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# Iterative Trefftz method for three-dimensional electromagnetic waves simulation

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Oslo, Norway, June 6, 2022



## Adimensional Maxwell problem

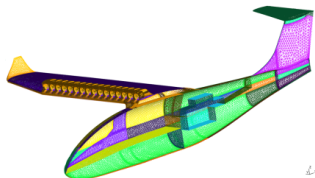
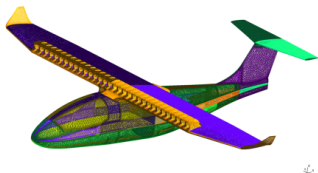
**Studied problem:**

$$\begin{cases} \nabla \times \mathbf{H} = ik\varepsilon_r \mathbf{E} \\ \nabla \times \mathbf{E} = -ik\mu_r \mathbf{H}, \text{ on } \Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^3$  is a three-dimensional domain,  $\varepsilon_r$  and  $\mu_r$  are the relative permittivity and permeability.

**Objective:** computation on very large domains

→ **up to 200 wavelengths =  $200\lambda = 8 \times 10^6$  elements in the mesh.**



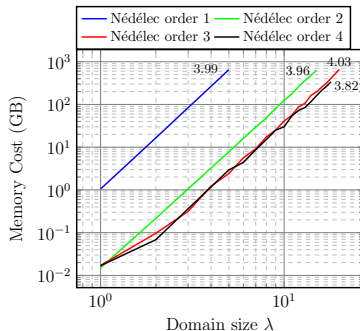
Ampere plane geometry (DEMR ONERA).

## Memory limits of a classic solver<sup>i</sup>

**Numerical dispersion** when large domain size with respect to the wavelength  $\Leftrightarrow L \gg \lambda$

→ Efficient optimisation cannot overcome the **memory issue**.

→ Need to develop **low memory cost methods** based on **domain decomposition**.



Memory used by the Nédélec solver in function of the size of the domain, **memory** =  $\lambda^4$ .

### Example of high order Nédélec finite elements

	Memory	Sockets	Domain size	Power	Cost	Hours.Sockets
Laptop	32 GB	1	$10\lambda$	200 W	0.034 €/h	0.25
// Computing	1 TB	16	$24\lambda$	3.2 KW	0.544 €/h	19.9
HPC	320 TB	5000	$100\lambda$	1 MW	170 €/h	> 25000
HPC	5120 TB	80000	$200\lambda$	16 MW	2720 €/h	> 800000

<sup>i</sup>F. Ihlenburg and I. Babuska, *Finite element solution of the Helmholtz equation with high wave number part II: the hp version of the FEM*, *SIAM Journal on Numerical Analysis*, 34 (1), pp. 315–358 (1997).

## Trefftz method construction steps

Trefftz method advantages<sup>ii</sup>

- Non dispersive basis functions
- Adapted to domain decomposition

### 1. Virtual work principle on an element $K$

$$W_K = -ik_0 \int_K \varepsilon_r \mathbf{E}^K \cdot \overline{\mathbf{E}'^K} + \mu_r \mathbf{H}^K \cdot \overline{\mathbf{H}'^K} dK.$$

### 2. The solution and the basis functions satisfy Maxwell on each element $K$

$$\left\{ \begin{array}{l} W_K = - \int_K \underbrace{\nabla \times \mathbf{H}^K}_{ik_0 \varepsilon_r \mathbf{E}^K} \cdot \overline{\mathbf{E}'^K} - \underbrace{\nabla \times \mathbf{E}^K}_{-ik_0 \mu_r \mathbf{H}^K} \cdot \overline{\mathbf{H}'^K} dK, \\ W_K = \int_K \mathbf{E}^K \cdot \underbrace{\nabla \times \mathbf{H}'^K}_{ik_0 \varepsilon_r \mathbf{E}'^K} - \mathbf{H}^K \cdot \underbrace{\nabla \times \mathbf{E}'^K}_{-ik_0 \mu_r \mathbf{H}'^K} dK. \end{array} \right.$$

### 3. Thanks to the general Stokes formula, a variational formulation is imposed on boundaries: reciprocity formula,

$$\sum_K \int_{\partial K} \left( \mathbf{n}_K \times \gamma_T \mathbf{H}^K \right) \cdot \gamma_T \overline{\mathbf{E}'^K} + \gamma_T \mathbf{E}^K \cdot \left( \mathbf{n}_K \times \gamma_T \overline{\mathbf{H}'^K} \right) = 0.$$

<sup>ii</sup>R. Hiptmair, A. Moiola and I. Perugia, A survey of Trefftz methods for the Helmholtz equation, In Building bridges: connections and challenges in modern approaches to numerical partial differential equations, pp. 237–279. Springer (2016).

## Trefftz variational formulation

1. Find  $\mathbf{x} = (\mathbf{E}, \mathbf{H}) \in \mathbb{X}$  such as for all  $\mathbf{x}' = (\mathbf{E}', \mathbf{H}') \in \mathbb{X}$ ,

$$\mathbf{a}(\mathbf{x}, \mathbf{x}') = \ell(\mathbf{x}'),$$

with  $\mathbb{X}$  a **discontinuous space of local solutions of Maxwell system**.

2. Clever choices of **penalization forms combinations** ensure a **strict coercivity**
- Riemann solvers

$$\gamma_t \hat{\mathbf{E}} = \{\{\gamma_t \mathbf{E}\}\} - \frac{\llbracket \gamma \times \mathbf{H} \rrbracket}{2},$$

$$\gamma \times \hat{\mathbf{H}} = \{\{\gamma_t \mathbf{H}\}\} + \frac{\llbracket \gamma_t \mathbf{E} \rrbracket}{2},$$

- UWVF methods,
- Upwind schemes.

3. **Consistency**: accuracy of the best approximation is ensured by the choice of  $\mathbb{X}$ .
4. Trefftz method is **convergent** and provides **convergence theory for the iterative method** <sup>iii</sup>.

⇒ **How to construct  $\mathbb{X}$  ?**

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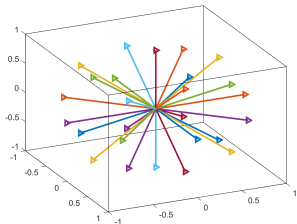
<sup>iii</sup>A. Buffa, P. Monk, Error estimates for the ultra weak variational formulation of the Helmholtz equation, Mathematical Modelling and Numerical Analysis 42 (6) (2008) 925940.

## Basis functions space

An electromagnetic plane wave  $\mathbf{E} = \vec{p} e^{ik\vec{d}\cdot\vec{x}}$  is defined by :

- a direction of propagation  $\vec{d} = (d_x, d_y, d_z)$ ,
- a polarization  $\vec{p} = (p_x, p_y, p_z)$  orthogonal to  $\vec{d}$ .

### 3D discrete space



A plane waves basis in a cube, composed of normalized basis functions  $\mathbf{w}^j$ , for  $j = 1, N$ .

- On an element  $K$ , the numerical wave  $\mathbf{x}^K := (\mathbf{E}^K, \mathbf{H}^K)$  is

$$\mathbf{E}^K = \sum_{j=1}^N u_j \underbrace{\vec{p}_j e^{ik\vec{d}_j\cdot\vec{x}}}_{\mathbf{w}_j},$$

$$\mathbf{H}^K = \sum_{j=1}^N u_j \left( \vec{p}_j \times \vec{d}_j \right) e^{ik\vec{d}_j\cdot\vec{x}}.$$

- The number of plane waves  $N$  is the number of degrees of freedom for each element  $K$ .
- What is the optimal number of plane waves ? <sup>i</sup> <sup>ii</sup>  
→ Basis reduction strategy

<sup>i</sup>T. Luostari, T. Huttunen, P. Monk, Improvements for the ultra weak variational formulation, Int. J. Numer. Meth. Engng 94 (2013) 598624.

<sup>ii</sup>S. Congreve, J. Gedicke, I. Perugia, Numerical investigation of the conditioning for plane wave discontinuous Galerkin methods, Vol. 126 of Lecture Notes in Computational Science and Engineering, Springer, 2019.

## General matrix shape

$$\mathbf{A}[\mathbf{x}] = \mathbf{F},$$

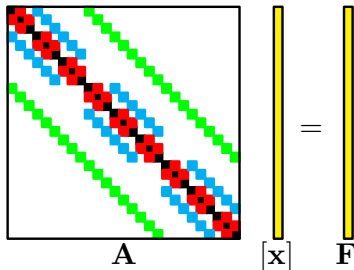
where

$$\mathbf{A}_{i,j} := \mathbf{a}(\mathbf{w}^j, \mathbf{w}^i)$$

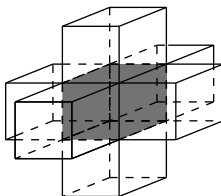
$$\mathbf{F}_i := \mathbf{l}(\mathbf{w}^i) \quad \text{for } i, j = 1, N_{\text{dof}}.$$

$$N_{\text{dof}} := N \times N_{\text{elem}}$$

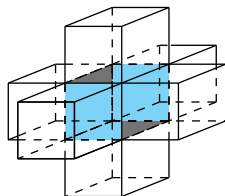
$$\text{size}(\mathbf{A}) = N^2 \times 7 \times N_{\text{elem}}$$



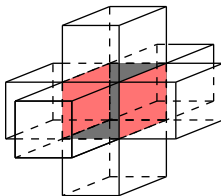
Element assembling of  $\mathbf{A}$  for  $N_{\text{elem}} = 27 = 3^3$  cubes.



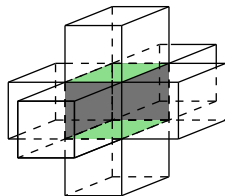
Self-interaction



Front and back interactions



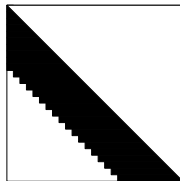
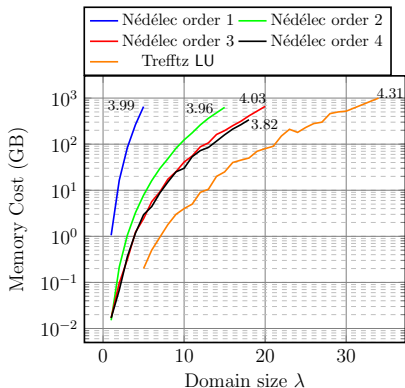
Left and right interactions



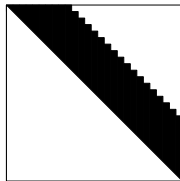
Bottom and top interactions



## Memory limit of the direct solver



L



U

Domain size	$5\lambda$	$10\lambda$	$20\lambda$	$100\lambda$	$200\lambda$
Number of elements $N_{\text{elem}}$	512	1000	8000	$10^6$	$8 \times 10^6$
LU memory	268 MB	4.3 GB	69 GB	43 TB	688 TB

Table: LU factorization memory in function of the domain size.

## GMRES method alternative

**Coercivity property**  $\operatorname{Re}(a(\mathbf{x}, \mathbf{x}')) > 0 \rightarrow$  suitable framework for GMRES.

$$[\mathbf{x}]^{N_{\text{kry}}} = \text{GMRES}(\mathbf{A}, \mathbf{F}, N_{\text{kry}})$$

General Minimal RESidual (GMRES) approximates the solution  $\mathbf{x}^{N_{\text{kry}}} \in \mathbb{K}^{N_{\text{kry}}}$  of

$$\text{Find } \mathbf{x}^{N_{\text{kry}}} \in \mathbb{K}^{N_{\text{kry}}}, \mathbf{a}(\mathbf{x}^{N_{\text{kry}}}, \mathbf{x}') = \mathbf{l}(\mathbf{x}'), \forall \mathbf{x}' \in \mathbb{K}^{N_{\text{kry}}}$$

$$\iff \text{Find } [\mathbf{x}^{N_{\text{kry}}}] \in [\mathbb{K}^{N_{\text{kry}}}], [\mathbf{x}']^* \mathbf{A} [\mathbf{x}^{N_{\text{kry}}}] = [\mathbf{x}']^* \mathbf{F}, \forall [\mathbf{x}'] \in [\mathbb{K}^{N_{\text{kry}}}],$$

where the **Krylov space** of dimension  $N_{\text{kry}}$  is

$$[\mathbb{K}^{N_{\text{kry}}}] := \text{span} \left\{ \underbrace{\left( \begin{array}{c} \vdots \\ \mathbf{F} \\ \vdots \end{array} \right), \left( \begin{array}{c} \vdots \\ \mathbf{A}\mathbf{F} \\ \vdots \end{array} \right), \dots, \left( \begin{array}{c} \vdots \\ \mathbf{A}^{N_{\text{kry}}-1}\mathbf{F} \\ \vdots \end{array} \right)}_{N_{\text{kry}}} \right\}^{N_{\text{dof}}},$$

$\implies \text{size}(\text{Krylov}) = N_{\text{kry}} \times N_{\text{dof}}$ , where  $N_{\text{dof}}$  is the dimension of  $\mathbf{F}$ .

## Trefftz GMRES method

- **GMRES convergence depends on the spectrum of the matrix  $A$ .**

The residual norm of a GMRES solver is given by<sup>iv</sup>

$$\|[\mathbf{x}]^{m+1} - [\mathbf{x}]^m\| = \|r^{m+1}\| \leq \| |\mathbf{P}| \| |\mathbf{P}^{-1}| \| r^0 \| \varepsilon^m,$$

with

$$\varepsilon^m = \min_{\substack{p \in \mathcal{P}^m \\ p(0)=1}} \max_{\lambda_i \in \rho(\mathbf{A})} |p(\lambda_i)| \quad \text{and} \quad \mathbf{P} \text{ the eigenvectors matrix.}$$

- **The consideration of huge cases depends on the memory cost of the method**

$$\text{size(Krylov)} + \text{size}(A) = N_{\text{kry}} \times N \times N_{\text{elem}} + N^2 \times 7 \times N_{\text{elem}}$$

**We propose strategies to**

1. **Remove the spectrum of  $A$  from 0.**
2. **Reduce the memory cost of the Trefftz GMRES method.**

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<sup>iv</sup>Saad, Youcef and Schultz, Martin H, GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, SIAM, 7, 3, 856–869, 1986

## GMRES method with a restart strategy

GMRES converges in theory in  $N_{\text{dof}}$  iterations, ie  $N_{\text{kry}} = N_{\text{dof}}$ .

- $N_{\text{elem}}$  the number of elements
- $N = 52$  the number of plane waves
- $N_{\text{dof}} = N \times N_{\text{elem}}$  the number of degrees of freedom

$$\rightarrow \text{size(Krylov)} = N_{\text{kry}} \times N_{\text{dof}} = 10^6 \times 52 \times 10^6 = 43264 \text{ TB.}$$

Problems when  $N_{\text{dof}}$  becomes large, ie when  $N_{\text{elem}}$  becomes large

- Huge memory cost to store  $\mathbb{K}^{N_{\text{kry}}}$
- Rounding errors accumulated at each step

Idea: truncate Krylov space  $\mathbb{K}^{N_{\text{kry}}}$  into smaller spaces of size  $N_{\text{kry}}^{\text{restart}} < N_{\text{kry}}$

$$[\mathbf{x}]^k = [\mathbf{x}]^{k-1} + \text{GMRES}(\mathbf{A}, \mathbf{F} - \mathbf{A}[\mathbf{x}]^{k-1}), \quad [\mathbf{x}]^0 = 0.$$

Example with  $N_{\text{kry}}^{\text{restart}} = 50$ :

$$\text{size(Krylov)} = N_{\text{kry}}^{\text{restart}} \times N_{\text{dof}} = 50 \times 52 \times 10^6 = 41.6 \text{ GB}$$

→ Strategy with restart reduces the memory cost

## Memory cost with matrix storage

- $N_{\text{kry}}^{\text{restart}} = 50$  the dimension of the Krylov space
- $N_{\text{elem}}$  the number of elements
- $N = 52$  the number of plane waves
- $N_{\text{dof}} = N \times N_{\text{elem}}$  the number of degrees of freedom

Memory cost to run GMRES method **with matrix storage** is

$$\text{size(Krylov)} + \text{size(A)} = 50 \times N \times N_{\text{elem}} + N^2 \times 7 \times N_{\text{elem}}$$

$$\frac{\text{size(A)}}{\text{size(Krylov)}} \simeq 7$$

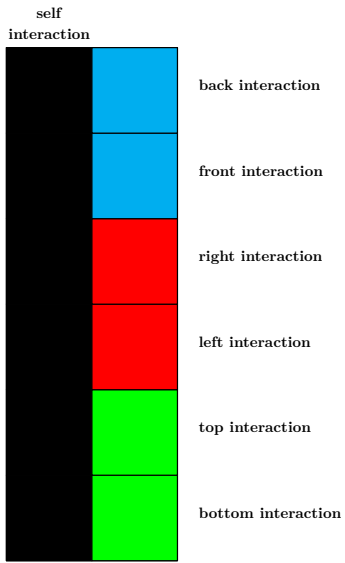
**With matrix storage: the blocking structure is  $N_{\text{elem}}$**

Domain size	$5\lambda$	$10\lambda$	$20\lambda$	$100\lambda$	$200\lambda$
$N_{\text{elem}}$	512	1000	8000	$10^6$	$8 \times 10^6$
<b>With matrix storage</b>	180 MB	344 MB	2.75 GB	344 GB	<b>2.75 TB</b>

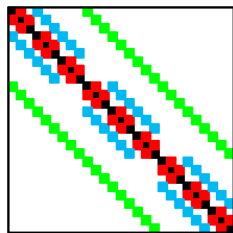
Table: Memory cost for the storage of the matrix  $\mathbf{A}$  in function of the domain size.

**Solution: run GMRES method **without matrix storage****

# Memory cost for matrix storage $7 \times N^2$



Cartesian structure



A

A is seeing as actions.

## Memory cost without matrix storage

- $N_{\text{kry}}^{\text{restart}} = 50$  the dimension of the Krylov space
- $N_{\text{elem}}$  the number of elements
- $N = 52$  the number of plane waves
- $N_{\text{dof}} = N \times N_{\text{elem}}$  the number of degrees of freedom

Memory cost to run GMRES method ~~with matrix storage~~ **without matrix storage** is

$$\text{size(Krylov)} + \text{size(A)} = 50 \times N \times N_{\text{elem}} + \cancel{N^2 \times 7 \times N_{\text{elem}}}$$

$$\text{size(Krylov)} + \text{size(interactions)} = 50 \times N \times N_{\text{elem}} + N^2 \times 7$$

Without matrix storage: only **interactions** between elements are stored

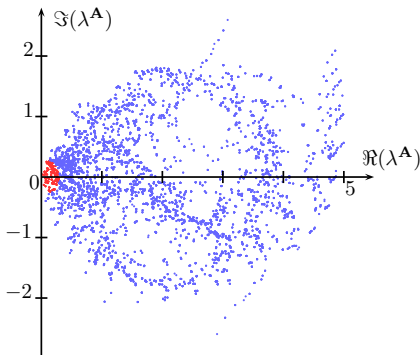
Domain size	$5\lambda$	$10\lambda$	$20\lambda$	$100\lambda$	$200\lambda$
$N_{\text{elem}}$	512	1000	8000	$10^6$	$8 \times 10^6$
<b>With matrix storage</b>	180 MB	344 MB	2.75 GB	344 GB	<b>2.75 TB</b>
<b>Without matrix storage</b>	21.6 MB	41.9 MB	0.33 GB	41.6 GB	<b>332 GB</b>
$\frac{\text{size(interactions)}}{\text{size(Krylov)}}$	0.01	$7 \times 10^{-3}$	$9 \times 10^{-4}$	$7 \times 10^{-6}$	$9 \times 10^{-7}$

Table: Comparison of a GMRES method with or without matrix storage.

## Cessenat-Després preconditioner

The GMRES method convergence depends on the **spectrum of  $A$**  and on the **eigenvalues close to 0**.

**Idea:** use a **preconditioned GMRES** method solving  $A\#A[x] = A\#F$ .

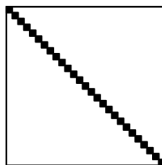


Real and complex parts of the spectrum  $\lambda^A$  of matrix  $A$ , for a  $6\lambda$  domain size.

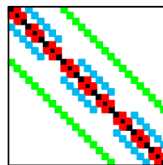
$$A = M - N$$

Cessenat-Després decomposition <sup>a</sup>

M



N



<sup>a</sup>O. Cessenat and B. Després, (1998). Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz problem. SIAM journal on numerical analysis, 35(1), 255-299.



## Cessenat-Després preconditioner

The GMRES method convergence depends on the **spectrum of  $\mathbf{A}$**  and on the **eigenvalues close to 0**.

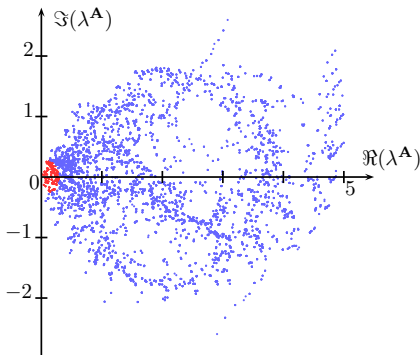
**Idea:** use a **preconditioned GMRES** method solving  $\mathbf{A}^\# \mathbf{A}[\mathbf{x}] = \mathbf{A}^\# \mathbf{F}$ .

$$\mathbf{A}^\# (\mathbf{M} - \mathbf{N})[\mathbf{x}] = \mathbf{A}^\# \mathbf{F}$$

$$\mathbf{A}^\# \mathbf{M}[\mathbf{x}]^{n+1} = \mathbf{A}^\# \mathbf{N}[\mathbf{x}]^n + \mathbf{A}^\# \mathbf{F}$$

→ a **preconditioner** can be  $\mathbf{A}^\# = \mathbf{M}^{-1}$

$$\tilde{\mathbf{A}} = \mathbf{M}^{-1} \mathbf{A} = \text{Id} - \mathbf{M}^{-1} \mathbf{N}$$

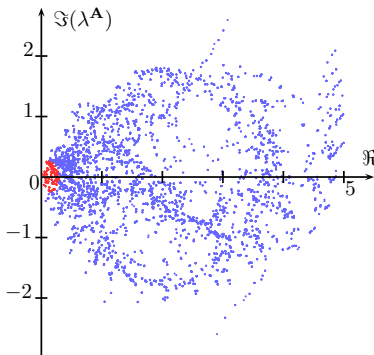


Real and complex parts of the spectrum  $\lambda^{\mathbf{A}}$  of matrix  $\mathbf{A}$ , for a  $6\lambda$  domain size.

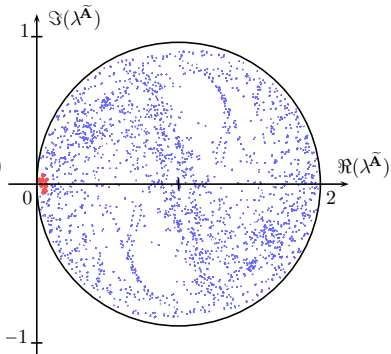
## Cessenat-Després preconditioner

The GMRES method convergence depends on the **spectrum of  $A$**  and on the **eigenvalues close to 0**.

$\tilde{A}$  is contractant



Real and complex parts of the spectrum  $\lambda^A$  of matrix  $A$ , for a  $6\lambda$  domain size.



Real and complex parts of the spectrum  $\lambda^{\tilde{A}}$  of matrix  $\tilde{A}$ , for a  $6\lambda$  domain size.

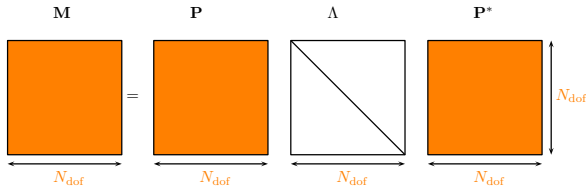
## Basis reduction strategy

**M is symmetric positive definite**

$$M_{\ell,k}^{i,j} = (\mathbf{w}_k^i, \mathbf{w}_\ell^i)_{L_t^2(\partial\mathcal{T})} \delta_{i,j}$$

for  $i, j = 1, N_{\text{elem}}$

and  $\ell, k = 1, N$ .



**P** is the orthogonal eigenvector matrix.

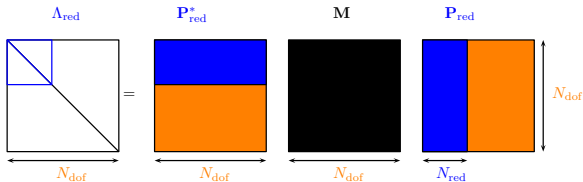
**$\Lambda$**  is the diagonal eigenvalue matrix satisfying

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N_{\text{dof}}} \quad \text{with } \lambda_i = \Lambda_{i,i}.$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{N_{\text{red}}} \\ \vdots \\ \lambda_{N_{\text{dof}}-1} \\ \lambda_{N_{\text{dof}}} \end{pmatrix} \rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{N_{\text{red}}} \end{pmatrix}$$

**Filtering**

$$\frac{\lambda_i}{\lambda_1} < \varepsilon$$



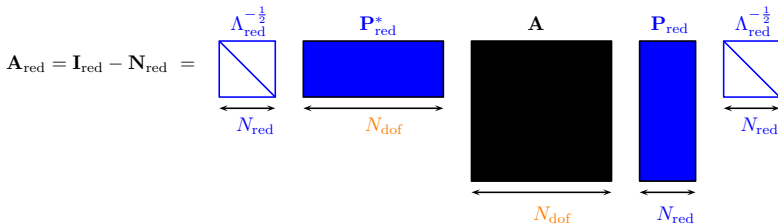
## Normalized reduced basis

The unknown has a new representation

$$\mathbf{x} = \sum_{i=1}^N [x]_i \mathbf{w}^i = \sum_{i=1}^N [y]_i \tilde{\mathbf{w}}^i \simeq \sum_{i=1}^{N_{\text{red}}} [y]_i \tilde{\mathbf{w}}^i, \quad \mathbf{Y}_{\text{red}} = [y]_{i=1, N_{\text{red}}}.$$

This approximation **does not lose information** since  $\frac{\lambda_i}{\lambda_1} < \varepsilon$ , with  $\varepsilon$  well chosen.

The **reduced basis** can be **normalized** and the new matrix is

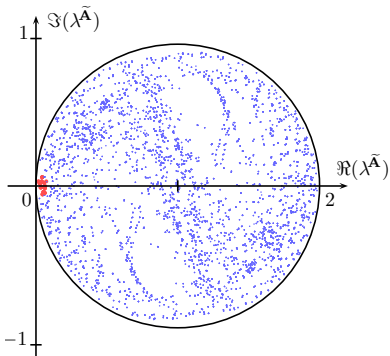


Computation of  $\mathbf{M}^{-1}$  is trivial  $\rightarrow$  **memory gain**

$$[\mathbf{x}^{n+1}] = \mathbf{N}_{\text{red}} [\mathbf{x}^n] + \mathbf{F}_{\text{red}}, \quad \text{with } \mathbf{N}_{\text{red}} := \Lambda_{\text{red}}^{-\frac{1}{2}} \mathbf{P}_{\text{red}}^* \mathbf{N} \mathbf{P}_{\text{red}} \Lambda_{\text{red}}^{-\frac{1}{2}} \quad \text{and} \quad [\mathbf{x}^0] = 0.$$

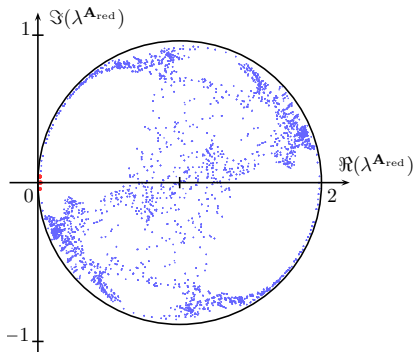
## Basis reduction improves matrix conditioning

Unreduced preconditioned matrix  $\tilde{\mathbf{A}}$



Real and complex parts of the spectrum  $\lambda^{\tilde{\mathbf{A}}}$  of matrix  $\tilde{\mathbf{A}}$ , for a  $6\lambda$  domain size.

Reduced preconditioned matrix  $\mathbf{A}_{\text{red}}$



Real and complex parts of the spectrum  $\lambda^{\mathbf{A}_{\text{red}}}$  of  $\mathbf{A}_{\text{red}}$ , for a  $6\lambda$  domain size and  $\epsilon = 10^{-5}$ .

## Basis reduction improves memory burden

$N_{\text{red}}$  depends on the size of an element  $K$  and of the threshold  $\lambda_1 \varepsilon$ .

$\lambda_1 \varepsilon$ \ Size of $K$	$10^{-13}$	$10^{-11}$	$10^{-9}$	$10^{-7}$	$10^{-5}$	$10^{-4}$	$10^{-3}$	$10^{-2}$
$0.25\lambda$	154	126	96	70	48	48	30	24
$0.5\lambda$	190	186	174	132	96	84	70	48
$1\lambda$	196	196	196	190	180	174	148	114

**Table:** Values of  $N_{\text{red}}$  when reducing the basis of size  $N = 196$  according to  $\varepsilon$  and the size of an element  $K$  in wavelength.

**0.25 $\lambda$  case,**  $N_{\text{kry}}^{\text{restart}} = 500$ .

$10^{-15}$	$\lambda_1 \varepsilon$	$10^{-13}$	$10^{-11}$	$10^{-9}$	$10^{-7}$	$10^{-5}$	$10^{-4}$	$10^{-3}$	$10^{-2}$
<b>N = 196</b>	$N_{\text{red}}$	154	126	96	70	48	36	30	16
<b>1.57 GB</b>	Memory cost (GB)	1.23	1.01	0.76	0.56	0.38	0.28	0.24	0.13

**Table:** Memory cost in GigaBytes (GB) for the storage of the non-reduced matrix  $\mathbf{A}$  and of the reduced matrix  $\mathbf{A}_{\text{red}}$  according to  $N_{\text{red}}$ , for a  $10\lambda$  domain size.

**GMRES does not converge without reduction** → improvement of the conditioning

## How far can we reduce the plane wave basis ?

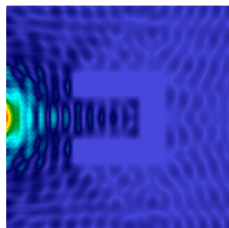
$\lambda_1 \varepsilon$	$10^{-7}$	$10^{-5}$	$10^{-4}$	$10^{-3}$	$10^{-2}$
Trefftz error	N/A	$3.4 \times 10^{-2}$	$9.8 \times 10^{-2}$	0.15	0.74
Memory cost (GB)	0.56	0.38	0.28	0.24	0.13
Time (hours)	55	27	11.3	7.23	0.58

Table: Trefftz error  $\frac{\sqrt{\|[\mathbf{x}_\varepsilon - \mathbf{x}_{\text{ref}}]^* \mathbf{M} [\mathbf{x}_\varepsilon - \mathbf{x}_{\text{ref}}]\|_z}}{\sqrt{\|[\mathbf{x}_{\text{ref}}]^* \mathbf{M} [\mathbf{x}_{\text{ref}}]\|_z}}$ , computation time and memory cost for a  $10^{-8}$  GMRES residual for  $[\mathbf{x}_\varepsilon]$  according to each  $\lambda_1 \varepsilon$ .

GMRES residual =  $10^{-8}$  but poor approximation when  $\lambda_1 \varepsilon$  is not well chosen.

Too much reduction

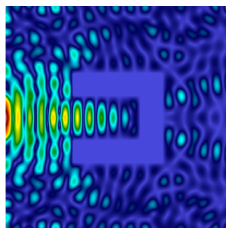
$$N_{\text{red}} = 16 \quad \lambda_1 \varepsilon = 10^{-2}$$



Electromagnetic Field Magnitude  
 0.0e+00    4    6    8    1.1e+01

A better compromise

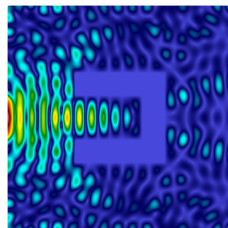
$$N_{\text{red}} = 36 \quad \lambda_1 \varepsilon = 10^{-4}$$



Electromagnetic Field Magnitude  
 0.0e+00    4    6    8    1.1e+01

Solution of reference

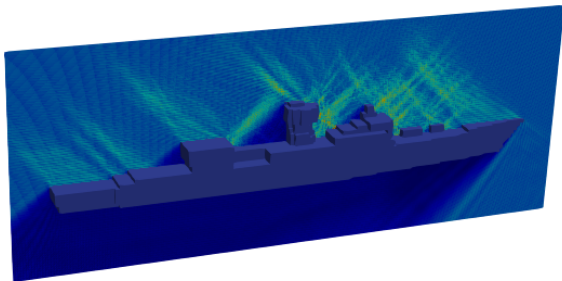
$$N_{\text{red}} = 70 \quad \lambda_1 \varepsilon = 10^{-7}$$



Electromagnetic Field Magnitude  
 0.0e+00    4    6    8    1.1e+01

## Heterogeneous visualization

**Interpretation: complex nearby electromagnetic field and shadow under the boat.**  
 Boat size =  $24 \times 61 \times 154$  meters.      Cartesian mesh =  $96 \times 244 \times 616$  elements.



Three-dimensional electromagnetic wave striking the top of a boat.

#elem	$N_{\text{red}}$	$N$	$\#\text{dof}_{\text{red}}$	$\#\text{dof}$
$> 14.4 \times 10^6$	46	52	$> 663 \times 10^6$	$> 750 \times 10^6$

GMRES iterations	Memory cost	Total duration	GMRES residual
800	389 GB	28.6 hours	$3.8 \times 10^{-2}$

Table: Numerical data and results associated to the boat case.



**Thank you for your attention !**

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