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M'hamed Gaïgi*, Vathana Ly Vath †, and Simone Scotti‡

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Abstract

Persistence in the literature is the perception of an inherent tradeoff between ecological conservation and economic harvesting goals. Overexploitation may lead to resource destruction, including extinction. Conservation measures should be decided and implemented. A standard ecological response is to impose no-take areas or marine reserves. In this paper, our objective is to study a harvesting management problem under the constraints of a no-take area, which we formulate mathematically as a bi-dimensional singular stochastic control problem. Using dynamic programming theory, we characterise our value function as the unique solution to a HJB equation. We also describe the optimal harvesting strategy by identifying the harvesting and non-harvesting regions. We show that setting up reserve area not only leads to a better ecological conservation but may also increase the economic benefit for fishing industry in the long run. We further enrich our studies with some numerical analysis, enabling us to get some insightful understanding on the size of the reserve area where no-take policy should be implemented.

Keywords : OR in natural resources, fisheries management, marine reserves, stochastic control, optimal harvesting.

Highlights :

- Optimal harvesting strategy is studied in a bio-economic stochastic growth model.
- The presence of a no-take area is considered to prevent extinction.

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- Value function is characterised theoretically using viscosity solution approach.
- Numerical analysis highlights the economic benefit of no-take areas.
- The optimal policy is to harvest all the net spillover from the reserve area when the fish population density exceeds a certain threshold.

1 Introduction

Fishing industry provides a highly valuable source of food and nutrition to a significant proportion of the world population. But overexploitation of living marine resources is causing a dangerous decline of fish populations. These resources are renewable, i.e. life cycles can replenish the population to overcome human consumption. However, biological reproduction is not an easy process and the question of fish resource sustainability is actually an open and primary question for biologists, industries and policy makers, see among others Scott (1955), Reed (1979), Alvarez and Shepp (1998), do Val, Guillotreau and Vallée (2019), Ni and Sandal (2019) and Kvamsdal, Maroto, Morán, and Sandal (2020). As a consequence, the economic and ecological dilemma of how to best catch fish plays a major role in fishery management.

Questions of interest include: how far can we harvest before putting a population that is currently healthy in danger of extinction? What is the optimal way to harvest fish in order to maximise yield while preserving its sustainability? Could we still harvest low-density populations? This is crucial in commercial fishing for two reasons. First, economically, we want the greatest catch to supply the demand for fish thus maximising yield. Second, ecologically, we do not want to deplete the population to keep diversity in the oceans and preserve its sustainability for future harvest. Similar questions arise for agricultural and forest management, see for instance Constantino and Martins (2018) and Alvarez-Miranda, Goycoolea, Ljubić, and Sinnl (2020).

Protected marine reserves are gaining popularity as a management option for marine conservation and fisheries. No-take marine reserves have been shown repeatedly to enhance the abundance, size, and diversity of species, see for instance Holland and Brazee (1996), Clemens, ReVelle, and Williams (1999), Sanchirico and Wilen (2001), Halpern (2003), Neubert (2003), Micheli, Halpern, Botsford, and Warner (2004), Halpern, Lester, and Kellner (2010) and Alvarez-Miranda, Goycoolea, Ljubić, and Sinnl (2020). A large literature exists about the effects of protected marine reserves which were highlighted at the 2002 World Summit on Sustainable Development. However, some

controversies remain about their effects. Different works, for instance Conrad (1999), indicate that the presence of a no-take marine reserve decreases the value function, i.e. the economic value of harvesting benefit, corresponding to the view of many fishers that reserves are unnecessary and costly. On the other side, Grafton, Kompas, and Lindenmayer (2005) highlight that the effect of protected areas becomes positive in presence of ecological uncertainty, i.e. a random evolution for fish populations.

Protected areas are predicted to benefit from non-protected areas through the main mechanism of net fish migration across borders. This effect is sometimes called spillover, see for instance Gell and Roberts (2003), Halpern (2003), Micheli, Halpern, Botsford and Warner (2004) and Halpern, Lester and Kellner (2010). Lester et al. (2009) compile a database of studies, based on a comprehensive survey of literature, in order to document biological effects of marine reserves and conclude with a statistically significance that *even small reserves can produce significant biological responses, although more data are needed to test whether reserve effects scale with reserve size.*

Motivated by studies on marine reserves, our goal is to study the problem of optimal harvesting policies maximising expected cumulative discounted harvest in presence of a no-take marine reserve under stochastic environment. The marine area is split into two parts, the no-take marine reserve where harvesting is strictly prohibited and no-poaching occurs, and a fishing area, i.e. a free-rule area where unconstrained harvesting is allowed. The evolution, without harvest, in each area is assumed to follow the most common Verhulst-Pearl logistic model, that is coherent with the previous literature, see for instance Alvarez and Shepp (1998). The spillover effect is modelled as a linear impact proportional to the unbalance between the density in the two areas, i.e. the total spillover can be split into the proper spillover from the marine reserve to the fishing area minus the proper spillover from the fishing area to the marine reserve, see Grafton, Kompas and Lindenmayer (2005).

On the mathematical point of view, our problem will be formulated as a bi-dimensional singular stochastic control problem. In terms of the existing literature, while there is a large literature on harvesting problem under no-take marine reserve constraints using deterministic approach, see for instance You and Zhao (2004), Costello and Polasky (2008) and Gonzalez-Olivares and Huincahue-Arcos (2011), few studies on optimal harvesting strategies are investigated under no-take marine reserve under stochastic uncertainty, to our best knowledge. Indeed, most stochastic harvesting problems are studied under the basic framework where no marine reserve is implemented, see for instance Alvarez and Shepp (1998), Lungu and Oksendal (2001), Yang and Liu (2004) and Alvarez and Koskella (2007). Alvarez and Shepp (1998) and Yang and Liu (2004) are both uni-dimensional

singular control problem while Alvarez and Koskella (2007) consider a harvesting problem under price uncertainty leading therefore to a bi-dimensional singular control problem. However, in Alvarez and Koskella (2007), both diffusion processes, i.e. fish population and price processes, are conveniently not coupled and driven by two independent Brownian motions. This will enable the authors to get full description of the optimal policies. Lungu and Oksendal (2001) study an optimal harvesting problem and formulate it mathematically as a bi-dimensional singular control problem. The authors obtain a very nice main result in their Theorem 2.1, but only by making some strong assumptions in order to get some explicit or quasi-explicit results. They also point out that indeed explicit solutions may be obtained but only in some degenerate cases. Grafton, Kompas and Lindenmayer (2005) is among the very few studies to investigate a bi-dimensional harvesting problem under no-take marine reserve using stochastic approach and is closely related to our problem. In their paper, the authors assume their set of harvesting strategies to be absolutely continuous leading therefore to a regular control problem with a standard associated Hamilton-Jacobi-Bellman (HJB) equation. They mainly focus on the study of the value function, i.e. the economic reward, but not on the optimal harvesting strategy, i.e. when/how to fish. As a consequence, their point of view is more economic than biological. In contrast, in our harvesting problem, which is formulated as a bi-dimensional singular control problem, our main focus is on the optimal harvesting strategy.

Most multi-dimensional non-degenerate singular control problems similar to our singular control problem are investigated through the studies of optimal dividend or investment policies, see for instance Jin, Yin, and Zhu (2012), Pierre, Villeneuve, and Warin (2016), Chevalier, Gaigi, and Ly Vath (2017), Chevalier, Ly Vath, and Roch (2020) and Zhu, Siu, and Yang (2020). It is well known that studying these above multi-dimensional non-degenerate singular control problems is a major challenge when one expects to get beyond the usual viscosity characterisation and obtain qualitative description of the optimal policies. There are, indeed, no general and easily applicable methods to completely solve this set of singular control problems as each problem is based on specific model features leading in general to non-standard HJB equation. Chevalier, Gaigi and Ly Vath (2017) tackle an optimal dividend problem under liquidity risk constraints via auxiliary functions which are defined recursively. Each auxiliary function may be characterised as a unique viscosity solution to its associated HJB equation. These recursive functions give them an implementable algorithm approximating their problem. Jin, Yin, and Zhu (2012) equally focus on numerical methods to solve their mixed regular and singular controls for regime-switching models. Chevalier, Ly Vath and Roch (2020) study an optimal dividend and capital structure problem for a firm,

which holds a certain amount of debt to which is associated a financial ratio covenant between the firm and its creditors. The authors are able to give a complete qualitative description of their problem by specifically introducing an optimal stopping time on a related reflected diffusion process. Solving this optimal stopping-time problem allows them to fully characterise the optimal issuance of capital policy. Pierre, Villeneuve, and Warin (2016) formulate their dividend and investment policies as a bi-dimensional singular control problem and solve their control problem by using a viscosity solution approach and a verification technique. Quasi-explicit solution may be obtained in special cases. Zhu, Siu, and Yang (2020) study singular dividend strategies with a stochastic quasi-hyperbolic discounting function in a linear diffusion model. They adopt a game-theoretic approach to establish economic equilibrium results.

In the study of our bi-dimensional singular control problem, our objective is twofold. First, we solve theoretically and rigorously our stochastic optimisation problem using viscosity approach, see Crandall, Ishii, and Lions (1992), Bayraktar and Young (2011), Chevalier, Ly Vath, Roch, and Scotti (2015), Cosso, Marazzina and Sgarra (2015), Pierre, Villeneuve, and Warin (2016), and more recently do Val, Guillotreau and Vallée (2019) and Oliveira and Perkowski (2020). Our contribution to the literature on the multi-dimensional singular control is our ability to obtain non-trivial results on the value function, in particular its concavity and more importantly on the optimal strategy, in particular the study around an optimal threshold $\bar{x}_y$ which in turn defines the optimal strategy. We may indeed characterise the optimal policy by showing that for any given level of population y in the no-take marine reserve, there exists a population threshold $\bar{x}_y$ in the fishing area above which it is optimal to harvest. This is clearly beyond the usual viscosity characterisation of the value function in most studies on multi-dimensional singular control problems.

Second, we solve our problem numerically, in order to characterise the optimal policy in an explicit framework. While Grafton, Kompas and Lindenmayer (2005) study the economic rewards of the problem and the optimal size of the protected area, we are focusing on the behaviour of the optimal level of fish population in the fishing area as function of the population density in the reserve area, i.e. the optimal harvesting strategy. Our findings are new in the literature and insightful as they clearly show that marine reserve policy may increase both ecological and economics interests. As such, they clearly contribute to a better understanding on optimal harvesting strategy under marine reserve constraints. Indeed, the obtained results are completely counter-intuitive as the optimal policy implies that when there is no extinction or overpopulation risk, i.e. when the fish population in the reserve area has some intermediate values, the less we have fish in the reserve

area, the more we have to harvest in the authorised area. This optimal strategy, which consists in harvesting all spillover from the protected area, clearly indicates that there is no longer any conflictual relation between ecological and economic interests. The reserve area plays the role of nursery habitat. Furthermore, comparative statics confirm these phenomena which are magnified when the transfer rate between the two regions increases, i.e. when the no-take area is highly connected with the harvesting area. Further computations on the value function show that the reserve not only has an evident ecological aim but also an economic surplus.

The paper is organised as follows. Section 2 presents the bio-economic bi-dimensional model and the singular control problem. Section 3 deals with the dynamic programming principle and the first properties of the value function. We show in particular that the value function is concave. Section 4 provides the characterisation of the value function as the unique viscosity solution to the associated HJB equation and defines the associated bang-bang optimal strategy. Section 5 discusses the numerical method and results, in particular on the optimal threshold beyond which, it is optimal to harvest. Three patterns are identified and the impact on fish population and economic revenue is discussed. Our analysis is enriched by detailed comparative statics.

2 The biological-economic model

In our study, we consider the case where a reserve policy is implemented in order to ensure that the fish stock will never be depleted due to overfishing. The reserve policy is simple: choose a proportion p, with $p \in [0,1]$, of the marine area and forbid any harvesting in that area. The fish population in the two areas, Y_t in the marine reserve and X_t in the fishing area where harvesting is allowed, are assumed to follow a coupled Stochastic Differential Equation (SDE) system. The reserve and fishing areas are assumed to hold respectively a population carrying capacity pK and $(1-p)K$. The biomass dynamics is assumed to follow a logistic growth, see for instance Alvarez and Shepp (1998), for both sub-populations. We consider a spillover function linking both non-reserve and reserve populations and a spillover parameter θ which determines the connectivity strength.

In order to formalise the problem, let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$. Let $W^{(1)}$ and $W^{(2)}$ be two correlated $\mathbb{F}$–Brownian motions, with correlation ρ, i.e. $d[W^{(1)}, W^{(2)}]_t = \rho dt$ for all t.

The interaction between both populations is due to the flow of fish transferring from one area to

another. We assume that the overall spillover is as follows, see Grafton, Kompas and Lindenmayer (2005):

- the flow from the reserve area to the fishing area is $\theta(1-p)\left(\frac{Y_t}{pK} - \frac{X_t}{(1-p)K}\right)dt$
- the flow from the fishing area to the reserve area is the opposite, i.e. $-\theta(1-p)\left(\frac{Y_t}{pK} - \frac{X_t}{(1-p)K}\right)dt$

with θ, the transfer rate of fish population from one area to another, assumed to be known. We now consider a harvesting strategy, H, a $\mathbb{F}$-adapted càd-làg non-decreasing process, which represents the total cumulative harvested fish, with $H_{0^-} = 0$. We recall that harvesting can be carried out only in the fishing area. As stated earlier, there is no constraint to the harvesting strategy in the fishing area, except that it is not possible to harvest more than the quantity of fish available in the fishing area, i.e. $H_t - H_{t^-} \leq X_t$. This last condition defines the set of admissible harvesting strategies, denoted by $\mathcal{A}$.

We now turn to the processes X and Y, associated to the control process H, which are assumed to be governed by the following SDE system:

$$\begin{cases} dX_t &= \left[rX_t\left(1 - \dfrac{X_t}{(1-p)K}\right) + \theta(1-p)\left(\dfrac{Y_t}{pK} - \dfrac{X_t}{(1-p)K}\right)\right]dt + \sigma X_t dW_t^{(1)} - dH_t \\ dY_t &= \left[rY_t\left(1 - \dfrac{Y_t}{pK}\right) - \theta(1-p)\left(\dfrac{Y_t}{pK} - \dfrac{X_t}{(1-p)K}\right)\right]dt + \sigma Y_t dW_t^{(2)}, \end{cases} \tag{2.1}$$

where r and σ are both constant and respectively represent the rate of mean-reverting and the volatility of both population processes.

Our objective is to optimise the expected profit that can be extracted from harvesting over an infinite time horizon:

$$J(x,y;H) = \mathbb{E}\left[\int_0^\infty e^{-\beta s}\, dH_s\right]$$

over all admissible harvesting strategies $H \in \mathcal{A}$, where $\beta > 0$ is the constant discount factor. Accordingly, the value function is defined as:

$$v(x,y) = \sup_{H \in \mathcal{A}} J(x,y;H) \tag{2.2}$$

3 Properties of the value function

This section deals with the characterisation and the first properties satisfied by the value function. In particular, we show that the value function is concave, see Proposition 3.2. This result is not

standard in a bi-dimensional setting and requires an ad hoc proof. The main issue is related to the absence of control in the variable y, i.e. the evolution of fish population in the marine reserve.

We also show that the value function is continuous, in particular on the domain boundary, see Proposition 3.2 point (iii). The continuity of a bi-dimensional function is well known to be an important issue compared to the one-dimensional case.

Finally, we give an explicit affine majorant of the value function, see Corollary 3.1 and the explicit value function if the discount factor β is bigger than the reproduction rate r, see Proposition 3.3.

We now introduce some notations. We denote by (U, V), the fish populations in respectively the fishing and the reserve areas in the absence of any harvesting activity, i.e. the solution to

$$\left\{\begin{array}{rcl} dU_t &=& \left[rU_t\left(1-\dfrac{U_t}{(1-p)K}\right)+\theta(1-p)\left(\dfrac{V_t}{pK}-\dfrac{U_t}{(1-p)K}\right)\right]dt+\sigma U_t dW_t^{(1)} \\ \\ dV_t &=& \left[rV_t\left(1-\dfrac{V_t}{pK}\right)-\theta(1-p)\left(\dfrac{V_t}{pK}-\dfrac{U_t}{(1-p)K}\right)\right]dt+\sigma V_t dW_t^{(2)}, \end{array}\right. \tag{3.1}$$

The associated second order differential operator is denoted $\mathcal{L}$:

$$\begin{array}{rcl} \mathcal{L}\phi &=& \left[rx\left(1-\dfrac{x}{(1-p)K}\right)+\theta(1-p)\left(\dfrac{y}{pK}-\dfrac{x}{(1-p)K}\right)\right]\phi_x+\dfrac{1}{2}\sigma^2x^2\,\phi_{xx} \\ && +\left[ry\left(1-\dfrac{y}{pK}\right)-\theta(1-p)\left(\dfrac{y}{pK}-\dfrac{x}{(1-p)K}\right)\right]\phi_y+\dfrac{1}{2}\sigma^2\,y^2\phi_{yy}+\rho\sigma^2xy\,\phi_{xy} \end{array}$$

To solve our optimisation problem, we shall assume that the following dynamic programming principle holds: For all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, and all stopping times τ, we have

$$v(x,y)=\sup_{H\in\mathcal{A}}\mathbb{E}\left[\int_0^\tau e^{-\beta s}dH_s+e^{-\beta\tau}v(X_\tau,Y_\tau)\right] \tag{3.2}$$

From the above dynamic programming principle, we may derive the HJB equation associated with the singular control problem (2.2):

$$\min\{\beta v-\mathcal{L}v,\ v_x-1\}=0,\quad \text{on } \mathbb{R}_+\times\mathbb{R}_+. \tag{3.3}$$

This divides the state-space $\mathbb{R}_+ \times \mathbb{R}_+$ into a *harvesting region*

$$\mathcal{H} \ = \ \{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+ : v_x(x,y)=1\}, \tag{3.4}$$

and a *no harvesting region* or a continuation region

$$\mathcal{NH} \ = \ \{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+ : v_x(x,y)>1\}. \tag{3.5}$$

Our objective is to characterise the value function as the unique viscosity solution to the HJB equation (3.3), then to use this characterisation to obtain either qualitatively or numerically the two sets of *harvesting region* $\mathcal{H}$ and *no harvesting region* $\mathcal{NH}$, see Sections 4 and 5.

Remark 3.1 The harvesting region $\mathcal{H}$ is the subset of $\mathbb{R}_+ \times \mathbb{R}_+$ where it is optimal to harvest. In other words, as long as (x, y), where x and y are respectively the fish population in the fishing area and the reserve area, belongs to $\mathcal{H}$, it is optimal to harvest. The other region $\mathcal{NH}$ is the subset of $\mathbb{R}_+ \times \mathbb{R}_+$ where it is not optimal to harvest. In other words, the optimal strategy is to stop harvesting and let the fish population evolve and grow until it reaches the harvesting region, in order to carry fishing activities again. As such, the knowledge of both $\mathcal{H}$ and $\mathcal{NH}$ will give us the optimal harvesting strategy.

We now state a standard comparison theorem, which says that any smooth function, which is a supersolution to the HJB equation (3.3), is a majorant of v.

Proposition 3.1 *Let φ be a nonnegative C^2 function, supersolution on $\mathbb{R}_+ \times \mathbb{R}_+$ i.e.*

$$\min\{\beta\varphi(x,y) - \mathcal{L}\varphi(x,y)\ ;\ \varphi_x(x,y) - 1\} \geq 0, \quad \forall (x,y)\ \in\ \mathbb{R}_+ \times \mathbb{R}_+, \tag{3.6}$$

Then $\quad v(x,y) \leq \varphi(x,y) \quad \forall (x,y)\ \in\ \mathbb{R}_+ \times \mathbb{R}_+.$

Proof. Let $\varphi\ \in\ \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$ satisfying (3.6). Set $\tau_n = \min\{\inf\{t \geq 0 : \max\{X_t, Y_t\} \geq n\}, n\}$.

Applying the Itô's formula for càd-làg semimartingales between finite stopping times 0 and τ_n, we obtain

$$\begin{aligned} e^{-\beta\tau_n}\varphi(X_{\tau_n}, Y_{\tau_n}) &= \varphi(x,y) + \int_0^{\tau_n} e^{-\beta s}(\mathcal{L}\varphi - \beta\varphi)(X_s, Y_s)ds \\ &\quad + \int_0^{\tau_n} e^{-\beta s}\sigma\left[X_s\varphi_x(X_s, Y_s)dW_s^{(1)} + Y_s\varphi_y(X_s, Y_s)dW_s^{(2)}\right] \\ &\quad - \int_0^{\tau_n} e^{-\beta s}\varphi_x(X_s, Y_s)dH_s^c + \sum_{0\leq s\leq\tau_n}\Big\{\varphi(X_s, Y_s) - \varphi(X_{s^-}, Y_s)\Big\}e^{-\beta s}, \end{aligned}$$

where H^c is the continuous part of the process H. Taking the expectation and noting that the integrand in the stochastic integral is bounded on $[0, \tau_n]$, we get

$$\begin{aligned} \mathbb{E}\left[e^{-\beta\tau_n}\varphi(X_{\tau_n}, Y_{\tau_n})\right] &= \varphi(x,y)\ +\ \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s}(\mathcal{L}\varphi - \beta\varphi)(X_s, Y_s)ds\right] \\ &\quad -\mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s}\varphi_x(X_s, Y_s)dH_s^c\right] \\ &\quad +\mathbb{E}\left[\sum_{0\leq s\leq\tau_n} e^{-\beta s}\Big\{\varphi(X_s, Y_s) - \varphi(X_{s^-}, Y_s)\Big\}\right]. \end{aligned} \tag{3.7}$$

Since φ is continuously differentiable in the first variable on $\mathbb{R}_+$, by using (3.6), we obtain

$$\varphi(X_{s^-}, Y_s) - \varphi(X_s, Y_s) = \int_{X_s}^{X_{s^-}} \varphi_x(u, Y_s)du \geq X_{s^-} - X_s = (H_s - H_{s^-}) = \Delta H_s \,. \tag{3.8}$$

Combining (3.6) and (3.7), we obtain

$$\begin{aligned} \mathbb{E}\left[e^{-\beta\tau_n}\varphi(X_{\tau_n}, Y_{\tau_n})\right] &\leq \varphi(x,y) - \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s} dH_s^c\right] - \sum_{0\leq s\leq \tau_n} \Delta H_s e^{-\beta s} \\ &\leq \varphi(x,y) - \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s} dH_s\right], \end{aligned}$$

which gives

$$\varphi(x,y) \geq \mathbb{E}\left[e^{-\beta\tau_n}\varphi(X_{\tau_n}, Y_{\tau_n})\right] + \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s} dH_s\right] \geq \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s} dH_s\right].$$

By letting n tend to infinity, we obtain that for any admissible harvesting strategy $\{H_t\}_{t\geq 0} \in \mathcal{A}$, and for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$\varphi(x,y) \geq \mathbb{E}\left[\int_0^{\infty} e^{-\beta s} dH_s\right].$$

As such $\varphi(x,y) \geq v(x,y)$, which ends the proof. □

The following result gives us an upper bound of the value function and is a direct result from Proposition 3.1.

Corollary 3.1 *The linear function*

$$\varphi_{_M}(x,y) = x + y + \frac{(\beta - r)^2 K}{4r\beta} \tag{3.9}$$

is a majorant of the value function.

Proof. We apply the previous Proposition 3.1. We have already $\dfrac{\partial \varphi_{_M}}{\partial x}(x,y) = 1$, we need to verify that $\left[\beta\varphi_{_M} - \mathcal{L}\varphi_{_M}\right](x,y) \geq 0$. A straightforward computation gives us

$$\begin{aligned} \left[\beta\varphi_{_M} - \mathcal{L}\varphi_{_M}\right](x,y) &= \beta\left[x + y + \frac{(\beta-r)^2K}{4r\beta}\right] - \left[rx\left(1 - \frac{x}{(1-p)K}\right) + ry\left(1 - \frac{y}{pK}\right)\right] \\ &= \frac{r}{(1-p)K}\left(x + \frac{(\beta-r)(1-p)K}{2r}\right)^2 + \frac{r}{pK}\left(y + \frac{(\beta-r)pK}{2r}\right)^2 \geq 0. \end{aligned}$$

□

Corollary 3.2 *In the case when $\beta \geq r$, we have $v(x,y) \leq x + y$.*

Proof. This result may be obtained using the same argument as in the Proof of the previous Corollary 3.1

We now establish a few properties of the value function v.

Lemma 3.1 *For all $x_1, x_2, y \geq 0$, such that $x_1 \leq x_2$, we have*

$$v(x_2, y) \geq x_2 - x_1 + v(x_1, y).$$

Proof. Using the dynamic programming principle (3.2) and choosing a harvesting strategy which consists in harvesting the quantity $x_2 - x_1$ at time 0, we obtain $v(x_2, y) \geq x_2 - x_1 + v(x_1, y)$, which ends the proof. □

Proposition 3.2 *The value function v*
(i.) is increasing function in x and y.
(ii.) is concave on $\mathbb{R}_+ \times \mathbb{R}_+$ and then locally Lipschitz on $\mathbb{R}_{++} \times \mathbb{R}_{++}$.
(ii.) is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$, in particular on the boundary $\{0\} \times [0, \infty)$ and $[0, \infty) \times \{0\}$.

Proof. (i.) We first prove the strict monotony of the value function.
- the monotonicity in x is a direct consequence of Lemma 3.1.
- Let $x \in \mathbb{R}_+$ and $y_a < y_b \in \mathbb{R}_+$. We study the processes $\left(X_t^{(x,y_a)}, Y_t^{(x,y_a)}\right)$ and $\left(X_t^{(x,y_b)}, Y_t^{(x,y_b)}\right)$ both governed by the same SDE (2.1) but with different initial values, i.e. $\left(X_{0^-}^{(x,y_a)}, Y_{0^-}^{(x,y_a)}\right) = (x, y_a)$ and $\left(X_{0^-}^{(x,y_b)}, Y_{0^-}^{(x,y_b)}\right) = (x, y_b)$.

Our objective is to prove that $v(x, y_a) < v(x, y_b)$. For that purpose, we consider the stopping time $\tau_{ab} = \inf\{t \geq 0 : Y_t^{(x,y_a)} \geq Y_t^{(x,y_b)}\}$. We have $\tau_{ab} > 0$, and $Y_t^{(x,y_a)} < Y_t^{(x,y_b)}$, for all $t \in [0, \tau_{ab})$. We set

$$\delta = \int_0^{\tau_{ab}} e^{-\beta s} \frac{\theta(1-p)}{p} \left(Y_t^{(x,y_b)} - Y_t^{(x,y_a)}\right) dt > 0.$$

We easily obtain that $v(x, y_b) \geq v(x, y_a) + \delta$ which is associated to the harvesting strategy for the initial state variable (x, y_b), consisting in taking the same optimal or ϵ-optimal harvesting strategy when the initial state variable is (x, y_a), plus an additional harvesting rate of $\frac{\theta(1-p)}{p}(Y_t^{(x,y_b)} - Y_t^{(x,y_a)})$. This additional harvesting rate corresponds to the extra flow coming from the reserve area with initial state (x, y_b) compared to the flow with initial state (x, y_a) . That ends the proof of (i).

(ii.) We now prove that the value function is concave. Let (x_a, y_a) and $(x_b, y_b) \in \mathbb{R}_+ \times \mathbb{R}_+$. We need to prove that for any $\alpha \in [0,1]$ we have

$$\alpha v(x_a, y_a) + (1-\alpha) v(x_b, y_b) \leq v\Big(\alpha x_a + (1-\alpha)x_b, \alpha y_a + (1-\alpha) y_b\Big).$$

Consider the processes $\left(X_t^{(x_a,y_a)}, Y_t^{(x_a,y_a)}\right)$ and $\left(X_t^{(x_b,y_b)}, Y_t^{(x_b,y_b)}\right)$ both governed by the same SDE (2.1) respectively using harvesting strategies H^a and H^b and with different following initial values, i.e. $\left(X_{0^-}^{(x_a,y_a)}, Y_{0^-}^{(x_a,y_a)}\right) = (x_a, y_a)$ and $\left(X_{0^-}^{(x_b,y_b)}, Y_{0^-}^{(x_b,y_b)}\right) = (x_b, y_b)$. From the definition the value function, we obtain:

$$\alpha v(x_a, y_a) + (1-\alpha) v(x_b, y_b) = \sup_{\substack{H^a \in \mathcal{A},\ X_t^{(x_a,y_a)} \geq 0, Y_t^{(x,y_a)} \geq 0; \\ H^b \in \mathcal{A},\ X_t^{(x_b,y_b)} \geq 0, Y_t^{(x_b,y_b)} \geq 0}} \alpha J(x_a, y_a, H^a) + (1-\alpha) J(x_b, y_b, H^b)$$

Let us introduce the auxiliary process $(\widehat{X}_t, \widehat{Y}_t)$ defined as follows

$$\left\{ \begin{array}{rcl} \widehat{X}_t &=& \alpha X_t^{(x_a,y_a)} + (1-\alpha) X_t^{(x_b,y_b)} \\ \widehat{Y}_t &=& \alpha Y_t^{(x_a,y_a)} + (1-\alpha) Y_t^{(x_b,y_b)} \end{array} \right.$$

After some recombinations, we may obtain its corresponding stochastic differential equation

$$\left\{ \begin{array}{rcl} d\widehat{X}_t &=& \left[r\widehat{X}_t \left(1 - \dfrac{\widehat{X}_t}{(1-p)K}\right) + \theta(1-p)\left(\dfrac{\widehat{Y}_t}{pK} - \dfrac{\widehat{X}_t}{(1-p)K}\right)\right] dt + \sigma \widehat{X}_t dW_t^{(1)} \\ && -dH_t^{(a,b)} - \dfrac{r\ \alpha\ (1-\alpha)}{(1-p)K}\left[\left(X_t^{(x_a,y_a)} - X_t^{(x_b,y_b)}\right)^2\right] dt \\ d\widehat{Y}_t &=& \left[r\widehat{Y}_t \left(1 - \dfrac{\widehat{Y}_t}{pK}\right) - \theta(1-p)\left(\dfrac{\widehat{Y}_t}{pK} - \dfrac{\widehat{X}_t}{(1-p)K}\right)\right] dt + \sigma \widehat{Y}_t dW_t^{(2)} \\ && -\dfrac{r\ \alpha\ (1-\alpha)}{p\ K}\left[\left(Y_t^{(x_a,y_a)} - Y_t^{(x_b,y_b)}\right)^2\right] dt, \end{array} \right. \tag{3.10}$$

where $H^{(a,b)} = \alpha H^a + (1-\alpha) H^b$.

We now consider the process $(\widetilde{X}_t^\alpha, \widetilde{Y}_t^\alpha)$ governed by the controlled stochastic differential equation (2.1), with the initial condition $\left(\widetilde{X}_{0^-}^\alpha, \widetilde{Y}_{0^-}^\alpha\right) = \Big(\alpha x_a + (1-\alpha)x_b, \alpha y_a + (1-\alpha) y_b\Big)$ and the harvesting policy $\widetilde{H}^\alpha$ defined as

$$\widetilde{H}_t^\alpha = H_t^{(a,b)} + \frac{r\ \alpha\ (1-\alpha)}{(1-p)K} \int_0^t \left[\left(X_s^{(x_a,y_a)} - X_s^{(x_b,y_b)}\right)^2\right] ds\,.$$

We need to compare the process $(\widetilde{X}_t^\alpha, \widetilde{Y}_t^\alpha)$ to $(\widehat{X}_t, \widehat{Y}_t)$. Both processes have the same initial condition and almost same stochastic differential equation except the presence, in the stochastic differential equation of $\widehat{Y}$, of the non-positive drift term

$$-\frac{r\ \alpha\ (1-\alpha)}{p}\left(Y_t^{(x_a,y_a)} - Y_t^{(x_b,y_b)}\right)^2 dt\,.$$

Applying to $(\widetilde{X}_t^\alpha, \widetilde{Y}_t^\alpha)$ the following harvesting strategy $d\overline{H}_t^\alpha = d\widetilde{H}_t^\alpha + \theta\frac{1-p}{p}(\widetilde{Y}_t^\alpha - \widehat{Y}_t)dt$, we obtain that , $\widetilde{X}_t^\alpha = \widehat{X}_t$ and $\widetilde{Y}_t^\alpha \geq \widehat{Y}_t \ \forall\, t \geq 0$ and then $\overline{H}^\alpha \in \mathcal{A}$.

From the definition of the value function, we may easily conclude that

$$v\Big(\alpha x_a + (1-\alpha)x_b, \alpha y_a + (1-\alpha)y_b\Big) \geq \alpha v(x_a, y_a) + (1-\alpha)v(x_b, y_b) + \mathbb{E}\left[\int_0^\infty e^{-\beta s} L_t dt\right].$$

with $L_t = \frac{r\,\alpha\,(1-\alpha)}{(1-p)K}\left(X_t^{(x_a,y_a)} - X_t^{(x_b,y_b)}\right)^2$, which is a nonnegative process.

(iii.) The concavity of the value function on $\mathbb{R}_+ \times \mathbb{R}_+$ gives us its continuity on $\mathbb{R}_{++} \times \mathbb{R}_{++}$. We therefore turn to the continuity on its boundary $\{0\} \times [0,\infty)$ and $[0,\infty) \times \{0\}$. Let $y_0 \in \mathbb{R}_{++}$, consider the point $(0, y_0)$. Recalling that v is locally Lipschitz on $\mathbb{R}_{++} \times \mathbb{R}_{++}$. As such, for a given $\varepsilon > 0$, there exists $\eta > 0$, such that $\forall$ (x_1, y_1) and $(x_2, y_2) \in B(0, y_0, 2\eta) \cap \mathbb{R}_{++} \times \mathbb{R}_{++}$, where $B(0, y_0, 2\eta)$ is a neighbourhood of $(0, y_0)$ with radius 2η, we have

$$|v(x_1, y_1) - v(x_2, y_2)| < \varepsilon \tag{3.11}$$

Let define $\tau^{(\eta)} = \inf\left\{t > 0, \left(U_t^{(0,y_0)}, V_t^{(0,y_0)}\right) \notin B(0, y_0, \eta) \cap \{\mathbb{R}_{++} \times \mathbb{R}_{++}\}\right\}$.

We now apply the dynamic programming principle for the harvesting strategy H^0 which consists in doing nothing between the initial time 0 and $\tau^{(\eta)}$. We may obtain

$$v(0, y_0) \geq \mathbb{E}\left[e^{-\beta\,\tau^{(\eta)}} v\left(X_{\tau^{(\eta)}}^{(0,y_0),H^0}, Y_{\tau^{(\eta)}}^{(0,y_0),H^0}\right)\right] \geq \mathbb{E}\left[e^{-\beta\,\tau^{(\eta)}} v\left(U_{\tau^{(\eta)}}^{(0,y_0)}, V_{\tau^{(\eta)}}^{(0,y_0)}\right)\right].$$

Given the fact that $\left(U_{\tau^{(\eta)}}^{(0,y_0)}, V_{\tau^{(\eta)}}^{(0,y_0)}\right) \in B(0, y_0, 2\eta) \cap \mathbb{R}_{++} \times \mathbb{R}_{++}$, and using (3.11), we may get

$$v(0, y_0) \geq \mathbb{E}\left[e^{-\beta\,\tau^{(\eta)}}\left(v\left(\eta, y_0\right) - \varepsilon\right)\right] \geq \left(v\left(\eta, y_0\right) - \varepsilon\right)\mathbb{E}\left[e^{-\beta\,\tau^{(\eta)}}\right].$$

For an η sufficiently small, we may have $v(0, y_0) \geq v\left(\eta, y_0\right) - 2\varepsilon$. Since v is non decreasing in the first variable, we have $v\left(\eta, y_0\right) - 2\varepsilon \leq v(0, y_0) \leq v\left(\eta, y_0\right)$. Given the local Lipschitz property of v as given in (3.11), we obtain, $\forall (x, y) \in B(0, y_0, \eta) \cap \mathbb{R}_{++} \times \mathbb{R}_{++}$

$$v(x, y) - 3\varepsilon \leq v(0, y_0) \leq v(x, y) + \varepsilon, \tag{3.12}$$

Recalling that v is a concave function on $\{0\} \times \mathbb{R}_{++}$, we equally have the continuity of the function $v(0, .) : y \to v(0, y)$ on $(0, \infty)$. We may therefore conclude that v is continuous on $\{0\} \times (0, \infty)$. Using the same argument, we may equally obtain the continuity of v on $(0, \infty) \times \{0\}$. □

The following Proposition gives the explicit solution when the discount rate β is higher than the reproduction rate r. The optimal policy consists in harvesting all the fish stock in the fishing area.

Proposition 3.3 *Let $\beta \geq r$, then the optimal strategy consists in immediately harvesting of all the existing population in the fishing area, i.e. $v(x,y) = x + g(y)$ with*

$$g(y) = g_0(y)\left[c - \frac{2\theta(1-p)}{\sigma^2 Kp}\int_0^y \int_0^z \left[u\, g_1(u)\, g_0(u)\right]^{-1} du\, dz\right] \tag{3.13}$$

where

$$\begin{aligned} g_0(y) &= y^l e^{-y}\mathcal{J}\left(l + \frac{2r}{\sigma^2} - 2\frac{\theta(1-p)}{\sigma^2 pK};\, 2l + \frac{2r}{\sigma^2} - 2\frac{\theta(1-p)}{\sigma^2 pK};\, -\frac{2r}{\sigma^2 pK}y\right) \\ l &= \sqrt{\frac{2\beta}{\sigma^2} + \left[\frac{r}{\sigma^2} - \frac{\theta(1-p)}{\sigma^2 pK} - \frac{1}{2}\right]^2} - \frac{r}{\sigma^2} + \frac{\theta(1-p)}{\sigma^2 pK} + \frac{1}{2} \\ g_1(y) &= y^{\frac{2}{\sigma^2}\left[\frac{\theta(1-p)}{pK} - r\right]} \exp\left\{\frac{2r}{pK\sigma^2}y - 2\int_0^y \frac{g_0'(u)}{g_0(u)}du\right\} \end{aligned}$$

where $\mathcal{J}$ is the confluent hypergeometric function of first kind and the constant c which depends on the model parameters.

Proof. The proof is split into two parts, we first deal with the computation of the reward function J according with the policy consisting in the immediate harvesting of all the existing population in the fishing area. Secondly, we will show that this function is a supersolution in the sense of Proposition 3.1 and then it coincides with the value function.

We consider the strategy $\widehat{H}$ consisting in the immediate harvesting of all the existing population in the fishing area, i.e. $\widehat{H}_0 = x$ and $d\widehat{H}_t = \frac{\theta(1-p)}{pK}\widehat{V}_t dt$ where $\widehat{V}$ is the process describing the evolution of the population in the reserve area, see the second equation in (2.1), under the policy $\widehat{H}$. It is easy to check that $\widehat{X}_t = 0$, and we have:

$$d\widehat{V}_t = \left[r\widehat{V}_t\left(1 - \frac{\widehat{V}_t}{pK}\right) - \theta(1-p)\frac{\widehat{V}_t}{pK}\right]dt + \sigma\widehat{V}_t dW_t,\ \widehat{V}_{0^-} = y,$$

The reward function associated with $\widehat{H}$ reads $J(x,y;\widehat{H}) = x + g(y)$ with

$$g(y) = \mathbb{E}\int_0^\infty \frac{\theta(1-p)}{pK}e^{-\beta t}\widehat{V}_t dt. \tag{3.14}$$

Applying Feynman-Kac formula, we have that g fulfills

$$\frac{1}{2}\sigma^2 y^2 g''(y) + \left[\left(r - \frac{\theta(1-p)}{pK}\right)y - \frac{r}{pK}y^2\right]g'(y) - \beta g(y) + \frac{\theta(1-p)}{pK}y = 0\,, \tag{3.15}$$

with $g(0) = 0$. Studying the ordinary differential equation (3.15), we start with the analysis of the autonomous term. It can be reduced to a confluent hypergeometric one using the transformation $g(y) = y^l w(y)$, where l is the positive solution of $\frac{1}{2}\sigma^2 l^2 + \left(r - \frac{\theta(1-p)}{pK} - \frac{1}{2}\sigma^2\right)l - \beta = 0$. We consider

only the positive solution since $g(0) = 0$ is well defined. We now search a particular solution of the non-homogenous ordinary differential equation (3.15) using the method of constant variation that gives, adapting the argument in Appendice B in Chevalier, Ly Vath, Roch, and Scotti (2015), the expression of $g(y)$ as in (3.13). Note that in computing the expectation $g(y)$ as defined in (3.14) by solving the second-order ODE (3.15) with only one explicit Dirichlet boundary condition, i.e. $g(0) = 0$, we obtain indeed an explicit solution in (3.13), with a "free" constant c.

We will now show that $J(x, y; \widehat{H})$ is a supersolution. Following Theorem V.39 page 312 in Protter (2004), we define $D_t(\omega, y) := \frac{\partial}{\partial y}\widehat{V}(t, \omega, y)$. Remarking that the explosion time, see the statement of Theorem V.39 in Protter (2004), is infinite in our case due to the negative drift for large values of Y. We have D satisfies the SDE

$$dD_t = D_t \left\{ \left[r - \frac{\theta(1-p)}{pK} - \frac{2r}{pK}\widehat{V}_t \right] dt + \sigma dW_t \right\} \quad D_0 = 1$$

which solution is

$$D_t = \exp\left\{ rt - \frac{\theta(1-p)}{pK}t - \frac{2r}{pK}\int_0^t \widehat{V}_s ds - \frac{1}{2}\sigma^2 t + \sigma W_t \right\}$$

that is controlled by

$$\widehat{D}_t = \exp\left\{ rt - \frac{\theta(1-p)}{pK}t - \frac{1}{2}\sigma^2 t + \sigma W_t \right\} = \exp\left\{ \left[r - \frac{\theta(1-p)}{pK} \right] t \right\} \mathcal{E}(\sigma W_t) \geq D_t$$

since $\widehat{V}$ is non negative and where $\mathcal{E}$ denotes the Dolean-Dade exponential. Applying Fatou's lemma and Fubini theorem we obtain that

$$g'(y) \leq \frac{\theta(1-p)}{pK} \frac{1}{\beta - r + \frac{\theta(1-p)}{pK}}$$

under the condition $\beta > r$. We will show that $J(x, y; \widehat{H})$ is a supersolution in the sense of (3.6). We have already $\frac{\partial J}{\partial x}(x, y; \widehat{H}) = 1$. We need to verify that $[\beta J - \mathcal{L}J](x, y; \widehat{H}) \geq 0$.

$$\begin{aligned}
[\beta J - \mathcal{L}J](x, y, \widehat{H}) &= \beta x + \beta g(y) - \left[rx\left(1 - \frac{x}{(1-p)K}\right) + \theta(1-p)\left(\frac{y}{pK} - \frac{x}{(1-p)K}\right)\right] \\
&\quad - \left[ry\left(1 - \frac{y}{pK}\right) - \theta(1-p)\left(\frac{y}{pK} - \frac{x}{(1-p)K}\right)\right] g'(y) - \frac{1}{2}\sigma^2 y^2 g''(y) \\
&= (\beta - r)x + \frac{r}{(1-p)K}x^2 + \frac{\theta}{K}x\left[1 - g'(y)\right] \\
&\quad + \beta g(y) - \frac{\theta(1-p)}{pK}y - \left[ry\left(1 - \frac{y}{pK}\right) - \frac{\theta(1-p)}{pK}y\right] g'(y) - \frac{1}{2}\sigma^2 y^2 g''(y).
\end{aligned}$$

The last line is equal to zero thanks to the relation (3.15) and the first line is positive, i.e.

$$\beta - r + \frac{\theta}{K}\left[1 - \frac{\theta(1-p)}{pK(\beta - r) + \theta(1-p)}\right] > 0$$

since $\beta \geq r$. □

Remark 3.2 Recalling Corollary 3.2, when $\beta \geq r$, we have $v(x,y) = x + g(y) \leq x + y$. This corollary implies that when the discount rate β is bigger than the reproduction rate r, then the economic value, $v(x,y)$, obtained from the optimal harvesting strategy under reserve constraint is lower than the economic value, $x + y$, obtained by depleting fish stocks in both areas. In this case, it may be argued that implementing a reserve policy prevents fishers from obtaining the optimal economic value. However, one may equally argue that obtaining optimal economic value by depleting the fish stocks is clearly short-termist. Indeed, the discount rate may actually fluctuate depending on the current economic environment. Under strong uncertainty, it may be higher than the reproduction rate in the short term, but not over the long run. As such, implementing a reserve constraint prevents the fishing industry from implementing damaging short-termism.

From now, we consider that $0 \leq \beta < r$.

4 Viscosity characterisation and some properties of the optimal policy

We now turn to the PDE characterisation of the value function using viscosity approach, see Theorem 4.1 and Theorem 4.2. This PDE characterisation, which states that the value function is the unique viscosity solution to associated the HJB equation (3.3), will allow us to obtain qualitative as well as numerical results to our control problem. The other main result of this section is to show that for any positive level of the fish population, y, in the marine reserve area, there exists a threshold $\bar{x}_y$ on the fish population in the fishing area, such that it is optimal to harvest any quantity of fish population exceeding $\bar{x}_y$, see Proposition 4.1. The explicit behaviour of the threshold $\bar{x}_y$ as function of y is studied numerically in Section 5.

Theorem 4.1 *The value function v is a continuous viscosity solution of the HJB equation* (3.3)*, i.e. v satisfies*

(i) Supersolution viscosity property: for any $(\overline{x},\overline{y}) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ and any $\mathcal{C}^2$ function φ in a neighbourhood of $(\overline{x},\overline{y})$ s.t. $(\overline{x},\overline{y})$ is a local minimum of $v - \varphi$ with $(v-\varphi)(\overline{x},\overline{y}) = 0$,

$$\min\{\beta v(\overline{x},\overline{y}) - \mathcal{L}v(\overline{x},\overline{y}),\ v_x(\overline{x},\overline{y}) - 1\} \geq 0 \tag{4.1}$$

(ii) Subsolution viscosity property: for any $(\overline{x},\overline{y}) \in \mathbb{R}_{+} \times \mathbb{R}_{+}$ and any $\mathcal{C}^2$ function φ in a neighbourhood of $(\overline{x},\overline{y})$ s.t. $(\overline{x},\overline{y})$ is a local maximum of $v - \varphi$ with $(v-\varphi)(\overline{x},\overline{y}) = 0$,

$$\min\{\beta v(\overline{x},\overline{y}) - \mathcal{L}v(\overline{x},\overline{y}),\ v_x(\overline{x},\overline{y}) - 1\} \leq 0 \tag{4.2}$$

We now turn to the comparison principle for constrained viscosity solutions to the HJB equation (3.3).

Theorem 4.2 *Suppose continuous functions u and w respectively viscosity subsolution and viscosity supersolution to the HJB equation* (3.3) *on $\mathbb{R}_{++} \times \mathbb{R}_{++}$, satisfying the boundary conditions*

$$\begin{aligned} u(0,y) &\leq w(0,y), \quad y \in \mathbb{R}_+ \\ u(x,0) &\leq w(x,0), \quad x \in \mathbb{R}_+ \end{aligned}$$

and the linear growth condition:

$$|u(x,y)| + |w(x,y)| \leq C(1+x+y), \quad \forall (x,y) \in \mathbb{R}_{++} \times \mathbb{R}_{++}. \tag{4.3}$$

for some positive constants C. Then

$$u \leq w \quad \text{on } \mathbb{R}_{++} \times \mathbb{R}_{++}.$$

Proof. The proof of Theorems 4.1 and 4.2 are postponed to Appendix. □

The following Proposition states that for any positive value of the fish population, y, in the marine reserve area, there exists a threshold $\bar{x}_y$ on the fish population in the fishing area, such that it is optimal to harvest any quantity of fish population exceeding $\bar{x}_y$. In other words, given any value y of the reserved fish Y, the optimal policy is to harvest all fish exceeding the threshold $\bar{x}_y$, i.e. harvesting the quantity $X - \bar{x}_y$, if $X \geq \bar{x}_y$. When X is below $\bar{x}_y$, it is optimal to wait until it exceeds the threshold $\bar{x}_y$ to start harvesting again. This results are in-line with the one-dimensional harvesting problem when no marine reserve is considered, see for instance Alvarez and Shepp (1998).

Proposition 4.1 *For all $y \in \mathbb{R}_{++}$, there exists a unique $\bar{x}_y \geq 0$ such that for all $x \geq \bar{x}_y$, we have $(x,y) \in \mathcal{H}$.*

Proof. We start proving the existence of $\bar{x}_y$. Defining $\mathcal{A}_C = \{(x,y) \,|\, px - (1-p)y > p(1-p)KC\}$ with $C > 1$ a constant large enough, we consider the function

$$\widehat{v}(x,y) = \begin{cases} x - (1-p)KC - \frac{1-p}{p}y + v\left((1-p)KC + \frac{1-p}{p}y, y\right) & \text{if } (x,y) \in \mathcal{A}_C \\ v(x,y) & \text{otherwise}. \end{cases} \tag{4.4}$$

For simplicity, we now set the drift of the process X and Y as follows:

$$\begin{aligned}\alpha_x(x,y) &:= rx\left(1-\frac{x}{(1-p)K}\right)+\theta(1-p)\left[\frac{y}{pK}-\frac{x}{(1-p)K}\right]\\ \alpha_y(x,y) &:= ry\left(1-\frac{y}{pK}\right)-\theta(1-p)\left[\frac{y}{pK}-\frac{x}{(1-p)K}\right].\end{aligned}$$

We remark that $\widehat{v}$ is the expected profit associated to the admissible harvesting strategy where the fisher applies the optimal strategy up to $(x,y)\in\mathcal{A}_C^c$ and harvests otherwise. Thanks to Lemma 3.1, we have $\widehat{v}\leq v$. To obtain the other inequality and therefore $\widehat{v}=v$, we will prove that $\widehat{v}$ is a viscosity supersolution of HJB equation (3.3).

This result is obtained by contradiction. Suppose that the claim is not true. There exist $(x_0,y_0)\in\mathbb{R}_{++}\times\mathbb{R}_{++}$ and a function $\widehat{\phi}\in C^2(\mathbb{R}_{++}\times\mathbb{R}_{++})$ such that $(\widehat{\phi}-\widehat{v})(x_0,y_0)=0$ and $\widehat{\phi}\leq\widehat{v}$ in a neighbourhood $\overline{B}_\varepsilon(x_0,y_0)=\{(x,y),\|(x,y)-(x_0,y_0)\|\leq\varepsilon\}$ of (x_0,y_0) and such that

$$\min\{\beta\widehat{\phi}(x_0,y_0)-\mathcal{L}\widehat{\phi}(x_0,y_0),\ \widehat{\phi}_x(x_0,y_0)-1\}<0 \tag{4.5}$$

Since $v=\widehat{v}$ on $\mathcal{A}_C^c$ and v is a supersolution to HJB (3.3), we have $(x_0,y_0)\in\mathcal{A}_C$

We have easily that $\widehat{v}$ restricted to $\mathcal{A}_C$ admits a partial derivative with respect to x and that this partial derivative is equal to 1. From (4.5), we obtain $(\beta-\mathcal{L})\widehat{\phi}(x_0,y_0)<0$.

Since the value function v is non-negative, we deduce that $\widehat{v}$ is non-negative and then we have $\widehat{\phi}(x_0,y_0)=v(x_0,y_0)>0$. We have

$$\begin{aligned}0<\mathcal{L}\widehat{\phi}(x_0,y_0) &= \alpha_x(x_0,y_0)+\alpha_y(x_0,y_0)\widehat{\phi}_y(x_0,y_0)\\ &\quad+\frac{1}{2}\sigma^2\left[x_0^2\widehat{\phi}_{xx}(x_0,y_0)+\rho x_0y_0\widehat{\phi}_{xy}(x_0,y_0)+y_0^2\widehat{\phi}_{yy}(x_0,y_0)\right].\end{aligned}$$

Recalling the definition of function $\widehat{v}$, i.e. equation (4.4), we have that $\widehat{v}$ is semi-differentiable with respect to y and the right and the left partial derivative of $\widehat{v}$ with respect to y are decreasing functions, always with respect to y, due to the concavity of the value function v. We could then assume that $\widehat{\phi}$ satisfies the same properties, i.e. it is semi-differentiable with respect to y and the right and the left partial derivative of $\widehat{\phi}$ with respect to y are decreasing functions, always with respect to y. Inside the domain $\mathcal{A}_C$, it is easy to see that $\widehat{\phi}_{xx}(x_0,y_0)=0=\widehat{\phi}_{xy}(x_0,y_0)$ due to the particular form of $\widehat{v}$ as defined in (4.4). Since the left partial derivative of $\widehat{v}$ with respect to y is decreasing $\widehat{\phi}_{yy}(x_0,y_0)$ is nonpositive, otherwise the inequality $\widehat{\phi}\leq\widehat{v}$ is not satisfied. In a similar way, we can show that the partial derivative of $\widehat{\phi}$ is controlled by the right and left partial derivative

of $\widehat{v}$, i.e.

$$\widehat{v_y^-}(x_0, y_0) \leq \widehat{\phi}_y(x_0, y_0) \leq \widehat{v_y^-}(x_0, y_0)\,. \tag{4.6}$$

Moreover, we have by concavity

$$\widehat{v_y^-}(x_0, y_0) = v_y^- \left((1-p)KC + \frac{1-p}{p} y_0, y_0 \right) \leq v_y^- \left((1-p)KC + \frac{1-p}{p} y_0, \eta \right), \tag{4.7}$$

where $\eta \leq y_0$. Resuming the previous relation can be satisfied only if

$$\alpha_y(x_0, y_0)\widehat{\phi}_y(x_0, y_0) > -\alpha_x(x_0, y_0)\,. \tag{4.8}$$

In the domain $\mathcal{A}_C$, we have $\alpha_x < 0$ and in particular

$$\begin{aligned} \alpha_x(x_0, y_0) &= r x_0 \left[1 - \frac{x_0}{(1-p)K} \right] + \frac{\theta}{pK} \left[(1-p)y_0 - p x_0 \right] \\ \alpha_y(x_0, y_0) &= r y_0 \left[1 - \frac{y_0}{pK} \right] - \frac{\theta}{pK} \left[(1-p)y_0 - p x_0 \right]. \end{aligned}$$

Thanks to relation (4.6) and the fact that the value function is increasing with respect to y, see Lemma 3.2, we deduce that $\widehat{\phi}_y(x_0, y_0)$ is non negative and then if $\alpha_y(x_0, y_0) \leq 0$, a contradiction appears.

We then consider the subset of $\mathcal{A}_C$ where $\alpha_y(x_0, y_0) > 0$. Replacing (4.6) and (4.7) into (4.8) and integrating with respect to η between $y_0/2$ and y_0 we obtain

$$\alpha_y(x_0, y_0) \left[v\left((1-p)\left(KC + \frac{y_0}{p} \right), y_0 \right) - v\left((1-p)\left(KC + \frac{y_0}{p} \right), \frac{y_0}{2} \right) \right] > -\frac{1}{2}\alpha_x(x_0, y_0) y_0.$$

Using Lemma 3.1, we have that $x \leq v(x, y) \leq \varphi_{_M}(x, y) < x + y + B$, where B is a constant. Then the previous inequality implies

$$\alpha_y(x_0, y_0)(y_0 + B) > -\frac{1}{2}\alpha_x(x_0, y_0) y_0.$$

Since α_x is quadratic in x_0 whereas α_y is linear, if C is large enough given $x_0 > (1-p)KC$, then the previous inequality is a contradiction. Therefore, we obtain that $\forall\ y \in \mathbb{R}_{++}$, there exists $\bar{x}_y \geq 0$ such that $v_x(\bar{x}_y, y) = 1$. Recalling that the function v is concave, see Lemma 3.2, we deduce that v_x is decreasing with respect to x. As a consequence $\forall x \geq \bar{x}_y$, we have $v_x(x, y) = 1$. That concludes the proof. □

5 Numerical results

This Section deals with numerical studies of our harvesting problem. In Section 4, we have characterised the value function as the unique solution in viscosity sense of the HJB equation (3.3).

Moreover, in Section 3, we have detailed some properties of the value function, in particular, it is non-decreasing and concave. Both properties are useful to obtain a robust numerical method to approximate the solution of the HJB equation (3.3).

5.1 Numerical scheme

To solve the HJB equation (3.3) arising from the stochastic control problem (3.2), we choose to use a deterministic approach based on a finite difference scheme which leads to the resolution of a controlled Markov chain problem. Such technique was widely popularised by Kushner and Dupuis (2013). The convergence of the solution of the numerical scheme towards the solution of the HJB equation, when the space step goes to zero, can be shown using standard arguments, i.e. it satisfies monotonicity, consistency and stability properties. Similar numerical schemes, involving a controlled Markov chain problem, are exploited in operational research, see for instance Cao, Li, and Yan (2012), Jin, Yin and Zhu (2012), Parpas et Webster (2014), Sesana, Marazzina and Fusai (2014), Cosso, Marazzina and Sgarra (2015) and Phelan, Marazzina, Fusai and Germano (2018). The concavity guarantees the existence of left derivative with respect to the variable x and this derivative is non-increasing. The harvesting region $\mathcal{H}$ is identified by the set where the left derivative with respect to x is equal to 1. Thanks to the concavity, if (x, y) belongs to the harvesting region then all (x', y) with $x' > x$ belongs to the harvesting region. The numerical method is then to identify, for all y, the smallest $\bar{x}_y$ such that $v_x(\bar{x}_y, y) = 1$.

We first localise the problem on a discretised grid. Let h and k be the discretisation steps along the directions x and y respectively. We define the space grid as $\mathcal{G}_{h,k} := \{0, h, 2h, .., x_{max}\} \times \{0, k, 2k, ..., y_{max}\}$, where x_{max} and y_{max} are nonnegative constants, that could be changed if the threshold $\bar{x}_y$ is not identified.

For sake of readability we introduce the following quantities:

$$\begin{aligned} z_1 &:= (x+h, y), & z_2 &:= (x, y+k), & z_3 &:= (x-h, y), & \sigma_x &:= \sigma x, \\ z_4 &:= (x, y-k), & z_5 &:= (x+h, y+k), & z_6 &:= (x-h, y-k), & \sigma_y &:= \sigma y, \end{aligned}$$

$$\mu_x := rx\left(1 - \frac{x}{(1-p)K}\right) + \theta\left(\frac{y}{pK} - \frac{x}{(1-p)K}\right), \quad \mu_y := ry\left(1 - \frac{y}{pK}\right) - \theta\left(\frac{y}{p} - \frac{x}{(1-p)K}\right).$$

For (x, y) in the space grid $\mathcal{G}_{h,k}$ we consider approximations of the following form:

$$\begin{aligned} \frac{\partial v}{\partial x}(x,y) &\approx \frac{v(x+h,y) - v(x,y)}{h}\mathbb{1}_{\mu_x \geq 0} - \frac{v(x-h,y) - v(x,y)}{h}\mathbb{1}_{\mu_x < 0}, \\ \frac{\partial v}{\partial y}(x,y) &\approx \frac{v(x,y+k) - v(x,y)}{k}\mathbb{1}_{\mu_y \geq 0} - \frac{v(x,y-k) - v(x,y)}{k}\mathbb{1}_{\mu_y < 0}, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 v}{\partial x^2}(x,y) &\approx \frac{v(x+h,y)+v(x-h,y)-2v(x,y)}{h^2},\\
\frac{\partial^2 v}{\partial y^2}(x,y) &\approx \frac{v(x,y+k)+v(x,y-k)-2v(x,y)}{k^2},\\
\frac{\partial^2 v}{\partial x \partial y}(x,y) &\approx \frac{2v(x,y)+v(x+h,y+k)+v(x-h,y-k)}{2hk}\\
&\quad -\frac{v(x+h,y)+v(x,y+k)+v(x-h,y)+v(x,y-k)}{2hk}.
\end{aligned}
$$

Thus, using the above notations and applying a finite difference scheme, the HJB equation (3.3) can be formulated as the following:

$$
v(x,y)=\max\left\{\frac{\sum_{i=1}^{6} p_i v(z_i)}{1+\beta\Delta t^{h,k}}, v(x-h,y)+h\right\}, \tag{5.1}
$$

where

$$
\begin{aligned}
p_1(x,y) &:= \frac{\sigma_x^2 k^2+2hk^2\mu_x^+-\rho hk\sigma_x\sigma_y}{2Q^{h,k}}, & p_2(x,y) &:= \frac{\sigma_y^2 h^2+2h^2 k\mu_y^+-\rho hk\sigma_x\sigma_y}{2Q^{h,k}},\\
p_3(x,y) &:= \frac{\sigma_x^2 k^2+2hk^2\mu_x^--\rho hk\sigma_x\sigma_y}{2Q^{h,k}}, & p_4(x,y) &:= \frac{\sigma_y^2 h^2+2h^2 k\mu_y^--\rho hk\sigma_x\sigma_y}{2Q^{h,k}},\\
p_5(x,y) &= p_6(x,y) := \frac{\rho hk\sigma_x\sigma_y}{2Q^{h,k}}, & \Delta t^{h,k}(x,y) &:= \frac{h^2k^2}{Q^{h,k}},
\end{aligned}
$$

$$
Q^{h,k}(x,y) := hk(|\mu_x|k+|\mu_y|h)+\sigma_x^2k^2+\sigma_y^2h^2-\rho hk\sigma_x\sigma_y.
$$

To compute explicitly the approximated solution of the discrete problem (5.1) we use the following iterative scheme:

$$
\begin{aligned}
v_{n+1}(x,y) &= \max\left\{\frac{\sum_{i=1}^{6} p_i v_n(z_i)}{1+\beta\Delta t^{h,k}}, v_n(x-h,y)+h\right\}, &(5.2)\\
v_0(x,y) &= 0. &(5.3)
\end{aligned}
$$

The above iterative scheme is explicit and fully implementable on the enlarged grid $\mathcal{G}_{h,k}^{+} := \{0,..,x_{max}+h\}\times\{-k,..,y_{max}+k\}$.

Using the following parameters, about 8 seconds are necessary to obtain the approximated value function and policy using Intel™Core i7 at 2.70 Ghz CPU with 8 Go of RAM.

5.2 Comparative statics

In this section, we study the sensitivity of the value function, i.e. the discounted reward from fishing, and the optimal policy, i.e. the optimal threshold $\bar{x}_y$, with respect to the main model parameters. Numerical tests are performed for different values of the parameters and with the following set of central values. The choice of the partition $p = 0.4$ will be justified by an economic justification and

r	β	σ	ρ	θ	K	p
0.3	0.05	0.01	0.01	3.5	1	0.4

Table 1: Parameters used for the numerical tests.

is coherent with the result of Grafton, Kompas and Lindenmayer (2005), since they obtain $p = 0.5$ but the numerical values for $p = 0.4$ are quite similar in their case.

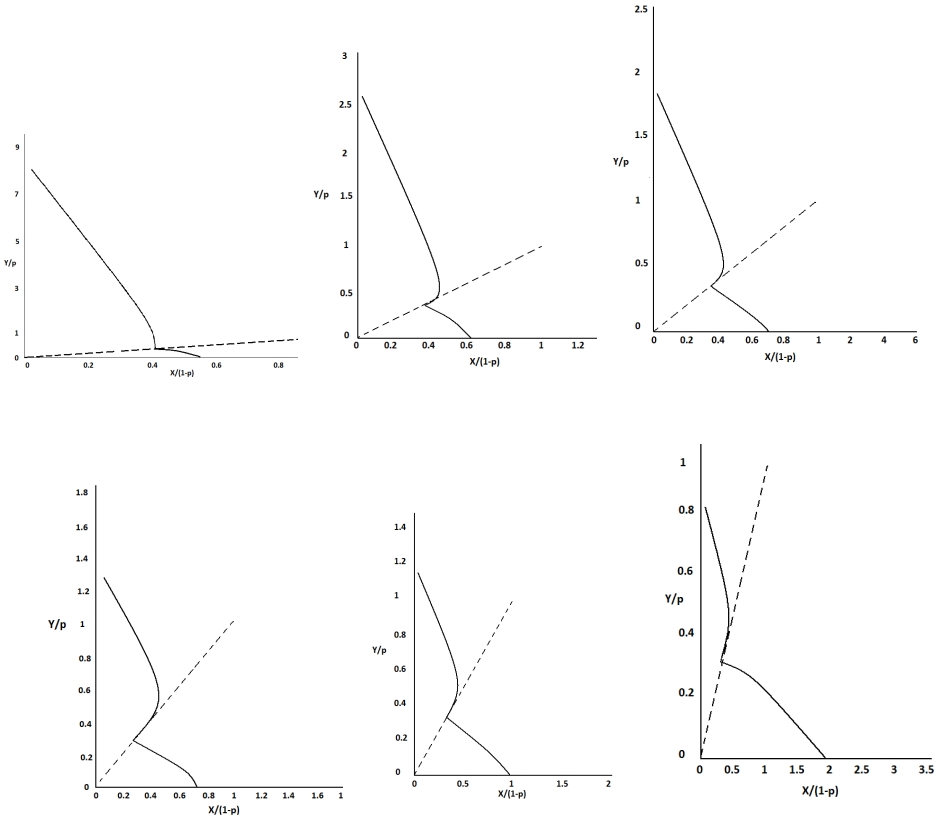

Figure 1: *Optimal harvesting threshold when the populations in reserve and unprotected area are renormalised by the proportion p. From top left to bottom right, graphs are for* $p = 0.1$; 0.3; 0.4; 0.5; 0.6; 0.8. *The dotted line indicates the states with same density in reserve and unprotected areas, i.e.* $y/p = x/(1-p)$.

We plot, in Figure 1, the shapes of the frontier which separate the *harvesting region* $\mathcal{H}$, which is on the right side of the frontier where $x \geq \bar{x}_y$, from the *no harvesting region* $\mathcal{NH}$, which is on the left side of the frontier where $x \leq \bar{x}_y$, characterising therefore the optimal harvesting policy for different values of p, i.e. for all y, the threshold $\bar{x}_y$ above which is optimal to harvest.

Our numerical results as plotted in Figure 1, allow us to identify the following three cases.

- First, for large values of y, i.e. $y > 0.5p$, the optimal threshold $\bar{x}_y$ is very small and get even smaller for higher values of y. The interpretation is that when the reserve area is overcrowded, there is little extinction risk. Then, the optimal policy is to harvest more in the authorised area, especially if the population overcomes the capacity of the environment, in order to rebalance as soon as possible the excess of population in the reserve .

- Second, for small values of y, i.e. $y < 0.3p$, the optimal threshold $\bar{x}_y$ is very large and gets higher for smaller values of y . The optimal policy consists in delaying harvesting and keeping a high spillover from the fishing area to the reserve in order to repopulate the reserve area quickly.

- A third case arises for intermediate values of y, i.e. $0.3p < y < 0.5p$. When the fish density decreases in the reserve area, i.e. y decreases, the optimal threshold $\bar{x}_y$ intuitively should increase as in the previous two cases, but this is no longer the case. Indeed, instead of delaying harvesting, we observe that the threshold $\bar{x}_y$ above which it is optimal to harvest, actually decreases. This is completely counter-intuitive as the optimal policy implies that the less we have fish in the reserve area, the more we have to harvest in the authorised area. Furthermore, we observe that the curve representing $\bar{x}_y$ with respect to y decreases for smaller value of y and becomes tangent and then coincides with the dotted line. We recall that the dotted line represents the line where fish density is the same in the reserve and authorised areas. Although it looks counter-intuitive, our findings state that for intermediate levels of fish population y, which represent no risk of extinction or overpopulation in the reserve area, the optimal policy is to harvest any excess of fish density in the authorised area, preventing therefore the fish net spillover from authorised area to the reserve. Indeed, spillover from authorised area to reserve area should only happen when the fish density in the reserve is very low, see the above second case. We will refer to this effect as the diagonal effect of the threshold $\bar{x}_y$.

In the economic point of view, we compute the value function for different values of p. We plot the following Figure 2 which compares the value function for different p and to the case when no reserve area is implemented, i.e. when $p = 0$. We can see that there is a clear economic surplus when a reserve area is implemented, with $p = 0.4$ delivering the highest value function, in other words, the best economic values. As such, these numerical results highlight that implementing a reserve policy not only has an evident ecological aim but also an economic surplus. These numerical

results are globally stable when we change other parameters, confirming therefore, numerically, the importance of no-take area in natural resources management.

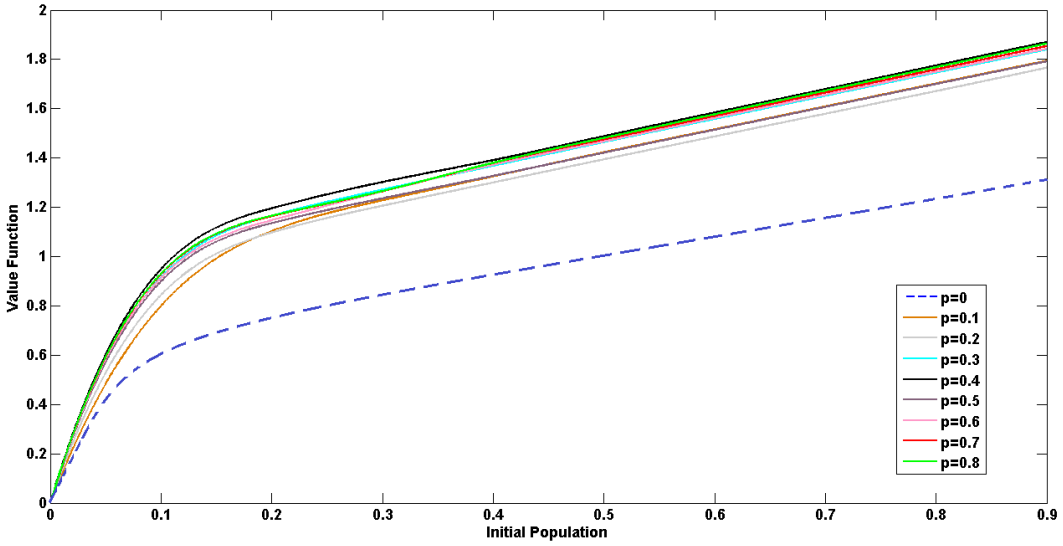

Figure 2: *Value function for different value of the proportion of the marine reserve. The case* $p = 0$ *indicates the absence of the marine reserve.*

We now evaluate the sensitivity with respect the growth rate r, the discount factor β and the volatility σ. We plot, in Figure 3, the resulting shapes of the frontier which separate the *harvesting region* $\mathcal{H}$, from the *no harvesting region* $\mathcal{NH}$, respectively for different values of r, β and σ. We may notice that the optimal threshold $\bar{x}_y$ is increasing with respect to the growth rate r but decreasing with respect the discount factor β and the volatility σ. These results are insightful and economically in-line with expectations. Indeed, our findings basically state that when the growth rate r gets larger, it is optimal to delay harvesting and let the system grows as we expect a better harvest later. The opposite findings are true with regard to the discount rate and the volatility. However, the diagonal effect of the threshold $\bar{x}_y$ appears for all the different sets of parameters, the only changes is to translate along the diagonal line. We also remark that the sensitivity with respect to the volatility σ is very small compared to the other parameters.

We also analyse, in Figure 4, the sensitivities of the value function with respect to the growth rate r, the discount factor β and the volatility σ. We may notice that the value function is increasing

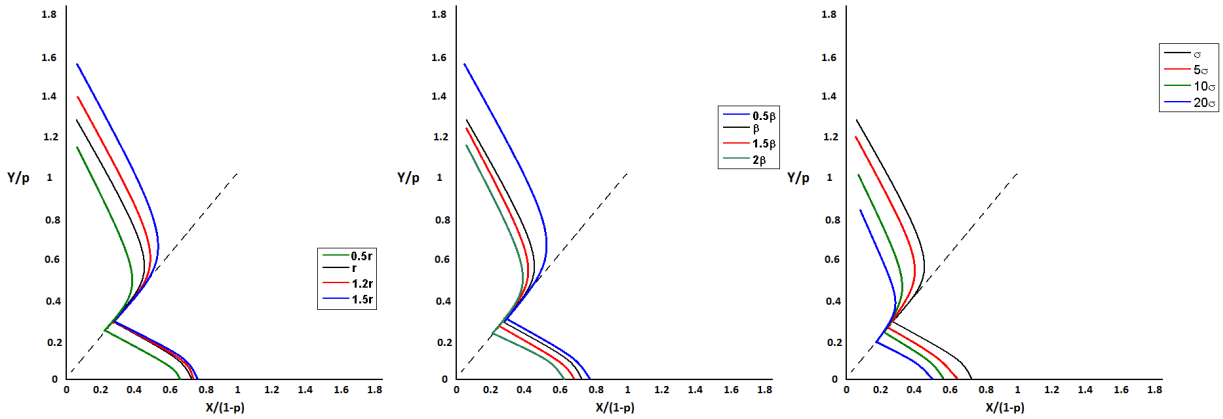

Figure 3: *Optimal harvesting threshold when the populations in reserve and unprotected area are renormalised by the proportion* $p = 0.4$. *From left to right, graphs are for the growth rate* r, *the discount factor* β *and the volatility* σ. *The dotted line indicates the states with same density in reserve and unprotected areas, i.e.* $y/p = x/(1-p)$.

in the growth rate r but decreasing with respect the discount factor β and the volatility σ. This findings is line with our economic intuitions. We also notice that the value function is much less sensitive to the change in σ compared to the other two parameters β and σ.

We finally focus on the parameter θ describing the transfer rate between the no-take reserve and the harvesting area. Figure 5 shows the sensitivity of the value function as well as the optimal threshold $\bar{x}_y$ with respect the transfer rate θ. We notice that the value function is mildly decreasing with respect to θ. The finding on the optimal threshold is insightful as it interestingly separates two situations when the fish density in the reserve area is higher or lower than that in the harvesting area, in other words, when the transfer flow, i.e. spillover, is positive (corresponding to the region above the doted line) or negative (corresponding to the region below the doted line) from the reserve area to the harvesting area. Indeed, when the transfer flow is positive, a bigger parameter θ optimally leads to a higher optimal threshold $\bar{x}_y$, i.e. it is optimal to delay harvesting and wait until enough fishes have moved into the harvesting area. On the other hand, when the transfer flow is negative, a bigger the parameter θ optimally leads to a lower optimal threshold $\bar{x}_y$, i.e. it is optimal to increase our harvest and not wait until too many fishes move out of the harvesting area. In other words, the diagonal effect is magnified when θ increases. We deduce that if the no-take reserve area is highly connected with the harvesting area, the diagonal effect will be really

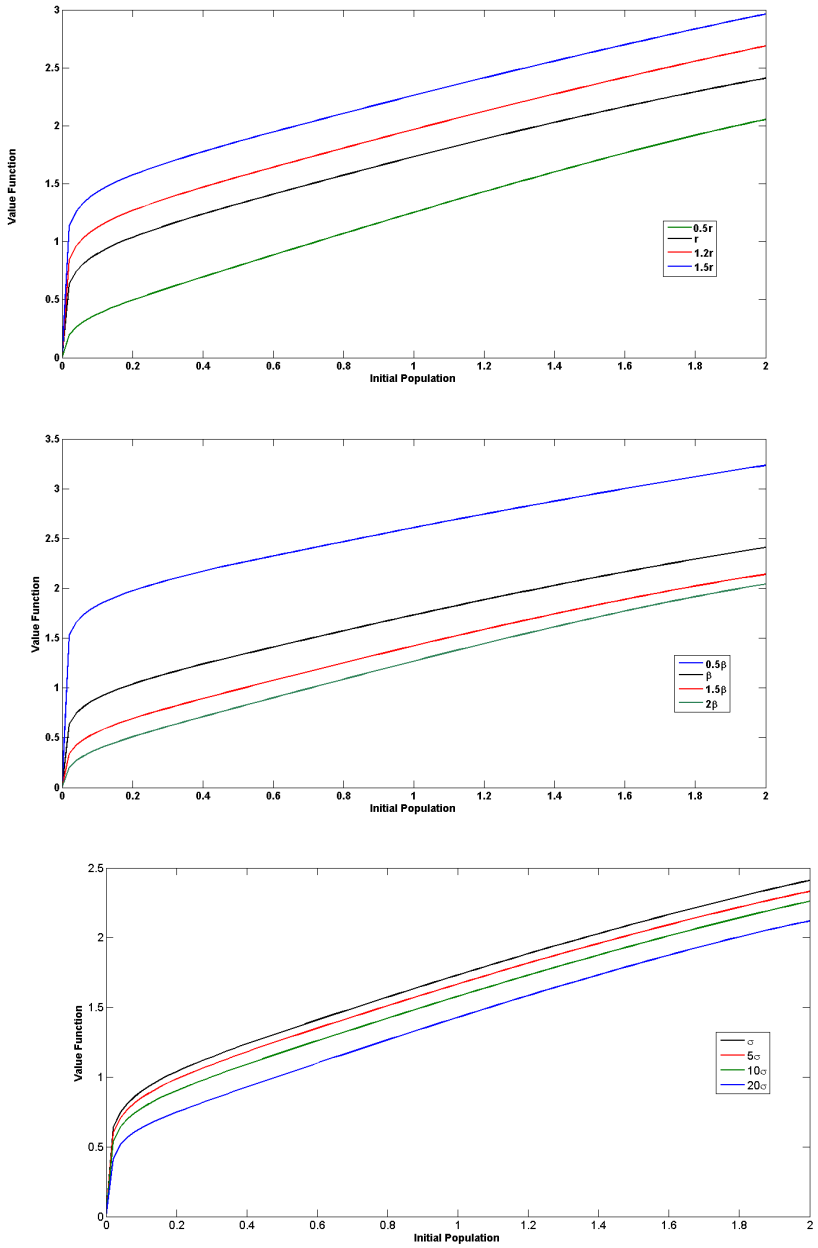

Figure 4: *Value function for different value of the proportion of the marine reserve. For different values of the growth rate r, the discount factor β and the volatility σ respectively.*

important. In contrast, when θ is small, for instance when the governments or regulators choose lowly connected areas, like a semi-enclosed sea connected via narrow straights, the diagonal effect will be minimal or negligible. To sum up, while the optimal economic reward may not be very sensitive to the parameter θ, i.e. to the way the border between the reserve and the harvesting area is designed, the optimal policy behaviour may change dramatically.

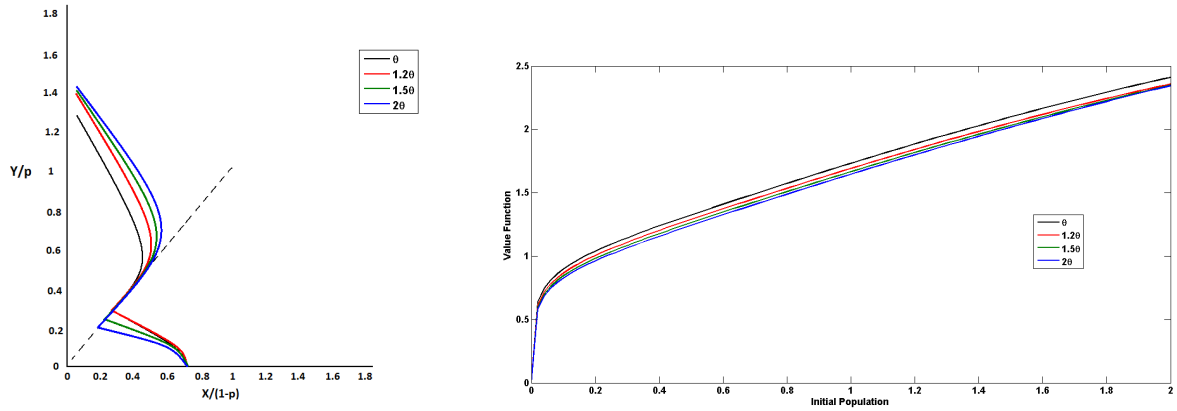

Figure 5: *Optimal harvesting threshold when the populations in reserve and unprotected area are renormalised by the proportion* $p = 0.4$. *From left to right, graphs are for the connection parameter* θ. *The dotted line indicates the states with same density in reserve and unprotected areas, i.e.* $y/p = x/(1-p)$. *Value function for different value of the proportion of the marine reserve. For different values of the connection parameter* θ.

6 General discussion and conclusions

We have studied an optimal harvesting problem under marine reserve constraints, in which the marine area is split into two parts, the no-take marine reserve where harvesting is strictly prohibited and a fishing area where unconstrained harvesting is allowed. In our harvesting problem, which is formulated as a bi-dimensional singular control problem, our main focus is on the optimal harvesting control strategy. We have addressed our problem in the following three mains aspects. First, the modelling aspect which includes important features and constraints characterising our harvesting problem. Then, we solve theoretically and rigorously our stochastic optimisation problem using viscosity approach. Finally, we solve our problem numerically, in order to characterise the optimal policy in an explicit framework. Our contribution to the literature on the multi-dimensional singular control is our ability to obtain non-trivial results on the value function and more importantly on the optimal strategy, in particular the study around an optimal threshold $\bar{x}_y$ which in turn defines the optimal strategy. This is clearly beyond the usual viscosity characterisation of the value function in most studies on non-degenerate multi-dimensional singular control problems. On the harvesting management aspect, our findings are new and insightful as they clearly show that marine reserve policy may increase both ecological and economics interests. Moreover, the obtained results are

completely counter-intuitive as the optimal policy implies that, when there is no extinction or overpopulation risk, the less we have fish in the reserve area, the more we have to harvest in the fishing area. This optimal strategy, which consists in harvesting all spillover from the protected area, clearly indicates that there is no longer any conflictual relation between ecological and economic interests. Furthermore, comparative statics confirm these phenomena which are magnified when the transfer rate between the two regions increases. As such, they clearly contribute to a better understanding on optimal harvesting strategy under marine reserve constraints.

However many operational questions and important realistic features in harvesting management problem remain open and currently unaddressed and need to be investigated in future research. Natural improvements to our model and control problem include interesting and distinctive characteristics of fish evolution such as seasonality, see for instance Ni and Sandal (2019) and Kvamsdal, Maroto, Morán and Sandal (2020), spatial distribution, see Costello and Polasky (2008), or multi-party exploitation of renewable resources, see Ekerhovd, Flam and Steinshamn (2021). Other interesting features to be considered in future work include the possibility of negative population shocks due to epidemics and/or ecological disasters. Previous literature models these shocks as Poisson events with constant intensity, see for instance, Grafton, Kompas and Lindenmayer (2005). However, based on the observations of the current pandemic, we may consider cluster structures, in which shocks are concentrated in short periods without events between each cluster. Such clustering effects may be incorporated by using Hawkes process or continuous state branching processes, see for instance Bernis, Brignone, Scotti and Sgarra (2021), Jiao, Ma, Scotti and Zhou (2021), and Sadoghi and Vecer (2021). Further realistic model features such as "dynamic" spillover may be incorporated to our model by using Markov chain leading to a regime-switching problem, therefore a system of variational inequalities, see for instance Chevalier, Ly Vath, Roch and Scotti (2015) and Oliveira and Perkowski (2020). Another realistic model feature which may be studied in future research is to consider an endogenous version of the discount rate β within our optimisation problem. The introduction of an endogenous β will certainly lead us to a better understanding in the degree of substitution between future and current consumption. Finally, a more theoretical issue arising in our study is the choice of the size of the no-take area. In this paper, we have consider a static problem, i.e. the regulator or the government fixes the proportion of the no-take area p. However, a more dynamic approach could also be investigated with a game between the social planner and the fishers. The arising natural setup is a singular control problem of McKean Vlasov dynamics, which could be addressed in a future research.

Appendix

The proofs of the following Theorems 4.1 and 4.2 could be obtained by mainly adapting the arguments used in Bayraktar and Young (2011) and Chevalier, Ly Vath, Roch, and Scotti (2015). We will therefore provide only the tricky parts of the proofs which include the subsolution property and a sketch of the proof of the comparison theorem.

Proof of the Theorem 4.1

Proof of the subsolution property on $\mathbb{R}_{++} \times \mathbb{R}_{++}$.

We prove the subsolution property by contradiction. Suppose that the claim is not true. Then, there exists $(\overline{x}, \overline{y})$ and a neighbourhood $\overline{B}_\varepsilon(\overline{x}, \overline{y}) = \{(x, y), \|(x, y) - (\overline{x}, \overline{y})\| \leq \varepsilon\}$ of $(\overline{x}, \overline{y})$, a $\mathcal{C}^2$ function φ with $(\varphi - v)(\overline{x}, \overline{y}) = 0$ and $\varphi \geq v$ on $\overline{B}_\varepsilon(\overline{x}, \overline{y})$, and $\eta > 0$, such that for all $(x, y) \in \overline{B}_\varepsilon(\overline{x}, \overline{y})$

$$(\beta\varphi - \mathcal{L}\varphi)(x, y) > \eta \tag{6.4}$$

$$\varphi_x(x, y) - 1 > \eta. \tag{6.5}$$

Let an admissible harvesting strategy $H \in \mathcal{A}$ and consider the exit time $\tau_\varepsilon = \inf\{t \geq 0, (X_t, Y_t) \notin \overline{B}_\varepsilon(\overline{x}, \overline{y})\}$. Applying the Itô's formula to the process $e^{-\beta t}\varphi(X_t, Y_t)$ between 0 and τ_ε^-, and taking the expectation, we obtain

$$\begin{aligned} \mathbb{E}\left[e^{-\beta(\tau_\varepsilon)}\varphi(X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon})\right] &= \varphi(\overline{x}, \overline{y}) + \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t}(-\beta\varphi + \mathcal{L}\varphi)(X_t, Y_t)dt\right] \\ &\quad - \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t}\varphi_x(X_t, Y_t)dH_t^c\right] \\ &\quad + \mathbb{E}\left[\sum_{0 \leq t < \tau_\varepsilon} e^{-\beta t}[\varphi(X_t, Y_t) - \varphi(X_{t^-}, Y_t)]\right]. \end{aligned} \tag{6.6}$$

From Taylor's formula and (6.5), and noting that $\Delta X_t = -\Delta H_t$, we have for all $0 \leq t < \tau_\varepsilon$

$$\varphi(X_t, Y_t) - \varphi(X_{t^-}, Y_t) \leq \Delta X_t \, \varphi_x(X_t + \Delta X_t, Y_t) \leq -(1 + \eta)\Delta H_t \,. \tag{6.7}$$

Using (6.5), (6.6) and (6.7), we have

$$\mathbb{E}\left[e^{-\beta\tau_\varepsilon}\varphi(X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon})\right] \leq \varphi(\overline{x}, \overline{y}) - \eta\,\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t}dt\right] - \,(1 + \eta)\,\mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-\beta t}dH_t\right]. \tag{6.8}$$

Notice that while $(X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon}) \in \overline{B}_\varepsilon(\overline{x}, \overline{y})\}$, $(X_{\tau_\varepsilon}, Y_{\tau_\varepsilon})$ is either on the boundary $\partial\overline{B}_\varepsilon(\overline{x}, \overline{y})$ or out of $\overline{B}_\varepsilon(\overline{x}, \overline{y})$. However, there's some random variable α valued in $[0, 1]$ and $X_{(\alpha)}$ such that

$$(X_{(\alpha)}, Y_{\tau_\varepsilon}) := (X_{\tau_\varepsilon^-} + \alpha\Delta X_{\tau_\varepsilon}, Y_{\tau_\varepsilon}) = (X_{\tau_\varepsilon^-} - \alpha\Delta H_{\tau_\varepsilon}, Y_{\tau_\varepsilon}) \;\in\; \partial\overline{B}_\varepsilon(\overline{x}, \overline{y})$$

Then similarly as in (6.7), we have

$$\varphi(X_{(\alpha)}, Y_{\tau_\varepsilon}) - \varphi(X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon}) \quad \leq \quad -\alpha(1+\eta)\Delta H_{\tau_\varepsilon} \tag{6.9}$$

Notice that $X_{\tau_\varepsilon} = X_{(\alpha)} - (1-\alpha)\Delta H_{\tau_\varepsilon}$, and from Lemma 3.1, we have

$$v(X_{(\alpha)}, Y_{\tau_\varepsilon}) \quad \geq \quad v(X_{\tau_\varepsilon}, Y_{\tau_\varepsilon}) + (1-\alpha)\Delta H_{\tau_\varepsilon} \tag{6.10}$$

Recalling that $\varphi(X_{(\alpha)}, Y_{\tau_\varepsilon}) \geq v(X_{(\alpha)}, Y_{\tau_\varepsilon})$, inequalities (6.9) and (6.10) imply

$$\varphi(X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon}) \geq v(X_{\tau_\varepsilon}, Y_{\tau_\varepsilon}) + (1+\alpha\eta)\Delta H_{\tau_\varepsilon}$$

Plugging this into (6.8), we obtain

$$\begin{aligned} v(\overline{x}, \overline{y}) \quad \geq \quad & \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t} dH_t\right] + \mathbb{E}\left[e^{-\beta\tau_\varepsilon} v(X_{\tau_\varepsilon}, Y_{\tau_\varepsilon})\right] \\ & + \eta\left\{\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t} dt\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-\beta t} dH_t\right] + \mathbb{E}\left[\alpha e^{-\beta\tau_\varepsilon}\Delta H_{\tau_\varepsilon}\right]\right\}. \end{aligned}$$

We now claim that there exists a constant $m_0 > 0$ such that for any admissible harvesting strategy

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t} dt\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-\beta t} dH_t\right] + \mathbb{E}\left[\alpha e^{-\beta\tau_\varepsilon}\Delta H_{\tau_\varepsilon}\right] \geq m_0. \tag{6.11}$$

We need to find some constant $M_0 > 0$ such that the $\mathcal{C}^2$ function $\psi(x,y) := M_0\left[(x-\overline{x})^2 + (y-\overline{y})^2 - \varepsilon^2\right]$ satisfies

$$\begin{aligned} \min\left\{\beta\psi - \mathcal{L}\psi + 1, 1 - |\psi_x|\right\} \quad & \geq \quad 0, \quad \text{on } \overline{B}_\varepsilon(\overline{x}, \overline{y}), \\ \psi \quad & = \quad 0, \quad \text{on } \partial\overline{B}_\varepsilon(\overline{x}, \overline{y}). \end{aligned}$$

It is sufficient to take M_0 small enough. Applying the Itô's formula, we then obtain

$$\mathbb{E}\left[e^{-\beta(\tau_\varepsilon)}\psi(X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon})\right] \quad \leq \quad \psi(\overline{x}, \overline{y}) + \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t} dt\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-\beta t} dH_t\right]. \tag{6.12}$$

Noticing that $\psi_x(x,y) \geq -1$, we have $\psi(X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon}) - \psi(X_{(\alpha)}, Y_{\tau_\varepsilon}) \geq -(X_{\tau_\varepsilon^-} - X_{(\alpha)}) = -\alpha\Delta H_{\tau_\varepsilon}$ Plugging into (6.12), we obtain

$$\begin{aligned} & \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\beta t} dt\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon^-} e^{-\beta t} dH_t\right] + \mathbb{E}\left[e^{-\beta(\tau_\varepsilon)}\alpha\Delta H_{\tau_\varepsilon}\right] \\ \geq \quad & \mathbb{E}\left[e^{-\beta(\tau_\varepsilon)}\psi(X_{(\alpha)}, Y_{\tau_\varepsilon})\right] - \psi(\overline{x}, \overline{y}) \geq -\psi(\overline{x}, \overline{y}) = M_0\varepsilon^2 \end{aligned}$$

Hence the claim (6.11) is true with $m_0 = M_0\varepsilon^2$.

Finally, by taking the supremum over all admissible harvesting strategies H, and using the dynamic programming principle (3.2), we have $v(\overline{x},\overline{y}) \geq v(\overline{x},\overline{y}) + \eta m_0$, which is a contradiction.
□

Proof of the Comparison Theorem 4.2

Let u be a continuous viscosity subsolution to the HJB equation (3.3) on $\mathbb{R}_{++} \times \mathbb{R}_{++}$, and w be a continuous viscosity supersolution to HJB equation (3.3) on $\mathbb{R}_{++} \times \mathbb{R}_{++}$, satisfying the boundary conditions

$$\begin{aligned} u(0^+, y) &\leq w(0^+, y), \quad y \in \mathbb{R}_+ \\ u(x, 0^+) &\leq w(x, 0^+), \quad x \in \mathbb{R}_+ \end{aligned}$$

and linear growth condition :

$$|u(x,y)| + |w(x,y)| \leq C(1 + x + y), \quad \forall (x,y) \in \mathbb{R}_{++} \times \mathbb{R}_{++}. \tag{6.13}$$

for some positive constants C.

We want to prove that

$$u \leq w \quad on\ \mathbb{R}_{++} \times \mathbb{R}_{++}.$$

Step 1. We first construct a strict supersolution to the HJB equation (3.3) with suitable perturbations of w. We consider the perturbation smooth function on $\mathbb{R}_{++} \times \mathbb{R}_{++}$,

$$H(x,y) = (x+y)^2 + (1-p)Kx + \delta x + pKy + B\,,$$

where B is a constant large enough, in particular larger than C. Then for any $\gamma \in (0,1)$, we show that the function $w^\gamma = (1-\gamma)w + \gamma H$ is a strict viscosity supersolution to HJB (3.3) in $\mathbb{R}_{++} \times \mathbb{R}_{++}$, i.e. there exists $\delta > 0$ such that

$$\min\left\{ \beta w^\gamma - \mathcal{L} w^\gamma;\ \frac{\partial w^\gamma}{\partial x} - 1 \right\} \geq \gamma\delta > 0. \tag{6.14}$$

A straightforward computation gives us the above strict inequality.

Step 2. In order to prove the comparison principle, it suffices to show that for all $\gamma \in (0,1)$:

$$\sup_{\mathbb{R}_{++} \times \mathbb{R}_{++}} (u - w^\gamma) \leq 0,$$

since the required result is obtained by letting γ to 0. We argue by contradiction and suppose that there exist some $\gamma \in (0,1)$, such that

$$\Theta \equiv \sup_{\mathbb{R}_{++} \times \mathbb{R}_{++}} (u - w^\gamma) > 0. \tag{6.15}$$

Notice that $u - w^\gamma$ goes to $-\infty$ when $||(x,y)||$ goes to infinity. We also have $\lim_{x\to 0^+} u(x,y) - \lim_{x\to 0^+} w^\gamma(x,y) \leq \gamma(C-B) \leq 0$. Hence, by continuity of the functions u and w^γ, there exists $(x_0,y_0) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ such that $\Theta = u(x_0,y_0) - w^\gamma(x_0,y_0)$. For sake of simplicity, we will denote $z \equiv (x,y)$, $z' \equiv (x',y')$ and $z_0 \equiv (x_0,y_0)$ For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned}\Phi_\varepsilon(z,z') &= u(z) - w^\gamma(z') - \Psi_\varepsilon(z,z'),\\ \Psi_\varepsilon(z,z') &= \frac{1}{4}||z-z_0||^4 + \frac{1}{2\varepsilon}||z-z'||^2,\end{aligned}$$

Now the rest of the proof follows, for instance the appendix A of Chevalier, Ly Vath, Roch, and Scotti (2015). □

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