an additional hypothesis. We consider the adjoint operator $A(\mu)^* \in \mathcal{L}(W, V')$, defined by ${}_{W'}\langle A(\mu)v, w \rangle_W = {}_{V'}\langle A(\mu)^*w, v \rangle_V$ for all $v \in V$ and for all $w \in W$. We recall that the adjoint operator (or conjugate-transposed operator) is automatically continuous since it satisfies $\|A(\mu)^*w\|_{V'} \leq \gamma(\mu)\|w\|_W$ for all $w \in W$, where $\gamma(\mu)$ is the continuity constant of $A(\mu)$ defined by Eq. (2.5). However, in order to ensure that the adjoint is a weakly coercive operator, we cannot rely on the fact that $A(\mu)$ is weakly coercive and must make one further assumption. Namely that,

$$\beta(\mu) = \inf_{w \in W} \sup_{v \in V} \frac{|\langle A(\mu)^*w, v \rangle|}{\|v\|_V \|w\|_W} > 0. \quad (3.1)$$

In the rest of this work, we shall assume that this stability condition is satisfied.

3.2 The dual natural norm

As in the case of the primal, we now introduce the dual supremizer operator $\Upsilon(\mu) \in \mathcal{L}(W, V)$, defined by

$$\Upsilon(\mu) = R_V^{-1} A(\mu)^*. \quad (3.2)$$

We can show that this operator satisfies the following supremizer property

$$\forall w \in W, \quad \frac{|\langle A(\mu)^*w, \Upsilon(\mu)w \rangle|}{\|\Upsilon(\mu)w\|_W} = \sup_{v \in V} \frac{|\langle A(\mu)^*w, v \rangle|}{\|v\|_V}.$$

We now use the dual supremizer operator to define a natural-norm on W , which shall be denoted $\|\cdot\|_{\mu, W}$. We define it as

$$\forall w \in W, \quad \|w\|_{\mu, W} = \|\Upsilon(\mu)w\|_V, \quad (3.3)$$

or equivalently, using the property on Riesz maps $\|\cdot\|_{V'} = \|R_V^{-1} \cdot\|_V$,

$$\forall w \in W, \quad \|w\|_{\mu, W} = \|A(\mu)^*w\|_{V'} = (\langle A(\mu)^*w, R_V^{-1}A(\mu)^*w \rangle)^{1/2}. \quad (3.4)$$

It is clear from the weak coercivity of $A(\mu)^*$ that $\|\cdot\|_{\mu,W}$ is indeed a norm on W equivalent to the $\|\cdot\|_W$ norm, with equivalence constants independent from the dimension \mathcal{N} .

With this natural-norm on W , we can define a natural-norm on the dual W' . The latter will be called the *dual natural-norm* and be denoted $\|\cdot\|_{\mu,W'}$. Namely, it is quite classically defined as

$$\forall \ell \in W', \quad \|\ell\|_{\mu,W'} = \sup_{w \in W} \frac{|\langle \ell, w \rangle|}{\|w\|_{\mu,W}}. \quad (3.5)$$

Thus defined, it is clear that $\|\cdot\|_{\mu,W'}$ is an equivalent norm to $\|\cdot\|_{W'}$. The following proposition gives us a more convenient formula for expressing the dual natural-norm norm.

Proposition 3.1. *The $\|\cdot\|_{\mu,W'}$ norm defined by (3.5) is equivalently*

$$\forall \ell \in W', \quad \|\ell\|_{\mu,W'} = \|A(\mu)^{-1}\ell\|_V = (\langle R_V A(\mu)^{-1}\ell, A(\mu)^{-1}\ell \rangle)^{1/2}.$$

Proof. Let $\ell \in W'$ and $w \in W$. By the adjoint property, we have

$$\langle \ell, w \rangle = \langle \ell, A(\mu)^{-*}A(\mu)^*w \rangle = \langle A(\mu)^{-1}\ell, A(\mu)^*w \rangle.$$

Thus,

$$\begin{aligned} \forall \ell \in W', \quad \|\ell\|_{\mu,W'} &= \sup_{w \in W} \frac{|\langle \ell, w \rangle|}{\|w\|_{\mu,W}} = \sup_{w \in W} \frac{\langle A(\mu)^{-1}\ell, A(\mu)^*w \rangle}{\|A(\mu)^*w\|_{V'}} \\ &= \sup_{\phi \in V'} \frac{\langle A(\mu)^{-1}\ell, \phi \rangle}{\|\phi\|_{V'}} \\ &= \|A(\mu)^{-1}\ell\|_V. \end{aligned} \quad (3.6)$$

□

We can show the following norm equivalence, leaving the proof reader.

Proposition 3.2. *For all $\ell \in W'$, there holds,*

$$\beta(\mu)\|\ell\|_{\mu,W'} \leq \|\ell\|_{W'} \leq \gamma(\mu)\|\ell\|_{\mu,W'}.$$

3.3 Error bound

We now arrive to our ultimate goal of deriving error estimates using the dual natural norm. To start with, note that the error norm is exactly the dual natural-norm of the residual. Indeed, for all $\mu \in \mathcal{D}$, the solution $u(\mu) \in V$ to (2.1) satisfies, for all $\tilde{v} \in V$,

$$\|u(\mu) - \tilde{v}\|_V = \|A(\mu)\tilde{v} - f(\mu)\|_{\mu, W'}. \quad (3.7)$$

Notice the symmetry with Eq. (2.11), repeated here for convenience,

$$\|u(\mu) - \tilde{v}\|_{\mu, V} = \|A(\mu)\tilde{v} - f\|_{W'}. \quad (3.8)$$

All is now set to derive the error estimate using the dual natural-norm.

Theorem 3. *Let $\bar{\mu} \in \mathcal{D}$. For all $\mu \in \mathcal{D}$, define*

$$\sigma_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\|A(\mu)v\|_{\bar{\mu}, W'}}{\|v\|_V} \left(= \inf_{v \in V} \frac{\|A(\bar{\mu})^{-1}A(\mu)v\|_V}{\|v\|_V} \right).$$

Then, for all $\mu \in \mathcal{D}$ the solution $u(\mu) \in V$ to (2.1) satisfies

$$\forall \tilde{v} \in V, \quad \|u(\mu) - \tilde{v}\|_V \leq \frac{1}{\sigma_{\bar{\mu}}(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{\bar{\mu}, W'}.$$

Furthermore, the inequality is an equality for $\mu = \bar{\mu}$ and $\sigma_{\bar{\mu}}(\bar{\mu}) = 1$.

Proof. Start by

$$\begin{aligned} \|A(\mu)\tilde{v} - f(\mu)\|_{\bar{\mu}} &= \|A(\bar{\mu})^{-1}(A(\mu)\tilde{v} - f(\mu))\|_V \\ &= \|A(\bar{\mu})^{-1}A(\mu)(\tilde{v} - u(\mu))\|_V \\ &\geq \left(\inf_{v \in V} \frac{\|A(\bar{\mu})^{-1}A(\mu)v\|_V}{\|v\|_V} \right) \|u(\mu) - \tilde{v}\|_V. \end{aligned}$$

It remains to justify that $\sigma_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\|A(\bar{\mu})^{-1}A(\mu)v\|_V}{\|v\|_V}$ is indeed > 0 . For this, we bound from below by using the norm equivalence of Proposition 3.2,

$$\sigma_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\|A(\bar{\mu})^{-1}A(\mu)v\|_V}{\|v\|_V} \geq \frac{1}{\gamma(\bar{\mu})} \inf_{v \in V} \frac{\|A(\mu)v\|_{W'}}{\|v\|_V} = \frac{\alpha(\mu)}{\gamma(\bar{\mu})}.$$

This lower bound is > 0 because the inf-sup constant $\alpha(\mu)$ is strictly positive.

To prove that the inequality is an equality when $\mu = \bar{\mu}$, we simply observe that $\sigma_{\bar{\mu}}(\bar{\mu}) = 1$ and come back to Eq. (3.7). \square

Let us now give a result on the potential sharpness of this error bound.

Proposition 3.3. *Let $\bar{\mu} \in \mathcal{D}$. For all $\mu \in \mathcal{D}$, define*

$$\Sigma_{\bar{\mu}}(\mu) = \sup_{v \in V} \frac{\|A(\mu)v\|_{\bar{\mu}, W'}}{\|v\|_V} \left(= \sup_{v \in V} \frac{\|A(\bar{\mu})^{-1}A(\mu)v\|_V}{\|v\|_V} \right).$$

Then, for all $\mu \in \mathcal{D}$ the solution $u(\mu) \in V$ to (2.1) satisfies

$$\forall \tilde{v} \in V, \quad 1 \leq \frac{\frac{1}{\sigma_{\bar{\mu}}(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{\bar{\mu}, W'}}{\|u(\mu) - \tilde{v}\|_V} \leq \frac{\Sigma_{\bar{\mu}}(\mu)}{\sigma_{\bar{\mu}}(\mu)},$$

furthermore $\Sigma_{\bar{\mu}}(\bar{\mu}) = \sigma_{\bar{\mu}}(\bar{\mu}) = 1$

Let us comment on the upper bound $\Sigma_{\bar{\mu}}(\mu)/\sigma_{\bar{\mu}}(\mu)$. This ratio can be interpreted as a *condition number*. It is $\mathcal{O}(1)$ for values of μ such that $A(\bar{\mu})^{-1}$ is a good left preconditioner for $A(\mu)$. With this understanding, the fact that both inequalities are equalities when $\mu = \bar{\mu}$ is due to the fact that $A(\bar{\mu})^{-1}$ is the ideal left preconditioner for $A(\bar{\mu})$. The sharpness of the bound provided by Theorem 3 is thus intimately linked to the properties of $A(\bar{\mu})^{-1}$ as left preconditioner for $A(\mu)$. In this sense, the dual natural-norm approach is a left preconditioning approach.

3.4 Re-interpretation of the primal natural-norm approach as a right-preconditioning approach

We now re-interpret the primal natural-norm approach reviewed in Sec. 2.3 as a right preconditioning approach. This shows the symmetry between the primal and dual natural-norm approaches. Notice that the arguments used in this section strongly rely on the fact that the Hilbert space V, W (and topological duals V', W') are finite dimensional.

Proposition 3.4. *Let $\bar{\mu} \in \mathcal{D}$. For all $\mu \in \mathcal{D}$, the primal natural norm constant $\alpha_{\bar{\mu}}(\mu)$ (defined in Theorem 2) can be equivalently defined as*

$$\alpha_{\bar{\mu}}(\mu) = \inf_{\ell_w \in W'} \frac{\|A(\mu)A(\bar{\mu})^{-1}\ell_w\|_{W'}}{\|\ell_w\|_{W'}},$$

and the constant $\gamma_{\bar{\mu}}(\mu)$ (defined in Proposition 2.2) can be equivalently defined as

$$\gamma_{\bar{\mu}}(\mu) = \sup_{\ell_w \in W'} \frac{\|A(\mu)A(\bar{\mu})^{-1}\ell_w\|_{W'}}{\|\ell_w\|_{W'}}.$$

Proof. Let $\ell_w \in W'$. Then there exists a unique solution $v \in V$ to the problem $A(\bar{\mu})v = \ell_w$. Thus,

$$\inf_{\ell_w \in W'} \frac{\|A(\mu)A(\bar{\mu})^{-1}\ell_w\|_{W'}}{\|\ell_w\|_{W'}} = \inf_{v \in V} \frac{\|A(\mu)v\|_{W'}}{\|A(\bar{\mu})v\|_{W'}} = \inf_{v \in V} \frac{\|v\|_{\mu, V}}{\|v\|_{\bar{\mu}, V}} = \alpha_{\bar{\mu}}(\mu).$$

We proceed analogously for $\gamma_{\bar{\mu}}(\mu)$, taking the supremum rather than infimum. \square

In the light of this Proposition, we can now re-interpret the ratio $\gamma_{\bar{\mu}}(\mu)/\alpha_{\bar{\mu}}(\mu)$ of Proposition 2.2 as the condition number from preconditioning $A(\mu)$ to the right using $A(\bar{\mu})^{-1}$ as preconditioner.

4 Practical natural-norm *a posteriori* error estimators

We now explain how the error bounds can be translated into practical *a posteriori* error estimators. The first concern is the derivation of practical (*i.e.*, computable) lower bounds for the μ -dependent stability constants (inf-sup and natural-norm constants) which represent a computational bottleneck.

The second concern is that error bounds based on the concept of natural-norm are only expected to be sharp locally in the neighborhood of a so-called fixed *anchor point* $\bar{\mu} \in \mathcal{D}$. Therefore, in order to estimate the error globally over \mathcal{D} , one must consider K local natural-norms based on a discrete set of K anchor points $\mathcal{C}^K = \{\bar{\mu}^1, \dots, \bar{\mu}^K\} \subset \mathcal{D}$ and an indicator function $\mathcal{I}^K : \mathcal{D} \rightarrow \mathcal{C}^K$ that maps each μ a unique "best" anchor point $\bar{\mu} \in \mathcal{C}^K$ in a sense that shall be defined shortly

4.1 Practical inf-sup based and primal natural-norm error estimators

Practical lower bounds for the inf-sup constant $\mu \in \mathcal{D} \mapsto \alpha(\mu)$ can be directly obtained using the Successive Constraints Method (SCM); *e.g.* see Refs. [12, 4, 23]. A variant – known as the natural-norm SCM; see Refs. [13, 5] – builds lower bounds for the inf-sup constant based on the following result, first shown in Ref. [22].

Proposition 4.1. *Let $\bar{\mu} \in \mathcal{D}$. Then for all $\mu \in \mathcal{D}$ the inf-sup constant $\alpha(\mu)$ (defined in (2.4)) can be bounded from below as*

$$\alpha(\mu) \geq \alpha(\bar{\mu})\alpha_{\bar{\mu}}(\mu),$$

where $\alpha_{\bar{\mu}}(\mu)$ is the primal natural-norm constant (defined in Theorem 2).

Proof. This is clear from

$$\begin{aligned} \alpha(\mu) &= \inf_{v \in V} \frac{\|v\|_{\mu,V}}{\|v\|_V} = \inf_{v \in V} \frac{\|v\|_{\bar{\mu},V} \|v\|_{\mu,V}}{\|v\|_V \|v\|_{\bar{\mu},V}} \\ &\geq \left(\inf_{v \in V} \frac{\|v\|_{\bar{\mu},V}}{\|v\|_V} \right) \left(\inf_{v \in V} \frac{\|v\|_{\mu,V}}{\|v\|_{\bar{\mu},V}} \right) = \alpha(\bar{\mu})\alpha_{\bar{\mu}}(\mu). \end{aligned}$$

□

In fact, the practical interest of this Proposition is very limited, because in practice the primal natural-norm constant is about as difficult to compute (or to approximate using SCM) as the inf-sup constant. The true interest of Proposition 4.1 is to replace the primal natural-norm constant by a more practical lower bound, provided by the following Proposition.

Proposition 4.2. *Let $\bar{\mu} \in \mathcal{D}$. For all $\mu \in \mathcal{D}$ define*

$$\bar{\alpha}_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\Re\{\langle A(\mu)v, R_W^{-1}A(\bar{\mu})v \rangle\}}{\|v\|_{\bar{\mu},V}^2}.$$

Then, for all $\mu \in \mathcal{D}$, $\alpha_{\bar{\mu}}(\mu) \geq \bar{\alpha}_{\bar{\mu}}(\mu)$. Furthermore $\bar{\alpha}_{\bar{\mu}}(\bar{\mu}) = 1$.

Proof. Recalling that $\|v\|_{\mu,V} = \|A(\mu)v\|_{W'} = \sup_{w \in W} \frac{|\langle A(\mu)v, w \rangle|}{\|w\|_W}$ we get

$$\alpha_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\|v\|_{\mu,V}}{\|v\|_{\bar{\mu},V}} = \inf_{v \in V} \sup_{w \in W} \frac{|\langle A(\mu)v, w \rangle|}{\|v\|_{\bar{\mu},V} \|w\|_W}.$$

We may choose the candidate supremizer $w = R_W^{-1}A(\bar{\mu})v$, yielding

$$\alpha_{\bar{\mu}}(\mu) \geq \inf_{v \in V} \frac{|\langle A(\mu)v, R_W^{-1}A(\bar{\mu})v \rangle|}{\|v\|_{\bar{\mu},V} \|R_W^{-1}A(\bar{\mu})v\|_W} = \inf_{v \in V} \frac{|\langle A(\mu)v, R_W^{-1}A(\bar{\mu})v \rangle|}{\|v\|_{\bar{\mu},V}^2} \geq \bar{\alpha}_{\bar{\mu}}(\mu),$$

where the last inequality simply stems from the fact that the modulus of a complex number is always greater than its real part. □

Remark. Proposition 4.2 would provide a sharper lower bound had the real part been replaced by the modulus in the definition of the constant $\bar{\alpha}_{\bar{\mu}}(\mu)$. We choose to consider the real part and not the modulus – thus accepting a less sharp lower bound – for purely practical reasons. Namely, $\bar{\alpha}_{\bar{\mu}}(\mu)$ can be computed as the smallest eigenvalue in the generalized hermitian eigenvalue problem: find $(\lambda, w) \in \mathbb{R} \times W$ such that $\frac{1}{2}(A(\mu)A(\bar{\mu})^{-1}R_W + R_W A(\bar{\mu})^* A(\mu)^*) w = \lambda R_W w$ in W . Justification for the form of this generalized hermitian eigenvalue problem will be provided in the proof of Proposition 4.4.

As shown in Ref. [22], under some regularity assumption on $\mu \mapsto A(\mu)$, the lower bound of Proposition 4.2 is second order accurate; in the sense

$$\alpha_{\bar{\mu}}(\mu) = \bar{\alpha}_{\bar{\mu}}(\mu) + \mathcal{O}(|\mu - \bar{\mu}|^2) \quad \text{as } \mu \rightarrow \bar{\mu}. \quad (4.1)$$

Combining the results from Proposition 4.2 and Proposition 4.1, we obtain the practical lower bound for the inf-sup constant: $\alpha(\mu) \geq \alpha(\bar{\mu})\bar{\alpha}_{\bar{\mu}}(\mu)$. Notice however that $\bar{\alpha}_{\bar{\mu}}(\mu)$ is not guaranteed to be positive (in opposition to $\alpha_{\bar{\mu}}(\mu)$, which is always > 0). It can typically turn negative when μ is "too distant" from $\bar{\mu}$ in some sense. When this is the case, our practical lower bound for the inf-sup constant becomes of no interest. For this reason, we define the *primal positivity coverage set* associated to any anchor point $\bar{\mu}$, as $\mathcal{D}_{\bar{\mu}}^{\text{pr}} = \{\mu \in \mathcal{D}, \bar{\alpha}_{\bar{\mu}}(\mu) > 0\}$. In this context, a "good" set of anchor points $\mathcal{C}^K = \{\bar{\mu}^1, \dots, \bar{\mu}^K\}$ is such that $\cup_{k=1}^K \mathcal{D}_{\bar{\mu}^k}^{\text{pr}} = \mathcal{D}$ and a possible associated indicator function \mathcal{I}^K maps each $\mu \in \mathcal{D}$ to the anchor point $\bar{\mu} \in \mathcal{C}^K$ such that the constant $\bar{\alpha}_{\bar{\mu}}(\mu)$ is largest, *i.e.*, $\mathcal{I}^K(\mu) = \underset{1 \leq k \leq K}{\operatorname{argmax}} \bar{\alpha}_{\bar{\mu}^k}(\mu)$.

All is now set to define our practical primal inf-sup based *a posteriori* error estimator

$$\forall \tilde{v} \in V, \quad \Delta_K^{\text{pr}}(\tilde{v}; \mu) = \frac{1}{\alpha(\bar{\mu})\bar{\alpha}_{\bar{\mu}}(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{W'}, \quad \text{with } \bar{\mu} = \mathcal{I}^K(\mu). \quad (4.2)$$

4.2 Practical error estimator based on dual natural-norm

In the same fashion, we now construct a practical error estimator based on the the dual natural norm.

Proposition 4.3. *Let $\bar{\mu} \in \mathcal{D}$. For all $\mu \in \mathcal{D}$, define*

$$\bar{\sigma}_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\Re\{\langle A(\mu)v, A(\bar{\mu})^{-*}R_V v \rangle\}}{\|v\|_V^2}.$$

Then for all $\mu \in \mathcal{D}$, $\bar{\sigma}_{\bar{\mu}}(\mu) \leq \sigma_{\bar{\mu}}(\mu)$. Furthermore $\bar{\sigma}_{\bar{\mu}}(\bar{\mu}) = 1$ and assuming $\mu \mapsto A(\mu)$ is differentiable in the neighborhood of $\bar{\mu}$ there holds

$$\sigma_{\bar{\mu}}(\mu) = \bar{\sigma}_{\bar{\mu}}(\mu) + \mathcal{O}(|\mu - \bar{\mu}|^2) \quad \text{as } \mu \rightarrow \bar{\mu}. \quad (4.3)$$

Proof. We start by the definition

$$\sigma_{\bar{\mu}}(\mu) = \inf_{v \in V} \frac{\|A(\bar{\mu})^{-1}A(\mu)v\|_V}{\|v\|_V} = \inf_{v \in V} \sup_{\ell \in V'} \frac{|\langle \ell, A(\bar{\mu})^{-1}A(\mu)v \rangle|}{\|v\|_V \|\ell\|_{V'}}. \quad (4.4)$$

Choose the candidate supremizer $\ell = R_V v$ and use the fact that the modulus of a complex number is always an upper bound for its real part.

In order to demonstrate the second order accuracy, we can refer to same arguments in the case of the primal, see Ref. [22]. We repeat the essential steps for completeness. Let us start from the definition of $\sigma_{\bar{\mu}}(\mu)$ and proceed as follows

$$(\sigma_{\bar{\mu}}(\mu))^2 = \inf_{v \in V} \frac{\|A(\bar{\mu})^{-1}A(\mu)v\|_V^2}{\|v\|_V^2} = \inf_{v \in V} \frac{\|v + A(\bar{\mu})^{-1}(A(\mu) - A(\bar{\mu}))v\|_V^2}{\|v\|_V^2}. \quad (4.5)$$

If $\mu \mapsto A(\mu)$ is differentiable in the neighborhood of $\bar{\mu}$, then $\|A(\bar{\mu})^{-1}(A(\mu) - A(\bar{\mu}))v\|_V^2 / \|v\|_V^2 = \mathcal{O}(|\mu - \bar{\mu}|^2)$. In this situation, developing the square in Eq. (4.5) yields

$$\begin{aligned} (\sigma_{\bar{\mu}}(\mu))^2 &= 1 + \inf_{v \in V} \frac{2\Re\{\langle R_V A(\bar{\mu})^{-1}(A(\mu) - A(\bar{\mu}))v, v \rangle\}}{\|v\|_V^2} + \mathcal{O}(|\mu - \bar{\mu}|^2) \\ &= 1 + 2 \left(\inf_{v \in V} \frac{\Re\{\langle R_V A(\bar{\mu})^{-1}A(\mu)v, v \rangle\}}{\|v\|_V^2} - 1 \right) + \mathcal{O}(|\mu - \bar{\mu}|^2). \end{aligned} \quad (4.6)$$

Thus, we have $\sigma_{\bar{\mu}}(\mu) = (1 + 2(\bar{\sigma}_{\bar{\mu}}(\mu) - 1) + \mathcal{O}(|\mu - \bar{\mu}|^2))^{1/2}$. We conclude by invoking the the formula $(1 + t)^{1/2} = 1 + \frac{1}{2}t + \mathcal{O}(t^2)$ for $t = 2(\bar{\sigma}_{\bar{\mu}}(\mu) - 1)$, combined with the fact that $(\bar{\sigma}_{\bar{\mu}}(\mu) - 1) = \mathcal{O}(|\mu - \bar{\mu}|)$. \square

Notice that we have defined the lower bound $\bar{\sigma}_{\bar{\mu}}(\mu)$ uses a real part and not a module, again for purely practical reasons (see Remark 4.1). Moreover, similar to its primal counterpart $\bar{\alpha}_{\bar{\mu}}(\mu)$, the constant $\bar{\sigma}_{\bar{\mu}}(\mu)$ is not guaranteed to be positive, so we must introduce the *dual positivity coverage set* associated to a given anchor point $\bar{\mu} \in \mathcal{C}^K$ as $\mathcal{D}_{\bar{\mu}}^{\text{du}} = \{\mu \in \mathcal{D}, \bar{\sigma}_{\bar{\mu}}(\mu) > 0\}$. Again, a "good" set of anchor points $\mathcal{C}^K = \{\bar{\mu}^1, \dots, \bar{\mu}^K\}$ is such that $\cup_{k=1}^K \mathcal{D}_{\bar{\mu}^k}^{\text{du}} = \mathcal{D}$ and a possible associated indicator function \mathcal{I}^K maps each $\mu \in \mathcal{D}$ to the anchor point $\bar{\mu} \in \mathcal{C}^K$ such that the constant $\bar{\sigma}_{\bar{\mu}}(\mu)$ is largest, *i.e.*, $\mathcal{I}^K(\mu) = \underset{1 \leq k \leq K}{\operatorname{argmax}} \bar{\sigma}_{\bar{\mu}^k}(\mu)$.

All is now set to define our practical dual natural-norm based *a posteriori* error estimator

$$\forall \tilde{v} \in V, \quad \Delta_K^{\text{du}}(\tilde{v}; \mu) = \frac{1}{\bar{\sigma}_{\bar{\mu}}(\mu)} \|A(\mu)\tilde{v} - f(\mu)\|_{\mu, W'}, \quad \text{with } \bar{\mu} = \mathcal{I}^K(\mu). \quad (4.7)$$

4.3 The self-adjoint case

Proposition 4.4. *In the self-adjoint case, (i.e., $V = W$ and $A(\mu) = A(\mu)^*$), there holds,*

$$\forall \mu \in \mathcal{D}, \quad \bar{\alpha}_{\bar{\mu}}(\mu) = \bar{\sigma}_{\bar{\mu}}(\mu),$$

where $\bar{\alpha}_{\bar{\mu}}(\mu)$ is defined in Proposition 4.2 and $\bar{\sigma}_{\bar{\mu}}(\mu)$ is defined in Proposition 4.3.

Proof. This is straightforward observing that $\bar{\alpha}_{\bar{\mu}}(\mu)$ is equivalently given by

$$\bar{\alpha}_{\bar{\mu}}(\mu) = \inf_{w \in W} \frac{\Re\{\langle A(\mu)A(\bar{\mu})^{-1}R_W w, w \rangle\}}{\|w\|_W^2}.$$

□

Thanks to the preliminary result given by Proposition 4.4, we can show that the dual natural-norm error estimator necessarily outperforms the inf-sup based error estimator in the self-adjoint case.

Theorem 4. *Let $\mathcal{C}^K = \{\bar{\mu}^1, \dots, \bar{\mu}^K\} \subset \mathcal{D}$ and consider the self-adjoint case. Denote*

$$\mathcal{D}^+ = \bigcup_{k=1}^K \mathcal{D}_{\bar{\mu}^k}^{\text{pr}} \left(= \bigcup_{k=1}^K \mathcal{D}_{\bar{\mu}^k}^{\text{du}} \right).$$

Then, for all $\mu \in \mathcal{D}^+$, the solution $u(\mu) \in V$ to (2.1) satisfies

$$\forall \tilde{v} \in V, \quad \|u(\mu) - \tilde{v}\|_V \leq \Delta_K^{\text{du}}(\tilde{v}; \mu) \leq \Delta_K^{\text{pr}}(\tilde{v}; \mu),$$

with Δ_K^{du} the dual natural-norm error estimator defined by (4.7) and Δ_K^{pr} the inf-sup based error estimator defined by (4.2).

Proof. Let $\mu \in \mathcal{D}$ and denote $\bar{\mu} = \mathcal{I}^K(\mu)$ the associated anchor point. Theorem 3 states that

$$\|u(\mu) - \tilde{v}\|_V \leq \frac{1}{\sigma_{\bar{\mu}}(\mu)} \|A(\bar{\mu})^{-1}(A(\mu)\tilde{v} - f(\mu))\|_V.$$

From the inequality $\frac{1}{\sigma_{\bar{\mu}}(\mu)} \leq \frac{1}{\bar{\sigma}_{\bar{\mu}}(\mu)}$, which stems from Proposition 4.3 combined to the fact that $\bar{\sigma}_{\bar{\mu}}(\mu) > 0$; we obtain the first inequality announced in the theorem.

The second inequality is a consequence of the equivalence of the $\|A(\bar{\mu})^{-1} \cdot\|_V$ norm and the $\|\cdot\|_{W'}$ norm, established in Proposition 3.2. Namely,

$$\|A(\bar{\mu})^{-1}(A(\mu)\tilde{v} - f(\mu))\|_V \leq \frac{1}{\beta(\bar{\mu})} \|A(\mu)\tilde{v} - f(\mu)\|_{W'}.$$

We have $\beta(\bar{\mu}) = \alpha(\bar{\mu})$ from the self-adjoint hypothesis. \square

5 Computational strategy

5.1 Offline/online strategy

In this section, we consider a Reduced Basis (RB) approximation space $\widetilde{V}_N \subset V$, with small dimension $N \ll \mathcal{N}$ and a RB approximation $\widetilde{u}_N(\mu)$ in this N -dimensional RB subspace. The resulting RB approximation error can be bounded using our dual natural-norm *a posteriori* error estimator (4.7) as

$$\|u(\mu) - \widetilde{u}_N(\mu)\|_V \leq \Delta_K^{\text{du}}(\widetilde{u}_N(\mu); \mu). \quad (5.1)$$

We now explain how $\Delta_K^{\text{du}}(\widetilde{u}_N(\mu); \mu)$ can be efficiently computed. There are two components in our error estimator: a stability constant (namely, $\bar{\sigma}_{\bar{\mu}}(\mu)$) which we propose to replace by a cheap SCM lower bound, and the dual natural-norm of the residual which can be efficiently computed following an offline/online strategy.

5.1.1 The affine hypothesis

Following the Reduced Basis Method standard [10, 20] we require that the operator $A(\mu) \in \mathcal{L}(V, W')$ is affinely parametrized, that is, that there exist $Q \geq 1$ μ -independent operators $A_q \in \mathcal{L}(V, W')$, $1 \leq q \leq Q$ and complex valued functions $\varsigma_q : \mathcal{D} \rightarrow \mathbb{C}$, $1 \leq q \leq Q$ such that

$$\forall \mu \in \mathcal{D}, \quad A(\mu) = \sum_{q=1}^Q \varsigma_q(\mu) A_q. \quad (5.2)$$

Similarly, we require the right-hand-side $f(\mu) \in W'$ to be affinely parametrized, that is, that there exist $Q^f \geq 1$ μ -independent linear forms $f_q \in W'$, $1 \leq q \leq Q^f$ and complex valued functions $\varsigma_q^f : \mathcal{D} \rightarrow \mathbb{C}$, $1 \leq q \leq Q^f$ such that

$$\forall \mu \in \mathcal{D}, \quad f(\mu) = \sum_{q=1}^{Q^f} \varsigma_q^f(\mu) f_q. \quad (5.3)$$

Note that if the operator or right-hand side do not satisfy the affine assumption, the Empirical Interpolation Method (EIM) can be employed to recover affinely parametrized approximations [2, 18].

5.1.2 Computing a lower bound for the dual natural norm constant

Let $\bar{\mu} \in \mathcal{D}$ be a given anchor point. We now explain how a lower bound for $\bar{\sigma}_{\bar{\mu}}(\mu)$ can be efficiently computed for any μ query. This is key to the success of the proposed method; otherwise our practical dual natural-norm error estimator would not be efficiently computable. By definition, $\bar{\sigma}_{\bar{\mu}}(\mu)$ is the smallest eigenvalue in the generalized hermitian eigenvalue problem: *find* $(\lambda, v) \in \mathbb{R} \times V$ such that

$$H_{\bar{\mu}}(\mu)v = \lambda R_V v \quad \text{in } V' \quad (5.4)$$

with $H_{\bar{\mu}}(\mu) \in \mathcal{L}(V, V')$ given by

$$\forall \mu \in \mathcal{D}, \quad H_{\bar{\mu}}(\mu) = \frac{1}{2} (A(\mu)^* A(\bar{\mu})^{-*} R_V + R_V A(\bar{\mu})^{-1} A(\mu)). \quad (5.5)$$

Note that $H_{\bar{\mu}}(\mu)$ is self-adjoint, but that it is not necessarily positive definite. An approach based on solving the generalized eigenvalue problem (5.4) for

each μ query would lead to prohibitive computational costs. We propose to use the SCM in order to compute cheap lower and upper bounds for $\mu \mapsto \bar{\sigma}_{\bar{\mu}}(\mu)$, using a computationally efficient offline/online strategy. Clearly, using the affine representation (5.2) of $A(\mu)$, the $H_{\bar{\mu}}(\mu)$ operator admits the following affine representation

$$H_{\bar{\mu}}(\mu) = \sum_{q=1}^Q \Re\{\varsigma_q(\mu)\} \frac{1}{2} (A_q^* A(\bar{\mu})^{-*} R_V + R_V A(\bar{\mu})^{-1} A_q) + \sum_{q=1}^Q \Im\{\varsigma_q(\mu)\} \frac{1}{2} (i A_q^* A(\bar{\mu})^{-*} R_V - i R_V A(\bar{\mu})^{-1} A_q). \quad (5.6)$$

Thus, we have $H_{\bar{\mu}}(\mu) = \sum_{q=1}^{2Q} \theta_q(\mu) H_{\bar{\mu},q}$ where $H_{\bar{\mu},q} \in \mathcal{L}(V, V')$, $1 \leq q \leq 2Q$ are μ -independent *self-adjoint* operators and $\theta_q : \mathcal{D} \rightarrow \mathbb{R}$, $1 \leq q \leq 2Q$ are *real-valued* functions. In this context, the SCM can be readily applied; *e.g.* see Refs. [12, 13, 23].

5.1.3 The dual natural-norm of the RB residual

Given an anchor point $\bar{\mu} \in \mathcal{D}$, we explain how the dual natural-norm of the residual $\|A(\mu)\widetilde{u}_N(\mu) - f(\mu)\|_{\bar{\mu}, W'}$ can be efficiently computed for all $\mu \in \mathcal{D}$. Let us assume the following decomposition for $\widetilde{u}_N(\mu)$ in the RB subspace $\widetilde{V}_N := \text{Span}\{\xi_1, \dots, \xi_N\} \subset V$,

$$\widetilde{u}_N(\mu) = \sum_{i=1}^N c_i(\mu) \xi_i. \quad (5.7)$$

With the RB method, the coefficients $c_i(\mu)$, $1 \leq i \leq N$ can be obtained very efficiently for any value of $\mu \in \mathcal{D}$ by solving a $N \times N$ linear system [20, 10]. For all $\mu \in \mathcal{D}$, there holds

$$\|A(\mu)\widetilde{u}_N(\mu) - f(\mu)\|_{\bar{\mu}, W'}^2 = \|A(\bar{\mu})^{-1}(A(\mu)\widetilde{u}_N(\mu) - f(\mu))\|_V^2 \quad (5.8)$$

Using the affine representations Eqs. (5.2) and (5.3) and the expression (5.7) for the RB approximation, we get from developing the square

$$\begin{aligned}
\|A(\mu)\widetilde{u}_N(\mu) - f(\mu)\|_{\bar{\mu}, W'}^2 &= \sum_{1 \leq q, p \leq Q^f} \varsigma_q^f(\mu) \overline{\varsigma_p^f(\mu)} \langle R_V A(\bar{\mu})^{-1} f_q, A(\bar{\mu})^{-1} f_p \rangle \\
&\quad \sum_{1 \leq i, j \leq N} \sum_{1 \leq q, p \leq Q} c_i(\mu) \overline{c_j(\mu)} \varsigma_q(\mu) \overline{\varsigma_p(\mu)} \langle R_V A(\bar{\mu})^{-1} A_q \xi_i, A(\bar{\mu})^{-1} A_p \xi_j \rangle \\
&\quad - 2 \sum_{1 \leq q \leq Q} \sum_{1 \leq p \leq Q^f} \sum_{1 \leq i \leq N} \Re \left\{ \varsigma_q(\mu) \overline{\varsigma_p^f(\mu)} c_i(\mu) \langle R_V A(\bar{\mu})^{-1} A_q \xi_i, A(\bar{\mu})^{-1} f_p \rangle \right\}.
\end{aligned} \tag{5.9}$$

Notice that none of the duality brackets depend on μ . Thus, these duality brackets can be computed once during the so-called *offline* phase. For each query $\mu \in \mathcal{D}$ (during the so-called *online* phase), the pre-computed duality brackets can be used to compute the dual natural-norm of the residual in $\mathcal{O}((Q^f)^2 + N^2 Q^2 + N Q Q^f)$ complexity using the formula (5.9).

Remark that during the offline phase, one must solve $NQ + Q^f$ problems of the form: *find* $y \in V$ *such that* $A(\bar{\mu})y = z$. Namely, for $z = A_q \xi_i$ ($1 \leq i \leq N$, $1 \leq q \leq Q$) and for $z = f_q$ ($1 \leq q \leq Q^f$). The number of problems to be solved is therefore $K(NQ + Q^f)$ when a set \mathcal{C}^K of K anchor points is considered.

5.2 Procedure for selecting anchor points

We now present a strategy for constructing the set of anchor points $\mathcal{C}^K = \{\bar{\mu}^1, \dots, \bar{\mu}^K\} \subset \mathcal{D}$. Let us adopt a discrete setting, by introducing an adequately fine discrete surrogate set $\Xi \subset \mathcal{D}$. Clearly, if we want to be able to estimate the error globally, the set of anchor points must be built in order that the following (discrete) *coverage property* holds

$$\forall \mu \in \Xi, \quad \exists \bar{\mu} \in \mathcal{C}^K, \quad \bar{\sigma}_{\bar{\mu}}(\mu) > \varrho, \tag{5.10}$$

or equivalently,

$$\forall \mu \in \Xi, \quad \max_{\bar{\mu} \in \mathcal{C}^K} \bar{\sigma}_{\bar{\mu}}(\mu) > \varrho, \tag{5.11}$$

where $\varrho \in [0, 1[$ is a prescribed threshold. Notice that with the strict inequalities, choosing $\varrho = 0$ will ensure that $\mu \mapsto \max_{\bar{\mu} \in \mathcal{C}^K} \bar{\sigma}_{\bar{\mu}}(\mu)$ remains strictly positive over Ξ , which is the minimum requirement to be able to estimate the

error over Ξ . We leave the possibility of considering $\varrho > 0$ if one is interested in sharper error estimates.

The procedure that we propose consists in building a sequence $\{\Xi_k\}_{k \geq 1}$, with $\Xi_1 = \Xi$ that will ultimately converge to \emptyset as follows:

1. Pick arbitrarily $\bar{\mu}^1$ in Ξ , set $\Xi_1 = \Xi$, $\mathcal{C}^0 = \emptyset$ and $k = 1$;
2. Update the set of anchor points $\mathcal{C}^k = \mathcal{C}^{k-1} \cup \{\bar{\mu}^k\}$;
3. Update training set $\Xi_{k+1} \leftarrow \Xi_k \setminus \{\mu \in \Xi_k, \bar{\sigma}_{\bar{\mu}^k}(\mu) > \varrho\}$;
4. If $\Xi_{k+1} \neq \emptyset$, then find

$$\bar{\mu}^{k+1} \leftarrow \operatorname{argmin}_{\mu \in \Xi_k} \max_{\bar{\mu} \in \mathcal{C}^k} \bar{\sigma}_{\bar{\mu}}(\mu), \quad (5.12)$$

set $k = k + 1$ and go back to (ii). Else, terminate.

At each iteration $k \geq 1$ such that $\Xi_k \neq \emptyset$, we consider a new anchor point $\bar{\mu}^k \in \Xi_k$, and construct the set $\Xi_k^+ = \{\mu \in \Xi_k, \bar{\sigma}_{\bar{\mu}^k}(\mu) > \varrho\}$ (this can be done efficiently using SCM). This set is guaranteed to be non-empty because it has at least one member: $\bar{\mu}^k$, using the fact that $\bar{\sigma}_{\bar{\mu}^k}(\bar{\mu}^k) = 1$ (see Proposition 4.3). Thus, the set $\Xi_{k+1} = \Xi_k \setminus \Xi_k^+$ is guaranteed to be a strict subset of Ξ_k . This demonstrates that the sequence $\{\Xi_k\}_{k \geq 1}$ converges to \emptyset in at most $\operatorname{Card}(\Xi)$ iterations and so the procedure terminates. Note that, in practice, much less than $\operatorname{Card}(\Xi)$ iterations will be required for convergence as we shall see in the numerical examples.

Furthermore, there is no difficulty in showing that at iteration $k \geq 1$, we have

$$\Xi = \Xi_{k+1} \cup \left(\bigcup_{\kappa=1}^k \Xi_{\kappa}^+ \right), \quad \Xi_{k+1} \cap \left(\bigcup_{\kappa=1}^k \Xi_{\kappa}^+ \right) = \emptyset, \quad (5.13)$$

thus the procedure terminates at iteration K such that $\Xi = \bigcup_{\kappa=1}^K \Xi_{\kappa}^+$, which means that the discrete coverage property holds.

6 Numerical results

6.1 Problem setting

Let $\Omega =]0, 1[\times]0, 1[$. The boundary of domain is divided into a Dirichlet boundary $\Gamma_D =]0, 1[\times \{0\}$ and a Neumann boundary $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Let

$f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma_N)$. We consider the 2D Helmholtz equation, parametrized by $\mu = (\mu_1, \mu_2) \in \mathcal{D}$: find $u^{\text{ex}}(\cdot; \mu) \in H^1(\Omega)$

$$\begin{cases} -\operatorname{div} \left(\begin{pmatrix} 1 & \nu \\ 0 & \mu_1 \end{pmatrix} \nabla u^{\text{ex}}(\mu) \right) - \mu_2 u^{\text{ex}}(\mu) = f, & \text{in } \Omega, \\ u^{\text{ex}}(\mu)|_{\Gamma_D} = 0, & \nabla u^{\text{ex}}(\mu) \cdot \mathbf{n}|_{\Gamma_N} = g. \end{cases} \quad (6.1)$$

The parameter μ_1 controls the anisotropy of the speed of sound, while the parameter μ_2 corresponds to the squared wavenumber. The constant ν (not a parameter) also controls the anisotropy of the speed of sound (isotropic speed of sound corresponds to $\nu = 0$, $\mu_1 = 1$). Notice that when $\nu = 0$ the problem is self-adjoint and corresponds to the benchmark proposed in Ref. [13] and addressed more recently in Ref. [24]. When $\nu > 0$, the problem is no longer self-adjoint.

We use the Finite Element (FE) method to discretize the weak form of (6.1). We define the Hilbert space V as the Lagrange P^1 approximation space, formed by globally continuous, piecewise first-order polynomial functions that vanish on the boundary Γ_D . This FE space being $H^1(\Omega)$ -conforming, the norm on V is simply the usual $\|\cdot\|_{H^1(\Omega)}$ norm. Using a triangulation of Ω , the dimension of this FE space is $\mathcal{N} = 3436$. The FE approximation $u(\mu) \in V$ is defined as the Galerkin projection of $u^{\text{ex}}(\mu)$ on V , which amounts to considering the test space $W = V$.

We further define a RB approximation $\widetilde{u}_N(\mu)$ as the Galerkin projection of $u(\mu)$ onto the RB space, meaning $\widetilde{u}_N(\mu) \in \widetilde{V}_N := \operatorname{Span}\{u(\mu_1), \dots, u(\mu_N)\}$, where the parameters μ_1, \dots, μ_N are selected in a greedy way based on the $\|\cdot\|_{W'}$ norm of the residual, following standard practice of the RB Method [10, 20]. Our goal will be able to estimate the RB error $\|u(\mu) - \widetilde{u}_N(\mu)\|_V$ for $\mu \in \mathcal{D}$.

Remark that, without any *a priori* knowledge on the possible location of resonant parameters, finding a "resonance-free" set \mathcal{D} is not an easy task. In our numerical tests, we shall consider $\mathcal{D} \subset \widetilde{\mathcal{D}} = [0.8, 1.2] \times [10, 50]$, as in Ref. [13]. We consider the two possible values $\nu = 0$ (self-adjoint case) and $\nu = 0.5$ (non self-adjoint case). We can see the norm of the FE solution for 2000 random points in $\widetilde{\mathcal{D}}$ on Figs. 1 and 2. We can visually see 4 resonance lines where the norm of the FE solution is maximal. Notice that the location of resonance lines slightly differ between the self-adjoint and non self-adjoint cases.

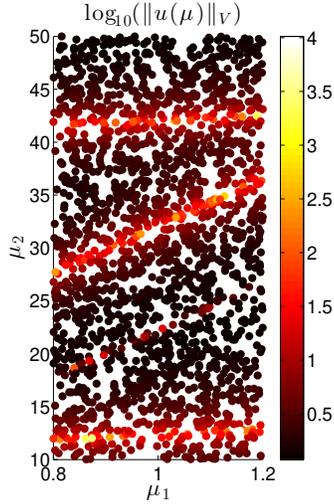


Figure 1: The norm of the FE solution, for 2000 random points in \mathcal{D} in self-adjoint case.

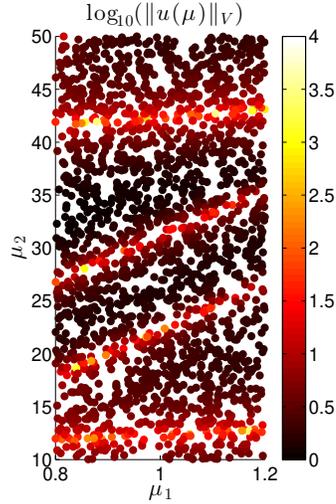


Figure 2: The norm of the FE solution, for 2000 random points in \mathcal{D} in non self-adjoint case.

Remark. As can be seen Figs. 1 and 2, there are some resonant parameter values in the compact set $\tilde{\mathcal{D}} = [0.8, 1.2] \times [10, 50]$. Denote \mathcal{D}^{res} the set of resonant values in $\tilde{\mathcal{D}}$ for which the Banach-Necas-Babuska assumptions are not satisfied. In this work, we choose to address the problem as if we had no a priori knowledge of the existence of this set \mathcal{D}^{res} . As we shall see, our method is constructive of a discrete surrogate set for the "resonance-free" set $\mathcal{D} \subset \tilde{\mathcal{D}}$ satisfying $\mathcal{D}^{\text{res}} \subset (\tilde{\mathcal{D}} \setminus \mathcal{D})$.

6.2 Self-adjoint case

6.2.1 The natural-norm constants

We test our anchor point selection procedure with $\varrho = 0$ and a surrogate set $\Xi \subset \tilde{\mathcal{D}}$ made of 2000 random points (uniformly distributed). The algorithm terminates with $K = 6$ anchor points. On Fig. 3, we have plotted the obtained SCM lower bounds $\mu \mapsto \exp(\bar{\sigma}_{\bar{\mu}}^{LB}(\mu))$ for each of the $K = 6$ selected anchor points $\bar{\mu} \in \mathcal{C}^K$. The reason for taking the $\exp(\cdot)$ is to obtain a better visualization, recalling that the lower bound for the dual natural-norm constant can become negative.

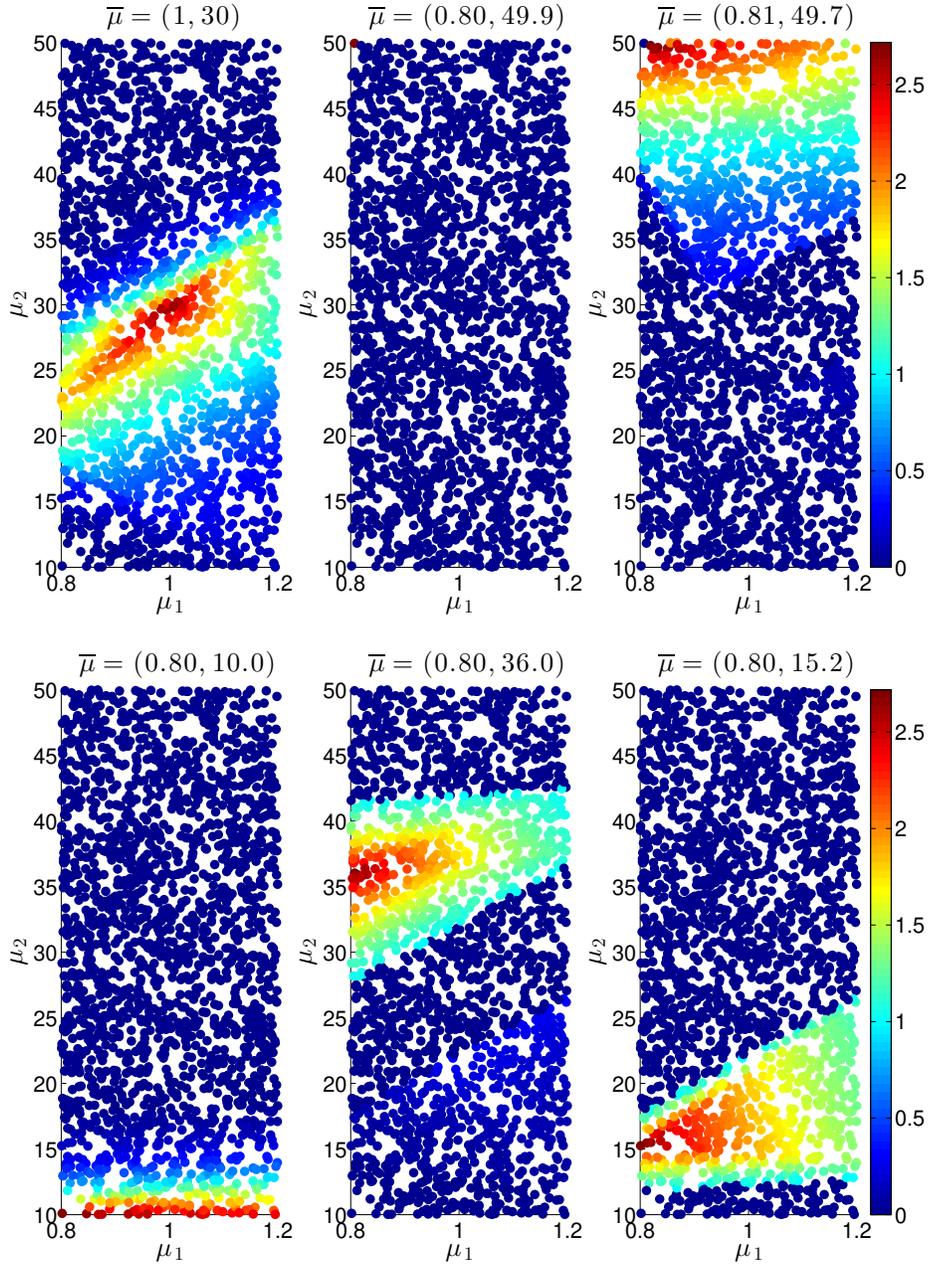


Figure 3: Self adjoint case: The SCM lower bound for the dual natural-norm constants $\mu \mapsto \exp(\bar{\sigma}_{\bar{\mu}}^{LB}(\mu))$ for the $K = 6$ successive values of $\bar{\mu}$ determined by the anchor point selection procedure.

A close comparison with Fig. 1 reveals that the dual natural-norm constant $\bar{\sigma}_{\bar{\mu}}(\mu)$ is only positive for values of μ such that μ and $\bar{\mu}$ can be joined without crossing any resonance lines. Interestingly, we have found $K = 6$, when $K = 5$ could have been expected from Fig. 1. In fact, we have checked that the anchor point $\bar{\mu} = (0.80, 49.9)$ is indispensable in order to achieve the positivity coverage, because there is indeed a resonance line to be crossed to reach this point starting from all previously selected anchor points. Let us now analyze the computational effort. At each iteration k of the anchor point selection procedure, a SCM algorithm is called in order to efficiently compute the lower bounds. In our numerical experiments, we have set the prescribed SCM tolerance to $tol = 0.9$. We have consigned in Table 1 the number of eigensolves of the generalized eigenvalue problem (5.4) performed during each call to the SCM.

k	1	2	3	4	5	6
Eigensolves	15	1	8	3	11	7

Table 1: Number of times that the generalized eigenvalue problem (5.4) must be solved at each iteration k of the anchor point selection procedure.

Comparing with Fig. 3, we find that the required number of eigensolves depends on the size of the positivity coverage. Typically, if the positivity coverage is vaster, then more eigensolves are needed.

Remark. *We notice a sensibility to the sampling of the surrogate set $\Xi \subset \tilde{\mathcal{D}}$. Namely, a different sampling of the 2000 uniformly distributed random points in which the point $(0.80, 49.9)$ was absent led to $K = 5$. In this situation, we have not been able to find an index $k = 1, \dots, 5$ such that $\bar{\sigma}_{\bar{\mu}^k}(\mu)$ for $\mu = (0.80, 49.9)$ was > 0 . This illustrates the potential risks that the discrete positivity coverage property (5.11) is dependent on the choice of surrogate set Ξ .*

6.2.2 Error estimates

We now consider a reduced basis approximation \tilde{V}_N of dimension $N = 15$. In order to assess the performance of our error estimators, we solve both FE and RB problems for all $\mu \in \Xi$, where $\Xi \subset \mathcal{D}$ is a random set of cardinality 2000. We have re-sampled the random points, thus this set Ξ is different from set the one used for selecting the anchor points. On Fig. 4, we have plotted two effectivity distributions; namely

- the effectivity distribution of the practical inf-sup based error estimator, that is $\{\Delta_K^{\text{pr}}(\widetilde{u}_N(\mu); \mu) / \|u(\mu) - \widetilde{u}_N(\mu)\|_V, \mu \in \Xi\}$ (top);
- the effectivity distribution of the practical dual natural-norm based error estimator, that is $\{\Delta_K^{\text{du}}(\widetilde{u}_N(\mu); \mu) / \|u(\mu) - \widetilde{u}_N(\mu)\|_V, \mu \in \Xi\}$ (bottom);

we have further consigned the essential statistics of these two effectivity distributions in Table 2.

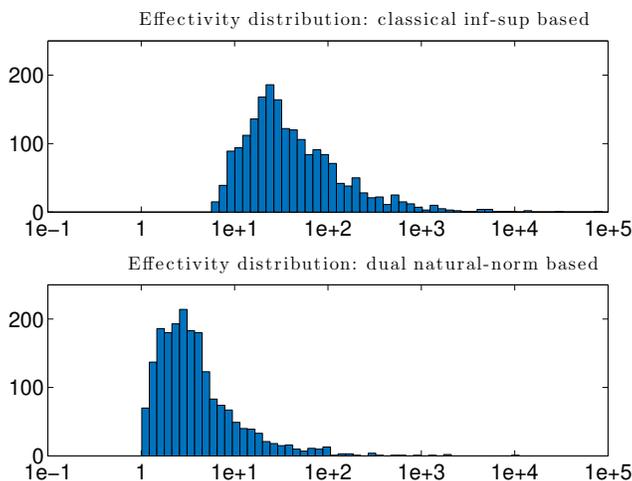


Figure 4: Self adjoint case: Effectivity distribution of the practical inf-sup based error estimator (top) and of the practical dual natural-norm based error estimator (bottom), obtained from 2000 random parameter samples in \mathcal{D} .

Error estimator	Max	85% quantile	Median	Mean
Inf-sup based	8.61×10^4	1.32×10^2	3.16×10^1	1.99×10^2
Dual natural-norm	1.11×10^4	9.51	3.17	1.74×10^1

Table 2: Effectivity statistics in self-adjoint case (based on the distribution shown on Fig. 4).

We find the inf-sup based error estimator to overestimate the error by at least one order of magnitude. On the contrary, the dual natural-norm based

error estimator captures the correct order of magnitude of the error for 85% of the considered values of $\mu \in \mathcal{D}$. For both error estimators, the amount of overestimation can become as large as 4 orders of magnitude. However, this phenomenon only occurs very locally; namely near the resonance lines. This is confirmed by Fig. 5, where we have plotted the effectivity in the (μ_1, μ_2) plane, and where we find all maximum values of effectivity to be located in the neighborhood of a resonant line. The tails of the distributions on Fig. 4 reflect the small probability of a random parameter value to be located very close to a resonant line.

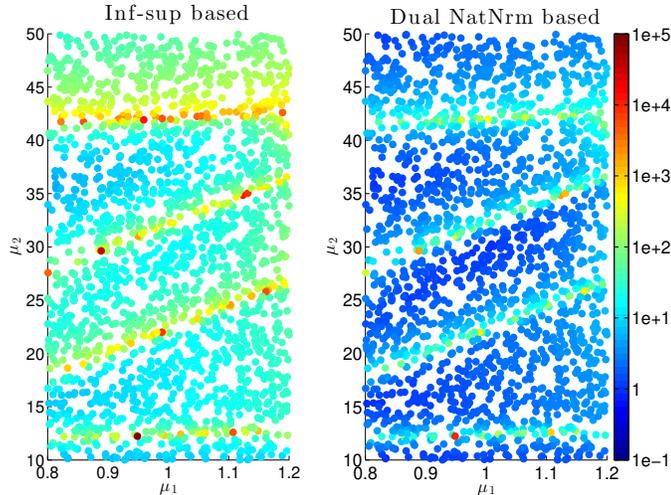


Figure 5: Self adjoint case: The effectivity of the inf-sup (left) and dual natural-norm (right) error estimators plotted as functions of $\mu = (\mu_1, \mu_2)$.

6.3 Non self-adjoint case

6.3.1 The natural-norm constants

We now address the non self-adjoint case. Recall that, in this case, there is a distinction between the primal natural-norm constant $\bar{\alpha}_{\bar{\mu}}(\mu)$ and the dual natural-norm constant $\bar{\sigma}_{\bar{\mu}}(\mu)$. We highlight the differences between these two constants in Fig. 6. It is worth noticing that the two coverage sets $\mathcal{D}_{\bar{\mu}}^{\text{pr}} = \{\mu \in \mathcal{D}, \bar{\alpha}_{\bar{\mu}}(\mu) > 0\}$ and $\mathcal{D}_{\bar{\mu}}^{\text{du}} = \{\mu \in \mathcal{D}, \bar{\sigma}_{\bar{\mu}}(\mu) > 0\}$ slightly differ.

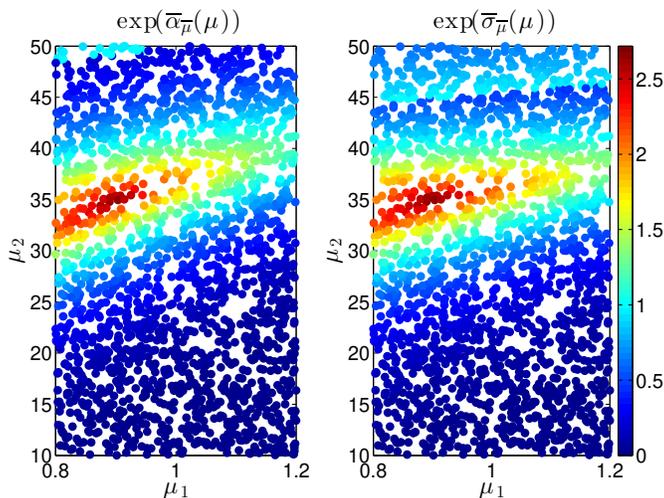


Figure 6: Non self-adjoint case: Comparison between the primal (left) and dual natural-norm (right) constants plotted as functions of $\mu = (\mu_1, \mu_2)$, with same anchor point $\bar{\mu} = (0.9, 35)$.

We test our anchor point selection procedure with $\varrho = 0$ and a surrogate set $\Xi \subset \tilde{\mathcal{D}}$ made of 2000 random points (uniformly distributed). In order for the algorithm to terminate, we had to slightly change the stopping criterion. Indeed, we found that stopping at the iteration k such that $\Xi_k = \emptyset$ was irrelevant in this situation due to the presence of resonances. We relaxed this criterion by stopping at the iteration k such that $\text{Card}(\Xi_k) \leq \lceil (1 - q)\text{Card}(\Xi) \rceil$, where $q \in]0, 1]$ is some fraction, $\text{Card}(\Xi)$ the number of points in the initial surrogate parameter set and $\lceil \cdot \rceil$ denotes the ceiling operation. Under this new criterion with $q = 0.95$, the algorithm converged in $K = 15$ iterations. This means $K = 15$ anchor points are enough to obtain the discrete coverage property over $\Xi^+ = \Xi \setminus \Xi^-$, where Ξ^- denotes the set comprised of the 5% of parameter points over which the discrete coverage property is not satisfied; *i.e.*, for all $\mu \in \Xi^-$, for all $1 \leq k \leq K$, $\bar{\sigma}_{\bar{\mu}^k}(\mu) \leq 0$. We have checked that the points in set Ξ^- correspond to points near the resonance lines. Thus, our method is constructive of the set Ξ^+ , which is a discrete surrogate set for the *a priori* unknown "resonance-free" set \mathcal{D} .

Looking at the converge curve on Fig. 7, we further find that the $K = 7$ first iterations already achieve 91.6% of the positivity coverage. The next iterations add small positivity coverage patches near the resonance lines to

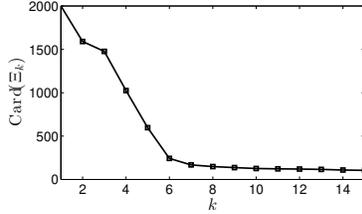


Figure 7: Convergence curve of anchor point selection procedure in the non self-adjoint case.

achieve the desired 95% positivity coverage property.

6.3.2 Error Estimates

We consider a reduced basis of size $N = 15$. In order to assess the performance of our dual natural-norm estimator, we solve both FE and RB problems for all $\mu \in \Xi$, where $\Xi \subset \mathcal{D}$ is a random set of cardinality 2000. We have re-sampled the random points, thus this set Ξ is different from set the one used for selecting the anchor points. On Fig. 8, we have plotted two effectivity distributions; namely

- the effectivity distribution of the exact inf-sup based error estimator, based on the computation of the exact inf-sup constant $\alpha(\mu)$, that is $\{\frac{1}{\alpha(\mu)}\|A(\mu)\widetilde{u}_N(\mu) - f(\mu)\|_{W'} / \|u(\mu) - \widetilde{u}_N(\mu)\|_V, \mu \in \Xi\}$ (top);
- the effectivity distribution of the practical dual natural-norm based error estimator, that is $\{\Delta_K^{\text{du}}(\widetilde{u}_N(\mu); \mu) / \|u(\mu) - \widetilde{u}_N(\mu)\|_V, \mu \in \Xi\}$ (bottom);

We find 114 points in Ξ for which the discrete positivity coverage property does not hold. This corresponds to 5.7% of our points, slightly above the expected 5% and we have checked these points are all located near the resonance lines. We remove these 114 points from the initial Ξ set, and thus obtain a set of 1886 points for which our error estimator can be computed. Notice at this stage that, estimating the error at these 114 using our dual natural-norm error estimator is impossible with $K = 15$, but this would be possible by consider more anchor points $K > 15$. Inspection of the two distributions reveals that the exact inf-sup based estimator never provides the correct order of magnitude of the error; while the dual natural-norm does for most parameter values.

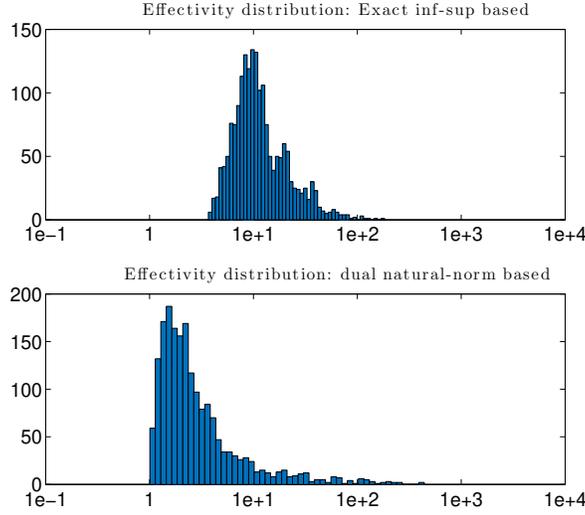


Figure 8: Non self-adjoint case: Effectivity distribution of the exact inf-sup based error estimator (top) and of the practical dual natural-norm based error estimator (bottom), obtained from 1886 random parameter samples in \mathcal{D} satisfying the positivity coverage property.

On Fig. 9, we have plotted the effectivity in the (μ_1, μ_2) plane. For the inf-sup based estimator, we find all maximum values of effectivity to be located in the neighborhood of a resonant line (as in the self-adjoint case). However, for the dual natural-norm based estimator, the maximum values are not always near a resonant line. Thus, the tail of the distribution on Fig. 8 (below) where the effectivity is large, does not necessarily correspond to parameter values located very near a resonance line.

In order to understand the origin of the tail of the distribution, we show on Fig. 10 the two stability constants at play: the inf-sup constant $\mu \mapsto \alpha(\mu)$ and the dual-natural norm constant $\mu \mapsto \max_{1 \leq k \leq K} \bar{\sigma}_{\mu^k}(\mu)$. While the minimas of the inf-sup constant clearly mark the resonance lines, this is not the case for the dual natural-norm constant. In fact, we find that the values of μ for which the dual natural-norm constant are minimal correspond to the values of μ for which the effectivities are maximal on Fig. 9 (right). This confirms the relevance of effectivity bound from Proposition 3.3, which suggests that a small dual natural-norm constant will deteriorate the effectivity. Of course,

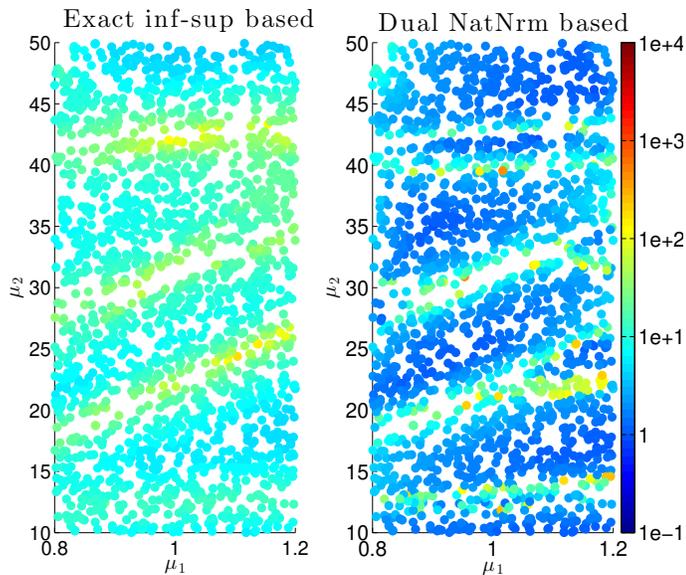


Figure 9: Self adjoint case: The effectivity of the inf-sup (left) and dual natural-norm (right) error estimators plotted as functions of $\mu = (\mu_1, \mu_2)$. Notice the logarithm scale.

since it is always possible to obtain a $\mathcal{O}(1)$ dual natural-norm constant by adding more anchor points.

7 Conclusions

In this work, we have developed both theoretical error bounds and practical *a posteriori* error estimators for reduced basis approximations to parametrized linear equations based on the concept of dual natural-norm. In comparison to the classical error bounds based on the inf-sup stability constant, the dual natural-norm error bounds are associated with a $\mathcal{O}(1)$ stability constant and are therefore very effective. Moreover, in opposition to the primal natural-norm approach, one is free to choose the norm $\|\cdot\|_V$ in which the error $u(\mu) - \widetilde{u}_N(\mu)$ should be measured, since the dual natural-norm is not a norm for measuring the error; but rather, a norm for measuring the residual.

We have shown a computational strategy for efficiently computing the proposed dual natural-norm error estimator in the context of reduced basis

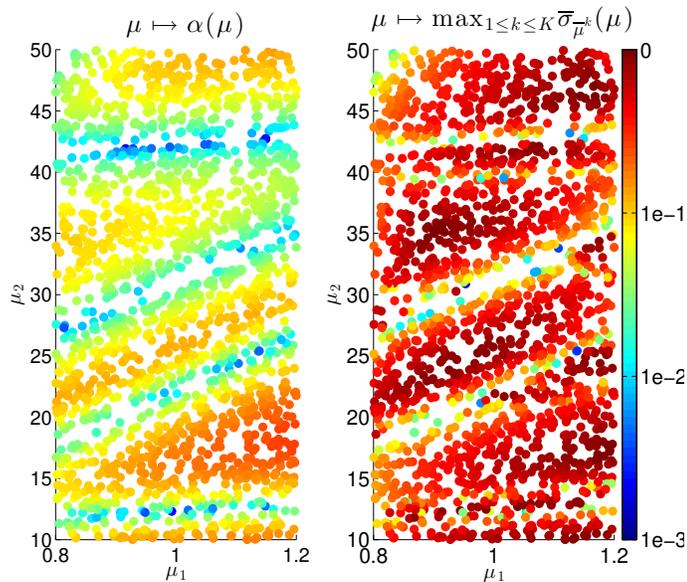


Figure 10: Self adjoint case: The inf-sup constant (left) and dual natural-norm constant (right) as functions of $\mu = (\mu_1, \mu_2)$. Notice the logarithm scale.

approximations. This strategy was successfully applied to a Helmholtz equation parametrized by the wavenumber and anisotropy parameter. Numerical results show great potential, especially in the case of challenging problems with resonant parameters. In this context, the proposed method also provides a very practical way to determine a "resonance-free" set of parameters \mathcal{D} out of a larger parameter set $\tilde{\mathcal{D}}$ which contains resonant parameters at *a priori* unknown locations.

References

- [1] O. Balabanov and A. Nouy. Preconditioners for model order reduction by interpolation and random sketching of operators. *arXiv preprint* arXiv:2104.12177, 2001.
- [2] M. Barrault, Y. Maday, N. C. Nguyen and A. T. Patera. An ‘empirical interpolation’ method: application to efficient reduced-basis discretiza-

- tion of partial differential equations. *Comptes Rendus Mathématique*, 339(9), 667-672, 2004.
- [3] G. Berkooz, P. Holmes, and J. L. Lumley. The proper orthogonal decomposition in the analysis of turbulent flows. *Annual review of fluid mechanics*, 25(1), 539-575, 1993.
- [4] Y. Chen, J. S. Hesthaven, Y. Maday and J. Rodríguez. A monotonic evaluation of lower bounds for inf-sup stability constants in the frame of reduced basis approximations. *Comptes Rendus Mathématique*, 346(23-24), 1295-1300, 2008.
- [5] Y. Chen. A certified natural-norm successive constraint method for parametric inf-sup lower bounds. *Applied Numerical Mathematics*, 99, 98-108, 2016.
- [6] S. Deparis. Reduced basis error bound computation of parameter-dependent Navier–Stokes equations by the natural norm approach. *SIAM journal on numerical analysis*, 46(4), 2039-2067, 2008.
- [7] A. Ern and J. L. Guermond. *Theory and practice of finite elements*, Springer, 2004.
- [8] M. A. Grepl and A. T. Patera. A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(1), 157-181, 2005.
- [9] S. Hain, M. Ohlberger, M. Radic and K. Urban. A hierarchical a posteriori error estimator for the reduced basis method. *Advances in Computational Mathematics*, 45(5), 2191-2214, 2019.
- [10] J. S. Hesthaven, G. Rozza, B. Stamm and others, *Certified reduced basis methods for parametrized partial differential equations*, Springer, 2016.
- [11] P. Houston, I. Perugia, and D. Schötzau. Energy norm a posteriori error estimation for mixed discontinuous Galerkin approximations of the Maxwell operator. *Computer methods in applied mechanics and engineering*, 194(2-5), 499-510, 2005.

- [12] D. B. P. Huynh, G. Rozza, S. Sen and A. T. Patera. A successive constraint linear optimization method for lower bounds of parametric coercivity and inf-sup stability constants. *Comptes Rendus Mathematique*, 345(8), 473-478, 2007.
- [13] D. B. P. Huynh, D. J. Knezevic, Y. Chen, J. S. Hesthaven and A. T. Patera. A natural-norm successive constraint method for inf-sup lower bounds. *Computer Methods in Applied Mechanics and Engineering*, 199(29-32), 1963-1975, 2010.
- [14] M. Kahlbacher and S. Volkwein. Galerkin proper orthogonal decomposition methods for parameter dependent elliptic systems. *Discussiones Mathematicae, Differential Inclusions, Control and Optimization*, 27(1), 95-117, 2007..
- [15] T. Lassila, A. Manzoni and G. Rozza. On the approximation of stability factors for general parametrized partial differential equations with a two-level affine decomposition. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46(6), 1555-1576, 2012.
- [16] Y. Maday, A. T. Patera and D. V. Rovas. A blackbox reduced-basis output bound method for noncoercive linear problems, 2001.
- [17] N. C. Nguyen, K. Veroy and A. T. Patera. Certified Real-Time Solution of Parametrized Partial Differential Equations, in *Handbook of Materials Modeling* (pp. 1529-1564). Springer, Dordrecht, 2005.
- [18] N. C. Nguyen. A posteriori error estimation and basis adaptivity for reduced-basis approximation of nonaffine-parametrized linear elliptic partial differential equations. *Journal of Computational Physics*, 227(2), 983-1006, 2007.
- [19] C. Prud'Homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera and G. Turinici. Reliable real-time solution of parametrized partial differential equations: Reduced-basis output bound methods. *Journal of Fluids Engineering*, 124(1), 70-80, 2002.
- [20] A. Quarteroni, A. Manzoni and F. Negri. *Reduced basis methods for partial differential equations: an introduction*, Springer, 2015.

- [21] G. Rozza, D. B. P. Huynh and A. T. Patera. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations, *Archives of Computational Methods in Engineering*, 15(3) 229–275, 2008.
- [22] S. Sen, K. Veroy, D. B. P. Huynh, S. Deparis, N. C. Nguyen and A. T. Patera. “Natural norm” a posteriori error estimators for reduced basis approximations. *Journal of Computational Physics*, 217(1), 37-62, 2006.
- [23] P. Sirkovic and D. Kressner. Subspace acceleration for large-scale parameter-dependent Hermitian eigenproblems. *SIAM Journal on Matrix Analysis and Applications*, 37(2), 695-718, 2016.
- [24] K. Smetana, O. Zahm, and A. T. Patera. Randomized residual-based error estimators for parametrized equations. *SIAM journal on scientific computing*, 41(2), A900-A926, 2019.
- [25] O. Zahm and A. Nouy. Interpolation of inverse operators for preconditioning parameter-dependent equations, *SIAM Journal on Scientific Computing*, 38(2) A1044–A1074, 2006.