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# Almost Kenmotsu Manifolds

*H. G. Nagaraja, U. Manjulamma*

**Abstract.** The object of this paper is to study generalized  $\phi$ -recurrent almost Kenmotsu manifolds with characteristic vector field  $\xi$  belonging to  $(k, \mu)'$ -nullity distribution. We have showed that these manifolds reduce to Kenmotsu manifolds with scalar curvature  $-1$ . Further we establish the relations among the associated 1-forms and proved the conditions under which gradient Ricci almost soliton reduce to gradient Ricci soliton.

## 1 Introduction

Dileo and Pastore [7] introduced the notion of  $(k, \mu)'$ -nullity distribution and established some classification results on almost Kenmotsu manifolds [17] with characteristic vector field  $\xi$  belonging to  $(k, \mu)$ -nullity distribution. As a weaker version of local symmetry Takahashi [16] introduced the idea of local  $\phi$ -symmetry on a Sasakian manifold and study extended to the locally  $\phi$ -symmetric  $\beta$ -Kenmotsu manifolds by Shaikh and Hui [13]. As a weaker version of local  $\phi$  symmetry, Dubey [8] introduced generalized recurrent manifolds and this notion with generalized Ricci recurrent manifolds has been extensively studied by De and Guha [3], [2]. Generalizing the idea of local  $\phi$ -symmetry, De et al. [4] introduced the notion of  $\phi$ -recurrent Sasakian manifolds and De et al. [5] extended the study to  $\phi$ -recurrent Kenmotsu manifolds. The investigation of almost Ricci solitons was presented by Pigola et al. [11] by including the condition the parameter  $\lambda$  to be a variable and changed the meaning of Ricci solitons.

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Motivated by above studies, in this paper, we study generalized  $\phi$ -recurrent almost Kenmotsu manifolds with characteristic vector field  $\xi$  belonging to  $(k, \mu)'$ -nullity distribution. In section 3, we proved the relations between associated 1-forms and found eigen value and the corresponding eigenvector of Ricci operator  $Q$ . Also we consider generalized concircularly  $\phi$ -recurrent and generalized projective  $\phi$ -recurrent almost Kenmotsu manifolds. In section 4, we show that almost gradient Ricci solitons in a generalized concircularly  $\phi$ -recurrent and generalized projective  $\phi$ -recurrent almost Kenmotsu manifolds reduce to gradient Ricci solitons.

## 2 Preliminaries

An almost contact manifold is an  $n$ -dimensional smooth manifold  $M$  endowed with a  $(1, 1)$ -tensor field  $\phi$ , a global vector field  $\xi$  and a one-form  $\eta$  on  $M$  such that

$$\phi^2 = -Id + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \quad (1)$$

In this case, such a manifold will be denoted by  $(M, \phi, \xi, \eta)$ . A Riemannian metric  $g$  on an almost contact manifold  $M$  is said to be compatible with the almost contact structure  $(\phi, \xi, \eta)$  if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

for any vector fields  $X, Y$  on  $M$ . An almost contact manifold  $(M, \phi, \xi, \eta)$  with a compatible Riemannian metric  $g$  is called an almost contact metric manifold and will be denoted by  $(M, \phi, \xi, \eta, g)$ . With this structure, one can associate to any vector fields  $X$  and  $Y$ , a 2-form  $\Phi$  such that  $\Phi(X, Y) = g(X, \phi Y)$ .

An almost contact metric manifold  $(M, \phi, \xi, \eta, g)$  is said to be an almost Kenmotsu manifold if the 1-form  $\eta$  is closed and  $d\Phi = 2\eta \wedge \Phi$ . It is well known that the normality of almost contact structure is expressed by the vanishing of the tensor  $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$ , where  $[\phi, \phi]$  is the Nijenhuis tensor of  $\phi$ . The normality of almost Kenmotsu manifold is expressed by [9]

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (3)$$

for any vector fields  $X, Y$  on  $M$ .

In [7], the authors introduced the idea of  $(k, \mu)'$  - nullity distribution on an almost Kenmotsu manifold  $(M, \phi, \xi, \eta, g)$ , which is defined for any  $p \in M$  and  $k, \mu \in R$  as follows:

$$N_p(k, \mu)' = \{Z \in T_p M : R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)h'X - g(X, Z)h'Y)\}, \quad (4)$$

where  $h' = h \circ \phi$  and  $h = \frac{1}{2}L_\xi \phi$  satisfying

$$h\xi = 0, \quad trh = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0. \quad (5)$$

In an almost Kenmotsu manifold with the characteristic vector field  $\xi$  belonging to  $(k, \mu)$ -nullity distribution, the following hold:

$$h'\xi = 0, \quad h'^2 = (\kappa + 1)\phi^2, \tag{6}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X, \tag{7}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{8}$$

where  $l = R(\cdot, \xi)\xi$ .

From equation (4), we obtain

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y]. \tag{9}$$

$$S(Y, Z) = (n - 1)kg(Y, Z) - \mu g(h'Y, Z). \tag{10}$$

$$QY = (n - 1)kY - \mu h'Y. \tag{11}$$

$$r = n(n - 1)k. \tag{12}$$

Also we have

$$(\nabla_X h')Y = -g(h'X + h'^2 X, Y)\xi - \eta(Y)(h'X + h'^2 X) - (\mu + 2)\eta(X)h'Y. \tag{13}$$

A Riemannian manifold  $(M, g)$  is called generalized recurrent [3] if its curvature tensor  $R$  of type (1,3) satisfies

$$\nabla R = A \otimes R + B \otimes G, \tag{14}$$

and  $(M, g)$  is called a generalized Ricci-recurrent manifold [2] if its Ricci tensor  $S$  of type (0,2) is not identically zero and satisfies the condition

$$\nabla S = A \otimes S + B \otimes S, \tag{15}$$

where  $A$  and  $B$  are non-vanishing 1-forms defined by  $A(\cdot) = g(\cdot, \rho_1)$ ,  $B(\cdot) = g(\cdot, \rho_2)$ , where  $\rho_1$  and  $\rho_2$  are unit vector fields.

Specially, if the 1-form  $B$  vanishes, then (14) turns into the notion of recurrent manifold introduced by Walker [18] and (15) reduced to the notion of Ricci-recurrent manifold introduced by Patterson [10].

A Riemannian manifold  $(M, g)$  is called a super generalized Ricci-recurrent manifold [12] if its Ricci tensor  $S$  of type (0,2) satisfies the condition

$$\nabla S = \alpha \otimes S + \beta \otimes g + \gamma \otimes \eta \otimes \eta, \tag{16}$$

where  $\alpha, \beta$ , and  $\gamma$  are non-vanishing unique 1-forms. In particular, if  $\beta = \gamma$ , then (16) reduces to the notion of quasi-generalized Ricci-recurrent manifold introduced by Shaikh and Roy [14].

A Riemannian manifold  $(M, g)$  is said to be an almost Ricci soliton, if there exist a complete vector field  $X$  and a smooth soliton function  $\lambda: M \rightarrow R$  satisfying

$$S(X, Y) + \frac{1}{2}\mathcal{L}_X g(X, Y) = \lambda g(X, Y), \tag{17}$$

where  $\mathcal{L}_X$  is the Lie derivative with respect to vector field  $X$ . An almost Ricci soliton  $(M, g, X, \lambda)$  will be called expanding, steady or shrinking, if  $\lambda < 0$ ,  $\lambda = 0$  or  $\lambda > 0$  respectively. If the vector field  $X$  is gradient of a smooth function  $f: M \rightarrow R$ , the manifold will be called a gradient almost Ricci soliton. In this case, the preceding equation becomes

$$S + \nabla^2 f = \lambda g, \quad (18)$$

where  $\nabla^2 f$  represents the Hessian of  $f$ .

### 3 Generalized $\phi$ -recurrent Almost Kenmotsu manifolds

**Definition 3.1.** An almost Kenmotsu manifold is said to be generalized  $\phi$ -recurrent if it satisfies the relation

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)(R(X, Y)Z) + B(W)(G(X, Y)Z), \quad (19)$$

for all  $X, Y, Z, W \in \chi(M)$ , where  $A(\cdot) = g(\cdot, \rho_1)$  and  $B(\cdot) = g(\cdot, \rho_2)$  are non-vanishing 1-forms,  $\rho_1$  and  $\rho_2$  are unit vector fields, and the tensor  $G$  of type (1,3) is given by

$$G(X, Y)Z = (X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y, \quad (20)$$

for all  $X, Y, Z \in \chi(M)$ ;  $\chi(M)$  being the lie algebra of smooth vector fields on  $M$  and  $\nabla$  denotes the operator of covariant differentiation with respect to the metric tensor  $g$ . The 1-forms  $A$  and  $B$  are called the associated 1-forms of the manifolds.

**Proposition 3.2.** *A generalized  $\phi$ -recurrent almost Kenmotsu manifold  $M$  with  $\xi$  belonging to  $(k, \mu)'$  nullity distribution, the vector fields of associated 1-forms are co-directional. Further in  $M$  the following are equivalent:*

- (i)  $\phi^2((\nabla_{\rho_1} R)(X, Y)Z) = 0$
- (ii)  $M$  is of constant sectional curvature  $k$ .

*Proof.* We suppose that the manifold  $M$  is a generalized  $\phi$ -recurrent almost Kenmotsu manifold.

Differentiating (9) covariantly with respect to  $W$ , we obtain

$$\begin{aligned} (\nabla_W R)(\xi, Y)Z &= k[\eta(Z)\eta(W)Y - g(Z, h'W)Y + \eta(\nabla_W Z)h'Y] \\ &\quad + \mu[g(Z, h'W)h'Y - \eta(Z)\eta(W)h'Y + \eta(Z)(\nabla_W h')(Y) - \eta(\nabla_W Z)h'Y]. \end{aligned} \quad (21)$$

By virtue of (1) and (19), we get

$$\begin{aligned} -(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi \\ = A(W)R(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (22)$$

Setting  $X = \xi$  in (22), we have

$$\begin{aligned}
 & -(\nabla_W R)(\xi, Y)Z + \eta((\nabla_W R)(\xi, Y)Z)\xi \\
 & = A(W)R(\xi, Y)Z + B(W)[g(Y, Z)\xi - g(\xi, Z)Y].
 \end{aligned}
 \tag{23}$$

Using (4), (21) in (23), we obtain

$$\begin{aligned}
 & k\{g(Z, h'W)Y - \eta(\nabla_W Z)Y - \eta(Z)\eta(W)Y\} + \mu\{\eta(Z)(\nabla_W h')Y - \eta(Z)\eta(W)h'Y \\
 & + g(Z, h'W)h'Y - \eta(\nabla_W Z)h'Y\} + k\{\eta(Z)\eta(W)\eta(Y)\xi - g(Z, h'W)\eta(Y)\xi \\
 & + \eta(\nabla_W Z)\eta(Y)\xi\} - \mu\{\eta(Z)\eta(\nabla_W h')\eta(Y)\xi\} \\
 & = A(W)[k\{g(Y, Z)\xi - \eta(Z)Y\} - \mu\{\eta(Z)h'Y\}] + B(W)[g(Y, Z)\xi - \eta(Z)Y].
 \end{aligned}
 \tag{24}$$

For any vector fields  $Y, Z$  orthogonal to  $\xi$ , (24) takes the form

$$\begin{aligned}
 & k[g(Z, h'W)Y - \eta(\nabla_W Z)Y] + \mu[g(Z, h'W)h'Y - \eta(\nabla_W Z)h'Y] \\
 & = A(W)\{kg(Y, Z)\xi\} + B(W)g(Y, Z)\xi.
 \end{aligned}
 \tag{25}$$

Contracting the above equation with  $\xi$ , we obtain

$$B(W) = -kA(W).
 \tag{26}$$

Putting (26) and (20) in (19), we obtain

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W) [R(X, Y)Z - k\{g(Y, Z)X - g(X, Z)Y\}].
 \tag{27}$$

Taking  $W = \rho_1$  in (27), we get

$$\phi^2((\nabla_{\rho_1} R)(X, Y)Z) = R(X, Y)Z - k\{g(Y, Z)X - g(X, Z)Y\}.
 \tag{28}$$

The result follows from (28). □

We have the following result due to Dileo and Pastore [6]:

**Theorem A.** *An almost Kenmotsu manifold of constant sectional curvature  $K$  is a Kenmotsu manifold and  $K = -1$ .*

*From Theorem A and Proposition 3.2, we have the following:*

**Corollary 3.3.** *If a generalized  $\phi$ -recurrent almost Kenmotsu manifold  $M$  with  $\xi$  belonging to  $(k, \mu)'$  nullity distribution satisfies  $\phi^2((\nabla_{\rho_1} R)(X, Y)Z) = 0$  then it reduces to a Kenmotsu manifold, where  $\rho_1$  is a vector field of associated 1-form of  $M$ . In this case sectional curvature  $K = k = -1$ .*

**Theorem 3.4.** *In a generalized  $\phi$ -recurrent almost Kenmotsu manifold  $(M, g)$ , the characteristic vector field  $\xi$  and the vector field  $\rho_1 k + \rho_2$  associated to the 1-form  $Ak + B$  are co-directional.*

*Proof.* Taking inner product of (22) with  $\xi$  and using first Bianchi's identity, we get

$$\begin{aligned} & A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) \\ & + B(W)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)] + B(X)[\eta(Y)g(W, Z) \\ & - \eta(W)g(Y, Z)] + B(Y)[\eta(W)g(X, Z) - \eta(X)g(W, Z)] = 0. \end{aligned} \quad (29)$$

Using (4) in (29), we obtain

$$\begin{aligned} & (A(W)k + B(W))[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ & + (A(X)k + B(X))[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \\ & + (A(Y)k + B(Y))[g(X, Z)\eta(W) - g(W, Z)\eta(X)] = 0, \end{aligned} \quad (30)$$

for any vector fields  $X, Y, Z, W$ . Let  $\{e_i, i = 1, 2, 3, \dots, n\}$  be a local orthonormal basis of tangent space at each point of the manifold  $M$ . Setting  $Y = U = e_i$  in the above equation and taking summation over  $i : 1 \leq i \leq n$ , we get

$$[A(W)k + B(W)]\eta(X) = [A(X)k + B(X)]\eta(W). \quad (31)$$

Putting  $X = \xi$  in (31), we obtain

$$[A(W)k + B(W)] = [\eta(\rho_1)k + \eta(\rho_2)]\eta(W), \quad (32)$$

for any vector field  $W$ . The result follows from equations (31) and (32).  $\square$

**Theorem 3.5.** *In a generalized  $\phi$ -recurrent almost Kenmotsu manifold  $\rho_1$  is an eigen vector of the ricci operator  $Q$  corresponding to the eigen value  $(\frac{-(n-1)}{2})$ .*

*Proof.* Taking cyclic sum of (22) in  $W, Y, X$  and then by virtue of Bianchi's second identity we have

$$\begin{aligned} & A(X)R(Y, W)Z + A(Y)R(W, X)Z + A(W)R(X, Y)Z \\ & + B(W)[g(Y, Z)X - g(X, Z)Y] \\ & + B(X)[g(W, Z)Y - g(Y, Z)W] \\ & + B(Y)[g(X, Z)W - g(W, Z)X] = 0. \end{aligned} \quad (33)$$

Contracting (33) with  $U$ , we obtain

$$\begin{aligned} & A(X)g(R(Y, W)Z, U) + A(Y)g(R(W, X)Z, U) + A(W)g(R(X, Y)Z, U) \\ & + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)] \\ & + B(X)[g(W, Z)g(Y, U) - g(Y, Z)g(W, U)] \\ & + B(Y)[g(X, Z)g(W, U) - g(W, Z)g(X, U)] = 0. \end{aligned} \quad (34)$$

Setting  $Y = U = e_i$  in (34), where  $\{e_i\}$  is an orthonormal basis of the tangent space at each point of the manifold and summing over  $i = 1, 2, 3, \dots, n$ , we get

$$\begin{aligned} & A(X)S(W, Z) + A(e_i)g(R(W, X)Z, e_i) + A(W)g(R(X, e_i)Z, e_i) \\ & + B(W)[g(X, Z)(1 - n)] + B(X)[(n - 1)g(W, Z)] \\ & + B(e_i)[g(X, Z)g(W, e_i) - g(W, Z)g(X, e_i)] = 0. \end{aligned} \quad (35)$$

Again taking  $Z = X = e_i$  in (35) and then taking summation over  $i$ ,  $1 \leq i \leq n$ , we get

$$S(W, \rho_1) = \frac{(n-1)^2}{2}B(W) + \frac{r}{2}A(W). \tag{36}$$

Using (26), (12) in (36), we obtain

$$Q\rho_1 = \left(\frac{-(n-1)}{2}\right)\rho_1. \tag{37}$$

In an almost Kenmotsu manifold for any (1,3) tensor  $K$ , we define

$$\phi^2((\nabla_W K)(X, Y)Z) = A(W)(K(X, Y)Z) + B(W)(G(X, Y)Z), \tag{37}$$

for all  $X, Y, Z, W \in \chi(M)$ , where  $A$  and  $B$  are defined as in (19).

The concircular curvature and projective curvature tensors of type (1,3) are respectively given by

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}G(X, Y)Z, \tag{38}$$

and

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}(S(Y, Z)X - S(X, Z)Y), \tag{39}$$

for all  $X, Y, Z \in \chi(M)$ .

**Theorem 3.6.** *A generalized concircularly  $\phi$ -recurrent almost Kenmotsu manifold  $M$  is a super generalized Ricci recurrent manifold.*

*Proof.* If  $M$  is an almost Kenmotsu manifold where (37) holds, then for a concircular curvature tensor  $\tilde{C}$ , we have

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = A(W)(\tilde{C}(X, Y)Z) + B(W)(G(X, Y)Z), \tag{40}$$

where  $A$  and  $B$  are defined as in (19).

Using (1) in (38), we get

$$-(\nabla_W \tilde{C})(X, Y)Z + \eta((\nabla_W \tilde{C})(X, Y)Z)\xi = A(W)(\tilde{C}(X, Y)Z) + B(W)[g(Y, Z)X - g(X, Z)Y]. \tag{41}$$

Contracting with  $U$  in (41), we obtain

$$\begin{aligned} & -g((\nabla_W \tilde{C})(X, Y)Z, U) + \eta((\nabla_W \tilde{C})(X, Y)Z)\eta(U) \\ & = A(W)[g(\tilde{C}(X, Y)Z, U) + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \end{aligned} \tag{42}$$

Setting  $X = U = e_i$  in (42), and using (38), we get

$$\begin{aligned} (\nabla_W S)(Y, Z) & = -A(W)S(Y, Z) + \left[\frac{(Wr) - n(n-1)^2B(W)}{n(n-1)}\right]g(Y, Z) \\ & \quad - \left[\frac{(Wr)}{n(n-1)}\right]\eta(Y)\eta(Z). \end{aligned} \tag{43}$$



Above equation can be written as

$$\nabla S = A_1 \otimes S + \psi \otimes g + H \otimes \eta \otimes \eta, \quad (44)$$

where

$$\begin{aligned} A_1(W) &= -A(W), \\ \psi(W) &= \frac{(Wr) - n(n-1)^2 B(W)}{n(n-1)}, \\ H(W) &= - \left[ \frac{(Wr)}{n(n-1)} \right]. \quad \square \end{aligned}$$

**Theorem 3.7.** *In a generalized projectively  $\phi$ -recurrent almost Kenmotsu manifold  $M$  with constant scalar curvature we have*

$$B(W) = \frac{-r}{n(n-1)} A(W).$$

*Proof.* We assume that an almost Kenmotsu manifold  $M$  is generalized projectively  $\phi$ -recurrent. Then from (37), we have

$$\phi^2((\nabla_W P)(X, Y)Z) = A(W)(P(X, Y)Z) + B(W)(G(X, Y)Z), \quad (45)$$

where  $A$  and  $B$  are defined as in (19).

By virtue of (1), it follows from (45), that

$$\begin{aligned} -(\nabla_W P)(X, Y)Z + \eta((\nabla_W P)(X, Y)Z)\xi &= A(W)(P(X, Y)Z) \\ &+ B(W)[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (46)$$

Taking inner product of (46) with  $U$ , we obtain

$$\begin{aligned} -g((\nabla_W P)(X, Y)Z, U) + \eta((\nabla_W P)(X, Y)Z)\eta(U) &= A(W)(g(P(X, Y)Z, U)) \\ &+ B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \end{aligned} \quad (47)$$

Setting  $X = U = e_i$  in (47) and summing over  $i = 1, 2, 3, \dots, n$ , we get

$$(\nabla_W S)(Y, Z) = -A(W)S(Y, Z) - [(n-1)B(W)]g(Y, Z). \quad (48)$$

Letting  $Y = Z = e_i$  in (48), we obtain

$$(Wr) = -(rA(W) + n(n-1)B(W)). \quad \square$$

### 4 Gradient Almost Ricci Solitons on Almost Kenmotsu manifolds

In this section we consider gradient almost Ricci soliton  $(M, g, X, \lambda)$ . i.e. an almost Ricci soliton  $(M, g, X, \lambda)$  when the vector field  $X$  is the gradient of a smooth function  $f \in C^\infty(M)$ . Accordingly equation (18) becomes

$$\nabla_Y Df = QY + \lambda Y, \tag{49}$$

where  $D$  is the gradient operator of  $g$  and  $Q$  is Ricci operator.

**Theorem 4.1.** *Let  $M$  be an almost Kenmotsu manifold admitting gradient almost Ricci soliton. If  $M$  is generalized concircularly  $\phi$ -recurrent or generalized projectively  $\phi$ -recurrent then the following are equivalent:*

- (i)  $X\lambda = (\xi\lambda)\eta(X)$  for any vector field  $X$ .
- (ii)  $f$  is constant along  $k\phi^2 X - \mu h' X$ .

*Proof.* By virtue of (49), we obtain

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X - (Y\lambda)X + (X\lambda)Y, \tag{50}$$

**Case (i):** Suppose  $M$  is generalized concircularly  $\phi$ -recurrent. From (43), we get

$$\begin{aligned} (\nabla_Y Q)X &= A(Y)QX + \left[ \frac{(Yr)}{n(n-1)} - \left( k - \frac{r}{n(n-1)A(Y)} + (n-2)B(Y) \right) \right] X \\ &\quad - \left[ \frac{(Yr)}{n(n-1)} - \left( k - \frac{r}{n(n-1)A(Y)} - B(Y) \right) \right] \eta(Y)\xi. \end{aligned} \tag{51}$$

Substituting (51) in (50), we get

$$\begin{aligned} R(X, Y)Df &= A(Y)QX - A(X)QY \\ &\quad + \left[ \frac{(Yr)}{n(n-1)} - \left( k - \frac{r}{n(n-1)A(Y)} + (n-2)B(Y) \right) \right] X \\ &\quad - \left[ \frac{(Yr)}{n(n-1)} - \left( k - \frac{r}{n(n-1)A(Y)} - B(Y) \right) \right] \eta(X)\xi \\ &\quad - \left[ \frac{(Xr)}{n(n-1)} - \left( k - \frac{r}{n(n-1)A(X)} + (n-2)B(X) \right) \right] Y \\ &\quad + \left[ \frac{(Xr)}{n(n-1)} - \left( k - \frac{r}{n(n-1)A(X)} - B(X) \right) \right] \eta(Y)\xi \\ &\quad - (Y\lambda)X + (X\lambda)Y. \end{aligned} \tag{52}$$

Taking inner product with  $\xi$  in (52), we obtain

$$\begin{aligned}
 g(R(X, Y)Df, \xi) &= A(Y)g(X, Q\xi) - A(X)g(Y, Q\xi) \\
 &+ \left[ \frac{(Yr)}{n(n-1)} - \left(k - \frac{r}{n(n-1)}\right)A(Y) + (n-2)B(Y) \right] \eta(X) \\
 &- \left[ \frac{(Yr)}{n(n-1)} - \left(k - \frac{r}{n(n-1)}\right)A(Y) - B(Y) \right] \eta(X) \\
 &- \left[ \frac{(Xr)}{n(n-1)} - \left(k - \frac{r}{n(n-1)}\right)A(X) + (n-2)B(X) \right] \eta(Y) \\
 &+ \left[ \frac{(Xr)}{n(n-1)} - \left(k - \frac{r}{n(n-1)}\right)A(X) - B(X) \right] \eta(Y) \\
 &- (Y\lambda)\eta(X) + (X\lambda)\eta(Y).
 \end{aligned} \tag{53}$$

Interchange  $Df$  by  $\xi$  and using (10) in (53), we get

$$g(R(X, Y)\xi, Df) = (Y\lambda)X - (X\lambda)Y. \tag{54}$$

Using (9), (11) in (54), we obtain

$$\begin{aligned}
 &k[g(X, Df)\eta(Y) - g(Y, Df)\eta(X)] + \mu[g(h'X, Df)\eta(Y) - g(h'Y, Df)\eta(X)] \\
 &= (Y\lambda)\eta(X) - (X\lambda)\eta(Y).
 \end{aligned} \tag{55}$$

Putting  $Y = \xi$  in (55), we get

$$k[g(X, Df) - \eta(X)\eta(Df)] + \mu[g(h'X, Df)] = (\xi\lambda)\eta(X) - (X\lambda). \tag{56}$$

The above equation may be rewritten in the form

$$(k\phi^2X - \mu h'X)f = X\lambda - (\xi\lambda)\eta(X). \tag{57}$$

**Case (ii):** Suppose  $M$  is generalized projectively  $\phi$ -recurrent. Then from (48), we have

$$\begin{aligned}
 (\nabla_Y Q)X &= A(Y)QX - \left[ A(Y)k\left(\frac{2n-3}{n-1}\right) + (n-2)B(Y) \right] X \\
 &+ \left[ A(Y)k\left(\frac{2n-3}{n-1}\right) + B(Y) \right] \eta(X)\xi \\
 &- (Y\lambda)X + (X\lambda)Y.
 \end{aligned} \tag{58}$$

Using (58) in (50) and proceeding as in case (i), we obtain (57). This completes the proof.  $\square$

**Remark 4.2.** In view of Theorem 4.1, we note that gradient almost Ricci soliton on a generalized concircularly  $\phi$ -recurrent (or generalized projectively  $\phi$ -recurrent) almost Kenmotsu manifold  $M$  reduces to gradient Ricci soliton if  $f$  is constant along  $k\phi^2X - \mu h'X$ .

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