Global in time collision-free flocking for the singular Cucker-Smale model (February 2022)

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***Abstract*— Following the ideas of J. Carillo, Y-P. Choi, P. Mucha and J. Peszek [1], we extend the result on the absence of collision for the singular Cucker-Smale (CS) model by proving the existence of a uniform (in time) minimal distance, which depends on the initial conditions and the different parameters of the model. This result is then applied to the optical flow control, which corresponds to a critical value ot the exponent in the singular CS model. Additionally we illustrate, through the implicit relation given by our result and numerical estimates, the difficulty to obtain an estimate of a lower bound on the minimal distance. Finally, we give some insight whether this uniform bound still holds or not for the singular CS model with additional control terms.**

***Index Terms*— Collision avoidance, Control, Optical flow, Singular Cucker-Smale model, Uniform minimal distance.**

I. Introduction

AGENT based models and population dynamics have been largely studied, modelling various systems ranging from animal groups to neuron networks or the evolution of an embryo. In 2007, Cucker and Smale proposed and analysed a simple second order model exhibiting flocking [2], which means individuals tend to reach a consensus of some sort. This is achieved by letting the force governing the system depend on the relative values of the consensus quantities. The most common application would be to a group of animals (say a sheep herd or a fish school), which will align their speed vectors over time. The original Cucker-Smale (CS) model is given by (4). It has since received a lot of attention : multi-cluster formation [3], multi-population [4], weighted interactions [5], low-connected graphs [6] and so on. However, this model does not always guarantee the absence of collisions. This feature is of paramount importance to describe biological behaviors such as bird flocks and applications to drone swarms for instance. Collision avoidance has been proved for suitable initial conditions [7], depending on the strength of interactions, or by adding a variety of forces ; some of them are studied in [8] and the references therein. Still, the CS model has a major advantage over its extensions, since we have a lot of routines allowing us to derive bounds and decay rates, which are detailed in the first section.

Besides, [9] observed that optical flow is a key component of flying animals and insects to safely navigate near obstacles. This property was used to stabilize a drone above a moving platform, and perform a safe landing [10]. It was also used to have flocking for nonholonomic robots [11]. Since optical flow provides a measure of the relative velocity weighted by the invert of the distance to obstacles in the local environment [12], it is very similar to the term used in the CS model with a singular communication rate $\psi(r) = 1/r$. Then, we can intuit from these observations that a singular communication rate seems more appropriate to achieve collision-free flocking.

Singular rates have been studied in [13], but in discrete time, and in [14] is proved the existence of a minimal distance for ψ such that there exist some positive Ψ satisfying $\Psi'(r) = -r\psi(r)$, which is not the case for $1/r$. Moreover, only simple cases with either two individuals in dimension 3, or any number of agents in dimension 2, were included. In [1], the authors proved the absence of collision in finite time for singular ψ for any initial condition. In this paper, we extend this result by proving the existence of a uniform minimal distance for all ψ functions that are non integrable both at the origin and infinity, given a non-collisional initial condition. These conditions include the optical flow case mentioned in the preceding paragraph. The proof relies on the use of the the flocking ability of the system, which gives a better estimate for the relative velocities and allows for the existence of a time-uniform minimal distance. An implicit relation giving a lower bound for this distance can also be inferred but, depending on the ψ function, might not give a usable solution. The result is further extended by adding extra Lipschitz controls over the speed and position vectors, in order to model target tracking. We eventually provide numerical simulations for the case $\psi(r) = 1/r$.

This article is organized as follows. In section II we introduce the notations used throughout and some useful results on the CS model, before tackling the main proof in section III. We then extend the result to other models in section IV, and illustrate numerically our estimations in section V.

II. Notations and Preliminaries

In this section, we introduce notations used throughout, and the Cucker-Smale model is recalled. All N agents are represented by a position $x_i \in \mathbb{R}^d$ and a speed $v_i \in \mathbb{R}^d$. We

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denote by r_{ij} the relative position between agent i and j, and simplify functions composed with r_{ij} as

$$r_{ij} := |x_i - x_j|, \qquad f_{ij} := f(r_{ij}). \tag{1}$$

We will also measure super-vectors $u = (u_1, \cdots, u_N) \in (\mathbb{R}^d)^N$ with the following quantity :

$$|u|_2 := \sqrt{\sum_{i,j=1}^{N} |u_i - u_j|^2}, \tag{2}$$

and consider a halved variant, namely we replace it with

$$\overline{|u|}_2 := \sqrt{\sum_{i<j=1}^{N} |u_i - u_j|^2} = \frac{1}{\sqrt{2}}|u|_2. \tag{3}$$

Then the CS model writes for $i = 1, \cdots, N$:

$$\begin{cases} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \dfrac{1}{N}\displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{N} \psi_{ij}(v_j - v_i)(t), \end{cases} \tag{4}$$

where ψ is the *communication rate* function. We say it has *long*, respectively *short, range* whenever $\int_1^\infty \psi(s)ds$, respectively $\int_0^1 \psi(s)ds$, is infinite. These two characteristics are sufficient conditions for the model to exhibit flocking and absence of collision, in the sense of the following usual definition :

Definition 1: We say model (4) exhibits

- *flocking* whenever speeds align at infinity, that is

$$\forall i, j \in [\![1, N]\!], |v_i - v_j|(t) \underset{t\to\infty}{\longrightarrow} 0.$$

- *no collision* if

$$\forall t \geq 0, \forall i, j \in [\![1, N]\!], r_{ij}(t) > 0.$$

Note that $\psi(r) = 1/r$ satisfies both conditions, streghtening our interest in ψ functions that are not integrable both at 0 and infinity. In particular, we have for long ranged potentials the existence of $r_M \in \mathbb{R}^+$ that bounds the relative distances, and gives an exponential decay rate of the speeds (see for instance [7]). More precisely, for any $t \geq 0$:

$$|x|_2(t) \leq r_M, \quad |v|_2(t) \leq |v_0|_2 \exp\Big(-\psi(r_M)t\Big). \tag{5}$$

r_M is defined as (see the proof of theorem 3 for details on another model) the largest real satisfying :

$$|v_0|_2 = \int_{|x_0|_2}^{r_M} \psi(s)ds. \tag{6}$$

Additionally, we can retrieve the exponential convergence for the position, since for any $t, s \geq 0$ and $i, j \in [\![1, N]\!]$

$$\begin{aligned} |(x_i - x_j)(s+t) - (x_i - x_j)(s)| &\leq \left|\int_s^{s+t} |v_i - v_j|(\tau)d\tau\right| \\ &\leq \frac{|v_0|_2}{\psi(r_M)} e^{-\psi(r_M)s}|e^{-\psi(r_M)t} - 1|. \end{aligned} \tag{7}$$

When s goes to ∞, we then have exponential convergence of the relative distances with the same rate as v.

On the other hand, short ranged potentials do not make agents move away from each other, but slows them down and makes them go in parallel directions. That is why nothing so far prevents them to collide at infinity. We will show in the following that we can improve the estimates used in [1] in order to grant the existence of a uniform minimal distance that only depends on ψ and the initial conditions.

III. Existence of a minimal distance

A. Main result

We now give the main result and its proof. We will discuss them in the next section :

Theorem 1: Let $\psi \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}^+)$ be non integrable both at 0 and infinity [1] and decreasing. Then for any non collisional initial condition, there exists for model (4) a lower bound $d_{min} > 0$ on all relative distances, that is independent of time :

$$\exists d_{min} > 0, \ \forall t \geq 0, \ \min_{i\neq j} r_{ij}(t) \geq d_{min}.$$

Moreover, we have $d_{min} \geq d_- \geq 0$ which satisfies

$$d_- = \frac{\sqrt{2}}{3N} \inf\Big(|\Psi|^{-1}\big(\{\alpha L^\psi(d_-) + C\}\big)\Big), \tag{8}$$

where Ψ is the anti derivative of ψ defined in (22), $f^{-1}(\{y\})$ is the pre-image of $\{y\}$ by f, $L^\psi(\mu)$ is the Lipschitz constant of ψ on $[\mu, \infty)$, and

$$\alpha = \frac{N-2}{\sqrt{2}} \frac{r_M}{\psi(r_M)} \overline{|v_0|}_2, \quad C = \frac{N}{2}\overline{|v_0|}_2.$$

Let us prove this result. Following the computations of [1], we already know that there is no collision in finite time, and that a unique global solution exists. From the hypothesis on the tail of ψ, we also have flocking and the exponential convergences given by (5) and (7). Suppose in order to get a contradiction that there is no such minimal distance. Then there exist a sequence of times $\{(t_n)\}_{\mathbb{N}}$ and a sequence of indices $\{(i, j)_n\}_{\mathbb{N}}$ such that $r^n_{ij}(t_n)$ converges to 0. Since the set of available indices is finite, we can assume, up to a subsequence, that this convergence is true for the pair (l, m). We then study the set of indices that lead to a collision with particle l at infinity :

$$\mathcal{L} := \{j \in [\![1, N]\!] \mid \exists \{(t_n)\}_{\mathbb{N}} \text{ s.t. } r_{lj}(t_n) \underset{n\to\infty}{\longrightarrow} 0\}. \tag{9}$$

In fact, since speeds align exponentially fast, we have more than this characterization :

Lemma 1: $\forall i, j \in \mathcal{L}$, $r_{ij}(t) \underset{t\to\infty}{\longrightarrow} 0$.

Proof: We start by proving that for all $i \in \mathcal{L}$, $r_{il}(t)$ converges to 0 at infinity. Suppose that there exist $\varepsilon > 0$ and a sequence $\{(s_n)\}_{\mathbb{N}}$ such that $r_{il}(s_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. By definition of $\mathcal{L}$, there exists a sequence $\{(t_p)\}_{\mathbb{N}}$ such that $r_{il}(t_p)$ converges to 0. Then by integrating r'_{il} between s_n and t_p with $s_n \geq t_p$, we have

$$r_{il}(s_n) \leq r_{il}(t_p) + \frac{|v_0|_2}{\psi(r_M)} \exp(-\psi(r_M)t_p).$$

[1] Or any condition on the tail of ψ and initial condition to obtain flocking.

Since the right hand side converges to 0, there exists p large enough so that it is inferior to $\varepsilon/2$. It remains to take n large so that $s_n \geq t_p$ to find $r_{il}(s_n) < \varepsilon/2$, which is a contradiction. To extend the result to any pair of indices, we simply use the triangle inequality. ∎

We also need a separation between the set $\mathcal{L}$ and the rest of agents later in the proof :

Lemma 2:

$$\exists \mu > 0,\ \exists T > 0,\ \forall t \geq T,\ \forall i \in \mathcal{L}, \forall j \notin \mathcal{L},\ r_{ij}(t) \geq \mu.$$

Proof: Since we have flocking, all r_{ij} converge at infinity. In particular, since $j \notin \mathcal{L}$, r_{jl} converges to some $l_j > 0$. Therefore, there exists some time T_j such that $r_{jl}(t) \geq l_j/2$ for $t \geq T_j$. Now r_{il} converges to 0, so that there is a $\tilde{T}_i$ allowing $r_{il}(t) \leq l_j/4$ for $t \geq \tilde{T}_i$. We thus have for $t \geq T := \max_{k=1,\cdots,N} \left\{T_k, \tilde{T}_k\right\}$

$$r_{ij}(t) \geq r_{jl}(t) - r_{il}(t) \geq l_j/4 \geq \mu := \min_{j=1,\cdots,N} l_j/4.$$

∎

Let us define the quantities we will control during the proof of our theorem :

$$|x|_{\mathcal{L}}(t) := \sqrt{\sum_{i,j\in\mathcal{L}} r_{ij}(t)^2}, \quad |v|_{\mathcal{L}}(t) := \sqrt{\sum_{i,j\in\mathcal{L}} |v_i - v_j|^2}, \tag{10}$$

which are $|\cdot|_2$ measurements restricted to $\mathcal{L}$. Similarly to (3), we define the $\overline{|\cdot|}_{\mathcal{L}}$ variant. By our assumptions, we have that $|\mathcal{L}| \geq 2$ and that $|x|_{\mathcal{L}}$ converges to 0 as t goes to infinity, which will be the source of the contradiction. Let us derive two inequalities on $|x|_{\mathcal{L}}$ and $|v|_{\mathcal{L}}$. We claim that

$$\frac{d}{dt}|x|_{\mathcal{L}} \leq |v|_{\mathcal{L}}. \tag{11}$$

Indeed,

$$\begin{aligned} 2|x|_{\mathcal{L}}\frac{d}{dt}|x|_{\mathcal{L}} = \frac{d}{dt}|x|_{\mathcal{L}}^2 &= 2\sum_{i,j\in\mathcal{L}} \langle x_i - x_j, v_i - v_j\rangle \\ &\leq 2\sum_{i,j\in\mathcal{L}} r_{ij}|v_i - v_j| \leq 2|x|_{\mathcal{L}}|v|_{\mathcal{L}}. \end{aligned}$$

We now study the time derivative of $|v|_{\mathcal{L}}$ for $t \geq T$:

$$\frac{d}{dt}|v|_{\mathcal{L}}^2 = A_{\mathcal{L}} + \tilde{A}_{\mathcal{L}}, \tag{12}$$

where

$$A_{\mathcal{L}} = \frac{2}{N}\sum_{\substack{i,j\in\mathcal{L}\\ k\in\mathcal{L}\setminus\{i,j\}}} \langle v_i - v_j, \psi_{ik}(v_k - v_i) - \psi_{jk}(v_k - v_j)\rangle,$$

$$\tilde{A}_{\mathcal{L}} = \frac{2}{N}\sum_{\substack{i,j\in\mathcal{L}\\ k\notin\mathcal{L}}} \langle v_i - v_j, \psi_{ik}(v_k - v_i) - \psi_{jk}(v_k - v_j)\rangle.$$

For the first term, we use a standard symmetrization technique in double sums twice, and the fact that $r_{ik} \leq |x|_{\mathcal{L}}$ for $i, k \in \mathcal{L}$:

$$\begin{aligned} A_{\mathcal{L}} &= \frac{4}{N}\sum_{\substack{i,j\in\mathcal{L}\\ k\in\mathcal{L}\setminus\{i,j\}}} \psi_{ik}\langle v_i - v_j, v_k - v_i\rangle \\ &= -\frac{2}{N}\sum_{\substack{i,j\in\mathcal{L}\\ k\in\mathcal{L}\setminus\{i,j\}}} \psi_{ik}|v_k - v_i|^2 \\ &\leq -\frac{2|\mathcal{L}|}{N}\psi(|x|_{\mathcal{L}})|v|_{\mathcal{L}}^2. \end{aligned} \tag{13}$$

For the second one, we first insert $\pm v_j$ in $v_k - v_i$. Then we use the fact that ψ is $L^\psi(\mu)$ Lipschitz continuous on $[\mu, \infty)$, coupled with Lemma 2. We conclude by using the exponential decay of both $|v_k - v_j|$ and $|x|_{\mathcal{L}}$ at rate $-\psi(r_M)$ from (5) and (7) :

$$\begin{aligned} \tilde{A}_{\mathcal{L}} &= -\frac{2}{N}\sum_{\substack{i,j\in\mathcal{L}\\ k\notin\mathcal{L}}} \underbrace{\psi_{ik}|v_i - v_j|^2}_{\geq 0} \\ &\quad + \frac{2}{N}\sum_{\substack{i,j\in\mathcal{L}\\ k\notin\mathcal{L}}} \left(\psi_{ik} - \psi_{jk}\right)\langle v_i - v_j, v_k - v_j\rangle \\ &\leq \frac{2}{N}\sum_{\substack{i,j\in\mathcal{L}\\ k\notin\mathcal{L}}} L^\psi(\mu) r_{ij}|v_i - v_j||v_k - v_j| \\ &\leq 2\frac{N - |\mathcal{L}|}{N} L^\psi(\mu)|v_0|_2\, e^{-\psi(r_M)t}\sum_{i,j\in\mathcal{L}} r_{ij}|v_i - v_j| \\ &\leq 2\frac{N - |\mathcal{L}|}{N} L^\psi(\mu)|v_0|_2\, e^{-\psi(r_M)t}|x|_{\mathcal{L}}|v|_{\mathcal{L}} \\ &\leq 2|\mathcal{L}|\frac{N - |\mathcal{L}|}{N} L^\psi(\mu)\frac{|v_0|_2^2}{\psi(r_M)}\, e^{-2\psi(r_M)t}|v|_{\mathcal{L}}. \end{aligned} \tag{14}$$

By combining (12), (13) and (14), we find

$$\frac{d}{dt}|v|_{\mathcal{L}} \leq -C_0 a(t)|v|_{\mathcal{L}} + C_1 b(t), \tag{15}$$

with

$$C_0 = \frac{|\mathcal{L}|}{N}, \quad C_1 = |\mathcal{L}|\frac{N - |\mathcal{L}|}{N}\frac{|v_0|_2^2}{\psi(r_M)}L^\psi(\mu), \tag{16}$$

$$a(t) = \psi(|x|_{\mathcal{L}}), \quad b(t) = \exp\big(-2\psi(r_M)t\big). \tag{17}$$

If we note $c(t) := e^{-C_0\int_T^t a(s)ds}$ and $v_T := v(T)$, then the usual Gronwall estimate yields for $t \geq T$:

$$|v|_{\mathcal{L}}(t) \leq \left[|v_T|_{\mathcal{L}} + C_1\int_T^t b(s)c^{-1}(s)ds\right]c(t). \tag{18}$$

We finally bring forth the contradiction. Let Ψ be an antiderivative of ψ. Remark that

$$a(t)c(t) = \frac{d}{dt}\left(-\frac{1}{C_0}c(t)\right). \tag{19}$$

We have for $t \geq T$ thanks to (11), (18), (19) and an integration

by part :

$$
\begin{aligned}
&|\Psi(|x_t|_{\mathcal{L}})| - |\Psi(|x_T|_{\mathcal{L}})| \leq \int_T^t \psi(|x|_{\mathcal{L}}) \left|\frac{d}{dt}|x|_{\mathcal{L}}\right| ds \\
&\leq \int_T^t \psi(|x|_{\mathcal{L}})|v|_{\mathcal{L}} ds \\
&\leq \int_T^t a(s)c(s)\left(|v_T|_{\mathcal{L}} + C_1\int_T^s b(\tau)c^{-1}(\tau)d\tau\right) ds \\
&= -\int_T^t -\frac{1}{C_0}c(s)C_1 b(s)c^{-1}(s) ds \\
&\quad + \left[-\frac{1}{C_0}c(s)\left(|v_T|_{\mathcal{L}} + C_1\int_T^s b(\tau)c^{-1}(\tau)d\tau\right)\right]_{s=T}^t \\
&\leq \frac{C_1}{C_0}\int_T^t \exp\Big(-2\psi(r_M)s\Big)ds + \frac{|v_T|_2}{C_0} \\
&\leq \frac{C_1}{2C_0\psi(r_M)}e^{-2\psi(r_M)T} + \frac{|v_0|_2}{C_0} < \infty. \qquad (20)
\end{aligned}
$$

As a result, we find a contradiction when we take the limit $t \to \infty$, and this gives us the existence of a minimal distance d_{min}.

B. Estimating the lower bound

In order to obtain the lower bound on our newly found minimal distance, we repeat the same computations, using a different set $\mathcal{L}$. Let T be a time were the minimal distance is realized for some r_{lm} (regardless of whether T is finite or not). We set

$$\mathcal{L} := \{j \in [\![1, N]\!] \mid r_{lj}(t) \underset{t\to T}{\to} d_{min} \text{ or } r_{mj}(t) \underset{t\to T}{\to} d_{min}\}. \qquad (21)$$

We repeat the same computations, excepted that we control $|x|_{\mathcal{L}}$ with r_M instead of a decreasing exponential, since $|x|_{\mathcal{L}}$ no longer converges to 0. Moreover, we work with halved quantities $\overline{|\cdot|}_2$ and $\overline{|\cdot|}_{\mathcal{L}}$ defined in (3). This allows to gain a factor $\sqrt{2}$ over the previous quantities in C_1 in (15). This manipulation does not impact (5). Moreover, we chose the constant of Ψ so that it vanishes at $\overline{|x_0|}_{\mathcal{L}}$. In other words, Ψ is precisely

$$\Psi(r) = \int_{\overline{|x_0|}_{\mathcal{L}}}^r \psi(s)ds. \qquad (22)$$

This time we can integrate all inequalities from 0, and we obtain using $|\mathcal{L}| \geq 2$

$$\left|\Psi(\overline{|x|}_{\mathcal{L}}(t))\right| \leq \Big(\frac{N-2}{\sqrt{2}}\frac{r_M}{\psi(r_M)}L^{\psi}(d_{min}) + \frac{N}{2}\Big)\overline{|v_0|}_2.$$

We invert $|\Psi|$, and use in $|x|_{\mathcal{L}}$ the fact that for $i, j \in \mathcal{L}$

$$
\begin{aligned}
r_{ij} \leq\ & (r_{il} + r_{lj})\mathbb{1}_{r_{il}\to d_{min} \cap\, r_{lj}\to d_{min}} \\
&+(r_{im} + r_{mj})\mathbb{1}_{r_{im}\to d_{min} \cap\, r_{mj}\to d_{min}} \\
&+(r_{il} + r_{lm} + r_{mj})\mathbb{1}_{r_{il}\to d_{min} \cap\, r_{mj}\to d_{min}} \\
&+(r_{im} + r_{ml} + r_{lj})\mathbb{1}_{r_{im}\to d_{min} \cap\, r_{lj}\to d_{min}},
\end{aligned}
$$

where $\mathbb{1}_{r_{il}\to d_{min}}$ denotes the fact that it is the index l that realizes the minimal distance for i the definition of $i \in \mathcal{L}$. It remains to remark that all cases presented are disjoint and cover all possibilities to conclude that $r_{ij}(\infty) \leq 3d_{min}$, to find (8) when we send t to T.

The bound we obtained in theorem 1 is pessimistic, since it takes into account pathological initial conditions. Moreover, while the theoretical minimal distance d_{min} always exists and is positive, (8) might not have positive solutions as seen in section V, therefore leading to an absence of estimation for d_{min}. We believe that it is due to the choice of $\mathcal{L}$ in (21). The set chosen here is the best one we found, but others might fit as well, leading to better estimates. In the one dimensional case, individuals can not cross each other. One could take advantage of this fact in order to obtain better estimates.

IV. Extension to other models

We now aim to generalize the result to additional command laws of the form

$$\dot{v}_i(t) = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N \psi_{ij}(v_j - v_i)(t) + F(t, x_i, v_i), \qquad (23)$$

where F is $2\gamma_x$ and $2\gamma_v$ Lipschitz with respect to the second and third variable respectively :

$$
\begin{aligned}
&\exists \gamma_x, \gamma_v > 0, \forall t \geq 0, \forall u, v, w \in \mathbb{R}^d, \\
&|F(t, u, v) - F(t, w, v)| \leq 2\gamma_x|u - w|, \qquad (24)\\
&|F(t, u, v) - F(t, u, w)| \leq 2\gamma_v|v - w|. \qquad (25)
\end{aligned}
$$

Some example are studied in more details in section IV-B.

A. Generic result

We have the following result :

Theorem 2: Let $\psi \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}^+)$ be non integrable at 0 and decreasing, and F satisfy (24)-(25). Then for any non collisional initial condition, there are no collision in finite time for model (23). Collisions may still occur at infinity.

Remark 1: As we will see in section IV-B, eventhough F satisfies (24)-(25), we may recover a minimal distance, or give an example of collision at infinity for some explicit forces. Therefore, this result is a bare minimum given by the singularity of ψ, and more can be derived from the nature of F.

Proof: Since we again have the same calculations, we refer to [1] or the proof of theorem 1 for details. We no longer have an exponential decay of speeds, so we will denote C_x (resp. $C_v(t)$) a bound on $|x|_2$ (resp. $|v|_2$). We let C_v depend on time to stay in the same framework as theorem 1. We suppose there is a collision with agent l at time T, and define the set $\mathcal{L}$ of agents colliding with l at T. We find similarly to (15)

$$
\begin{aligned}
\frac{d}{dt}|v|_{\mathcal{L}} \leq & -\left(\frac{|\mathcal{L}|}{N}\psi(|x|_{\mathcal{L}}) - \gamma_v\right)|v|_{\mathcal{L}} \\
&+\left(\frac{N - |\mathcal{L}|}{N}L^{\psi}(\mu)C_v + \gamma_x\right)C_x,
\end{aligned}
$$

which gives after the Gronwall argument and the integration of Ψ

$$
\begin{aligned}
|\Psi(|x|_{\mathcal{L}}(t))| \leq\ & C_x e^{\gamma_v T}(N-2)L^{\psi}(\mu)\int_0^T C_v(s)ds \\
&+\frac{N|v_0|_2}{2} + C_x N e^{\gamma_v T}\gamma_x T + |\Psi(|x_0|_{\mathcal{L}})|.
\end{aligned}
$$

We take the limit $t \to T$ to conclude. ■

Remark 2: We see that the addition of F brings a factor $e^{\gamma_v T}$ and a shift $\gamma_x T$. The argument suffices to prove the absence of collision in finite time as t goes to T, but in order to have a minimal distance, we would like to take the limit $T \to \infty$. For this, we need $C_v \in L^1(\mathbb{R}^+)$ (typically exponentially decreasing), and a way to get rid of γ_v. There is no hope for the argument to work if γ_x is present. That is what we will see in the next section.

B. Applications

Let us apply the previous result on two specific command laws, namely a control on the center of mass and on the center of speed, respectively given by (26) and (27) :

$$\dot{v}_i(t) = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \psi_{ij}(v_j - v_i) + \gamma_x(x_{tar}(t) - x_i), \qquad (26)$$

$$\dot{v}_i(t) = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \psi_{ij}(v_j - v_i) + \gamma_v(v_{tar}(t) - v_i), \qquad (27)$$

where x_{tar} and v_{tar} are continuous position and speed targets to reach, that may evolve through time. Both models satisfy the hypothesis of theorem 2. As stated earlier, this result prevents collisions in finite time, thanks to ψ's singularity. However, we can not ask for more without knowing the exact form of the additional force. Indeed, we are about to see that we can recover a minimal distance for model (27), while we have an example of collision at infinity for model (26), even if ψ has a heavy tail.

1) Speed center control: Here is the result for model (27) :

Theorem 3: Let $\psi \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}^+)$ be non integrable both at 0 and infinity [2] and decreasing. Then for any non collisional initial condition, there exists for model (27) a lower bound $d_{min} > 0$ on all relative distances, that is independent of time :

$$\exists d_{min} > 0,\ \forall t \geq 0,\ \min_{i\neq j} r_{ij}(t) \geq d_{min}.$$

Moreover, we have $d_{min} \geq d_- \geq 0$ which satisfies

$$d_- = \frac{\sqrt{2}}{3N} \inf\left(|\Psi|^{-1}\big(\{\alpha L^{\psi}(d_-) + C\}\big)\right), \qquad (28)$$

where Ψ is the anti-derivative of ψ defined in (22), $L^{\psi}(\mu)$ is the Lipschitz constant of ψ on $[\mu, \infty)$, and

$$\alpha = \frac{N-2}{\sqrt{2}} \frac{r_M}{\psi(r_M) + \gamma_v} \overline{|v_0|}_2, \quad C = \frac{N}{2}\overline{|v_0|}_2.$$

Proof: We know from theorem 2 that a global solution exists, without collision in finite time. In order to use the same computations as in theorem 1 and conclude, we need to prove that speeds align exponentially fast. We have the usual system of differential inequalities :

$$\begin{cases} |x|_2' \leq |v|_2, \\ |v|_2' \leq -(\psi(|x|_2) + \gamma_v)|v|_2. \end{cases} \qquad (29)$$

[2]Or any condition on the tail of ψ and initial condition to obtain flocking.

Then define the energy $E(t) = |v|_2 + \Psi(|x|_2)$, where Ψ is an antiderivative of ψ, so that $E'(t) \leq -\gamma_v |v|_2 \leq 0$. As a result,

$$|v_0|_2 \geq |v(t)|_2 + \int_{|x_0|_2}^{|x(t)|_2} \psi(s)ds.$$

If $|x|_2$ were not bounded, we would have a contradiction with the non integrability of ψ. We deduce that for all time t, $|x(t)|_2 \leq r_M$, leading to $|v|_2' \leq -(\psi(r_M) + \gamma_v)|v|_2$ i.e. $|v|_2$ converges exponentially fast to 0. ■

2) Mass center control: For model (26), we have nothing more than theorem 2. Indeed, nothing makes agents move away from each other, and they are attracted to a point. We would need some repulsion force in order to prevent collisions (see section IV-B.3). We provide here an explicit example of collision at infinity. Let us consider the following one dimensional simple configuration. Take two individuals, one at position $-r_0$ and speed v_0, the other at position r_0 and speed $-v_0$, where $r_0, v_0 > 0$. Add a frozen target at position 0. This means we have two agents facing their objective, attracted by it and slown down because of the alignment force. Let us note v resp. x the difference of the speed resp. position of the two agents. By the choice of the configuration, we have $x \geq 0$ and $v \leq 0$. These quantities solve

$$\dot{v}(t) = -\psi(x(t))v(t) - \gamma_x x(t).$$

Let us suppose there exists a minimal distance δ. Then for all $t \geq 0$,

$$\dot{v}(t) \leq -\psi(\delta)v(t) - \gamma_x \delta.$$

We now consider the following dichotomy : either for all time t, $v(t) \in [-\frac{\gamma_x \delta}{\psi(\delta)}, 0)$, or there exists some time $t_0 > 0$ such that $v(t_0) < -\frac{\gamma_x \delta}{\psi(\delta)}$, which implies that $v(t) \leq -\frac{\gamma_x \delta}{\psi(\delta)}$ for $t \geq t_0$. In both cases, we have the existence of some $c > 0$ and $t_* > 0$ such that $v(t) \leq -c$ for $t \geq t_*$, which contradicts the fact that x is lower bounded. We conclude there is no minimal distance, and that particles collide at infinity.

3) Attraction and repulsion: We have seen that model (26) needs a repulsion force of some sort to prevent collisions at infinity. That is why our last example will be the CS model with an attraction-repulsion force, also called three zone model (3ZM). It is given by

$$\dot{v}_i(t) = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \psi_{ij}(v_j - v_i) + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_{ij}\frac{x_j - x_i}{r_{ij}}. \qquad (30)$$

If $\alpha_{ij}(t) = \varphi_{ij}'(t)$, we retrieve the usual attraction-repulsion term deriving from some potential φ. The most common example would be $\varphi(r) = (r-\eta)^2/2$, for some $\eta > 0$. Unfortunately, we do not have a general collisionless result. We can prove there are no collision for well prepared initial conditions (see [8] for instance), and we can retrieve theorem 2 with the same computations as before. In order to make the proof for the minimal distance we studied work, we need a few other elements that are yet to be proved. Indeed, if we repeat the calculations with the same $\mathcal{L}$ defined in (9), we get

$$|\Psi(|x|_{\mathcal{L}}(t))| \leq |\Psi(|x_0|_{\mathcal{L}})| + \frac{1}{C_0}\left(\int_0^t b(s)ds + |v_0|_2\right), \qquad (31)$$

for some constant C_0, where

$$b(t) := \frac{N-|\mathcal{L}|}{N} L^{\psi}(\mu)|x|_{\mathcal{L}} C_v(t) + \frac{2|\mathcal{L}|}{N} F_{\mathcal{L}}(t),$$
$$F_{\mathcal{L}}(t) := \sqrt{\sum_{i\in\mathcal{L}} |F_i|^2},$$
$$F_i := \sum_{k=1}^{N} \alpha_{ik} \frac{x_k - x_i}{r_{ik}}.$$

That is enough to conclude that there are no collision in finite time, but in order to establish the existence of a minimal distance, insights on the integrability of both C_v and $F_{\mathcal{L}}$ are still needed. Whether these properties are guaranteed without any more assumption on the model is a topic which is still under investigation.

V. Application to optical flow, numerical simulations

In this last section, we perform some numerical experiment for $\psi(r) = 1/r$. This choice is motivated by the fact that it modelizes a system where individuals recover the information from their neighbors through optical flow, as discussed in the next subsection. We then discuss the application of theorem 1 to this case, before presenting the numerical results.

A. From optical flow to the singular CS model

Optical flow is a dense measure of the velocities of all visible objects in the environment captured in the image surface of an agent's onboard visual sensor (e.g., an insect eye or a robot camera). Consider that the image surface is a non-rotating sphere [10], then the velocity of a point P_{ij} on agent j projected onto the image surface of agent i gives the optical flow equation

$$\dot{p}_{ij} = \left(I_d - p_{ij}\, p_{ij}^{\top}\right) \frac{v_j - v_i}{|P_{ij}|}, \tag{32}$$

where $p_{ij} = P_{ij}/|P_{ij}|$.

Equation (32) provides only a velocity information of dimension $(d-1)$ since it is a projected measure on a surface sphere. To recover a full dimensional information, it can be integrated over the set of visible points P_{ij} on the agent j (we assume here that agents have a non-empty volume), as described in [10], to extract a weighted measure

$$w_{ij} = \frac{v_j - v_i}{r_{ij}}.$$

Consider now that the dynamics of each agent can be modeled as a double integrator:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases}$$

where, for $i = 1, \dots, N$, $u_i \in \mathbb{R}^d$ is a control input. For all i, consider that agent i applies the following control:

$$u_i(t) = \frac{1}{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} k\, w_{ij}, \tag{33}$$

where $k > 0$ is a constant gain parameter. Then, the derivative of $v_i(t)$ verifies the equation of the singular CS model with $\psi(r) = k/r$:

$$\dot{v}_i(t) = \frac{1}{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{k}{r_{ij}} (v_j - v_i)(t). \tag{34}$$

B. Theoretical results for $\psi(r) = 1/r$

Since $\psi(r) = 1/r$ satisfies the hypothesis of theorem 1, we know there exists some minimal distance between agents, and we can compute the bound given by (8). If we rewrite the last inequality in the proof of theorem 1 we have :

$$\frac{3|\mathcal{L}|}{\sqrt{2}} d_{min} \exp\left(\frac{\alpha}{d_{min}^2} + C\right) \geq \overline{|x_0|}_{\mathcal{L}} \geq 3\sigma \frac{|\mathcal{L}|}{\sqrt{2}}$$

with

$$\sigma = \frac{1}{3} \min_{i\neq j} r_{ij}(0), \quad C = \frac{N}{2} \overline{|v_0|}_2,$$
$$\alpha = \frac{N-2}{\sqrt{2}} \overline{|v_0|}_2 \left(\overline{|x_0|}_2 \exp(\overline{|v_0|}_2)\right)^2,$$

since $L^{1/r}(s) = 1/s^2$, $|\Psi(s)| = |\log(s/\overline{|x_0|}_{\mathcal{L}})|$, $\overline{|x_0|}_{\mathcal{L}} \geq \min_{i\neq j} r_{ij}(0)|\mathcal{L}|/\sqrt{2}$ and $r_M = \overline{|x_0|}_2 \exp(\overline{|v_0|}_2)$ (according to (6)). Therefore we take d_- as the smallest real satisfying

$$d_- = \sigma \exp\left(-\frac{\alpha}{d_-^2} - C\right). \tag{35}$$

We see that α is large, and is responsible for the high sensitivity of d_- on $|x_0|_2, |v_0|_2$ and N. In particular we are going to prove that (35) has no positive solution, for this particular choice of ψ, when parameters start to grow. Indeed, we want to solve $d = a\exp(-c/d^2)$ with $a, c > 0$. Let $f(s) := \exp(-h/s^2)$ with $h := c/a^2$. Then our problem is equivalent to

$$f(s) = s. \tag{36}$$

Remark that

$$\frac{d}{ds} f(s) = \frac{2h}{s^3} f(s).$$

We observe a phase transition as h goes from $+\infty$ to 0, where we start with no positive solution to (36), to a pair of solutions. The transition is characterized by $\dot{f}_{h_0}(s_0) = 1$ and $f_{h_0}(s_0) = s_0$. These two equations yield $h_0 = 1/2e$. Therefore, as soon as

$$\alpha \geq \sigma^2 \exp(-2C) h_0, \tag{37}$$

d_- is not defined.

We perform experiments with the following initial condition. We equally space N agents on a circle of radius R centered at the origin of the plane. For speeds, we take $v_0 = -(\lambda/R)x_0$, $\lambda \geq 0$, which means that speeds point towards the center of the circle, and are of amplitude λ. Therefore, we look at a case where all individuals can be sent toward the center of the circle, which yields a small theoretical minimal distance. It thus allows to see if the bound given by (8) is good under such non advantageous initial conditions. We do our simulations on the interval $[0, 5]$ using an explicit RK2 scheme, with a time step of 0.001. We fix $R = 1$ and try different values of λ and N.

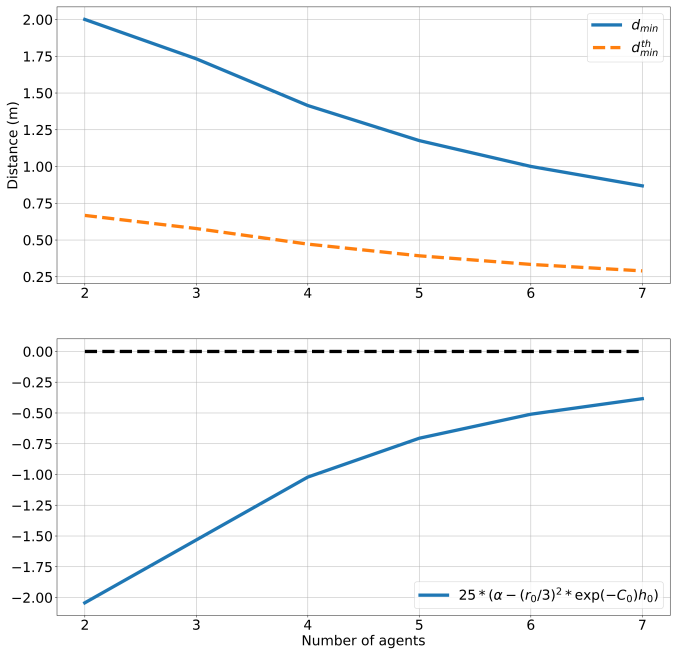

Fig. 1. Minimal distance measured in experiments $\boldsymbol{d_{min}}$ and its theoretical lower bound $\boldsymbol{d_{min}^{th}}$ in terms of the number of agents (above), with the condition of existence $\boldsymbol{\alpha - \sigma^2 \exp(-2C)h_0 \leq 0}$ (below).

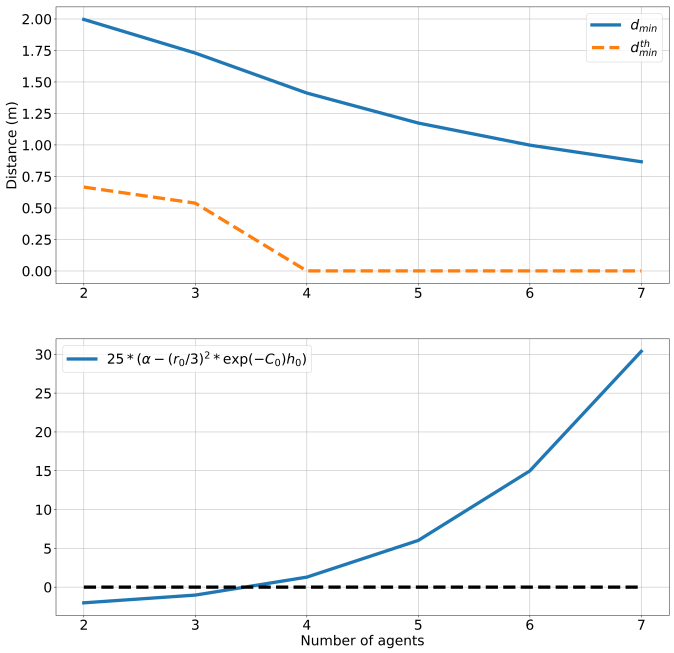

Fig. 2. Minimal distance measured in experiments $\boldsymbol{d_{min}}$ and its theoretical lower bound $\boldsymbol{d_{min}^{th}}$ in terms of the number of agents (above), with the condition of existence $\boldsymbol{\alpha - \sigma^2 \exp(-2C)h_0 \leq 0}$ (below).

C. Experiments

1) First experiment : $\lambda = 0$: We take $\lambda = 0$, so that $\alpha = C = 0$. In Fig.1, we plot in the top part the minimal distance at equilibrium from the simulation d_{min}, as well as the bound (35) d_{min}^{th}, in term of N for $N = 2, \cdots, 7$. In the bottom part, we plot the condition of existence (37) : as long as the quantity is negative, our lower bound exists. In this case, (35) reduces to $d_{min}^{th} = \min_{i \neq j} r_{ij}(0)/3 = d_{min}/3$, since individuals are not moving. Therefore $d_{min}/d_{min}^{th} = 3$, so that the bound stays of the same order for large groups of agents. However, taking $\lambda = 0$ means individuals already are polarized, and the minimal distance is known. As mentioned above, it will get incredibly worse for $\lambda > 0$.

2) Second experiment : $\lambda > 0$: We now take $\lambda = 0.001$, so that all agents are launched toward the same position. In Fig.2, we plot the same data as above. We see that even for a small λ, d_{min}^{th} is not defined, since condition (37) is not met. As it gets worse as $|x_0|_2, |v_0|_2$ and N increase, it is clearly not suited as a realistic practical estimate.

3) Third experiment : using experiment-specific constants: In the previous experiment, it is known that the group of agents is contracting. Therefore we can replace $r_M = |x_0|_2 \exp(|v_0|_2)$ with $\max_{i \neq j} r_{ij}(0)$ in order to improve our estimation. The resulting plot is given in Fig.3. We see that the change of constant helped, but is not enough. It means estimations need to be reworked.

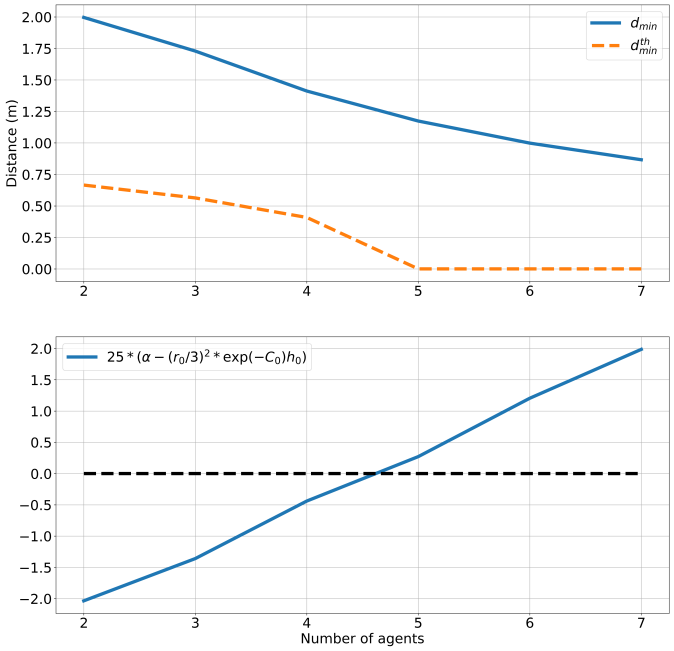

Fig. 3. Minimal distance measured in experiments $\boldsymbol{d_{min}}$ and its theoretical lower bound $\boldsymbol{d_{min}^{th}}$ in terms of the number of agents (above), with the condition of existence $\boldsymbol{\alpha - \sigma^2 \exp(-2C)h_0 \leq 0}$ (below).

VI. Conclusion

In this paper we proved that there always exists a uniform minimal distance between individuals subject to a singular Cucker-Smale model (4) if the alignment rate ψ also provides flocking. In consequence, we showed that a swarm whose individuals use optical flow as a control law is collisionless for all time given that these individuals are small enough. The extension to more complex control laws with additional effects has also been investigated, covering repulsion or attraction effects, or speed / mass center control of the swarm. Deriving an efficient estimate of a lower bound for this minimal distance still remains a challenge. Some other extensions to more forms of control overs speeds which also exhibits an exponential

convergence rate to equilibrium, such as leadership, are also worthy of investigation since they might be prone to preserve the uniform minimal distance. These points will be the focus of future work.

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