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1 GLOBAL EXISTENCE OF SMOOTH SOLUTIONS FOR A 2 NON-CONSERVATIVE BITEMPERATURE EULER MODEL*

DENISE AREGBA-DRIOLLET[†], STÉPHANE BRULL[†], AND YUE-JUN PENG[‡]

Abstract. The bitemperature Euler model describes a crucial step of Inertial Confinement 4 Fusion (ICF) when the plasma is quasineutral while ionic and electronic temperatures remain distinct. 5 6 The model is written as a first-order hyperbolic system in non-conservative form with partially dissipative source terms. We consider the polytropic case for both ions and electrons with different γ -law pressures. The system does not fulfill the Shizuta-Kawashima condition and the physical 8 entropy, which is a strictly convex function, does not provide a symmetrizer of the system. In this 9 paper we exhibit a symmetrizer to apply the result on the local existence of smooth solutions in 11 several space dimensions. In the one-dimensional case we establish energy and dissipation estimates 12leading to global existence for small perturbations of equilibrium states.

13 **Key words.** non-conservative hyperbolic system, partial dissipation, symmetrization, energy 14 estimates, Euler type model for plasmas

15 **AMS subject classifications.** 35L60, 35F55, 35Q31, 76N10, 76W05

3

1. Introduction. This paper is devoted to the study of the global existence of 16 smooth solutions near constant equilibrium states for a bitemperature Euler system. 1718 This fluid model describes the interaction of a mixture of one species of ions and one species of electrons in thermal nonequilibrium, with applications in the field of 19Inertial Confinement Fusion (ICF). It was derived from a kinetic model by using a 20hydrodynamic limit and the Boltzmann entropy. For this kinetic model, a Discrete 2122 Velocity Model (DVM) method with an asymptotic preserving discretization toward 23 Euler equations was obtained. The kinetic approach also allows to design numerical schemes for the bit mperature Euler equations. See [1, 5]. 24

We denote by ρ_e and ρ_i the electronic and ionic densities, $\rho = \rho_e + \rho_i$ the total density, m_e and m_i the related masses, c_e and c_i the mass fractions. These variables satisfy

28 (1.1)
$$\rho_e = m_e n_e = c_e \rho, \quad \rho_i = m_i n_i = c_i \rho, \quad m_e > 0, \quad m_i > 0, \quad c_e + c_i = 1.$$

Quasineutrality is assumed, so that the ionization ratio $Z = n_e/n_i$ is a constant. This implies that the electronic and ionic mass fractions are constant and given by

31 (1.2)
$$c_e = \frac{Zm_e}{m_i + Zm_e}, \quad c_i = \frac{m_i}{m_i + Zm_e}$$

We suppose that the ionic and electronic velocities are equal: $u_e = u_i = u$, and the pressure of each species satisfies a gamma-law with its own γ exponent :

34 (1.3)
$$p_e = (\gamma_e - 1)\rho_e\varepsilon_e = n_ek_BT_e, \quad p_i = (\gamma_i - 1)\rho_i\varepsilon_i = n_ik_BT_i, \quad \gamma_e > 1, \quad \gamma_i > 1,$$

where k_B is the Boltzmann constant $(k_B > 0)$, ε_{α} and T_{α} represent respectively the internal specific energy and the temperature of species α for $\alpha = e, i$.

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Denoting by $|\cdot|$ the Euclidean norm in \mathbb{R}^d , the total energies for the particles are defined by

39 (1.4)
$$\mathcal{E}_{\alpha} = \rho_{\alpha}\varepsilon_{\alpha} + \frac{1}{2}\rho_{\alpha}|u|^{2} = c_{\alpha}\left(\rho\varepsilon_{\alpha} + \frac{1}{2}\rho|u|^{2}\right), \quad \alpha = e, i.$$

We denote by $\nu \geq 0$ the interaction coefficient between the electronic and ionic tem-40 peratures. Physically this coefficient is a complicated function of the electronic and 41 ionic temperatures and of ρ , see the NRL plasma formulary [11]. A rigorous derivation 42 of ν is obtained via a kinetic underlying formulation [1]. It gives $\nu(\rho) = K\rho$ where 43 K is a positive constant. This expression of ν implies that more dense is the plasma, 44 faster it reaches the thermal equilibrium. In order to simplify the notation, we as-45 sume that ν is a sufficiently smooth function of ρ , denoted by $\nu = \nu(\rho)$, and satisfies 46 $\nu(\rho) > 0$ for $\rho > 0$. In particular, it suffices to assume that $\nu(1) > 0$ in the study of 47 the global existence of smooth solutions for ρ near 1. From the proof of the main the-48orem, we will see easily that global existence still holds when ν is a smooth function 49 of (T_e, T_i, ρ) and remains positive at an equilibrium point $(T_e, T_i, \rho) = (\overline{T}, \overline{T}, 1)$ for a 50positive constant \overline{T} .

52 The model consists of two conservative equations for mass and momentum and 53 two non-conservative equations for each energy:

54 (1.5)
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla(p_e + p_i) = 0, \\ \partial_t \mathcal{E}_e + \operatorname{div}\left(u(\mathcal{E}_e + p_e)\right) - u \cdot (c_i \nabla p_e - c_e \nabla p_i) = \rho \nu(T_i - T_e), \\ \partial_t \mathcal{E}_i + \operatorname{div}\left(u(\mathcal{E}_i + p_i)\right) + u \cdot (c_i \nabla p_e - c_e \nabla p_i) = -\rho \nu(T_i - T_e), \end{cases}$$

where " \cdot " stands for the inner product in \mathbb{R}^d . This is a non-conservative hyperbolic system which can be written in the synthetic form

57 (1.6)
$$\partial_t \mathcal{W} + \sum_{j=1}^d \mathcal{C}_j(\mathcal{W}) \partial_{x_j} \mathcal{W} = F(\mathcal{W}).$$

58 Now we introduce

59 (1.7)
$$\begin{cases} \eta(\rho,\rho u, \mathcal{E}_e, \mathcal{E}_i) = -\sum_{\alpha=e,i} \frac{\rho_\alpha}{b_\alpha} \ln\left(\frac{(\gamma_\alpha - 1)\rho_\alpha \varepsilon_\alpha}{\rho_\alpha^{\gamma_\alpha}}\right),\\ \phi(\rho,\rho u, \mathcal{E}_e, \mathcal{E}_i) = \eta(\rho,\rho u, \mathcal{E}_e, \mathcal{E}_i)u, \end{cases}$$

60 where

61 (1.8)
$$b_{\alpha} = \frac{(\gamma_{\alpha} - 1)m_{\alpha}}{k_B} > 0, \ \alpha = e, i.$$

62 It was proved in [1] (see Theorem 2.9) that the functions (η, ϕ) defined in (1.7) are a

pair of entropy-entropy flux of (1.5), and η is strictly convex in the set of state space Ω given by

65
$$\Omega = \{ (\rho, u, \varepsilon_e, \varepsilon_i) \in \mathbb{R}^{d+3} \mid \rho > 0, \, \varepsilon_e > 0, \, \varepsilon_i > 0 \}.$$

66 Moreover, any smooth solution of the system satisfies the entropy equality

67 (1.9)
$$\partial_t \eta(\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i) + \operatorname{div} \phi(\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i) = -\frac{\nu \rho}{T_e T_i} (T_e - T_i)^2,$$

which is a partially dissipative condition of the system. It is known that the secondorder derivative of a strictly convex entropy provides a symmetrizer of a hyperbolic system in conservative form (see [9, 3]). Unfortunately, the equations for \mathcal{E}_e and \mathcal{E}_i in (1.5) are not in conservative form. As already noticed in [2], $\eta''(\mathcal{W})$ is not a symmetrizer of system (1.5). For the sake of completeness we prove this result in the Appendix of the present article.

According to the theory on the symmetrizable hyperbolic system [14, 12, 15], the existence of a symmetrizer is very important to study smooth solutions in Sobolev spaces. Such a symmetrizer for (1.5) is constructed in Section 2 in any space dimension. It implies the local existence of smooth solutions. See $\mathcal{B}_0(\mathcal{V})$ defined in (2.10) and Proposition 2.1.

In order to study global existence, we may introduce the total energy \mathcal{E} and the total pressure p defined by

81
$$\mathcal{E} = \mathcal{E}_e + \mathcal{E}_i, \quad p = p_e + p_i,$$

82 From (1.3) and (1.5), we have

83
$$\mathcal{E} = \frac{p_e}{\gamma_e - 1} + \frac{p_i}{\gamma_i - 1} + \frac{1}{2}\rho|u|^2, \quad p = \rho\left[(\gamma_e - 1)c_e\varepsilon_e + (\gamma_i - 1)c_i\varepsilon_i\right]$$

84 and

85 (1.10)
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ \partial_t \mathcal{E} + \operatorname{div}\left(u(\mathcal{E} + p)\right) = 0, \quad t > 0, \quad x \in \mathbb{R}^d. \end{cases}$$

The last equation in (1.10) shows that the total energy is a conservative variable. If $\gamma_e = \gamma_i$, we introduce a total internal specific energy ε by $\varepsilon = c_e \varepsilon_e + c_i \varepsilon_i$. Then

88
$$\mathcal{E} = \rho \varepsilon + \frac{1}{2} \rho |u|^2, \quad p = (\gamma_e - 1) \rho \varepsilon.$$

Therefore, (1.10) becomes the gas dynamics equations. In this case, system (1.5) is 89 decoupled and contains (1.10). It is known that smooth solutions to the gas dynamics 90 equations blow up in finite time [13, 23]. Hence, global existence is not expected. 91 In physically realistic situations, one can define a weak solution containing shocks. 93 Existence and uniqueness of weak entropy solutions is rather well understood for onedimensional strictly hyperbolic systems of conservation laws, see [4] and references 94therein. For systems with non-conservative products, the authors of [8] gave a def-95 inition of shocks, but to our knowledge there is no result on the existence of such 96 solutions for (1.5). 97

In what follows, we consider the Cauchy problem for (1.5) near constant equilibrium states in case $\gamma_e \neq \gamma_i$. Let us introduce

100
$$\mathcal{V} = \left(\rho, u^T, \varepsilon_e, \varepsilon_i\right)^T$$
.

101 An equilibrium state $\bar{\mathcal{V}}$ is a constant solution of (1.5). We consider in particular an 102 equilibrium state with zero velocity. Let

103
$$\overline{\mathcal{V}} = (1, 0, \overline{\varepsilon}_e, \overline{\varepsilon}_i)^T,$$

104 be such an equilibrium state with $\bar{\varepsilon}_e > 0$ and $\bar{\varepsilon}_i > 0$.

105 System (1.5) is supplemented by an initial condition

106 (1.11)
$$t = 0: \quad \mathcal{V} = \mathcal{V}_0(x) \stackrel{def}{=} \left(\rho_0(x), u_0^T(x), \varepsilon_{e0}(x), \varepsilon_{i0}(x)\right)^T, \quad x \in \mathbb{R}^d$$

For a positive integer m we denote by $H^m(\mathbb{R}^d)$ the usual Sobolev space equipped with the norm $\|\cdot\|_m$. The result of the global existence of solutions holds in one space dimension and can be stated as follows.

110 THEOREM 1.1. Let d = 1 and $m \ge 2$. Assume $\mathcal{V}_0 - \bar{\mathcal{V}} \in H^m(\mathbb{R})$ and $\gamma_e \ne \gamma_i$. 111 There are two positive constants c and κ_0 such that if $\|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m \le \kappa_0$, then the 112 Cauchy problem (1.5) and (1.11) admits a unique global solution \mathcal{V} satisfying $\mathcal{V} - \bar{\mathcal{V}} \in$ 113 $C(\mathbb{R}^+; H^m(\mathbb{R})) \cap C^1(\mathbb{R}^+; H^{m-1}(\mathbb{R}))$. Moreover,

114 (1.12)
$$\sup_{t\in\mathbb{R}^+} \|\mathcal{V}(t,\cdot) - \bar{\mathcal{V}}\|_m \le c \|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m.$$

For conservative hyperbolic systems with source terms, the global existence of 115smooth solutions near constant equilibrium states was proved in [10, 26] in a general 116 framework under two main conditions. A typical example in this framework can be 117118 seen in [24, 7] for the gas dynamics equations with damping. The first condition required in [10, 26] is an entropy dissipation near an equilibrium state. It implies in 119particular an L^2 energy estimate of solutions. The second one is the classical Shizuta-120 Kawashima condition (SK) at the equilibrium state [22]. Unfortunately, these two 121conditions are not satisfied by system (1.5). The first condition obviously fails because 122 123 (1.5) is not a conservative system. However, it is known that (SK) is not a necessary condition for the global existence of smooth solutions. There do exist conservative 124 systems for which global existence holds without this condition. We refer the reader 125to [27, 6, 19, 17] for examples in which different techniques are employed to avoid 126condition (SK). 127

Thus, it is important to establish a global existence result for a class of systems 128 129including at least one of these examples. In [16] the authors studied energy estimates of smooth solutions near non-constant equilibrium states for conservative systems. In 130one space dimension, they obtained global existence for systems violating condition 131(SK) but admitting a very special structure. This allows them to give a proof of global 132existence by using only a partially dissipative condition via an entropy dissipation. 133 This situation is different from that of the present paper. In the proof of Theorem 1.1, 134we not only need a partially dissipative condition but also a dissipation estimate for 135 other variables (see Lemma 3.5). In [21] the authors tried to explore a link between the 136linear degeneracy of characteristic fields and condition (SK) for conservative systems. 137 Under restrictive conditions, they obtained time-decay estimates of solutions which 138139 imply global existence. One can check that the conditions in [16] and [21] are not fulfilled by (1.5) and the systems in [27, 6, 19, 17]. 140

Up to our knowledge, Theorem 1.1 provides a first result on the global existence 141 of smooth solutions for a non-conservative partially dissipative hyperbolic system 142 with source terms without condition (SK). The proof of this theorem is based on 143the local existence of solutions and uniform energy estimates with respect to time 144through Lagrangian coordinates. It consists of three steps. The first step concerns 145an L^2 energy estimate. For this purpose, the entropy equality (1.9) is not sufficient 146 because the system is not in conservative form. We need further to prove equilibrium 147conditions between the system and the entropy η given in (1.7) at the equilibrium 148 state. The verification of these conditions is very complicated and tedious for (1.5). 149150To avoid this, we turn to consider the Cauchy problem in Lagrangian coordinates

where these conditions can be easily checked (see Lemma 3.1). The second step is to

152 establish higher-order energy estimates with a dissipation estimate for $T_e - T_i$. This is

a classical step which is done by choosing an appropriate symmetrizer of the system (see Lemma 3.4). In the last step, we prove a dissipation estimate for $(\nabla u, \nabla p)$

(see Lemma 3.5). In view of special structures of the system, these estimates are sufficient to obtain the global existence of solutions in Lagrangian coordinates. Then Theorem 1.1 follows from the equivalence result for the solutions between Eulerian

and Lagrangian coordinates. Remark that in the proof of Theorem 1.1, we need to use different independent unknown variables in different energy estimates. The difficulty on the lack of condition (SK) for system (1.5) is overcome by choosing appropriate

161 variables connected by C^{∞} -diffeomorphisms.

Finally, we point out that there exists a result on the global existence of solutions for partially dissipative hyperbolic systems in non-conservative form which satisfy condition (SK). However, the space dimension is required to be bigger than 3 [20] (see Theorem 2.4). System (1.5) is not included in this framework since it does not satisfy condition (SK). So far, global existence in several space dimensions is an open problem for (1.5).

168 This paper is organized as follows. In the next section, we first exhibit a symmetrizer to apply the result on the local existence of smooth solutions in several space 169dimensions. Then we study the structure of the system in one space dimension in Eu-170 lerian and Lagrangian coordinates. In particular, we show that system (1.5) does not 171satisfy condition (SK). We also state a result on the global existence of solutions for 172173the system in Lagrangian coordinates (see Theorem 2.3). Section 3 is devoted to the proof of the energy estimates in the three steps mentioned above. In the last section, 174 we give the proof of Theorem 2.3 and then the proof of Theorem 1.1 by using a result 175on the equivalence of solutions for the Cauchy problem in Eulerian and Lagrangian 176coordinates. 177

178 **2.** Study of the bitemperature Euler model.

179 **2.1. Symmetrization of the system.** System (1.5) can be written in the form

$$(2.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ \partial_t \mathcal{E}_e + \operatorname{div}\left(u(\mathcal{E}_e + p_e)\right) - u \cdot (c_i \nabla p_e - c_e \nabla p_i) = \nu(\rho)v, \\ \partial_t \mathcal{E}_i + \operatorname{div}\left(u(\mathcal{E}_i + p_i)\right) + u \cdot (c_i \nabla p_e - c_e \nabla p_i) = -\nu(\rho)v, \quad t > 0, \ x \in \mathbb{R}^d, \end{cases}$$

181 with relations (1.1)-(1.4) and (1.8) and

182
$$v = \rho(T_i - T_e), \quad T_\alpha = b_\alpha \varepsilon_\alpha, \quad \alpha = e, i$$

183 Now we write the system with variables $(\rho, u, \varepsilon_e, \varepsilon_i)$. We first remark that

184 (2.2)
$$\operatorname{div}(\rho u \otimes u) = \rho(u \cdot \nabla)u + u \operatorname{div}(\rho u), \quad p = p_e + p_i.$$

185 Then, for $\rho > 0$, the first two equations in (2.1) give

186 (2.3)
$$\partial_t u + (u \cdot \nabla)u + \rho^{-1} \nabla p = 0.$$

187 By the definition of \mathcal{E}_{α} and the first two equations in (2.1) together with (2.3), we

188 have

189
$$\frac{1}{c_{\alpha}} \left[\partial_t \mathcal{E}_{\alpha} + \operatorname{div}(u\mathcal{E}_{\alpha}) \right] = \frac{1}{2} \rho u \cdot \partial_t u + \frac{1}{2} u \cdot \partial_t (\rho u) + \rho \partial_t \varepsilon_{\alpha} + \varepsilon_{\alpha} \partial_t \rho + \operatorname{div}\left(\frac{1}{2} \rho |u|^2 u + \rho u \varepsilon_{\alpha}\right)$$

190
$$= -\frac{1}{2}\rho u \cdot \left[(u \cdot \nabla)u + \rho^{-1}\nabla p \right] - \frac{1}{2}u \cdot \left[\operatorname{div}(\rho u \otimes u) + \nabla p \right] + \rho \partial_t \varepsilon_\alpha$$

191
$$- \varepsilon_\alpha \operatorname{div}(\rho u) + \operatorname{div}\left(\frac{1}{2}\rho |u|^2 u + \rho u\varepsilon_\alpha\right).$$

192 Since

193
$$-\rho u \cdot \left[(u \cdot \nabla)u\right] = -\frac{1}{2}\rho u \cdot \nabla(|u|^2),$$

194
$$\operatorname{div}\left(\frac{1}{2}\rho|u|^2u\right) = \frac{1}{2}|u|^2\operatorname{div}(\rho u) + \frac{1}{2}\rho u \cdot \nabla(|u|^2),$$

195 using (2.2), we obtain

196
$$-\frac{1}{2}\rho u \cdot \left[(u \cdot \nabla)u + \rho^{-1}\nabla p \right] - \frac{1}{2}u \cdot \left[\operatorname{div}(\rho u \otimes u) + \nabla p \right] + \operatorname{div}\left(\frac{1}{2}\rho|u|^{2}u\right)$$
197
$$= -\frac{1}{2}\rho u \cdot \nabla(|u|^{2}) - u \cdot \nabla p - \frac{1}{2}|u|^{2}\operatorname{div}(\rho u) + \operatorname{div}\left(\frac{1}{2}\rho|u|^{2}u\right)$$

198 $= -u \cdot \nabla p.$

199 We also have

200

$$-\varepsilon_{\alpha}\operatorname{div}(\rho u) + \operatorname{div}(\rho u\varepsilon_{\alpha}) = \rho u \cdot \nabla \varepsilon_{\alpha}.$$

201 These equalities imply that

202
$$\frac{1}{c_{\alpha}} \left[\partial_t \mathcal{E}_{\alpha} + \operatorname{div}(u\mathcal{E}_{\alpha}) \right] = \rho \partial_t \varepsilon_{\alpha} + \rho u \cdot \nabla \varepsilon_{\alpha} - u \cdot \nabla p.$$

203 Moreover,

204 (2.4)
$$\begin{cases} \operatorname{div}(up_e) - u \cdot (c_i \nabla p_e - c_e \nabla p_i) = p_e \operatorname{div} u + c_e u \cdot \nabla p, \\ \operatorname{div}(up_i) + u \cdot (c_i \nabla p_e - c_e \nabla p_i) = p_i \operatorname{div} u + c_i u \cdot \nabla p. \end{cases}$$

205 It follows that

$$\begin{array}{l}
206 \qquad \frac{1}{c_e} \left[\partial_t \mathcal{E}_e + \operatorname{div} \left(u(\mathcal{E}_e + p_e) \right) - u \cdot \left(c_i \nabla p_e - c_e \nabla p_i \right) \right] = \rho \partial_t \varepsilon_e + \rho u \cdot \nabla \varepsilon_e + \frac{1}{c_e} p_e \operatorname{div} u, \\
207 \\
208 \qquad \frac{1}{c_i} \left[\partial_t \mathcal{E}_i + \operatorname{div} \left(u(\mathcal{E}_i + p_i) \right) + u \cdot \left(c_i \nabla p_e - c_e \nabla p_i \right) \right] = \rho \partial_t \varepsilon_i + \rho u \cdot \nabla \varepsilon_i + \frac{1}{c_i} p_i \operatorname{div} u.
\end{array}$$

209 Finally, by the expression of p_{α} and the last two equations in (2.1), we obtain

210
210

$$\partial_t \varepsilon_e + u \cdot \nabla \varepsilon_e + (\gamma_e - 1) \varepsilon_e \operatorname{div} u = \nu(\rho) (c_e \rho)^{-1} v,$$

211
 $\partial_t \varepsilon_i + u \cdot \nabla \varepsilon_i + (\gamma_i - 1) \varepsilon_i \operatorname{div} u = -\nu(\rho) (c_i \rho)^{-1} v,$

212 which are the equations for ε_e and ε_i . Thus, system (2.1) is equivalent to

213 (2.5)
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + (u \cdot \nabla)u + \rho^{-1} \nabla p = 0, \\ \partial_t \varepsilon_e + u \cdot \nabla \varepsilon_e + (\gamma_e - 1)\varepsilon_e \operatorname{div} u = \nu(\rho)(c_e \rho)^{-1} v, \\ \partial_t \varepsilon_i + u \cdot \nabla \varepsilon_i + (\gamma_i - 1)\varepsilon_i \operatorname{div} u = -\nu(\rho)(c_i \rho)^{-1} v, \quad t > 0, \ x \in \mathbb{R}^d, \end{cases}$$

214

215 (2.6)
$$\begin{cases} p = \rho[c_e(\gamma_e - 1)\varepsilon_e + c_i(\gamma_i - 1)\varepsilon_i], \\ v = \rho(b_i\varepsilon_i - b_e\varepsilon_e). \end{cases}$$

216 Let

217

$$\mathcal{V} = (
ho, u^T, arepsilon_e, arepsilon_i)^T, \quad arepsilon_1 = c_e(\gamma_e - 1)arepsilon_e + c_i(\gamma_i - 1)arepsilon_i,$$

where the superscript T denotes the transpose of a vector. Since $p = \rho \varepsilon_1$ and 218

219
$$\rho^{-1}\nabla p = \rho^{-1}\varepsilon_1\nabla\rho + c_e(\gamma_e - 1)\nabla\varepsilon_e + c_i(\gamma_i - 1)\nabla\varepsilon_i,$$

system (2.5) is written in the form 220

221 (2.7)
$$\partial_t \mathcal{V} + \sum_{j=1}^d \mathcal{B}_j(\mathcal{V}) \partial_{x_j} \mathcal{V} = \mathcal{H}(\mathcal{V}), \quad t > 0, \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d,$$

222 where

223 (2.8)
$$\mathcal{B}_{j}(\mathcal{V}) = \begin{pmatrix} u_{j} & \rho e_{j}^{T} & 0 & 0\\ \rho^{-1}\varepsilon_{1}e_{j} & u_{j}I_{d} & c_{e}(\gamma_{e}-1)e_{j} & c_{i}(\gamma_{i}-1)e_{j}\\ 0 & (\gamma_{e}-1)\varepsilon_{e}e_{j}^{T} & u_{j} & 0\\ 0 & (\gamma_{i}-1)\varepsilon_{i}e_{j}^{T} & 0 & u_{j} \end{pmatrix},$$

1

0

224and

225 (2.9)
$$\mathcal{H}(\mathcal{V}) = \begin{pmatrix} 0 \\ 0 \\ \nu(\rho)(c_e \rho)^{-1} v \\ -\nu(\rho)(c_i \rho)^{-1} v \end{pmatrix},$$

with $u = (u_1, \dots, u_d)^T$, I_d being the unit matrix and (e_1, \dots, e_d) being the standard 226basis of \mathbb{R}^d . 227

By a symmetrizer $\mathcal{B}_0(\mathcal{V})$ for system (2.7) we mean that $\mathcal{B}_0(\mathcal{V})$ is a symmetric 228 positive definite matrix such that $\mathcal{B}_0(\mathcal{V})\mathcal{B}_j(\mathcal{W})$ is symmetric for all $j \in \{1, 2, \cdots, d\}$ 229 (see [15]). Now we introduce a diagonal matrix 230

231 (2.10)
$$\mathcal{B}_0(\mathcal{V}) = \operatorname{diag}(\varepsilon_1 \varepsilon_e \varepsilon_i, \rho^2 \varepsilon_e \varepsilon_i I_d, c_e \rho^2 \varepsilon_i, c_i \rho^2 \varepsilon_e).$$

Obviously, $\mathcal{B}_0(\mathcal{V})$ is symmetric positive definite in Ω . Moreover, 232

$$233 \qquad = \begin{pmatrix} a_j \varepsilon_1 \varepsilon_e \varepsilon_e \\ \rho \varepsilon_1 \varepsilon_e \varepsilon_i e \\ 0 \\ 0 \end{pmatrix}$$

 $\begin{aligned} \mathcal{B}_{0}(\mathcal{V})\mathcal{B}_{j}(\mathcal{V}) \\ &= \begin{pmatrix} u_{j}\varepsilon_{1}\varepsilon_{e}\varepsilon_{i} & \rho\varepsilon_{1}\varepsilon_{e}\varepsilon_{i}e_{j}^{T} & 0 & 0\\ \rho\varepsilon_{1}\varepsilon_{e}\varepsilon_{i}e_{j} & \rho^{2}u_{j}\varepsilon_{e}\varepsilon_{i}I_{d} & c_{e}(\gamma_{e}-1)\rho^{2}\varepsilon_{e}\varepsilon_{i}e_{j} & c_{i}(\gamma_{i}-1)\rho^{2}\varepsilon_{e}\varepsilon_{i}e_{j}\\ 0 & c_{e}(\gamma_{e}-1)\rho^{2}\varepsilon_{e}\varepsilon_{i}e_{j}^{T} & c_{e}\rho^{2}u_{j}\varepsilon_{i} & 0\\ 0 & c_{i}(\gamma_{i}-1)\rho^{2}\varepsilon_{e}\varepsilon_{i}e_{j}^{T} & 0 & c_{i}\rho^{2}u_{j}\varepsilon_{e} \end{pmatrix} \end{aligned}$ 234 which is a symmetric matrix. Therefore, $\mathcal{B}_0(\mathcal{V})$ is a symmetrizer and system (2.7) is symmetrizable hyperbolic in the sense of Friedrichs. According to Lax [14] or Kato 235[12] (see also Majda [15]), for smooth initial data, the Cauchy problem for (2.1) admits 236 a unique smooth solution, locally in time. This result is stated as follows and it holds 237in any space dimension. 238

239 PROPOSITION 2.1. Let m > d/2 + 1 be an integer and $\bar{\varepsilon}_e > 0$ and $\bar{\varepsilon}_i > 0$ be two 240 constants. We suppose that $\mathcal{V}_0 - \bar{\mathcal{V}} \in H^m(\mathbb{R}^d)$ and

241 (2.11)
$$\inf_{x \in \mathbb{R}^d} \rho_0(x) > 0, \quad \inf_{x \in \mathbb{R}^d} \varepsilon_{e0}(x) > 0, \quad \inf_{x \in \mathbb{R}^d} \varepsilon_{i0}(x) > 0.$$

242 Then, there exist T > 0 and a unique smooth solution \mathcal{V} to the Cauchy problem (1.5) 243 and (1.11). This solution satisfies $\mathcal{V} - \overline{\mathcal{V}} \in C([0,T]; H^m(\mathbb{R}^d)) \cap C^1([0,T]; H^{m-1}(\mathbb{R}^d))$ 244 and

245
$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^d}\rho(t,x)>0,\quad\inf_{(t,x)\in[0,T]\times\mathbb{R}^d}\varepsilon_e(t,x)>0,\quad\inf_{(t,x)\in[0,T]\times\mathbb{R}^d}\varepsilon_i(t,x)>0.$$

246 Remark 2.2.

247 Condition $\|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m \le \kappa_0$ in Theorem 1.1 with κ_0 being sufficiently small implies 248 (2.11).

249 2.2. The system in one space dimension. In one space dimension, systems 250 (2.1) and (2.5) are written as :

251 (2.12)
$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p) = 0, \\ \partial_t \mathcal{E}_e + \partial_x (u(\mathcal{E}_e + p_e)) - u(c_i \partial_x p_e - c_e \partial_x p_i) = \nu(\rho) v, \\ \partial_t \mathcal{E}_i + \partial_x (u(\mathcal{E}_i + p_i)) + u(c_i \partial_x p_e - c_e \partial_x p_i) = -\nu(\rho) v, \quad t > 0, \ x \in \mathbb{R} \end{cases}$$

252 and

253 (2.13)
$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t u + u \partial_x u + \rho^{-1} \partial_x p = 0, \\ \partial_t \varepsilon_e + u \partial_x \varepsilon_e + (\gamma_e - 1) \varepsilon_e \partial_x u = \nu(\rho) (c_e \rho)^{-1} v, \\ \partial_t \varepsilon_i + u \partial_x \varepsilon_i + (\gamma_i - 1) \varepsilon_i \partial_x u = -\nu(\rho) (c_i \rho)^{-1} v, \quad t > 0, \ x \in \mathbb{R} \end{cases}$$

254 respectively. From (2.6) and (2.13), we further obtain

255
$$\begin{cases} \partial_t(\rho^2\varepsilon_e) + \partial_x(\rho^2\varepsilon_e u) + \gamma_e\rho^2\varepsilon_e\partial_x u = \frac{1}{c_e}\nu(\rho)\rho v,\\ \partial_t(\rho^2\varepsilon_i) + \partial_x(\rho^2\varepsilon_i u) + \gamma_i\rho^2\varepsilon_i\partial_x u = -\frac{1}{c_i}\nu(\rho)\rho v, \end{cases}$$

256 which imply that

257 (2.14)
$$\begin{cases} \partial_t(\rho p) + \partial_x(\rho p u) + \rho \mu_1 \partial_x u = (\gamma_e - \gamma_i)\nu(\rho)\rho v, \\ \partial_t(\rho v) + \partial_x(\rho v u) + \rho \mu_2 \partial_x u = -\left(\frac{b_i}{c_i} + \frac{b_e}{c_e}\right)\nu(\rho)\rho v, \end{cases}$$

258 where

259 (2.15)
$$\begin{cases} \mu_1 = \rho [c_e \gamma_e (\gamma_e - 1)\varepsilon_e + c_i \gamma_i (\gamma_i - 1)\varepsilon_i], \\ \mu_2 = \rho (b_i \gamma_i \varepsilon_i - b_e \gamma_e \varepsilon_e). \end{cases}$$

By (2.6) and the expression of μ_2 above, we see that μ_1 and μ_2 can further be expressed as linear functions of p and v as

262 (2.16)
$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = M \begin{pmatrix} p \\ v \end{pmatrix},$$

263 where M is a constant invertible matrix given by

264 (2.17)
$$M = \frac{1}{(m_e c_i + m_i c_e)k_B} \begin{pmatrix} k_B(m_e c_i \gamma_i + m_i c_e \gamma_e) & c_e c_i k_B^2(\gamma_i - \gamma_e) \\ m_e m_i(\gamma_i - \gamma_e) & k_B(m_e c_i \gamma_e + m_i c_e \gamma_i) \end{pmatrix}.$$

By the expression of \mathcal{B}_j given in (2.8), we can calculate the eigenvalues λ_i and the eigenvectors r_i of (2.13). They are given by

267
$$\lambda_1(\mathcal{V}) = u - a, \quad \lambda_2(\mathcal{V}) = \lambda_3(\mathcal{V}) = u, \quad \lambda_4(\mathcal{V}) = u + a,$$

269

271

$$r_{1}(\mathcal{V}) = \begin{pmatrix} \rho \\ -a \\ (\gamma_{e} - 1)\varepsilon_{e} \\ (\gamma_{i} - 1)\varepsilon_{i} \end{pmatrix}, \quad r_{2}(\mathcal{V}) = \begin{pmatrix} 0 \\ 0 \\ -(\gamma_{i} - 1)c_{i} \\ (\gamma_{e} - 1)c_{e} \end{pmatrix},$$
$$r_{3}(\mathcal{V}) = \begin{pmatrix} -\rho \\ 0 \\ \varepsilon_{e} \\ \varepsilon_{i} \end{pmatrix}, \quad r_{4}(\mathcal{V}) = \begin{pmatrix} \rho \\ a \\ (\gamma_{e} - 1)\varepsilon_{e} \\ (\gamma_{i} - 1)\varepsilon_{i} \end{pmatrix},$$

270 where

$$a(\varepsilon_e, \varepsilon_i) = \sqrt{c_e \gamma_e(\gamma_e - 1)\varepsilon_e + c_i \gamma_i(\gamma_i - 1)\varepsilon_i}.$$

272 Moreover, by (2.9), we have

273
$$\mathcal{H}'(\bar{\mathcal{V}}) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{b_e}{c_e}\nu(1) & \frac{b_i}{c_e}\nu(1)\\ 0 & 0 & \frac{b_e}{c_i}\nu(1) & -\frac{b_i}{c_i}\nu(1) \end{pmatrix}.$$

It is known that condition (SK) is invariant under a change of unknown variables by a C^1 -diffeomorphism [10]. This condition shows a coupling property between the eigenvectors and the source terms of the system. At a given equilibrium state $\bar{\mathcal{V}}$, it means that $\mathcal{H}'(\bar{\mathcal{V}})r_i(\bar{\mathcal{V}}) \neq 0$ for all i = 1, 2, 3, 4. From (3.3), we see easily that $\mathcal{H}'(\bar{\mathcal{V}})r_3(\bar{\mathcal{V}}) = 0$. This shows that condition (SK) is not satisfied for system (2.13).

279 **2.3. The system in Lagrangian coordinates.** Let $(\rho, u) \in C^1(\mathbb{R}^+ \times \mathbb{R})$ sat-280 isfying $\rho \geq \text{const} > 0$ in $\mathbb{R}^+ \times \mathbb{R}$ and

281 (2.18)
$$\partial_t \rho + \partial_x (\rho u) = 0.$$

282 The Euler-Lagrange change of variables from
$$(t, x)$$
 to (t', y) is defined by

283
$$t' = t, \quad dy = \rho dx - \rho u dt,$$

284 or equivalently for y:

285
$$y = \int_{X_1(t)}^x \rho(t,\xi) d\xi, \quad \text{with } X_1'(t) = u(t,X_1(t)).$$

It is clear that this change of variables is a diffeomorphism from $\mathbb{R}^+ \times \mathbb{R}$ to itself. For simplicity, we use the same notation for unknown variables in Eulerian coordinates (t, x) and in Lagrangian coordinates (t, y). 289 Consider smooth solutions for (2.12). Let

290 (2.19)
$$\tau = \rho^{-1}, \quad E_{\alpha} = \tau \mathcal{E}_{\alpha} = \frac{1}{2} c_{\alpha} u^2 + c_{\alpha} \varepsilon_{\alpha}, \quad \alpha = e, i.$$

291 Given a first-order partial differential equation

292 (2.20)
$$\partial_t w + \partial_x z_1 + b \partial_x z_2 = f.$$

293 By (2.18), in Lagrangian coordinates this equation is written equivalently as

294 (2.21)
$$\partial_t(\tau w) + \partial_y(z_1 - wu) + b\partial_y z_2 = \tau f.$$

Applying this to (2.12), we obtain

296 (2.22)
$$\begin{cases} \partial_t \tau - \partial_y u = 0, \\ \partial_t u + \partial_y p = 0, \\ \partial_t E_e + p_e \partial_y u + c_e u \partial_y p = \nu \tau v, \\ \partial_t E_i + p_i \partial_y u + c_i u \partial_y p = -\nu \tau v. \end{cases}$$

297 Similarly to (2.14), by (2.4), we obtain

298 (2.23)
$$\begin{cases} \partial_t p + \tau^{-1} \mu_1 \partial_y u = (\gamma_e - \gamma_i) \nu v, \\ \partial_t v + \tau^{-1} \mu_2 \partial_y u = -\left(\frac{b_i}{c_i} + \frac{b_e}{c_e}\right) \nu v, \end{cases}$$

299 where μ_1 and μ_2 are given in (2.16).

300 Regarding p_{α} and p as functions of (τ, u, E_e, E_i) , we have

301
$$p_{\alpha} = \frac{1}{\tau} (\gamma_{\alpha} - 1) \left(E_{\alpha} - \frac{1}{2} c_{\alpha} u^2 \right), \quad \alpha = e, i, \quad p = p_e + p_i.$$

302 Hence, system (2.22) can be written as

303 (2.24)
$$\partial_t \mathcal{U} + \mathcal{A}(\mathcal{U})\partial_y \mathcal{U} = \mathcal{G}(\mathcal{U}), \quad t > 0, \quad y \in \mathbb{R}, \quad \mathcal{U} = (\tau, u, E_e, E_i)^T,$$

304 which is supplemented by an initial condition

305 (2.25)
$$t = 0: \quad \mathcal{U} = \mathcal{U}_0(y) \stackrel{def}{=} (\tau_0(y), u_0(y), E_{e0}(y), E_{i0}(y)), \quad y \in \mathbb{R}.$$

306 Here,

307
$$\mathcal{A}(\mathcal{U}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \partial_{\tau}p & \partial_{u}p & \partial_{E_{e}}p & \partial_{E_{i}}p \\ c_{e}u\partial_{\tau}p & p_{e} + c_{e}u\partial_{u}p & c_{e}u\partial_{E_{e}}p & c_{e}u\partial_{E_{i}}p \\ c_{i}u\partial_{\tau}p & p_{i} + c_{i}u\partial_{u}p & c_{i}u\partial_{E_{e}}p & c_{i}u\partial_{E_{i}}p \end{pmatrix}, \quad \mathcal{G}(\mathcal{U}) = \begin{pmatrix} 0 \\ 0 \\ \nu\tau\nu \\ -\nu\tau\nu \end{pmatrix},$$

308 with

which is an equilibrium state of (2.24) with $\bar{E}_e > 0$ and $\bar{E}_i > 0$. The result of global existence of solutions to (2.24) and (2.25) can be stated as follows.

THEOREM 2.3. Let $m \ge 2$ and $\mathcal{U}_0 - \bar{\mathcal{U}} \in H^m(\mathbb{R})$. Assume $\gamma_e \ne \gamma_i$. There are two positive constants c and κ_1 such that if $\|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m \le \kappa_1$, then the Cauchy problem (2.24) and (2.25) admits a unique global solution \mathcal{U} satisfying $\mathcal{U} - \bar{\mathcal{U}} \in C(\mathbb{R}^+; H^m(\mathbb{R})) \cap$ $C^1(\mathbb{R}^+; H^{m-1}(\mathbb{R}))$. Moreover,

318 (2.26)
$$\sup_{t \in \mathbb{R}^+} \|\mathcal{U}(t, \cdot) - \bar{\mathcal{U}}\|_m^2 + \int_0^{+\infty} \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2 \right) dt' \le c \|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m^2.$$

319 **3. Energy estimates in Lagrangian coordinates.** We study energy esti-320 mates for the Cauchy problem (2.24) and (2.25). Let $m \ge 2$ be an integer and T > 0321 such that the local smooth solution \mathcal{U} is defined on time interval [0,T]. We denote 322 by $\|\cdot\|$, $\|\cdot\|_{\infty}$ and $\|\cdot\|_{l}$ the usual norms of $L^{2}(\mathbb{R})$, $L^{\infty}(\mathbb{R})$ and $H^{l}(\mathbb{R})$ for $l \in \mathbb{N}$, 323 respectively. We also denote

324
$$\mathcal{U}_T = \max_{t \in [0,T]} \|\mathcal{U}(t, \cdot) - \bar{\mathcal{U}}\|_m.$$

We consider a smooth solution \mathcal{U} near $\overline{\mathcal{U}}$, namely, \mathcal{U}_T is small. In the proof below, we denote by C > 0 and $c_0 > 0$ generic constants independent of t and T.

The global existence of smooth solutions to (2.24) and (2.25) will be proved in the three steps shown in Introduction.

3.1. An L^2 estimate. We first look at the entropy equality (1.9) in Lagrangian coordinates. From (2.6), we have

331
$$\begin{cases} \varepsilon_e = \frac{(m_i p - c_i \kappa_B v)\tau}{(\gamma_e - 1)(m_e c_i + m_i c_e)},\\ \varepsilon_i = \frac{(m_e p + c_e k_B v)\tau}{(\gamma_i - 1)(m_e c_i + m_i c_e)}, \end{cases}$$

which are strictly positive in a neighborhood of v = 0 when $\tau > 0$ and p > 0. It follows from the definition of b_{α} , T_{α} and v that

334
$$-\frac{\nu\rho}{T_e T_i} (T_i - T_e)^2 = -\nu_1 \rho v^2,$$

335 where $\nu_1 = \nu_1(\tau, p, v)$ given by

336 (3.1)
$$\nu_1 = \frac{k_B^2 (m_e c_i + m_i c_e)^2 \nu}{m_e m_i (m_i p - c_i k_B v) (m_e p + c_e k_B v)}$$

It is clear that, for all $\tau > 0$ and p > 0, $\nu_1 > 0$ in a neighborhood of v = 0. We introduce a new variable

339
$$s = \tau \eta = -\sum_{\alpha = e,i} \frac{c_{\alpha}}{b_{\alpha}} \ln\left[\left(\frac{(\gamma_{\alpha} - 1)\tau^{\gamma_{\alpha} - 1}}{c_{\alpha}^{\gamma_{\alpha}}}\right) \left(E_{\alpha} - \frac{c_{\alpha}}{2}u^{2}\right)\right],$$

which is a function of variable \mathcal{U} . According to the equivalence of equations (2.20) and (2.21) in two coordinates, the entropy equality (1.9) in variables (t, y) becomes

342 (3.2)
$$\partial_t s = -\nu_1 v^2,$$

which means that s is an entropy of system (2.24) with 0 as its entropy-flux.

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Recall that an equilibrium state with zero velocity is of the from

345
$$\overline{\mathcal{V}} = (1, 0, \overline{\varepsilon}_e, \overline{\varepsilon}_i)^T.$$

By definition, $\overline{\mathcal{V}}$ is an equilibrium state for (2.7) if $\mathcal{H}(\overline{\mathcal{V}}) = 0$. Since $\nu > 0$, by the definition of \mathcal{H} and v, we have

348 (3.3)
$$b_e \bar{\varepsilon}_e = b_i \bar{\varepsilon}_i.$$

349 Combining this with (2.19) yields

$$\frac{b_e E_e}{c_e} = \frac{b_i E_i}{c_i} \stackrel{def}{=} \bar{E}_* > 0.$$

351 LEMMA 3.1. For all \mathcal{U} in the domain under consideration, it holds

352 (3.4)
$$\nabla s(\bar{\mathcal{U}})\mathcal{G}(\mathcal{U}) = 0$$

353 and

12

354 (3.5)
$$\nabla s(\bar{\mathcal{U}})\mathcal{A}(\mathcal{U}) = \nabla \mathcal{F}(\mathcal{U}),$$

355 where

356
$$\mathcal{F}(\mathcal{U}) = \frac{1}{\bar{E}_*} up - k_B \left(\frac{c_e}{m_e} + \frac{c_i}{m_i}\right) u.$$

357 Proof. A straightforward calculation gives

$$-\partial_{\tau}s(\mathcal{U}) = \frac{k_B}{\tau} \Big(\frac{c_e}{m_e} + \frac{c_i}{m_i}\Big),$$

$$-\partial_u s(\mathcal{U}) = -\left(\frac{c_e}{b_e \varepsilon_e} + \frac{c_i}{b_i \varepsilon_i}\right) u,$$

360
$$-\partial_{E_e} s(\mathcal{U}) = \frac{1}{b_e \varepsilon_e}, \quad -\partial_{E_i} s(\mathcal{U}) = \frac{1}{b_i \varepsilon_i},$$

361 where

362

$$\varepsilon_{\alpha} = \frac{1}{c_{\alpha}} E_{\alpha} - \frac{1}{2} u^2, \quad \alpha = e, i.$$

363 Therefore,

$$-\partial_{\tau}s(\bar{\mathcal{U}}) = k_B \Big(\frac{c_e}{m_e} + \frac{c_i}{m_i}\Big),$$

$$-\partial_u s(\bar{\mathcal{U}}) = 0,$$

$$-\partial_{E_e} s(\bar{\mathcal{U}}) = -\partial_{E_i} s(\bar{\mathcal{U}}) = \frac{1}{\bar{E}_*}.$$

 $_{367}$ Hence, it is easy to check that (3.4) and (3.5) are satisfied.

368 LEMMA 3.2. In a neighborhood of $\overline{\mathcal{U}}$, it holds

369 (3.6)
$$\|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|^2 + \int_0^t \|v(t',\cdot)\|^2 dt' \le C \|\mathcal{U}_0 - \bar{\mathcal{U}}\|^2, \quad \forall t \in [0,T].$$

370 *Proof.* We introduce

371

$$S(\mathcal{U}) = s(\mathcal{U}) - s(\bar{\mathcal{U}}) + \nabla s(\bar{\mathcal{U}})(\mathcal{U} - \bar{\mathcal{U}}).$$

Since η is a strictly convex entropy for (2.1), by a result in [25], s is a strictly convex entropy for (2.24). Hence, by Taylor formula, in a neighborhood of $\overline{\mathcal{U}}$, these exist two constants $c_2 \ge c_1 > 0$ such that

375
$$c_1 |\mathcal{U} - \bar{\mathcal{U}}|^2 \le S(\mathcal{U}) \le c_2 |\mathcal{U} - \bar{\mathcal{U}}|^2.$$

376 Using (2.24) and (3.2), we have

377
$$\partial_t S - \nabla s(\bar{\mathcal{U}}) \mathcal{A}(\mathcal{U}) \partial_y \mathcal{U} = -\nu_1 v^2 - \nabla s(\bar{\mathcal{U}}) \mathcal{G}(\mathcal{U}).$$

378 It follows from Lemma 3.1 that

$$\partial_t S - \partial_y \mathcal{F}(\mathcal{U}) = -\nu_1 v^2.$$

In a neighborhood of $\overline{\mathcal{U}}$, there is a constant $\overline{\nu}_1 > 0$ such that $\nu_1 \ge \overline{\nu}_1$. Thus, integrating this equality over $[0, t] \times \mathbb{R}$ with $t \in [0, T]$, we obtain (3.6).

3.2. Higher-order energy estimates. Let $U = (u, p, v, s)^T$. We use variable U in higher-order energy estimates. From (2.22), (2.23), and (3.2), we have

(3.7)
$$\begin{cases} \partial_t u + \partial_y p = 0, \\ \partial_t p + \tau^{-1} \mu_1 \partial_y u = -(\gamma_i - \gamma_e) \nu v, \\ \partial_t v + \tau^{-1} \mu_2 \partial_y u = -b \nu v, \\ \partial_t s = -\nu_1 v^2, \quad t > 0, \ x \in \mathbb{R}, \end{cases}$$

where ν_1 is defined in (3.1), μ_1 and μ_2 are defined in (2.16) and (2.17), and

$$b = \frac{b_i}{c_i} + \frac{b_e}{c_e} > 0.$$

In particular, μ_1 and μ_2 are linear functions of p and v. This system can be written as

$$\partial_t U + A(U)\partial_y U = G(U), \quad t > 0, \ x \in \mathbb{R},$$

390 where

391
$$A(U) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \tau^{-1}\mu_1 & 0 & 0 & 0 \\ \tau^{-1}\mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G(U) = \begin{pmatrix} 0 \\ -(\gamma_i - \gamma_e)\nu\nu \\ -b\nu\nu \\ -\nu_1\nu^2 \end{pmatrix},$$

and τ is regarded as a function of U. By the definition in (2.6), the equilibrium state for U is $\overline{U} = (0, \overline{p}, \overline{v}, \overline{s})$ with

394
$$\bar{p} = (\gamma_e - 1)\bar{E}_e + (\gamma_i - 1)\bar{E}_i > 0, \quad \bar{v} = 0, \quad \bar{s} = s(\bar{\mathcal{U}}).$$

395 We first prove the following useful property.

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396 LEMMA 3.3. Let $\delta(p, v)$ be defined by

397 (3.9)
$$\delta(p,v) = b\mu_1(p,v) - (\gamma_i - \gamma_e)\mu_2(p,v).$$

398 There is a constant $\overline{\delta} > 0$ such that $\delta(p, v) \ge \overline{\delta}$ in a neighborhood of $(\overline{p}, 0)$.

399 Proof. By continuity, it is sufficient to prove that $\delta(\bar{p}, 0) > 0$. From (1.2), we have

$$\frac{c_e}{c_i} = \frac{Zm_e}{m_i}.$$

401 It follows from the definition of b_{α} in (1.8) that

402
$$\frac{b_i c_e(\gamma_e - 1)}{c_i} = Z b_e(\gamma_i - 1), \quad \frac{b_e c_i(\gamma_i - 1)}{c_e} = \frac{b_i}{Z} (\gamma_e - 1).$$

403 Since $\bar{\rho} = 1$, from (2.15) we obtain

404
$$\delta(\bar{p},0) = \left(\gamma_i - 1 + \frac{\gamma_e - 1}{Z}\right) \left(Z\gamma_e b_e \bar{\varepsilon}_e + \gamma_i b_i \bar{\varepsilon}_i\right) - \left(\gamma_i - \gamma_e\right) \left(\gamma_i b_i \bar{\varepsilon}_i - \gamma_e b_e \bar{\varepsilon}_e\right).$$

405 Using the fact that $b_e \bar{\varepsilon}_e = b_i \bar{\varepsilon}_i > 0$ (see (3.3), $\delta(\bar{p}, 0) > 0$ if and only if

406
$$\left(\gamma_i - 1 + \frac{\gamma_e - 1}{Z}\right) (Z\gamma_e + \gamma_i) > (\gamma_i - \gamma_e)^2,$$

407 or equivalently,

408

$$\gamma_e Z(\gamma_i - 1) + \gamma_i(\gamma_e - 1) > 0$$

409 Lemma 3.3 is proved since Z > 0, $\gamma_i > 1$ and $\gamma_e > 1$.

410 LEMMA 3.4. Let the conditions of Theorem 2.3 hold. If $\|\mathcal{U} - \bar{\mathcal{U}}\|_m$ is sufficiently 411 small, for all $t \in [0, T]$, we have

412 (3.10)
$$\begin{aligned} & \|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|_m^2 + \int_0^t \|v(t',\cdot)\|_m^2 dt' \\ & \leq C \|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m^2 + C \int_0^t \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\right) \|\mathcal{U} - \bar{\mathcal{U}}\|_m dt'. \end{aligned}$$

413 Proof. Let

414
$$A_0(U) = \begin{pmatrix} \mu_0 & 0 & 0 & 0\\ 0 & \frac{b\mu_2}{\gamma_i - \gamma_e} & -\mu_2 & 0\\ 0 & -\mu_2 & \mu_1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$

415 where

416
$$\mu_0(\tau, p, v) = \frac{1}{(\gamma_i - \gamma_e)\tau} \mu_2(p, v) \delta(p, v),$$

417 and $\delta(p, v)$ is defined in (3.9). By Lemma 3.3, in a neighborhood of $\overline{\mathcal{U}}$, there are 418 positive constants $\overline{\mu}_1$, $\overline{\mu}_2$ and $\overline{\mu}_0$ such that

419
$$\mu_1(p,v) \ge \bar{\mu}_1, \quad (\gamma_i - \gamma_e)\mu_2(p,v) \ge \bar{\mu}_2, \quad \mu_0(\tau, p, v) \ge \bar{\mu}_0.$$

Then it is easy to check that, in a neighborhood of \mathcal{U} , $A_0(U)$ is a symmetrizer of system (3.8), namely, $A_0(U)$ is symmetric positive definite and $A_0(U)A(U)$ is symmetric. In particular,

424 Let $1 \le k \le m$ be an integer. We denote $U_k = \partial_y^k U$. From (3.8), we have

425 (3.11)
$$\partial_t U_k + A(U)\partial_y U_k = \partial_y^k G(U) + J_k,$$

426 where

427

$$J_k = A(U)\partial_y U_k - \partial_y^k (A(U)\partial_y U)$$

Taking the inner product of (3.11) with $A_0(U)U_k$ in $L^2(\mathbb{R})$, we obtain the Friedrichs energy equality

430 (3.12)
$$\frac{d}{dt} \langle A_0(U)U_k, U_k \rangle = 2 \langle A_0(U)\partial_y^k G(U), U_k \rangle + 2 \langle A_0(U)J_k, U_k \rangle + \langle \operatorname{div} \vec{A}(U)U_k, U_k \rangle,$$

431 where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\mathbb{R})$ and

.

432
$$\operatorname{div} \vec{A}(U) = \partial_t A_0(U) + \partial_y \tilde{A}(U), \quad \tilde{A} = A_0 A.$$

433 By the definition of A_0 and \tilde{A} , we have

434
$$\operatorname{div} \vec{A}(U) = \begin{pmatrix} \partial_t \mu_0 & \partial_y \mu_0 & 0 & 0\\ \partial_y \mu_0 & \frac{b}{\gamma_i - \gamma_e} \partial_t \mu_2 & -\partial_t \mu_2 & 0\\ 0 & -\partial_t \mu_2 & \partial_t \mu_1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

435 with

436
$$\partial_y \mu_0(U) = \mu'_0(U) \partial_y U,$$

437
$$\partial_t \mu_i(U) = \mu'_i(U)\partial_t U = \mu'_i(U)\big(G(U) - A(U)\partial_y U\big), \quad i = 0, 1, 2.$$

438 Since G(U) = O(v) and the imbedding from $H^m(\mathbb{R})$ to $W^{1,\infty}(\mathbb{R})$ is continuous, we 439 obtain

440 (3.13)
$$\langle \operatorname{div} \vec{A}(U)U_k, U_k \rangle \leq C \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|\partial_y v\|_{m-1}^2 \right) \|U - \bar{U}\|_m.$$

441 Next, a direct calculation yields

442
$$A_0(U)J_k = \begin{pmatrix} 0 \\ \left[\tau^{-1}\left(\frac{b}{\gamma_i - \gamma_e}\mu_1 - \mu_2\right)\partial_y^{k+1}u - \partial_y^k(\tau^{-1}\left(\frac{b}{\gamma_i - \gamma_e}\mu_1 - \mu_2\right)\partial_y u\right)\right]\mu_2 \\ \mu_2\partial_y^k(\tau^{-1}\mu_1\partial_y u) - \mu_1\partial_y^k(\tau^{-1}\mu_2\partial_y u) \\ 0 \end{pmatrix}.$$

By the Moser-type inequalities [15], we have 443

444 (3.14)
$$2\langle A_0(U)J_k, U_k \rangle \le C \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|\partial_y v\|_{m-1}^2 \right) \|U - \bar{U}\|_m.$$

445Moreover,

446
$$\partial_y^k G(U) = \begin{pmatrix} 0 \\ -(\gamma_i - \gamma_e)\nu\partial_y^k v \\ -b\nu\partial_y^k v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (\gamma_i - \gamma_e)(\nu\partial_y^k v - \partial_y^k(\nu v)) \\ b\nu\partial_y^k v - b\partial_y^k(\nu v) \\ -\partial_y^k(\nu_1 v^2) \end{pmatrix} \stackrel{def}{=} G_1 + G_2,$$

Ω

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447 with

448
$$A_0(U)G_1 = \begin{pmatrix} 0 \\ 0 \\ -\delta\nu\partial_y^k v \\ 0 \end{pmatrix}$$

and 449

450
$$A_0(U)G_2 = \begin{pmatrix} 0 \\ a_{22}(\gamma_i - \gamma_e) \left(\nu \partial_y^k v - \partial_y^k (\nu v)\right) - \mu_2 \left[b\nu \partial_y^k v - b \partial_y^k (\nu v)\right] \\ \mu_1 \left[b\nu \partial_y^k v - b \partial_y^k (\nu v)\right] - \mu_2 (\gamma_i - \gamma_e) \left(\nu \partial_y^k v - \partial_y^k (\nu v)\right) \\ - \partial_y^k (\nu_1 v^2) \end{pmatrix}$$

where 451

$$452 a_{22} = -\frac{b\mu_2}{2}$$

$$r = \frac{1}{\gamma_i - \gamma_e}$$

These equalities imply that 453

454
$$A_0(U)G_1 \cdot U_k = -\delta\nu |\partial_y^k v|^2$$

and 455

$$A_{0}(U)G_{2} \cdot U_{k} = \left(a_{22}(\gamma_{i} - \gamma_{e})\left(\nu\partial_{y}^{k}v - \partial_{y}^{k}(\nu v)\right) - \mu_{2}\left[b\nu\partial_{y}^{k}v - b\partial_{y}^{k}(\nu v)\right]\right)\partial_{y}^{k}u$$

$$+ \left(\mu_{1}\left[b\nu\partial_{y}^{k}v - b\partial_{y}^{k}(\nu v)\right] - \mu_{2}(\gamma_{i} - \gamma_{e})\left(\nu\partial_{y}^{k}v - \partial_{y}^{k}(\nu v)\right)\right)\partial_{y}^{k}p$$

$$- \partial_{y}^{k}(\nu_{1}v^{2})\partial_{y}^{k}s.$$

Observe that each of three terms on the right-hand side of (3.15) is quadratic in 457variables (u, p, v) with coefficients depending on derivatives of $U - \overline{U}$ up to order m. 458 Moreover, using Lemma 3.3, we have $\delta \nu > \bar{\delta} \bar{\nu}$ in a neighborhood of $\bar{\mathcal{U}}$, where $\bar{\nu} > 0$ is 459a constant. Thus, the Moser-type inequalities imply that 460

461 (3.16)
$$\langle A_0(U)\partial_y^k G(U), U_k \rangle + \bar{\delta}\bar{\nu} \|\partial_y^k v\|^2 \leq C (\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2) \|U - \bar{U}\|_m$$

Since $A_0(U)$ is positive definite, $\langle A_0(U)U_k, U_k \rangle$ is equivalent to $||U_k||^2$. Combining 462(3.12)-(3.16) and integrating (3.12) over [0, t] with $t \in [0, T]$, we have 463

464
$$\|U_k\|^2 + \int_0^t \|\partial_y^k v(t', \cdot)\|^2 dt'$$

465
$$\leq C \|U_0 - \bar{U}\|_m^2 + C \int_0^t \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2 \right) \|U - \bar{U}\|_m dt',$$

where U_0 is the initial data of U. Finally, the change of variables $U \mapsto \mathcal{U}$ is a C^{∞} -466 diffeomorphism in a neighborhood of \bar{U} . Then, $\|U - \bar{U}\|_l$ is equivalent to $\|U - \bar{U}\|_l$ for 467 all $l \in \mathbb{N}$. Summing up this inequality for all $k = 1, 2, \dots, m$, and using Lemma 3.2, 468 we obtain (3.10). 469

3.3. Dissipation estimates. 470

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473

LEMMA 3.5. Let the conditions of Theorem 2.3 hold. If $\|\mathcal{U} - \bar{\mathcal{U}}\|_m$ is sufficiently 471 small, for all $t \in [0, T]$, we have 472

(3.17)
$$\int_{0}^{t} \left(\|\partial_{y}u\|_{m-1}^{2} + \|\partial_{y}p\|_{m-1}^{2} \right) dt' \\ \leq C \|\mathcal{U}_{0} - \bar{\mathcal{U}}\|_{m}^{2} + C \int_{0}^{t} \left(\|\partial_{y}u\|_{m-1}^{2} + \|\partial_{y}p\|_{m-1}^{2} + \|v\|_{m}^{2} \right) \|\mathcal{U} - \bar{\mathcal{U}}\|_{m} dt'$$

Proof. Let k be an integer with $0 \le k \le m-1$. Applying ∂_y^k to the first three 474 475equations in (3.7) yields

476 (3.18)
$$\begin{cases} \partial_t \partial_y^k u + \partial_y^{k+1} p = 0, \\ \partial_t \partial_y^k p + \tau^{-1} \mu_1 \partial_y^{k+1} u = \tau^{-1} \mu_1 \partial_y^{k+1} u - \partial_y^k (\tau^{-1} \mu_1 \partial_y u) - (\gamma_i - \gamma_e) \partial_y^k (\nu v), \\ \partial_t \partial_y^k v + \tau^{-1} \mu_2 \partial_y^{k+1} u = \tau^{-1} \mu_2 \partial_y^{k+1} u - \partial_y^k (\tau^{-1} \mu_2 \partial_y u) - b \partial_y^k (\nu v). \end{cases}$$

We multiply the third equation in (3.18) by $(\gamma_i - \gamma_e)$ and take the inner product with $\partial_y^{k+1}u$ in $L^2(\mathbb{R})$. Using $(\gamma_i - \gamma_e)\tau^{-1}\mu_2 \geq 3c_0$ it yields 477478

479

$$3c_0 \|\partial_y^{k+1}u\|^2 \leq -(\gamma_i - \gamma_e) \langle \partial_t \partial_y^k v, \partial_y^{k+1}u \rangle + (\gamma_i - \gamma_e) \langle \tau^{-1}\mu_2 \partial_y^{k+1}u - \partial_y^k (\tau^{-1}\mu_2 \partial_y u) - b \partial_y^k (\nu v), \partial_y^{k+1}u \rangle.$$

By the Young inequality and the Moser-type inequalities, the last term above is 480 bounded by 481

482
$$c_0 \|\partial_y^{k+1}u\|^2 + C\|v\|_m^2 + C\|\partial_y u\|_{m-1}^2 \|\mathcal{U} - \bar{\mathcal{U}}\|_m$$

Moreover, by the first equation in (3.18) and an integration by parts, we have 483

$$484 \qquad -(\gamma_i - \gamma_e) \langle \partial_t \partial_y^k v, \partial_y^{k+1} u \rangle = -(\gamma_i - \gamma_e) \frac{d}{dt} \langle \partial_y^k v, \partial_y^{k+1} u \rangle + (\gamma_i - \gamma_e) \langle \partial_y^{k+1} v, \partial_y^{k+1} p \rangle$$

$$485 \qquad \qquad \leq -(\gamma_i - \gamma_e) \frac{d}{dt} \langle \partial_y^k v, \partial_y^{k+1} u \rangle + \beta \|\partial_y^{k+1} p\|^2 + C \|v\|_m^2,$$

where $\beta > 0$ is a small constant to be chosen. This implies that 486

487 (3.19)
$$2c_0 \|\partial_y^{k+1}u\|^2 \le -(\gamma_i - \gamma_e) \frac{d}{dt} \langle \partial_y^k v, \partial_y^{k+1}u \rangle + \beta \|\partial_y^{k+1}p\|^2 + C \|v\|_m^2 + C \|\partial_y u\|_{m-1}^2 \|\mathcal{U} - \bar{\mathcal{U}}\|_m$$

Similarly, taking the inner product of the first equation in (3.18) with $\partial_u^{k+1}p$ in 488 $L^2(\mathbb{R})$ and using an integration by parts, we have 489

490
$$\|\partial_y^{k+1}p\|^2 = -\frac{d}{dt} \langle \partial_y^k u, \partial_y^{k+1}p \rangle - \langle \partial_y^{k+1}u, \partial_y^k \partial_t p \rangle.$$

By the second equation in (3.18), we obtain as above 491

492
$$-\langle \partial_y^{k+1}u, \partial_y^k \partial_t p \rangle = \langle \partial_y^k (\tau^{-1}\mu_1 \partial_y u) - \tau^{-1}\mu_1 \partial_y^{k+1}u + (\gamma_i - \gamma_e) \partial_y^k (\nu v), \partial_y^{k+1}u \rangle$$

493
$$+ \langle \tau^{-1}\mu_1 \partial_y^{k+1}u, \partial_y^{k+1}u \rangle$$

493
$$+ \langle \tau^{-1} \mu_1 \partial_y^{\kappa+1} u, \partial_y^{\kappa} \rangle$$

494
$$\leq C \|\partial_y^{k+1}u\|^2 + C \|v\|_m^2 + C \|\partial_y u\|_{m-1}^2 \|\mathcal{U} - \mathcal{U}\|_m.$$

Hence, 495

496 (3.20)
$$\|\partial_y^{k+1}p\|^2 \le -\frac{d}{dt} \langle \partial_y^k u, \partial_y^{k+1}p \rangle + C \|\partial_y^{k+1}u\|^2 + C \|v\|_m^2 + C \|\partial_y u\|_{m-1}^2 \|\mathcal{U} - \bar{\mathcal{U}}\|_m.$$

Combining (3.19) and (3.20), and choosing $\beta > 0$ to be sufficiently small, it yields 497

498 (3.21)
$$c_{0} \|\partial_{y}^{k+1}u\|^{2} + \beta \|\partial_{y}^{k+1}p\|^{2} \leq -\frac{d}{dt} [(\gamma_{i} - \gamma_{e}) \langle \partial_{y}^{k}v, \partial_{y}^{k+1}u \rangle + 2\beta \langle \partial_{y}^{k}u, \partial_{y}^{k+1}p \rangle] + C \|v\|_{m}^{2} + C \|\partial_{y}u\|_{m-1}^{2} \|\mathcal{U} - \bar{\mathcal{U}}\|_{m}.$$

Finally, since $0 \le k \le m - 1$, we have 499

,

500
$$\left|\left\langle \partial_y^k v, \partial_y^{k+1} u \right\rangle\right| + \left|\left\langle \partial_y^k u, \partial_y^{k+1} p \right\rangle\right| \le C \|\mathcal{U} - \bar{\mathcal{U}}\|_m^2.$$

Integrating (3.21) over [0, t] with $t \in [0, T]$, we obtain 501

502
$$\int_{0}^{t} \left(\|\partial_{y}^{k+1}u\|^{2} + \|\partial_{y}^{k+1}p\|^{2} \right) dt' \leq C \|\mathcal{U} - \bar{\mathcal{U}}\|_{m}^{2} + C \|\mathcal{U}_{0} - \bar{\mathcal{U}}\|_{m}^{2}$$

503
$$+ C \int_{0}^{t} \left(\|v\|_{m}^{2} + \|\partial_{y}u\|_{m-1}^{2} \|\mathcal{U} - \bar{\mathcal{U}}\|_{m} \right) dt'.$$

Summing this inequality for all $k = 0, 1, \dots, m-1$ and using Lemma 3.4, we obtain 504 505 (3.17).

3.4. Proof of Theorem 2.3. From (3.10) and (3.17), we have 506

507
$$\|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|_m^2 + \int_0^t \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2 \right) dt'$$

508
$$\leq C \|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m^2 + C \mathcal{U}_T \int_0^t \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2 \right) dt', \quad \forall t \in [0, T].$$

Since \mathcal{U}_T is sufficiently small, we further obtain 509

510
$$\|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|_m^2 + \int_0^t \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2 \right) dt' \le C \|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m^2, \quad \forall t \in [0,T]$$

This estimate together with a bootstrap argument implies (2.26) and the global ex-511istence of a solution \mathcal{U} to (2.24) and (2.25), provided that $\|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m$ is sufficiently 512small. 513

4. Proof of Theorem 1.1. For the Cauchy problem for (2.12) with initial data 514given in (1.11), we first define 515

516
$$Y_0(x) = \int_0^x \rho_0(\xi) d\xi.$$

Then $Y'_0 = \rho_0$. By the condition in Theorem 1.1, we have $\inf_{x \in \mathbb{R}} \rho_0(x) > 0$ and $\rho_0 - \rho_0(x) = 0$ 517 $1 \in H^m(\mathbb{R})$. Therefore, the continuous imbedding from $H^m(\mathbb{R})$ to $C^{m-1}(\mathbb{R})$ implies 518that Y_0 is a C^m -diffeomorphism from \mathbb{R} to \mathbb{R} . We denote by X_0 the inverse C^m -519diffeomorphism of Y_0 and define 520

521
$$\mathcal{U}_0(y) = \left(\frac{1}{\rho_0}, u_0, \frac{1}{2}c_e u_0^2 + c_e \varepsilon_{e0}, \frac{1}{2}c_i u_0^2 + c_i \varepsilon_{i0}\right) (X_0(y)).$$

Then condition $\mathcal{V}_0 - \bar{\mathcal{V}} \in H^m(\mathbb{R})$ implies that $\mathcal{U}_0 - \bar{\mathcal{U}} \in H^m(\mathbb{R})$ and condition $\|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m \leq \kappa_0$ with κ_0 being sufficiently small implies that $\|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m$ is sufficiently small. According to Theorem 2.3, there exists a global smooth solution $\mathcal{U}(t, y) = (\tau(t, y), u(t, y), E_e(t, y), E_i(t, y))^T$ to the Cauchy problem (2.24) and (2.25). Then, we define

527
$$\rho(t,y) = \frac{1}{\tau(t,y)}, \quad \varepsilon_{\alpha}(t,y) = \frac{1}{c_{\alpha}}E_{\alpha}(t,y) - \frac{1}{2}u^{2}(t,y), \quad \alpha = e, i.$$

528 On the other hand, the result in Theorem 2.3 also implies that $\mathcal{U} \in C^1(\mathbb{R}^+ \times \mathbb{R})$ 529 and \mathcal{U} is globally Lipschitzian on \mathbb{R} with respect to y (in particular for τ and u). Then 530 the Cauchy problem to the following ordinary differential equation

531
$$Y'_1(t) = u(t, Y_1(t)), \quad Y_1(0) = 0,$$

admits a unique global solution $Y_1 \in C^2(\mathbb{R}^+)$. Let us further define a function X by

533
$$X(t,y) = \int_{Y_1(t)}^y \tau(t,\eta) d\eta$$

Then, $X \in C^1(\mathbb{R}^+ \times \mathbb{R})$. Similarly to Y_0 , for all $t \in \mathbb{R}^+$, $X(t, \cdot)$ is a C^m -diffeomorphism from \mathbb{R} to \mathbb{R} . Let us denote by $Y(t, \cdot)$ the inverse C^m -diffeomorphism of $X(t, \cdot)$. It is easy to see that

537
$$X(0,y) = X_0(y), \quad Y(0,x) = Y_0(x).$$

538 Finally, we define

539

$$\mathcal{V}(t,x) = (\rho, u, \varepsilon_e, \varepsilon_i)^T (t, Y(t,x)).$$

It is proved in [18] (see also [25]) that entropy solutions of the Cauchy problem for a hyperbolic system of conservation laws are equivalent in Eulerian and Lagrangian coordinates. Moreover, there are explicit formulations of the solutions between two coordinates. Since the solutions studied here are smooth, it is obvious that this equivalence result holds for non-conservative systems. Applying this result, we see that \mathcal{V} is a smooth solution to the Cauchy problem (2.13) and (1.11). Estimate (1.12) follows from (2.26) together with Moser-type inequalities.

547 **Appendix A. Strictly convex entropy and symmetrizer.** There is a well-548 known result showing that the second-order derivative of a strictly convex entropy is 549 a symmetrizer for the hyperbolic system of conservation laws [9, 3]. In general, this 550 result does not hold for a non-conservative system. In this Appendix, we want to show 551 that the bitemperature Euler model, which is a non-conservative system, provides a 552 good example on this topic.

553 More precisely, we consider the system in the form (1.5) or equivalently (1.6). De-554 note $\mathcal{W} = (\rho, \rho u^T, \mathcal{E}_e, \mathcal{E}_i)$. Since η defined in (1.7) is a strictly convex entropy, $\eta''(\mathcal{W})$ 555 is a symmetric positive definite matrix. The result below implies that $\eta''(\mathcal{W})\mathcal{C}_j(\mathcal{W})$ 556 is not symmetric in one space dimension.

Proposition. Consider the one dimensional system (2.12) and denote by $C_1(W) = C(W)$ the related matrix. Then $\eta''(W)C(W)$ is symmetric if and only if $T_e = T_i$.

Proof. We denote $\Gamma = c_e \gamma_e + c_i \gamma_i$. A straightforward calculation using (1.5) gives 559

$$560 \quad \mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2}(\Gamma - 3)u^2 & -(\Gamma - 3)u & \gamma_e - 1 & \gamma_i - 1 \\ -\frac{\gamma_e u p_e}{(\gamma_e - 1)\rho} + \frac{1}{2}c_e(\Gamma - 2)u^3 & \frac{\gamma_e p_e}{(\gamma_e - 1)\rho} + c_e(\frac{3}{2} - \Gamma)u^2 & (\gamma_e c_e + c_i)u & c_e(\gamma_i - 1)u \\ -\frac{\gamma_i u p_i}{(\gamma_i - 1)\rho} + \frac{1}{2}c_i(\Gamma - 2)u^3 & \frac{\gamma_i p_i}{(\gamma_i - 1)\rho} + c_i(\frac{3}{2} - \Gamma)u^2 & c_i(\gamma_e - 1)u & (\gamma_i c_i + c_e)u \end{pmatrix}.$$

562 Let $q = \rho u$. From (1.3) and (1.4), we may write p_{α} in variable \mathcal{W} as

563
$$p_{\alpha} = (\gamma_{\alpha} - 1) \Big(\mathcal{E}_{\alpha} - \frac{c_{\alpha} q^2}{2\rho} \Big).$$

Then η defined in (1.7) can be expressed as 564

565
$$\eta = \eta_e + \eta_i, \quad \eta_\alpha = -\frac{c_\alpha \rho}{b_\alpha} \ln\left(\frac{p_\alpha}{c_\alpha^{\gamma_\alpha} \rho^{\gamma_\alpha}}\right), \qquad \alpha = e, i.$$

Obviously, 566

567

$$\begin{aligned} \eta'_e(\mathcal{W}) &= \left(\frac{\eta_e}{\rho} - \frac{k_B c_e^2 q^2}{2m_e \rho p_e} + \frac{\gamma_e c_e}{b_e}, \frac{k_B c_e^2 q}{m_e p_e}, -\frac{k_B c_e \rho}{m_e p_e}, 0\right),\\ \eta'_i(\mathcal{W}) &= \left(\frac{\eta_i}{\rho} - \frac{k_B c_i^2 q^2}{2m_i \rho p_i} + \frac{\gamma_i c_i}{b_i}, \frac{k_B c_i^2 q}{m_i p_i}, 0, -\frac{k_B c_i \rho}{m_i p_i}\right),\\ \eta'(\mathcal{W}) &= \eta'_e(\mathcal{W}) + \eta'_i(\mathcal{W}). \end{aligned}$$

Since $\partial^2_{\mathcal{E}_e \mathcal{E}_i} \eta = 0$, the hessian matrix of η is of the following form : 568

569
$$\eta''(\mathcal{W}) = \begin{pmatrix} \partial^2_{\rho\rho}(\eta_e + \eta_i) & \partial^2_{\rho q}(\eta_e + \eta_i) & \partial^2_{\rho \mathcal{E}_e} \eta_e & \partial^2_{\rho \mathcal{E}_i} \eta_i \\ \partial^2_{\rho q}(\eta_e + \eta_i) & \partial^2_{q q}(\eta_e + \eta_i) & \partial^2_{q \mathcal{E}_e} \eta_e & \partial^2_{q \mathcal{E}_i} \eta_i \\ \partial^2_{\rho \mathcal{E}_e} \eta_e & \partial^2_{q \mathcal{E}_e} \eta_e & \partial^2_{\mathcal{E}_e \mathcal{E}_e} \eta_e & 0 \\ \partial^2_{\rho \mathcal{E}_i} \eta_i & \partial^2_{q \mathcal{E}_i} \eta_i & 0 & \partial^2_{\mathcal{E}_i \mathcal{E}_i} \eta_i \end{pmatrix},$$

with 570

571
$$\partial_{\rho\rho}^{2}\eta_{\alpha} = \frac{\gamma_{\alpha}c_{\alpha}}{b_{\alpha}\rho} + \frac{c_{\alpha}u^{4}}{4b_{\alpha}\rho\varepsilon_{\alpha}^{2}}, \quad \partial_{\rho q}^{2}\eta_{\alpha} = -\frac{c_{\alpha}u^{3}}{2b_{\alpha}\rho\varepsilon_{\alpha}^{2}}, \quad \partial_{\rho\mathcal{E}_{\alpha}}^{2}\eta_{\alpha} = -\frac{1}{b_{\alpha}\rho\varepsilon_{\alpha}} + \frac{u^{2}}{2b_{\alpha}\rho\varepsilon_{\alpha}^{2}},$$
572

573
$$\partial_{qq}^2 \eta_{\alpha} = \frac{c_{\alpha}}{b_{\alpha} \rho \varepsilon_{\alpha}} + \frac{c_{\alpha} u^2}{b_{\alpha} \rho \varepsilon_{\alpha}^2}, \quad \partial_{q}^2 \varepsilon_{\alpha} \eta_{\alpha} = -\frac{u}{b_{\alpha} \rho \varepsilon_{\alpha}^2}$$

574
$$b_{\alpha}\rho\varepsilon_{\alpha} + b_{\alpha}\rho\varepsilon_{\alpha}^{2}$$

$$\partial^2_{\mathcal{E}_{\alpha}\mathcal{E}_{\alpha}}\eta_{\alpha} = rac{1}{c_{\alpha}b_{\alpha}\rho\varepsilon^2_{\alpha}}, \qquad \alpha = e, i.$$

576Hence we obtain

1

579
$$\partial_{qq}^2 \eta = \sum_{\alpha = e,i} \frac{c_{\alpha}}{b_{\alpha} \rho \varepsilon_{\alpha}^2} \left(\varepsilon_{\alpha} + u^2 \right), \quad \partial_{q \mathcal{E}_{\alpha}}^2 \eta = -\frac{u}{b_{\alpha} \rho \varepsilon_{\alpha}^2},$$

580

575

581
$$\partial^2_{\mathcal{E}_{\alpha}\mathcal{E}_{\alpha}}\eta = \frac{1}{b_{\alpha}c_{\alpha}\rho\varepsilon_{\alpha}^2}, \qquad \alpha = e, i.$$

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582 The entry in the 3-th row and 1-th column of $\eta'' \mathcal{C}$ is

$$(\eta''\mathcal{C})_{31} = \partial_{q\mathcal{E}_e}^2 \eta_e \times \frac{1}{2} (c_e \gamma_e + c_i \gamma_i - 3) u^2 + \partial_{\mathcal{E}_e \mathcal{E}_e}^2 \eta_e [\frac{1}{2} c_e (c_e \gamma_e + c_i \gamma_i - 2) u^3 - c_e \gamma_e u \varepsilon_e]$$
$$= \frac{u^3}{2b_e \rho \varepsilon^2} - \frac{\gamma_e u}{b_e \rho \varepsilon_e}.$$

583

585

$$(\eta''\mathcal{C})_{13} = \frac{u^3}{2b_e\rho\varepsilon_e^2} - \left(\frac{\gamma_e c_e + c_i}{b_e\varepsilon_e} + \frac{c_i(\gamma_e - 1)}{b_i\varepsilon_i}\right)\frac{u}{\rho}.$$

Therefore, $(\eta''\mathcal{C})_{31} = (\eta''\mathcal{C})_{13}$ if and only if $b_e\varepsilon_e = b_i\varepsilon_i$, that is to say $T_e = T_i$. This 586 proves that η is not a symmetrizer of the system. In a same way, we can show that 587 588 $\eta''\mathcal{C}$ is symmetric if and only if $T_i = T_e$.

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