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# Synchronisation of Elementary Cellular Automata with a Small Initial Error. Application to Rule 18.

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**Abstract.** In this paper, we study how synchronization and state estimation are related in the context of elementary cellular automata. We first characterize the synchronization error between two 1D elementary cellular automata implementing Wolfram’s 18<sup>th</sup> rule. Then we propose a simple approach to statistically model the transient phase of the synchronization error spread. We finally present a way to utilize this model of the error spread to place mobile sensors in order to reduce the overall synchronization error when the initial error is small.

**Keywords:** cellular automata · synchronization · mobile sensors

## 1 Introduction

In control theory, monitoring physical systems which are distributed in space is based on the construction of an estimate from measurements and the dynamics of the system. Measurements which come from potentially mobile sensors. The problem of positioning these sensors is crucial to make it possible to estimate the state of the system. This state estimation problem is widely studied by classical control theory [5,7] and it follows from the verification of observability, a notion that ensures that the sensors are well placed. This notion of observability can be applied to cellular automata (CA) [4,6,3] (and by extension to Boolean networks [10] which can be seen as a generalization of CA) but its evaluation has proven to be extremely complicated when it comes to non-linear CA [6].

The synchronization problem consists in converging a system called replica to another one called driver by means of a unidirectional coupling between the two. In the case of CA, the state of some cells of the driver are copied to these same cells of the replica. The coupling between the two can be realized with a single cell [2], with fixed cells [8] or with cells chosen randomly at each time step [1]. In the first case, Dogaru et al showed that a strong condition regarding

the chaoticity of the system is needed to synchronize the driver and the replica. In the second case, Urías et al propose a necessary and sufficient condition concerning the cell position to ensure the synchronization of linear elementary cellular automata. Finally, Bagnoli and Rechtman propose a statistical approach to synchronization with a critical probability  $p_c$  that ensures synchronization.

The problem of synchronization of two CA can also be seen as a state estimation problem. Indeed, the driver can be seen as the system to be observed, the replica as the state estimator and the synchronized cells as sensors. For the purpose of monitoring physical systems, the conditions on the system imposed by Dogaru et al (chaoticity) and by Urías et al (linearity) make it difficult to apply to this type of system. The approach of Bagnoli and Rechtman, on the contrary, is not based on a specific type of system. Moreover, it allows to include the notion of mobile sensor through the random choice of synchronized cells.

The main objective of this paper is to study a synchronized CA as a state estimator for the observation of distributed parameter system with spatio-temporal dynamics. We focus on synchronization with a small initial error because in some physical system monitoring, only a small portion of the system is unknown. For example, when monitoring forest fires spread, the topology of the forest is known but only the ignition points are unknown. Throughout this article, we focus on a single elementary rule that exhibits spatio-temporal dynamics so that the obtained results may be transferred to other CA and in particular to physical systems. Therefore, we chose to study the elementary rule 18 because it is the smallest chaotic, symmetric, and nonlinear rule [9].

The article starts by studying the differences in synchronization performance as a function of the initial synchronization error. Then, we model the spreading of the initial error within the CA using basic geometry. We finish by presenting an improvement of the synchronization algorithm for systems with a small initial error.

## 2 Influence of Initial Error on Synchronization

In order to study the impact of the initial error on the synchronization performance, we need to define the synchronization method but also to express it in terms of the initial error. For this purpose, we chose the definition proposed by Bagnoli and Rechtman [1] which expresses the synchronization of two 1D CA of  $N$  cells:  $x$  and  $y$ . The synchronisation of  $y$  with  $x$  is done by copying some of  $x$  cells' state in the matching cells in  $y$ , at each time step. A diagonal matrix  $P$  indicates which cells are coupled. A value of 1 in this matrix indicates that the corresponding cells are coupled. The position of the coupled cells are determined randomly at each time step with a probability  $p$ , called control strength. The expression for synchronization is:

$$\begin{cases} x_{t+1} = f(x_t) \\ y_{t+1} = (I - P) \cdot f(y_t) + P \cdot f(x_t) \\ e_{t+1} = x_{t+1} \oplus y_{t+1} \\ \epsilon_{t+1} = \frac{1}{N} \cdot \sum_i e_{t+1}^{(i)} \end{cases} \quad (1)$$

The synchronization error  $e_t$  is the difference between  $x_t$  and  $y_t$ , and  $\epsilon_t$  its normalized mean error value. Since we are studying the influence of the initial error, we initialize  $x_0$  and  $e_0$  randomly and set  $y_0 = x_0 \cdot e_0$ . We will note  $e$  the proportion of cells in  $y_0$  that are different from cells in  $x_0$ , in percentage.

In [1], Bagnoli and Rechtman discuss the notion of critical control strength  $p_c$  (determined statistically or analytically using maximum Liapunov exponents) which guarantees that the synchronization is total during a random synchronization. This critical parameter insures, for a state estimator, that the estimated state correctly corresponds to the state of the observed system. In section 4, we present an improvement of this synchronization algorithm in order to reduce this critical power control for a total synchronization.

To get relevant results, we conducted a large number of simulations for each initial conditions. This is required by the fact that some initial conditions lead very quickly to a convergence that biases the results. More on this later.

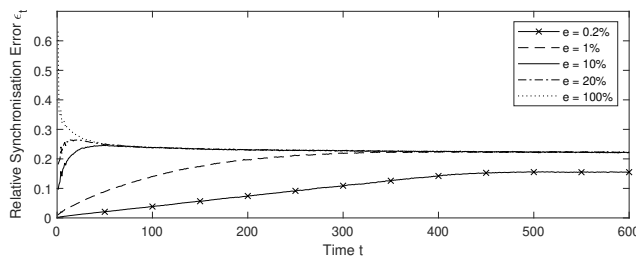


Fig. 1: Mean synchronization error as a function of time for different initial error  $\epsilon_0$ .

Figure 1 presents mean synchronization error  $\epsilon_t$  as a function of time for different initial error  $\epsilon_0$  values. These results were obtained by taking the mean of the synchronization error  $\epsilon_t$  over 500 simulations for the elementary rule 18 with 500 cells and a control strength  $p = 0.1$ . The initial configuration  $x_0$  was randomly initialized at each simulation, same for the initial unsynchronized cells,  $\epsilon_0$ . Initial error has a clear impact on the performances of synchronization. Its first influence is on the speed of convergence towards the asymptote. Indeed, the 10% curve seems to converge faster than the 20% and 100% curves which converges earlier than the 1% and 0.2% curves. The second effect of the initial error is on the value of the asymptote when the error is small enough. For sufficiently large errors, all simulations converge towards the same asymptote value, around 0.23. But if  $\epsilon_0$  is sufficiently small, the reached asymptote is lower than this "generic" one.

To understand the difference in value between the two asymptotes, we studied the evolution of the error as a function of time for the particular case of a single cell of initial error ( $e = 0.2\%$ ). As we can see on Figure 2, there are two very different kind of evolution of the synchronization error  $e_t$ . On one hand,

in [Figure 2a](#), the error spreads until it covers the whole CA and reaches the asymptotic non-zero value. On the other hand, in the very specific case depicted on [Figure 2b](#), the synchronization quickly becomes total and the error reaches zero. Therefore, when we average these two cases, which we did for [Figure 1](#), we obtain a lower asymptotic value than the generic case will give. For the remaining of the study, we chose to dissociate the two cases and to not consider the fast total synchronization cases when we study the asymptotic value.



Fig. 2: Evolution of the synchronization error for elementary rule 18 with 500 cells from a single cell error ( $e = 0.2\%$ ). The time is represented on the vertical axis. A black pixel is an erroneous cell in the synchronized CA.

To characterize the influence of the initial error  $\epsilon_0$  on the ability of the synchronized CA to be considered as a state estimate, we will only consider the mean of the asymptotic value of the synchronization error. [Figure 3](#) represents this mean asymptotic synchronization error as a function of the initial error. First, if we consider only the asymptotic synchronization (without special cases of early complete synchronization), the value of the asymptote does not depend on the initial error. Second, the value from which the average error with and without total synchronization become different depends on the strength of the control  $p$ : the stronger it is, the more the chances of total synchronization increase.

### 3 Modeling of the Error Spreading Dynamics

In order to explain the dynamics of the evolution of the synchronization error, we will study how the error propagates within the CA as a function of the control strength. To do so, we will start by studying the propagation of the error with the simple case of a single erroneous cell, and then generalize these results.

Typical error propagation dynamics from one erroneous cell are depicted in [Figure 4](#). We adopted a triangle as a simple geometric model for these dynamics. It appears that the top angle of the triangle is inversely proportional to the control strength  $p$ .

To describe how the synchronization error spreads, two parameters will be used: the first being the aperture angle of the propagation triangle, and the

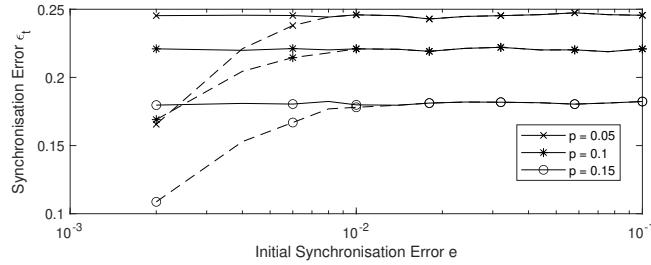


Fig. 3: The asymptotic value of the mean synchronization error as a function of initial error  $\epsilon_0$ . This was obtained by taking the mean of the synchronization error  $\epsilon_t$  as a function of time over 200 iterations. The continuous lines consider only the asymptotic synchronization while the dashed lines include both asymptotic and total synchronization.

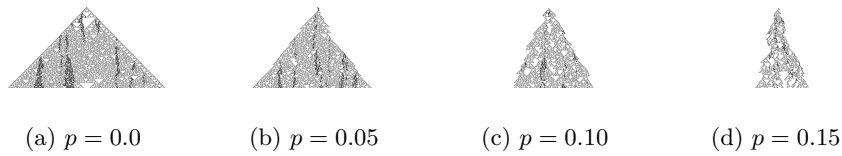


Fig. 4: Evolution of the error for elementary rule 18 with 500 cells from a single erroneous cell ( $e = 0.2\%$ ). The time is represented on the vertical axis.

second being the shift angle between the altitude and the median of the triangle. Indeed, the median of the triangle seems to vary from one simulation to another. Figure 5 describes the geometry associated with these angles  $\alpha$  and  $\beta$  which describe respectively the aperture angle and the shift angle.

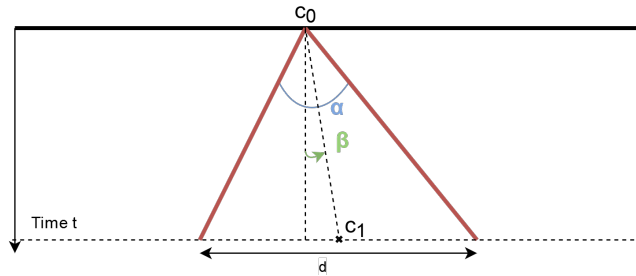


Fig. 5: Schematic of the theoretical spread of the synchronization error from a single initial error cell.

For the purpose of this article, we will not use directly the  $\alpha$  and  $\beta$  angles but their tangents, which represents spread velocities. We will simply call  $\alpha$  and  $\beta$  the velocities associated to the angles and not the angles themselves. Therefore, the **error spreading ratio**  $\alpha$  represents the mean number of cells by which the triangle base increases at each time step and **error shift ratio**  $\beta$  the mean number of cell shift at each time step.

We can calculate the mean value of the error spreading ratio by measuring the area of the error at time  $T$  and divide by current time to obtain the tangent of  $\alpha$ . [Figure 6](#) describes the evolution of the average spread ratio as a function of the control strength. This one is a linear function of which we experimentally obtained the equation  $\alpha = -8.23.p + 1.93$  using linear regression.

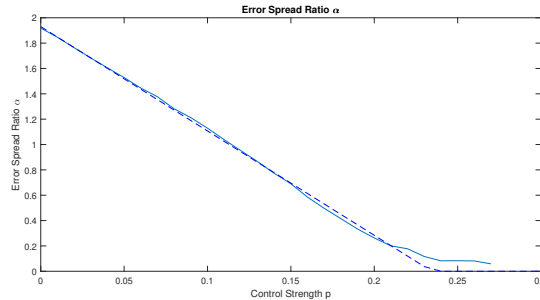


Fig. 6: The error propagation coefficient  $\alpha$  as a function of control strength. The continuous curve is obtained by taking the mean of  $\alpha$  over 1000 simulations. The dashed curve is the curve obtained by linear regression.

The error spreading ratio  $\alpha$  has, for a given control strength, a normal distribution whose average is presented on [Figure 6](#). The standard deviation related to  $\alpha$  can be calculated in order to have a better representation of  $\alpha$ . The error shift ratio  $\beta$  also follows a normal distribution. Based on these two parameters, we can express the width and the center of the error at time  $T$  by  $c_1 = c_0 + \beta \cdot T$  and  $d = \alpha \cdot T$ .

**Corollary 1.** *The synchronization error  $\epsilon_t$  can be estimated from the parameter  $\alpha$  as well as the value of the asymptote  $\gamma$  associated to the control strength  $p$ . Thus, the synchronization error  $\epsilon_t$  is defined by :*

$$\epsilon_T = \frac{\gamma}{N} \cdot \max(\alpha \cdot T, N)$$

Indeed,  $\alpha \cdot T$  gives the width of the error in number of cells. When dividing by  $N$ , we obtain the normalized error width and then multiplying by the asymptote  $\gamma$ , the value of  $\epsilon_t$  when the error is present on the whole CA, we obtain the synchronization error  $\epsilon_T$ .

Using the [Corollary 1](#), we can make an example for  $p = 0.1$ . For this control strength, the error spreading ratio  $\alpha$  follows a normal distribution with mean 1.127 and standard deviation 0.1376. On [Figure 7](#), the synchronization error  $\epsilon_T$  is displayed as well as the estimated error with an  $\alpha$  fixed at the mean and an  $\alpha$  that follows the normal distribution. We quickly notice that the use of the normal distribution in the calculation of the error allows to explain the rounded curve when the error approaches the asymptote. However, the two theoretical curves have a difference with the real curve which is explained by a faster increase of the error during the first iterations which is caused by a higher  $\alpha$  as the error is not yet detected, and therefore controlled, by the sensor.

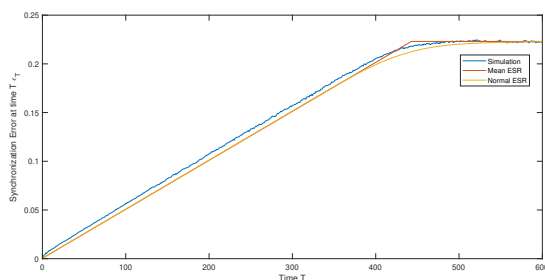


Fig. 7: Evolution of the average synchronization error as a function of time for the real case as well as the theoretical cases with the error spreading ratio which is constant and follows a normal distribution.

This method allows us to simply represent the propagation of the error in the case where a single cell is erroneous in the initial configuration. If we consider two or more erroneous cells then the modeling becomes more complex. Indeed, the two errors propagate independently until they *collide*, in this case we must consider that the errors merge in a single (larger) source of error. Thus, considering that the collision takes place at time  $t_1$ , we can consider that the error spreading ratio  $\alpha$  is expressed as:

$$\alpha(t) = \begin{cases} \alpha_0 + \alpha_1 & , \text{ if } t \leq t_1 \\ \frac{(\alpha_0 - \beta_0 + \alpha_1 + \beta_1)}{2} & , \text{ if } t \geq t_1 \end{cases}$$

The time  $t_1$  of the collision depends on the initial distance between the two initial errors, whose probability distribution depends on the boundary conditions used. Moreover, since each of these initial errors is subject to the total and fast synchronization (of probability  $\tau$ ), the model must include, with probability  $2\tau(1 - \tau)$ , a propagation with only one initial error using the model of [Figure 5](#). With more than two erroneous cells, the operation is the same but it is necessary to take into account several collisions at different times.



#### 4 Optimization of Algorithm for a Single Erroneous Cell

If we consider that the synchronized cellular automaton has only a few errors at initialization, then it is possible to adapt the synchronization algorithm so as to concentrate the sensors only on the area that contains errors. To do this, we must first identify the areas that possibly contain errors and then distribute the sensors over those.

To identify the error area, a sensor must already detect an error. Then, with a method similar to the one shown in Figure 5, it is possible to backpropagate the error measured at time  $t$  to obtain the possible error area  $\hat{e}_0$  at time 0 which could lead to the initial error. Propagating an error from this initial estimate, we can obtain the possible current error area  $\hat{e}_t$ . Figure 8 represents the backpropagation of the error with a ratio  $\alpha_{max}$  which corresponds to a ratio  $\alpha$  large enough to include all (or a large part) of the possible spread ratios. The maximum ratio is 2 because it is not possible for the error to spread to more than one cell on each side (this results from the size of the neighbourhood) but if the strength of the control  $p$  is strong enough  $\alpha_{max}$  can be chosen smaller. As  $\alpha$  follows a normal distribution, a ratio  $\alpha_{max} = \alpha_{mean} + 3\sigma$  encompasses 99.9% of the possible spreading ratios.

As new errors are detected, the initial error area can be refined using the intersection of all the initial error areas of all the errors detected by the sensors. In this way, it is possible to reduce the size of the error zone at time  $t$  but also to locate the position of the initial error.

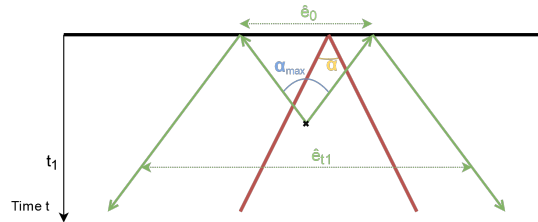


Fig. 8: Schematic of the backpropagation of the synchronization error to find the initial error area  $\hat{e}_0$

Now that error area can be estimated, it remains to position the sensors. The method consists in placing the sensors only in the area where the error could be present. The number of sensors will remain the same but the control strength (the sensor density) of the error area will increase proportionally to the smallness of the error area resulting in a lower critical control strength  $p_c$  as shown in Figure 9. The control strength in the error area is described by:

$$p_{error} = p \cdot \frac{N}{\hat{e}_t}$$

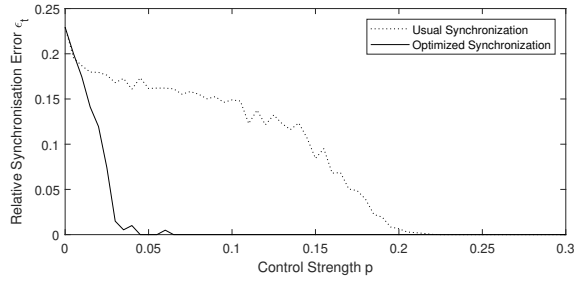


Fig. 9: Evolution of the average synchronization error as a function of control strength for a single cell error.

As shown in Figure 9, the optimized synchronization performs better than the usual one with a critical control strength  $p_c$  at 0.05 instead of 0.21. However, when the control is too weak, the difference between the two is negligible because the first error cell is detected too late by the sensors and therefore the optimized control strength  $p_{error}$  is not sufficient to synchronize the two systems. In Figure 10, we have compared these two synchronization methods on other elementary rules belonging to different classes [9]. The results obtained are minimal in that the error spreading ratio used for the backpropagation is  $\alpha_{max} = 2$ , smaller values according to the distribution of probability could have been chosen to further increase the performances. The optimized synchronization performs better but the difference between the two seems to depend on the class. Class 2 CA (presented by Wolfram as "filters") seem to exhibit lower error propagation coefficients  $\alpha$  than class 3 and 4 CA. Without control,  $\alpha$  is 0.048 for rule 37, 0.54 for rule 110 and 1.9 for rule 126. However, a systematic study on elementary CA would be necessary to confirm this conjecture.

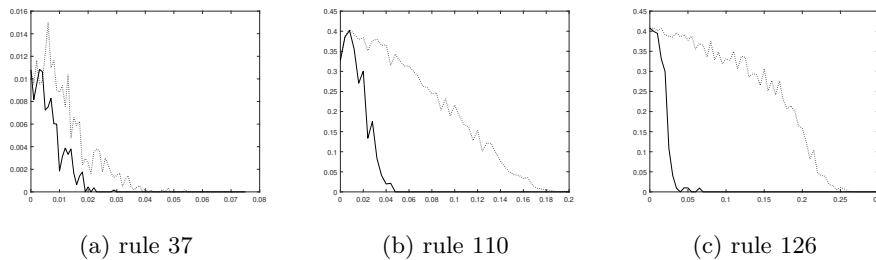


Fig. 10: Evolution of the average synchronization error as a function of control strength  $p$  for a single cell error for different rules. From left to right: rule 37 class 2; rule 110 class 4; and rule 126 class 3. Continuous line represents the optimized synchronization and dotted line the usual synchronization.

## 5 Conclusion and Perspectives

In this paper we studied how CA synchronization relates to state estimation of distributed parameter system in the context of Wolfram's 18<sup>th</sup> rule. In order to understand how a synchronized CA can be seen as an estimated state, we studied the dynamics of the synchronization error spreading. To do so we proposed a simple geometric model of this propagation. Finally, we present a sensors placement algorithm utilizing this geometric model in order to reduce the synchronization error and improve the accuracy of the synchronized CA as an estimate of the original CA representing the studied system. This optimized synchronization has been studied in the case of different elementary rules of classes 2, 3 and 4 whose performance increase in comparison to the usual synchronization is particularly important in the case of classes 3 and 4.

In the future, we will be able to adapt this algorithm to more than a single initial error cell. Furthermore, a systematic study on the elementary automata could be carried out in order to refine the algorithm of synchronization according to damage (error) spreading.

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