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► **To cite this version:**

Nadia Belmabrouk, Mondher Damak, Mohsen Miraoui. Stochastic Nicholson's blowflies model with delays. *International Journal of Biomathematics*, 2022, 16, 10.1142/s1793524522500656. hal-03945925

**HAL Id: hal-03945925**

**<https://hal.science/hal-03945925>**

Submitted on 18 Jan 2023

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1 International Journal of Biomathematics  
 2 2250065 (32 pages)  
 3 © World Scientific Publishing Company  
 4 DOI: 10.1142/S1793524522500656



5 **Stochastic Nicholson's blowflies model with delays**

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14 Received  
 15 Accepted  
 16 Published

10 October 2021

29 March 2022

AQ: Please check whether author name, email ID, affiliation details, indication of corresponding author are correct and complete as set. Also please check keywords, left and right running heads.

17 In this research work, we introduce a new concept of double measure ergodic processes  
 18 to identify the space of double measure pseudo almost periodic (or pap) processes in  
 19 the  $p$ th mean sense. We display some findings regarding the completeness as well as the  
 20 composition theorems and the invariance of the space consisting in double measure pap  
 21 processes. Motivated by the above-mentioned results, the Banach fixed point theorem  
 22 and the stochastic analysis techniques, we prove the existence, uniqueness and the global  
 23 exponential stability of doubly measure pseudo almost periodic mild solution for a class  
 24 of stochastic Nicholson's blowflies model with mixed delays in a separable real Hilbert  
 25 space. The concluding part of the work is crowned with an example to confirm the  
 26 reliability and feasibility of our findings.

27 *Keywords:* Pseudo almost periodic solution; measure theory; fixed point theorem;  
 28 stochastic Nicholson's blowflies model.

29 Mathematics Subject Classification 2020: 60H15, 60G51, 34C27, 35R60

30 **1. Introduction**

31 In this work, we take into consideration the following Nicholson's blowflies model  
 32 with multiple harvesting terms and infinite delays, driven by Brownian motion in  
 33 a separable Hilbert space  $\mathbb{H}$ :

$$dx(t) = -a(t)x(t)dt + \sum_{i=1}^m \alpha_i(t)x(t - \tau_i(t))e^{-\beta_i(t) \int_{-\infty}^0 k_i(s)x(t+s)ds} dt + f(t, x(t - \tau))d\mathbb{W}(t), \quad t \in \mathbb{R}, \quad (1.1)$$

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1 Let  $\mathbb{R}^+ = [0, +\infty)$ ,  $\mathbb{R}_- = (-\infty, 0)$ , and  $\underline{a} = \inf_{t \in \mathbb{R}} a(t)$ .  $a, \alpha_i, \beta_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ ,  $\tau_i \in$   
 2  $\mathcal{B}_C(\mathbb{R}, \mathbb{R}^+)$ ,  $\tau \geq 0$ , the delay kernel  $k_i \in \mathcal{C}(\mathbb{R}_-, \mathbb{R}^+)$ ,  $f$  is a stochastic process and  
 3  $\mathbb{W}(t)$  is a two-sided standard Brownian motion with values in  $\mathbb{H}$ . This equation  
 4 models the population of the Australian sheep blowfly:  $x(t)$  represents the size of  
 5 the population at time  $t$ ,  $a$  is the per capita delay death rate,  $\alpha_i$  is the maximum per  
 6 capita delay egg production,  $\frac{1}{\beta_i}$  represents the size at which the blowfly population  
 7 reproduces at its maximum rate and  $f$  is the process of harvesting population.

8 In [7], Chrif investigates the existence of the quadratic-mean pseudo almost  
 9 periodic solutions of the following equation:

$$dx(t) = Ax(t)dt + f(t, x(t - \tau))dt + \varphi(t, x(t - \tau))d\mathbb{W}(t), \quad t \in \mathbb{R}, \quad (1.2)$$

10 where  $A$  is the infinitesimal generator of a  $C_0$ -semi-group  $(T(t))_{t \geq 0}$  exponentially  
 11 stable,  $f, \varphi$  are two stochastic processes. In [11], the authors obtained sufficient  
 12 condition for the existence of  $p$ th mean  $\mu$ -pseudo almost periodic mild solutions to  
 13 the following class of nonlinear stochastic evolution equations driven by a fractional  
 14 Brownian motion in a separable Hilbert space  $\mathbb{H}$ :

$$dx(t) = A(t)x(t)dt + f(t, x(t))dt + \theta(t, x(t))d\mathbb{W}(t) \\ + \psi(t)dB^H(t), \quad t \in \mathbb{R}, \quad (1.3)$$

15 where  $(A(t))_{t \in \mathbb{R}}$  is a family of densely defined closed linear operators satisfy-  
 16 ing Acquistapace–Terreni conditions;  $f, \theta$  are two stochastic processes and  $\psi$  a  
 17 function deterministic. The notion of pseudo almost periodicity with measure  
 18 (see [2, 6, 8, 10, 14–17]) is a generalization of the almost periodicity and pseudo-  
 19 almost periodicity introduced by Zhang [20]; it is also a generalization of weighted  
 20 pseudo almost periodicity firstly introduced by Diagana [9]. The concept of measure  
 21 almost periodicity is of great importance in probability for investigating stochastic  
 22 processes. Such a notion is also very interesting for applications arising in many  
 23 branches of mathematical biology, physics and statistics. Moreover, for Nicholson’s  
 24 Blowflies model, many results were proved, (see for example [3, 12, 13, 19]). Moti-  
 25 vated by the above discussion, we introduce the concept of doubly measure pseudo  
 26 almost periodicity in the stochastic case, we give some fundamental properties and  
 27 we investigate the existence, uniqueness and stability of  $(\mu, \nu)$ -pseudo almost peri-  
 28 odic mild solutions in  $p$ th mean sense for Eq. (1.1).

29 This paper is laid out as follows: In Sec. 2, we introduce the concept of doubly  
 30 measure pseudo almost periodicity. Some new developments on the completeness  
 31 and composition of measure pseudo almost periodic functions are exhibited in Sec. 3.  
 32 In Sec. 4, we address the existence and uniqueness and exponential stability of  
 33  $(\mu, \nu)$ -pseudo almost periodic mild solutions in  $p$ th mean sense for Eq. (1.1). Finally,  
 34 in the closing section, we set forward an example to assess the effectiveness of the  
 35 results.

1 **2.  $(\mu, \nu)$ -Pseudo-Almost Periodic Processes**

2 We denote by  $\mathcal{B}$  the Lebesgue  $\sigma$ -field of  $\mathbb{R}$  and by  $M$  the set of all positive measures  
 3  $\mu$  on  $\mathcal{B}$  satisfying  $\mu(\mathbb{R}) = +\infty$  and  $\mu([a, b]) < +\infty$  for all  $a, b \in \mathbb{R}$  ( $a < b$ ). We  
 4 introduce the following new space of  $(\mu, \nu)$ -ergodic functions:

5 **Definition 2.1.** Let  $\mu, \nu \in M$ . A bonded continuous function  $f : \mathbb{R} \rightarrow \mathbb{H}$  is said to  
 6 be  $(\mu, \nu)$ -ergodic in  $p$ th ( $p \geq 2$ )mean sense, if

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \|f(t)\|^p d\mu(t) = 0.$$

7 We denote the space of such all functions by  $\xi(\mathbb{R}, \mathbb{H}, \mu, \nu)$ .

8 We give the following hypothesis:

9 (H<sub>1</sub>) Let  $\mu, \nu \in M$ ,

$$\limsup_{r \rightarrow +\infty} \frac{\mu([-r, r])}{\nu([-r, r])} = \alpha < \infty.$$

10 **Proposition 2.1.** Let  $\mu, \nu \in M$  satisfy (H<sub>1</sub>). Then  $(\xi(\mathbb{R}, \mathbb{H}, \mu, \nu), \|\cdot\|_\infty)$  is a Banach  
 11 space, where  $\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|$ .

12 **Proof.** It is enough to prove that  $\xi(\mathbb{R}, \mathbb{H}, \mu, \nu)$  is closed in  $\mathcal{B}_C(\mathbb{R}, \mathbb{H})$ . Let  $(f_n)_n$  be a  
 13 sequence in  $\xi(\mathbb{R}, \mathbb{H}, \mu, \nu)$  such that  $\lim_{n \rightarrow +\infty} \|f_n - f\|_\infty = 0$ . Since  $\|f\|^p$  is a convex  
 14 function, then for  $r > 0$ , we have

$$\int_{-r}^r \|f(t)\|^p d\mu(t) \leq 2^{p-1} \int_{-r}^r \|f_n(t) - f(t)\|^p d\mu(t) + 2^{p-1} \int_{-r}^r \|f_n(t)\|^p d\mu(t).$$

15 It implies

$$\begin{aligned} & \frac{1}{\nu([-r, r])} \int_{-r}^r \|f(t)\|^p d\mu(t) \\ & \leq \frac{2^{p-1}}{\nu([-r, r])} \int_{-r}^r \|f_n(t) - f(t)\|^p d\mu(t) + \frac{2^{p-1}}{\nu([-r, r])} \int_{-r}^r \|f_n(t)\|^p d\mu(t). \end{aligned}$$

16 It follows that

$$\begin{aligned} & \limsup_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \|f(t)\|^p d\mu(t) \\ & \leq \limsup_{r \rightarrow +\infty} 2^{p-1} \frac{\mu([-r, r])}{\nu([-r, r])} \sup_{t \in \mathbb{R}} \|f_n(t) - f(t)\|^p. \end{aligned}$$

17 Moreover,

$$\sup_{t \in \mathbb{R}} \|f_n(t) - f(t)\|^p \leq \left( \sup_{t \in \mathbb{R}} \|f_n(t) - f(t)\| \right)^p = \|f_n - f\|_\infty^p,$$

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1 and from (H<sub>1</sub>), we obtain

$$\limsup_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \|f(t)\|^p d\mu(t) \leq \alpha 2^{p-1} \|f_n - f\|_\infty^p.$$

2 Since  $\lim_{n \rightarrow +\infty} \|f_n - f\|_\infty = 0$ , we deduce that

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \|f(t)\|^p d\mu(t) = 0. \quad \square$$

3 **Definition 2.2** ([11]). A stochastic process  $x : \mathbb{R} \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$  is said to be stochas-  
4 tically bounded in  $p$ th mean sense, if there exists  $C > 0$  such that

$$\mathbb{E}\|x(t)\|^p \leq C, \quad \forall t \in \mathbb{R},$$

5 and a stochastic process  $x : \mathbb{R} \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$  is said to be stochastically continuous in  
6  $p$ th mean sense, if

$$\lim_{t \rightarrow s} \mathbb{E}\|x(t) - x(s)\|^p = 0, \quad \forall s \in \mathbb{R}.$$

7 Denote by  $\mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  the collection of all the stochastically bounded con-  
8 tinuous processes. We can verify that  $(\mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H})), \|\cdot\|_\infty)$  is a Banach space,  
9 where

$$\|x\|_\infty = \sup_{t \in \mathbb{R}} (\mathbb{E}\|x(t)\|^p)^{1/p}.$$

10 **Definition 2.3.** Let  $\mu, \nu \in M$ . A stochastic process  $x$  is said to be  $(\mu, \nu)$ -ergodic  
11 in  $p$ th mean sense, if  $x \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and it satisfies

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E}\|x(t)\|^p d\mu(t) = 0.$$

12 Denote by  $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  the set of all such stochastic processes.

13 **Proposition 2.2.** Let  $\mu, \nu \in M$  satisfy (H<sub>1</sub>). Then  $(\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu), \|\cdot\|_\infty)$  is  
14 a Banach space.

15 To prove this proposition, we just have to use the same arguments as in the  
16 proof of Proposition 2.1. The following lemma gives some properties of ergodicity.

17 **Lemma 1.** Let  $\mu, \nu \in M$ , satisfy (H<sub>1</sub>), and let  $I$  be a bounded interval (eventually  
18  $I = \emptyset$ ). Suppose that  $x \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ . Then the following assertions are  
19 equivalent:

- 20 (i)  $x \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ ;  
21 (ii)  $\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \mathbb{E}\|x(t)\|^p d\mu(t) = 0$ ;  
22 (iii) for any  $\epsilon > 0$ ,

$$\lim_{r \rightarrow +\infty} \frac{\mu\{t \in [-r, r] \setminus I : \mathbb{E}\|x(t)\|^p > \epsilon\}}{\nu\{t \in [-r, r] \setminus I\}} = 0.$$

1 **Proof.** The proof is based on the same arguments as in the proof of [11, Theorem  
2 2.22].

3 (i)  $\Leftrightarrow$  (ii) We denote by  $A = \nu(I)$ ,  $B = \int_I \mathbb{E}\|x(t)\|^p d\mu(t)$  and  $C = \mu(I)$ . Since  
4 the interval  $I$  is bounded and  $x \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ , then  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are finite.

5 For  $r > 0$  such that  $I \subset [-r, r]$  and  $\nu([-r, r] \setminus I) > 0$ ,  
6 we have

$$\begin{aligned} & \frac{1}{\nu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \mathbb{E}\|x(t)\|^p d\mu(t) \\ &= \frac{1}{\nu([-r, r]) - A} \left( \int_{-r}^r \mathbb{E}\|x(t)\|^p d\mu(t) - B \right) \\ &= \frac{\nu([-r, r])}{\nu([-r, r]) - A} \left( \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E}\|x(t)\|^p d\mu(t) - \frac{B}{\nu([-r, r])} \right). \end{aligned}$$

7 Since  $\nu(\mathbb{R}) = +\infty$ , we obtain that

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E}\|x(t)\|^p d\mu(t) = 0.$$

8 Thus, (i) and (ii) are equivalent.

9 (ii)  $\Rightarrow$  (iii) Let  $A_r^\varepsilon = \{t \in [-r, r] \setminus I : \mathbb{E}\|x(t)\|^p > \varepsilon\}$  and  $B_r^\varepsilon = \{t \in$   
10  $[-r, r] \setminus I : \mathbb{E}\|x(t)\|^p \leq \varepsilon\}$ .

11 Assume that (ii) holds. Then, we have

$$\begin{aligned} \frac{1}{\nu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \mathbb{E}\|x(t)\|^p d\mu(t) &\geq \frac{1}{\nu([-r, r] \setminus I)} \int_{A_r^\varepsilon} \mathbb{E}\|x(t)\|^p d\mu(t) \\ &\geq \varepsilon \frac{\mu(A_r^\varepsilon)}{\nu([-r, r] \setminus I)}. \end{aligned}$$

12 Therefore, for  $r$  large enough, we obtain (iii).

13 (iii)  $\Rightarrow$  (ii) Assume that (iii) holds and let  $\varepsilon > 0$ , then from  $(H_1)$ , we have

$$\begin{aligned} & \frac{1}{\nu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \mathbb{E}\|x(t)\|^p d\mu(t) \\ &= \int_{A_r^\varepsilon} \mathbb{E}\|x(t)\|^p d\mu(t) + \int_{B_r^\varepsilon} \mathbb{E}\|x(t)\|^p d\mu(t) \\ &\leq \|x\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r, r] \setminus I)} + \varepsilon \frac{\mu(B_r^\varepsilon)}{\nu([-r, r] \setminus I)} \\ &\leq \|x\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r, r] \setminus I)} + \varepsilon \frac{\mu([-r, r] \setminus I)}{\nu([-r, r] \setminus I)} \end{aligned}$$

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$$\begin{aligned}
&\leq \|x\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r, r] \setminus I)} + \varepsilon \frac{\mu([-r, r]) - C}{\nu([-r, r]) - A} \\
&\leq \|x\|_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r, r] \setminus I)} + \varepsilon \frac{\mu([-r, r])}{\nu([-r, r])} \frac{1 - \frac{C}{\mu([-r, r])}}{1 - \frac{A}{\nu([-r, r])}} \\
&\leq \alpha \varepsilon \frac{1 - \frac{C}{\mu([-r, r])}}{1 - \frac{A}{\nu([-r, r])}}.
\end{aligned}$$

1 Since,  $\mu(\mathbb{R}) = \nu(\mathbb{R}) = +\infty$ , then we get

$$\limsup_{r \rightarrow +\infty} \frac{1}{\nu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \mathbb{E} \|x(t)\|^p d\mu(t) \leq 2\alpha\varepsilon, \quad \forall \varepsilon > 0. \quad (2.1)$$

2 Therefore, (ii) holds.  $\square$

3 **Definition 2.4.** Let  $\mu, \nu \in M$ . A function  $f: \mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}) \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$ ,  $(t, x) \mapsto$   
4  $f(t, x)$  is said to be  $(\mu, \nu)$ -ergodic in  $p$ th sense in  $t \in \mathbb{R}$  uniformly with respect to  
5  $x \in \mathcal{K}$ , if  $f \in \mathcal{B}_C(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}))$  and it satisfies

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|f(t, x)\|^p d\mu(t) = 0,$$

6 where  $\mathcal{K} \subset \mathcal{L}^p(\Omega, \mathbb{H})$  is a compact.

7 We denote by

$$\begin{aligned}
&\xi_p(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu) \\
&= \{f(\cdot, x) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu) \text{ for any } x \in \mathcal{L}^p(\Omega, \mathbb{H})\}
\end{aligned}$$

8 the set of all such functions.

9 **Definition 2.5 ([4]).** Let  $x: \mathbb{R} \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$  be a continuous stochastic process.  $x$   
10 is said to be almost periodic process in  $p$ th mean sense if for each  $\varepsilon > 0$  there exists  
11  $l > 0$  such that for all  $\alpha \in \mathbb{R}$ , there exists  $\tau \in [\alpha, \alpha + l]$  satisfying

$$\sup_{t \in \mathbb{R}} \mathbb{E} \|x(t + \tau) - x(t)\|^p < \varepsilon.$$

12 We denote by  $\mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  the space of all such stochastic processes. It is  
13 easy to verify that  $(\mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H})), \|\cdot\|_\infty)$  is a Banach space.

14 **Definition 2.6 ([11]).** Let  $f: \mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}) \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$  be continuous.  $f$  is said  
15 to be almost periodic in  $p$ th mean sense in  $t \in \mathbb{R}$  uniformly in  $x \in \mathcal{K}$ , where  
16  $\mathcal{K} \subset \mathcal{L}^p(\Omega, \mathbb{H})$  is a compact, if for each  $\varepsilon > 0$ , there exists  $l(\varepsilon, \mathcal{K}) > 0$  such that for  
17 all  $\alpha \in \mathbb{R}$ , there exists  $\tau \in [\alpha, \alpha + l(\varepsilon, \mathcal{K})]$  satisfying

$$\sup_{t \in \mathbb{R}} \mathbb{E} \|f(t + \tau, x) - f(t, x)\|^p < \varepsilon,$$

18 for each stochastic process  $x: \mathbb{R} \rightarrow \mathcal{K}$ .

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1 We denote by  $\mathbf{AP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H})) = \{f(\cdot, x) \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$   
2 for any  $x \in \mathcal{L}^p(\Omega, \mathbb{H})\}$  the space of such stochastic processes.

**Definition 2.7.** Let  $\mu, \nu \in M$ . A continuous stochastic process  $x$  is said to be  $(\mu, \nu)$ -pseudo almost periodic in  $p$ th mean sense, if it can be written as

$$x = x_1 + x_2,$$

3 where  $x_1 \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and  $x_2 \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

4 Denote by  $\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  the set of all such stochastic processes.

5 We can verify that

$$\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu) \subset \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H})).$$

6 We introduce the following new space of doubly measure pseudo almost periodic  
7 functions:

8 **Definition 2.8.** Let  $\mu, \nu \in M$ . A continuous function  $f: \mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}) \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$   
9 is said to be  $(\mu, \nu)$ -pseudo almost periodic in  $p$ th mean sense, if it can be written as

$$f = g + h,$$

10 where  $g \in \mathbf{AP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}))$  and  $h \in \xi_p(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

11 Denote by  $\mathbf{PAP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  the set of all such functions.

12 **Definition 2.9 ([6]).** Let  $\mu_1, \mu_2 \in M$ . If there exist positive constants  $\alpha, \beta$  and  
13 a bounded interval  $I$  (eventually  $I = \emptyset$ ) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A)$$

14 for  $A \in \mathcal{B}$  satisfying  $A \cap I = \emptyset$ , then we say that  $\mu_1$  and  $\mu_2$  are equivalent  $\mu_1 \sim \mu_2$ .

15 **Proposition 2.3.** Let  $\mu_1, \mu_2, \nu_1$  and  $\nu_2 \in M$ . If  $\mu_1$  and  $\nu_1$  are equivalent,  
16 respectively, to  $\mu_2$  and  $\nu_2$ , then

$$\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2), \quad (2.2)$$

17 and

$$\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2). \quad (2.3)$$

18 **Proof.** First, we will prove (2.2). Since  $\mu_1 \sim \mu_2$ ,  $\nu_1 \sim \nu_2$  and  $\mathcal{B}$  is a Lebesgue  
19  $\sigma$ -field, then there exist  $\alpha, \beta, \gamma, \theta > 0$ , such that

$$\alpha\mu_1 \leq \mu_2 \leq \beta\mu_1,$$

20 and

$$\gamma\nu_1 \leq \nu_2 \leq \theta\nu_1.$$



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1 It implies that

$$\begin{aligned} & \frac{\alpha \mu_1\{t \in [-r, r] \setminus I : \mathbb{E}\|f(t)\|^p > \varepsilon\}}{\theta \nu_1\{t \in [-r, r] \setminus I\}} \\ & \leq \frac{\mu_2\{t \in [-r, r] \setminus I : \mathbb{E}\|f(t)\|^p > \varepsilon\}}{\nu_2\{t \in [-r, r] \setminus I\}} \\ & \leq \frac{\beta \mu_1\{t \in [-r, r] \setminus I : \mathbb{E}\|f(t)\|^p > \varepsilon\}}{\gamma \nu_1\{t \in [-r, r] \setminus I\}}. \end{aligned}$$

2 From Lemma 1, we get

$$\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2).$$

3 Using the definition of  $(\mu, \nu)$ -pseudo almost periodicity, we conclude that

$$\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2).$$

4 For  $\mu \in M$  and  $\sigma \in \mathbb{R}$ , we define the positive measure  $\mu_\sigma$  on  $(\mathbb{R}, \mathcal{B})$  by

$$\mu_\sigma(A) = \mu(a + \sigma : a \in A), \quad A \in \mathcal{B}.$$

5 We give the following hypothesis:

6  $(H_2)$  For all  $\sigma \in \mathbb{R}$ , there exist  $\alpha > 0$  and a bounded interval  $I$  such that

$$\mu_\sigma(A) \leq \alpha \mu(A),$$

7 where  $A \in \mathcal{B}$  satisfies  $A \cap I = \emptyset$ . □

8 **Lemma 2** ([6]). *Let  $\mu \in M$ . Then  $\mu$  satisfies  $(H_2)$  if and only if  $\mu$  is equivalent*  
9 *to  $\mu_\sigma$  for all  $\sigma \in \mathbb{R}$ .*

10 **Lemma 3** ([6]). *It follows from hypothesis  $(H_2)$  that*

$$\forall \delta > 0, \quad \limsup_{r \rightarrow +\infty} \frac{\mu([-r - \delta, r + \delta])}{\mu([-r, r])} < +\infty.$$

11 Let  $f \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ . For all  $\alpha \in \mathbb{R}$ , we define  $f_\alpha$  by

$$f_\alpha(t) = f(t + \alpha).$$

12 We say that the subset  $\mathcal{S}$  of  $\mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  is translation invariant, if  $\forall f \in \mathcal{S}$ , we  
13 have  $f_\alpha \in \mathcal{S}$ .

14 **Theorem 2.4.** *Let  $\mu, \nu \in M$  satisfy  $(H_2)$ . Then  $\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  is trans-*  
15 *lation invariant.*

16 **Proof.** First, we need to prove that  $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  is translation invariant. In  
17 other words, if  $f \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then  $f_\tau \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ ,  $\forall \tau \in \mathbb{R}$ .

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1 Let  $\mu_\tau = \mu(\{t + \tau; t \in A\}) \forall A \in \mathcal{B}$ .

$$\begin{aligned} & \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|f(t + \tau)\|^p d\mu(t) \\ &= \frac{\nu[-r + \tau, r + \tau]}{\nu([-r, r])} \cdot \frac{1}{\nu[-r + \tau, r + \tau]} \int_{-r}^r \mathbb{E} \|f(t + \tau)\|^p d\mu(t) \\ &= \frac{\nu[-r + \tau, r + \tau]}{\nu([-r, r])} \cdot \frac{1}{\nu[-r + \tau, r + \tau]} \int_{-r + \tau}^{r + \tau} \mathbb{E} \|f(t)\|^p d\mu_{-\tau}(t) \\ &\leq \frac{\nu[-r - |\tau|, r + |\tau|]}{\nu([-r, r])} \cdot \frac{1}{\nu[-r + \tau, r + \tau]} \int_{-r + \tau}^{r + \tau} \mathbb{E} \|f(t)\|^p d\mu_{-\tau}(t). \end{aligned}$$

2 Since  $\mu$  and  $\nu$  satisfy  $(H_2)$  and according to Lemma 3 we obtain that

$$\begin{aligned} & \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|f(t + \tau)\|^p d\mu(t) \\ &\leq Cte. \frac{\nu[-r - |\tau|, r + |\tau|]}{\nu([-r, r])} \cdot \frac{1}{\nu[-r + \tau, r + \tau]} \int_{-r + \tau}^{r + \tau} \mathbb{E} \|f(t)\|^p d\mu(t). \end{aligned}$$

3 We get

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|f(t + \tau)\|^p d\mu(t) = 0.$$

4 Therefore,  $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  is translation invariant. Since  $\mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  is  
5 translation invariant, then  $\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  is also translation invariant.

6 □

### 7 3. Completeness and Composition Theorem

8 **Theorem 3.1.** *Let  $\mu, \nu \in M$ . Assume that  $f \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  can be  
9 written as  $f = g + h$ , where  $g \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and  $h \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . If  
10  $\mu$  and  $\nu$  satisfy  $(H_2)$ , then*

$$\{g(t), t \in \mathbb{R}\} \subset \overline{\{f(t), t \in \mathbb{R}\}}. \quad (3.1)$$

11 **Proof.** For the proof, we use the same arguments given in [6]. Assume that (3.1)  
12 does not hold. Then, there exists  $t_0 \in \mathbb{R}$  such that

$$g(t_0) \notin \overline{\{f(t), t \in \mathbb{R}\}}.$$

13 Since  $\mu, \nu$  satisfy  $(H_2)$ , and from Theorem 2.4, we deduce that  $\mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$   
14 and  $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  are translation invariant. We can suppose that  $t_0 = 0$ ,  
15 then there exists  $\varepsilon > 0$  such that

$$\mathbb{E} \|f(t) - g(0)\|^p \geq 2^p \varepsilon \quad \forall t \in \mathbb{R}.$$

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1 Note,

$$\mathbb{E}\|f(t) - g(0)\|^p \leq 2^{p-1}\mathbb{E}\|f(t) - g(t)\|^p + 2^{p-1}\mathbb{E}\|g(t) - g(0)\|^p.$$

2 For all  $t \in C_\varepsilon := \{t \in \mathbb{R} : \mathbb{E}\|g(t) - g(0)\|^p < \varepsilon\}$ , we obtain that

$$\begin{aligned} \mathbb{E}\|h(t)\|^p &= \mathbb{E}\|f(t) - g(t)\|^p \\ &\geq 2^{1-p}\mathbb{E}\|f(t) - g(0)\|^p - \mathbb{E}\|g(t) - g(0)\|^p \\ &\geq 2^{1-p}\mathbb{E}\|f(t) - g(0)\|^p \\ &\geq \varepsilon. \end{aligned}$$

3 For all  $i \in \{1, 2, \dots, n\}$  and for all  $t \in \alpha_i + C_\varepsilon$ , where  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  such that

$$\mathbb{R} = \bigcup_{i=1}^n (\alpha_i + C_\varepsilon),$$

4 we have

$$\mathbb{E}\|h(t - \alpha_i)\|^p \geq \varepsilon. \quad (3.2)$$

5 Let  $\psi$  be the function defined by

$$\psi(t) = \sum_{i=1}^n \mathbb{E}\|h(t - \alpha_i)\|^p \quad (3.3)$$

6 (3.2) and (3.3) imply that

$$\psi(t) \geq \varepsilon \quad \forall t \in \mathbb{R}. \quad (3.4)$$

7 Since  $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  is translation invariant, then we get

$$[t \rightarrow h(t - \alpha_i)] \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu), \quad \forall i \in \{1, \dots, n\}.$$

8 Therefore,  $\psi \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , which is absurd by (3.4).  $\square$

9 **Theorem 3.2.** *Let  $\mu, \nu \in M$  satisfy  $(H_2)$ . Then the decomposition of  $(\mu, \nu)$ -pseudo*  
 10 *almost periodic function in the form  $f = g + h$ , where  $g \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and*  
 11  *$h \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , is unique.*

12 This theorem can be proved with the same steps developed in the proof of  
 13 [7, Theorem 2].

14 **Theorem 3.3.** *Let  $\mu, \nu \in M$  satisfy  $(H_1)$  and  $(H_2)$ . Then  $(\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu,$   
 15  $\nu, \|\cdot\|_\infty)$  is a Banach space.*

16 **Proof.** We use the same arguments of the proof of [10, Theorem 2.19], where we  
 17 take the Banach space  $X := \mathcal{L}^p(\Omega, \mathbb{H})$  equipped with the norm  $\|x\|_{\mathcal{L}^p} = (\mathbb{E}\|x\|^p)^{1/p}$ .

18  $\square$

*Stochastic Nicholson's blowflies model with delays*

1 **Theorem 3.4** ([5]). Let  $f: \mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}) \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$ ,  $(t, x) \mapsto f(t, x)$ , be an  
 2 almost periodic process in  $t$  uniformly in  $x \in \mathcal{K}$ , where  $\mathcal{K} \subset \mathcal{L}^p(\Omega, \mathbb{H})$  is a compact.  
 3 Assume that  $f$  satisfies the Lipschitz condition: there exists  $L > 0$  such that for any  
 4  $x, y \in \mathcal{L}^p(\Omega, \mathbb{H})$

$$\mathbb{E}\|f(t, x) - f(t, y)\|^p \leq L \mathbb{E}\|x - y\|^p.$$

5 Then  $t \mapsto f(t, x(t)) \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  for any  $x \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ .

6 **Theorem 3.5.** Let  $\mu, \nu \in M$  satisfy  $(H_2)$ . Assume that  $f \in \mathbf{PAP}(\mathbb{R} \times$   
 7  $\mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . If  $f$  satisfies the Lipschitz condition in the second variable,  
 8 that is, there exists  $L > 0$  such that, for any  $x, y \in \mathcal{L}^p(\Omega, \mathbb{H})$ ,

$$\mathbb{E}\|f(t, x) - f(t, y)\|^p \leq L \mathbb{E}\|x - y\|^p, \quad \forall t \in \mathbb{R}.$$

9 Then  $t \mapsto f(t, x(t)) \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  for any  $x \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

10 **Proof.** Let  $f \in \mathbf{PAP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  and  $x \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}),$   
 11  $\mu, \nu)$ .

12 Then, we can write  $f = g + h$ , where  $g \in \mathbf{AP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}))$  and  
 13  $h \in \xi_p(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . And

$$x = x_1 + x_2,$$

14 with  $x_1 \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ , and  $x_2 \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . We decomposed  $f$  as

$$\begin{aligned} f(t, x(t)) &= g(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + [f(t, x_1(t)) - g(t, x_1(t))] \\ &= g(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + h(t, x_1(t)). \end{aligned}$$

15 To prove this theorem, we need to verify

- 16 (i)  $g(\cdot, x_1(\cdot)) \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ .
- 17 (ii)  $f(\cdot, x(\cdot)) - f(\cdot, x_1(\cdot)) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .
- 18 (iii)  $h(\cdot, x_1(\cdot)) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

19 To demonstrate (i), we use the similar arguments of Step (1) in the proof of [11,  
 20 Theorem 5.7].

21 (ii) Let  $x, x_1 \in \mathcal{L}^p(\Omega, \mathbb{H})$ . By using the Lipschitz condition, we obtain

$$\begin{aligned} &\frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E}\|f(t, x(t)) - f(t, x_1(t))\|^p d\mu(t) \\ &\leq \frac{1}{\nu([-r, r])} L \int_{-r}^r \mathbb{E}\|x(t) - x_1(t)\|^p d\mu(t) \\ &\leq \frac{1}{\nu([-r, r])} L \int_{-r}^r \mathbb{E}\|x_2(t)\|^p d\mu(t). \end{aligned}$$

22 Since  $x_2 \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then  $\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E}\|x_2(t)\|^p d\mu(t) = 0$ .

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1 We deduce that

$$\limsup_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|f(t, x(t)) - f(t, x_1(t))\|^p d\mu(t) = 0.$$

2 Therefore,

$$f(\cdot, x(\cdot)) - f(\cdot, x_1(\cdot)) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu).$$

3 (iii) It remains to demonstrate the ergodicity of  $h(\cdot, x_1(\cdot))$ . First, we have

$$\begin{aligned} \|h(t, x) - h(t, y)\|^p &= \|f(t, x) - g(t, x) - f(t, y) + g(t, y)\|^p \\ &\leq 2^{p-1} \|f(t, x) - f(t, y)\|^p + 2^{p-1} \|g(t, x) - g(t, y)\|^p. \end{aligned}$$

4 By using the Lipschitz condition, we obtain that

$$\begin{aligned} \mathbb{E} \|h(t, x) - h(t, y)\|^p &\leq 2^{p-1} \mathbb{E} \|f(t, x) - f(t, y)\|^p + 2^{p-1} \mathbb{E} \|g(t, x) - g(t, y)\|^p \\ &\leq 2^p .L \mathbb{E} \|x - y\|^p. \end{aligned}$$

5 Since  $\mathcal{K} = \overline{\{x_1(t), t \in \mathbb{R}\}}$  is a compact. Then, for  $\varepsilon > 0$  there exists  $x_1, \dots, x_m \in \mathcal{K}$ ,  
6 such that

7  $\mathcal{K} \subset \cup_{i=1}^m (Bx_i, \frac{\varepsilon}{2^{2p-1}L})$ , where  $B(x_i, \frac{\varepsilon}{2^{2p-1}L}) = \{x \in \mathcal{K}; \|x_i - x\|^p \leq \frac{\varepsilon}{2^{2p-1}L}\}$ .

8 It implies that  $\mathcal{K} \subset \cup_{i=1}^m \{x \in \mathcal{K}, \forall t \in \mathbb{R}, \mathbb{E} \|h(t, x) - h(t, x_i)\|^p \leq \frac{\varepsilon}{2^{p-1}}\}$ .

9 Let  $t \in \mathbb{R}$  and  $x \in \mathcal{K}$ . Then, there exists  $i_0 \in \{1, \dots, m\}$  such that

$$\mathbb{E} \|h(t, x) - h(t, x_{i_0})\|^p \leq \frac{\varepsilon}{2^{p-1}}.$$

10 We get

$$\begin{aligned} \mathbb{E} \|h(t, x_1(t))\|^p &\leq 2^{p-1} \mathbb{E} \|h(t, x_1(t)) - h(t, x_{i_0})\|^p + 2^{p-1} \mathbb{E} \|h(t, x_{i_0})\|^p \\ &\leq \varepsilon + 2^{p-1} \sum_{i=1}^m \mathbb{E} \|h(t, x_i)\|^p. \end{aligned}$$

11 Since  $\forall i \in \{1, \dots, m\}$  we have

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|h(t, x_i)\|^p d\mu(t) = 0.$$

12 It follows that

$$\limsup_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|h(t, x_1(t))\|^p d\mu(t) \leq \varepsilon, \quad \forall \varepsilon > 0.$$

13 We deduce that

$$\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|h(t, x_1(t))\|^p d\mu(t) = 0.$$

14 Finally,

$$t \mapsto h(t, x_1(t)) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu),$$

15 which ends the proof.  $\square$

1 **4. Existence of  $(\mu, \nu)$ -pseudo Almost Periodic Mild Solutions for**  
 2 **Nonlinear Stochastic Differential Equations With Delay**

3 In this section, we state to prove the existence and exponential stability of  $(\mu, \nu)$ -  
 4 pseudo almost periodic solutions in  $p$ th mean sense of Eq. (1.1). Together with  
 5 Eq. (1.1), we consider

$$x'(t) = -a(t)x(t), \quad t \in \mathbb{R}. \quad (4.1)$$

6 Denote by  $w(t, s) = e^{-\int_s^t a(r)dr}$  for  $s \leq t$ . The solutions of (4.1) satisfying the initial  
 7 condition  $x(t_0) = x_0$  are in the form  $x(t) = w(t, t_0)x_0$ .

8 **Definition 4.1.** A  $\mathcal{F}_t$ -progressively measurable stochastic process  $\{x(t)\}_{t \in \mathbb{R}}$  is  
 9 called a mild solution of Eq. (1.1) if it satisfies the corresponding stochastic integral  
 10 equation

$$x(t) = w(t, t_0)x_0 + \int_{t_0}^t w(t, s)h_x(s)ds + \int_{t_0}^t w(t, s)f(s, x(s - \tau))dW(s), \quad (4.2)$$

11 for all  $t, t_0 \in \mathbb{R}$  such that  $t \geq t_0$ , where  $h_x(s) = \sum_{i=1}^m \alpha_i(s)x(s -$   
 12  $\tau_i(s))e^{-\beta_i(s) \int_{-\infty}^0 k_i(r)x(s+r)dr} ds$ .

13 **Lemma 4 ([18]).** Let  $S: [0, T] \times \Omega \rightarrow \ell(\mathcal{L}^p(\Omega, \mathbb{H}))$  be a  $\mathcal{F}_t$ -adapted measurable  
 14 stochastic process satisfying

$$\int_0^T \mathbb{E} \|S(t)\|^2 dt < \infty \quad a.s,$$

15 where  $\ell(\mathcal{L}^p(\Omega, \mathbb{H}))$  designates the space of all continuous linear operators from  
 16  $\mathcal{L}^p(\Omega, \mathbb{H})$  to itself. Then  $\forall p \geq 1$ , there exists a constant  $C_p > 0$  such that

$$\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t S(s)dW(s) \right\|^p \leq C_p \mathbb{E} \left( \int_0^T \|S(s)\|^2 ds \right)^{p/2}, \quad T > 0.$$

17 We introduce the following assumptions:

- 18 (A<sub>1</sub>)  $a: \mathbb{R} \rightarrow \mathbb{R}_+$  is almost periodic,  
 19 (A<sub>2</sub>)  $\alpha_i, \beta_i: \mathbb{R} \rightarrow \mathbb{R}_+$  are  $(\mu, \nu)$ -pseudo almost periodic in  $p$ th-mean sense, for  $i =$   
 20  $1, 2, \dots, n$ ,  
 21 (A<sub>3</sub>)  $f: \mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}) \rightarrow \mathcal{L}^p(\Omega, \mathbb{H})$  is uniformly  $(\mu, \nu)$ -pseudo almost periodic pro-  
 22 cess,  
 23 (A<sub>4</sub>) the delay kernel  $k_i(t) \in C(\mathbb{R}^-, \mathbb{R}_+)$  and  $\int_{-\infty}^0 k_i(t)dt = 1$ ,  
 24 (A<sub>5</sub>) there exists  $L > 0$  such that

$$\mathbb{E} \|f(t, x) - f(t, y)\|^p \leq L \mathbb{E} \|x - y\|^p, \quad \forall t \in \mathbb{R}, \quad x, y \in \mathcal{L}^p(\Omega, \mathbb{H}),$$

- 25 (A<sub>6</sub>) there exists a positive constant  $K$  such that  $\|f\|_\infty \leq \frac{1}{C_p} K^p \left(\frac{\alpha}{2}\right)^{\frac{p}{2}}$  and  
 26  $(\sum_{i=1}^m \bar{\alpha}_i)^p \leq (2\underline{\alpha})^p$ .

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1 **Theorem 4.1.** *If  $f \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then*

2 (i)  $t \mapsto \int_{-\infty}^t w(t, s)f(s)ds \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

3 (ii)  $t \mapsto \int_{-\infty}^t w(t, s)f(s - \tau)dW(s) \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

4 **Proof.** (i) We know that  $f \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then it can be decomposed  
5 as  $f = g + \Phi$  where  $g \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ , and  $\Phi \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

6 Denote by  $(\Lambda x)(t) = \int_{-\infty}^t w(t, s)g(s)ds$  and  $(\Gamma x)(t) = \int_{-\infty}^t w(t, s)\Phi(s)ds$ .

7 We need to verify that  $(\Lambda x)(t) \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and  $(\Gamma x)(t) \in$   
8  $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

9 • Our first step consists of proving that  $(\Lambda x)(t)$  and  $(\Gamma x)(t)$  are stochastically  
10 continuous.

11 Let  $t_0 \in \mathbb{R}$  fixed. Assume that  $\alpha = s - t + t_0$ , and from Holder's inequality,  
12 we get

$$\begin{aligned}
& \mathbb{E}\|(\Lambda x)(t) - (\Lambda x)(t_0)\|^p \\
&= \mathbb{E}\left\|\int_{-\infty}^t w(t, s)g(s)ds - \int_{-\infty}^{t_0} w(t_0, s)g(s)ds\right\|^p \\
&= \mathbb{E}\left\|\int_{-\infty}^{t_0} w(t, \alpha + t - t_0)g(\alpha + t - t_0)d\alpha - \int_{-\infty}^{t_0} w(t_0, s)g(s)ds\right\|^p \\
&\leq 2^{p-1}\mathbb{E}\left\|\int_{-\infty}^{t_0} [w(t, s + t - t_0) - w(t_0, s)]g(s + t - t_0)ds\right\|^p \\
&\quad + 2^{p-1}\mathbb{E}\left\|\int_{-\infty}^{t_0} w(t_0, s)[g(s + t - t_0) - g(s)]ds\right\|^p \\
&= I_1 + I_2 \\
I_1 &\leq 2^{p-1}\mathbb{E}\left[\int_{-\infty}^{t_0} \|w(t, s + t - t_0) - w(t_0, s)\|^{\frac{p-1}{p}} \|w(t, s + t - t_0)\right. \\
&\quad \left. - w(t_0, s)\|^{\frac{1}{p}} \|g(s + t - t_0)\| ds\right]^p \\
&\leq 2^{p-1}\mathbb{E}\left[\left(\int_{-\infty}^{t_0} (\|w(t, s + t - t_0) - w(t_0, s)\|^{\frac{p-1}{p}})^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}}\right. \\
&\quad \left.\times \left(\int_{-\infty}^{t_0} (\|w(t, s + t - t_0) - w(t_0, s)\|^{\frac{1}{p}} \|g(s + t - t_0) - g(s)\|)^p ds\right)^{\frac{1}{p}}\right]^p \\
&\leq 2^{p-1}\left(\int_{-\infty}^{t_0} \|w(t, s + t - t_0) - w(t_0, s)\| ds\right)^{p-1}
\end{aligned}$$

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$$\begin{aligned}
& \times \int_{-\infty}^{t_0} \|w(t, s+t-t_0) - w(t_0, s)\| \cdot \mathbb{E} \|g(s+t-t_0) - g(s)\|^p ds \\
& \leq 2^{2p-2} \left( \int_{-\infty}^{t_0} e^{-\underline{a}(t_0-s)} ds \right)^{p-1} \\
& \quad \times \int_{-\infty}^{t_0} \|w(t, s+t-t_0) - w(t_0, s)\| \mathbb{E} \|g(s+t-t_0)\|^p ds \\
& \leq 2^{2p-2} \left( \frac{1}{\underline{a}} \right)^{p-1} \times \int_{-\infty}^{t_0} e^{-\underline{a}(t_0-s)} \mathbb{E} \|g(s+t-t_0)\|^p ds.
\end{aligned}$$

- 1 Let  $\{t_n\}$  be a real arbitrary sequence such that  $t_n \rightarrow t_0$  as  $n \rightarrow +\infty$ . Since  $g \in$   
2  $\mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ , we have, for  $n$  large enough, one has

$$e^{-\underline{a}(t_0-s)} \mathbb{E} \|g(s+t_n-t_0)\|^p \leq e^{-\underline{a}(t_0-s)} \|g\|_\infty^p.$$

- 3 Furthermore,

$$\int_{-\infty}^{t_0} e^{-\underline{a}(t_0-s)} \|g\|_\infty^p ds < \infty.$$

- 4 In addition,

$$\|w(t_n, s+t_n-t_0) - w(t_0, s)\|^2 \rightarrow 0, \quad n \rightarrow +\infty.$$

- 5 Then, from Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n \rightarrow +\infty} \int_{-\infty}^{t_0} \|w(t_n, s+t_n-t_0) - w(t_0, s)\| \|g(s+t-t_0)\|^p ds = 0.$$

- 6 Hence,

$$\lim_{t \rightarrow t_0} \int_{-\infty}^{t_0} \|w(t, s+t-t_0) - w(t_0, s)\| \|g(s+t-t_0)\|^p ds = 0.$$

- 7 For the second integral:

$$\begin{aligned}
I_2 &= 2^{p-1} \mathbb{E} \left\| \int_{-\infty}^{t_0} w(t_0, s) [g(s+t-t_0) - g(s)] ds \right\|^p \\
&\leq \mathbb{E} \left[ \int_{-\infty}^{t_0} \|w(t_0, s)\|^{\frac{p-1}{p}} \|w(t_0, s)\|^{\frac{1}{p}} \|g(s+t-t_0) - g(s)\| ds \right]^p \\
&\leq \mathbb{E} \left[ \left( \int_{-\infty}^{t_0} (\|w(t_0, s)\|^{\frac{p-1}{p}})^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \right. \\
&\quad \left. \times \left( \int_{-\infty}^{t_0} (\|w(t_0, s)\|^{\frac{1}{p}} \|g(s+t-t_0) - g(s)\|)^p ds \right)^{\frac{1}{p}} \right]^p
\end{aligned}$$



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$$\begin{aligned} &\leq \left( \int_{-\infty}^{t_0} \|e^{-\underline{a}(t_0-s)}\| ds \right)^{p-1} \\ &\quad \times \int_{-\infty}^{t_0} \|e^{-\underline{a}(t_0-s)}\| \cdot \mathbb{E} \|g(s+t-t_0) - g(s)\|^p ds \\ &\leq \frac{1}{\underline{a}^{p-1}} \int_{-\infty}^{t_0} e^{-\underline{a}(t_0-s)} \mathbb{E} \|g(s+t-t_0) - g(s)\|^p ds. \end{aligned}$$

1 Using similar arguments, we obtain

$$\lim_{t \rightarrow t_0} \int_{-\infty}^{t_0} e^{-\underline{a}(t_0-s)} \mathbb{E} \|g(s+t-t_0) - g(s)\|^p ds = 0.$$

2 Hence

$$\lim_{t \rightarrow t_0} \mathbb{E} \|(\Lambda x)(t) - (\Lambda x)(t_0)\|^p = 0.$$

3 By analogous steps, we verify that  $(\Gamma x)$  is also stochastically continuous.

4 • We now prove the almost periodicity of  $\Lambda x$ . Note that the integral

$$\int_{-\infty}^t w(t,s)g(s)ds$$

5 is absolutely convergent, and

$$\left\| \int_{-\infty}^t w(t,s)g(s)ds \right\| \leq \frac{1}{\underline{a}} \|g\|_{\infty}.$$

6 Since  $a$  and  $g$  are two almost periodic functions then  $\forall \varepsilon > 0, \exists l > 0, \forall \alpha \in$   
7  $\mathbb{R}$ , there exists  $\delta \in [\alpha, \alpha + l]$  satisfying

$$\sup_{t \in \mathbb{R}} \mathbb{E} \|g(t+\delta) - g(t)\|^p < \frac{\varepsilon \cdot \underline{a}^p}{2^p}, \text{ and } \sup_{t \in \mathbb{R}} |a(t+\delta) - a(t)| < \frac{\varepsilon \cdot \underline{a}^2}{2 \cdot \|g\|_{\infty}}.$$

8 In other hand, assume that  $\sigma = s - \delta$  and using the same steps developed above,  
9 we obtain

$$\begin{aligned} &\mathbb{E} \|(\Lambda x)(t+\delta) - (\Lambda x)(t)\|^p \\ &= \mathbb{E} \left\| \int_{-\infty}^{t+\delta} w(t+\delta, s)g(s)ds - \int_{-\infty}^t w(t, s)g(s)ds \right\|^p \\ &= \mathbb{E} \left\| \int_{-\infty}^t w(t+\delta, \sigma+\delta)g(\sigma+\delta)d\sigma - \int_{-\infty}^t w(t, s)g(s)ds \right\|^p \end{aligned}$$

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$$\begin{aligned}
&= \mathbb{E} \left\| \int_{-\infty}^t [w(t+\delta, s+\delta) - w(t, s)]g(s+\delta)ds \right. \\
&\quad \left. + \int_{-\infty}^t w(t, s)[g(s+\delta) - g(s)]ds \right\|^p \\
&\leq 2^{p-1} \mathbb{E} \left\| \int_{-\infty}^t [w(t+\delta, s+\delta) - w(t, s)]g(s+\delta)ds \right\|^p \\
&\quad + 2^{p-1} \mathbb{E} \left\| \int_{-\infty}^t w(t, s)[g(s+\delta) - g(s)]ds \right\|^p \\
&\leq 2^{p-1} \|g\|_{\infty}^p \left( \int_{-\infty}^t \|w(t+\delta, s+\delta) - w(t, s)\| ds \right)^p \\
&\quad + 2^{p-1} \left( \int_{-\infty}^t w(t, s) ds \right)^{p-1} \\
&\quad \times \int_{-\infty}^t w(t, s) \mathbb{E} \|g(s+\delta) - g(s)\|^p ds.
\end{aligned}$$

1 There exists  $c \in [0, 1]$  such that

$$\begin{aligned}
&\mathbb{E} \|(\Lambda x)(t+\delta) - (\Lambda x)(t)\|^p \\
&\leq 2^{p-1} \|g\|_{\infty}^p \left( \int_{-\infty}^t e^{-\int_{s+\delta}^{t+\delta} a(r)dr} - e^{-\int_s^t a(r)dr} ds \right)^p \\
&\quad + 2^{p-1} \left( \int_{-\infty}^t e^{-\int_s^t a(r)dr} ds \right)^{p-1} \times \int_{-\infty}^t e^{-\int_s^t a(r)dr} \mathbb{E} \|g(s+\delta) - g(s)\|^p ds \\
&\leq 2^{p-1} \|g\|_{\infty}^p \left[ \int_{-\infty}^t e^{-(\int_s^t a(r+\delta)dr + c(\int_s^t a(r)dr - \int_s^t a(r+\delta)dr))} \right. \\
&\quad \left. \times \left( \int_s^t a(r+\delta)dr - \int_s^t a(r)dr \right) ds \right]^p \\
&\quad + 2^{p-1} \left( \int_{-\infty}^t e^{-\underline{a}(t-s)} ds \right)^{p-1} \times \int_{-\infty}^t e^{-\underline{a}(t-s)} \mathbb{E} \|g(s+\delta) - g(s)\|^p ds.
\end{aligned}$$

2 Using the almost periodicity of  $a$  and  $g$ , we get

$$\begin{aligned}
&\mathbb{E} \|(\Lambda x)(t+\delta) - (\Lambda x)(t)\|^p \\
&\leq 2^{p-1} \|g\|_{\infty}^p \left( \int_{-\infty}^t e^{-\underline{a}(t-s)} e^{-c \cdot \varepsilon \cdot \frac{\underline{a}^2}{2 \cdot \|g\|_{\infty}} (t-s)} \varepsilon \cdot \frac{\underline{a}^2}{2 \cdot \|g\|_{\infty}} (t-s) ds \right)^p \\
&\quad + 2^{p-1} \varepsilon \left( \int_{-\infty}^t e^{-\underline{a}(t-s)} ds \right)^p
\end{aligned}$$

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$$\begin{aligned} &\leq 2^{p-1} \|g\|_\infty^p \frac{1}{\underline{a}^{2p}} \cdot \varepsilon^p \cdot \frac{\underline{a}^{2p}}{2^p \cdot \|g\|_\infty^p} + 2^{p-1} \frac{1}{\underline{a}^p} \cdot \frac{\varepsilon \cdot \underline{a}^p}{2^p} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

- 1 • It remains to verify that  $(\Gamma x) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . First, we have

$$\begin{aligned} &\frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\Gamma x)(t)\|^p d\mu(t) \\ &= \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \left\| \int_{-\infty}^t w(t, s) \Phi(s) ds \right\|^p d\mu(t) \\ &\leq \frac{1}{\nu([-r, r])} \int_{-r}^r \left[ \mathbb{E} \int_{-\infty}^t \|w(t, s) \Phi(s)\|^p ds \right]^p d\mu(t). \end{aligned}$$

- 2 From Holder's inequality and Fubini's theorem, we get

$$\begin{aligned} &\frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\Gamma x)(t)\|^p d\mu(t) \\ &\leq \frac{1}{\nu([-r, r])} \int_{-r}^r \left( \int_{-\infty}^t e^{\int_s^t e^{-a(r)dr} ds} \right)^{p-1} \\ &\quad \times \left( \int_{-\infty}^t e^{\int_s^t e^{-a(r)dr} ds} \mathbb{E} \|\Phi(s)\|^p ds \right) d\mu(t) \\ &\leq \frac{1}{\underline{a}^{p-1}} \frac{1}{\nu([-r, r])} \int_{-r}^r \int_{-\infty}^t e^{-\underline{a}(t-s)} \mathbb{E} \|\Phi(s)\|^p ds d\mu(t) \\ &\leq \frac{1}{\underline{a}^{p-1}} \cdot \frac{1}{\nu([-r, r])} \int_{-r}^r \int_{\mathbb{R}} \mathbf{1}_{]-\infty, t]}(s) e^{-\underline{a}(t-s)} \mathbb{E} \|\Phi(s)\|^p ds d\mu(t) \\ &\leq \frac{1}{\underline{a}^{p-1}} \cdot \frac{1}{\nu([-r, r])} \int_{\mathbb{R}} \int_{-r}^r \mathbf{1}_{]-\infty, t]}(s) e^{-\underline{a}(t-s)} \mathbb{E} \|\Phi(s)\|^p d\mu(t) ds. \end{aligned}$$

- 3 Let  $v = t - s$ . Then we obtain

$$\begin{aligned} &\frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\Gamma x)(t)\|^p d\mu(t) \\ &\leq \frac{1}{\underline{a}^{p-1}} \int_0^{+\infty} \frac{e^{-\underline{a}v}}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|\Phi(t-v)\|^p d\mu(t) dv. \end{aligned}$$

- 4 Moreover, we have

$$\left| \frac{e^{-\underline{a}v}}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|\Phi(t-v)\|^p d\mu(t) \right| \leq e^{-\underline{a}v} \|\Phi\|_\infty^p.$$

1 Since  $\Phi \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  which is translation invariant, and from Lebesgue's  
2 dominated convergence theorem, we deduce that

$$\begin{aligned} & \lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\Gamma x)(t)\|^p d\mu(t) \\ & \leq \frac{1}{a^{p-1}} \int_0^{+\infty} \left( e^{-av} \lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|\Phi(t-v)\|^p d\mu(t) \right) dv \\ & = 0. \end{aligned}$$

3 Thus,  $(\Gamma x)(t) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . Finally, (i) holds.

4 (ii) Since  $f \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then it can be written as  $f = \Theta + \varphi$   
5 where  $\Theta \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ , and  $\varphi \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

6 Let  $\mathcal{S}_1 x(t) = \int_{-\infty}^t w(t, s) \Theta(s - \tau) dW(s)$  and  $\mathcal{S}_2 x(t) = \int_{-\infty}^t w(t, s) \varphi(s - \tau) dW(s)$ . We  
7 must demonstrate that  $\mathcal{S}_1 x \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and  $\mathcal{S}_2 x \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

8 •• On the first time, we verify that  $(\mathcal{S}_1 x)(t)$  is stochastically continuous, we  
9 take an arbitrary number  $t_0 \in \mathbb{R}$ . Let  $\alpha = s - t + t_0$ , then we obtain

$$\begin{aligned} & \mathbb{E} \|(\mathcal{S}_1 x)(t) - (\mathcal{S}_1 x)(t_0)\|^p \\ & = \mathbb{E} \left\| \int_{-\infty}^t w(t, s) \Theta(s - \tau) dW(s) \right. \\ & \quad \left. - \int_{-\infty}^{t_0} w(t_0, s) \Theta(s - \tau) dW(s) \right\|^p \\ & = \mathbb{E} \left\| \int_{-\infty}^{t_0} w(t, \alpha + t - t_0) \Theta(\alpha + t - t_0 - \tau) dW(\alpha + t - t_0) \right. \\ & \quad \left. - \int_{-\infty}^{t_0} w(t_0, s) \Theta(s - \tau) dW(s) \right\|^p. \end{aligned}$$

10 Suppose that

$$\widetilde{W}(\alpha) = W(\alpha + t - t_0) - W(t - t_0).$$

11 We note that  $W$  and  $\widetilde{W}$  are two Wiener processes and have the same distribution.

12 Using Lemma 4, we obtain

$$\begin{aligned} & \mathbb{E} \|(\mathcal{S}_1 x)(t) - (\mathcal{S}_1 x)(t_0)\|^p \\ & = \mathbb{E} \left\| \int_{-\infty}^{t_0} w(t, \alpha + t - t_0) \Theta(\alpha + t - t_0 - \tau) d\widetilde{W}(\alpha) \right. \\ & \quad \left. - \int_{-\infty}^{t_0} w(t_0, s) \Theta(s - \tau) dW(s) \right\|^p \end{aligned}$$

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$$\begin{aligned}
&= \mathbb{E} \left\| \int_{-\infty}^{t_0} w(t, \alpha + t - t_0) \Theta(\alpha + t - t_0 - \tau) d\widetilde{W}(\alpha) \right. \\
&\quad \left. - \int_{-\infty}^{t_0} w(t_0, s) \Theta(s - \tau) d\widetilde{W}(s) \right\|^p \\
&\leq 2^{p-1} \mathbb{E} \left\| \int_{-\infty}^{t_0} [w(t, s + t - t_0) - w(t_0, s)] \Theta(s + t - t_0 - \tau) d\widetilde{W}(s) \right\|^p \\
&\quad + 2^{p-1} \mathbb{E} \left\| \int_{-\infty}^{t_0} w(t_0, s) [\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)] d\widetilde{W}(s) \right\|^p \\
&\leq 2^{p-1} C_p \mathbb{E} \left[ \int_{-\infty}^{t_0} \|w(t, s + t - t_0) - w(t_0, s)\|^2 \|\Theta(s + t - t_0 - \tau)\|^2 ds \right]^{\frac{p}{2}} \\
&\quad + 2^{p-1} C_p \mathbb{E} \left[ \int_{-\infty}^{t_0} \|w(t_0, s)\|^2 \|\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)\|^2 ds \right]^{\frac{p}{2}} \\
&= I_1 + I_2.
\end{aligned}$$

1 From Holder's inequality, we get

$$\begin{aligned}
I_1 &\leq 2^{p-1} C_p \mathbb{E} \left( \int_{-\infty}^{t_0} \|w(t, s + t - t_0) - w(t_0, s)\|^{2 \cdot \frac{p-2}{p}} \cdot \|w(t, s + t - t_0) \right. \\
&\quad \left. - w(t_0, s)\|^{\frac{4}{p}} \times \|\Theta(s + t - t_0 - \tau)\|^2 ds \right)^{\frac{p}{2}} \\
&\leq 2^{p-1} C_p \mathbb{E} \left[ \left( \int_{-\infty}^{t_0} (\|w(t, s + t - t_0) - w(t_0, s)\|^{2 \cdot \frac{p-2}{p}})^{\frac{p}{p-2}} ds \right)^{\frac{p-2}{p}} \right. \\
&\quad \left. \times \left( \int_{-\infty}^{t_0} (\|w(t, s + t - t_0) - w(t_0, s)\|^{\frac{4}{p}} \|\Theta(s + t - t_0 - \tau)\|^2)^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \right]^{\frac{p}{2}} \\
&\leq 2^{2p-3} C_p \left( \int_{-\infty}^{t_0} e^{-2a(t_0-s)} ds \right)^{\frac{p-2}{2}} \\
&\quad \times \int_{-\infty}^{t_0} \|w(t, s + t - t_0) - w(t_0, s)\|^2 \mathbb{E} \|\Theta(s + t - t_0 - \tau)\|^p ds \\
&\leq 2^{2p-3} C_p \left( \frac{1}{2a} \right)^{\frac{p-2}{2}} \times \int_{-\infty}^{t_0} e^{-2a(t_0-s)} \mathbb{E} \|\Theta(s + t - t_0 - \tau)\|^p ds.
\end{aligned}$$

2 Let  $\{t_n\}$  be a real arbitrary sequence such that  $t_n \rightarrow t_0$  as  $n \rightarrow +\infty$ . Since  $\Theta \in$   
3  $\mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ , we have, for  $n$  large enough one has

$$e^{-2a(t_0-s)} \mathbb{E} \|\Theta(s + t_n - t_0 - \tau)\|^p \leq e^{-2a(t_0-s)} \|\Theta\|_{\infty}^p.$$

1 Furthermore,

$$\int_{-\infty}^{t_0} e^{-2\underline{a}(t_0-s)} \|\Theta\|_{\infty}^p ds < \infty.$$

2 In addition,

$$\|w(t_n, s + t_n - t_0) - w(t_0, s)\|^2 \rightarrow 0, \quad n \rightarrow +\infty.$$

3 Then, from Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n \rightarrow +\infty} \int_{-\infty}^{t_0} \|w(t_n, s + t_n - t_0) - w(t_0, s)\|^2 \|\Theta(s + t - t_0 - \tau)\|^p ds = 0.$$

4 Hence,

$$\lim_{t \rightarrow t_0} \int_{-\infty}^{t_0} \|w(t, s + t - t_0) - w(t_0, s)\|^2 \|\Theta(s + t - t_0 - \tau)\|^p ds = 0.$$

5 Next, we have to evaluate  $I_2$ .

$$\begin{aligned} I_2 &\leq 2^{p-1} C_p \mathbb{E} \left( \int_{-\infty}^{t_0} \|w(t_0, s)\|^{2 \cdot \frac{p-2}{p}} \cdot \|w(t_0, s)\|^{\frac{4}{p}} \right. \\ &\quad \left. \times \|\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)\|^2 ds \right)^{\frac{p}{2}} \\ &\leq 2^{p-1} C_p \mathbb{E} \left[ \left( \int_{-\infty}^{t_0} (\|w(t_0, s)\|^{2 \cdot \frac{p-2}{p}})^{\frac{p}{p-2}} ds \right)^{\frac{p-2}{p}} \right. \\ &\quad \left. \times \left( \int_{-\infty}^{t_0} (\|w(t_0, s)\|^{\frac{4}{p}} \|\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)\|^2)^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \right]^{\frac{p}{2}} \\ &\leq 2^{p-1} C_p \left( \int_{-\infty}^{t_0} e^{-2\underline{a}(t_0-s)} ds \right)^{\frac{p-2}{2}} \\ &\quad \times \int_{-\infty}^{t_0} e^{-2\underline{a}(t_0-s)} \mathbb{E} \|\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)\|^p ds \\ &\leq 2^{p-1} C_p (2\underline{a})^{\frac{2-p}{2}} \int_{-\infty}^{t_0} e^{-2\underline{a}(t_0-s)} \mathbb{E} \|\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)\|^p ds. \end{aligned}$$

6 By the similar arguments as above, we obtain

$$\lim_{t \rightarrow t_0} \int_{-\infty}^{t_0} e^{-2\underline{a}(t_0-s)} \mathbb{E} \|\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)\|^p ds = 0.$$

7 Therefore,

$$\lim_{t \rightarrow t_0} \mathbb{E} \|(\mathcal{S}_1 x)(t) - (\mathcal{S}_1 x)(t_0)\|^p = 0.$$

8 In this way, we have shown that  $(\mathcal{S}_1 x)(t)$  is stochastically continuous. By an analogous argument, we verify that  $(\mathcal{S}_2 x)(t)$  is also continuous.

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- 1     •• We know that  $\Theta \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and  $a \in \mathbf{AP}(\mathbb{R}, \mathbb{R}_+)$ . So,  $\forall \varepsilon > 0, \exists l >$   
 2      $0, \forall \alpha \in \mathbb{R}, \exists \delta \in [\alpha, \alpha + l]$  satisfying

$$\sup_{t \in \mathbb{R}} \mathbb{E} \|\Theta(t + \delta) - \Theta(t)\|^p < \varepsilon \quad \text{and} \quad \sup_{t \in \mathbb{R}} |a(t + \delta) - a(t)| < \varepsilon.$$

- 3     Let  $\sigma = s - \delta$ , and  $\widetilde{W}(\sigma) = W(\sigma + \delta) - W(\delta)$ . From Lemma 4, we obtain

$$\begin{aligned} & \mathbb{E} \|(\mathcal{S}_1 x)(t + \delta) - (\mathcal{S}_1 x)(t)\|^p \\ &= \mathbb{E} \left\| \int_{-\infty}^{t+\delta} w(t + \delta, s) \Theta(s - \tau) dW(s) - \int_{-\infty}^t w(t, s) \Theta(s - \tau) dW(s) \right\|^p \\ &= \mathbb{E} \left\| \int_{-\infty}^t w(t + \delta, \sigma + \delta) \Theta(\sigma + \delta - \tau) dW(\sigma + \delta) \right. \\ & \quad \left. - \int_{-\infty}^t w(t, s) \Theta(s - \tau) dW(s) \right\|^p \\ &= \mathbb{E} \left\| \int_{-\infty}^t w(t + \delta, \sigma + \delta) \Theta(\sigma + \delta - \tau) d\widetilde{W}(\sigma) \right. \\ & \quad \left. - \int_{-\infty}^t w(t, s) \Theta(s - \tau) d\widetilde{W}(s) \right\|^p \\ &\leq 2^{p-1} C_p \mathbb{E} \left[ \int_{-\infty}^t \|w(t + \delta, s + \delta) - w(t, s)\|^2 \|\Theta(s + \delta - \tau)\|^2 ds \right]^{\frac{p}{2}} \\ & \quad + 2^{p-1} C_p \mathbb{E} \left[ \int_{-\infty}^t \|w(t, s)\|^2 \|\Theta(s + \delta - \tau) - \Theta(s - \tau)\|^2 ds \right]^{\frac{p}{2}} \\ &\leq 2^{p-1} C_p \left( \int_{-\infty}^t \|w(t + \delta, s + \delta) - w(t, s)\|^2 ds \right)^{\frac{p-2}{2}} \\ & \quad \times \int_{-\infty}^t \|w(t + \delta, s + \delta) - w(t, s)\|^2 \mathbb{E} \|\Theta(s + \delta - \tau)\|^p ds \\ & \quad + 2^{p-1} C_p \left( \int_{-\infty}^t \|w(t, s)\|^2 ds \right)^{\frac{p-2}{2}} \\ & \quad \times \int_{-\infty}^t \|w(t, s)\|^2 \mathbb{E} \|\Theta(s + \delta - \tau) - \Theta(s - \tau)\|^p ds \\ &\leq 2^{p-1} C_p \|\Theta\|_\infty^p \left( \int_{-\infty}^t \|w(t + \delta, s + \delta) - w(t, s)\|^2 ds \right)^{\frac{p}{2}} \end{aligned}$$

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$$+ 2^{p-1} C_p \left( \int_{-\infty}^t \|w(t, s)\|^2 ds \right)^{\frac{p-2}{2}}$$

$$\times \int_{-\infty}^t \|w(t, s)\|^2 \cdot \mathbb{E} \|\Theta(s + \delta - \tau) - \Theta(s - \tau)\|^p ds.$$

- 1 Using similar steps as above and the almost periodicity of  $a$  and  $\Theta$ , we deduce there  
2 exists  $c \in [0, 1]$  such that

$$\mathbb{E} \|(\Lambda x)(t + \delta) - (\Lambda x)(t)\|^p$$

$$\leq 2^{p-1} C_p \|\Theta\|_{\infty}^p \mathbb{E} \left[ \int_{-\infty}^t e^{-2(\int_s^t a(r+\delta)dr + c(\int_s^t a(r)dr - \int_s^t a(r+\delta)dr))} \right. \\ \left. \times \left( \int_s^t a(r + \delta)dr - \int_s^t a(r)dr \right)^2 ds \right]^{\frac{p}{2}}$$

$$+ 2^{p-1} C_p \left( \int_{-\infty}^t e^{-2a(t-s)} ds \right)^{\frac{p-2}{2}}$$

$$\times \int_{-\infty}^t e^{-2a(t-s)} \cdot \mathbb{E} \|\Theta(s + \delta - \tau) - \Theta(s - \tau)\|^p ds$$

$$\leq 2^{p-1} C_p \|\Theta\|_{\infty}^p \frac{1}{2^{\frac{p}{2}}} \frac{1}{(\underline{a})^p} \cdot \varepsilon^p + 2^{p-1} C_p \frac{1}{(2\underline{a})^{\frac{p}{2}}} \varepsilon$$

$$\leq 2^{\frac{p-2}{2}} C_p \left( \frac{\|\Theta\|_{\infty}^p}{\underline{a}^p} + \frac{1}{\underline{a}^{\frac{p}{2}}} \right) \varepsilon.$$

- 3 Consequently,  $(\mathcal{S}_1 x)(t)$  is almost periodic.  
4 **••** In order to complete the proof we still have to show the  $(\mu, \nu)$ -ergodicity of  
5  $(\mathcal{S}_2 x)(t)$ . By Lemma 4, we get

$$\frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|\mathcal{S}_2 x(t)\|^p d\mu(t)$$

$$= \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \left\| \int_{-\infty}^t w(t, s) \varphi(s - \tau) dW(s) \right\|^p d\mu(t)$$

$$\leq C_p \frac{1}{\nu([-r, r])} \int_{-r}^r \left[ \mathbb{E} \int_{-\infty}^t \|w(t, s) \varphi(s - \tau)\|^2 ds \right]^{\frac{p}{2}} d\mu(t).$$

- 6 From Holder's inequality and Fubini's theorem, we infer that

$$\frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\mathcal{S}_2 x)(t)\|^p d\mu(t)$$

$$\leq C_p \frac{1}{\nu([-r, r])} \int_{-r}^r \left( \int_{-\infty}^t \|w(t, s)\|^2 ds \right)^{\frac{p-2}{2}}$$



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$$\begin{aligned}
& \times \int_{-\infty}^t \|w(t, s)\|^2 \cdot \mathbb{E} \|\varphi(s - \tau)\|^p ds d\mu(t) \\
& \leq C_p \frac{1}{\nu([-r, r])} \int_{-r}^r \left( \int_{-\infty}^t e^{-2\underline{a}(t-s)} ds \right)^{\frac{p-2}{2}} \\
& \quad \times \left( \int_{-\infty}^t e^{-2\underline{a}(t-s)} \mathbb{E} \|\varphi(s - \tau)\|^p ds \right) d\mu(t) \\
& \leq C_p \frac{1}{(2\underline{a})^{\frac{p-2}{2}}} \frac{1}{\nu([-r, r])} \int_{-r}^r \int_{-\infty}^t e^{-2\underline{a}(t-s)} \mathbb{E} \|\varphi(s - \tau)\|^p ds d\mu(t) \\
& \leq C_p \frac{1}{(2\underline{a})^{\frac{p-2}{2}}} \cdot \frac{1}{\nu([-r, r])} \int_{-r}^r \int_{\mathbb{R}} \mathbf{1}_{]-\infty, t]}(s) e^{-2\underline{a}(t-s)} \mathbb{E} \|\varphi(s - \tau)\|^p ds d\mu(t) \\
& \leq C_p \frac{1}{(2\underline{a})^{\frac{p-2}{2}}} \cdot \frac{1}{\nu([-r, r])} \int_{\mathbb{R}} \int_{-r}^r \mathbf{1}_{]-\infty, t]}(s) e^{-2\underline{a}(t-s)} \mathbb{E} \|\varphi(s - \tau)\|^p d\mu(t) ds.
\end{aligned}$$

1 Let  $v = t - s$ . Then we obtain

$$\begin{aligned}
& \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\mathcal{S}_2 x)(t)\|^p d\mu(t) \\
& \leq C_p \frac{1}{(2\underline{a})^{\frac{p-2}{2}}} \int_0^{+\infty} \frac{e^{-2\underline{a}v}}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|\varphi(t - v - \tau)\|^p d\mu(t) dv.
\end{aligned}$$

2 Add to that, we have

$$\left| \frac{e^{-2\underline{a}v}}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|\varphi(t - v - \tau)\|^p d\mu(t) \right| \leq e^{-2\underline{a}v} \|\varphi\|_{\infty}^p.$$

3 Since,  $\varphi \in \xi_p(\mathbb{R}, \mathbb{H}, \mu, \nu)$  which is translation invariant, and from the Lebesgue  
4 dominated convergence theorem, we deduce that

$$\begin{aligned}
& \lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\mathcal{S}_2 x)(t)\|^p d\mu(t) \\
& \leq C_p \frac{1}{(2\underline{a})^{\frac{p-2}{2}}} \int_0^{+\infty} e^{-2\underline{a}v} \\
& \quad \times \lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|\varphi(t - v, x(t - v - \tau))\|^p d\mu(t) dv \\
& = 0,
\end{aligned}$$

5 which completes the proof.  $\square$

6 **Lemma 5.** *If  $f, g \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then  $f \times g \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .*

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**Proof.** By definition of  $(\mu, \nu)$ -almost periodic functions,  $f = f_1 + f_2$ ,  $g = g_1 + g_2$  where  $f_1, f_2 \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$  and  $f_2, g_2 \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . Evidently,

$$fg = f_1g_1 + f_1g_2 + f_2g_1 + f_2g_2.$$

1 In order to demonstrate the almost periodicity of  $f_1g_1$ , we need only to invest similar  
2 arguments to those in [8, Lemma 2] with the function  $t \rightarrow \mathbb{E}\|f(t)\|^p$ . Moreover, by  
3 the  $(\mu, \nu)$ -ergodicity of functions  $f_2$  and  $g_2$

$$\begin{aligned} & \lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E}\|f_1g_2(t) + f_2g_1(t) + f_2g_2(t)\|^p d\mu(t) \\ &= \frac{3^{p-1}}{\nu([-r, r])} \int_{-r}^r [\|f_1\|_\infty \mathbb{E}\|g_2(t)\|^p \\ & \quad + \|g_1\|_\infty \mathbb{E}\|f_2(t)\|^p + \|f_2\|_\infty \mathbb{E}\|g_2(t)\|^p] d\mu(t) \\ &= 0 \quad \text{as } r \rightarrow +\infty. \end{aligned} \quad \square$$

4 **Lemma 6.** *If  $(H_2)$ ,  $(A_4)$  hold and  $x(\cdot) \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then*

$$t \rightarrow \int_{-\infty}^0 k(s)x(t+s)ds \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu).$$

5 **Proof.** Since  $x \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ , then there exist  $x_1 \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$   
6 and  $x_2 \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  such that  $x = x_1 + x_2$ . Let  $\psi(t) = \int_{-\infty}^0 k(s)x_1(t+s)$   
7  $ds + \int_{-\infty}^0 k(s)x_2(t+s)ds = \psi_1(t) + \psi_2(t)$ . We can prove the almost periodicity  
8 of  $\psi_1$  along the direction of the proof of [1, Lemma 4.3] where we take the Banach  
9 space  $\mathcal{L}^p(\Omega, \mathbb{H})$  equipped with the norm  $\|\cdot\|_{\mathcal{L}^p(\Omega, \mathbb{H})}$ . Next, we have to show the  
10  $(\mu, \nu)$ -ergodicity of  $\psi_2$ . From Holder's inequality and Fubini's theorem, we get

$$\begin{aligned} & \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E}\|(\psi_2)(t)\|^p d\mu(t) \\ &= \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \left\| \int_{-\infty}^0 k(s)x(t+s)ds \right\|^p d\mu(t) \\ &\leq \frac{1}{\nu([-r, r])} \int_{-r}^r \left[ \mathbb{E} \int_{-\infty}^0 \|k(s)x(t+s)\|^p ds \right]^p d\mu(t). \\ &\leq \frac{1}{\nu([-r, r])} \int_{-r}^r \left( \int_{-\infty}^0 k(s)ds \right)^{p-1} \\ & \quad \times \left( \int_{-\infty}^0 k(s)\mathbb{E}\|x(t+s)\|^p ds \right) d\mu(t) \\ &\leq \frac{1}{\sigma^{p-1}} \frac{1}{\nu([-r, r])} \int_{-r}^r \int_{-\infty}^0 k(s)\mathbb{E}\|x(t+s)\|^p ds d\mu(t) \end{aligned}$$

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$$\begin{aligned} &\leq \frac{1}{\sigma^{p-1}} \cdot \frac{1}{\nu([-r, r])} \int_{-r}^r \int_{\mathbb{R}} \mathbf{1}_{]-\infty, 0]}(s) k(s) \mathbb{E} \|x(t+s)\|^p ds d\mu(t) \\ &\leq \frac{1}{\nu([-r, r])} \int_{\mathbb{R}} \int_{-r}^r \mathbf{1}_{]-\infty, 0]}(s) k(s) \mathbb{E} \|x(t+s)\|^p d\mu(t) ds \\ &\leq \int_{-\infty}^0 \frac{k(s)}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|x(t+s)\|^p d\mu(t) ds. \end{aligned}$$

1 Moreover, we have

$$\left| \frac{k(s)}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|x(t+s)\|^p d\mu(t) \right| \leq k(s) \|x\|_{\infty}^p.$$

2 Since  $x \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$  which is translation invariant, and from Lebesgue's  
3 dominated convergence theorem, we deduce that

$$\begin{aligned} &\lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|(\psi_2)(t)\|^p d\mu(t) \\ &\leq \int_0^{+\infty} \left( k(s) \lim_{r \rightarrow +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^r \mathbb{E} \|x(t+s)\|^p d\mu(t) \right) ds \\ &= 0. \end{aligned}$$

4 Thus,  $\psi_2(t) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ . □

5 **Theorem 4.2.** *Let  $\mu, \nu \in M$  satisfy (H<sub>1</sub>) and (H<sub>2</sub>). If the conditions (A<sub>1</sub>)–(A<sub>5</sub>)*  
6 *are met then*

- 7 (1) *There exists a unique  $(\mu, \nu)$ -pseudo almost periodic solution  $x(t)$  of Nichol-*  
8 *son's blowflies model with infinite delays (1.1) in the region  $T^* = \{x \in$*   
9  *$\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu), \mathbb{E} \|x(t)\|^p \leq K^p, t \in \mathbb{R}\}$ .*  
10 (2) *The solution  $x(t)$  in the region  $T^*$  is exponentially stable.*

11 **Proof.** We define the nonlinear operator  $F$  in  $T^*$

$$\begin{aligned} Fx(t) &= \int_{-\infty}^t w(t, s) h_x(s) ds + \int_{-\infty}^t w(t, s) f(s - \tau) dW(s) \\ &= F_1 x(t) + F_2 x(t), \end{aligned}$$

12 where  $h_x(s) = \sum_{i=1}^m \alpha_i(s) x(s - \tau_i(s)) e^{-\beta_i(s)} \int_{-\infty}^0 k_i(r) x(s+r) dr ds$ .

1 **Step 1.** We will show that  $F$  is self-mapping from  $T^*$  to  $T^*$ .

$$\begin{aligned} \mathbb{E}\|Fx(t)\|^p &\leq 2^{p-1}\mathbb{E}\left\|\int_{-\infty}^t w(t,s)h_x(s)ds\right\|^p \\ &\quad + 2^{p-1}\mathbb{E}\left\|\int_{-\infty}^t w(t,s)f(s,x(s-\tau))dW(s)\right\|^p \\ &\leq 2^{p-1}[R_1 + R_2]. \end{aligned}$$

2 From Holder's inequality, we get

$$\begin{aligned} R_1 &\leq \mathbb{E}\left\|\int_{-\infty}^t w(t,s)\sum_{i=1}^m\alpha_i(s)x(s-\tau_i(s))e^{-\beta_i(s)}\int_{-\infty}^0 k_i(r)x(s+r)dr ds\right\|^p \\ &\leq \mathbb{E}\left[\int_{-\infty}^t\left\|w(t,s)\sum_{i=1}^m\alpha_i(s)x(s-\tau_i(s))\right\| ds\right]^p \\ &\leq \mathbb{E}\left[\int_{-\infty}^t\|w(t,s)\|^{\frac{p-1}{p}}\|w(t,s)\|^{\frac{1}{p}}\left\|\sum_{i=1}^m\alpha_i(s)x(s-\tau_i(s))\right\| ds\right]^p \\ &\leq \left[\int_{-\infty}^t e^{-\underline{a}(t-s)} ds\right]^{p-1}\int_{-\infty}^t e^{-\underline{a}(t-s)}\mathbb{E}\left\|\sum_{i=1}^m\alpha_i(s)x(s-\tau_i(s))\right\|^p ds \\ &\leq \frac{1}{\underline{a}^p}\left(\sum_{i=1}^m\bar{\alpha}_i\right)^p K^p. \end{aligned}$$

3 Similarly, using Lemma 4 and Holder's inequality, we have

$$\begin{aligned} R_2 &= \mathbb{E}\left\|\int_{-\infty}^t w(t,s)f(s,x(s-\tau))dW(s)\right\|^p \\ &\leq C_p\mathbb{E}\left[\int_{-\infty}^t\|w(t,s)\|^2\|f(s,x(s-\tau))\|^2 ds\right]^{p/2} \\ &\leq C_p\left[\int_{-\infty}^t e^{-2\underline{a}(t-s)}\right]^{\frac{p-2}{2}}\int_{-\infty}^t e^{-2\underline{a}(t-s)}\mathbb{E}\|f(s,x(s-\tau))\|^p \\ &\leq C_p\left(\frac{1}{2\underline{a}}\right)^{\frac{p-2}{2}}\cdot\frac{1}{2\underline{a}}\|f\|_{\infty}^p \\ &\leq C_p\left(\frac{1}{2\underline{a}}\right)^{\frac{p}{2}}\|f\|_{\infty}^p. \end{aligned}$$

4 By (A<sub>6</sub>), one has

$$\mathbb{E}\|Fx(t)\|^p \leq 2^{p-1}\frac{1}{\underline{a}^p}\left(\sum_{i=1}^m\bar{\alpha}_i\right)^p K^p + 2^{p-1}C_p\left(\frac{1}{2\underline{a}}\right)^{\frac{p}{2}}\|f\|_{\infty}^p \leq K^p.$$

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1 Now, we will prove that  $F_1x(t) = \int_{-\infty}^t w(t, s) \sum_{i=1}^m \alpha_i(s)x(s - \tau_i(s))$   
 2  $e^{-\beta_i(s) \int_{-\infty}^s k_i(r)x(s+r)dr} ds \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

3 According to Theorem 2.4  $x(\cdot - \tau) \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu) \forall \tau \in \mathbb{R}$ . Using  
 4 Lemma 6 and the fact that  $(x \mapsto e^{-x})$  is Lipschitzian, then by Theorem 3.4, we  
 5 have

$$t \mapsto \sum_{i=1}^m \alpha_i(t)x(t - \tau_i(t))e^{-\beta_i(t) \int_{-\infty}^t k_i(s)x(t+s)ds}$$

× belongs to  $\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ .

6  $\forall x, y \in T^*$  we have

$$\begin{aligned} & \mathbb{E}\|(Fx)(t) - (Fy)(t)\|^p \\ &= \mathbb{E}\left\| \int_{-\infty}^t w(t, s)h_x(s)ds + \int_{-\infty}^t w(t, s)f(s, x(s - \tau))dW(s) \right. \\ & \quad \left. - \int_{-\infty}^t w(t, s)h_y(s)ds - \int_{-\infty}^t w(t, s)f(s, y(s - \tau))dW(s) \right\|^p \\ &= \mathbb{E}\left\| \int_{-\infty}^t w(t, s)[h_x(s) - h_y(s)]ds \right. \\ & \quad \left. + \int_{-\infty}^t w(t, s)[f(s, x(s - \tau)) - f(s, y(s - \tau))]dW(s) \right\|^p \\ &\leq 2^{p-1}\mathbb{E}\left\| \int_{-\infty}^t w(t, s)[h_x(s) - h_y(s)]ds \right\|^p \\ & \quad + 2^{p-1}\mathbb{E}\left\| \int_{-\infty}^t w(t, s)[f(s, x(s - \tau)) - f(s, y(s - \tau))]dW(s) \right\|^p \\ &= 2^{p-1}(F^1 + F^2). \end{aligned}$$

$$\begin{aligned} F^1 &= \mathbb{E}\left\| \int_{-\infty}^t w(t, s)[h_x(s) - h_y(s)]ds \right\|^p \\ &\leq \left( \int_{-\infty}^t \|w(t, s)\| ds \right)^{p-1} \\ & \quad \times \int_{-\infty}^t \|w(t, s)\| \cdot \mathbb{E}\|h_x(s) - h_y(s)\|^p ds \\ &\leq \frac{1}{\underline{a}^{p-1}} \cdot \int_{-\infty}^t e^{-\underline{a}(t-s)} \mathbb{E}\|h_x(s) - h_y(s)\|^p ds. \end{aligned}$$

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1 Moreover,  $\forall u, v \in [0, +\infty[$  we have  $|e^u - e^v| < |u - v|$ , so we have

$$\begin{aligned} & \mathbb{E} \| [h_x(s) - h_y(s)] \|^p \\ &= \mathbb{E} \left\| \sum_{i=1}^m \alpha_i(s) [x(s - \tau_i(s)) e^{-\beta_i(s) \int_{-\infty}^0 k_i(r) x(s+r) dr} \right. \\ & \quad \left. - y(s - \tau_i(s)) e^{-\beta_i(s) \int_{-\infty}^0 k_i(r) y(s+r) dr}] \right\|^p \\ &= \mathbb{E} \left\| \sum_{i=1}^m \alpha_i(s) [(x(s - \tau_i(s)) - y(s - \tau_i(s))) e^{-\beta_i(s) \int_{-\infty}^0 k_i(r) x(s+r) dr} \right. \\ & \quad \left. + y(s - \tau_i(s)) (e^{-\beta_i(s) \int_{-\infty}^0 k_i(r) x(s+r) dr} - e^{-\beta_i(s) \int_{-\infty}^0 k_i(r) y(s+r) dr})] \right\|^p \\ &\leq \left[ \sum_{i=1}^m \bar{\alpha}_i \|x - y\|_{\infty} (1 + K \bar{\beta}_i) \right]^p. \end{aligned}$$

2 Then,

$$F^1 \leq \frac{1}{\underline{a}^p} \left[ \sum_{i=1}^m \bar{\alpha}_i (1 + K \bar{\beta}_i) \right]^p \|x - y\|_{\infty}^p.$$

3 Second, using Lemma 4, Holder theorem and (A<sub>5</sub>), we get

$$\begin{aligned} F^2 &= \mathbb{E} \left\| \int_{-\infty}^t w(t, s) [f(s, x(s - \tau)) - f(s, y(s - \tau))] dW(s) \right\|^p \\ &\leq C_p \mathbb{E} \left( \int_{-\infty}^t \|w(t, s) [f(s, x(s - \tau)) - f(s, y(s - \tau))]\|^2 ds \right)^{\frac{p}{2}} \\ &\leq C_p \left( \int_{-\infty}^t e^{-2\underline{a}(t-s)} ds \right)^{\frac{p-2}{2}} \\ & \quad \times \int_{-\infty}^t e^{-2\underline{a}(t-s)} \mathbb{E} \|f(s, x(s - \tau)) - f(s, y(s - \tau))\|^p ds \\ &\leq C_p \cdot \frac{1}{(2\underline{a})^{p/2}} \cdot L \sup_{t \in \mathbb{R}} \mathbb{E} \|x(t - \tau) - y(t - \tau)\|^p. \end{aligned}$$

4 Consequently,

$$\mathbb{E} \|(Fx)(t) - (Fy)(t)\|^p \leq \left[ \frac{2^{p-1}}{\underline{a}^p} \left( \sum_{i=1}^m \bar{\alpha}_i (1 + K \bar{\beta}_i) \right)^p + \frac{2^{\frac{p-2}{2}}}{\underline{a}^{p/2}} C_p \cdot L \right] \|x - y\|_{\infty}^p.$$

5 Then,  $F$  is a contracting operator in  $T^*$ . Consequently, by the fixed point theorem,  
6 there exists a unique  $(\mu, \nu)$ -almost periodic solution  $x(t)$  of Eq. (1.1) in  $T^*$ . Let  $y(t)$

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1 be a solution of Eq. (1.1) with an initial condition  $y(t_0) = y_0$ , we prove that all  
2 solutions converge exponentially to the  $(\mu, \nu)$ -pseudo almost periodic solution  $x(t)$ .

$$\begin{aligned}
& \mathbb{E}\|x(t) - y(t)\|^p \\
& \leq 3^{p-1} e^{-p\underline{a}(t-t_0)} \mathbb{E}\|x_0 - y_0\|^p + 3^{p-1} \frac{1}{\underline{a}^{p-1}} \\
& \quad \cdot \int_{t_0}^t e^{-\underline{a}(t-s)} \mathbb{E}\|h_x(s) - h_y(s)\|^p ds \\
& \quad + 3^{p-1} C_p \cdot \frac{1}{(2\underline{a})^{\frac{p-2}{2}}} \\
& \quad \times \int_{t_0}^t e^{-2\underline{a}(t-s)} \mathbb{E}\|f(s, x(s-\tau)) - f(s, y(s-\tau))\|^p ds \\
& \leq 3^{p-1} e^{-p\underline{a}(t-t_0)} \mathbb{E}\|x_0 - y_0\|^p \\
& \quad + 3^{p-1} \frac{1}{\underline{a}^{p-1}} \cdot \left[ \sum_{i=1}^m \bar{\alpha}_i \left( 1 + K \bar{\beta}_i \frac{1}{\sigma} \right) \right]^p \\
& \quad \times \int_{t_0}^t e^{-\underline{a}(t-s)} \mathbb{E}\|x(s - \tau_i(s)) - y(s - \tau_i(s))\|^p ds \\
& \quad + 3^{p-1} C_p \cdot \frac{1}{(2\underline{a})^{\frac{p-2}{2}}} \cdot L \times \int_{t_0}^t e^{-2\underline{a}(t-s)} \mathbb{E}\|x(s - \tau) - y(s - \tau)\|^p ds.
\end{aligned}$$

3 Put  $\mathbb{E}\|v(t)\|^p = \mathbb{E}\|x(t) - y(t)\|^p e^{p\underline{a}t}$ . According to the Gronwall–Bellman lemma,  
4 we obtain

$$\begin{aligned}
& \mathbb{E}\|x(t) - y(t)\|^p \\
& \leq 3^{p-1} \mathbb{E}\|x_0 - y_0\|^p e^{[-p\underline{a} + (\frac{3}{\underline{a}})^{p-1} \cdot (\sum_{i=1}^m \bar{\alpha}_i (1 + K \bar{\beta}_i \frac{1}{\sigma}))^p + \frac{3^{p-1} C_p}{(2\underline{a})^{(p-2)/2}} \cdot L](t-t_0)}.
\end{aligned}$$

5 The unique  $(\mu, \nu)$ -pseudo almost periodic solution of (1.1)  $x(t)$  is exponentially  
6 stable.  $\square$

## 7 5. Example

8 Consider the following stochastic Nicholson's blowflies model with a harvesting  
9 term:

$$\begin{aligned}
dx(t) = & \left[ -(9 + \cos^2 t)x(t) + (\sin t + \sin 2\pi\sqrt{2}t + e^{-|t|})x(t - \sin\sqrt{3}t) \right. \\
& \left. \times e^{-(1 + |\sin\sqrt{3}t| + e^{-|t|})} \int_{-\infty}^0 2e^{2(s-|s|)x(t+s)} ds \right]
\end{aligned}$$

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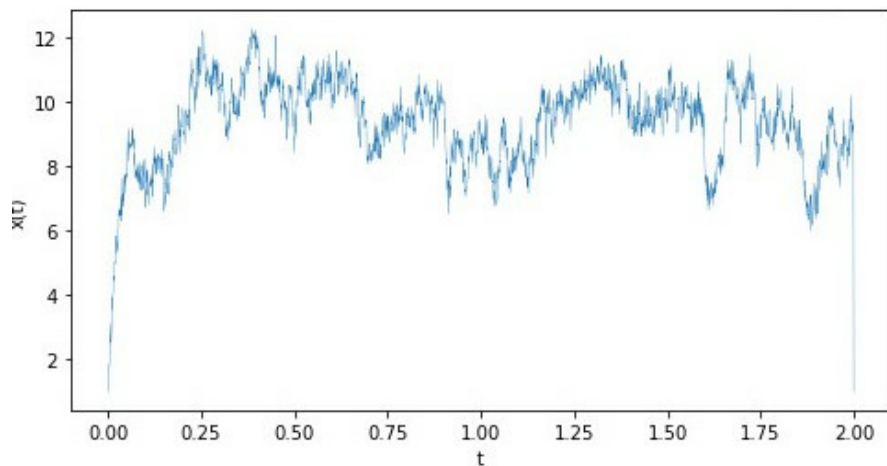


Fig. 1. Double measure pseudo almost periodic solution of stochastic Nicholson's blowflies model (5.1).

$$\begin{aligned}
 & + \left( \frac{\sin 2\pi t + \sin \sqrt{2}t}{4} + \frac{1}{2}e^{-|t|} \right) x(t - \sin \sqrt{11}t) \\
 & \times e^{-(1+|\sin \sqrt{11}t|+e^{-|t|})} \int_{-\infty}^0 3e^{3(s-|s|x(t+s))} ds \Big] dt \\
 & - \left( \frac{(23 + \cos^2 t)|\sin t|}{3000} + e^{-|t|} \right) x(t - e)dw(t). \tag{5.1}
 \end{aligned}$$

1 Suppose that  $\mu$  is a positive measure, where its Randon–Nikodym derivative is

$$\rho(t) = \begin{cases} \exp(t) & \text{if } t \leq 0, \\ 1 & \text{if } t > 0, \end{cases} \tag{5.2}$$

2 and  $\nu$  is the Lebesgue measure. Then from [6]  $\mu$  and  $\nu$  satisfy  $(H_1)$  and  $(H_2)$ .

3 From Theorem 4.2, we deduce that Eq. (5.1) has a unique  $(\mu, \nu)$ -pseudo almost

4 periodic solution in  $T^*$ , where

$$T^* = \{x \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu), \mathbb{E}\|x(t)\|^p \leq e^3, t \in \mathbb{R}\}.$$

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