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ARTICLE TEMPLATE

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ARTICLE HISTORY

ABSTRACT
The aim of this article is to provide several ways to assess the observability and the reconstructibility of bounded cellular automata. First, generic definitions for the considered cellular automata and their associated sensors are given, as well as the definitions for observability and reconstructibility of these measured dynamical systems. Criteria of observability and reconstructibility are proposed for linear and affine cellular automata through an extension of the Kalman criterion and for non-linear cellular automata from an extension of the results on observability and reconstructibility of boolean networks. A method for decentralized observability calculation is presented for non-linear cellular automata in order to reduce the algorithmic complexity. These criteria are analyzed and illustrated through several examples, either with fixed or mobile sensors. Numerical methods and related complexity issues are discussed.

KEYWORDS
cellular automata; observability; reconstructibility; mobile sensors; decentralized observability

1. Introduction

An increasing number of fundamental problems in natural and engineering applications involves complex systems consisting of a population of similar units that interact directly with their nearest neighbours. Although the microscopic rules are well defined, these systems exhibit emergent macroscopic behaviours that may be difficult to predict from the knowledge of their individual elements and their interactions (D'Andrea & Dullerud, 2003). An important aspect of many of these systems is that measurement and control is often available at the unit level. These units can then be considered as agents with limited sensing, data processing and communication capabilities (Bamieh, Paganini, & Dahleh, 2002). Recent technological advances in the fields of microelectronics and network communication have greatly facilitated the practical implementations of this kind of spatially distributed systems (Chu, 1974).

In the analysis, design, and synthesis of systems, a fundamental issue is to determine

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their *controllability* and *observability*: Is it feasible to drive the internal variables of the system from an initial state to a desired target state? And is it possible to estimate the values of these internal variables from external measurements?

For spatially distributed systems the classical theoretical concepts of controllability and observability are, in general, difficult to apply (Curtain & Zwart, 2012; D'Andrea, 1998; D'Andrea, Beck, & Dullerud, 1999). In addition, the complexity of the systems under consideration, which involve a large number of units or sub-systems, is such that it is very difficult to verify the controllability and observability criteria. The main difficulty coming from the prohibitive computational cost of existing methods even though efforts to reduce this costs are underway (Kang & Xu, 2009).

Another issue is related to the system's spatial heterogeneity (i.e. state variables and parameters are varying with the spatial coordinates) and partial state measurement, through fixed or mobile regional sensors. In this case, only part of the state may be measured, only on a part of the domain (zone sensors) and possibly only part-time (mobile sensors). This prevents a complete characterisation of the system behaviour, directly from measurements (Gauthier & Kupka, 1994; Liu, Slotine, & Barabási, 2013; Zuazua, 2007). The measurements are then processed (usually) through state estimators (dynamical observers, smart/software sensors) which usually make use of system properties (e.g. through a dynamical model) to reconstruct the state.

In this paper, we will consider dynamical systems described by cellular automata (CA) models that have been developed to describe life-like and physical phenomena. CA exhibit complex behaviours and provide powerful models usually viewed as a counterpart of partial differential equations (PDEs) for modelling distributed parameters systems (Bagnoli, Rechtman, & El Yacoubi, 2012; El Yacoubi, 2008; El Yacoubi & Mingarelli, 2011). CA consists of a regular grid in which each cell is in a finite state at each time step. The state of each cell evolves over time based on a set of rules that take into account the state of the considered cell and its neighbouring cells. These local interactions between neighbouring cells yield global results, i.e. state change, over the entire grid that can be observed over time (Chaudhuri, Chowdhury, Nandi, & Chattopadhyay, 1997; Chopard & Droz, 1998; El Yacoubi & Jacewicz, 2007; Garzon, 2012). In the scope of our study, we will limit ourselves to the study of the observability and reconstructibility of bounded cellular automata.

The aim of this work is to introduce and characterize the observability and reconstructibility properties for bounded cellular automata with output operators, describing spatially distributed systems with zone sensors, either fixed or mobile. Roughly speaking, the question adressed by observability (resp. reconstructibility) is the possibility to estimate the system's initial state (resp. the current state) value from a sequence of measurements. Observability is a fundamental property of dynamical systems with an extensive literature. Some interesting work investigating distributed parameters systems that are usually described by (PDEs), in the linear, nonlinear, or stochastic cases and other related work can be found in (Cohn & Dee, 1988; El Jai, 1991; Liu et al., 2013; Sarachik & Kreindler, 1965; Zhou, 2015).

The observability and reconstructibility problem of cellular automata is only very little studied (El Yacoubi, 2008; El Yacoubi et al., 2021). However, bounded cellular automata can be seen as a special case of boolean networks which, on the other hand, are widely studied. Several approaches exist to study the observability of Boolean networks, including algebraic (Cheng, Qi, & Li, 2010; Yu, Meng, & Feng, 2020) and graph-theoretic (Laschov, Margaliot, & Even, 2013) approaches. The problem of observability and reconstructibility is also widely studied for Boolean control networks (Fornasini & Valcher, 2012) as well as for probabilistic Boolean networks (Fornasini & Valcher,

2019).

This paper focuses first on the linear and affine CA where an extension of the well known Kalman criterion is given to prove theoretically their observability and reconstructibility properties. A first study was carried out (Plénet, El Yacoubi, Raïevsky, & Lefèvre, 2020) where the observability for cellular automata was defined and characterised using an extension of the Kalman criterion for affine cellular automata. We go further in this paper by considering more general case and exploring other method to prove observability compared with reconstructibility issue. The second part focuses on non-linear cellular automata by applying the results of observability of Boolean networks (Cheng et al., 2010) to cellular automata. The criteria are then adjusted according to the constraints of bounded cellular automata in order to reduce the computational complexity of observability and reconstructibility. Finally, a decomposition of the observability and reconstructibility problem is presented in order to allow the study of large cellular automata despite the computational complexity.

This paper is structured as follows: In Section 2, preliminary on CA, sensors, and observability are discussed. In Section 3, some theoretical results using an extension of the Kalman condition are obtained in order to prove observability and reconstructibility properties for affine and linear CA. In Section 4, nonlinear CA are considered where the representation of the transition rule is given through a binary relation in order to focus on the global relation between CA configurations instead of working with local interactions. A numerical method to compute logical matrices representing the binary relations is given. Two examples are given in Section 5 in order to show the applicability of the observability and reconstructibility criteria in the case of traffic flow monitoring and the observation of a forest fire spread. Finally, Section 6 concludes the paper and gives some perspectives for future work.

2. Definition of Observability and Reconstructibility for Cellular Automata

2.1. Cellular Automata Model

Cellular automata (CA) are useful to model discrete-time distributed parameter systems (DPS). There is no universal formal description of CA and we therefore present here a general definition of a bounded cellular automata and sensors within the scope of observation and reconstruction. For the purpose of this article, we will simply refer to bounded cellular automata as cellular automata or CA. Such a definition have been given in (El Yacoubi, 2008), but was designed to investigate controllability (and more particularly regional controllability) issues. We will extend it with sensor models and output operators in order to cope with observability and reconstructibility analysis.

Definition 2.1. A bounded cellular automata is defined by a quadruple $A = (\mathcal{L}, \mathcal{S}, \mathcal{N}, f)$ where:

- $\mathcal{L}$ is a d-dimensional finite lattice of cells (a generic cell will be denoted c, in the sequel) which are spatially arranged according their shape. We denote by $N = |\mathcal{L}| \in \mathbb{N}$ the cardinal of $\mathcal{L}$.
- $\mathcal{S}$ denotes a discrete set of state $\mathcal{S} = \{0, 1, \ldots, k-1\}$. In the section 3, it will be necessary that $\mathcal{S}$ is a field, i.e. that $\mathcal{S}$ is a finite commutative ring with a prime number of states. This set may also be called $\mathbb{Z}/k\mathbb{Z}$.
- $\mathcal{N}$ is a mapping which defines the cell's neighbourhood. This neighbourhood is

usually the same for all cells but it can vary through space and time. It is given by:

$$\begin{aligned} \mathcal{N} &: \mathcal{L} \to \mathcal{L}^n \\ & c \mapsto \mathcal{N}(c) = \{c' \in \mathcal{L} | \left\| c' - c \right\|_i \leq r\} \end{aligned} \tag{1}$$

where $\|c\|_i, i \in \{1, \infty\}$ represents the taxicab norm ($i = 1$) or the infinity norm ($i = \infty$). These norms correspond respectively to the Von Neumann and Moore neighbourhoods of radius r. We denote by $n = |\mathcal{N}(c)|$ the cardinal of $\mathcal{N}(c)$.

- f is a transition function which determine the cell's state at time $t+1$ given the state of the neighbouring cells at time t. It is defined by:

$$\begin{aligned} f &: \mathcal{S}^n \to \mathcal{S} \\ & s_t(\mathcal{N}(c)) \mapsto s_{t+1}(c) = f(s_t(\mathcal{N}(c))) \end{aligned} \tag{2}$$

where $s_t(c)$ represents the state of the cell c at time t and $s_t(\mathcal{N}(c)) = \{s_t(c') | c' \in \mathcal{N}(c)\}$ is the neighbourhood state at time t.

Remark 1. As we consider only finite lattices of cells, we need to define *boundaries*, that is to say the state of the cells that are directly "outside" the lattice, which will be used to compute states of the cells at the border of $\mathcal{L}$. We will consider the three types of boundaries which are the most commonly used: periodic, reflective, and fixed (El Yacoubi & El Jai, 2002; LuValle, 2019).

This definition describes a CA from a local point of view, but in our case, we need a global definition to apply the results from system theory. This global definition will be given in term of the global CA configuration s_t and a global transition function F which are defined as:

$$\begin{aligned} s_t &: \mathcal{L} \to \mathcal{S} \\ & c \mapsto s_t(c) \end{aligned} \tag{3}$$

$$\begin{aligned} F &: \mathcal{S}^{\mathcal{L}} \to \mathcal{S}^{\mathcal{L}} \\ & s_t \mapsto F(s_t) = s_{t+1} \end{aligned} \tag{4}$$

The global evolution of the CA starting from an initial configuration s_0 is defined by the following state equation:

$$\begin{cases} s_{t+1} = F(s_t) \\ s_0 \in \mathcal{S}^{\mathcal{L}} \end{cases} \tag{5}$$

which can also be written:

$$s_t = F^t(s_0), s_0 \in \mathcal{S}^{\mathcal{L}} \tag{6}$$

where F^t represents the t-th composition of F.

2.2. Sensor Model and Output Operator

To represent the *measure* of a physical system modelled by a CA, we extend the previous local and global CA dynamical models with *sensors* and and *output operator*.

Definition 2.2. We denote q_i, $i \in \{1, 2, \ldots, Q\}$, the sensor measuring the states of a set of cell $\mathcal{L}_{q_i}$ and q, the set of Q sensors that observes the set of cells $\mathcal{L}_q \subset \mathcal{L}$ with:

$$\mathcal{L}_q = \bigcup_{i=1}^{Q} \mathcal{L}_{q_i} \tag{7}$$

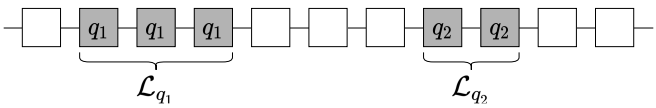


Figure 1. Two sensors measuring a one-dimensional CA. Grey cells are measured by a sensor, either q_1 or q_2.

Remark 2. The sensor set q can contain mobile sensors, for which observed cell set can change over time. In this case, we will denote $\mathcal{L}_{q_{i,t}}$ the set of cells observed by the sensor i at time t and $\mathcal{L}_{q_t}$ the set of all observed cells at time t.

Given this formalisation of sensors' position and motion, we are able to build an operator which associates the state of the system with the measurements obtained by whole set of sensors.

Definition 2.3. The *output operator* H_t maps, at each time t, the state s_t of the cellular automata $\mathcal{A}$ to the *output vector* $\theta_t \in \mathcal{O} = \mathcal{S}^{\mathcal{L}_{q_t}}$:

$$\begin{aligned} H_t \colon \mathcal{S}^{\mathcal{L}} &\to \mathcal{O} \\ s_t &\mapsto \theta_t := s_t|_{\mathcal{L}_{q_t}} \end{aligned} \tag{8}$$

where $s_t|_{\mathcal{L}_{q_t}}$ denotes the restriction of the state s_t to the set $\mathcal{L}_{q_t}$ of cells effectively measured at time t

Remark 3. The vector composed of the successive output vectors (measurements) generated by the output operator H_t is called an *output sequence*. The sequence of measurements spanning from t_0 to t_T is noted $\Theta_{0,T} = (\theta_0, \theta_1, ..., \theta_{T-1})$. An output sequence can be computed from the initial state s_0 using the operator:

$$\Theta_T \colon s_0 \mapsto \Theta_{0,T} = (H_0(s_0), H_1 \circ F(s_0), H_2 \circ F^2(s_0), ..., H_{T-1} \circ F^{T-1}(s_0)) \tag{9}$$

2.3. Observability and Reconstructiblity

Observability, as defined by Kalman (Kalman, 1960, 1963), determines if it is possible to reconstruct the state of a system based on the measurements obtained from one or more sensors. Observability focuses on reconstructing the *initial* state of the system whereas reconstructibility focuses on reconstructing the *current* state of the system. In

the case of discrete time systems, both of these are not equivalent. With a deterministic system, observability is a more general concept as knowing the initial state implies knowing all the evolution of the system. Reconstructibility, on the other hand, is less general but can be easier to assess.

Definition 2.4 (State Observability)**.** A *state* $s_0 \in \mathcal{S}^{\mathcal{L}}$ of a cellular automata $\mathcal{A}$ is **observable** by an output operator H at time T if and only if it is possible to reconstruct this initial state s_0 from the corresponding output sequence $\Theta_{0,T}$.

Definition 2.5 (Global Observability)**.** A *cellular automata* $\mathcal{A}$ is **observable** by an output operator H at time T if and only if all states $s \in \mathcal{S}^{\mathcal{L}}$ are observable by this output operator.

Proposition 2.6. *The definition 2.5 of the global observability is equivalent to the following proposition:*

$$\forall s'_0, s''_0 \in \mathcal{S}^{\mathcal{L}}, \Theta_T(s'_0) = \Theta_T(s''_0) \implies s'_0 = s''_0 \tag{10}$$

Definition 2.7 (State Reconstructiblity)**.** A state $s_T \in \mathcal{S}^{\mathcal{L}}$ of a cellular automata $\mathcal{A}$ is **reconstructible** by an output operator H at time T if and only if it is possible to reconstruct the current state s_T from the corresponding output sequence $\Theta_{0,T}$.

Definition 2.8 (Global Reconstructiblity)**.** A cellular automata $\mathcal{A}$ is **reconstructible** by an output operator H at time T if and only if all states $s \in \mathcal{S}^{\mathcal{L}}$ are **reconstructible** by this output operator.

Proposition 2.9. *The definition 2.8 of the global reconstructibility is the equivalent to the following proposition:*

$$\forall s'_0, s''_0 \in \mathcal{S}^{\mathcal{L}}, \Theta_T(s'_0) = \Theta_T(s''_0) \implies F^T(s'_0) = F^T(s''_0) \tag{11}$$

3. Observability and Reconstructibility for Affine Cellular Automata

This section presents the extension of the Kalman criterion, usually used to determine the observability of continuous time or discrete time linear systems, to linear cellular automata and more generally to *affine* CA.

3.1. Affine and Linear Automata

Before presenting the extension of the Kalman criterion to CA, we need first to express the state of a CA in the form of a state vector and to define the properties of a CA that make it linear or affine.

To express a CA configuration as a state vector, the CA must have a finite lattice and its cells must be arranged in an arbitrary order. Under this assumption, an isomorphism ξ between the CA configuration s_t and a state vector representation x_t can be defined. This isomorphism can be applied to the global transition function F to obtain another function $\tilde{F}$ which computes x_{t+1} from x_t such that:

$$\tilde{F} = \xi \circ F \circ \xi^{-1} \tag{12}$$

Definition 3.1 (Affine Cellular Automata)**.** A cellular automata is *affine* if and only if its transition function $\tilde{F}$ is an affine map. Moreover, this affine map can be written in the form of a linear map A plus a constant $\eta = \tilde{F}(0)$, which can therefore be written as:

$$\begin{cases} x_{t+1} &= \tilde{F}(x_t) = Ax_t + \eta \\ x_0 &= \xi\left(s_0\right), s_0 \in \mathcal{S}^{\mathcal{L}} \end{cases} \tag{13}$$

Remark 4. The trajectory of the CA state may be explicitely written in terms of the initial state x_0, as:

$$\begin{cases} x_{t+1} &= A^{t+1}x_0 + J_t\eta \\ x_0 &= \xi\left(s_0\right), s_0 \in \mathcal{S}^{\mathcal{L}} \end{cases} \tag{14}$$

with $J_t = \sum_{k=0}^{t} A^k$

Remark 5. An affine CA that has a null constant is said *linear*. In the theory of elementary CA (Wolfram, 1983), linear CA are called *additive* CA. Also, affine CA are the complement of additive CA. For example, rules 90 and 150 are linear rules and their complementary rules (respectively 165 and 105) are affine rules.

Remark 6. It is possible to apply the same approach to *output sequences*: if the observation space is finite (i.e. $|\mathcal{L}_q| = Q \in \mathbb{N}$) then there is an isomorphism χ which makes it possible to write the output sequence θ_t as a vector y_t. Moreover, the output operator is affine if and only if the $\tilde{H}$ function (i.e. the H function composed with the χ isomorphism) is affine. The system can then be written as $y_t = Cx_t + \gamma_t$ where C_t is the matrix of the linear map and γ_t is the affine constant.

Definition 3.2. The output sequence Θ_T can also be represented in affine form using the affine forms of the transition function and the output operator. We note Y_T the output sequence for T outputs where O_T is the linear map and Γ_T the constant vector.

$$Y_T = \begin{bmatrix} y_0 \\ y_1 \\ \dots \\ y_{T-1} \end{bmatrix} = \underbrace{\begin{bmatrix} C_0 \\ C_1A \\ \dots \\ C_{T-1}A^{T-1} \end{bmatrix}}_{O_T} x_0 + \underbrace{\begin{bmatrix} \gamma_0 \\ C_1J_0\eta + \gamma_1 \\ \dots \\ C_{T-1}J_{T-2}\eta + \gamma_{T-1} \end{bmatrix}}_{\Gamma_T} = O_Tx_0 + \Gamma_T \tag{15}$$

We call O_T the observability matrix as this linear map carries the injectivity property of the output sequence and therefore the observability property.

3.2. Extension of the Kalman Criterion

The Kalman condition (Kalman, 1960, 1963) was initially derived to assess the controllability (resp. observability) of continuous-time linear dynamical systems when they are controlled (resp. observed). It has been generalised to discrete-time systems (Sarachik & Kreindler, 1965) and this paper proposes a generalisation to cellular automata.

Theorem 3.3 (Kalman Condition)**.** *Let $\mathcal{A}$ and H be an affine cellular automata and affine output operator, with associated matrices respectively denoted A and C.*

The pair $(\mathcal{A}, H)$ (i.e. the automata $\mathcal{A}$ with the output operator H) is ***observable*** *if and only if there exists a time $T \in \mathbb{N}$ such that:*

$$rank\ O_T = \begin{bmatrix} C_0 \\ C_1 A \\ \cdots \\ C_{T-1} A^{T-1} \end{bmatrix} = N \tag{16}$$

In linear algebra, the Cayley-Hamilton theorem states that any square matrix over a commutative ring (in our case $\mathcal{S}$) satisfies its characteristic equation. It can be coupled with the Kalman theorem here above to give an upper bound on the time horizon T.

Proposition 3.4. *Suppose a CA observed by a time invariant output operator C (*i.e. *by a static sensor), the observability matrix O_T is of size $N \times Q.T$. For the matrix O_T to be of full column rank, the time horizon T has to be bounded by:*

$$\frac{N}{Q} \leq T \leq N \tag{17}$$

Proof. The two inequalities are proven by the following:

- $T \leq N$: The Cayley-Hamilton theorem guarantees that A^N is a linear combination of lower powers, the rank of O_T will not increase beyond T = N.
- $T \geq N/Q$: For O_T to have a rank of N, it needs at least N rows and columns, thus $Q.T \geq N$.

□

Remark 7. In the case of a mobile sensor (*i.e.* a time variant output operator C_t), only the lower bound of the inequation (17) stands true. The Cayley-Hamilton theorem does not apply because $C_T A^T$ is a linear combination of $C_T + C_T A + \cdots + C_T A^{T-1}$ and not of $C_0 + C_1 A + \cdots + C_{T-1} A^{T-1}$

Proof. Let $\mathcal{A}$ an affine CA and A and η its associated matrix and constant. Let H_t be a time dependant affine output operator associated with the matrix C_t and the constant γ_t. Then, let $x_0 \in \mathcal{S}^N$ the initial state and $Y_T = \begin{bmatrix} y_0 & y_1 & \dots & y_{T-1} \end{bmatrix}$ the output sequence generated by the output operator such that $Y_T = O_T x_0 + \Gamma_T$.

Considering the definition 2.5 of global observability, then:

$$\begin{aligned}
(\mathcal{A}, H_t) \text{ is observable} &\iff \forall s'_0, s''_0 \in \mathcal{S}^{\mathcal{L}}, \Theta_T(s'_0) = \Theta_T(s''_0) \implies s'_0 = s''_0 \\
&\iff \forall x'_0, x''_0 \in \mathcal{S}^N, O_T x'_0 + \Gamma_T = O_T x''_0 + \Gamma_T \implies x'_0 = x''_0 \\
&\iff \forall x'_0, x''_0 \in \mathcal{S}^N, O_T(x'_0 - x''_0) = 0 \implies (x'_0 - x''_0) = 0 \\
&\iff ker\ O_T = \{0\} \\
&\iff rank\ O_T = dim\ O_T - dim(ker\ O_T) = N
\end{aligned}$$

□

Corollary 3.5. *If the Kalman criterion is verified, then it is possible to reconstruct the initial state by inverting the observability matrix. Indeed, based on the formulation (15) we obtain:*

$$x_0 = O_T^{\dagger}(Y_T - \Gamma_T) \tag{18}$$

Proof. Consider an affine CA $\mathcal{A}$ observable by an affine output operator H_t such that $\forall x_0 \in \mathcal{S}^N, Y_T = O_T x_0 + \Gamma_T$ and $rank\ O_T = N$. To simplify the notations, we shall simply note O and Γ to respectively represent O_T and Γ_T.

As $rank\ O = N$, it means that it exists P such that $PO = I$ (but not necessarily $OP \neq I$ because O is full column rank not full row rank). We can find $P = O^{\dagger}$ by computing the pseudo-inverse of O. Using the equation (15) we find that:

$$Y_T = Ox_0 + \Gamma \iff O^{\dagger}(Y_T - \Gamma) = O^{\dagger}Ox_0 \iff O^{\dagger}(Y_T - \Gamma) = x_0 \tag{19}$$

As O is full column rank, $O^{\dagger} = (O^t O)^{-1} O^t$. If O is square then $O^{\dagger} = O^{-1}$. □

Remark 8. In definition 2.1, one of the conditions for $\mathcal{S}$ to be a field is that the state number k is prime. If it is not, the inverse of a number does not necessarily exist however the inverse is required to reconstruct the initial state with corollary 3.5. Moreover, in modular arithmetic, this inverse is not computed from the usual division but from multiplication and modulo:

$$a^{-1} = b \implies b \cdot a = 1 \pmod{k} \tag{20}$$

In the case of linear systems, observability and reconstructibility are equivalent concepts (Van Willigenburg & De Koning, 2008). The Kalman criterion ensure observability, however there are some systems which are reconstructible but not observable (See example 5.1), for this reason we propose a new theorem which assesses the reconstructibility of affine CA.

Theorem 3.6 (Reconstructibility Criterion)**.** *Let $\mathcal{A}$ and H be an affine CA and affine output operator; A, C their matrix form; and η and γ their constants.*

The pair $(\mathcal{A}, H)$ (i.e. the automata $\mathcal{A}$ with the output operator H) is reconstructible if and only if there exists $T \in \mathbb{N}$ such that:

$$ker\ O_T \subset ker\ A^T \tag{21}$$

Remark 9. If the CA is reversible (Kari, 2005) then A^T is full rank thus reconstructibility is equivalent to observability. Indeed, $ker\ O_T \subset ker\ A^T = \{0\} \iff rank\ O_T = N$.

Proof. Let $\mathcal{A}$ an affine CA and A and η its associated matrix and constant. Let H_t be a time dependant affine output operator associated to the matrix C_t and the constant γ_t. Then, let $x_0 \in \mathcal{S}^N$ be the initial state and $Y_T = \begin{bmatrix} y_0 & y_1 & \dots & y_{T-1} \end{bmatrix}$ the output sequence generated by the output operator such that $Y_T = O_T x_0 + \Gamma_T$.

Consider the definition 2.8 of the global observability, then:

$$
\begin{aligned}
&(\mathcal{A}, H_t) \text{ is reconstructible} \\
&\iff \forall s'_0, s''_0 \in \mathcal{S}^{\mathcal{L}}, \Theta_T(s'_0) = \Theta_T(s''_0) \implies F^T(s'_0) = F^T(s''_0) \\
&\iff \forall x'_0, x''_0 \in \mathcal{S}^N, O_T x'_0 + \Gamma_T = O_T x''_0 + \Gamma_T \implies A^T x'_0 + J_{T-1}\eta = A^T x''_0 + J_{T-1}\eta \\
&\iff \forall x'_0, x''_0 \in \mathcal{S}^N, O_T(x'_0 - x''_0) = 0 \implies A^T(x'_0 - x''_0) = 0 \\
&\iff ker\ O_T \subset ker\ A^T
\end{aligned}
$$

□

Corollary 3.7. *If the reconstructibility criterion is verified, then it is possible to find a matrix R such that:*

$$x_T = R(Y_T - \Gamma_T) + J_{T-1}\eta \tag{22}$$

Proof. Consider an affine CA $\mathcal{A}$ (with a matrix A and a constant η) reconstructible by an affine output operator H_t such that $Y_T = O_T x_0 + \Gamma_T$ and $ker\ O_T \subset ker\ A^T$.

As $ker\ O_T \subset ker\ A^T$, it means there exists a matrix R such that $A^T = RO_T$. With this property, we can find that:

$$
\begin{aligned}
Y_T = O_T x_0 + \Gamma_T &\iff R(Y_T - \Gamma_T) = RO_T x_0 \\
&\iff R(Y_T - \Gamma_T) + J_{T-1}\eta = A^T x_0 + J_{T-1}\eta \\
&\iff x_T = R(Y_T - \Gamma_T) + J_{T-1}\eta
\end{aligned}
$$

□

3.3. Numerical Method for Assessing Observability

When cellular automata are very large, it becomes necessary to use a computer to construct the A, C, and O matrices and to evaluate the rank of the observability matrix. This section presents some heuristics to numerically construct these matrices and to evaluate observability.

Before constructing the matrices, the operators ξ and χ must be determined according to the order in which the CA's cells are arranged. Any order would do but to simplify the construction of the matrices it is wise to take the usual order.

Using this method, the matrix A appears as a circulant (or a nearly circulant depending on the boundaries conditions) matrix with several diagonals that depends on the neighbourhood.

In most cases, the state of a cell is observed directly, i.e. each element of the output vector y is the state of one of the cells of the automata. If this is the case, then the matrix C has only one non-zero element in each row at the position corresponding to the observed cell.

The following matrix is designed to measure the 2^{nd} cell as well as the 5^{th} cell :

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Once the matrices A and C have been constructed, it is possible to calculate the

matrix O from its definition. Then, the observability of the CA can be determined by computing the rank of O.

The rank of a matrix is usually calculated with Gaussian elimination (Bunch & Hopcroft, 1974) but new and better methods exist (Cheung, Kwok, & Lau, 2013). The algorithmic complexity of Gaussian elimination of a $M \times N$ matrix is $O(NM^{\omega-1})$, with $\omega \leq 2.38$. When the Gaussian elimination is applied to the rank of the observability matrix O_T of size $N \times Q.T$, the complexity is $O(QTN^{\omega-1})$.

4. Observability and Reconstructibility of Non-Linear Cellular Automata

In the previous part, we have studied the observability and the reconstructibility of affine CA but this represents a very small part of the existing automata: only 16 rules out of the 256 Wolfram's elementary rules are affine.

To study non-linear cellular automata, we chose not to focus on the algebraic expression of the transition function, as we were able to do with the linear case. Indeed, this method does not allow us to study all types of automata independently of the properties of their transition function. The idea is to study the relation between configurations and not between cells.

This method has already been studied many times for the computation of the observability and reconstructibility of Boolean (Control) Networks (Cheng et al., 2010; Laschov et al., 2013). In this section, we adapt these formulations to bounded cellular automata and propose numerical methods to verify their observability and reconstructibility. Moreover, we present a method for computing observability in a decentralized way that aims at reducing the algorithmic complexity at the expense of the number of sensors.

4.1. Binary Relation Representation

Bounded cellular automata can be seen as a particular case of Boolean networks (Gershenson, 2002) where the topology of the network depends on the position of the cell (or variable) and the transition function does not depend on the cell (for homogeneous CA only). This great similarity between the two representations makes it possible to apply the observability and reconstructibility results of BN to CA, but their differences permit to adapt these theorems in order to reduce their algorithmic complexity.

Cheng et al. developed an algebraic state representation for BCN that uses the semi-tensor product (Cheng et al., 2010), a matrix product derived from the Kronecker product. This method allows to have a state representation for nonlinear cellular automata as we have done in the previous section.

Definition 4.1 (Semi-tensor product (Cheng, 2005))**.** Let $A \in M_{p\times q}$ and $B \in M_{m\times n}$. The semi-tensor product is define by:

$$A \ltimes B = A(B \otimes I_k), \text{ if } q = km$$
$$A \ltimes B = (A \otimes I_k)B, \text{ if } kq = m$$

where $\otimes$ is the Kronecker product.

Definition 4.2 (Cheng et al. (2010))**.** Let $\mathcal{A}$ and H be a cellular automata and output operator, with associated binary matrices respectively denoted M_F and M_H.

The evolution of the CA can be written as :

$$\begin{cases} x_{t+1} &= M_F \ltimes x_t \\ y_t &= M_H \ltimes x_t \end{cases} \tag{23}$$

with $x_t = x_t^1 \ltimes \cdots \ltimes x_t^N$

With this representation, the observability matrix M_O can be constructed in a similar way to the definition (3.2) by replacing the usual matrix product by the semi-tensor product. Once the observability matrix M_O is obtained, we can study its injectivity property by looking at the different columns rather than the rank of the matrix.

Theorem 4.3 (Laschov et al. (2013))**.** *Let $\mathcal{A}$ and H be a cellular automata and output operator, with associated binary matrices respectively denoted M_F and M_H.*

The pair $(\mathcal{A}, H)$ is observable if and only if there exists a time $T \in N$ such that no two columns of the observability matrix M_O are identicals.

The binary matrix M_F describes the behavior of the CA at the configuration level, in other words M_F plays the role of the adjacency matrix of the state transition diagram of the CA (Laschov et al., 2013; Martin, Odlyzko, & Wolfram, 1984). Thus the matrix M_F is not of size $N \times N$ as for affine CA but $S^N \times S^N$, where N is the number of cells and S^N the number of configurations. This exponential size of the matrix makes it difficult to apply to CA with a large number of cells.

We have the possibility to reduce the size of the matrices by using sparse matrices (Buluç, Fineman, Frigo, Gilbert, & Leiserson, 2009), an advantageous formulation (in terms of memory and algorithmic complexity) for matrices that have few non-zero elements. Moreover, by using the Yale format (or Compressed Row Format) observability can be evaluated by checking that a 1D array does not contain duplicates. Therefore, the algorithmic complexity to evaluate the observability of a CA with S^N configurations is $O(S^N)$. Laschov et al. have shown that in the general case, the computation of the observability of BN is a NP-Hard problem (Laschov et al., 2013), so without simplification on the CA (as we have done with affine CA) it will be difficult (or even impossible) to find an algorithm with polynomial complexity.

Even using sparse matrices, the calculation of observability or reconstructibility for cellular automata becomes difficult when these exceed a hundred cells. The objective is therefore to benefit from some properties of cellular automata to simplify these computations of observability and reconstructibility. The first property concerns in particular cellular automata with constraints on the initial configurations. Thus, many initial configurations can be excluded from the observability computation because they are impossible in the concerned application case. If we take the example of section 5.2, as we know the topology of the forest and that the fire can only start on one cell of the CA, the number of initial configurations to study goes from 4^{225} to only 158. The second property takes advantage of the ease of distribution of the computation of the CA (Millán, Wolovick, Piccoli, Garino, & Bringa, 2017). Indeed, when the number of cells is large, it is more efficient to compute several times the local function f than the global function F once. Thus the computation of the observability matrix is no longer performed from the matrix product of F and H but from a large number of simulations.

In order to benefit from those properties, we propose a definition of observability and reconstructibility based on binary relations. These allow to associate two elements, here the initial configuration s_0 (or current configuration s_T if we study the

reconstructibility) and the associated output sequence $\Theta_T(s_0)$. When we come to build this binary relation, we add the pair $(s_0, \Theta_T(s_0))$ only if the initial configuration s_0 is part of the initial configurations to study. In this formulation, the calculation of Θ_T can be done by simulation or by calculation using F and H according to what is more efficient. Therefore, we can define the **observability binary relation** $\mathcal{R}_\Theta$ and the **reconstructibility binary relation** $\tilde{\mathcal{R}}_\Theta$ by:

Definition 4.4. The observability and reconstructibility binary relation $\mathcal{R}_\Theta$ and $\tilde{\mathcal{R}}_\Theta$ associated to the output sequence Θ_T is defined as follows:

$$\mathcal{R}_\Theta = \{(s_0, \Theta_{0,T}) | s_0 \in \mathcal{W} \text{ and } \Theta_{0,T} = \Theta_T(s_0)\} \tag{24}$$

$$\tilde{\mathcal{R}}_\Theta = \{(s_T, \Theta_{0,T}) | s_0 \in \mathcal{W} \text{ and } \Theta_{0,T} = \Theta_T(s_0) \text{ and } s_T = F^T(s_0)\} \tag{25}$$

where $\mathcal{W} \subset \mathcal{S}^{\mathcal{L}}$ represents the set of admissible initial configurations.

Remark 10. If we consider the whole set of initial configurations (i.e. $\mathcal{W} = \mathcal{S}^{\mathcal{L}}$) then $\mathcal{R}_\Theta$, M_O, Θ_T represent the same relation except that $\mathcal{R}_\Theta$ is in set form and M_O in binary matrix form.

Theorem 4.5. *A cellular automata $\mathcal{A}$ is **observable** (resp. **reconstructible**) by an output operator H at time T if and only if the binary relation $\mathcal{R}_\Theta$ (resp. $\tilde{\mathcal{R}}_\Theta$) is injective.*

Proof. For the observability condition, the theorem follows directly from proposition 2.6 which represents the writing of the injectivity of Θ and $\mathcal{R}_\Theta$ is the relational writing of Θ.

For the reconstructibility condition, the theorem follows directly from proposition 2.9 which can be written as follows:

$$\begin{aligned}
(11) &\iff \forall s'_0, s''_0 \in \mathcal{W}, \Theta_T(s'_0) = \Theta_T(s''_0) \implies s'_T = s''_T \\
&\iff \forall s'_T, s''_T \in F^T(\mathcal{W}), \tilde{\mathcal{R}}_\Theta(s'_T) = \tilde{\mathcal{R}}_\Theta(s''_T) \implies s'_T = s''_T \\
&\iff \tilde{\mathcal{R}}_\Theta \text{ is injective}
\end{aligned}$$

□

In order to numerically verify the observability or reconstructibility of the CA, we must first construct the relations $\mathcal{R}_\Theta$ and $\tilde{\mathcal{R}}_\Theta$ and then verify their injectivity. Therefore we propose the algorithm 1 that allows to construct simulataneously the relations $\mathcal{R}_\Theta$ and $\tilde{\mathcal{R}}_\Theta$ but also to check their injectivity during their construction.

In order to construct the observability relation, the output sequence is computed for each of the possible initial configurations $s_0 \in \mathcal{W}$. However, to obtain the output sequence $\Theta_T(s_0)$ we need the configurations from s_0 to s_{T-1}, so we can take advantage of s_{T-1} to compute s_T in order to build the reconstructibility relation. As for the evaluation of the injectivity, we use a specific data strcuture called *Hashmap* to record the relations $\mathcal{R}_\Theta$ and $\tilde{\mathcal{R}}_\Theta$. This type of data structure allows to store a value with a key and to guarantee the uniqueness of the key. In this manner, by using the output sequence as a key and the initial configuration (or current configuration) as a value we are able to verify two things. The first is to make sure that each output sequence is associated to only one configuration and thus to check the injectivity of the observability and reconstructibility relations. The second is to be able to easily

find the configuration related to the output sequence and therefore know the initial configuration when measurements are performed on the system.

Algorithm 1: Verification of observability and reconstructibility of $(\mathcal{A}, H)$

```
AssessObsRec (W, F, Θ_T)
    inputs : A set of admissible initial configuration W;
             a global transition function F;
             an output sequence operator Θ_T
    output: The observability Hashmap Obs;
             the reconstructibility Hashmap Rec;
             the observability status is_Obs;
             the reconstructibility status is_Rec
    Declare Obs and Rec as two empty Hashmap;
    Declare is_Obs and is_Rec as two boolean variables initialized to true;
    foreach s_0 ∈ W do
        (Θ_{0,T}, s_{T-1}) ← Θ_T(s_0);
        s_T ← F(s_{T-1});
        if Obs[Θ_{0,T}] already exists then
            is_Obs ← false;
        else
            Obs[Θ_{0,T}] ← s_0;
        if Rec[Θ_{0,T}] already exists then
            if Rec[Θ_{0,T}] ≠ s_T then
                is_Rec ← false;
        else
            Rec[Θ_{0,T}] ← s_T;
    return Obs, Rec, is_Obs, is_Rec
```

This algorithm is separated into two parts, the first where the output sequence and the current configuration are computed and the second where these values are stored in the Hashmap. Thus the first part has a complexity of $O(TG)$ where T is the time horizon and G is the complexity of the global transition function F, then the second part has a complexity of $O(1)$. Thus, the total algorithmic complexity is $O(WTG)$ where W is the number of admissible configurations.

If we consider that all initial configurations are studied (i.e. $\mathcal{W} = \mathcal{S}^{\mathcal{L}}$) and that F has linear complexity with respect to the number of cells N, because the local function f is applied separately to each cell, (so logarithmic with respect to the number of configurations S^N) then the complexity of the alogrithm is $\mathcal{O}(S^N TNlog(S))$.

Compared to the algorithm that verifies observability for affine CAs presented in section 3.3, this algorithm has a greater complexity. The algorithmic complexity for verifying observability is polynomial for affine CA $\mathcal{O}(TQN^{\omega-1})$ and exponential for non-linear CA $\mathcal{O}(S^N TNlog(S))$, with N representing the number of cells, Q the number of sensors, S the number of states and T the time horizon.

4.2. Decomposition of the Observability and Reconstructibility Problem

In order to be able to use the results of the previous subsection, we need to find a method that will drastically reduce the algorithmic complexity necessary to compute the observability. As cellular automata are not very connected (few links between cells), it is possible to decompose the observability problem into smaller ones which would drastically reduce the algorithmic complexity. In this way, the observability is computed only on a part of the CA which allows to move the complexity factor from the size of the CA to the size of this part. To do this, sensors and subdivisions must be judiciously placed to ensure observability or reconstructibility at any point in the cellular automata.

The subdivisions do not represent a whole CA and are no longer isolated, so the boundaries are cells of the CA and do not respect the rules on boundaries that we defined at the beginning of the paper. We can consider that these boundaries take random states at each time step, in this way, we have to define a new notion of observability and reconstructibility for automata with **randomly valued boundaries**. On the observation horizon T, the boundaries will have a sequence of values called boundary trajectory and it will be necessary to check the observability for each trajectory therefore if the output sequence and the boundary trajectory are known then the initial state of the subdivision can be reconstructed.

Theorem 4.6. *A cellular automata $\mathcal{A}$ with **randomly valued boundaries** is observable (reconstructible) by an output operator H at time T if and only if the cellular automata is observable (reconstructible) for every **boundary trajectories**.*

To ensure the observability of a part of CA, it is necessary to know the value of boundary trajectory, so we must place sensors directly outside the CA but also inside it. If we ensure the observability only for a part of the CA we speak then of regional observability (Fekih & Jai, 2006). But in our case, we will study the observability of the whole CA through multiple parts of it.

Remark 11. This method works very well for one-dimensional automata because the size of the boundary is independent of the number of cells in the CA, there are always 2 boundaries. However, in two dimensions (or more) the number of boundary cells increases greatly. Indeed, to obtain the observability of a 3×3 CA it is necessary to know the state of the 16 boundary cells (i.e. more than the CA itself).

For one-dimensional CA, the sensors can be placed directly on the edges of the CA of the sub-observer and thus act as a sensor for the sub-observer and as a boundary cell sensor for the neighbouring sub-observers (see figure 2). In this way, each sensor has the dual purpose of measuring the boundary cells and estimating the cells of the CA. Of course, if this sensor topology is not sufficient to ensure observability or reconstructibility, other sensors can be added inside the sub-observer but their only purpose will be to estimate the state of the sub-observer.

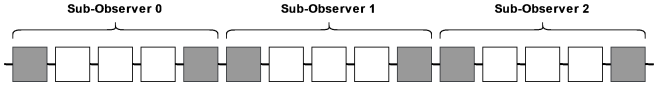


Figure 2. A one-dimensional CA subdivided into 3 parts. Grey cells are measured by a sensor.

This method significantly reduces the complexity of the observability of one-

dimensional CA because the subdivision reduces the number of cells whose observability we seek to verify. If we split a CA of size N in k equal parts then the number of initial configuration is not S^N but $S^{\frac{N}{k}}$ but we also have to evaluate for the 2^{S^T} limit trajectories so there are $S^{\frac{N}{k}} \times 2^{S^T}$. However, if the sub-observers are not of the same size, then observability must be checked independently for each sub-observer size.

5. Example on Several Application

In this section, we will apply the criteria of observability and reconstructibility to two separate examples. For the first example, we will use a Wolfram's non-linear rule which is used to model a traffic flow: the rule 184. This rule also makes it possible to show, on certain systems, that observability is difficult to ensure in contrast to reconstructibility which may be simpler. In the second example we study the observability problem for a two-dimensional forest fire spread model. This example will also show that observability may sometimes be easier to ensure with mobile sensors.

5.1. Traffic Flow Example

Elementary Rule 184 is one of the most used of Wolfram's elementary rules. It is mainly used to model in a simple way the traffic flows (Maerivoet & De Moor, 2005) and the particle deposition (Krug & Spohn, 1988). The model is deterministic, boolean and one-dimensional.

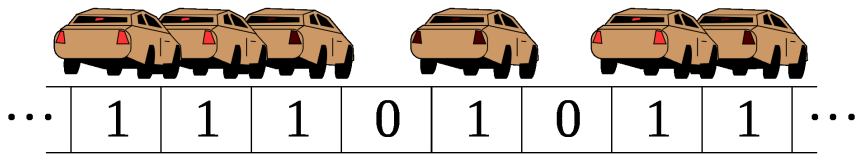


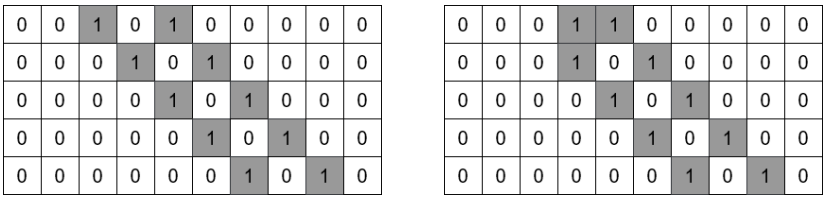
Figure 3. Rule 184 interpreted as a simulation of traffic flow. Each 1 cell corresponds to a vehicle, and each vehicle moves forward only if it has open space in front of it. Image from David Epstein at English Wikipedia

For this example, we will study the elementary rule 184 which is a one-dimensional Boolean cellular automata. Because of the algorithmic complexity of the relational method we will only consider a few cells. We will consider 10 cells for a total of 1024 configurations and we will use periodic boundaries, so the CA can be defined as:

- $\mathcal{L} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\mathcal{S} = \{0, 1\}$
- $\mathcal{N} : c \mapsto \{c - 1, c, c + 1\}$ with respect to the periodic boundaries, thus $\mathcal{N}(0) = \{0, 1, 9\}$ and $\mathcal{N}(9) = \{0, 8, 9\}$.
- $f : \{s_t(c-1), s_t(c), s_t(c+1)\} \mapsto s_t(c-1) + s_t(c-1) \cdot s_t(c) + s_t(c) \cdot s_t(c+1)$ with respect to the modular arithmetic of $\mathcal{S}$.

Since the transition function of this CA is non-linear, its observability cannot be assessed with the Kalman criterion.

This CA poses another problem regarding observability: it cannot be evaluated with few sensors. Indeed, to differentiate the two initial configurations presented in figure 4, it would be necessary to observe 2 of the 3 cells that are non-zero among the two initial configurations. This problem can be shifted over the whole CA and it would



(a) Configuration 0010100000 (b) Configuration 0001100000

Figure 4. Evolution of the rule 184 for two similar initial configurations. Time is going down on the vertical axis.

thus be necessary to place a sensor every two cells to resolve this issue. Conversely, this convergence of two initial configurations does not pose any difficulty with respect to the reconstructibility of the cellular automata. Because of this, we won't study the observability but focus on finding a sensor to ensure the reconstructibility.

We have the intuition that reconstructibility can be provided by a single sensor, wherever it is. Indeed, because the system propagates in one direction only, a sensor could see all the non-zero states pass and deduce their future positions. To carry on with the traffic flow analogy, this sensor could be a tollbooth which, at a certain moment, measures all the cars. In this case, if the cell 0 is observed by the sensor then the output operator is defined as:

- $\mathcal{L}_q = \{0\}$
- $H : s_t \mapsto s_t(0)$

To verify the reconstructibility of the cellular automaton, we will use algorithm 1 in order to construct the reconstructibility relation R_T as well as to verify its injectivity. For this application case, there is no constraint on the configurations since each location can be occupied by a car or not, therefore $\mathcal{W} = S^L$. We cannot apply the Cayley-Hamilton theorem, thus we start assessing the reconstructibility with a minimum time horizon $T = 1$ and will increase it gradually until the reconstructibility is verified. In addition, we modify the algorithm 1 so that it counts the number of unobservable and unreconstructible configurations and plot this number as a function of the time horizon T in figure 5. Finally, the system is reconstructable from $T = 14$.

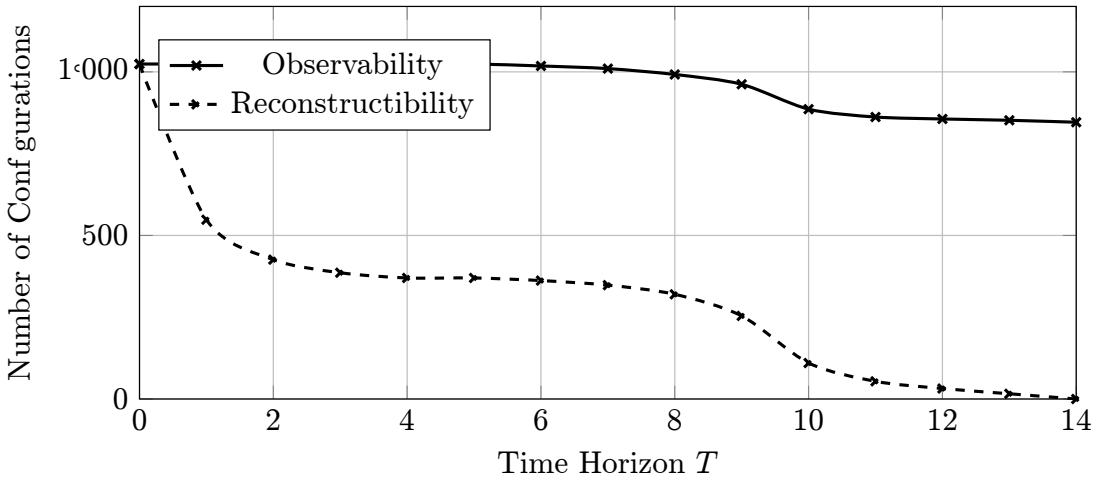


Figure 5. Number of unobservable/unreconstructible configurations as a function of the time horizon T.

The number of unreconstructible configurations converges to 0 while the number of unobservable configurations is not zero. Reconstructibility is well ensured at time 14 but not observability. The reduction in the number of configurations between time 9 and 10 is specific to the system, it is the minimum time necessary for the sensor to measure the impact of all the cells through the measured cell. Indeed, with 10 cells and periodic boundaries, it takes 10 time steps for the cell on the far right to have an impact on the cell on the far left (the measured cell).

Finally, we aim to extend our model to observe a larger road network. For this, we intend to observe a CA of $1,000$ cells but the algorithmic complexity of algorithm 1 is far too important. Therefore, we study a CA of size 1000 but using 100 sub-observers of size 10. We use the sensor topology shown in Figure 2, one sensor at each end of the lattice. In the same way as for the study of reconstructibility, we have no constraint on the time horizon T from which the CA is reconstructible for randomly valued boundaries. To do this, we start with a low time horizon T and increase it gradually until reconstructibility is ensured. By doing simulations, we verify that the reconstructibility is assessed for $T = 7$.

This method with the sub-observers allowed to ensure the reconstructibility for the whole CA which would have been impossible without it. Indeed, with 1000 cells the number of operations to evaluate the reconstructibility is 2^{1000} with the usual method against $2^{10} \cdot 2^{2*7} = 2^{24}$ for the reconstructibility with the sub-observers.

5.2. Forest Fire Example

In this example, we will study the observability and reconstructibility property for a two-dimensional cellular automata. To observe the state of this system, we will use a single sensor that can move. This type of sensor can be similar to a drone that flies over an environment and perceives only part of it. For this second example we will study the observability and reconstructibility of a two-dimensional CA of size 15×15 whose goal is to model the spread of a fire in a non homogeneous forest.

The state of the cells represents the state of the forest, which can correspond to four different states: an empty cell, a forest cell, a burning forest cell and a burned forest cell. The behavior of the cells is quite simple: an empty cell does not change state; a forest cell becomes a burning forest cell if at least one of its neighbors is a burning forest cell; a burning forest cell becomes a burned forest cell; and a burned forest cell does not change state. This behavior can be represented by the figure 6. The cellular automata is defined as follows:

- $\mathcal{L} = \{0, 1, 2, \ldots, 13, 14\}$
- $\mathcal{S} = \{0, 1, 2, 3\}$
- $\mathcal{N}$ is the Von Neumann neighbourhood of radius 1 and we consider fixed to 0 boundaries, thus:

$$\mathcal{N} \colon c \mapsto \mathcal{N}(c) = \{c' \in \mathcal{L} | \left\| c' - c \right\|_1 \leq 1\}$$

The presented CA is non-linear, so we will use the results of part 4 to prove observability. However, due to the number of configurations, it is not possible to check the observability, so we will drastically reduce the number of possible initial configurations while respecting the coherence with the model.

In the case of the forest fire spread, we can consider that the initial configuration represents a forest (i.e. neither burning nor burned forest) and only one burning forest

Figure 6. State Transition Diagram for the Cell.

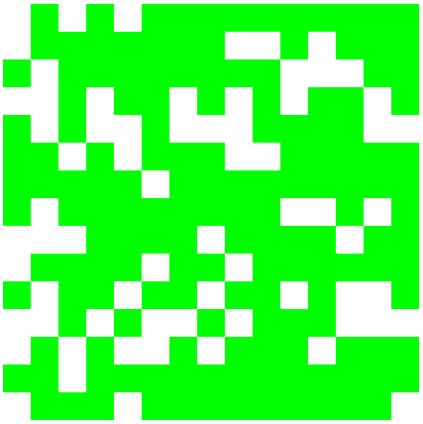


Figure 7. Topology of the forest without fire outbreak.

cell to represent the ignition point of the fire. Moreover, the topology of the forest is known, i.e. only one initial configuration of forest will be studied. For our example, we will use the forest presented on figure 7 with 157 forest cells and 68 empty cells. By making these simplifications, the number of possible initial configurations increases from 4^{225} to 158.

Some parts of the forest are not connected to the rest and therefore cannot be ignited. In order to determine the state of these parts, these cells must be directly measured by the sensors. For the chosen topology, there are 4 cells isolated from the rest of the forest which would require 4 sensors but a single measurement would be enough to detect the fire in these areas. The mobile sensors could measure the state of these cells and directly go and measure other cells. For this kind of property, mobile sensors seem to be more efficient. Indeed, if we consider a random distribution of 15 sensors then the average number of unobservable configurations is 18.2 for mobile sensors (the position is determined randomly at each time step) against 47.4 for fixed sensors. Although we have not been able to find a trajectory for mobile sensors that ensures observability, they seem to be more efficient than fixed sensors.

Finding positions or trajectories for the sensor network to observe the system is a complex problem as there is no "simple" method to produce an observer from the system and the constraints. Moreover, for given constraints there is not necessarily a position or trajectory that ensures observability or reconstructibility.

The calculation of a trajectory is more complex than the calculation of a position because it is a question of finding a position per sensor and per time step. Moreover, new constraints are added such as obstacle avoidance or physical constraints of speed or acceleration. Some researchers are directly studying the problem of observability with

mobile sensors. Noticeably Demetriou *et al* (Demetriou, 2010; Demetriou & Hussein, 2009) or Hussein *et al* (Hussein & Stipanovic, 2007) are considering networks of mobile sensors. Related problems occurred when mobile sensors (or actuators) are considered such as obstacle avoidance or formation coordination to get maximum coverage.

6. Conclusion

At the beginning of this article, the notions of observability and reconstructibility were defined for cellular automata. They have been presented theoretically for finite-state and finite-lattice cellular automata. They also allow the observation of cellular automata with a network of static and mobile sensors.

Two criteria were then presented to assess observability and reconstructibility of CA. The first one applies only to affine CA and derives from an extension of the Kalman criterion. The second one derives from results on Boolean networks and applies more generally to CA, at the cost of a greater complexity. Despite its greater complexity, this second criterion has been utilized in conjonction with CA-specific constraints to, first, design an algorithm that verifies observability and reconstructibility, and, second, to decompose the observability and reconstructiblity problems and solve them for CA with many more cells.

Finally, the observability and reconstructibility criteria have been applied to two very different examples. The first one is the elementary rule 184 use to model traffic flow. We have shown the benefit of recontructibility compare to observability for this particular example as well as the decomposition of the reconstructibility problem for a thousand cells. The second example is a two dimensional forest fire spread model in which we have address the observability problem by largely reducing the number of inital configurations and by using a network of mobile sensors.

This work on observability and reconstructibility will make it possible in the future to develop a systemic method for constructing an observer. For the moment, observers are found by trial and error, which, along with the complexity of relational methods, poses problems of computation time. The observability and reconstructibility criteria can also be used to coordinate a sensor network by using the classical method such predictive controller at the cost of a higher computation time or by using a decentralized method such as the multi agent paradigm at the cost of optimality loss.

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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