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# Two multifidelity kriging-based strategies to control discretization error in reliability analysis exploiting a priori and a posteriori error estimators

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## Abstract

This paper presents two approaches to tackle the issue of discretization error in the reliability assessment of structures. The first method (AGSK-MCS for Adaptive Guaranteed State Kriging Monte Carlo Sampling) uses discretization error bounds to guarantee the state safe or failed of the points used to build the Kriging metamodel of the limit state function. Two kriging metamodels interpolating lower and upper bounds can be constructed. These metamodels allow to compute discretization error bounds on the probability of failure through Monte Carlo sampling, which can then be used to validate the choice of the mesh. However, discretization error bounds are not available for any solver and any mechanical problem. In that case, a Mesh Size parameterized Kriging (MSK) metamodel can be used to check mesh convergence of the probability of failure. First, finite element simulations are spread on different mesh sizes. Second, the metamodel is used to compute the probability of failure for a given set of mesh sizes using Monte Carlo estimation. The mesh convergence of the probability of failure can be checked and may guide the user toward remeshing. These two strategies are illustrated on two 2-D mechanical problems.

*Keywords:* Reliability, Kriging, Finite Element Method, Discretization Error

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## 1. Introduction

In structural engineering, both geometry and solicitations are usually complex. Modelling such structure leads to a mechanical problem whose solution may only be approximated using discretized methods such as the finite element method. The discretization error may be reduced by choosing a fine discretization which increases computational cost. A compromise between precision and cost has to be found. Moreover, lack of knowledge (epistemic uncertainty) and inherent variability (intrinsic uncertainty) of structural model parameters often make deterministic frameworks impractical. Indeed, the variability of model parameters has to be propagated.

Reliability analysis consists in studying the lifetime of the structure in a stochastic framework. The scenario of failure is defined through a limit state function delimiting failure from safety for negative values. The probability of failure is one of the outputs of the reliability study. Estimating this probability with precision is no trivial task and another estimate called reliability index has been developed. First, approximation methods such as FORM [26], SORM [11] are relatively cheap to compute but do not allow to fully control the approximation. Also, these methods may require the calculation of gradients of the limit state function. Secondly, stochastic Finite Elements [52] give the structural response for any value of the random variable but may be intrusive, which makes them impractical for complex problems. Finally, sampling techniques such as the Monte Carlo method are fully non-intrusive but require many calls to the limit state function to compute the probability of failure with precision [39]. Several variance

17 reduction techniques may be used to reduce the number of calls to the solver [4, 10]. Among them, multilevel Monte  
18 Carlo [24] is a multifidelity method exploiting different levels of fidelity of the limit state function to estimate the  
19 probability of failure. Reliability-based mesh convergence analysis using adjusted control variates [23] is a variance  
20 reduction technique based on multilevel Monte Carlo. It allows to check mesh convergence on the probability of failure  
21 and validate mesh choice by computing the probability of failure at different mesh sizes. Yet, variance reduction  
22 techniques still require many calls to the limit state function and can be costly.

23 In order to reduce the computational cost, a surrogate model can be built from a few evaluations of the original  
24 model and is cheaper to evaluate. Three families of surrogates to be used together with sampling techniques may be  
25 encountered. First, reduced basis methods aims at approximating the structural response for any input parameter  
26 with reduced cost. Controlling the approximation due to the use of the reduced basis is a difficult task but methods  
27 do exist to obtain bounds allowing to control it [19]. Second, it is possible to use classifiers such as Support Vector  
28 Machines [55] as the reliability problem may be seen as a classification problem between safe and failed sampling points  
29 [41]. However, using such techniques does not benefit from the knowledge of the value of the limit state function at  
30 each evaluation. Finally, metamodels that aims at approximating the limit state function are of particular interest.  
31 Classical metamodels that are used in the reliability context are polynomial functions [12], generalized response  
32 function [48], kriging [16], splines [49] and neural networks [49]. Among these techniques, kriging [27] is a flexible  
33 metamodel with interpolating properties. Thanks to its stochastic formulation, kriging provides an estimation of the  
34 variance at any unexplored point of the metamodel. This estimation allows to enrich the metamodel with new points  
35 in unexplored areas : the surrogate then becomes an adaptative metamodel [16].

36 Exploiting different levels of fidelity in a multifidelity framework is an efficient way to reduce computational  
37 cost but also to control the discretization error introduced by the mesh. To this day, the use of multifidelity in  
38 metamodel-based estimation of the probability of failure is limited. A response surface built with the mesh size as an  
39 input variable was used in [3] for the purpose of response surface methodology. The adaptation of kriging to correlated  
40 discrete levels of fidelity, called multifidelity co-kriging was used with FORM in [40]. The multifidelity cokriging  
41 metamodel was also used with crude Monte Carlo simulations in [33] but no adaptive enrichment of the metamodel  
42 was used. It was only until recently [59] that this metamodel was used in an adaptative framework to reduce the  
43 computational burden. Choosing a sufficiently converged mesh (that is to say the highest fidelity level) is usually  
44 done prior to the computation of the probability of failure. However, it is shown in [38, 22] that a small error on  
45 the value of the performance function may lead to a large error on the probability of failure. Several recent works  
46 propose to estimate and control the discretization error on the probability of failure. First, the knowledge on the  
47 mesh convergence of the limit state function may be used to estimate a converged probability of failure. Richardson  
48 extrapolation is used in [3] with a polynomial metamodel to compute the value of the probability of failure when  
49 mesh size tends to zero. In [36], the same Richardson extrapolation is used with FORM to compute a converged  
50 probability of failure. However, neither of them allow to control the degree of approximation introduced by the use  
51 of Richardson extrapolation. Second, it is possible to estimate the discretization error estimators introduced by the  
52 mesh on the value of the limit state function [31, 1, 6, 61]. Among these techniques, the error bounds based on the  
53 error in constitutive relation [31] provide guaranteed error bounds. In [18], this discretization error estimator is used  
54 with FORM to compute bounds on the probability of failure. In [38], a kriging metamodel is built using points that

55 are guaranteed to be rightly classified by the discretization error bounds. Bounds on the probability of failure are  
 56 also obtained when the performance function is monotonic against random variables on a numerical example with a  
 57 unique random variable.

58 The objective of this paper is to propose adaptive numerical strategies to control the discretization error during  
 59 the estimation of the probability of failure thanks to multi-fidelity kriging-based metamodels. The first method in  
 60 this paper is developed for users having access to a discretization error estimator. Note that, it may not be the case  
 61 for non-linear mechanics, non-linear quantities of interest and/or with industrial finite element softwares in which  
 62 the error estimators is not available. This method is an adaptation of the method presented in [38]. The novelties to  
 63 this method presented in this paper are the extension to multiple random variables and the derivation of bounds on  
 64 the probability of failure. The second method developed in this paper consists in building a mesh size parameterized  
 65 kriging metamodel to be able to compute the probability of failure for any mesh size. This property will allow to  
 66 check *a posteriori* the convergence of the probability of failure. This method does not require any discretization error  
 67 estimator and is only based on *a priori* convergence properties and evaluations of the solution on different meshes.

68 The structure of this paper is the following. In Section 2, mechanical and reliability formulations are presented.  
 69 Then, classical kriging-based computation of the probability of failure is explained in Section 3. In Subsection 4.1,  
 70 guaranteed state kriging as introduced in [38] is presented together with its extension to multiple variables. In  
 71 Subsection 4.2, mesh size-parameterized kriging is presented. Finally, both techniques are tested on two numerical  
 72 problems in Section 5.

## 73 2. Mechanical and reliability formulation

74 In order to define the reliability problem, the deterministic continuous mechanical problem is presented in this  
 75 section. Then, the stochastic reliability problem is given: random variables and the limit state function are defined.  
 76 Finally, the estimation of the probability of failure is presented.

### 77 2.1. Continuous mechanical problem

78 First, the structure is defined as occupying an open domain  $\Gamma \subset \mathbb{R}^d$ , where  $\mathbb{R}^d$  represents the physical space.  
 79 Let us consider the static equilibrium of this (polyhedral) structure subject to a given body force  $\underline{f}$  within  $\Gamma$ , to  
 80 a given traction force  $\underline{F}$  on  $\partial_F \Gamma$  and to a given displacement field  $\underline{u}_d$  on the complementary part  $\partial_u \Gamma \neq \emptyset$ . Let  
 81 us make the hypothesis that the structure undergoes small perturbations and that the material has linear elastic  
 82 properties, characterized by Hooke's elasticity fourth order tensor  $\mathbb{H}$ . Let  $\underline{u}$  be the unknown displacement field,  $\underline{\underline{\sigma}}(\underline{u})$   
 83 the symmetric part of the gradient of  $\underline{u}$ ,  $\underline{\underline{\sigma}}$  the Cauchy stress tensor.

84 We first introduce two affine subspaces and a positive form :

- The affine subspace of kinematically admissible fields (KA-fields)

$$85 \quad \text{CA} = \left\{ \underline{u} \in (\text{H}^1(\Gamma))^d, \underline{u} = \underline{u}_d \text{ on } \partial_u \Gamma \right\} \quad (1)$$

86 where  $\text{H}^1$  designates the space of square-integrable with its derivative also being square-integrable. Let us note  
 86  $\text{CA}^0$  the associated vectorial space.

- Affine subspace of statically admissible fields (SA-fields)

$$\text{SA} = \left\{ \underline{\underline{\tau}} \in (\text{L}^2(\Gamma))_{\text{sym}}^{d \times d}; \forall \underline{v} \in \text{CA}^0, \right. \\ \left. \int_{\Gamma} \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}(\underline{v}) d\Gamma = \int_{\Gamma} \underline{f} \cdot \underline{v} d\Gamma + \int_{\partial_F \Gamma} \underline{F} \cdot \underline{v} dS \right\} \quad (2)$$

87 Where  $\text{L}^2$  represents the set of square-integrable fields.

- Error in constitutive equation

$$e_{CR_{\Gamma}}(\underline{u}, \underline{\underline{\sigma}}) = \|\underline{\underline{\sigma}} - \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u})\|_{\mathbb{H}^{-1}, \Gamma} \quad (3)$$

88 where  $\|\underline{\underline{\cdot}}\|_{\mathbb{H}^{-1}, \Gamma} = \sqrt{\int_{\Gamma} (\underline{\underline{\cdot}} : \mathbb{H}^{-1} : \underline{\underline{\cdot}}) d\Gamma}$

The continuous problems is:

$$\left\{ \begin{array}{l} \text{Find a displacement field } \underline{u} \text{ and a stress field } \underline{\underline{\sigma}} \text{ such that} \\ \underline{u} = \underline{u}_d \text{ on } \partial\Gamma \cap \partial_u\Gamma \text{ and } \underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2}(\underline{\underline{\text{grad}}}(\underline{u}) + \underline{\underline{\text{grad}}}^T(\underline{u})) \text{ on } \Gamma \\ \underline{\text{div}}(\underline{\underline{\sigma}}) + \underline{f} = \underline{0} \text{ on } \Gamma \text{ and } \underline{\underline{\sigma}}\underline{n} = \underline{F} \text{ on } \partial_F\Gamma \\ \underline{\underline{\sigma}} = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}) \text{ on } \Gamma \end{array} \right. \quad (4)$$

89 The solution to this problem exists and is unique according to the Kirchhoff uniqueness theorem. This solution  
90 is a couple of exact displacement field and exact stress field denoted as  $(\underline{u}_{ex}, \underline{\underline{\sigma}}_{ex})$ .

## 91 2.2. Definition of the probability of failure of structures

92 Let us define a performance function defining the domain of failure and of safety. This performance function is also  
93 called limit state function and is usually written as a margin between a resistance  $R$  and a solicitation  $S_{ex} = S(\underline{u}_{ex})$ :

$$g_{ex} = R - S_{ex} = R - S(\underline{u}_{ex}) \quad (5)$$

94 The structure is considered as failed if  $g_{ex} \leq 0$  and safe if  $g_{ex} > 0$ .

Let us consider that the uncertainties on the mechanical problem (geometry, loading, ...) are modeled by random variables. Let us gather the  $n$  random variables of the mechanical problem in  $\underline{X} \in \Omega$  with joint probability density  $f_{\underline{X}}(\underline{x})$ . Therefore, the quantities  $\underline{u}_{ex}$ ,  $S(\underline{u}_{ex})$  or even  $R$  are also random. The exact probability of failure  $P_f$  reads :

$$P_f = \int_{g_{ex}(\underline{X}) \leq 0} f_{\underline{X}}(\underline{x}) dx_1 \dots dx_n \quad (6)$$

95 Usually, the exact values  $\underline{u}_{ex}$  and  $S_{ex}$  are unknown. Therefore, a discretization technique is employed to solve  
96 the mechanical problem, which introduces a discretization error. In this paper, we consider that the finite element  
97 (FE) method is used. In the next subsection, we detail the discrete mechanical problem for reliability assessment.

98 *2.3. Discrete mechanical problem for reliability assessment*

Let  $\Gamma_h$  be a tessellation of  $\bar{\Gamma}$  by triangles. The subscript  $h$  denotes the mesh size chosen as an input for meshing the structure. The finite element method consists in searching for a displacement field in the finite subspace  $\text{CA}_h$  of  $\text{CA}$  where  $\text{CA}_h$  reads:

$$\text{CA}_h = \left\{ \underline{u} \in (\text{H}^1(\Gamma))^d, \underline{u} = \underline{u}_d \text{ on } \partial_u \Gamma_h \right\} \quad (7)$$

99  $\text{CA}_h^0$  is the associated vectorial space.

The discrete problem can be formulated as:

$$\begin{aligned} &\text{Find } \underline{u}_h \in \text{CA}_h \text{ such that} \\ &\underline{\underline{\sigma}}_H = \underline{\underline{\mathbb{H}}} : \underline{\underline{\varepsilon}}(\underline{u}_h) \\ &\int_{\Gamma_h} \underline{\underline{\sigma}}_H : \underline{\underline{\varepsilon}}(\underline{v}_h) d\Gamma = \int_{\Gamma_h} \underline{f} \cdot \underline{v}_h d\Gamma + \int_{\partial_F \Gamma_h} \underline{F} \cdot \underline{v}_h dS, \forall \underline{v}_h \in \text{CA}_h^0 \end{aligned} \quad (8)$$

The solution of this discrete problem exists and is unique according to the Lax-Milgram theorem. The discrete solution  $\underline{u}_h$  usually does not coincide with the exact solution  $\underline{u}_{ex}$ . Let note  $g_h$  the limit state function computed from the finite element solution  $\underline{u}_h$  and defined by:

$$g_h = R - S_h = R - S(\underline{u}_h) \quad (9)$$

It is possible to use  $g_h$  to compute an estimation of the probability of failure:

$$P_{f,h} = \int_{g_h(\underline{X}) \leq 0} f_{\underline{X}}(\underline{x}) dx_1 \dots dx_n \quad (10)$$

100 Because of the discretization error,  $P_f \neq P_{f,h}$ .

101 *2.4. Estimation of the discretization error for structural reliability assessment*

102 The finite element method introduces a discretization error as the discrete displacement field  $\underline{u}_h$  differs from the  
103 exact solution  $\underline{u}_{ex}$ . *A priori* estimation of this error is made possible by exploiting the convergence rate of the FE  
104 problem. Theoretically, the rate of convergence  $\gamma$  against mesh size is known to follow [5]:

$$\|\underline{u}_{ex} - \underline{u}_h\|_{\mathcal{L}^2(\Gamma)} \leq Ch^{-\gamma} \quad (11)$$

105 If the quantity of interest  $S$  is a linear form of the displacement field, then :

$$|S(\underline{u}_{ex}) - S(\underline{u}_h)| \leq [Ch^{-\gamma}]^2 = C^2 h^{-2\gamma} \quad (12)$$

$\gamma$  depends on the regularity of the problem and the degree of interpolation of the finite element shape functions. Note that this inequality involves constants that are not computable. *A posteriori* estimation of the discretization error is also possible. Such techniques rely on a post-process of the finite element solution [2]. In this paper, estimators based on the error in constitutive relation (3) are chosen as they provide guaranteed error bounds [31]. First, let us define the energy norm of the displacement  $\|\cdot\|_{\Gamma}$ :

$$\|\underline{v}\|_{\Gamma} = \|\underline{\underline{\varepsilon}}(\underline{v})\|_{\underline{\underline{\mathbb{H}}}, \Gamma} \quad (13)$$

106 These estimators of the discretization error are based on the fundamental Pythagore theorem applied to this  
 107 norm:

$$\begin{aligned} \forall(\hat{\underline{u}}, \hat{\underline{\sigma}}) \in \text{CA} \times \text{SA}, \\ \|\underline{\underline{\varepsilon}}(\underline{u}_{ex}) - \underline{\underline{\varepsilon}}(\hat{\underline{u}})\|_{\mathbb{H}, \Gamma}^2 + \|\underline{\underline{\sigma}}_{ex} - \hat{\underline{\sigma}}\|_{\mathbb{H}^{-1}, \Gamma}^2 = e_{CR\Gamma}^2(\hat{\underline{u}}, \hat{\underline{\sigma}}) \end{aligned} \quad (14)$$

Choosing  $\hat{\underline{u}} = \underline{u}_h \in \text{CA}$  allows to obtain the following upper bound for the error  $e_{discr} = \underline{u}_{ex} - \underline{u}_h$ :

$$e_{discr} := \|\underline{e}_{discr}\|_{\Gamma} \leq e_{CR\Gamma}(\underline{u}_h, \hat{\underline{\sigma}}) \quad (15)$$

108 While being a complex task, several methods have been developed to compute a statically admissible stress field  
 109  $\hat{\underline{\sigma}} \in \text{SA}$  (see [30], [42], [44] and [46]).

110 By using extractors [7] in the context of goal-oriented error estimation, it is possible to obtain guaranteed bounds  
 111 on the discretization error on the quantity of interest  $|S(\underline{u}_{ex}) - S(\underline{u}_h)|$ . Note that specific methods to calculate  
 112 guaranteed bounds exist for non linear quantities of interest [51, 47]. If not, the quantity of interest would have to be  
 113 linearized at the cost of the loss of guaranteed bounding. Goal-oriented error estimation requires to solve an adjoint  
 114 problem defined by :

$$\text{Find } (\tilde{\underline{u}}_{ex}, \tilde{\underline{\sigma}}_{ex}) \in \text{CA}^0(\Gamma) \times \widetilde{\text{SA}}(\Gamma) \text{ such that } e_{CR\Gamma}(\tilde{\underline{u}}_{ex}, \tilde{\underline{\sigma}}_{ex}) = 0 \quad (16)$$

where

$$\widetilde{\text{SA}}(\Gamma) = \left\{ \underline{\underline{\tau}} \in (\mathbf{L}^2(\Gamma))_{\text{sym}}^{d \times d}; \quad \forall \underline{v} \in \text{CA}^0(\Gamma), \quad \int_{\Gamma} \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}(\underline{v}) d\Gamma = S(\underline{v}) \right\} \quad (17)$$

115 The adjoint problem is usually solved using the finite element method with a mesh of size  $\tilde{h}$  whose output is  $\tilde{\underline{u}}_{\tilde{h}}$ .  
 116 The mesh does not need to be the same as for the forward problem, however, in this paper, the same mesh is used  
 117 ( $\tilde{h} = h$ ) as it saves computational time in stiffness matrix factorization. Solving the finite element problem is in  
 118 fact simplified to a multiple (double) right-hand side linear system resolution. Let us note  $\hat{\underline{\sigma}}_{\tilde{h}}$  a statically admissible  
 119 stress field built from  $\tilde{\underline{u}}_{\tilde{h}}$  thanks to [30], [42], [44] or [46].

120 Exploiting the results from [28, 29], it is possible to derive guaranteed bounds on the exact value of the limit  
 121 state function:

$$g^- \leq g_{ex} \leq g^+ \quad (18)$$

where

$$\begin{aligned} g^+ &= g_h - \frac{1}{2} \int_{\Gamma} (\hat{\underline{\sigma}}_{\tilde{h}} + \mathbb{H} : \underline{\underline{\varepsilon}}(\tilde{\underline{u}}_h)) : \mathbb{H}^{-1} : (\hat{\underline{\sigma}}_{\tilde{h}} - \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_h)) d\Gamma + \frac{1}{2} e_{CR\Gamma}(\underline{u}_h, \hat{\underline{\sigma}}_{\tilde{h}}) e_{CR\Gamma}(\tilde{\underline{u}}_h, \hat{\underline{\sigma}}_{\tilde{h}}) \\ g^- &= g_h - \frac{1}{2} \int_{\Gamma} (\hat{\underline{\sigma}}_{\tilde{h}} + \mathbb{H} : \underline{\underline{\varepsilon}}(\tilde{\underline{u}}_h)) : \mathbb{H}^{-1} : (\hat{\underline{\sigma}}_{\tilde{h}} - \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_h)) d\Gamma - \frac{1}{2} e_{CR\Gamma}(\underline{u}_h, \hat{\underline{\sigma}}_{\tilde{h}}) e_{CR\Gamma}(\tilde{\underline{u}}_h, \hat{\underline{\sigma}}_{\tilde{h}}) \end{aligned} \quad (19)$$

122 Let us note that  $g_h$  is not guaranteed to lie within the interval defined in (18). The middle of these bounds  $g^m = \frac{g^+ + g^-}{2}$   
 123 was found to be a much better approximation of the true value  $g_{ex}$  than  $g_h$  in [37].

## 124 2.5. Computation of the probability of failure thanks to the Monte Carlo method

125 Computing the probability of failure  $P_{f,h}$  from equation (10) is not a trivial task. Indeed, the limit state function  
126  $g_h$  is not known explicitly but only implicitly as it requires to call the FE solver. Therefore, sampling techniques  
127 such as the Monte Carlo method are usually used and the probability of failure is approximated as:

$$P_{f,h,MC} = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \text{ind}_{g_h < 0}(x_i) \quad (20)$$

128 It requires the generation of  $n_{MC}$  realizations of the random variable  $\underline{X}$ . Those realizations are gathered in a  
129 set denoted  $P_{MC}$  which is the Monte Carlo population. The Monte Carlo method introduces an estimation error  
130 as  $P_{f,h,MC} \neq P_{f,h}$ . It is possible to calculate *a posteriori* the coefficient of variation on the probability of failure  
131 and check if the size of the Monte Carlo population is sufficient to compute a converged probability of failure. This  
132 criterion reads:

$$\text{COV}_{P_{f,h,MC}} = \sqrt{\frac{P_{f,h,MC}}{n_{MC}(1-P_{f,h,MC})}} < \text{COV}_{P_{f,h,MC},\text{target}} \quad (21)$$

133 If this criterion is not satisfied, the Monte Carlo population has to be enriched with new points to satisfy (21).  
134 Unfortunately, the Monte Carlo method is known for having a slow convergence rate. Thus, computing the probability  
135 of failure of the structure through crude Monte Carlo sampling is too expensive computationally as it would require too  
136 many expensive calls to the FE solver. This is the reason why metamodels are often used to reduce the computational  
137 burden. Kriging-based metamodels are very briefly presented in the next Section.

## 138 3. Adaptive kriging for computing a probability of failure

### 139 3.1. General presentation

140 Classical metamodels used to compute a probability of failure with Monte Carlo estimation are polynomial chaos  
141 [12], generalized response surfaces [48], neural networks [49], splines [49], kriging [16] and evofusion [38]. Using a  
142 meta-model built from FE computations combined with Monte Carlo sampling to compute the probability of failure  
143 introduces 5 sources of error:

- 144 • The **statistical error** due to the limited statistics for identifying the joint probability density function of  $\underline{X}$
- 145 • The **model error** due to the model choice of the mechanical problem
- 146 • The **discretization error** due to the choice of the mesh size
- 147 • The **approximation error** or **meta-modelling error** due to the metamodel
- 148 • The **estimation error** due to the Monte Carlo population being finite

149 The statistical error may be reduced by increasing the database for variables selected from the sensitivity analysis.  
150 The model error may be evaluated by checking each individual hypotheses of the mechanical model and by doing  
151 experiments for example [53]. Both statistical and model errors are outside the scope of this paper. Conducting  
152 the convergence study and choosing a mesh guaranteeing a small discretization error on the limit state function

153 may lead to a large error on the probability of failure as shown in [38, 22]. However, expert knowledge or standard  
 154 and recommended practices may guide mesh choice *a priori*. The estimation error may be controlled by using the  
 155 coefficient of variation of the Monte Carlo estimator to verify that the Monte Carlo population is large enough. The  
 156 metamodeling error may be reduced or even negated by the use of a learning function and criterion (adaptive kriging).  
 157 Kriging-based metamodels are appealing meta-models as they offer an estimation of uncertainty together with the  
 158 estimation of the quantity of interest at any input location. In the rest of the paper, as the exact mechanical solution  
 159 is unknown and as the observations can only be obtained from FE simulations, the subscript  $h$  will be omitted.

160 The following subsections aim at briefly explaining the principle of kriging and how it is used for estimating a  
 161 probability of failure.

### 162 3.2. Kriging meta-model

163 The objective of kriging is to build a meta-model  $\hat{g} : \underline{x} \rightarrow \hat{g}(\underline{x})$  from  $n_{obs}$  observations  $(g(\underline{x}_j))_{j=1..n_{obs}}$  by assuming  
 164 that  $\hat{g}$  is the realization of a stationary Gaussian process  $G$  [27]. In this paper, we use ordinary kriging, which means  
 165 that the mean of the stationary process is an unknown constant. This constant is estimated during the construction  
 166 of the meta-model. Moreover, it is necessary to postulate the form of the correlation of  $G$ . Here, a gaussian correlation  
 167 function is selected as it was observed to give the best approximation between most standard correlation functions on  
 168 several test cases [35]. In order to determine the variance of the gaussian process, we used the maximum likelihood  
 169 estimation as it is considered for its asymptotical optimality [58, p. 124]. Note that a good initial guess may be  
 170 obtained through mean square estimation [13]. Once that the variance is determined,  $\hat{g}$  is searched as the best linear  
 171 unbiased predictor which leads to the solving of an optimization problem. In addition to the estimator of the mean  
 172  $\hat{g}$ , the kriging metamodel provides the variance  $\sigma_{\hat{g}}$ . Intereted readers on the implementation of kriging can refer to  
 173 [14].

### 174 3.3. Learning function

175 In order to reduce the metamodeling error by adding new points to the metamodel  $\hat{g}$ , one needs a learning function  
 176 that identifies uncertain points within the Monte Carlo population and a learning criterion to identify whether the  
 177 metamodel is sufficiently converged or not.

178 The first learning function that was introduced is the Expected Feasibility Function (EFF) in [9]. It measures  
 179 the expectation that the stationary gaussian process belongs to  $[-\epsilon, +\epsilon]$ , where  $\epsilon$  is chosen as proportional to  $\sigma_{\hat{g}}$ .  
 180 The next learning point is chosen as the point within the Monte Carlo population  $P_{MC}$  maximizing the EFF. For  
 181 multifidelity cokriging and to select both the next learning point and the level of fidelity on which the point will be  
 182 calculated, only the EFF has been adapted [59].

Since then, several learning functions have been introduced [60, 8, 45, 9, 43] but the U function [16] remains  
 a reference in reliability analysis [34]. The next learning point  $\underline{x}_{next}$  is chosen as the point in the Monte Carlo  
 population maximizing the probability of predicting the wrong sign for  $\hat{g}(\underline{x}_{next})$ :

$$\underline{x}_{next} = \arg \min_{\underline{x} \in P_{MC}} U(\underline{x}) = \frac{|\hat{g}|}{\sigma_{\hat{g}}} \quad (22)$$

183 The U learning function was used with guaranteed state kriging in [38] and will be used in this paper as well.

184 *3.4. Approximation error and learning criterion*

185 In [16], it is considered that the learning process can stop when  $U(\underline{x}_{next}) > 2$ . However this criterion is known to  
 186 be too conservative as new points keep being added while the estimation of the probability of failure has converged  
 187 [21, 54]. A possibility introduced in [59] is to propagate the probability of wrong sign to the estimation of the  
 188 approximation error :

$$e_{approx} = \frac{|N_f - \hat{N}_f|}{N_f} \quad (23)$$

189 with  $N_f = \text{card}(\{\underline{x} \in P_{MC} | g(\underline{x}) \leq 0\})$  and  $\hat{N}_f = \text{card}(\{\underline{x} \in P_{MC} | \hat{g}(\underline{x}) \leq 0\})$ . The estimated number of false negative  
 190 points  $\hat{N}_r$  is an approximated upper bound for  $|N_f - \hat{N}_f|$  which represents the error on the number of points in  
 191 the failure domain, as some false positives might compensate for the false negatives. Note that other estimates of  
 192 the approximation error are available in the literature [60, 56, 57]. They all rely on the construction of confidence  
 193 intervals. It is possible to estimate  $\hat{N}_r$  by assuming that points with wrong sign are uncorrelated :

$$\hat{N}_r = \sum_{i=1}^{n_{MC}} \text{Ind}_{\hat{g} \leq 0}(\underline{x}_i) \times \Phi(-U(\underline{x}_i)) \quad (24)$$

Inserting (24) into (23) gives an approximate upper bound on the approximation error :

$$e_{approx} \lesssim \frac{\hat{N}_r}{|\hat{N}_f - \hat{N}_r|} = \tilde{e}_{approx} \quad (25)$$

194 The learning criterion may be written as a maximal approximation error. In this paper, this maximal approxi-  
 195 mation error is taken as 1%:

$$\tilde{e}_{approx} < 1\% \quad (26)$$

196 Finally, the probability of failure is estimated using Monte Carlo method on the meta-model:

$$\hat{P}_{f,MC} = \frac{1}{n_{MC}} \sum_{\underline{x} \in P_{MC}} \text{Ind}_{\hat{g} \leq 0}(\underline{x}) \quad (27)$$

197 *3.5. Articulation of both criteria*

198 In AK-MCS [16], the learning criterion is the first criterion to be satisfied on the Monte Carlo population. The  
 199 estimation error is controlled by enriching the Monte Carlo population only when the learning criterion is already  
 200 satisfied. Indeed, if the approximation error is not sufficiently controlled and the probability of failure is found too  
 201 low, it might entail enriching the Monte Carlo population too much. It would result in unnecessary calls to the  
 202 metamodel  $\hat{g}$  but also to  $g$  as a large number of points would have to satisfy the learning criterion.

203 *3.6. Initialization of the metamodel*

204 Twelve calls to the FEM solver are used for initialization in [16]. However, this number should be dependent  
 205 on the number of random variables  $n$  and their correlation while being greater than  $n + 1$  in order to allow the  
 206 construction of the kriging hypersurface  $\hat{g}$ . In this paper, twelve calls are used for initialization as the two problems

207 have two random variables. The method to select these input points should be designed in order to spread them  
 208 and obtain a good first approximation of the limit state  $g = 0$ . The selection of these points may be done by expert  
 209 judgement or by using available simulations. More systematically, Latin Hypercube Sampling (LHS) may be used  
 210 to pick these points according to the random variables distributions while guaranteeing spread as done in [16]. For  
 211 bounded random variables, a factorial experiment may also be used in order to explore the whole domain  $\Omega$ .

#### 212 4. New adaptive multi-fidelity kriging strategies

213 In this section, two methods are presented to perform multi-fidelity adaptive kriging in order to compute the  
 214 probability of failure. A first method based on [38] can be employed by users having access to a finite element  
 215 discretization error estimator. This method is based on the bounding (18). It enables to compute bounds on the  
 216 exact probability of failure. In the second subsection, a method to compute the probability of failure thanks to a  
 217 mesh size-parameterized kriging meta-model is presented. This method does not require error estimators. It enables  
 218 to compute the value of the probability of failure for any mesh size.

##### 219 4.1. Guaranteed state kriging using discretization error estimator

###### 220 4.1.1. General principle from [38]

221 Guaranteed state kriging (GSK) consists in building a kriging meta-model from calls to different levels of fidelity,  
 222 here different mesh sizes. First, a set of possible meshes ranked from coarsest to finest is generated. The idea is to  
 223 build a kriging metamodel using points which the discretization error bounds enables to guarantee its state (failed  
 224 or safe). If it is not the case, the FE simulation is computed on the next finer mesh until the finest mesh is reached.  
 225 The subroutine that selects this mesh size is presented in Algorithm 1. The advantage of this method is that coarse  
 226 mesh can be used for points far from the limit state as long as their state is certain, which results in computational  
 227 time reduction.

---

#### Algorithm 1: Mesh selection with guaranteed state kriging

---

**Input** :  $\underline{x}_{next}$  : Point selected by the learning criterion or by the initialization

$h_{vec} = [h_{min}, \dots, h_{max}]$  : Possible mesh sizes on which to compute  $\underline{x}_{next}$

**Mesh Choice:**

$h_{next} = h_{max}$

$i = 1$

Compute  $g^m(\underline{x}_{next}, h_{next})$ ,  $g^+(\underline{x}_{next}, h_{next})$ ,  $g^-(\underline{x}_{next}, h_{next})$  using (19);

**while**  $g^-(\underline{x}_{next}, h_{next}) \times g^+(\underline{x}_{next}, h_{next}) < 0 \ \&\& \ h_{next} > h_{min}$  **do**

$i = i + 1$

$h_{next} = h_{vec}(i)$

    Compute  $g^m(\underline{x}_{next}, h_{next})$ ,  $g^+(\underline{x}_{next}, h_{next})$ ,  $g^-(\underline{x}_{next}, h_{next})$

**end**

**Result:**  $g^m(\underline{x}_{next}, h_{next})$ ,  $g^-(\underline{x}_{next}, h_{next})$  and  $g^+(\underline{x}_{next}, h_{next})$

---

228 In this paper, only two levels of fidelity are used. Indeed, it was shown in [38] that it is a good compromise between  
 229 using a single level of fidelity or intermediate meshes between coarse and fine meshes. On the one hand, using a single

230 level of fidelity would result in always calling the FEM solver on a costly fine mesh (standard monofidelity approach).  
 231 On the other hand, using too many levels of fidelity would result in calling the FEM solver on too many useless  
 232 intermediate mesh sizes which would increase the computational cost. Figure 1 shows an artificial example of kriging  
 233 metamodel built with the above remeshing strategy, with  $g = g^m$  the middle of discretization error bounds and one  
 234 random variable  $x$ . To build the meta-model, 9 calls to the FE solver were done for the seven points :  $x = 1.4$ ,  
 235  $x = 1.5$ ,  $x = 2.11$ ,  $x = 3.2$ ,  $x = 3.53$ ,  $x = 4.63$  and  $x = 4.69$  (for two points, 2 FE simulations were done: one on  
 236 the coarse mesh, one on the fine mesh). The notations  $g^-$  and  $g^+$  denote respectively lower and upper discretization  
 237 error bounds on  $g_{ex}$ . We can observe that close to the limit state  $\hat{g} = 0$ ,  $\hat{g}$  is built from computations on the fine  
 238 mesh.

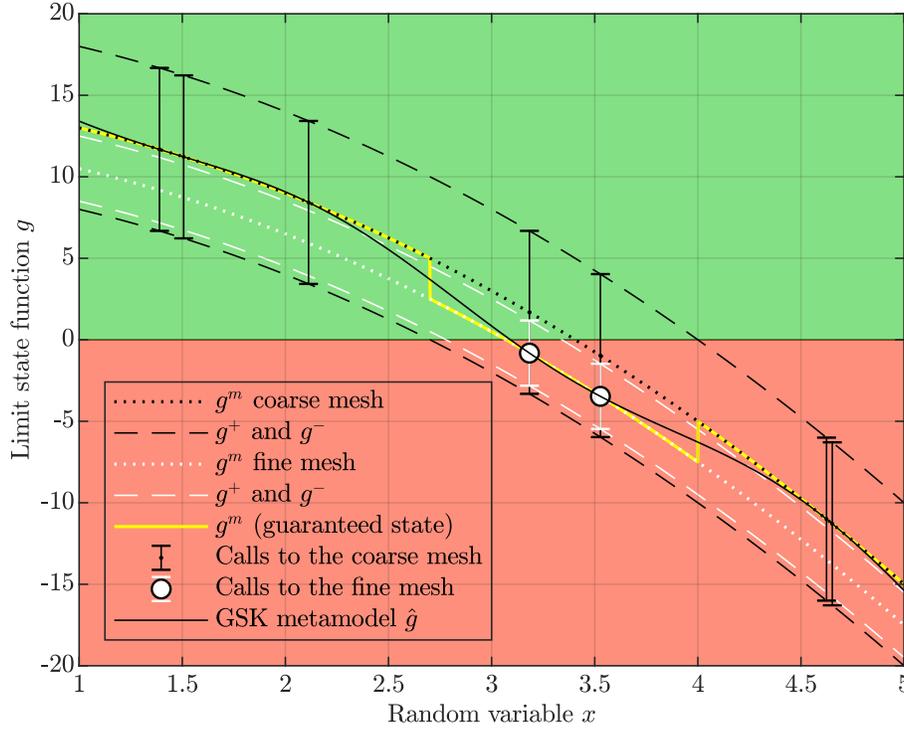


Figure 1: Example of guaranteed state kriging metamodel

#### 239 4.1.2. Estimating discretization error bounds on the probability of failure

240 In [38], bounds were propagated on the probability of failure by assuming the monotonicity of the limit state  
 241 function against the value of the random variable. Let refer to Figure 1. For  $x^+ = 3.53$ ,  $g_{ex}(x^+)$  is strictly negative  
 242 since it lies between the upper and lower bounds computed on the fine mesh that are strictly negative. Therefore,  
 243 the point  $x^+$  is in the failure domain. Adding the hypothesis of monotonicity allows to guarantee the state of any  
 244 realization of the random variable greater than  $x^+$ . Using  $F_X$ , the cumulative distribution function of  $X$ , allows to  
 245 compute a lower bound on the probability of failure:

$$\hat{P}_f^- = F_X(x^+); \quad (28)$$

246 It is also possible to compute an upper bound on the probability of failure. Indeed, for  $x^- = 2.15$ ,  $g_{ex}(x^-)$  is  
 247 strictly positive since it lies between the upper and lower bounds that are strictly positive. Therefore, the point  $x^-$

248 is in the safe domain.

$$\hat{P}_f^+ = 1 - F_X(x^-) \quad (29)$$

249 The bounds may also be computed in 2D using a similar methods as developed in [15]. Such method provides  
 250 strict error bounds only relying on the hypotheses that the limit state function is monotonic. Note that under the  
 251 hypothesis of linear quasi-static mechanical problem, the limit state function is monotonic if it is defined from a  
 252 classical mechanical failure scenario corresponding to the difference between resistance and sollicitation. For time-  
 253 dependant mechanical problems, it is possible to define scenario of failure leading to non monotonic limite state  
 254 function.

255 In this paper, we propose an approach to the estimation of the discretization error bounds on the probability of  
 256 failure not relying on the monotonicity hypothesis. To do so, two kriging metamodells  $\hat{g}^+$  and  $\hat{g}^-$  may be constructed  
 257 using respectively  $\underline{g}^{obs+} = [g^+(x_1), g^+(x_2) \dots g^+(x_{n_{obs}})]$  and  $\underline{g}^{obs-} = [g^-(x_1), g^-(x_2) \dots g^-(x_{n_{obs}})]$ . Finally,  
 258 bounds on  $P_f$  are computed using the same Monte Carlo population as for the initial adaptive strategy. It is not  
 259 necessary to use an adaptive strategy to compute  $P_f^-$  and  $P_f^+$  precisely as it is possible to estimate the approximation  
 260 error due to the use of non-converged metamodells  $\hat{g}^+$  and  $\hat{g}^-$ . It is also possible to take advantage of the property  
 261  $\hat{g}^+ > \hat{g} > \hat{g}^-$ . Let us partition  $P_{MC}$  into two subpopulations  $P_{MC}^+$  and  $P_{MC}^-$ :

$$\begin{aligned} P_{MC}^+ &= \{\underline{x} \in P_{MC}, \hat{g}(\underline{x}) > 0\} \\ P_{MC}^- &= P_{MC} - P_{MC}^+ = \{\underline{x} \in P_{MC}, \hat{g}(\underline{x}) \leq 0\} \end{aligned} \quad (30)$$

It eases the computation of the following bounds on the probability of failure:

$$\begin{aligned} \hat{P}_{f,MC}^+ &= \frac{1}{n_{MC}} \sum_{\underline{x} \in P_{MC}^+} \text{Ind}_{\hat{g}^+ \leq 0}(\underline{x}) \\ \hat{P}_{f,MC}^- &= \frac{1}{n_{MC}} \left[ \sum_{\underline{x} \in P_{MC}^-} \text{Ind}_{\hat{g}^- \leq 0}(\underline{x}) + \sum_{\underline{x} \in P_{MC}^+} \text{Ind}_{\hat{g} \leq 0}(\underline{x}) \right] \end{aligned} \quad (31)$$

262 It is also possible to estimate the approximation error by rewriting (25):

$$\begin{aligned} \tilde{e}_{approx}^+ &\leq \frac{\hat{N}_r^+}{|\hat{N}_f^+ - \hat{N}_r^+|} \\ \tilde{e}_{approx}^- &\leq \frac{\hat{N}_r^- + \hat{N}_r}{|\hat{N}_f^- + \hat{N}_r - \hat{N}_r^- - \hat{N}_r|} \end{aligned} \quad (32)$$

263 Where the notations  $\hat{N}_r^+$ ,  $\hat{N}_f^+$ ,  $\hat{N}_f^-$  and  $\hat{N}_r^-$  are straightforwardly adapted from (28).

264 This strategy is illustrated in section 5.2.

### 265 4.1.3. Algorithm

266 Algorithm 2 presents the implementation of Adaptive Guaranteed State Kriging with Monte Carlo Simulations  
 267 (AGSK-MCS).

### 268 4.2. Mesh size-parameterized kriging meta-model

269 It is not always possible to compute discretization error bounds as it may be intrusive to industrial finite element  
 270 codes. To this end, a novel approach is presented that only requires calls to FE solver with different mesh sizes and  
 271 exploits the convergence rate of the FE method.

---

**Algorithm 2:** Adaptive Guaranteed State Kriging with Monte Carlo Simulations (AGSK-MCS)

---

**Input** :

$COV_{target}$ : Target value of the coefficient of variation;

$n_{MC}$  : Initial size of the Monte Carlo population;

$h_{vec} = \llbracket h_{max}, \dots, h_{min} \rrbracket$ : possible mesh sizes;

**Initialization** :

Generate  $\underline{x}^{obs}$  to initialize the metamodel ;

Use the remeshing strategy from Algorithm 1 for each point in  $\underline{x}^{obs}$ ;

Build the metamodel  $\hat{g}$  and compute it for the whole Monte Carlo population  $P_{MC}$ ;

Compute  $\hat{P}_{f,MC}$ ,  $\tilde{e}_{approx}$  and  $COV_{\hat{P}_{f,MC}}$ ;

**Enrichment** :

**while**  $\tilde{e}_{approx} > 1\%$  and  $COV_{\hat{P}_{f,MC}} > COV_{target}$  **do**

**if**  $\tilde{e}_{approx} > 1\%$  **then**

        Find  $\underline{x}_{next} \in P_{MC}$ , such that  $U(\underline{x}_{next}) = \min(U(P_{MC}))$ ;

        Append  $\underline{x}^{obs}$  with  $\underline{x}_{next}$

        Compute the remeshing strategy from Algorithm 1 for  $\underline{x}_{next}$  and  $h_{vec}$  to obtain  $g^m(\underline{x}_{next}, h_{next})$ ,  $g^-(\underline{x}_{next}, h_{next})$  and  $g^+(\underline{x}_{next}, h_{next})$

        Append  $\underline{g}^{obs}$  with  $g^m(\underline{x}_{next}, h_{next})$ ,  $\underline{g}^{obs-}$  with  $g^-(\underline{x}_{next}, h_{next})$  and  $\underline{g}^{obs+}$  with  $g^+(\underline{x}_{next}, h_{next})$

        Build the metamodel  $\hat{g}$

**else**

        Compute  $n_{MC} = \frac{1 - \hat{P}_{f,MC}}{\hat{P}_{f,MC} COV_{target}^2}$

        Enrich the Monte Carlo population so that:  $\text{size}(P_{MC}) = n_{MC}$

**end**

    Compute the metamodel  $\hat{g}$  on the whole Monte Carlo population  $P_{MC}$ ;

    Estimate  $\hat{P}_{f,MC}$ ,  $COV_{\hat{P}_{f,MC}}$ ,  $\tilde{e}_{approx}$  and  $U(P_{MC})$

**end**

**Post-processing:** Separate  $P_{MC}$  into two subpopulations  $P_{MC}^+$  and  $P_{MC}^-$  using (30);

Build  $\hat{g}^+$  and  $\hat{g}^-$  respectively with  $\underline{g}^{obs+}$  and  $\underline{g}^{obs-}$

Compute  $\hat{P}_{f,MC}^+$  and  $\hat{P}_{f,MC}^-$  using (31) together with  $COV_{\hat{P}_{f,MC}^+}$  and  $COV_{\hat{P}_{f,MC}^-}$ ;

Compute  $\tilde{e}_{approx}^+$  and  $\tilde{e}_{approx}^-$  using (32);

**Result:**

$\hat{P}_{f,MC}$ ,  $\tilde{e}_{approx}$ ,  $COV_{\hat{P}_{f,MC}}$

$\hat{P}_{f,MC}^-$ ,  $\tilde{e}_{approx}^-$ ,  $COV_{\hat{P}_{f,MC}^-}$

$\hat{P}_{f,MC}^+$ ,  $\tilde{e}_{approx}^+$ ,  $COV_{\hat{P}_{f,MC}^+}$

---

272 *4.2.1. General presentation*

273 Here an extended kriging metamodel is developed and called mesh size-parameterized kriging (MSK). The input  
274 variables of this metamodel are  $\underline{x}$  and the mesh size of the numerical simulation:

$$\begin{aligned} \hat{g} : \quad [\underline{x}, h] &\rightarrow \hat{g}(\underline{x}, h) \\ \Omega \times \mathbb{R}^{+*} &\rightarrow \mathbb{R} \end{aligned} \quad (33)$$

275 The key ingredient is to take advantage from the known convergence rate of  $g_h$  as shown in (12):

$$\exists [\alpha, \gamma] \in \mathbb{R}^+ \times \mathbb{R} \text{ such that } \gamma > \alpha, \text{ and that } \forall \underline{x} \in \Omega, \quad g_{ex}(\underline{x}) - g_h(x) = \beta_1 h^\alpha + O(h^\gamma) \quad (34)$$

276 to build a universal kriging process defined as:

$$g(\underline{x}, h) = \beta_0 + \beta_1 h^\alpha + Z(\underline{x}, h) \quad (35)$$

277 The parameter  $\alpha$  has to be computed prior to the construction of the kriging metamodel. The parameters  $\beta_0$   
278 and  $\beta_1$  are found during the construction of the meta-model. One will notice that  $Z$  still depends on  $h$  to account  
279 for spatial correlation that depends on  $h$ .

280 The great advantage of this method is that once the meta-model is built, it can be used to compute the probability  
281 of failure for any mesh size with crude Monte Carlo sampling since evaluations of the meta-model are cheap.

282 *4.2.2. Augmented U learning function*

283 As the mesh size is now a parameter of the kriging meta-model, we propose to adapt the  $U$  learning function in  
284 order to obtain the next point to evaluate and the optimal mesh size on which it will be evaluated.

285 First, the next observation point  $\underline{x}_{next}$  is defined as

$$\underline{x}_{next} = \arg \min_{\underline{x} \in P_{MC}} \frac{|\hat{g}_0(\underline{x}, h_{min})|}{\sigma_{\hat{g}_0}(\underline{x}, h_{min})} \quad (36)$$

286 where  $\hat{g}_0$  denotes the metamodel before the addition of a new point. Indeed, our objective is that the learning  
287 criterion should be fulfilled on the fine mesh  $h_{min}$ .

288 Then, the mechanical problem is solved for  $\underline{x} = \underline{x}_{next}$  on the coarsest mesh  $h = h_{max}$ , which allows to obtain  
289 information at low computationnal cost for  $\underline{x} = \underline{x}_{next}$ . The metamodel is updated thanks to this new observation.  
290 We note  $\hat{g}_1$  the obtained metamodel.

291 Finally, the mesh size  $h_{next}$  is sought as:

$$h_{next} \in [h_{min}, h_{max}], \text{ such that } \frac{|\hat{g}_{h_{next}}(\underline{x}_{next}, h_{min})|}{\sigma_{\hat{g}_{h_{next}}}(\underline{x}_{next}, h_{min})} = U_{target} \quad (37)$$

where  $\hat{g}_{h_{next}}$  is the meta-model obtained after the update at the learning point  $\underline{x}_{next}$  thanks to the call to the FE code  
at the mesh size  $h_{next}$ . Note that  $\sigma_{\hat{g}_{h_{next}}}$  is the associated variance estimator. The previous equation is equivalent  
to:

$$h_{next} = \arg \min_{h \in [h_{min}, \dots, h_{max}]} \left| \frac{\hat{g}_h}{\sigma_{\hat{g}_h}} - U_{target} \right| \quad (38)$$

292 In this paper, the value  $U_{\text{target}} = 2$  is used as it was seen to be sufficiently demanding in [21]. The method to select  
 293 the mesh size  $h_{\text{next}}$  is summarised in Algorithm 4. This Algorithm relies on a function  $f$  that is defined in Algorithm  
 294 3. Minimizing (38) can be done with any standard minimization algorithm. This learning process is more expensive  
 295 than standard learning process as it requires to build a kriging metamodel at each step of the minimization process  
 296 as shown in Algorithm 4. Finally, the algorithm enforces that  $h_{\text{min}} \leq h_{\text{next}} \leq h_{\text{max}}$ . Indeed, if the minimization  
 297 of  $f$  over  $h$  is done with a standard minimization algorithm, it is possible that a standard minimization algorithm  
 298 finds a minimum for  $h < h_{\text{min}}$ . However,  $h_{\text{min}}$  is the smallest mesh size on which we accept to do finite element  
 299 computations. Therefore, if the minimization leads to  $h < h_{\text{min}}$ , we set  $h_{\text{next}} = h_{\text{min}}$ . On the contrary, it is possible  
 300 that adding the  $y_0$  to the meta-model is already sufficient to reach the learning objective so that the minimization on  
 301  $f$  leads to  $h_{\text{next}} > h_{\text{max}}$ . Therefore, we set  $h_{\text{next}} = h_{\text{max}}$  and in this particular case, the construction of the updated  
 302 meta-model has already been realized on the third step of Algorithm 4.

---

**Algorithm 3:** Definition of  $f$  function

---

**Function  $f(h)$ :**

Compute  $g_{\text{next}} = \hat{g}_1(\underline{x}_{\text{next}}, h)$

Construct  $\hat{g}_h$  by adding  $[\underline{x}_{\text{next}}, h_{\text{next}}]$  with value  $g_{\text{next}}$  to the metamodel  $\hat{g}_1$ .

Compute  $obj = \left| \frac{\hat{g}_h}{\sigma_{\hat{g}_h}} - U_{\text{target}} \right|$

**Result:**  $obj$

**return**

---



---

**Algorithm 4:** Selection of the new learning point and mesh size using mesh size parameterized kriging

---

**Input :**  $U_{\text{target}}$  : Fixed to 2 in this paper

Select  $\underline{x}_{\text{next}} = \arg \min_{\underline{x} \in P_{MC}} \frac{|\hat{g}_0(\underline{x}, h_{\text{min}})|}{\sigma_{\hat{g}_0(\underline{x}, h_{\text{min}})}}$

Call the mechanical solver to compute  $y_0 = g(\underline{x}_{\text{next}}, h_{\text{max}})$

Construct  $\hat{g}_1$  by adding  $[\underline{x}_{\text{next}}, h_{\text{max}}]$  with value  $y_0$  to the metamodel  $\hat{g}_0$ .

Obtain  $h_{\text{next}}$  by minimizing  $f$  over  $h$  with starting point  $h = h_{\text{min}}$

**if**  $h_{\text{next}} > h_{\text{max}}$  **then**

|  $h_{\text{next}} = h_{\text{max}}$

**end**

**if**  $h_{\text{next}} < h_{\text{min}}$  **then**

|  $h_{\text{next}} = h_{\text{min}}$

**end**

**Result:**  $\underline{x}_{\text{next}}$  and  $h_{\text{next}}$

---

303 *4.2.3. Initialization of the mesh size parameterized kriging metamodel*

When choosing the points  $[\underline{x}^{\text{obs}}, h^{\text{obs}}]$  to initialize the mesh size parameterized kriging metamodel, it is necessary to select points distributed on different mesh sizes for two reasons : to be able to calculate a correlation length in the direction of  $h$  and to allow the computation of  $\alpha$ , the mesh convergence rate. In order to compute  $\alpha$ , we choose to do these computations at a given point  $\underline{x}^*$  but for different  $h$ . To this end, a vector of five different mesh sizes is

defined with a factor of two between each mesh size:

$$\underline{h}_{vec} = [h_{min}, 2h_{min}, 4h_{min}, 16h_{min}] \quad (39)$$

304 First, a set of  $m$  observations of  $g$  are computed on the coarse mesh of size  $16h_{min}$ . The selection of these points  
 305 is problem dependent but follows the same idea as in 3.6. Second, the point  $\underline{x}^*$  is defined:

$$\underline{x}^* = \underset{\underline{x} \in \underline{x}^{obs}}{\operatorname{argmin}}(|g(\underline{x})|) \quad (40)$$

306 The FE solver is called for the realization  $\underline{x}^*$  on all the mesh sizes in  $h_{vec}$  except  $h_{min}$ . The factor of two between  
 307 each mesh size allows to compute the mesh convergence rate :

$$\alpha = \frac{\log\left(\frac{g(\underline{x}^*, 4h_{min}) - g(\underline{x}^*, 8h_{min})}{g(\underline{x}^*, 2h_{min}) - g(\underline{x}^*, 4h_{min})}\right)}{\log(2)} \quad (41)$$

#### 308 4.2.4. Algorithm

309 Algorithm 5 presents the methods developed throughout this section 4.2. The input of this algorithm are :  
 310  $COV_{target}$  the target value of the coefficient of variation,  $n_{MC}$  the initial size of the Monte Carlo population,  $m$  the  
 311 number of calls to the coarse mesh at initialisation (updated during enrichment),  $\underline{h}_{vec} = [h_{min}, 2h_{min}, 4h_{min}, 16h_{min}]$   
 312 the vector of mesh sizes used for initialization,  $\underline{h}_{prob}$  the vector gathering the  $n_{prob}$  mesh sizes on which the probability  
 313 of failure is extrapolated.  $h_{vec}(i)$  denotes the  $i^{th}$  value of the vector  $\underline{h}_{vec}$ .  $h_{prob}(i)$  denotes the  $i^{th}$  value of the vector  
 314  $\underline{h}_{prob}$ .

### 315 5. Numerical assessments

316 In this section, we apply the two new methods on two mechanical examples: the so-called Gamma-shaped structure  
 317 (GSS) and the cracked plate (CP). Both mechanical problem rely on the same hypotheses listed in Section 2 and  
 318 are presented in the first subsection. All numerical illustration were done on Dell laptop with Intel Core processor  
 319 (2.20GHz×12) and RAM of 8.1Go.

320 All metamodels were built using the OODace toolbox [14] with a gaussian kernel. Guaranteed state kriging uses  
 321 ordinary kriging while the mesh size parameterized kriging metamodel uses universal kriging. Default parameters of  
 322 the kriging metamodel are defined in Table 1 where  $\operatorname{var}(x_1^{obs})$  denotes the variance estimated on the observations  
 323 at initialization for the first variable  $x_1$  and  $\operatorname{var}(x_2^{obs})$  for  $x_2$ . The correlation length is chosen as different for each  
 324 direction in  $\underline{x}$  (anisotropic kriging). The initial value of that hyperparameter is updated after each construction of  
 325 the kriging metamodel: the value found at a given construction of the metamodel is used as the initial one for the  
 326 next construction. The learning criterion is fixed to  $\tilde{e}_{approx} < 1\%$ . While the number of initial points in  $P_{MC}$  is  
 327 dependent on the problem itself (50,000 for the gamma shaped structure, 5,000 for the cracked plate problem), the  
 328 target coefficient of variation on the probability is fixed to  $2 \times 10^{-2}$  for both problems. To attest about the robustness  
 329 of each approach, they were all tested on 5 different Monte Carlo populations for each problem.

---

**Algorithm 5:** Adaptive mesh size parameterized kriging with Monte Carlo Simulations (AMsK-MCS)

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**Input** : $COV_{target}$ : Target value of the coefficient of variation; $n_{MC}$  : Initial size of the Monte Carlo population ; $m$ : Number of calls to the coarse mesh at initialisation (updated during enrichment); $\underline{h}_{vec} = [h_{min}, 2h_{min}, 4h_{min}, 16h_{min}]$  : Vector of mesh sizes used for initialization; $\underline{h}_{prob}$ : The vector of size  $n_{prob}$  on which the probability of failure is extrapolated.**Initialization** :Generate  $\underline{x}^{obs}$  the points used to initialize the metamodel by any of the methods described in 3.6;Call the FE solver with the coarse mesh (mesh size:  $h_{max}$ ) for each points in  $\underline{x}^{obs}$ :  $\underline{g}^{obs} = g(\underline{x}_1^{obs}, h_{max})$ ;Compute  $\underline{x}^*$  using (40);**for**  $i=2:4$  **do**| Call the mechanical solver with the mesh size  $h_{vec}(i)$  for  $\underline{x}^*$  and append  $\underline{g}^{obs}$  with it;**end**Build a first mesh size parameterized kriging metamodel  $\hat{g}$ .Compute  $\alpha$  using (41).Compute the metamodel on the whole Monte Carlo population and for the fine mesh:  $\hat{g}(P_{MC}, h_{min})$  ;**Enrichment** :**while**  $\tilde{e}_{approx} > 1\%$  and  $COV_{\hat{P}_f} > COV_{target}$  **do**| **if**  $\tilde{e}_{approx} > 1\%$  **then**| | Use Algorithm 4 to select the new learning point  $\underline{x}_{next}$  and mesh size  $h_{next}$ .| | Obtain  $g_{next}$  by calling the mechanical solver with the mesh size  $h_{next}$  for  $\underline{x}_{next}$ .| | Append  $\underline{x}^{obs}$  with  $\underline{x}_{next}$ ,  $\underline{g}^{obs}$  with  $g_{next}$  and  $h^{obs}$  with  $h_{next}$ ;| | Compute the metamodel on the whole Monte Carlo population and for the fine mesh:  $\hat{g}(P_{MC}, h_{min})$  ;| **else**| |  $n_{MC} = \frac{1 - \hat{P}_f(h_{min})}{\hat{P}_{f,MC}(h_{min})COV_{target}^2}$  Enrich the Monte Carlo population so that: size( $P_{MC}$ ) =  $n_{MC}$ | | Compute the metamodel  $\hat{g}$  on the whole Monte Carlo population  $P_{MC}$ ;| | Estimate  $\hat{P}_{f,MC}(h_{min})$  and  $COV_{\hat{P}_{f,MC}}(h_{min})$ .| **end****end****Post-processing:****for**  $i = 1 : n_{prob}$  **do**| Compute  $\hat{g}(P_{MC}, h_{prob}(i))$ | Compute  $\hat{P}_{f,MC}(h_{prob}(i))$  and  $\tilde{e}_{approx}(h_{prob}(i))$ **end****Result:**  $\hat{P}_{f,MC}(h_{prob})$ ,  $\tilde{e}_{approx}(h_{prob})$  and  $COV_{\hat{P}_{f,MC}}(h_{prob})$ 

---

Hyperparameter	Initial	Minimal	Maximal
Correlation length $x_1$	$\text{var}(x_1^{obs})$	0	$\text{var}(x_1^{obs})$
Correlation length $x_2$	$\text{var}(x_2^{obs})$	0	$\text{var}(x_2^{obs})$
Correlation length $h$ (for mesh size kriging only)	$\text{var}(h^{obs})$	0	$\text{var}(h^{obs})$

Table 1: Default value of kriging hyperparameters

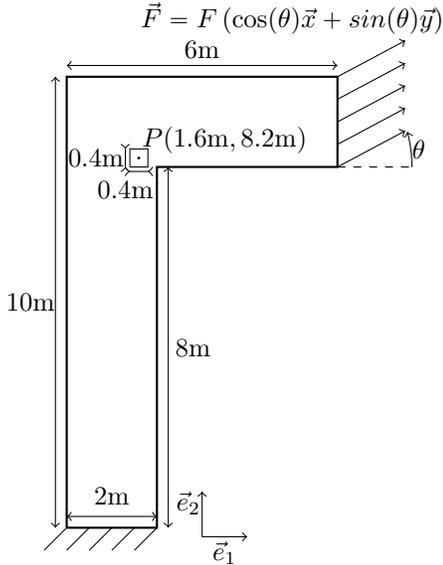


Figure 2: Gamma structure layout

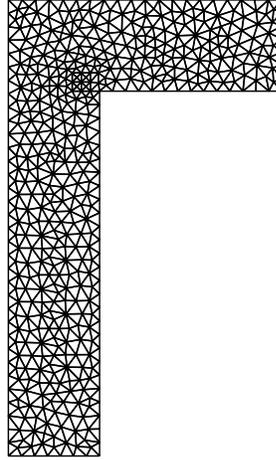


Figure 3: Mesh for mesh size fixed to 0.3m

### 330 5.1. Definition of the problems

#### 331 5.1.1. First example: gamma shaped structure

332 The gamma shaped structure layout is described in Figure 2: it is blocked on its base and subjected to a constant  
333 force  $\vec{F}$  on the right-upper part of its boundary. The Young's modulus is  $E = 210\text{GPa}$  and the Poisson's ratio is  
334  $\nu = 0.3$ . Plane strain hypothesis is made. The quantity of interest defining the limit state function of this problem  
335 is:

$$g = \sigma_{res} - \sigma_{11} \quad (42)$$

with  $\sigma_{res} = 53\text{Pa}$  that leads to a probability of failure around  $10^{-2}$  or  $\sigma_{res} = 95\text{Pa}$  that leads to a probability of failure around  $10^{-5}$  and where  $\sigma_{11}$  is the stress component in the  $\vec{e}_1$  direction computed on a square zone  $\omega$  of size  $0.4\text{m} \times 0.4\text{m}$  as depicted on 2:

$$\sigma_{11} = \frac{1}{\text{mes}(\omega)} \int_{\omega} \sigma_{11}(e_1, e_2) de_1 de_2 \quad (43)$$

336 Where  $e_1$  and  $e_2$  are the spatial coordinates of a point on the borders of the square  $\omega$ . As depicted on Figure 3, the  
337 mesh is chosen as homogenous using T6 elements.

338 The random variables of this problem are chosen as  $x_1 = F$  and  $x_2 = \theta$  defined as both the modulus and the  
339 angle of the force applied to the structure. Their distribution are described in Table 2 and their joint probability  
340 density function is plotted in Figure 6.

Random variable	Distribution Type	Low. Bound	Upp. Bound
$\theta$	Uniform	0 rad	$\frac{\pi}{4}$ rad
Random variable	Distribution Type	Mean	Variance
$F$	Log normal	10.2 Pa.m	2.2 Pa <sup>2</sup> .m <sup>2</sup>

Table 2: Distributions of random variable for the gamma shaped structure

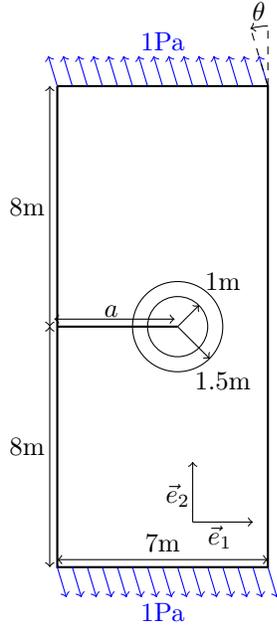


Figure 4: Cracked plate layout

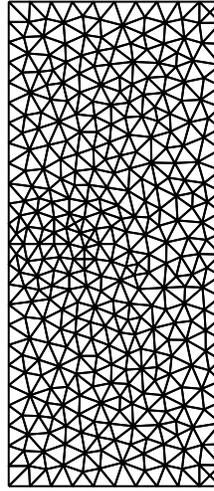


Figure 5: Mesh for mesh size fixed to 0.7m

### 341 5.1.2. Second example: crack propagation

342 The cracked plate layout is presented in Figure 4: the plate is solicited in both traction and shear. The Young's  
343 modulus is  $E = 1\text{Pa}$  and the Poisson's ratio is  $\nu = 0.3$ . Plane strain hypothesis is done.

344 The scenario of failure is crack propagation based on the Griffith criterion [25]:

$$g = K_{I,res} - K_I \quad (44)$$

345 Where  $K_{I,res} = 22\text{Pa}\sqrt{\text{m}}$  is the critical stress intensity factor of the material and  $K_I$  is the stress intensity factor  
346 for the first mode of propagation. The stress intensity factor and discretization error bounds are computed using an  
347 integral defined on the crown shown in 4 as introduced in [50, 20]. As depicted on Figure 5, the mesh is chosen as  
348 homogenous using T3 elements. The two random variables were chosen as the crack length  $x_1 = a$  and the angle of  
349 the force  $x_2 = \theta$ . The random variables are bounded by the beta distributions as described in Table 3.

350 Figure 6 shows the value of the probability density function against the value of the random variable for both the  
351 gamma shaped structure problem and the cracked plate.

### 352 5.1.3. Computation of the probability of failure using a standard monofidelity approach

353 Let the standard monofidelity approach be the AK-MCS algorithm from [16] with the learning criterion changed  
354 from  $U > 2$  to  $\tilde{e}_{approx} < 1\%$ . The standard monofidelity approach is applied to both problems to obtain reference

Random variable	Distribution type	Low. Bound	Upp. Bound	Shape par. 1	Shape par. 2
$a$	Beta	2 m	5 m	2	2
Random variable	Distribution type	Low. Bound	Upp. Bound	Shape par. 1	Shape par. 2
$\theta$	Beta	$-\frac{\pi}{2}$ rad	$\frac{\pi}{2}$ rad <sup>2</sup>	3	2

Table 3: Distributions of random variable for the cracked plate

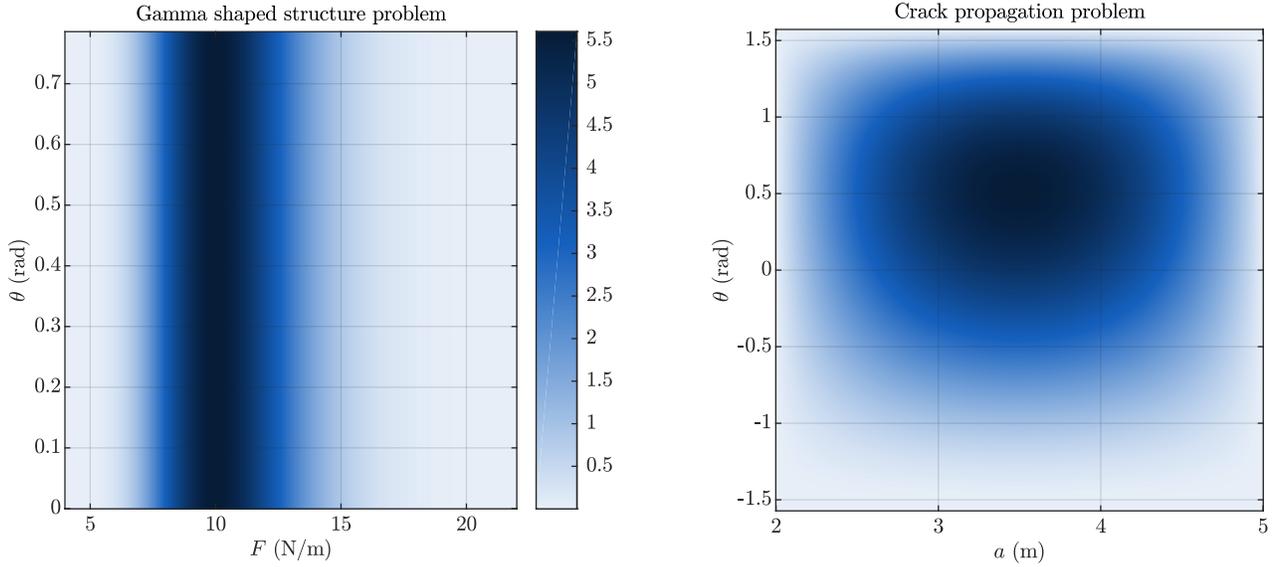


Figure 6: Colour map of the probability density function for each problem (same color bar for both color maps)

355 monofidelity results that are gathered in Tables 4, 5 and 6. Let  $t_{Algo}$  measure the time spent building the metamodel,  
356 evaluating it and computing the probability of failure. Let  $t_{FEM}$  measure the time spent building and solving the  
357 FE problem. As 5 different Monte Carlo populations were tested for each problem, each line in these tables represent  
358 a different population  $P_{MC}$ . The limit-state was plotted on Figures 7, 8 and 9 for each first line of the Tables 4, 5  
359 and 6. In this figure, the zone for which the sign of the meta-model is not the same sign as the exact limit state  
360 function was plotted. As the exact limit state function is unknown, an overkill metamodel was built with a much  
361 more demanding learning criterion :  $\tilde{e}_{approx} < 0.01\%$  (to be compared to  $\tilde{e}_{approx} < 1\%$  in the rest of the paper).

$\hat{P}_{f,MC}(h)$ ( $\times 10^{-2}$ )	Nb. calls $h = 0.01m$	$t_{FEM}$ (s)	$t_{Algo}$ (s)
3.54	31	1653	<1
3.83	35	1795	<1
4.04	38	1911	<1
3.72	40	2022	<1
3.75	34	1851	<1

Table 4: GSS problem with  $\sigma_{res} = 53Pa$  - Results with standard monofidelity approach

362 We can observe that the probability of failure is about  $\hat{P}_{f,MC} = 4.10^{-2}$  for the gamma shaped structure with

$\hat{P}_{f,MC}(h)$ ( $\times 10^{-5}$ )	Nb. calls $h = 0.01m$	$t_{FEM}$ (s)	$t_{Algo}$ (s)
1.69	27	1460	55.4
1.75	23	1196	50.9
1.86	25	1338	57.7
1.73	25	1329	53.7
1.77	29	1627	71.2

Table 5: GSS problem with  $\sigma_{res} = 95Pa$  - Results with standard monofidelity approach

$\hat{P}_{f,MC}(h)$ ( $\times 10^{-3}$ )	Nb. calls $h = 0.03m$	$t_{FEM}$ (s)	$t_{Algo}$ (s)
5.75	19	382	1
5.49	19	375	1
5.60	19	370	1
5.38	20	439	1
5.61	20	388	1

Table 6: Crack problem - Results with standard monofidelity approach

363  $\sigma_{res} = 53Pa$ ,  $\hat{P}_{f,MC} = 1.7 \cdot 10^{-5}$  for the gamma shaped structure with  $\sigma_{res} = 95Pa$  and  $\hat{P}_{f,MC} = 5.10^{-3}$  for the  
364 cracked plate. Looking at Figures 7 and 8, the limit state for the gamma shaped structure seems to present a large  
365 zone with wrong sign. Note that the objective of the standard monofidelity approach is not to compute the best  
366 meta-model of the limit state function but to obtain an estimation of the probability of failure. The learning function  
367 does not trigger the computation of observations in zones where the probability density function is small, which is  
368 the case for the zone with wrong sign.

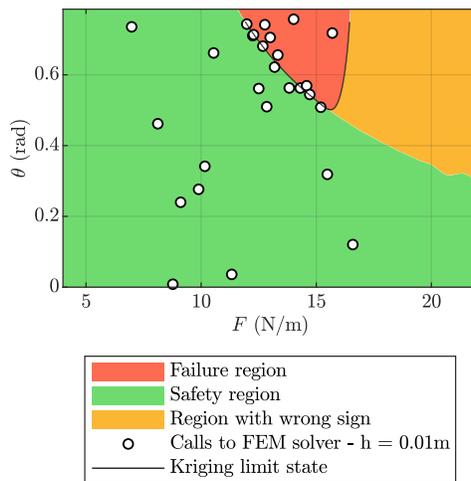


Figure 7: GSS problem with  $\sigma_{res} = 53Pa$  - Limit state using a standard monofidelity approach - results correspond to the first Monte Carlo population

369 Calls to the FE code represent the main part of the total computational cost: it is found around 1840s for the

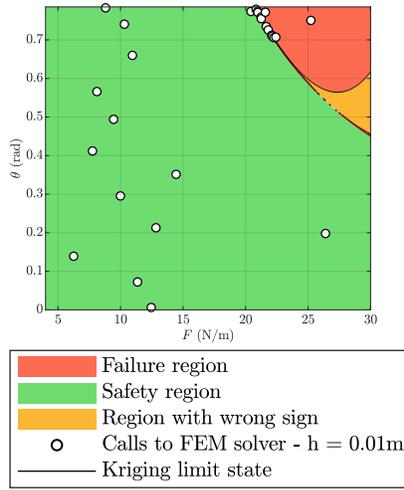


Figure 8: GSS problem with  $\sigma_{res} = 95\text{Pa}$  - Limit state using a standard monofidelity approach - results correspond to the first Monte Carlo population

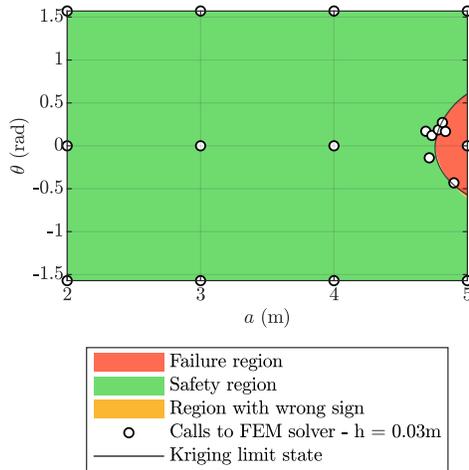


Figure 9: Crack problem - Limit state using a standard monofidelity approach - results correspond to the first Monte Carlo population

370 gamma shaped structure whereas 390s for the cracked plate. Note that for the GSS problem, the number of FE  
 371 calls is slightly smaller for  $\sigma_{res} = 95\text{Pa}$ . This could be explained by the fact that the limit state is located in a  
 372 zone in which the Monte Carlo population is less dense so that the learning process is less triggered by the learning  
 373 criterion. For the GSS problem with  $\sigma_{res} = 95\text{Pa}$ , the algorithm cost is much higher du to the very large Monte Carlo  
 374 population required for the computation of a statistically converged probability of failure as the required memory to  
 375 store the Monte Carlo population is of the same order of magnitude as the RAM memory. We observe that this cost  
 376 is almost fully due to the calls to the FEM solver (around 35 calls for the gamma shaped structure whereas 20 calls  
 377 for the cracked plate). Therefore, the use of multifidelity observations is an opportunity to reduce computational  
 378 cost.

### 379 5.2. Application of the AGSK-MCS algorithm

380 In this subsection, the method AGSK-MCS presented in Section 4.1 is used to compute the probability of failure  
 381  $\hat{P}_{f,MC}$ . The algorithm 2 is used with the coarse and fine mesh chosen as  $[0.16\text{m}, 0.04\text{m}]$  for the gamma shaped

382 structure and as  $[0.5\text{m}, 0.1\text{m}]$  for the cracked plate. The results are presented in Tables 7, 8 and 9. Let  $t_{error}$   
 383 measure the computational time required to compute the discretization error bounds on the limit state function.

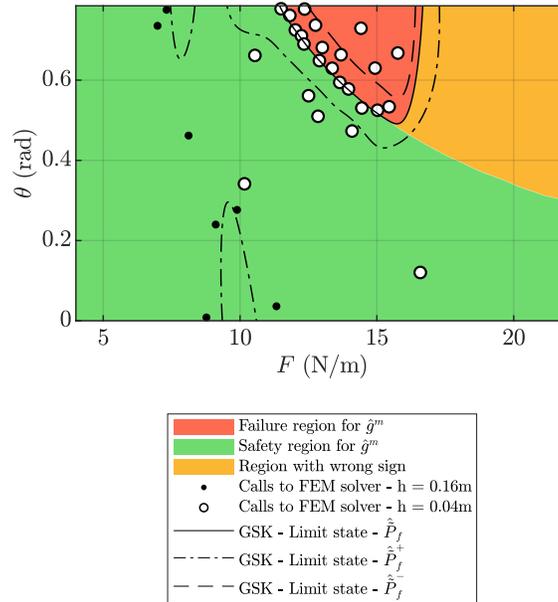


Figure 10: GSS problem with  $\sigma_{res} = 53\text{Pa}$  - Limit state using AGSK-MCS algorithm - results correspond to the first Monte Carlo population

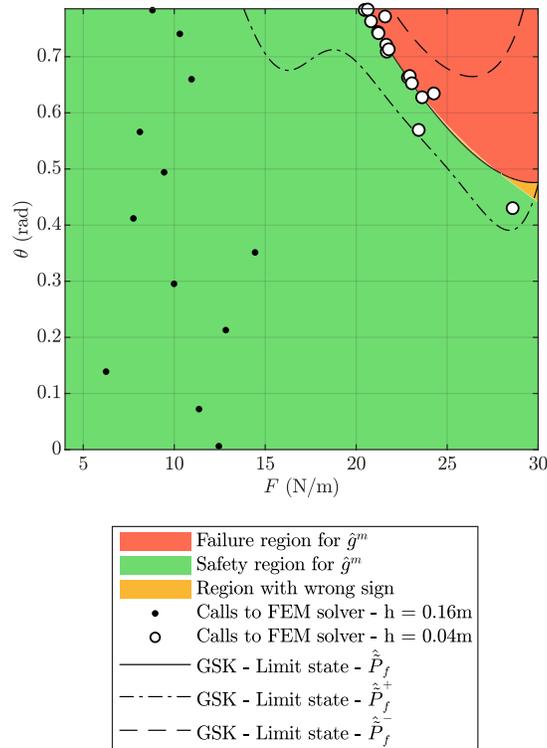


Figure 11: GSS problem with  $\sigma_{res} = 95\text{Pa}$  - Limit state using AGSK-MCS algorithm - results correspond to the first Monte Carlo population

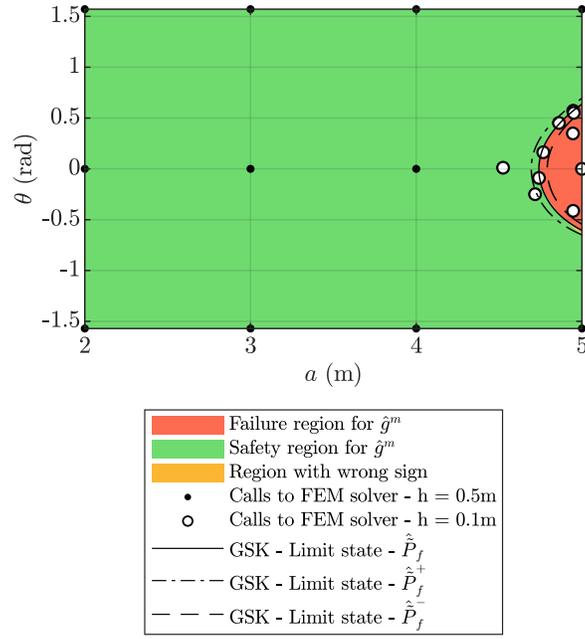


Figure 12: Crack problem - Limit state using AGSK-MCS algorithm - results correspond to the first Monte Carlo population

			Nb. calls								
$\hat{P}_{f,MC}^-$	$\hat{P}_{f,MC}$	$\hat{P}_{f,MC}^+$	Mesh size (m)		$t_{FEM}$	$t_{error}$	$t_{Algo}$				
$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	0.16	0.04	(s)	(s)	(s)	$2 \cdot 10^{-2}$	$4 \cdot 10^{-2}$	$10^{-1}$	$2 \cdot 10^{-1}$
1.63	4.06	17.09	32	25	111	4329	0.1				
1.85	4.08	14.62	33	27	124	4509	0.1				
2.05	4.30	7.71	36	30	132	4832	0.1				
1.88	4.20	7.47	37	32	141	5105	0.1				
1.90	4.30	8.46	34	29	123	4560	0.1				

Table 7: GSS problem with  $\sigma_{res} = 53\text{Pa}$  - Results with AGSK-MCS, two levels of fidelity

384 For each problem, the converged limit state on the first Monte Carlo population is plotted in Figures 10, 11  
385 and 12. For the gamma shaped structure, with  $\sigma_{res} = 53\text{Pa}$ , we observe that the upper bound is very large only  
386 for the first two Monte Carlo populations  $P_{MC}$ . Looking at Figure 10, it seems that the limit state giving  $\hat{P}_{f,MC}^+$   
387 is disrupted around  $[F = 7 \text{ N/m}, \theta = 0.7 \text{ rad}]$  and  $[F = 10 \text{ N/m}, \theta = 0.3 \text{ rad}]$ . These failed points being situated in  
388 densely populated zones of  $P_{MC}$  according to Figure 6, they increase drastically the value of  $\hat{P}_{f,MC}^+$ . However, the  
389 estimation of the approximation error is very high, if the value of the upper bound on  $\hat{P}_{f,MC}$  was important for  
390 reliability analysis, it would be necessary to continue the enrichment strategy until convergence on the limit-state of  
391  $\hat{P}_{f,MC}^+$  as a post-process of the one on  $\hat{P}_{f,MC}$ . For the case where  $\sigma_{res} = 95\text{Pa}$ , we observe the same behaviour for  
392 all Monte Carlo populations except the second one. We can observe that the limit state giving  $\hat{P}_{f,MC}^+$  is quite far  
393 from the reference limit state at the vicinity of  $[F = 17 \text{ N/m}, \theta = 0.7 \text{ rad}]$ . The limit-states for the cracked plate  
394 problem seem to behave better than for the gamma shaped structure. It results in lower, middle and upper bounds

			Nb. calls					
$\hat{P}_{f,MC}^-$	$\hat{P}_{f,MC}$	$\hat{P}_{f,MC}^+$	Mesh size (m)		$t_{FEM}$	$t_{error}$	$t_{Algo}$	
$\times 10^{-5}$	$\times 10^{-5}$	$\times 10^{-5}$	0.5	0.1	(s)	(s)	(s)	
0.43	2.25	492.26	28	16	149	4429	41.9	
0.49	2.26	8.37	25	14	431	13004	42.1	
0.32	2.36	975.8	23	19	163	4772	54.8	
0.43	2.21	371.73	25	13	97	3180	43.9	
0.36	2.29	818.51	27	15	147	3960	48	

Table 8: GSS problem with  $\sigma_{res} = 95\text{Pa}$  - Results with AGSK-MCS, two levels of fidelity

			Nb. calls					
$\hat{P}_{f,MC}^-$	$\hat{P}_{f,MC}$	$\hat{P}_{f,MC}^+$	Mesh size (m)		$t_{FEM}$	$t_{error}$	$t_{Algo}$	
$\times 10^{-3}$	$\times 10^{-3}$	$\times 10^{-3}$	0.5	0.1	(s)	(s)	(s)	
4.17	6.99	10.47	21	10	40	1034	0.7	
4.14	6.90	10.31	19	8	33	786	0.7	
4.22	7.04	10.50	20	9	34	898	0.7	
3.96	6.70	10.46	20	9	56	1625	0.7	
3.93	6.74	10.29	19	8	33	897	0.7	

Table 9: Crack problem - Results with AGSK-MCS, two levels of fidelity

395 on  $\hat{P}_{f,MC}$  being very close between each population in Table 9.

396 Let us now look at the spatial distribution of the learning points and the value of the mesh size. For the gamma  
397 shaped structure, with  $\sigma_{res} = 53\text{Pa}$ , the transition between calls done on the coarse mesh and calls done on the fine  
398 mesh is far from the limit state for  $\hat{P}_{f,MC}$ . It suggests that a third level of fidelity may be added between the current  
399 coarse and fine meshes to reduce computational cost. On the contrary, for  $\sigma_{res} = 95\text{Pa}$ , the two levels of fidelity  
400 are satisfactory as the finer mesh is only used close to the limit state. The reader must not be surprised that some  
401 observations are guaranteed for the case  $\sigma_{res} = 95\text{Pa}$  and not for the case  $\sigma_{res} = 53\text{Pa}$ . Indeed, the resistance is  
402 different and the discretization error interval  $[g^-; g^+]$  has not a constant measure on the domain of random variables.  
403 For the cracked plate, this transition is close to the limit state, which is more satisfactory: the computational effort  
404 is made where it is relevant.

405 Compared to the monofidelity approach, the computational cost is very high around 4500s for the gamma shaped  
406 structure and 900s for the cracked plate due to the expensive computation of error bounds. However, using the middle  
407 of bounds  $g^m$  instead of the the FEM solution  $g_h$  provides further precision on  $\hat{P}_{f,MC}$  compared to a monofidelity

408 approach with an even finer mesh. In fact, the value of  $\hat{P}_{f,MC}$  for the cracked plate using AGSK-MCS seems to be  
 409 close to the one with AMSK-MCS for mesh size extrapolated to zero in Figure 15. This emphasizes that the middle  
 410 of bounds is a more precise output than the FEM solution. Finally, one will see that error bounds on the probability  
 411 of failure allow to know the order of magnitude of  $P_f$ . If thinner error bounds are desired for reliability analysis, one  
 412 could reduce the size of the finer mesh or increase the polynomial degree of the basis function in the FEM problem.

### 413 5.3. Application of the AMSK-MCS algorithm

414 To start with, the state-of-the-art multifidelity approach from [59] exploiting multifidelity cokriging is computed  
 415 to provide reference results. As prescribed in [59], the initialization uses  $25 > 10 \times n$  calls to the coarse mesh (selected  
 416 by LHS for the gamma shaped structure and by factorial experiment  $5 \times 5$  for the cracked plate) and 6 calls to the  
 417 fine mesh selected by the exchange algorithm from [17]. Results are shown in Tables 10 and 11.

$\hat{P}_{f,MC}$ ( $\times 10^{-2}$ )	Nb. calls - h :		$t_{FEM}$ (s)	$t_{Algo}$ (s)
	0.16m	0.01m		
4.03	25	6	329	<1
3.87	25	6	333	<1
3.98	25	6	343	<1
3.70	25	6	340	<1
3.93	26	7	438	<1

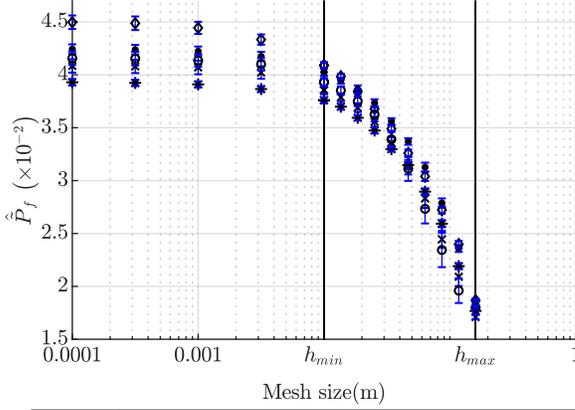
$\hat{P}_{f,MC}$ ( $\times 10^{-3}$ )	Nb. calls - h :		$t_{FEM}$ (s)	$t_{Algo}$ (s)
	0.64m	0.03m		
5.67	27	8	181	2
5.66	29	10	213	2
5.63	27	8	174	2
5.37	28	9	195	2
5.82	27	8	177	2

Table 10: GSS problem with  $\sigma_{res} = 53\text{Pa}$  - multifidelity cokriging      Table 11: Crack propagation problem - multifidelity cokriging

418 This strategy shows a great reduction in computational time: around 80% for the gamma shaped structure and  
 419 50% for the cracked plate.

420 Then, the mesh size parameterized kriging strategy is used for both structures. First, the fine mesh size  $h_{min}$  is  
 421 chosen as the mesh size used with the monofidelity approach in 5.1.3. Three of them are plotted on Figures 16, 17  
 422 and 18 together with the distribution of calls to several mesh sizes. The probability of failure can also be computed  
 423 for any mesh size.

424 First, for all problems, the algorithm converges to a probability of failure very close to the one obtained with  
 425 the monofidelity approach. Furthermore, the wrongly classified zone is drastically shrunk using the mesh size  
 426 parametrized kriging strategy compared to monofidelity approach. Let  $t_{post}$  represent computational time due to  
 427 the computation of probability for each of 15 mesh sizes plotted on Figures 13, 14 and 15. As depicted on those  
 428 three Figures, the extrapolation of the probability of failure for  $h = 0\text{m}$  provides similar results independently of the  
 429 Monte Carlo population  $P_{MC}$ . Regarding the computational cost, compared to the monofidelity and the multifidelity  
 430 cokriging approaches,  $t_{post}$  concerns an additional output of the reliability analysis. Therefore, it should not be part  
 431 of the comparison of CPU time with monofidelity and multifidelity cokriging approaches. Algorithm AMSK-MCS  
 432 succeeds in reducing computational time by around 75% for the GSS with  $\sigma_{res} = 53\text{Pa}$ , 50% for the GSS with  
 433  $\sigma_{res} = 95\text{Pa}$  and 70% for the CP compared to the monofidelity approach. This reduction is approximately equivalent  
 434 to the CPU time reduction obtained with multifidelity cokriging for the GSS with  $\sigma_{res} = 53\text{Pa}$  but 20% higher for  
 435 the CP.



$\hat{P}_{f,MC}(h) (\times 10^{-2})$	Nb	$t_{FEM}$	$t_{Algo}$	$t_{post}$
$0m$	$h_{min}$	calls	(s)	(s)
4.25	4.03	59	210	90
4.09	3.86	67	585	81
4.50	4.09	65	276	92
3.93	3.76	59	218	72
4.16	3.93	63	592	66

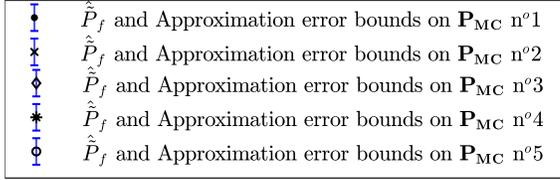
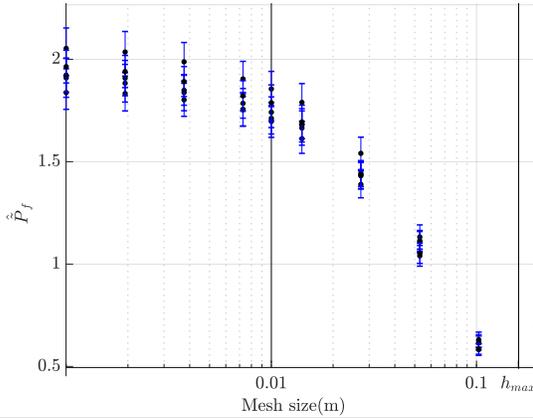


Figure 13: GSS problem with  $\sigma_{res} = 53\text{Pa}$  -results using AMSK-MCS with  $h = [0.16; 0.08; 0.04; 0.02; 0.01]$



$\hat{P}_{f,MC}(h) (\times 10^{-5})$	Nb	$t_{FEM}$	$t_{Algo}$	$t_{post}$
$0m$	$h_{min}$	calls	(s)	(s)
1.93	1.70	55	643	93
1.85	1.71	49	639	84
2.08	1.86	45	547	75
1.95	1.74	45	519	77
2.00	1.79	67	1016	114

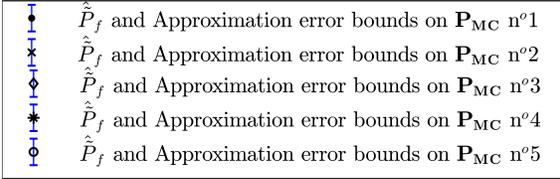
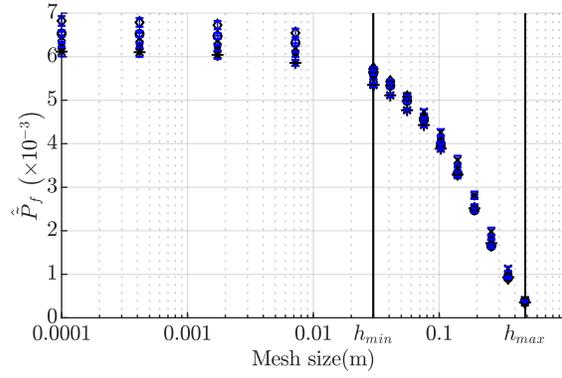


Figure 14: GSS problem with  $\sigma_{res} = 95\text{Pa}$  -results using AMSK-MCS with  $h = [0.16; 0.08; 0.04; 0.02; 0.01]$

436 To highlight the extrapolation capacity of the MSK metamodel, the fine mesh  $h_{min}$  was increased by a factor of  
437 two and the AMSK-MCS method was applied again on the GSS problem with  $\sigma_{res} = 53\text{Pa}$  and on the CP problem.  
438 Results are shown in Figure 19. For the GSS, the probability of failure at  $h = 0m$  is very similar to the ones in  
439 Figure 13 except for the third Monte Carlo population  $P_{MC}$ . For this particular simulation, the large error bounds  
440 are sufficient to judge that the probability of failure is miscalculated. For the CP, the probability of failure at  $h = 0m$   
441 is less clustered than in Figure 15. In Figure 19, the approximation error bounds are larger for  $h = 0m$  for the GSS  
442 than in Figure 13.

443 Finally, one can realize that choosing  $h_{max}$  to compute the probability of failure is expected to underestimate



$\hat{P}_{f,MC}(h) (\times 10^{-3})$	Nb	$t_{FEM}$	$t_{Algo}$	$t_{post}$	
$0m$	$h_{min}$	calls	(s)	(s)	(s)
6.27	5.52	39	76	23	74
6.45	5.65	41	99	30	81
6.83	5.72	41	67	26	75
6.12	5.35	37	71	28	73
6.54	5.64	41	91	34	78

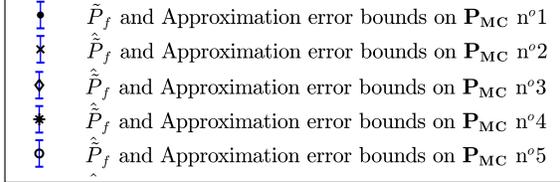


Figure 15: Crack problem - results using AMSK-MCS with  $h = [0.48; 0.24; 0.12; 0.06; 0.03]$

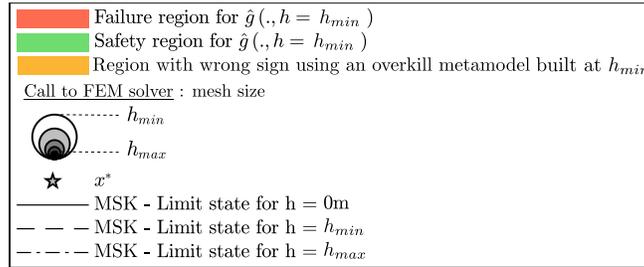
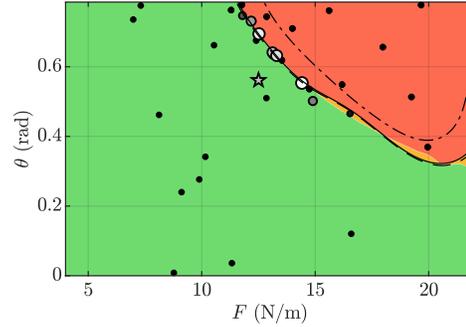


Figure 16: GSS problem with  $\sigma_{res} = 53\text{Pa}$  - Limit state for different mesh sizes using AMSK-MCS

444 the probability of failure by a factor of 2.3 for the GSS and 17.0 for the CP according to Figure 15. This underpins  
 445 the main advantage of the AMSK-MCS method compared to a monofidelity approach for which the mesh size may  
 446 be badly chosen. AMSK-MCS allows to check *a posteriori* if the meshes are well chosen. If the extrapolation is  
 447 found uncertain, then the user should select a suitably converged mesh in terms of probability of failure. Then, the  
 448 AMSK-MCS method could be computed a second time using already computed points to initialize the metamodel.  
 449 It is a useful guide to treat new problems for which the sensitivity to mesh size is unknown.

## 450 Conclusion

451 In this paper, we proposed two methods to build multifidelity kriging-based meta-models to compute the proba-

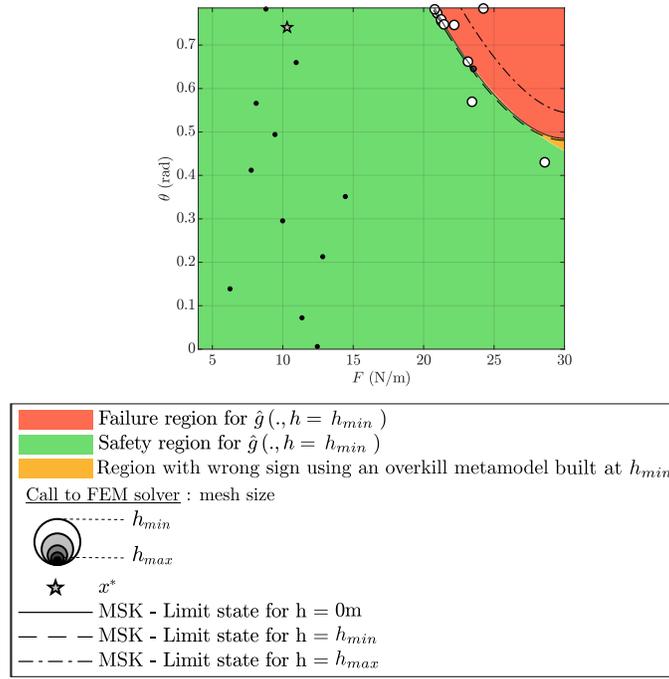


Figure 17: GSS problem with  $\sigma_{res} = 95Pa$  -Limit state for different mesh sizes using AMSK-MCS

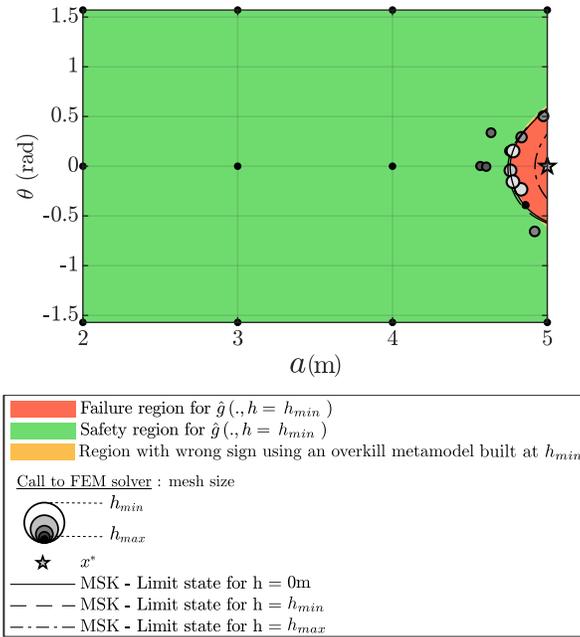
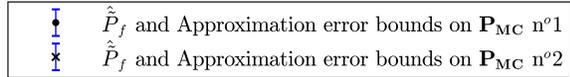
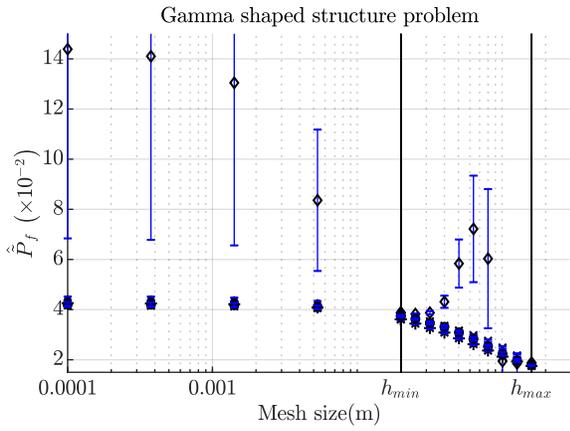


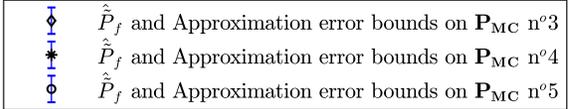
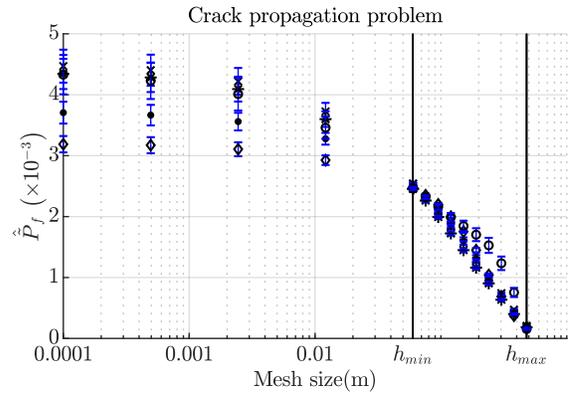
Figure 18: CP problem - Limit state for different mesh sizes using AMSK-MCS

452 bility of failure while controlling the discretization error introduced by the use of a finite element code. The method  
 453 AGSK-MCS uses *a posteriori* discretization error bounds that are available as a post-process of the finite element  
 454 simulations to guarantee the state of points used to build the kriging metamodel. The points far from the limit state  
 455 are computed on a coarse mesh while reserving the fine mesh for evaluations of points close to the limit state. Two  
 456 additional kriging metamodels interpolating upper and lower discretization error bounds can be built. It allows to  
 457 compute discretization error bounds on the probability of failure which helps validate the choice of the finer mesh  
 458 size and guide enrichment of the kriging metamodel to improve the estimation of the probability of failure. However,



$\hat{P}_{f,MC}(h) (\times 10^{-2})$			Nb	$t_{FEM}$	$t_{Algo}$	$t_{post}$
0m	0.01m	$h_{min}$	calls	(s)	(s)	(s)
4.43	4.15	3.82	62	167	69	16
4.17	3.97	3.78	62	154	71	17
14.47	5.49	3.88	96	359	143	31
4.26	3.93	3.61	64	153	71	18
4.20	3.99	3.75	60	152	66	15

GSS problem with  $\sigma_{res} = 53\text{Pa}$  -  $h = [0.16; 0.08; 0.04; 0.02]$



$\hat{P}_{f,MC}(h) (\times 10^{-3})$			Nb	$t_{FEM}$	$t_{Algo}$	$t_{post}$
0m	0.03m	$h_{min}$	calls	(s)	(s)	(s)
7.44	6.65	4.98	44	57	40	99
9.00	7.60	5.08	42	65	37	94
6.40	5.92	4.93	48	61	44	112
8.74	7.36	4.91	38	44	26	83
8.76	7.09	4.93	100	153	166	252

Crack problem -  $h = [0.48; 0.24; 0.12; 0.06; 0.03]$

Figure 19: Summary of results using AMSK-MCS and a fine mesh that is twice coarser as previously

459 a posteriori error estimators are not always available, especially in commercial codes. Therefore, a second strategy  
 460 called AMSK-MCS is proposed in this paper to tackle this issue. By including the mesh size as a parameter of the  
 461 kriging metamodel, it is possible to compute the limit state at any mesh size. The enrichment of the meta-model  
 462 is done at points close to the limit state and for the optimal mesh size required to fulfill the learning criterion.  
 463 This strategy allows to drastically reduce computational time compared to a monofidelity approach. In addition, it  
 464 enables to compute the probability of failure at any mesh size. This property allows to check *a posteriori* the mesh  
 465 convergence of the probability of failure and either validate mesh choice or guide remeshing.

466  
 467 For mechanical problems for which it would not be possible to parametrize the mesh with a unique mesh size,  
 468 only the AGSK-MCS method can still be used. Indeed, mesh splitting could be a way to answer this issue as long as  
 469 it is possible to defined a coarsest mesh and a finest mesh on which we consider running finite element simulations.  
 470 For instance, the finest mesh could be the result of the coarse mesh that has been refined by splitting it three times.  
 471 This operation can be done easily by mesh generators from an initial connectivity. Note that unrefinement is more  
 472 complicated. AMSK-MCS is not applicable if the mesh cannot be described by a unique scalar mesh size.

473

474 The two algorithms are derived regardless the number of random variables. Therefore their application on  
475 problems with more random variables is straightforward. However, it is well-known that computing the kriging  
476 meta-model is challenging when the number of random variables is larger than 12. Therefore, the methods we pro-  
477 pose inherit this drawback and are therefore limited to a small number of random variables.

478

479 Complex limit states usually arise from time-dependant reliability problems or system reliability which are not  
480 in the scope of our paper. A posteriori error estimators do exist for non-linear problems (see [32] for more details).  
481 The construction of guaranteed error bounds on  $g(\underline{x}_i)$  for a realization  $\underline{x}_i$  of the random variables is more expensive  
482 due to the time-dependency but still possible. Note that for time-reliability methods using estimation of stationary  
483 probability of failure, the methods we proposed could be easily adapted. However, the use of *a priori* estimator (that  
484 is to say convergence rate) for time-dependant problem seems not possible as the multiplicative constant depends on  
485 the time discretization. Finally, we highlight that the time discretization introduces an additional source of error.

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