



HAL
open science

An implicit representation for the analysis of piecewise affine discrete-time systems

Leonardo Broering Groff, Giorgio Valmorbida, João Manoel Gomes da Silva

► **To cite this version:**

Leonardo Broering Groff, Giorgio Valmorbida, João Manoel Gomes da Silva. An implicit representation for the analysis of piecewise affine discrete-time systems. *Automatica*, 2023, 147, pp.110730. 10.1016/j.automatica.2022.110730 . hal-03946015

HAL Id: hal-03946015

<https://hal.science/hal-03946015v1>

Submitted on 18 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An implicit representation for the analysis of piecewise affine discrete-time systems [★]

Leonardo B. Groff¹ and Giorgio Valmorbida² and Joao M. Gomes da Silva Jr.¹

¹ Universidade Federal do Rio Grande do Sul (UFRGS), Av. Osvaldo Aranha, 103, 90035-190 Porto Alegre- RS, Brazil

{leonardo.groff, jmgomes}@ufrgs.br

² Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France.

giorgio.valmorbida@centralesupelec.fr

Abstract

This paper presents a new framework for stability assessment of discrete-time piecewise affine systems. An implicit representation for piecewise functions based on ramp nonlinearities is proposed. Instead of the usual sector inequalities adopted in the study of Lurie systems, we directly exploit properties of the ramp function, which are given by a particular set of identities and inequalities. These properties are then used to derive stability conditions associated to piecewise quadratic Lyapunov functions. These functions are implicitly defined from quadratic forms involving ramp functions. These conditions can be cast as LMIs and their numerical solution is illustrated by examples highlighting the advantages of the proposed method over existing stability analysis methods for PWA systems.

Key words: piecewise affine systems, discrete-time systems, stability, piecewise quadratic Lyapunov functions

1 Introduction

Models for Piecewise Affine (PWA) systems have been proposed to study engineered systems such as nonlinear circuits [15, 17], where simple piecewise affine nonlinearities may lead to complex behavior, or in the context of hybrid systems [8]. Moreover, systems presenting some static nonlinearities, such as saturation or deadzone, can also be studied in this framework since these functions are indeed piecewise affine [9, 18]. The practical interest on PWA continuous functions in discrete-time systems also appears in the context of Receding Horizon Optimal Control (ROHC) [2], in which multi-parametric linear or quadratic programs can be solved offline to obtain PWA control laws associated to a specific partition of the state space, termed explicit Model Predictive Control.

Studies on this class of systems trace back to [21], where the following explicit representation was introduced for a PWA function $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n_f}$, $f = f_j, \forall x \in \Gamma_j \subset \mathbb{R}^n$,

$$f_j(x) = A_j x + b_j, \quad (1)$$

[★] This work was supported by STIC-AmSud 18-STIC-01. This research is funded in part by ANR via project HANDY, number ANR-18-CE40-0010.

$j = 1, \dots, n_s$, with disjoint sets Γ_j defining a partition of \mathbb{R}^n , i.e. $\cup_{j=1}^{n_s} \Gamma_j = \mathbb{R}^n$.

The stability analysis of PWA continuous-time systems using the explicit representation (1) has been studied with continuous piecewise quadratic Lyapunov functions (LF) [14, 13, 12]. In [14], the sets Γ_j are described by the intersections of half spaces. In [13] the particular case of conewise linear systems is addressed. In [12], a representation of the polyhedral regions of the state space partition by vertices and cone rays is considered. In this case, to guarantee the positivity of the Lyapunov function and the negativity of its derivative along the trajectories of the system, sufficient conditions based on the cone rays and the vertices that represent each region have to be tested.

On the other hand, the stability analysis of discrete-time PWA systems has been studied for instance in [7, 8, 10, 20]. Similar to [14], polyhedral partitions described by the intersections of half spaces are considered in these works. Differently from the continuous-time case, discontinuous piecewise quadratic Lyapunov functions can be studied for discrete-time systems. However, a drawback of the results using a different quadratic function for each set Γ_j in the partition appears when

assessing the decrease of the LF during a transition. Indeed, one needs to enumerate all possible transitions between sets in the partition and evaluate the associated decrease of the LF.

The goal of this paper is to investigate stability conditions for discrete-time PWA systems. With this aim we introduce an implicit representation of PWA functions. The proposed representation allows to avoid some shortcomings of the explicit representation (1). In particular, by adopting the proposed representation we show that it is possible to parametrize continuous piecewise quadratic Lyapunov functions by considering generalized quadratic forms involving ramp functions. The stability of PWA systems can thus be assessed by evaluating Lyapunov stability conditions through linear matrix inequalities (LMI) tests, and does not require the enumeration of transitions between the sets of the partition.

This paper is organized as follows: Section 2 presents the proposed implicit representation of PWA systems, based on vector-valued ramp functions, which is the basis of the stability analysis framework. Properties of vector-valued ramp functions and conditions to certify the positivity of generalized quadratic forms involving such functions are introduced in Section 3. In Section 4 we apply the positivity verification to formulate conditions for assessing the stability of discrete-time PWA systems using PWQ Lyapunov functions. Finally we illustrate the obtained results with numerical examples in Section 5 and present concluding remarks and perspectives in Section 6.

Notation For a vector $v \in \mathbb{R}^n$, v_i denotes its i th entry, for a matrix $M \in \mathbb{R}^{n \times m}$, $M_{i,j}$ denotes its (i,j) entry. Define $\mathbb{D}^n = \{M \in \mathbb{R}^{n \times n} \mid M_{i,j} = 0, i \neq j\}$, $\mathbb{P}^{n \times m} = \{M \in \mathbb{R}^{n \times m} \mid M_{i,j} \geq 0, \forall i, j\}$, and $\mathbb{S}^n = \{M \in \mathbb{R}^{n \times n} \mid M = M^\top\}$. For $M \in \mathbb{R}^{n \times n}$ we define $He(M) := M + M^\top$. The set of non-negative real numbers is denoted $\mathbb{R}_{\geq 0}$. The absolute value of a scalar is denoted $|\cdot|$ and for $M \in \mathbb{R}^{m \times n}$, $\|M\|$ denotes the largest singular value of matrix M . We use $\mathbf{0}$ (boldface) to denote a matrix of zeros of suitable dimensions. For a vector $z \in \mathbb{R}^d$, $diag(z)$ corresponds to the diagonal matrix of dimension $d \times d$ with its diagonal given by vector z .

2 Problem Statement

In this paper we study the stability of piecewise affine discrete-time dynamical systems given by

$$x^+ = f(x), \quad (2)$$

where x is the state, $f(x)$ is a continuous piecewise affine vector function, and x^+ denotes the next time-step value for the discrete-time system. With this aim, we introduce now an implicit representation for a generic piecewise

affine vector function $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n_f}$ as follows:

$$f(x) = F_1 x + F_2 \phi(y(x)) \quad (3a)$$

$$y(x) = F_3 x + F_4 \phi(y(x)) + f_5 \quad (3b)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n_y}$, $F_1 \in \mathbb{R}^{n_f \times n}$, $F_2 \in \mathbb{R}^{n_f \times n_y}$, $F_3 \in \mathbb{R}^{n_y \times n}$, $F_4 \in \mathbb{R}^{n_y \times n_y}$, $f_5 \in \mathbb{R}^{n_y}$ and the vector function $\phi : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_y}$ is defined elementwise by the ramp function as

$$\phi_i(y) = r(y_i) := \begin{cases} 0 & \text{if } y_i < 0 \\ y_i & \text{if } y_i \geq 0 \end{cases}, i = 1, \dots, n_y \quad (4)$$

as depicted in Figure 1.

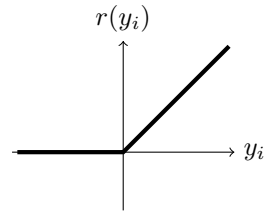


Fig. 1. Ramp function $r(y_i)$.

Using (3)-(4) as a model for continuous PWA functions avoids the explicit definition of partitions and the corresponding affine functions, as in the standard representation (1). In fact, with (3)-(4), it is the vector function $\phi(y(x))$ and the regions where its arguments are not negative that define the PWA partition of \mathbb{R}^n . For this reason, we refer to (3)-(4) as an *implicit representation* of PWA functions. Examples of this representation and its relation with the explicit representation (1) are given in Section 5.

Note that in this implicit representation (3b) is, in general, an implicit equation (see Appendix A for a discussion on the well-posedness and solution of this equation). However, for particular structures of matrix F_4 , explicit solutions to (3b) can be obtained. Henceforth we assume that (3b) is well posed, i.e. for each $x \in \mathbb{R}^n$ there exists a unique solution $y(x)$ for the equation. We should also observe that $f(x)$ is continuous thanks to the continuity of ϕ .

The main feature of (3) that will be exploited in the formulation of stability conditions for PWA discrete-time systems is the characterization of the vector-valued ramp function ϕ in terms of identities and inequalities, presented below in Section 3. With these relations we follow a similar path to the one adopted to study stability of systems with sector bounded nonlinearities using sector inequalities. These properties will be key to obtain numerically tractable conditions for the verification of Lyapunov inequalities.

Differently from approaches based on the explicit representation (1) (e.g. [7, 8]), the proposed implicit representation also simplifies the stability analysis since the partition and possible transitions between sets do not have to be explicitly accounted for in the piecewise quadratic Lyapunov inequalities. Another important aspect is that handling uncertainties in the partition induced by (3) can be simpler than with an explicit representation since these uncertainties can be cast as uncertainties on the matrices F_3 , F_4 and f_5 . These uncertainties can be described by matrix sets such as polytopic or norm-bounded ones [3].

3 Conditions for Positivity of Generalized Quadratic Forms Involving Vector-Valued Ramp Functions

Note that ϕ is a sector bounded nonlinearity as $\phi_i(y) = r(y_i)$ belongs to the sector $[0, 1]$, $\forall i = 1, \dots, n_y$. Several results to verify the positivity of generalized quadratic forms involving sector nonlinearities rely on sector inequalities that hold either globally or locally [22, 11]. These standard sector inequalities cover a broad class of nonlinearities lying in the considered sector. Hence, stability conditions based on these inequalities are conservative if we are interested in the study of a specific nonlinearity. In the following, we provide a set of relations that are verified only for the vector-valued ramp function ϕ . These relations are based on an exact characterization of the ramp function (4), by using some identities and inequalities.

Lemma 1 *For any matrix $T \in \mathbb{D}^{n_y}$ the function ϕ in (4) satisfies the identity*

$$s_1(T, y) := \phi^\top(y) T \phi(-y) = 0, \quad \forall y \in \mathbb{R}^{n_y}. \quad (5)$$

Proof. Since for $\theta < 0$ we have $r(\theta) = 0$, and for $\theta \geq 0$ we have that $r(-\theta) = 0$, the relation

$$r(\theta)r(-\theta) = 0. \quad (6)$$

holds for all $\theta \in \mathbb{R}$. Since the elements of ϕ are defined as ramp functions we have

$$s_1(T, y) = \sum_{i=1}^{n_y} T_{i,i} r(y_i) r(-y_i)$$

which, using (6) and (4), gives (5). ■

Lemma 2 *For any vector $\zeta \in \mathbb{R}^{n_\zeta}$ and matrix $R \in \mathbb{R}^{n_\zeta \times n_y}$ the function ϕ in (4) satisfies the identity*

$$s_2(R, \zeta, y) := \zeta^\top R (y - (\phi(y) - \phi(-y))) = 0, \quad (7)$$

$\forall y \in \mathbb{R}^{n_y}$.

Proof.

Let us first note that the ramp function r satisfies the identity

$$\theta - (r(\theta) - r(-\theta)) = 0. \quad (8)$$

Indeed, if $\theta < 0$, we have $r(\theta) = 0$ and $r(-\theta) = -\theta$. Thus, for $\theta < 0$, $\theta - r(\theta) + r(-\theta) = \theta - 0 - \theta = 0$. If $\theta \geq 0$ we have $r(\theta) = \theta$ and $r(-\theta) = 0$, thus, for $\theta \geq 0$, $\theta - r(\theta) + r(-\theta) = \theta - \theta + 0 = 0$. Since the elements of vector $y - (\phi(y) - \phi(-y))$ are given by $y_i - (r(y_i) - r(-y_i))$ we have that (7) follows from (8). ■

Lemma 3 *For any matrix $M \in \mathbb{P}^{(1+2n_y) \times (1+2n_y)}$ the vector function ϕ in (4) satisfies the inequality*

$$s_3(M, y) := \begin{bmatrix} 1 \\ \phi(y) \\ \phi(-y) \end{bmatrix}^\top M \begin{bmatrix} 1 \\ \phi(y) \\ \phi(-y) \end{bmatrix} \geq 0. \quad (9)$$

$\forall y \in \mathbb{R}^{n_y}$.

Proof. As the ramp function satisfies

$$r(\theta) \geq 0, \quad \forall \theta \in \mathbb{R} \quad (10)$$

from (4) one has that $\phi_i(y) = r(y_i) \geq 0$ and $\phi_i(-y) = r(-y_i) \geq 0$, $\forall i = 1, \dots, n_y$. Thus, if all entries of M are nonnegative, it follows that $s_3(M, y)$ is a nonnegative scalar. ■

We now use the above lemmas to set conditions to verify the positivity of generalized quadratic forms of the type

$$h(x) = \chi(x)^\top H \chi(x). \quad (11)$$

with $\chi(x) = \begin{bmatrix} 1 & x^\top & \phi^\top(y(x)) & \phi^\top(-y(x)) \end{bmatrix}^\top$ and ϕ defined as in (4).

Proposition 1 *Given a generalized quadratic form h as in (11), if there exist matrices $T \in \mathbb{D}^{n_y}$, $R \in \mathbb{R}^{(1+n+2n_y) \times n_y}$, $M \in \mathbb{P}^{(1+2n_y) \times (1+2n_y)}$ such that*

$$h(x) + s_1(T, y(x)) + s_2(R, \chi, y(x)) - s_3(M, y(x)) \geq 0 \quad (12)$$

then

$$h(x) \geq 0 \quad \forall x \in \mathbb{R}^n. \quad (13)$$

Proof. From Lemmas 1 and 2, which hold for all $y(x)$, if (12) is satisfied it follows that

$$h(x) \geq s_3(M, y(x)), \quad \forall x \in \mathbb{R}^n.$$

With Lemma 3 we conclude that $h(x) \geq 0, \forall x \in \mathbb{R}^n$. ■

Remark 1 Setting conditions to verify the non-negativity of a generalized quadratic form as (11) by solving the inequality (12) makes possible the solution to the Lyapunov inequalities related to the stability of PWA systems. These inequalities are studied in the next section.

Since we consider an expression h with terms $\phi(y)$ and $\phi(-y)$ the use of the relations in Lemmas 1, 2 and 3 is important to restrict the validity of the test for the non-negativity of h in Proposition 1 only to functions ϕ of which the components are ramp functions. Thus, by using Lemmas 1, 2 and 3, we restrict the elements of ϕ in the generalized quadratic forms treated in Proposition 1 to ramp functions and not a broad set of functions.

┘

Remark 2 It is possible to use the Karush-Kuhn-Tucker (KKT) optimality conditions to implicitly characterize nonlinearities in terms of identities and inequalities. Such an idea was detailed in [19] for the saturation nonlinearity. We illustrate this approach for the ramp function, which can be expressed as the solution to the optimization problem parameterized in θ as follows

$$\underset{r}{\text{minimize}} \quad \frac{1}{2}(r - \theta)^2 \quad \text{subject to} \quad r \geq 0. \quad (14)$$

With the Lagrangian associated to the optimization problem, $\mathcal{L}(r, \lambda) = \frac{1}{2}(r - \theta)^2 - \lambda r$, we obtain the KKT conditions

$$(r - \theta) - \lambda = 0; \quad \lambda r = 0; \quad r \geq 0; \quad \lambda \geq 0$$

which are necessary for optimality. Note that these relations offer a characterization in terms of linear and quadratic identities and inequalities in three variables (θ, r, λ) . To obtain a description in the variables (θ, r) one can use $\lambda = (r - \theta)$ above to obtain

$$(r - \theta)r = 0 \quad (16a)$$

$$r \geq 0 \quad (16b)$$

$$(r - \theta) \geq 0. \quad (16c)$$

These relations can also be obtained using (6), (8) and (10). Indeed, note that (16a) corresponds to (6), since, from (8), $r(\theta) - \theta = r(-\theta)$. Also, (16b) corresponds to (10), and again, using $r(\theta) - \theta = r(-\theta)$ we have that (10), gives (16c). ┘

4 Stability Analysis of PWA Systems with PWQ Lyapunov Functions

In this section we apply the results for the verification of non-negativity of generalized quadratic forms presented in the previous section to study the stability of the origin

of a discrete-time systems (2) with $f(x)$ defined by the implicit representation (3), with $n_f = n$.

We assume that $\phi(y(0)) = 0$ and thus the origin is an equilibrium point, since $f(0) = 0$ in this case. If $\phi(y(0)) = 0$, we have that (3b) leads to $y(0) = f_5$. Since $\phi(y(0)) = 0$ implies $y_i(0) \leq 0$, we have that $f_{5i} \leq 0$, $i = 1, \dots, n_y$.

The stability of the origin of system (2) is studied with a continuous piecewise quadratic Lyapunov function, given by a generalized quadratic form on x and the function $\phi(y(x))$. Hence, differently from previous approaches, the definition of an explicit quadratic form on x for each set of the partition is not required. More precisely, we consider Lyapunov candidate functions $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $V(0) = 0$ given by

$$V(x) = \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}^\top P \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}. \quad (17)$$

with $P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix}$, $P_1 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^{n \times n_y}$ and $P_3 \in \mathbb{S}^{n_y}$.

We obtain below a quadratic upper bound for $V(x)$ which will be used in the proof of exponential stability of the origin of (2). To this end, we first compute an upper bound for $\|\phi\|^2$. With $\bar{y} := y - f_5$, (3b) gives

$$\bar{y} = F_3 x + F_4 \phi(\bar{y} + f_5). \quad (18)$$

Since for $f(0) = 0$ we have $f_{5i} \leq 0$ and $0 \leq r(y_i + f_5) \leq r(y_i)$, for $i = 1, \dots, n_y$, we thus obtain $\phi(\bar{y} + f_5) = \Delta \bar{y}$ with $\Delta \in \mathcal{D} = \{\Delta \in \mathbb{D}^{n_y} \mid \Delta_{i,i} \in [0, 1]\}$. From the well-posedness assumption (see Appendix A), we have that $(I - F_4 \Delta)$ is invertible for all $\Delta \in \mathcal{D}$, thus using (18) we obtain

$$\bar{y} = (I - F_4 \Delta)^{-1} F_3 x$$

and $\phi(y) = \phi(\bar{y} + f_5) = \Delta \bar{y} = \Delta (I - F_4 \Delta)^{-1} F_3 x$, yielding

$$\|\phi(y(x))\| \leq \sigma \|x\|,$$

with $\sigma = \max_{\Delta \in \mathcal{D}} \|\Delta (I - F_4 \Delta)^{-1} F_3\|$. Hence, from (17), it follows that

$$\begin{aligned} V(x) &\leq \|P_1\| \|x\|^2 + 2 \|P_2\| \|x\| \|\phi\| + \|P_3\| \|\phi\|^2 \\ &\leq (\|P_1\| + 2\sigma \|P_2\| + \sigma^2 \|P_3\|) \|x\|^2 = \epsilon_2 \|x\|^2. \end{aligned} \quad (19)$$

The theorem below presents conditions for the global exponential stability of the origin of (2) using (17) as a Lyapunov function candidate.

Theorem 1 *If there exist matrices $P \in \mathbb{S}^{(n+n_y)}$, $T \in \mathbb{D}^{n_y}$, $R \in \mathbb{R}^{(1+n+2n_y) \times n_y}$, $M \in \mathbb{P}^{(1+2n_y) \times (1+2n_y)}$ and a*

positive scalar ϵ_1 such that

$$(V(x) - \epsilon_1 x^\top x) + s_1(T, y) + s_2(R, \chi, y) - s_3(M, y) \geq 0 \quad (20)$$

and matrices $\tilde{T} \in \mathbb{D}^{2n_y}$, $\tilde{R} \in \mathbb{R}^{1+n+4n_y \times 2n_y}$, $\tilde{M} \in \mathbb{P}^{(1+4n_y) \times (1+4n_y)}$ and a scalar $\eta \in (0, 1)$ such that

$$-(V(x^+) - (1 - \eta)V(x)) + s_1(\tilde{T}, \tilde{y}) + s_2(\tilde{R}, \tilde{\chi}, \tilde{y}) - s_3(\tilde{M}, \tilde{y}) \geq 0 \quad (21)$$

with $\tilde{y} = [y^\top y^{+\top}]^\top$ and $\tilde{\chi} = [1 x^\top \phi(\tilde{y})^\top \phi(-\tilde{y})^\top]^\top$ then the origin of (2) is globally exponentially stable.

Proof. From Proposition 1 and (19), if (20) and (21) hold it respectively follows that

$$\epsilon_1 \|x\|^2 \leq V(x) \leq \epsilon_2 \|x\|^2 \quad (22a)$$

$$V(x^+) \leq (1 - \eta)V(x). \quad (22b)$$

Thus, (22) allows to conclude that $\|x(k)\| \leq ce^{\delta k} \|x(0)\|$ with $c = (\frac{\epsilon_2}{\epsilon_1})^{\frac{1}{2}}$, $\delta = \ln(\sqrt{1 - \eta})$, $\forall x(0) \in \mathbb{R}^n$. Moreover, (22a) implies that $V(x)$ is radially unbounded. ■

The generalized quadratic form involving the the state and a nonlinearity as in (17) has been studied in the context of stability analysis of *continuous-time* linear complementarity systems [16]. Here, the generic formulation presented in [16] is used considering ramp functions, that also satisfy complementarity condition (16a) as discussed in Remark 2. In that paper the authors suggest that their stability conditions could benefit from a numerical formulation exploiting co-positivity conditions. The co-positivity is here accounted for by considering the inequalities of Lemma 3 in Theorem 1. The numerical formulation we propose is detailed in the next section.

4.1 LMI conditions

The relations (20) and (21) can be written in the generic quadratic form given by (11)-(12), where the corresponding matrices H present an affine dependence on the elements of matrix P . Hence, conditions in LMI form can be obtained to ensure (20) and (21). This is formalized in the following Corollary to Theorem 1.

Corollary 1 *If there exist matrices $P_1 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^{n \times n_y}$, $P_3 \in \mathbb{S}^{n_y}$, $T \in \mathbb{D}^{n_y}$, $\tilde{T} \in \mathbb{D}^{2n_y}$, symmetric matrices $M \in \mathbb{P}^{(1+2n_y) \times (1+2n_y)}$ and $\tilde{M} \in \mathbb{P}^{(1+4n_y) \times (1+4n_y)}$, matrices $R \in \mathbb{R}^{(1+n+2n_y) \times n_y}$ and $\tilde{R} \in \mathbb{R}^{(1+n+4n_y) \times 2n_y}$ and positive scalars $\eta < 1$ and ϵ_1 such that the following LMIs*

are verified

$$H + \frac{1}{2}He\{\Psi + R\Xi\} - \mathcal{I}^\top M \mathcal{I} \geq \mathbf{0} \quad (23a)$$

$$\tilde{H} + \frac{1}{2}He\{\tilde{\Psi} + \tilde{R}\tilde{\Xi}\} - \tilde{\mathcal{I}}^\top \tilde{M} \tilde{\mathcal{I}} \geq \mathbf{0} \quad (23b)$$

where

$$H = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_1 - \epsilon_1 I_n & P_2 & \mathbf{0} \\ \mathbf{0} & P_2^\top & P_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathcal{I}^\top = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n_y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_y} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \Xi^\top = \begin{bmatrix} f_5^\top \\ F_3^\top \\ F_4^\top - I_{n_y} \\ I_{n_y} \end{bmatrix},$$

$$\tilde{H} = - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & N_1 & N_2 & \mathbf{0} \\ \mathbf{0} & N_2^\top & N_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathcal{I}}^\top = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{2n_y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{2n_y} \end{bmatrix},$$

$$\tilde{\Psi} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{T} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \tilde{\Xi}^\top = \begin{bmatrix} \tilde{f}_5^\top \\ \tilde{F}_3^\top \\ \tilde{F}_4^\top - I_{2n_y} \\ I_{2n_y} \end{bmatrix}$$

with

$$\tilde{F}_3 = \begin{bmatrix} F_3 \\ F_3 F_1 \end{bmatrix}, \quad \tilde{F}_4 = \begin{bmatrix} F_4 & \mathbf{0} \\ F_3 F_2 & F_4 \end{bmatrix}, \quad \tilde{f}_5 = \begin{bmatrix} f_5 \\ f_5 \end{bmatrix},$$

$$N_1 = F_1^\top P_1 F_1 - (1 - \eta)P_1,$$

$$N_2 = \begin{bmatrix} F_1^\top P_1 F_2 - (1 - \eta)P_2 & F_1^\top P_2 \\ F_2^\top P_1 F_2 - (1 - \eta)P_3 & F_2^\top P_2 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} P_2^\top F_2 & P_3 \end{bmatrix}.$$

then the origin of (2) is globally exponentially stable.

Proof. Consider $V(x)$ defined as in (17). We show now that LMIs (23a) and (23b) are equivalent to the satisfaction of conditions (20) and (21) in Theorem 1.

Recalling that $\chi = [1 x^\top \phi(y)^\top \phi(-y)^\top]^\top$, it follows that $\chi^\top H \chi = V(x) - \epsilon_1 x^\top x$, $\chi^\top \Psi \chi = s_1(T, y)$ and

$\chi^\top \mathcal{I}^\top M \mathcal{I} \chi = s_3(M, y)$. Moreover, taking into account (3b), one has that $\chi^\top R \Xi \chi = s_2(R, \chi, y)$. Hence, by left and right multiplying (23a) respectively by χ^\top and χ , it follows that (20) holds.

Considering now $\tilde{\chi} = \left[1 \ x^\top \ \phi(\tilde{y})^\top \ \phi(-\tilde{y})^\top \right]^\top$, with $\tilde{y} = \left[y^\top \ y^{+\top} \right]^\top$, it follows that $\tilde{\chi}^\top \tilde{H} \tilde{\chi} = -(V(x^+) - (1 - \eta)V(x))$. Furthermore, as $y^+ = F_3 x^+ + F_4 \phi(y^+) + f_5 = F_3(F_1 x + F_2 \phi(y)) + F_4 \phi(y^+) + f_5$, one has $\tilde{y} = \tilde{F}_3 x + \tilde{F}_4 \phi(\tilde{y}) + f_5$ and it follows that $\tilde{\chi}^\top \tilde{\Psi} \tilde{\chi} = s_1(\tilde{T}, \tilde{y})$, $\tilde{\chi}^\top \tilde{I}^\top \tilde{M} \tilde{I} \tilde{\chi} = s_3(\tilde{M}, \tilde{y})$ and $\tilde{\chi}^\top \tilde{R} \tilde{\Xi} \tilde{\chi} = s_2(\tilde{R}, \tilde{\chi}, \tilde{y})$. Thus by left and right multiplying (23b) respectively by $\tilde{\chi}^\top$ and $\tilde{\chi}$, it follows that (21) holds, which concludes the proof. \blacksquare

Remark 3 Differently from results in the literature considering the description of the polyhedral partitions by hyperplanes (e.g. [14, 7, 8]) or by vertices and cone rays (e.g [13, 12]), in our approach only two LMIs have to be tested to assess the stability of the PWA system. Note that in the aforementioned works, an LMI is associated to each possible state transition from a region Γ_j to a region Γ_i . These inequalities are required to enforce the strictly decrease of the LF. Moreover, for each region Γ_j , an LMI constraint is needed to ensure the positivity of the piecewise quadratic Lyapunov function. Thanks to the proposed representation, all possible transitions are implicitly taken into account and no enumeration is needed. \lrcorner

Remark 4 The LMIs in Corollary 1 correspond to the expressions of the V and its one-step variation ΔV with the additional terms from Lemma 1 to Lemma 3 as in (20) and (21). Note that, as detailed in Section 3 and discussed in Remark 2, the ramp function is specified when using the relations $r(\theta)r(-\theta) = 0$ from Lemma 1, $\theta - (r(\theta) - r(-\theta)) = 0$ from Lemma 2, and $r(\theta) \geq 0$, $r(-\theta) \geq 0$ from Lemma 3. These Lemmas introduce respectively matrices (T , R and M) and (\tilde{T} , \tilde{R} and \tilde{M}) in the LMIs in Corollary 1. Except for very particular cases, all these matrices need to be free variables whenever solving (23). This way, we are restricting the analysis of generalized quadratic forms described only by the vector valued ramp functions ϕ , and not another nonlinear function that satisfies a subset of the relations in Section 3. Moreover, we stress that matrices T and \tilde{T} , related to the complementarity condition detailed in Lemma 1 are sign-free variables. This is in contrast to the sector condition where a set of nonlinearities is described using the same expression as (5) however with an inequality and a positive semi-definite matrix T (without the terms from Lemmas 2 and 3). Whether the sign of the terms in these T and \tilde{T} matrices from the solutions to (23) is positive or negative will depend on the problem data (mainly on the matrix F_4).

Similar LMIs, without matrix R as in Corollary 1, can be obtained if the algebraic relation $\theta - (r(\theta) - r(-\theta)) = 0$ in Lemma 2 is used in both Lemma 1, and Lemma 3. Namely by using $r(-\theta) = \theta - r(\theta)$ thus allowing to replace χ by vector $\left[1 \ x^\top \ \phi(y)^\top \right]^\top$. On the other hand, the formulation of Corollary 1 avoids products between matrices T and M and the matrices F_3, F_4 and f_5 defining y . This feature will be instrumental to study polytopic uncertain system with parameter dependent multipliers T and M . \lrcorner

5 Numerical Examples

In this section, we illustrate the results of Theorem 1 with four numerical examples. In the first, we demonstrate the global stability (of the origin) of a generic piecewise linear system. In the second one, we analyze the global stability of a linear system subject to actuator saturation. A third example treats a benchmark example borrowed from the explicit MPC literature. In all examples we have used the values for parameters $\eta = 10^{-3}$ and $\epsilon_1 = 10^{-5}$.

Example 1. Consider a piecewise linear system given by (2) with $f(x)$ described by the implicit representation (3) with

$$F_1 = \begin{bmatrix} 0.5 & 0.1 + \kappa \\ -1 & 0.5 \end{bmatrix}, \quad F_2 = \kappa \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & -\frac{2}{3} \\ -1 & 0 \end{bmatrix}, \quad f_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The corresponding partition of \mathbb{R}^2 defined from equation (3b) is given by the following sets

$$\begin{aligned} \Gamma_1 &= \{x \in \mathbb{R}^2 | \phi_1(y(x)) = 0, \phi_2(y(x)) = 0\} \\ &= \{x \in \mathbb{R}^2 | -x_1 \leq x_2; x_1 \leq x_2\}, \\ \Gamma_2 &= \{x \in \mathbb{R}^2 | \phi_1(y(x)) \geq 0, \phi_2(y(x)) = 0\} \\ &= \{x \in \mathbb{R}^2 | x_1 \leq 0; x_2 \leq -x_1\}, \\ \Gamma_3 &= \{x \in \mathbb{R}^2 | \phi_1(y(x)) = 0, \phi_2(y(x)) \geq 0\} \\ &= \{x \in \mathbb{R}^2 | x_2 \leq x_1; x_2 \geq -5x_1\}, \\ \Gamma_4 &= \{x \in \mathbb{R}^2 | \phi_1(y(x)) \geq 0, \phi_2(y(x)) \geq 0\} \\ &= \{x \in \mathbb{R}^2 | 0 \leq x_1; x_2 \leq -5x_1\}. \end{aligned}$$

and is depicted in Figure 2. The sets in the partition do not satisfy $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. However, f is uniquely defined since the functions are continuous and coincide on the boundary of the sets.

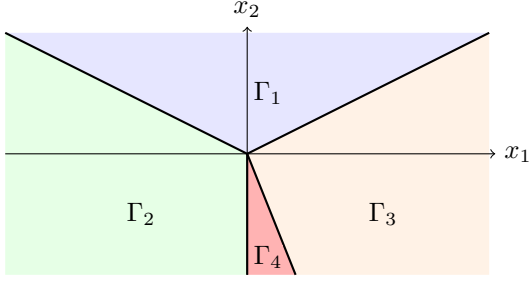


Fig. 2. Example I - partition of \mathbb{R}^2

Note that a corresponding explicit representation for $f(x)$ as in (1) can be obtained as follows:

$$x^+ = A_j x = (F_1 + F_2 \Delta_j (I - F_4 \Delta_j)^{-1} F_3) x, \quad \text{if } x \in \Gamma_j,$$

for $j = 1, \dots, 4$ with $\Delta_1 = \text{diag}((0, 0))$, $\Delta_2 = \text{diag}((1, 0))$, $\Delta_3 = \text{diag}((0, 1))$, $\Delta_4 = \text{diag}((1, 1))$.

Applying the conditions of Theorem 1 through the LMI formulations in Corollary 1, we can show that the origin of the system is globally stable for $\kappa = 0.699$, and (17) is a Lyapunov function for the system with

$$P = \begin{bmatrix} 2.2172 & -0.0151 & -0.4494 & 0.0094 \\ -0.0151 & 1.6462 & 0.0094 & 0.3570 \\ -0.4494 & 0.0094 & -1.2060 & -0.8242 \\ 0.0094 & 0.3570 & -0.8242 & -0.4758 \end{bmatrix}.$$

Note that the matrix P is not positive definite. Indeed the positive definiteness of matrix P is not imposed by the conditions in Theorem 1. However, since (20) holds we have that the Lyapunov function is guaranteed to be positive definite. Some trajectories of the system are shown in Figure 3, along with the level sets of the decreasing Lyapunov function. For comparison, the dual problem presented in [7, Section II] demonstrate that there does not exist a quadratic Lyapunov function, that is $V(x) = x^\top P_1 x$, with $P_1 \in \mathbb{R}^{n \times n}$, that certifies the stability for $\kappa \geq 0.357$, and through simulation, we find that the origin of the system is stable for $-0.35 < \kappa < 0.7$. It should also be pointed out that with the method proposed in [7], using a piecewise quadratic Lyapunov function, it is not possible to certify the stability of the system for $\kappa \geq 0.51$ (considering known all the admissible transitions between regions), which shows that our conditions lead to less conservative results.

As pointed out in Remark 4, testing the LMIs (23) for this example without all variables, for instance by imposing $R = 0$ and $\tilde{R} = 0$ resulted in no feasible solution for any value of κ . This highlights the importance of the term introduced by s_2 in Lemma 2.

Example II. Consider the following system taken from [6], discretized with a sampling period of 100ms,

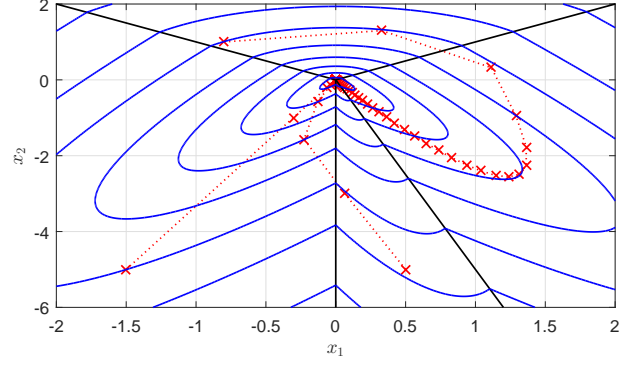


Fig. 3. System trajectories and Lyapunov function level sets for Example I.

and subject to asymmetric actuator saturation

$$x^+ = Ax + B \text{sat}_{[-1, 15]}(Kx) \quad (24)$$

with $A = \begin{bmatrix} 0.9464 & 0.0957 \\ -0.9568 & 0.9033 \end{bmatrix}$, $B = \begin{bmatrix} 0.0049 \\ 0.0959 \end{bmatrix}$ and $K = \begin{bmatrix} 9.9000 & 0.4950 \end{bmatrix}$.

First, note that the saturation function is a piecewise affine function defined as follows:

$$\text{sat}_{[-1, 15]}(Kx) = \begin{cases} -1 & \text{if } Kx \leq -1 \\ Kx & \text{if } -1 \leq Kx \leq 15 \\ 15 & \text{if } Kx \geq 15. \end{cases} \quad (25)$$

From (25), it follows that (24) can be cast as a PWA system (2) with an implicit representation (3) defined with:

$$F_1 = A + BK, \quad F_2 = \begin{bmatrix} -B & B \end{bmatrix}$$

$$F_3 = \begin{bmatrix} K \\ -K \end{bmatrix}, \quad F_4 = \mathbf{0}, \quad f_5 = \begin{bmatrix} -15 \\ -1 \end{bmatrix}$$

leading to the following partition of \mathbb{R}^2 in terms of ϕ

$$\begin{aligned} \Gamma_1 &= \{x \in \mathbb{R}^2 \mid \phi_2(y(x)) \geq 0\} = \{x \in \mathbb{R}^2 \mid Kx \leq -1\}, \\ \Gamma_2 &= \{x \in \mathbb{R}^2 \mid \phi_1(y(x)) = \phi_2(y(x)) = 0\} \\ &= \{x \in \mathbb{R}^2 \mid -1 \leq Kx \leq 15\}, \\ \Gamma_3 &= \{x \in \mathbb{R}^2 \mid \phi_1(y(x)) \geq 0\} = \{x \in \mathbb{R}^2 \mid 15 \leq Kx\} \end{aligned}$$

which is depicted in Figure 4.

It can be shown (see [7, Section II]) that there does not exist a common quadratic Lyapunov function for the

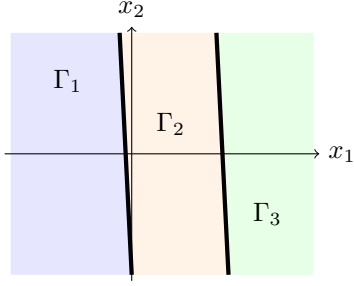


Fig. 4. Example II - partition of \mathbb{R}^2 .

linear systems defined by A and $(A + BK)$. Since the quadratic global stability of a linear system subject to a saturating linear state feedback imposes the existence of a common Lyapunov function for the open-loop system and the closed-loop system without saturation, we conclude that there is no quadratic function to assess the global stability of the origin of system [4]. However, considering a piecewise quadratic Lyapunov function as in (17) and applying Theorem 1, we can certify that the origin is globally exponentially stable with

$$P = \begin{bmatrix} 0.1372 & 0.1684 & -0.0030 & -0.0241 \\ 0.1684 & 1.0349 & -0.0241 & 0.0668 \\ -0.0030 & -0.0241 & 0.1042 & -0.0073 \\ -0.0241 & 0.0668 & -0.0073 & 0.0934 \end{bmatrix}.$$

This matrix was obtained from the solution to the LMIs described in Corollary 1.

In Figure 5, a trajectory of the system and the level sets of the decreasing Lyapunov function are depicted.

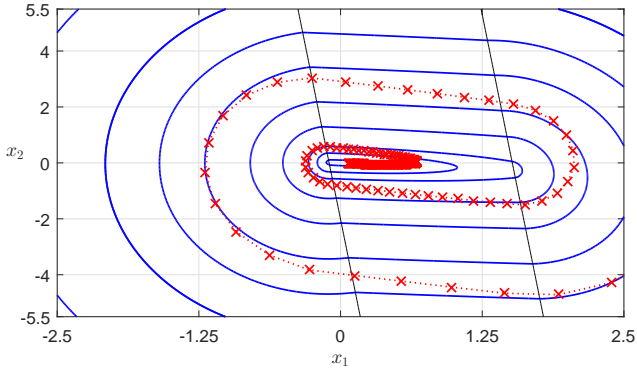


Fig. 5. System trajectory and Lyapunov function level sets for Example II.

Example III. Consider the following closed-loop system

$$x^+ = Ax + Bu,$$

$$A = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix},$$

with u given by the explicit MPC law computed in [2] leading to the explicit PWA representation in Table 1.

Region i	Control u_i
$\begin{bmatrix} -5.9220 & -6.8883 \\ 5.9229 & 6.8883 \\ -1.5379 & 6.8296 \\ 1.5379 & -6.8296 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -5.9220 & -6.8883 \end{bmatrix} x$
$\begin{bmatrix} -6.4159 & -4.6953 \\ -0.0275 & 0.1220 \\ 6.4159 & 4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x + 0.6423$
$\begin{bmatrix} 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x - 0.6423$
$\begin{bmatrix} -3.4155 & 4.6452 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	2
$\begin{bmatrix} 0.0679 & -0.0924 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	2
$\begin{bmatrix} -0.0679 & 0.0924 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	-2
$\begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	-2

Table 1
Explicit MPC law: inequalities defining the sets of the partition and the corresponding affine control law.

The closed-loop system can therefore be described by a PWA system (2), with $f(x)$ in (3) given by the following matrices:

$$F_1 = A + BK_1, \quad F_2 = B \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \phi(y)$$

$$F_3 = \begin{bmatrix} K_2 - K_1 \\ K_1 - K_2 \\ -K_1 \\ K_1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix},$$

$$f_5^T = \begin{bmatrix} -0.6423 & -0.6423 & -2 & -2 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} -5.9220 & -6.8883 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix}.$$

By applying Theorem 1, we can find a quadratic Lyapunov function that certifies the global stability of the origin. A quadratic function is a particular case of the generic form (17), in which we consider $P_2 = 0$ and

$P_3 = 0$, in this case $V(x) = x^\top P_1 x$ with

$$P_1 = \begin{bmatrix} 0.9262 & 0.4674 \\ 0.4674 & 1.0815 \end{bmatrix}.$$

A trajectory and the level sets of the obtained Lyapunov function are shown in Figure 6.

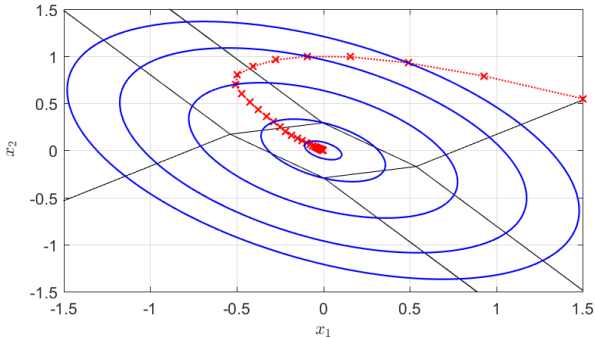


Fig. 6. System trajectory and Lyapunov function level sets for Example III.

As observed in Remark 3, since in this case there are 7 sets in the partition, if we consider the results in references [7, 8], which are based on the explicit representation (1), we have to test $7^2 = 49$ LMIs, regarding all the possible transitions between the sets of the partition, plus 7 LMIs to ensure the positivity of the PWQ Lyapunov function on each set. This number of LMIs can be reduced if we perform a reachability analysis, that requires the solution of 49 linear programming feasibility problems. In contrast, our conditions requires only the solution of two LMIs. It can also be shown that the number of variables is smaller with our approach.

6 Conclusion and future work

We have presented a new framework for the stability analysis of discrete-time PWA systems. To this end we introduced a novel implicit representation of PWA functions based on the use of ramp functions. By exploiting some properties of ramp functions as a set of identities and inequalities, we obtain Lyapunov inequalities related to piecewise quadratic Lyapunov functions candidates. These inequalities are then expressed as LMIs.

In the proposed framework: *a)* there is no need of defining the quadratic function associated to each set of the partition since this is implicitly obtained with a generalized quadratic form; *b)* there is no need to enumerate possible transitions between regions; *c)* the conditions for stability are cast in two LMIs and can be efficiently tested with standard optimization packages; *d)* the use of properties associated to ramp functions applies only

this class of function and therefore are less conservative than generic sector bounded conditions; *e)* from the novel proposed representation, the matrices describing the partition of the system appear affinely on the stability LMI conditions, which allows to directly consider uncertainties in the partition.

Future work includes the extension of the framework to study local (regional) stability, including efficient ways of estimating the region of attraction of the origin of PWA systems, and the synthesis of stabilizing feedback control laws also in PWA form. The proposition of stability conditions for continuous-time system is also being studied.

A Conditions for well-posedness

Note that in Example II above, the solution to equation (3b) is explicit since $F_4 = 0$, giving $y = F_3 x + f_5$. It is then straightforward to compute $f(x)$ using the value of y . Explicit solutions can also be obtained in case matrix F_4 is structured, for instance for a strictly lower triangular structure as in Example III.

In general, with $F_4 \neq 0$, as in Example I above, (3b) becomes an implicit equation and the existence of a unique solution y for all $x \in \mathbb{R}^n$ must be ensured. With this aim, below we provide a condition for the well-posedness of (3b), that is, the existence and uniqueness of solutions to

$$\tilde{f}(y(x)) = F_3 x + f_5 \quad \forall x \in \mathbb{R}^n, \quad (\text{A.1})$$

with $\tilde{f}(y(x)) = y(x) - F_4 \phi(y(x))$. In [24, Proposition 2] it is shown that for a locally Lipschitz function $\tilde{f}(y)$ such that the Jacobian satisfies $J_y \tilde{f}(y) \in \mathcal{M} \subset \mathbb{R}^{n_y \times n_y}$ for almost all $y \in \mathbb{R}^{n_y}$, where \mathcal{M} is a compact, convex set, with each of its elements being non-singular, there exists a unique globally Lipschitz function $y(\xi)$ satisfying $\tilde{f}(y(\xi)) = \xi$. Such a result is used in [24] to obtain a condition for the well-posedness of an algebraic loop involving saturation and deadzone functions.

Using the definition of the ramp function in (4), we have that the Jacobian $\tilde{f}(y)$ in (A.1) with respect to y of is given by $J_y \tilde{f}(y) = (I - F_4 \Delta)$ with $\Delta \in \mathcal{D} = \{\Delta \in \mathbb{D}^{n_y} \mid \Delta_{(i,i)} \in [0, 1]\}$, which is a compact and convex set of matrices. Thus, following [24, Proposition 2] a unique solution to (A.1) exists if $(I - F_4 \Delta)$ is non-singular for all $\Delta \in \mathcal{D}$. A condition for the well-posedness is an LMI (see [23, 24]) as in the proposition below.

Proposition 2 ([24, Proposition 1]) *If there exist a matrix $W \in \mathbb{D}^{n_y}$, $W > 0$ such that $-2W + W F_4 + F_4^\top W < 0$ then $(I - F_4 \Delta)$ is non-singular $\forall \Delta \in \mathcal{D}$.*

For implementation purposes, for instance when the PWA function has to be computed to generate a control

input, a well posed equation (A.1) can be solved from the solution of a Linear Complementarity Problem [5].

Note that from (16) and (4), we have

$$\begin{aligned}(\phi_i - y_i)\phi_i &= 0 \\ \phi_i &\geq 0 \\ (\phi_i - y_i) &\geq 0,\end{aligned}$$

$i = 1, \dots, n_y$. Setting now $\xi = F_3x + f_5$ in equation (3b), and using $y_i = (F_4\phi + \xi)_i$ in the above expressions one obtains respectively

$$((I - F_4)\phi - \xi)_i\phi_i = 0 \quad (\text{A.3a})$$

$$\phi_i \geq 0 \quad (\text{A.3b})$$

$$((I - F_4)\phi - \xi)_i \geq 0. \quad (\text{A.3c})$$

$i = 1, \dots, n_y$. The problem of solving on ϕ the inequalities (A.3c), (A.3b) affine in ϕ , and equations (A.3a), quadratic in ϕ , is called a mixed Linear Complementarity Problem (LCP). For a given ξ , the solution ϕ to (A.3) thus provides a solution to the implicit equation $y - F_4\phi(y) = \xi$. Please refer to the Lemke algorithm presented in [1, Section 5.1] for a strategy to solve LCPs yielding solutions to algebraic loops. Also, as one should expect, the condition for the well posedness of LCPs in [1, Proposition 7.1] applied to (A.3) holds if the condition in Proposition 2 is satisfied.

References

- [1] A. A. Adegbege and W. P. Heath. A framework for multivariable algebraic loops in linear anti-windup implementations. *Automatica*, 83:81–90, 2017.
- [2] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, 2002.
- [3] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. SIAM, Philadelphia, 1994.
- [4] C. Burgat and S. Tarbouriech. Stability and control of saturated linear systems. In A.J. Fossard and D. Normand-Cyrot D., editors, *Nonlinear Systems*. Springer, Boston, MA, USA, 1996.
- [5] R. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, Upper Saddle River, NJ, USA, 1992.
- [6] R. Drummond, G. Valmorbidia, and S. R. Duncan. Generalized absolute stability using Lyapunov functions with relaxed positivity conditions. *IEEE Control Systems Letters*, 2(2):207–212, 2018.
- [7] G. Feng. Stability analysis of piecewise discrete-time linear systems. *IEEE Transactions on Automatic Control*, 47(7):1108–1112, 2002.
- [8] G. Ferrari-Trecate, F. A. Cuzzola, D. Mignone, and M. Morari. Analysis of discrete-time piecewise affine and hybrid systems. *Automatica*, 38(12):2139–2146, 2002.
- [9] J. M. Gomes da Silva Jr. and S. Tarbouriech. Polyhedral regions of local stability for linear discrete-time systems with saturating controls. *IEEE Transactions on Automatic Control*, 44(11):2081–2085, Nov 1999.
- [10] M. Hovd and S. Olaru. Relaxing PWQ Lyapunov stability criteria for PWA systems. *Automatica*, 49(2):667–670, 2013.
- [11] T. Hu, A. R. Teel, and L. Zaccarian. Stability and Performance for Saturated Systems via Quadratic and Nonquadratic Lyapunov Functions. *IEEE Transactions on Automatic Control*, 51(11):1770–1786, nov 2006.
- [12] R. Iervolino, D. Tancredi, and F. Vasca. Lyapunov stability for piecewise affine systems via cone-copositivity. *Automatica*, (81):22–29, 2002.
- [13] R. Iervolino, F. Vasca, and L. Iannelli. Cone-copositive piecewise quadratic Lyapunov functions for conewise linear systems. *IEEE Transactions on Automatic Control*, 60(11):3077–3082, 2015.
- [14] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):555–559, 1998.
- [15] P. Julian, A. Desages, and O. Agamennoni. High-level canonical piecewise linear representation using a simplicial partition. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 46(4):463–480, 1999.
- [16] M. K. Camlibel, J. S. Pang, and J. Shen. Lyapunov stability of complementarity and extended systems. *SIAM Journal on Optimization*, 17(4):1056–1101, 2006.
- [17] C. Kahlert and L. O. Chua. The complete canonical piecewise-linear representation. I. The geometry of the domain space. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 39(3):222–236, 1992.
- [18] T. Lathuilière, G. Valmorbidia, and E. Panteley. Periodic orbits in planar linear systems with input saturation. *IEEE Control Systems Letters*, 2(3):435–440, July 2018.
- [19] J. A. Primbs and M. Giannelli. Kuhn-Tucker-based stability conditions for systems with saturation. *IEEE Transactions on Automatic Control*, 46(10):1643–1647, Oct 2001.
- [20] M. Rubagotti, L. Zaccarian, and A. Bemporad. A Lyapunov method for stability analysis of piecewise-affine systems over non-invariant domains. *International Journal of Control*, 89(5):950–959, 2016.
- [21] E. Sontag. Nonlinear regulation: the piecewise linear approach. *IEEE Transactions on Automatic Control*, 26(2):346–358, 1981.
- [22] S. Tarbouriech, G. Garcia, J. M. Gomes da Silva Jr,

and I. Queinnec. *Stability and stabilization of linear systems with saturating actuators*. Springer-Verlag, London, 2011.

[23] G. Valmorbida, R. Drummond, and S. R. Duncan. Regional analysis of slope-restricted Lurie systems. *IEEE Transactions on Automatic Control*,

64(3):1201–1208, 2019.

[24] L. Zaccarian and A. R. Teel. A common framework for anti-windup, bumpless transfer and reliable designs. *Automatica*, 38(10):1735–1744, 2002.