



HAL
open science

FUNCTIONAL INEQUALITIES AND APPLICATIONS TO DOUBLY NONLINEAR DIFFUSION EQUATIONS

Iwona Chlebicka, Nikita Simonov

► **To cite this version:**

Iwona Chlebicka, Nikita Simonov. FUNCTIONAL INEQUALITIES AND APPLICATIONS TO DOUBLY NONLINEAR DIFFUSION EQUATIONS. 2023. hal-03945247

HAL Id: hal-03945247

<https://hal.science/hal-03945247>

Preprint submitted on 18 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FUNCTIONAL INEQUALITIES AND APPLICATIONS TO DOUBLY NONLINEAR DIFFUSION EQUATIONS

IWONA CHLEBICKA AND NIKITA SIMONOV

ABSTRACT. We study weighted inequalities of Hardy and Hardy–Poincaré type and find necessary and sufficient conditions on the weights so that the considered inequalities hold. Examples with the optimal constants are shown. Such inequalities are then used to quantify the convergence rate of solutions to doubly nonlinear fast diffusion equation towards the Barenblatt profile.

1. INTRODUCTION

We investigate functional inequalities of Hardy and Poincaré type. Our main objective is to provide new constructive methods of derivation of such inequalities with explicit constants that are optimal in some cases. Our study starts with finding necessary and sufficient conditions on the weights so that the following *Poincaré inequality* on the real line is valid for any compactly supported $f \in W^{1,\infty}(\mathbb{R})$

$$(P) \quad \int_{\mathbb{R}} |f - (f)_{w_1}|^q w_1(s) ds \leq C_P \int_{\mathbb{R}} |f'|^q w_2(s) ds, \quad \text{where } 1 < q < \infty,$$

$(f)_{w_1}$ is a weighted average with respect to w_1 , and $C_P = C_P(q, w_1, w_2) > 0$, see Theorem 1. Having *Poincaré inequality* on the real line, we prove some compatibility conditions to be necessary and sufficient for *Hardy–Poincaré inequality on \mathbb{R}^N* , $N > 2$, of a form

$$(HP) \quad \int_{\mathbb{R}^N} |\varphi - \bar{\varphi}|^q w_1(x) dx \leq C_{HP} \int_{\mathbb{R}^N} |\nabla \varphi|^q w_2(x) dx, \quad \text{where } 1 < q < N,$$

to hold for all compactly supported $\varphi \in W^{1,\infty}(\mathbb{R}^N)$, where $\bar{\varphi} = \frac{1}{H_1} \int_{\mathbb{R}^N} \varphi w_1(x) dx$, and $C_{HP} = C_{HP}(q, w_1, w_2) > 0$, see Theorem 2. Our next result is Theorem 3 providing a constructive PDE-based method of obtaining admissible weights to the following *Hardy inequality*

$$(H) \quad \int_{\mathbb{R}^N} |\varphi|^q w_1(x) dx \leq C_H \int_{\mathbb{R}^N} |\nabla \varphi|^q w_2(x) dx, \quad \text{where } 1 < q < \infty,$$

for every compactly supported $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ and $C_H = C_H(q, w_1, w_2) > 0$.

The research of I. Chlebicka was partially funded by NCN Grant 2019/34/E/ST1/00120. N. Simonov was partially supported by the Spanish Ministry of Science and Innovation, through the FPI-grant BES-2015-072962, associated to the project MTM2014-52240-P (Ministry of Science and Innovation, Spain), by the project MTM2017-85757-P (Ministry of Science and Innovation, Spain), by the E.U. H2020 MSCA programme, grant agreement 777822, by the Project EFI (ANR-17-CE40-0030) of the French National Research Agency (ANR), and by the DIM Math-Innov of the Region Île-de-France.

The name *Hardy-Poincaré inequalities* has been introduced in [11]. In that paper the authors consider a family of inequalities which interpolates between the classical Hardy and Poincaré inequalities, see [11, Proposition 5]. Since then, the name Hardy-Poincaré has been popularized and several authors used it, see for instance [12, 4, 37, 38, 15, 62]. In the present paper, we preferred to distinguish between two different situations: when the left-hand side weight is integrable (Hardy-Poincaré inequality (HP)) and when it is not true (Hardy inequality (H)).

The investigation of the inequalities mentioned above is motivated by the study of nonlinear diffusion equations where they play a critical role in understanding the asymptotic behaviour of solutions to Cauchy problems. The constants in the inequalities dictate the convergence rates of a solution towards a self-similar one. We apply such an inequality to investigate the long-time asymptotics of solutions to the Cauchy problem

$$(1) \quad u_t = \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m), \quad u(0, x) = u_0(x).$$

We point out that the inequalities (HP) and (P) retrieve the Hardy–Poincaré and Hardy inequalities applied in [12, 16] in order to obtain precise rates of convergence of solutions to (1) for $p = 2$ and $m < 1$. Inequalities (P), (HP), and (H) are far better understood for $q = 2$ or in bounded domains, cf. [6, 10, 13, 43]. We are interested in providing them for general q and under possibly general assumptions on the weights. Despite our ultimate goal is to provide a handy tool for analysis of an evolution equation, let us mention that such inequalities play an important role in other branches of analysis. Hardy-type inequalities are used in functional analysis, probability theory, interpolation theory, and PDEs [8, 10, 12, 24, 25, 42, 45, 47, 50, 57, 66]. We refer to [6, 10, 13, 56] for information on the relation between (H), (HP) and Log-Sobolev inequalities. The latter are of great interest in geometry in finite and infinite dimension, probability theory, and statistical mechanics.

In order to present our first result let us define the *median* of the measure $w_1(s) ds$ being a point $\eta \in \mathbb{R}$ such that $\int_{-\infty}^{\eta} w_1(s) ds = \|w_1\|_{L^1(\mathbb{R})}/2$. For any $m \in \mathbb{R}$ we define

$$(2) \quad \begin{aligned} B_m^+ &:= \sup_{t \geq m} \left\{ \int_t^{\infty} w_1(s) ds \left(\int_m^t w_2^{\frac{1}{1-q}}(s) ds \right)^{q-1} \right\}, \\ B_m^- &:= \sup_{t \leq m} \left\{ \int_{-\infty}^t w_1(s) ds \left(\int_t^m w_2^{\frac{1}{1-q}}(s) ds \right)^{q-1} \right\}. \end{aligned}$$

The following theorem characterizes the measures which satisfy inequality (P).

Theorem 1 (Poincaré inequality on \mathbb{R}). *Let $1 < q < \infty$, $0 \leq w_1, w_2$, $\|w_1\|_{L^1(\mathbb{R})} < \infty$, and let η be a median of the measure $w_1(s) ds$. Then inequality (P) holds if and only if $B_\eta^+, B_\eta^- < \infty$. Moreover, the optimal constant C_p satisfies*

$$\frac{(2^{\frac{q-1}{q}} - 1)^q}{2^{q-1}} \max(B_\eta^+, B_\eta^-) \leq C_p \leq \frac{(2q)^q}{(q-1)^{q-1}} \max(B_\eta^+, B_\eta^-)$$

In the case $q = 2$, the result was proven in [6, Théorème 6.2.2] and [13]. See also [56] for a discussion on the bounds on the optimal constant if $q = 2$. The method of the proof is inspired by the above mentioned papers.

As an application of Theorem 1 we prove our next result – sufficient and necessary conditions for inequality (HP) to hold. In order to state our result we define for any $m > 0$ and for any function $0 \leq h \in L^1([0, \infty))$ the quantity

$$(3) \quad H_2(m) := \max \left\{ \sup_{t > m} \left(\int_t^\infty r^{N-1} h(r) dr \left[\int_m^t (r^{N-1+q} h(r))^{-\frac{1}{q-1}} dr \right]^{q-1} \right), \right. \\ \left. \sup_{t \in (0, m)} \left(\int_0^t r^{N-1} h(r) dr \left[\int_t^m (r^{N-1+q} h(r))^{-\frac{1}{q-1}} dr \right]^{q-1} \right) \right\}.$$

With a slight abuse of notation, for a nonnegative, radial N -dimensional measure $h(r)r^{N-1}dr$ we define a *median* to be $\eta \in [0, \infty)$ such that $\int_0^\eta h(r)r^{N-1}dr = \|h\|_{L^1(\mathbb{R}^N)}/2$. Our result reads as follows.

Theorem 2 (General Hardy–Poincaré inequality). *Let $1 < q \leq 2$ if $N = 2$ and $1 < q < N$ if $N \geq 3$. Assume $h : [0, \infty) \rightarrow [0, \infty)$ is such that*

$$(4) \quad H_1 := \|h\|_{L^1(\mathbb{R}^N)} < \infty,$$

and let η be a median of the measure $h(r)r^{N-1}dr$, $w_1(x) = h(|x|)$, and $w_2(x) = |x|^q h(|x|)$.

Then inequality (HP) holds if and only if $H_2(\eta) < \infty$. Moreover, the optimal constant C_{HP} satisfies

$$(5) \quad \frac{(2^{\frac{q-1}{q}} - 1)^q}{2^{q-1}} H_2(\eta) \leq C_{HP} \leq \frac{(2q)^q}{(q-1)^{q-1}} H_2(\eta).$$

Let us concentrate on inequalities of Hardy type (H). It is known that such inequalities can be proven with weights depending on solutions to elliptic PDE, see [9, 26, 27, 28, 44, 61, 62]. The principal idea is to re-interpret certain versions of Caccioppoli estimate for a solution as Hardy inequalities for test functions. In our result weights w_1, w_2 in (H) are determined by the means of the gradient and the θ -Laplacian of a sub- or super- θ -harmonic function. Let us define the operator

$$(6) \quad \Delta_\theta g = \operatorname{div}(|\nabla g|^{\theta-2} \nabla g) \quad \text{for } \theta > 1.$$

Note that for $\theta = 2$, we have $\Delta_\theta g = \Delta g$.

Theorem 3 (General Hardy inequality). *Suppose $1 < q, \theta < \infty$, $g \in W_{loc}^{1,1}(\mathbb{R}^N)$ is positive and such that $\Delta_\theta g \in L_{loc}^1(\mathbb{R}^N)$ has constant sign. Then for every compactly supported $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ inequality (H) holds with $w_1(x) = |\Delta_\theta g|$, $w_2(x) = |\nabla g|^{q(\theta-1)} |\Delta_\theta g|^{1-q}$, and $C_H = q^q$.*

The above theorem has surprisingly strong consequences as for how easy is its proof. See [27, Theorem 2.5] and [61, Theorem 3.1] for our inspiration. The precision of the reasoning is illustrated by in Section 3.2 with a family of inequalities, where the constant is proven to be optimal.

As an application of Theorem 2 we study the *asymptotic behaviour* of solutions to doubly nonlinear equation (1). Equations as (1) have been investigated since the 70's due to their intrinsic mathematical difficulties and the wide range of applications, for instance in glaciology and non-Newtonian fluids [7, 48, 53]. For more details on cornerstones of the field we refer to [51, 65] and also the monographs [64, 63, 68]. Special cases of equation (1) are the heat equation ($p = 2, m = 1$), the porous medium equation ($p = 2, m > 1$), and the fast diffusion equation ($p = 2, m < 1$). In the case $m = 1$, (1) involves the already classical p -Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. What is more, then evolution equation (1) requires different methods for $1 < p < 2$ and $2 < p$ being called singular and degenerate, respectively. Studies on solutions to equations like (1) attract deep attention of various groups developing their theory from different points of view [2, 4, 14, 39, 40, 41, 49, 54, 60, 67]. The issue of convergence of solutions to a self-similar profile to nonlinear diffusion equations has been studied e.g. in [12, 20, 33, 29, 63] and in the doubly nonlinear evolution equation in [1, 4].

We are interested in (1) with nonnegative initial datum with $u_0 \in L^1(\mathbb{R}^N)$. Our result is proven for $p > 1, m > 0$, and

$$(7) \quad \frac{N-p}{p} < m(p-1) < \frac{N-p+1}{N},$$

which fits in the *fast diffusion range* where the mass of the solution is conserved along time. Under such regime infinite speed of propagation holds, i.e., if $u_0 \geq 0$ then $u(t) > 0$ for all times $t > 0$. Within (7) the asymptotic behaviour of a solution is controlled by a *self-similar* one called the *Barenblatt* solution

$$(8) \quad B_D(t, x) = R(t)^{-N} \mathfrak{B}_D(x/R(t)),$$

where

$$(9) \quad R(t) := (1 + \vartheta t)^{\frac{1}{\vartheta}} \quad \text{and} \quad \mathfrak{B}_D(x) = \left(D + \frac{1 - m(p-1)}{mp} |x|^{\frac{p}{p-1}} \right)^{\frac{p-1}{m(p-1)-1}}.$$

with $\vartheta = p - N[1 - m(p-1)] > 0$. The parameter D depends on the mass of \mathfrak{B}_D . In particular, it is known that solutions to (1), within the range (7) converge to B_D in the $L^1(\mathbb{R}^N)$ -topology as $t \rightarrow \infty$. Here, we provide an effective alternative way for one important step of the proof of [4, Theorem 1.1] and supply it with an estimate on a rate of convergence.

Theorem 4. *Let $N \geq 3, m > 0, p > 1$ as in (7) and let nonnegative $u_0 \in L^1(\mathbb{R}^N)$ be such that*

$$\mathfrak{B}_{D_0}(y) \leq u_0(y) \leq \mathfrak{B}_{D_1}(y)$$

for some $D_0, D_1 > 0$. Let $D_ > 0$ be such that $\int_{\mathbb{R}^d} u_0 dx = \int_{\mathbb{R}^d} B_{D_*}(0, x) dx$. Then there exist nonnegative constants C and λ , and a time t_0 such that, for any solution to (1) with initial datum u_0 , the following holds*

$$(10) \quad \|u(t) - B_{D_*}(t)\|_{L^1(\mathbb{R}^d)} \leq C t^{-\lambda/2}, \quad \forall t \geq t_0.$$

In [4] the authors apply the entropy method, where a Hardy–Poincaré inequality (HP) is used to compare the linear entropy with a linearised version version of it. In order to prove such an inequality they invoke the Persson's Theorem as it was done in [11], which one may avoid

entirely. Our main contribution is to prove the needed Hardy–Poincaré inequality by the constructive method of Theorem 2.

Our last result is motivated by study of the asymptotic behaviour of the fast diffusion range of the *evolution p -Laplace equation*, i.e. equation (1) with $m = 1$ and $1 < p < 2$. Only few results are available in this range, see [3, 18]. Therefore, in a forthcoming paper, we will give a more detailed analysis for a natural class of initial data. In this work we provide the relevant Hardy–Poincaré inequality for this range of parameters. For the proof see Section 3.3.

Corollary 1.1. *Let $1 < p < 2$, $N > 2$, $N > \frac{p}{(2-p)(p-1)}$. Then for all compactly supported $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ it holds that*

$$(11) \quad \int_{\mathbb{R}^N} |\varphi|^2 |x|^{-\frac{p}{p-1}} \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{2-p}} dx \leq C_H \int_{\mathbb{R}^N} |\nabla \varphi|^2 |x|^{-\frac{2-p}{p-1}} \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{2-p}} dx$$

with a positive finite constant C_H . If we additionally assume that $N \leq 7$ **OR** $p \notin (p_-, p_+)$ for

$$p_- = \frac{3}{2} - \frac{1}{2} \sqrt{\frac{N-7}{N+1}} \quad \text{and} \quad p_+ = \frac{3}{2} + \frac{1}{2} \sqrt{\frac{N-7}{N+1}},$$

then the constant C_H is optimal and reads

$$C_H = 4 \left(N - \frac{p}{(2-p)(p-1)} \right)^{-2}.$$

The range $N < \frac{p}{(2-p)(p-1)}$ is not considered in the above result since the left-hand-side weight becomes integrable, so different techniques are needed in order to obtain an inequality, see Theorem 2. From the point of view of asymptotics, in this last range the difference of two Barenblatt profiles is integrable, while in the range considered in the above corollary is not. The case $N = \frac{p}{(2-p)(p-1)}$ happens only if $N \geq 6$ and for $p = \frac{3}{2} - \frac{1}{2N} \pm \frac{\sqrt{N^2 - 6N + 1}}{2N}$. These two last exponents play the same role played by $m_* = (N-4)/(N-2)$ in the better understood case of the fast diffusion equation (eq. (1) with $p = 2$, $0 < m < 1$), where particularly sophisticated techniques were needed to deal with this strange exponent, see [12, 16].

Organization of the paper. Section 2 concerns proofs of inequalities (P), (HP), and (H). In Section 3.1 we show examples of (HP) and in Section 3.2 of (H); here Corollary 1.1 is proven. The proof of Theorem 4 is presented in Section 4.

2. FUNCTIONAL INEQUALITIES

2.1. Poincaré inequality on \mathbb{R} . Proof of Theorem 1. We will employ the classical result of Muckenhoupt.

Lemma 2.1 (Muckenhoupt’s inequality, Theorem 1, [58]). *Let $1 < q < \infty$, $0 \leq w_1, w_2$ such that $w_1 \in L^1_{loc}([0, \infty))$. There exists $0 < C_M < \infty$ for which*

$$\int_0^\infty |f - f(0)|^q w_1(r) dr \leq C_M \int_0^\infty |f'|^q w_2(r) dr$$

is true for every compactly supported $f \in W^{1,\infty}([0, \infty))$ if and only if

$$H_M := \sup_{\rho > 0} \left(\int_{\rho}^{\infty} w_1(r) dr \left[\int_0^{\rho} (w_2(r))^{-\frac{1}{q-1}} dr \right]^{q-1} \right) < \infty$$

Moreover, then the optimal constant C_M satisfies $H_M \leq C_M \leq q^q (q-1)^{1-q} H_M$.

We are in a position to prove Poincaré inequality (P) on the real line.

Proof of Theorem 1. We assume first that B_{η}^+ and B_{η}^- are both finite. For any m we have that

$$\begin{aligned} \int_{-\infty}^{\infty} |f - (f)_{w_1}|^q w_1(s) ds &\leq 2^q \int_{-\infty}^{\infty} |f - f(m)|^q w_1(s) ds \\ &= 2^q \int_{-\infty}^m |f - f(m)|^q w_1(s) ds + 2^q \int_m^{\infty} |f - f(m)|^q w_1(s) ds \\ &=: 2^q (L_1(m) + L_2(m)). \end{aligned}$$

Let us define

$$J_1(m) := \int_{-\infty}^m |f'|^q w_2(s) ds \quad \text{and} \quad J_2(m) := \int_m^{\infty} |f'|^q w_2(s) ds,$$

and let us call A_m^- (A_m^+ resp.) the optimal constant of the inequality $L_1(m) \leq A_m^- J_1(m)$ (of inequality $L_2(m) \leq A_m^+ J_2(m)$ resp.). In particular, if we consider $m = \eta$, we obtain that

$$B_{\eta}^- \leq A_{\eta}^- \leq q(q')^{q-1} B_{\eta}^- \quad \text{and} \quad B_{\eta}^+ \leq A_{\eta}^+ \leq q(q')^{q-1} B_{\eta}^+$$

as a consequence of a change of variables and Lemma 2.1. By summing up the previous inequalities, we find that

$$\begin{aligned} \int_{-\infty}^{\infty} |f - (f)_{w_1}|^q w_1(s) ds &\leq 2^q (L_1(m) + L_2(m)) \\ &\leq 2^q \max\{A_{\eta}^+, A_{\eta}^-\} (J_1(m) + J_2(m)) \\ &\leq 2^q q(q')^{q-1} \max(B_{\eta}^+, B_{\eta}^-) \int_{\mathbb{R}} |f'|^q w_2(s) ds, \end{aligned}$$

which proves inequality (P) with the wanted upper bound on the constant C_p .

Assume now that inequality (P) holds with a finite constant $C_p > 0$. Let us first restrict our attention to the case when $B_{\eta}^+, B_{\eta}^- < \infty$ (so that, by the optimality of A_{η}^+ and A_{η}^- , $\infty > A_{\eta}^+ \geq B_{\eta}^+$ and $\infty > A_{\eta}^- \geq B_{\eta}^-$) and will to prove the lower bound on the constant C_p . Without any loss of generality we can assume that $\|w_1\|_{L^1(\mathbb{R})} = 1$. Let us recall that A_{η}^+ is the optimal constant for inequality $L_2(\eta) \leq A_{\eta}^+ J_2(\eta)$. By optimality, for any $\varepsilon > 0$ there exists a function f_{ε} and such

that

$$(12) \quad \int_{\eta}^{\infty} \left| \int_{\eta}^s f_{\varepsilon}(t) dt \right|^q w_1(s) ds \geq (A_{\eta}^+ - \varepsilon) \int_{\eta}^{\infty} |f_{\varepsilon}(s)|^q w_2(s) ds.$$

Note that, without loss of generality, we can assume that $f_{\varepsilon} \geq 0$. Let us define $F_{\varepsilon}(x) = 0$ if $x \leq \eta$ and $F_{\varepsilon}(x) = \int_{\eta}^x f_{\varepsilon}(t) dt$ if $x \geq \eta$. Therefore, by the property of the median, we have $w_1(\{F_{\varepsilon} > 0\}) \leq 1/2$. Then, by Hölder inequality, we obtain

$$\int_{\mathbb{R}} F_{\varepsilon}(t) w_1(t) dt \leq \left(\int_{\mathbb{R}} |F_{\varepsilon}(t)|^q w_1(t) dt \right)^{\frac{1}{q}} w_1(\{F_{\varepsilon} > 0\})^{\frac{q-1}{q}} \leq 2^{-\frac{q-1}{q}} \left(\int_{\mathbb{R}} |F_{\varepsilon}(t)|^q w_1(t) dt \right)^{\frac{1}{q}}.$$

Then, by using the above inequality with the triangle inequality and inequality (12), we find

$$\begin{aligned} \int_{\mathbb{R}} |F_{\varepsilon}(t) - (F_{\varepsilon})_{w_1}|^q w_1(t) dt &\geq \left(\left(\int_{\mathbb{R}} |F_{\varepsilon}(t)|^q w_1(t) dt \right)^{\frac{1}{q}} - (F_{\varepsilon})_{w_1} \right)^q \\ &\geq \frac{(2^{\frac{q-1}{q}} - 1)^q}{2^{q-1}} \int_{\mathbb{R}} |F_{\varepsilon}(t)|^q w_1(t) dt \\ &\geq \frac{(2^{\frac{q-1}{q}} - 1)^q}{2^{q-1}} (A_{\eta}^+ - \varepsilon) \int_{\eta}^{\infty} |f_{\varepsilon}(s)|^q w_2(s) ds \\ &\geq \frac{(2^{\frac{q-1}{q}} - 1)^q}{2^{q-1}} (B_{\eta}^+ - \varepsilon) \int_{\eta}^{\infty} |f_{\varepsilon}(s)|^q w_2(s) ds \\ &\geq \frac{(2^{\frac{q-1}{q}} - 1)^q}{2^{q-1}} \frac{(B_{\eta}^+ - \varepsilon)}{C_p} \int_{\mathbb{R}} |F_{\varepsilon}(t) - (F_{\varepsilon})_{w_1}|^q w_1(t) dt, \end{aligned}$$

where we used that A_{η}^+ is the optimal constant and inequality (P). This proves that

$$C_p \geq \frac{(2^{\frac{q-1}{q}} - 1)^q}{2^{q-1}} (B_{\eta}^+ - \varepsilon)$$

for any $\varepsilon > 0$. The same construction can be used for the case of B_{η}^- . It remains to prove that if inequality (P) holds for a finite $C_p > 0$ then $B_{\eta}^+, B_{\eta}^- < \infty$. To justify this one may argue by contradiction using the same lines as above. \square

When we apply Theorem 1 for $w_1(r) = r^{N-1}h(r)\mathbb{1}_{[0,\infty)}(r)$ and $w_2(r) = r^{N-1+q}h(r)\mathbb{1}_{[0,\infty)}(r)$, and denote $(f) = (f)_{w_1}$, we get the following consequence.

Corollary 2.1. *Suppose $1 < q < \infty$ and $0 \leq h \in L_{loc}^1([0, \infty))$ is such that and there exists $m > 0$ for which $H_2(m)$ given by (3) is finite. There exists $0 < C < \infty$ such that*

$$(13) \quad \int_0^{\infty} |f - (f)|^q r^{N-1} h(r) dr \leq C \int_0^{\infty} |f'|^q r^{N-1+q} h(r) dr$$

holds for every compactly supported $f \in W^{1,\infty}([0, \infty))$. Moreover, $C \leq 2^q q(q')^{q-1} H_2(m)$.

2.2. Hardy–Poincaré inequality on \mathbb{R}^N . We prove the inequality with radial weights as an application of the Poincaré inequality (P). For this we introduce the standard change of variables from Cartesian to spherical coordinates, that is

$$r = |x| \quad \text{and} \quad \vartheta = x/|x|.$$

In these coordinates, the gradient can be written as $(\partial_r, \frac{1}{r}\nabla_{\vartheta})$ where $\partial_r = \frac{x}{r} \cdot \nabla$ is the partial derivative with respect to the radial variable r and ∇_{ϑ} is the derivative with respect to the angular variables. Then

$$(14) \quad |\partial_r f(r, \vartheta)|^2 + \frac{1}{r^2} |\nabla_{\vartheta} f(r, \vartheta)|^2 = |\nabla f(x)|^2.$$

By $S^{N-1} \subset \mathbb{R}^N$ we denote the unit sphere and parametrize it with the variable ϑ . Moreover, for every compactly supported function $f \in W^{1,\infty}(\mathbb{R}^N)$ we denote the directional average as

$$(15) \quad \tilde{f}(\vartheta) := \frac{|S^{N-1}|}{H_1} \int_0^\infty f(r, \vartheta) r^{N-1} h(r) dr.$$

Recall that the global average with respect to h is given by

$$(16) \quad \bar{f} = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \tilde{f}(\vartheta) d\vartheta = \frac{1}{H_1} \int_{\mathbb{R}^N} f(x) h(|x|) dx.$$

We recall the Poincaré inequality on the sphere.

Lemma 2.2 (Spherical Poincaré inequality). *Let $1 \leq q < N$ if $N \geq 3$, and $1 \leq q \leq 2$ when $N = 2$. Then there exists $C_{\text{sph}} > 0$ such that*

$$(17) \quad \int_{S^{N-1}} |\tilde{f}(\vartheta) - \bar{f}|^q d\vartheta \leq C_{\text{sph}} \int_{S^{N-1}} |\nabla_{\vartheta} \tilde{f}(\vartheta)|^q d\vartheta$$

for every $f \in W^{1,q}(S^{N-1})$.

In the case $N \geq 3$, inequality (17) is a direct consequence of [46, Theorem 2.10] formulated for a smooth and compact Riemannian manifold. In the case $q = 2$ and $N = 2$, it follows from a decomposition in spherical harmonics which then coincides with the Fourier basis, see [5]. To our best knowledge its optimal value of the constant from (17) is known only in the case of $q = 2$, when $C_{\text{sph}} = \frac{1}{N-1}$, see [23, Chapter 4, Proposition 1].

The following result would be instrumental in the proof of Theorem 2.

Proposition 2.1. *Suppose assumptions of Theorem 2 are satisfied. If for every compactly supported function $f \in W^{1,\infty}(\mathbb{R}^N)$ the following inequality holds*

$$(18) \quad \int_0^\infty \int_{S^{N-1}} |f(r, \vartheta) - \tilde{f}(\vartheta)|^q r^{N-1} h(r) d\vartheta dr \leq C_1 \int_0^\infty \int_{S^{N-1}} |\partial_r f(r, \vartheta)|^q r^{N-1+q} h(r) d\vartheta dr,$$

then there exist $C_{HP} > 0$, such that for the same f one has (HP) with $w_1(x) = h(|x|)$, $w_2(x) = |x|^q h(|x|)$, and a positive constant $C_{HP} \leq 2^{q-1} \max\{C_1, H_1^q C_{\text{sph}} |S^{N-1}|^{-q}\}$.

Proof. By (16) and Jensen's inequality it holds

$$\begin{aligned} \int_{\mathbb{R}^N} |f(x) - \bar{f}|^q h(|x|) dx &= \int_0^\infty \int_{S^{N-1}} |f(r, \vartheta) - \bar{f}|^q r^{N-1} h(r) d\vartheta dr \\ &\leq 2^{q-1} \int_0^\infty \int_{S^{N-1}} |f(r, \vartheta) - \tilde{f}(\vartheta)|^q r^{N-1} h(r) d\vartheta dr \\ &\quad + 2^{q-1} \int_0^\infty \int_{S^{N-1}} |\tilde{f}(\vartheta) - \bar{f}|^q r^{N-1} h(r) d\vartheta dr =: I_1 + I_2. \end{aligned}$$

Since the first term on the right-hand side above can be estimated due to (18) and then by properties of the gradient (14), we get

$$I_1 \leq 2^{q-1} C_1 \int_0^\infty \int_{S^{N-1}} |\partial_r f(r, \vartheta)|^q r^{N-1+q} h(r) d\vartheta dr \leq 2^{q-1} C_1 \int_{\mathbb{R}^N} |\nabla f(x)|^q |x|^q h(|x|) dx.$$

As for the second one we use (4) to obtain

$$I_2 = 2^{q-1} \frac{H_1}{|S^{N-1}|} \int_0^\infty |\tilde{f}(\vartheta) - \bar{f}|^q d\vartheta$$

and we apply the spherical Poincaré inequality (Lemma 2.2) to get

$$I_2 = 2^{q-1} \frac{H_1}{|S^{N-1}|} \int_{S^{N-1}} |\tilde{f}(\vartheta) - \bar{f}|^q d\vartheta \leq 2^{q-1} \frac{H_1}{|S^{N-1}|} C_{\text{sph}} \int_{S^{N-1}} |\nabla_{\vartheta} \tilde{f}(\vartheta)|^q d\vartheta =: J.$$

As $r^{N-1} h(r) |S^{N-1}| / H_1 dr$ is a probability measure by Jensen's inequality we have that

$$\begin{aligned} J &= 2^{q-1} \frac{H_1}{|S^{N-1}|} C_{\text{sph}} \int_{S^{N-1}} \left| \nabla_{\vartheta} \int_0^\infty f(r, \vartheta) r^{N-1} h(r) dr \right|^q d\vartheta \\ &= 2^{q-1} \left(\frac{H_1}{|S^{N-1}|} \right)^{q+1} C_{\text{sph}} \int_{S^{N-1}} \left| \int_0^\infty \nabla_{\vartheta} f(r, \vartheta) r^{N-1} h(r) \frac{|S^{N-1}|}{H_1} dr \right|^q d\vartheta \\ &\leq 2^{q-1} \left(\frac{H_1}{|S^{N-1}|} \right)^q C_{\text{sph}} \int_{S^{N-1}} \int_0^\infty |\nabla_{\vartheta} f(r, \vartheta)|^q r^{N-1} h(r) dr d\vartheta =: K. \end{aligned}$$

Due to (14) we have $r^{-q} |\nabla_{\vartheta} f(r, \vartheta)|^q \leq |\nabla f(x)|^q$ and we infer that

$$K \leq 2^{q-1} \left(\frac{H_1}{|S^{N-1}|} \right)^q C_{\text{sph}} \int_{\mathbb{R}^N} |\nabla f(x)|^q |x|^q h(|x|) dx.$$

By summing up the previous computations we have the claim. \square

We are in the position to prove the Hardy–Poincaré inequality.

Proof of Theorem 2. By Corollary 2.1 with

$$(f) = \frac{|S^{N-1}|}{H_1} \int_0^\infty f(r) r^{N-1} h(r) dr,$$

and H_1 from (4), we have the following inequality

$$\int_0^\infty |f - (f)|^q r^{N-1} h(r) dr \leq 2^q C_M \int_0^\infty |f'|^q r^{N-1+q} h(r) dr$$

for all compactly supported $f \in W^{1,\infty}([0, \infty))$. Hence, also the following radial inequality follows

$$\int_0^\infty |f(r, \vartheta) - \tilde{f}(\vartheta)|^q r^{N-1} h(r) dr \leq C_1 \int_0^\infty |\partial_r f(r, \vartheta)|^q r^{N-1+q} h(r) dr \quad \text{for a.e. } \vartheta \in S^{N-1}$$

with $C_1 = 2^q C_M$. The above inequality implies (18), which by Proposition 2.1 completes the proof of (HP) with $w_1(x) = h(|x|)$ and $w_2(x) = |x|^q h(|x|)$. Bounds on the constant result from the estimates on c_p provided in Theorem 1. \square

2.3. Hardy inequality on \mathbb{R}^N . Proof of Theorem 3.

Proof of Theorem 3. Let us observe that integration by parts and the Cauchy-Schwartz inequality imply

$$\int_{\mathbb{R}^N} |\varphi|^q |\Delta_\theta g| dx \leq q \int_{\mathbb{R}^N} |\varphi|^{q-1} |\nabla \varphi| |\nabla g|^{\theta-1} dx =: I.$$

By the Hölder inequality, we have

$$\begin{aligned} I &= q \int_{\mathbb{R}^N} \left(|\varphi|^{q-1} |\Delta_\theta g|^{\frac{q-1}{q}} \right) \left(\frac{|\nabla \varphi| |\nabla g|^{\theta-1}}{|\Delta_\theta g|^{\frac{q-1}{q}}} \right) dx \\ &\leq q \left(\int_{\mathbb{R}^N} |\varphi|^q |\Delta_\theta g| dx \right)^{(q-1)/q} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^q \frac{|\nabla g|^{q(\theta-1)}}{|\Delta_\theta g|^{q-1}} dx \right)^{1/q}. \end{aligned}$$

Summing up the above remarks, we get

$$\left(\int_{\mathbb{R}^N} |\varphi|^q |\Delta_\theta g| dx \right)^{1/q} \leq q \left(\int_{\mathbb{R}^N} |\nabla \varphi|^q \frac{|\nabla g|^{q(\theta-1)}}{|\Delta_\theta g|^{q-1}} dx \right)^{1/q}.$$

\square

3. EXAMPLES

3.1. Hardy–Poincaré inequalities. As a consequence of Theorem 2, we get the following version of weighted Hardy–Poincaré inequality on \mathbb{R}^N .

Example 3.1. Let $1 < q \leq 2$ if $N = 2$ and $1 < q < N$ if $N \geq 3$. Assume $\alpha < 0 < \beta$, $N + \gamma > 0$, $N + \gamma + \alpha\beta < 0$. Then there exists a finite constant $C = C(N, \alpha, \beta, \gamma, q) > 0$, such that for every compactly supported $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ the following inequality holds true

$$(19) \quad \int_{\mathbb{R}^N} |\varphi - \bar{\varphi}|^q |x|^\gamma (1 + |x|^\beta)^\alpha dx \leq C \int_{\mathbb{R}^N} |\nabla \varphi|^q |x|^{\gamma+q} (1 + |x|^\beta)^\alpha dx,$$

where $\bar{\varphi}$ is an average of φ with respect to $|x|^\gamma (1 + |x|^\beta)^\alpha dx$.

Proof. Note that for

$$(20) \quad h(|x|) = |x|^\gamma(1 + |x|^\beta)^\alpha.$$

it holds that $H_1 < \infty$ since $N + \gamma > 0$ and $N + \gamma + \alpha\beta < 0$. We will show that $H_2(m) < \infty$ for any $m > 0$. We notice that

$$H_2(m) = \max\{A[m], B[m]\},$$

where

$$A[m] := \sup_{t>m} A_1(t)A_2(t) = \sup_{t>m} \left(\int_t^\infty r^{N-1+\gamma}(1+r^\beta)^\alpha dr \left[\int_m^t (r^{N-1+\gamma+q}(1+r^\beta)^\alpha)^{-\frac{1}{q-1}} dr \right]^{q-1} \right)$$

$$B[m] := \sup_{t \in (0,m)} \left(\int_0^t r^{N-1+\gamma}(1+r^\beta)^\alpha dr \left[\int_t^m (r^{N-1+\gamma+q}(1+r^\beta)^\alpha)^{-\frac{1}{q-1}} dr \right]^{q-1} \right).$$

Since $A_1(t) \leq c(N, \alpha, \beta, \gamma, q)t^{N+\gamma+\alpha\beta}$ and $A_2(t) \leq c(N, \alpha, \beta, \gamma, q)(m^{-(N+\gamma+\alpha\beta)} + m^{-(N+\gamma)})$, we have $A[m] < \infty$. On the other hand

$$B[m] \leq \sup_{t<m} \left(|t^{N+\gamma}(1+m^\beta)^{-\frac{\alpha}{q-1}}(t^{-\frac{N+\gamma}{q-1}})^{q-1} \right) \leq c(\alpha, \beta, \gamma, q, m)$$

being finite under the assumed regime. In turn $H_2 = \max\{A[m], B[m]\} < \infty$ for any (fixed) $m > 0$. Therefore, we can apply Theorem 2 to deduce (19) with a constant depending additionally on m . If one chooses m to be a median of $r^{N-1+\gamma}(1+r^\beta)^\alpha dr$, the constant can be estimated by Theorem 3. Median η is estimated in Remark 3.1. \square

Remark 3.1. One can estimate a median η of the weight of Example 3.1, namely $|x|^\gamma(1 + |x|^\beta)^\alpha dx$, as follows

$$\left(\frac{N+\gamma}{2^{|\alpha+1|N+\gamma+\alpha\beta}} \right)^{\frac{1}{N+\gamma}} \leq \eta \leq 2 \frac{|\alpha+1|}{|N+\gamma+\alpha\beta|}.$$

For this it is enough to observe that

$$H_1 \geq \int_1^\infty r^{N-1+\gamma}(1+r^\beta)^\alpha dr \geq 2^\alpha \int_1^\infty r^{N-1+\gamma+\alpha\beta} dr = \frac{2^\alpha}{|N+\gamma+\alpha\beta|},$$

$$\frac{H_1}{2} = \int_\eta^\infty r^{N-1+\gamma}(1+r^\beta)^\alpha dr \leq \int_\eta^\infty r^{N-1+\gamma+\alpha\beta} dr = \frac{\eta^{N+\gamma+\alpha\beta}}{|N+\gamma+\alpha\beta|},$$

and

$$\frac{H_1}{2} = \int_0^\eta r^{N-1+\gamma}(1+r^\beta)^\alpha dr \leq \int_0^\eta r^{N-1+\gamma} dr = \frac{\eta^{N+\gamma}}{|N+\gamma|}.$$

Remark 3.2. We are particularly interested in the special case of Example 3.1 with the choice

$$\gamma = 0, \quad \beta = \frac{p}{p-1}, \quad \text{and} \quad \alpha = -\frac{1}{2-p},$$

for $1 < p < 2$, which find application in asymptotics of fast diffusion equation, see Section 4.

3.2. Hardy inequalities. As a consequence of Theorem 3 we get the following family of Hardy-type inequalities on \mathbb{R}^N . We also refer to [47, 11] where the optimal constant of some of the inequalities in the following family were already computed.

Example 3.2. Let $q > 1$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha < 0 < \beta$, $\gamma + N > 0$, $|\alpha\beta + \gamma + 2| \geq |\gamma + 2|$,

$$(21) \quad \operatorname{sgn}(\alpha\beta + \gamma + 2) = \operatorname{sgn}(\gamma + 2), \quad \text{and} \quad \alpha\beta + \gamma + N > 0.$$

Then for all compactly supported $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ it holds that

$$(22) \quad \int_{\mathbb{R}^N} |\varphi|^q |x|^\gamma (1 + |x|^\beta)^\alpha dx \leq C_H \int_{\mathbb{R}^N} |\nabla \varphi|^q |x|^{\gamma+q} (1 + |x|^\beta)^\alpha dx$$

with a positive, finite constant C_H . If we additionally assume that

$$(23) \quad \alpha\beta + 2(\gamma + 1) + N \leq 0,$$

then the constant C_H is optimal and reads

$$C_H = \left(\frac{q}{\alpha\beta + \gamma + N} \right)^q.$$

Proof. We will apply Theorem 3 with $\theta = 2$. For $g(x) = |x|^{\gamma+2}(1 + |x|^\beta)^\alpha$, it holds

$$\begin{aligned} |\Delta g(x)| &= |x|^\gamma (1 + |x|^\beta)^{\alpha-2} \left| \left[|x|^\beta (\alpha\beta + \gamma + 2)(\alpha\beta + \gamma + N) + (\gamma + 2)(\gamma + N) \right] (1 + |x|^\beta) - |x|^\beta \alpha\beta(\alpha - 1) \right|, \\ |\nabla g(x)|^q |\Delta g(x)|^{1-q} &= |x|^{\gamma+q} (1 + |x|^\beta)^{\alpha-2+q} \left| |x|^\beta (\alpha\beta + \gamma + 2) + \gamma + 2 \right|^q \\ &\cdot \left| \left[|x|^\beta (\alpha\beta + \gamma + 2)(\alpha\beta + \gamma + N) + (\gamma + 2)(\gamma + N) \right] (1 + |x|^\beta) - |x|^\beta \alpha\beta(\alpha - 1) \right|^{1-q}. \end{aligned}$$

Note that $|\Delta_\theta g| \in L^1_{loc}(\mathbb{R}^N)$ since $\gamma + N > 0$. Furthermore, then $c_1 h(|x|) \leq |\Delta g(x)|$ for

$$\begin{aligned} c_1 &= \inf_{|x| \geq 0} \frac{\left| \left[|x|^\beta (\alpha\beta + \gamma + 2)(\alpha\beta + \gamma + N) + (\gamma + 2)(\gamma + N) \right] (|x|^\beta + 1) - |x|^\beta \alpha\beta(\alpha - 1) \right|}{(|x|^\beta + 1)^2} \\ &= \inf_{s \geq 0} \frac{|s^2(\eta + 2)(\eta + N) + s((\eta + 2)(\eta + N) + (\gamma + 2)(\gamma + N) - \alpha\beta(\alpha - 1)) + (\gamma + 2)(N + \gamma)|}{(s + 1)^2} \\ &=: \inf_{s \geq 0} \mathcal{C}_1(s), \end{aligned}$$

where $\eta = \alpha\beta + \gamma$. Notice that under (21) we have $(\eta + 2)(\eta + N) < 0$, $(\gamma + 2)(\gamma + N) < 0$ and $\alpha\beta(\alpha - 1) > 0$. So the numerator of $\mathcal{C}_1(s)$ is separated from 0 and c_1 is well defined. Under additional assumption

$$\alpha\beta + 2(\gamma + 1) + N \leq 0$$

we have

$$c_1 = \mathcal{C}_1(\infty) = (-\alpha\beta - \gamma - 2)(\alpha\beta + \gamma + N).$$

Indeed, let us consider

$$\xi(s) = \log(\mathcal{C}_1(s))$$

and denote the numerator of $\mathcal{C}_1(s)$ by

$$P(s) = s^2(-\eta - 2)(\eta + N) + s((-\eta - 2)(\eta + N) + (-\gamma - 2)(\gamma + N) + \alpha\beta(\alpha - 1)) + (-\gamma - 2)(N + \gamma) := as^2 + bs + c$$

$$\xi'(s) = \frac{2as+b}{P(s)} - \frac{2}{s+1} = \frac{s(2a-b)+(b-2c)}{P(s)(s+1)}.$$

We observe that

$$\begin{aligned} 2a - b &= (-\eta - 2)(\eta + N) - (-\gamma - 2)(\gamma + N) - \alpha\beta(\alpha - 1) \\ &\leq (-\eta - 2)(\eta + N) - (-\gamma - 2)(\gamma + N) \leq 0, \end{aligned}$$

where last inequality is equivalent to $\alpha\beta + 2(\gamma + 1) + N \leq 0$ Therefore ξ' is decreasing, so ξ and further \mathcal{C}_1 attain minimum at 0 or at ∞ . But

$$c_1 = \min\{\mathcal{C}_1(0), \mathcal{C}_1(\infty)\} = \min\{|\gamma + 2||\gamma + N|, |\eta + 2||\eta + N|\} = (-\eta - 2)(\eta + N).$$

On the other hand $|\nabla g(x)|^q |\Delta g(x)|^{1-q} \leq c_2 h(|x|)|x|^q$ for

$$c_2 = \sup_{|x| \geq 0} \frac{|\nabla g(x)|^q |\Delta g(x)|^{1-q}}{h(|x|)|x|^q} = \sup_{|x| \geq 0} \left(\frac{|\nabla g(x)|}{h(|x|)|x|} \right)^q \mathcal{C}_1(|x|)^{1-q}$$

Consider

$$f(s) = \frac{s(-\eta-2)+(-\gamma-2)}{1+s}$$

and

$$(\log f)'(s) = \frac{-\eta-2}{s(-\eta-2)+(-\gamma-2)} - \frac{1}{s+1} = \frac{(-\eta-2)-(-\gamma-2)}{(s+1)(s(-\eta-2)+(-\gamma-2))} \geq 0$$

Therefore $f(s)$ is increasing and under our assumption it holds $\inf_{s \geq 0} \mathcal{C}_1(s) = \mathcal{C}_1(\infty)$

$$c_2 = \sup_{s \geq 0} f(s)^q \mathcal{C}_1(s)^{1-q} = f(\infty)^q \mathcal{C}_1(\infty)^{1-q} = |\alpha\beta + \gamma + 2| |\alpha\beta + \gamma + N|^{1-q}$$

and by Theorem 3 we have $C_H \leq q^q \frac{c_2}{c_1}$. To motivate optimality of C_H in the special case we compare it with the optimal constant in the classical Hardy inequality. Recall that we are in the regime when $c_1 = \mathcal{C}_1(\infty)$ and, consequently, $C_H = q^q (\alpha\beta + \gamma + N)^{-q}$. Let us consider the family of rescaled functions $\varphi_s(x) = \varphi(sx) \in C_c^\infty(\mathbb{R}^N)$ with $s > 0$. We multiply both sides of (11) by $t^{\alpha\beta+\gamma}$ and rearrange them to get

$$(24) \quad \int_{\mathbb{R}^N} |\varphi_s(x)|^q |tx|^\gamma (t^\beta + |tx|^\beta)^\alpha dx \leq C_H \int_{\mathbb{R}^N} |\nabla \varphi_s(x)|^q |tx|^{\gamma+q} (t^\beta + |tx|^\beta)^\alpha dx.$$

After the change of variables $y = xt$ we obtain

$$\int_{\mathbb{R}^N} |\varphi(y)|^q |y|^\gamma (t^\beta + |y|^\beta)^\alpha dy \leq C_H \int_{\mathbb{R}^N} |\nabla \varphi(y)|^q |y|^{\gamma+q} (t^\beta + |y|^\beta)^\alpha dy.$$

Using the Lebesgue Monotone Convergence Theorem, we let $t \searrow 0$ and get

$$(25) \quad \int_{\mathbb{R}^N} |\varphi(y)|^q |y|^{\alpha\beta} dy \leq C_H \int_{\mathbb{R}^N} |\nabla \varphi(y)|^q |y|^{\alpha\beta+\gamma+q} dy,$$

with classical Hardy inequality with power weights where the optimal constant is $q^q(\alpha\beta + \gamma + N)^{-q}$. Therefore, C_H cannot be improved. \square

By the same arguments as in the proof of Example 3.2 one can show the following consequence of Theorem 3.

Example 3.3. *Let $q > 1$, $\alpha \in \mathbb{R}$, $\beta > 0$, $\alpha\beta + N > 0$. Then for all compactly supported $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ it holds that*

$$(26) \quad \int_{\mathbb{R}^N} |\varphi|^q (1 + |x|^\beta)^\alpha dx \leq C_H \int_{\mathbb{R}^N} |\nabla \varphi|^q |x|^q (1 + |x|^\beta)^\alpha dx$$

with a positive, finite constant C_H .

Note that the above example for $q = 2 = \beta$ relates to the main result of [11] used there and in [12] in the study of the asymptotics to fast diffusion equation. Indeed, within such a choice of parameters the left-hand side weight in (26) is the same as in the inequality (1) of [11], but the right-hand weight of [11, (1)] reads $(1 + |x|^2)^{\alpha+1}$.

3.3. Proof of Corollary 1.1.

Proof. We verify assumptions of Example 3.2 applied with $q = 2$, $\alpha = -\frac{p-1}{2-p}$, $\beta = \frac{p}{p-1} > 0$, and $\gamma = -\frac{p}{p-1}$. We see that $\alpha\beta + \gamma + 2 = 2 - \frac{p}{(2-p)(p-1)} < 0$, $\gamma + 2 = 2 - \frac{p}{p-1} < 0$. Since $\alpha < 0$, also $|\alpha\beta + \gamma + 2| \geq |\gamma + 2|$. Moreover, $\alpha\beta + \gamma + N = N - \frac{p}{(2-p)(p-1)} > 0$, which in particular implies that $p > N/(N-1)$ equivalent to $\gamma + N > 0$. Then by Example 3.2 we know that $C_H \in (0, \infty)$. The optimality of C_H follows from the fact that if $N \leq 7$ or $p \notin (p_-, p_+)$, condition (23) is satisfied. \square

4. LONG-TERM ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO DNLE

Before we present the proof of Theorem 4, let us recall important properties of solutions to (1). For an initial datum $u_0 \in L^1(\mathbb{R}^N)$, existence and uniqueness to (1) is settled, see [54, 2]. Moreover, in the range of parameters under consideration, we have that $u(t) \in C^{1,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$, see [67, 39, 40, 34] and mass is conserved, i.e.,

$$\int_{\mathbb{R}^N} u(t, x) dx = \int_{\mathbb{R}^N} u_0(x) dx \quad \text{for } t > 0,$$

For further information about basic properties of solutions to (1) we refer to the monographs [35], [64, Part III] and references therein.

Rates of convergence like (10) with the use of different norms for equation (1) has been a problem attracting a lot of attention. In the case $p = 2$ and $m > 1$, optimal rates of convergence were obtained independently by Carrillo and Toscani [21], and if $m > 1 - 1/N$ by Del Pino and Dolbeault [29] and by Otto [59]. When $p = 2$ and $1 - 1/N < m < 1$, see also [15, 17, 22] for rates of convergence in the stronger norm of uniform relative error. In the case of $p = 2$ and $1 - N/(N+2) < m < 1$ improved convergence rate were obtained by Dolbeault and Toscani

using the *best matching* Barenblatt profile, see [37, 38]. For $p = 2$ and $0 < m < 1 - 1/N$ rates of convergence (optimal on a subrange) were computed, independently, by Carrillo and Vazquez in [22], McCann and Slepcev in [55], Kim and McCann in [52] and, for the whole range $m < 1$ (negative values of m are admitted) by Blanchet, Bonforte, Dolbeault, Grillo and Vázquez in [12]. The aforementioned results are often based on the study of a linearised problem performed in [33], see also the monograph [32].

In the case $m = 1$ and $p \neq 2$ less is known. In the range $2N/(N+1) + 1/(N+1) \leq p < N$, (non-optimal) rates of convergence were obtain by Del Pino and Dolbeault in [30]. Similarly, in the case of the doubly nonlinear equation (non-optimal) rates of convergence were obtain by Del Pino and Dolbeault in [31], by Agueh [3] and by Agueh, Blanchet and Carrillo in [4].

Here we only sketch the proof of Theorem 4 and emphasize where Theorem 2 is applied. For details we refer to [4].

Proof of Theorem 4. We restrict our analysis to the case $p \neq 2$ which is fully covered by [12].

By the scaling properties of the equation (1), we can assume that $D_\star = 1$ and let us define $\mathfrak{B} = \mathfrak{B}_1$. Let us define the *self-similar* change of variables

$$(27) \quad v(\tau, y) := R(t)^N u(t, x), \quad \text{where } \tau := \log(R(t)), \quad y := x/R(t).$$

The main advantage of the change of variables is that the Barenblatt profile $B_1(t, x)$ is transformed into \mathfrak{B} which is stationary in time. The equation satisfied by v is now

$$(28) \quad \frac{\partial v}{\partial \tau} = \operatorname{div} \left(m^{p-1} v^{(1-m)(1-p)} |\nabla v|^{p-2} \nabla v + v y \right).$$

The *relative entropy* with respect to the Barenblatt profile \mathfrak{B} is defined by

$$(29) \quad \mathcal{E}[v|\mathfrak{B}] := \frac{m}{\sigma(\sigma-1)} \int_{\mathbb{R}^N} \left(v^\sigma(y) - \mathfrak{B}^\sigma(y) - \sigma \mathfrak{B}^{\sigma-1}(v(y) - \mathfrak{B}(y)) \right) dy$$

for any $0 \leq v \in L^1(\mathbb{R}^N)$ where $\sigma = m + (p-2)/(p-1)$. Notice that, by the convexity of the function $u \mapsto m u^\sigma / \sigma(\sigma-1)$, the relative entropy is nonnegative functional $\mathcal{E}[v|\mathfrak{B}] \geq 0$. The derivative of $\mathcal{E}[v(\tau), \mathfrak{B}]$ along the flow (28) is called *Fisher information* and is formally given by

$$(30) \quad \begin{aligned} \mathcal{I}[v|\mathfrak{B}] &:= -\frac{d\mathcal{E}[v(\tau)|\mathfrak{B}]}{d\tau} \\ &= m^p \int_{\mathbb{R}^N} v(\tau) \left(\frac{\nabla v(\tau)}{v(\tau)^{2-\sigma}} - \frac{\nabla \mathfrak{B}}{\mathfrak{B}^{2-\sigma}} \right) \cdot \left(v(\tau)^{(\sigma-2)(p-1)} \frac{\nabla v}{|\nabla v|^{2-p}} - \mathfrak{B}^{(\sigma-2)(p-1)} \frac{\nabla \mathfrak{B}}{|\nabla \mathfrak{B}|^{2-p}} \right) dy, \end{aligned}$$

where $a \cdot b$ is the standard scalar product between $a, b \in \mathbb{R}^N$. Since $(a-b) \cdot (|a|^{p-2}a - |b|^{p-2}b) \geq 0$ for $p > 1$ and any $a, b \in \mathbb{R}^N$, we infer that $\mathcal{I}[v(\tau)|\mathfrak{B}] \geq 0$.

The convergence of $v(\tau)$ to \mathfrak{B} as $\tau \rightarrow \infty$ follows from the decaying in time of the entropy functional. Once the inequality

$$(31) \quad \mathcal{E}[v(\tau)|\mathfrak{B}] \leq e^{-\mu\tau} \mathcal{E}[v(0)|\mathfrak{B}] \quad \forall \tau \geq \tau_0$$

is established for some $\mu, \tau_0 > 0$, then inequality (10) can be obtained by combining (31) with the Csiszár-Kullback inequality

$$\|v(\tau) - \mathfrak{B}\|_{L^1(\mathbb{R}^N)}^2 \leq C \mathcal{E}[v(\tau)|\mathfrak{B}],$$

for which we refer to [19, 3]. Finally, to obtain the algebraic rate in the t variable, one should take into account the self-similar change of variables (27) which also gives the relation $\lambda = \mu/\vartheta$.

In order to prove (31), it suffices to have

$$(32) \quad \mu \mathcal{E}[v(\tau)|\mathfrak{B}] \leq \mathcal{J}[v(\tau)|\mathfrak{B}].$$

for τ large enough. Indeed, (31) follows by combining a Gronwall-type argument with inequality (32) and (30). In the range of parameters under consideration, inequality (32) does not hold for any function $f \in C_c^\infty(\mathbb{R}^N)$, due to scaling arguments. This is very different from the case, $1 > m(p-1) > 1 - (p-1)/N$, where (32) is equivalent to a class of Gagliardo–Nirenberg–Sobolev inequalities, see [3, 15, 36, 29]. However, when $v(\tau)$ is close enough to \mathfrak{B} , then (32) holds. In order to prove so, let us introduce the following weights:

$$w_1(x) = \frac{1}{m} \left(1 + \frac{(1-\sigma)(p-1)}{pm} |x|^{\frac{p}{p-1}} \right)^{\frac{2-\sigma}{\sigma-1}}, \quad w_2(x) = |x|^{\frac{p-2}{p-1}} \left(1 + \frac{(1-\sigma)(p-1)}{pm} |x|^{\frac{p}{p-1}} \right)^{\frac{1}{\sigma-1}}$$

and for any $\varepsilon \in (0, 1)$

$$w_{2,\varepsilon}(x) = \left(1 + \frac{(1-\sigma)(p-1)}{pm} |x|^{\frac{p}{p-1}} \right)^{\frac{1}{\sigma-1}} \left(\varepsilon + |x|^{\frac{1}{p-1}} \right)^{-(2-p)}.$$

Let us define the *linearised relative entropy* as

$$E[\varphi] := \int_{\mathbb{R}^N} |\varphi - \bar{\varphi}|^2 w_1(x) dx$$

and the *linearised Fisher information* that for $p > 2$ takes a form

$$I[\varphi] := \int_{\mathbb{R}^N} |\nabla \varphi|^2 w_2(x) dx.$$

When $1 < p < 2$, the role of the *linearised Fisher information* is played by the quantity $I_\varepsilon[\varphi]$ defined as $I[\varphi]$ with the weight $w_{2,\varepsilon}$ instead of w_2 . The inequalities

$$(33) \quad C_{p,m}^{(1)} E[\varphi] \leq I[\varphi] \quad \text{and} \quad C_{p,m}^{(2)} E[\varphi] \leq I_\varepsilon[\varphi]$$

hold within the range of parameters (7) and for any function $\varphi \in C^{1,\alpha}(\mathbb{R}^N)$ and $\alpha \in (0, 1)$. The inequalities in (33) follow from Example 3.1 with $\gamma = 0$, $\beta = p/(p-1)$ and $\alpha = (2-\sigma)/(\sigma-1)$ and the fact that $c_1 w_1(|x|) \leq (1 + |x|^\beta)^\alpha$, $|x|^2(1 + |x|^\beta)^\alpha \leq c_2 w_2(|x|)$ and $|x|^2(1 + |x|^\beta)^\alpha \leq c_{2,\varepsilon} w_{2,\varepsilon}(|x|)$, for some finite $c_1, c_2, c_{2,\varepsilon} > 0$ depending on m, p .

By [4, Proposition 4.2], if $p > 2$ there exist $\tau_0, \kappa_1, \kappa_2 > 0$ such that, for all $\tau > \tau_0$

$$(34) \quad I[v(\tau) - \mathfrak{B}] \leq \kappa_1 \mathcal{J}[v(\tau)|\mathfrak{B}] + \kappa_2 E[v(\tau) - \mathfrak{B}].$$

For $1 < p < 2$, the same inequality holds but with $I_\varepsilon[v(\tau) - \mathfrak{B}]$ instead of $I[v(\tau) - \mathfrak{B}]$ in the left-hand-side. The constant κ_2 can be taken arbitrary small provided that τ_0 is large enough. By [4, Proposition 4.1], it holds for any $\tau > 0$ that

$$(35) \quad \frac{1}{m} D_1^{\frac{1}{m(p-1)-1}} \mathcal{E}[v(\tau)|\mathfrak{B}] \leq E[v(\tau) - \mathfrak{B}].$$

Combining inequalities (35), (34), (33) and provided τ_0 is large enough, we find (32) with $\mu = D_1^{\frac{1}{m(p-1)-1}} (C_{p,m}^{(i)} - \kappa_2) / m\kappa_1$ with $i = 1$ when $p > 2$ and $i = 2$ when $1 < p < 2$. Choosing τ_0 large enough so that $C_{p,m}^{(i)} - \kappa_2 \geq C_{p,m}^{(i)}/2$, we conclude that the rate of convergence λ can be taken as

$$\lambda = \frac{D_1^{\frac{1}{m(p-1)-1}} C_{p,m}^{(i)}}{2m\kappa_2(p - N[1 - m(p-1)])}.$$

The proof is concluded. □

ACKNOWLEDGEMENTS

Part of this project was carried out in Univesidad Autónoma de Madrid, when IC was visiting Matteo Bonforte. Both authors are grateful to him for guidance, patience, and invaluable help. Additionally, IC would like to thank Michał Strzelecki for insightful discussions and Błażej Miasojedow for essential help with computations.

© 2021 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

REFERENCES

- [1] M. Agueh. Asymptotic behavior for doubly degenerate parabolic equations. *C. R. Math. Acad. Sci. Paris*, 337(5):331–336, 2003.
- [2] M. Agueh. Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory. *Adv. Differential Equations*, 10(3):309–360, 2005.
- [3] M. Agueh. Rates of decay to equilibria for p -Laplacian type equations. *Nonlinear Anal.*, 68(7):1909–1927, 2008.
- [4] M. Agueh, A. Blanchet, and J. A. Carrillo. Large time asymptotics of the doubly nonlinear equation in the non-displacement convexity regime. *J. Evol. Equ.*, 10(1):59–84, 2010.
- [5] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.
- [6] C. Ané, S. Blachère, D. Chafai, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.
- [7] D. G. Aronson. Regularity properties of flows through porous media: The interface. *Arch. Rational Mech. Anal.*, 37:1–10, 1970.
- [8] P. Baras and J. A. Goldstein. The heat equation with a singular potential. *Trans. Amer. Math. Soc.*, 284(1):121–139, 1984.
- [9] G. Barbatis, S. Filippas, and A. Tertikas. A unified approach to improved L^p Hardy inequalities with best constants. *Trans. Amer. Math. Soc.*, 356(6):2169–2196, 2004.
- [10] F. Barthe and C. Roberto. Modified logarithmic Sobolev inequalities on \mathbb{R} . *Potential Anal.*, 29(2):167–193, 2008.

- [11] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.-L. Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. *C. R. Math. Acad. Sci. Paris*, 344(7):431–436, 2007.
- [12] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. *Arch. Ration. Mech. Anal.*, 191(2):347–385, 2009.
- [13] S. G. Bobkov and F. Götze. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.*, 163(1):1–28, 1999.
- [14] V. Bögelein, F. Duzaar, P. Marcellini, and C. Scheven. Doubly nonlinear equations of porous medium type. *Arch. Ration. Mech. Anal.*, 229(2):503–545, 2018.
- [15] M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov. Stability in Gagliardo–Nirenberg–Sobolev inequalities: flows, regularity and the entropy method, <https://arxiv.org/abs/2007.03674>, 2021.
- [16] M. Bonforte, G. Grillo, and J. L. Vázquez. Special fast diffusion with slow asymptotics: Entropy method and flow on a riemannian manifold. *Arch. Ration. Mech. Anal.*, 196(2):631–680, 2009.
- [17] M. Bonforte and N. Simonov. Fine properties of solutions to the cauchy problem for a fast diffusion equation with Caffarelli–Kohn–Nirenberg weights, <https://arxiv.org/abs/2002.09967>, 2020.
- [18] M. Bonforte, N. Simonov, and D. Stan. The Cauchy problem for the fast p -Laplacian evolution equation. Characterization of the global Harnack principle and fine asymptotic behaviour. 2021.
- [19] J. A. Carrillo, A. Jüngel, P. A. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatsh. Math.*, 133(1):1–82, 2001.
- [20] J. A. Carrillo, C. Lederman, P. A. Markowich, and G. Toscani. Poincaré inequalities for linearizations of very fast diffusion equations. *Nonlinearity*, 15(3):565–580, 2002.
- [21] J. A. Carrillo and G. Toscani. Asymptotic L^1 -decay of solutions of the porous medium equation to self-similarity. *Indiana Univ. Math. J.*, 49(1):113–142, 2000.
- [22] J. A. Carrillo and J. L. Vázquez. Fine asymptotics for fast diffusion equations. *Comm. Partial Differential Equations*, 28(5-6):1023–1056, 2003.
- [23] I. Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
- [24] I. Chlebicka and A. Zatorska-Goldstein. Existence to nonlinear parabolic problems with unbounded weights. *J. Evol. Equ.*, 19(1):1–19, 2019.
- [25] S.-K. Chua. On weighted Sobolev interpolation inequalities. *Proc. Amer. Math. Soc.*, 121(2):441–449, 1994.
- [26] L. D’Ambrosio. Hardy inequalities related to Grushin type operators. *Proc. Amer. Math. Soc.*, 132(3):725–734, 2004.
- [27] L. D’Ambrosio. Hardy-type inequalities related to degenerate elliptic differential operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(3):451–486, 2005.
- [28] L. D’Ambrosio and S. Dipierro. Hardy inequalities on Riemannian manifolds and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):449–475, 2014.
- [29] M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9)*, 81(9):847–875, 2002.
- [30] M. Del Pino and J. Dolbeault. Nonlinear diffusions and optimal constants in Sobolev type inequalities: asymptotic behaviour of equations involving the p -Laplacian. *C. R. Math. Acad. Sci. Paris*, 334(5):365–370, 2002.
- [31] M. Del Pino and J. Dolbeault. Asymptotic behavior of nonlinear diffusions. *Mathematical Research Letters*, 10(4):551–557, 2003.
- [32] J. Denzler, H. Koch, and R. J. McCann. Higher-order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach. *Memoirs of the American Mathematical Society*, 234(1101):vi+81, 2015.
- [33] J. Denzler and R. J. McCann. Fast diffusion to self-similarity: complete spectrum, long-time asymptotics, and numerology. *Arch. Ration. Mech. Anal.*, 175(3):301–342, 2005.
- [34] E. DiBenedetto. *Degenerate parabolic equations*. New York, NY: Springer-Verlag, 1993.
- [35] E. DiBenedetto, U. Gianazza, and V. Vespi. *Harnack’s Inequality for Degenerate and Singular Parabolic Equations*. Springer New York, 2012.
- [36] J. Dolbeault. Functional inequalities: nonlinear flows and entropy methods as a tool for obtaining sharp and constructive results, <https://arxiv.org/abs/2107.08219>, 2021.

- [37] J. Dolbeault and G. Toscani. Fast diffusion equations: matching large time asymptotics by relative entropy methods. *Kinetic and Related Models*, 4(3):701–716, 2011.
- [38] J. Dolbeault and G. Toscani. Improved interpolation inequalities, relative entropy and fast diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(5):917–934, 2013.
- [39] F. G. Düzgün, S. Mosconi, and V. Vespri. Harnack and pointwise estimates for degenerate or singular parabolic equations. In *Contemporary research in elliptic PDEs and related topics*, volume 33 of *Springer INdAM Ser.*, pages 301–368. Springer, Cham, 2019.
- [40] S. Fornaro, E. Henriques, and V. Vespri. Regularity results for a class of doubly nonlinear very singular parabolic equations. *Nonlinear Anal.*, 205:112213, 30, 2021.
- [41] S. Fornaro, M. Sosio, and V. Vespri. Harnack type inequalities for some doubly nonlinear singular parabolic equations. *Discrete Contin. Dyn. Syst.*, 35(12):5909–5926, 2015.
- [42] J. P. García Azorero and I. Peral Alonso. Hardy inequalities and some critical elliptic and parabolic problems. *J. Differential Equations*, 144(2):441–476, 1998.
- [43] N. Ghoussoub. *Functional inequalities : new perspectives and new applications*. American Mathematical Society, Providence, Rhode Island, 2013.
- [44] N. Ghoussoub and A. Moradifam. Bessel pairs and optimal Hardy and Hardy-Rellich inequalities. *Math. Ann.*, 349(1):1–57, 2011.
- [45] C. E. Gutiérrez and R. L. Wheeden. Sobolev interpolation inequalities with weights. *Trans. Amer. Math. Soc.*, 323(1):263–281, 1991.
- [46] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.
- [47] X. Huang and D. Ye. First order Hardy inequalities revisited, <http://arxiv.org/pdf/2109.05471v1>:PDF. 2021.
- [48] K. Hutter. Mathematical foundation of ice sheet and ice shelf dynamics. A physicist’s view. In *Free boundary problems: theory and applications (Crete, 1997)*, volume 409 of *Chapman & Hall/CRC Res. Notes Math.*, pages 192–203. Chapman & Hall/CRC, Boca Raton, FL, 1999.
- [49] A. V. Ivanov, P. Z. Mkrtychyan, and V. Yaeger. Existence and uniqueness of a regular solution of the first initial-boundary value problem for a class of doubly nonlinear parabolic equations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 213(Kraev. Zadachi Mat. Fiz. Smezh. Voprosy Teor. Funktsii. 25):48–65, 224–225, 1994.
- [50] A. Kałamańska and K. Pietruska-Pałuba. On a variant of the Gagliardo-Nirenberg inequality deduced from the Hardy inequality. *Bull. Pol. Acad. Sci. Math.*, 59(2):133–149, 2011.
- [51] A. S. Kalashnikov. Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations. *Russian Mathematical Surveys*, 42(2):169–222, 1987.
- [52] Y. J. Kim and R. J. McCann. Potential theory and optimal convergence rates in fast nonlinear diffusion. *J. Math. Pures Appl. (9)*, 86(1):42–67, 2006.
- [53] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969. Second English edition, revised and enlarged, Translated from the Russian by Richard A. Silverman and John Chu.
- [54] J. Li. Cauchy problem and initial trace for a doubly degenerate parabolic equation with strongly nonlinear sources. *J. Math. Anal. Appl.*, 264(1):49–67, 2001.
- [55] R. J. McCann and D. Slepčev. Second-order asymptotics for the fast-diffusion equation. *International Mathematics Research Notices*, pages Art. ID 24947, 22, 2006.
- [56] L. Miclo. Quand est-ce que des bornes de Hardy permettent de calculer une constante de Poincaré exacte sur la droite? *Ann. Fac. Sci. Toulouse Math. (6)*, 17(1):121–192, 2008.
- [57] E. Mitidieri and S. I. Pokhozhaev. Absence of positive solutions for quasilinear elliptic problems in \mathbf{R}^N . *Tr. Mat. Inst. Steklova*, 227(Issled. po Teor. Differ. Funkts. Mnogikh Perem. i ee Prilozh. 18):192–222, 1999.
- [58] B. Muckenhoupt. Hardy’s inequality with weights. *Studia Math.*, 44:31–38, 1972.

- [59] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.
- [60] L. Schätzler. The obstacle problem for singular doubly nonlinear equations of porous medium type. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 31(3):503–548, 2020.
- [61] I. Skrzypczak. Hardy-type inequalities derived from p -harmonic problems. *Nonlinear Anal.*, 93:30–50, 2013.
- [62] I. Skrzypczak. Hardy-Poincaré type inequalities derived from p -harmonic problems. In *Calculus of variations and PDEs*, volume 101 of *Banach Center Publ.*, pages 225–238. Polish Acad. Sci. Inst. Math., Warsaw, 2014.
- [63] J. L. Vázquez. *Smoothing and decay estimates for nonlinear diffusion equations*, Equations of porous medium type, volume 33 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2006.
- [64] J. L. Vázquez. *The porous medium equation*, Mathematical theory. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.
- [65] J. L. Vázquez. The mathematical theories of diffusion: Nonlinear and fractional diffusion. In *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions*, pages 205–278. Springer International Publishing, 2017.
- [66] J. L. Vázquez and E. Zuazua. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. *J. Funct. Anal.*, 173(1):103–153, 2000.
- [67] V. Vespi and M. Vestberg. An extensive study of the regularity of solutions to doubly singular equations. *Advances in Calculus of Variations*, 2021.
- [68] Z. Wu, J. Zhao, J. Yin, and H. Li. *Nonlinear diffusion equations*. Singapore: World Scientific, 2001.

IWONA CHLEBICKA, FACULTY OF MATHEMATICS, INFORMATICS AND MECHANICS, UNIVERSITY OF WARSAW, UL. BANACHA 2, 02-097 WARSAW, POLAND

Email address: i.chlebicka@mimuw.edu.pl

NIKITA SIMONOV, CEREMADE, UMR CNRS N° 7534, UNIVERSITÉ PARIS-DAUPHINE, PSL RESEARCH UNIVERSITY, PLACE DE LATTRE DE TASSIGNY, 75775 PARIS CEDEX 16, FRANCE.

Email address: simonov@ceremade.dauphine.fr