HAL open science

Thermodynamic Formalism and Perturbation Formulae for Quenched Random Open Dynamical Systems

Jason Atnip, Gary Froyland, Cecilia González-Tokman, Sandro Vaienti

To cite this version:

Jason Atnip, Gary Froyland, Cecilia González-Tokman, Sandro Vaienti. Thermodynamic Formalism and Perturbation Formulae for Quenched Random Open Dynamical Systems. 2023. hal-03942121

HAL Id: hal-03942121

https://hal.science/hal-03942121v1

Preprint submitted on 16 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Thermodynamic Formalism and Perturbation Formulae for Quenched Random Open Dynamical Systems

Jason Atnip

Gary Froyland

Cecilia González-Tokman

Sandro Vaienti

School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia
Email address: j.atnip@unsw.edu.au

School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia
Email address: g.froyland@unsw.edu.au

School of Mathematics and Physics, The University of Queensland, St Lucia, QLD 4072, Australia
Email address: cecilia.gt@uq.edu.au

Aix Marseille Université, Université de Toulon, CNRS, CPT, 13009 Marseille, France
Email address: vaienti@cpt.univ-mrs.fr

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

Abstract. We develop a quenched thermodynamic formalism for open random dynamical systems generated by finitely branched, piecewise-monotone mappings of the interval. The openness refers to the presence of holes in the interval, which terminate trajectories once they enter; the holes may also be random. Our random driving is generated by an invertible, ergodic, measure-preserving transformation σ on a probability space $(\Omega, \mathscr{F}, m)$. For each $\omega \in \Omega$ we associate a piecewise-monotone, surjective map $T_\omega : I \to I$, and a hole $H_\omega \subset [0,1]$; the map T_ω, the random potential φ_ω, and the hole H_ω generate the corresponding open transfer operator $\mathcal{L}_\omega$. The paper is divided into two chapters. In the first chapter we prove, for a contracting potential, that there exists a unique random probability measure ν_ω supported on the survivor set $X_{\omega,\infty}$ satisfying $\nu_{\sigma(\omega)}(\mathcal{L}_\omega f) = \lambda_\omega \nu_\omega(f)$. Correspondingly, we also prove the existence of a unique (up to scaling and modulo ν) random family of functions ϕ_ω that satisfy $\mathcal{L}_\omega \phi_\omega = \lambda_\omega \phi_{\sigma(\omega)}$. Together, these provide an ergodic random invariant measure $\mu = \nu\phi$ supported on the global survivor set $\mathcal{X}_\infty$, while ϕ combined with the random closed conformal measure yields a random absolutely continuous conditional invariant measure (RACCIM) η supported on $[0,1]$. Further, we prove quasi-compactness of the transfer operator cocycle generated by $\mathcal{L}_\omega$ and exponential decay of correlations for μ. The escape rates of the random closed conformal measure and the RACCIM η coincide, and are given by the difference of the expected pressures for the closed and open random systems. Finally, we prove that the Hausdorff dimension of the surviving set $X_{\omega,\infty}$ is equal to the unique zero of the expected pressure function for almost every fiber $\omega \in \Omega$. We provide examples, including a large class of random Lasota-Yorke maps with holes, for which the above results apply.

In the second chapter of the paper we consider quasi-compact linear operator cocycles $\mathcal{L}^n_{\omega,0} := \mathcal{L}_{\sigma^{n-1}\omega,0} \circ \cdots \circ \mathcal{L}_{\sigma\omega,0} \circ \mathcal{L}_{\omega,0}$, and their small perturbations $\mathcal{L}^n_{\omega,\varepsilon}$. The operators $\mathcal{L}_{\omega,0}$ and $\mathcal{L}_{\omega,\varepsilon}$ need not be transfer operators. We prove an abstract ω-wise first-order formula for the leading Lyapunov multipliers $\lambda_{\omega,\varepsilon} = \lambda_{\omega,0} - \theta_\omega \Delta_{\omega,\varepsilon} + o(\Delta_{\omega,\varepsilon})$, where $\Delta_{\omega,\varepsilon}$ quantifies the closeness of $\mathcal{L}_{\omega,\varepsilon}$ and $\mathcal{L}_{\omega,0}$. We then consider the situation where $\mathcal{L}^n_{\omega,0}$ is a transfer operator cocycle for a closed random map cocycle T^n_ω and the perturbed transfer operators $\mathcal{L}_{\omega,\varepsilon}$ are defined by the introduction of small random holes $H_{\omega,\varepsilon}$ in $[0,1]$, creating a random open dynamical system. We obtain a first-order perturbation formula in this setting, which reads $\lambda_{\omega,\varepsilon} = \lambda_{\omega,0} - \theta_\omega \mu_{\omega,0}(H_{\omega,\varepsilon}) + o(\mu_{\omega,0}(H_{\omega,\varepsilon}))$, where $\mu_{\omega,0}$ is the unique equivariant random measure (and equilibrium state) for the original closed random dynamics. Our new machinery is then deployed to create a spectral approach for a quenched extreme value theory that considers random dynamics and random observations. An extreme value law is derived using the first-order terms θ_ω. Further, in the setting of random piecewise expanding interval maps, we establish the existence of random equilibrium states and conditionally invariant measures for random open systems with small holes via a random perturbative approach, in contrast to the cone-based arguments of the first chapter. Finally we prove quenched statistical limit theorems for random equilibrium states arising from contracting potentials. We illustrate all of the above theory with a variety of examples.

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

Contents

Chapter 0. Introduction 5
0.1. Preliminaries on random open systems 14

Chapter 1. Thermodynamic formalism for random interval maps with holes 21
1.1. Preliminaries of random interval maps with holes 21
1.2. Random conditionally invariant probability measures 25
1.3. Functionals and partitions 27
1.4. Random Birkhoff cones and Hilbert metrics 31
1.5. Lasota-Yorke inequalities 33
1.6. Cone invariance on good fibers 39
1.7. Density estimates and cone invariance on bad fibers 41
1.8. Further properties of Λ_ω 47
1.9. Finding finite diameter images 50
1.10. Conformal and invariant measures 52
1.11. Decay of correlations 63
1.12. Expected pressures and escape rates 67
1.13. Equilibrium states 69
1.14. Bowen's formula 69
1.15. Examples 74

Chapter 2. Perturbation formulae for quenched random dynamics with applications to open systems and extreme value theory 82
2.1. Sequential perturbation theorem 82
2.2. Random open systems 88
2.3. Quenched perturbation theorem and escape rate asymptotics for random open systems 90
2.4. Quenched extreme value law 98
2.5. Quenched thermodynamic formalism for random open interval maps via perturbation 111
2.6. Limit theorems 122
2.7. Examples 130

Chapter 2 Appendices 139
2.A. A version of [33] with general weights 139
2.B. A summary of checks that relevant results from [22] can be applied to BV_1 141
2.C. A var–$\nu_{\omega,0}(|\cdot|)$ Lasota–Yorke inequality 142
2.D. Proof of Claim (6) of Theorem 2.5.12 144

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

Acknowledgments 146

Bibliography 147

Glossary of Assumptions 151

Index 158

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

CHAPTER 0

Introduction

In this paper we develop a quenched thermodynamic formalism for random open dynamical systems, and their perturbations. This work may be seen as a successor to [3], which was devoted to the construction of conformal measures and equilibrium states for random potentials associated to countably branched random maps of the unit interval. We now extend such a formalism to the more challenging setting of random *open* dynamical systems. A random dynamical system is qualified to be open if it contains holes which terminate trajectories once they enter.

The current paper is divided into two chapters. In the first chapter, the random holes are allowed to be large ensuring that the asymptotic dynamics take place on a surviving set and the objective will be to define equilibrium states and their variational principles. A key step in this program will be the construction of (random) conditionally invariant probability measures. Moreover we will give an explicit formula for the escape rate. In the second chapter, the introduction of small random holes are seen as a perturbation of a closed transfer operator. We will first develop a perturbation theory for (nonautonomous) cocycles of transfer operators, which we will then use to study recurrence properties in the small set with particular attention to extreme value theory. Then we will perturb with the introduction of small random holes with measure tending to zero.

Summary of Chapter 1. Thermodynamic formalism for random interval maps with holes

Deterministic closed transitive dynamics $T : [0,1] \to [0,1]$ with enough expansivity enjoy a "thermodynamic formalism": the transfer operator with a sufficiently regular potential φ has a unique absolutely continuous invariant measure (ACIM) μ, absolutely continuous with respect to the conformal measure ν. Furthermore, μ arises as an equilibrium state, i.e. a maximiser of the sum of the integral of the potential φ and the metric entropy $h(\mu)$.

Continuing with the deterministic setting, if one introduces a hole $H \subset [0,1]$, the situation becomes considerably more complicated. In the simplest case where the potential is the usual geometric potential $\varphi = -\log|T'|$, because of the lack of mass conservation, one expects at best an absolutely continuous *conditionally* invariant measure (ACCIM) μ, conditioned according to survival from the infinite past. Absolute continuity is again with respect to a conformal measure ν, which is supported on the survivor set X_∞, the set of points whose infinite forward trajectories remain in $[0,1]$. Early work on the existence of the ACCIM and exponential convergence of non-equilibrium densities, includes [64, 20]. The paper [53] handles general potentials that are contracting [54] for the closed system, demonstrating exponential decay for μ. There has been further work on the Lorentz gas

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

and billiards [23, 24], intermittent maps [25, 27], and multimodal maps [26, 28]. In the setting of diffeomorphisms with SRB measures, following the introduction of a hole, relations between escape rates and the pressures have been studied in [29].

Looking to the fractal dimension of the surviving set X_∞, the machinery of thermodynamic formalism was first employed by Bowen [13] to find the Hausdorff dimension of the limit sets of quasi-Fuchsian groups in terms of the pressure function, and then pioneered in the setting of open dynamical systems in [63].

In the random setting, repeated iteration of a single deterministic map is replaced with the composition of maps $T_\omega : X \to X$ drawn from a collection $\{T_\omega\}_{\omega\in\Omega}$. A driving map $\sigma : \Omega \to \Omega$ on a probability space $(\Omega, \mathscr{F}, m)$ creates a map cocycle $T^n_\omega := T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega$. The authors recently developed a complete, quenched thermodynamic formalism for random, countably-branched, piecewise monotonic interval maps [3], enabling the treatment of discontinuous, non-Markov T_ω.

The situation of random open dynamics is relatively untouched. For a single piecewise expanding map $T : [0, 1] \to [0, 1]$ with holes H_ω randomly chosen in an i.i.d. fashion, [9] consider escape rates for the annealed (averaged) transfer operator in the small hole limit (the Lebesgue measure of the H_ω goes to zero). In a similar setting, now assuming T to be Markov and considering non-vanishing holes, [8] show existence of equilibrium states, again for the annealed transfer operator. For the first time in [41] the authors consider escape rates for quenched random open interval maps where they are able to show that the escape rate is bounded above by the Lyapunov exponent of a Perron-Frobenius cocycle. In [5], the authors consider random, full-branched interval maps with negative Schwarzian derivative. The maps are allowed to have critical points, but the partition of monotonicity and holes, made up of finitely many open intervals, are fixed and non-random. In this setting the existence of a unique invariant random probability measure is proven as well as a formula for the Hausdorff dimension of the surviving set. In our current setting, we do not allow the existence of critical points, however our maps may have non-full branches, and our partitions of monotonicity as well as our holes are allowed to vary randomly from fiber to fiber.

Sequential systems with holes have been considered in [42], where a cocycle T_ω of open maps is generated by a single ω orbit. The maps (which include the hole) must be chosen in a small neighborhood of a fixed map (with hole), in contrast to our setting where our cocycle may include very different maps. Moreover in [42], Lebesgue is used as a reference measure and the specific potential $-\log|\det DT|$ is used. The theory is developed for uniformly expanding maps in higher dimensions and the main goal is to establish the "conditional memory loss", a concept analogous to exponential decay of correlations for closed dynamics.

Other related work includes [2], which considered non-transitive random interval maps with holes. In [2] we proved a complete thermodynamic formalism for random interval maps (i) containing sufficiently many full branches and (ii) random potentials satisfying a strong contracting potential assumption. In the current work we treat maps that contain no full branches and potentials that satisfy a significantly weaker contracting potential assumption at the cost of assuming the closed system satisfies a mild covering condition. This allows us to obtain results for a large class of random Lasota-Yorke maps (Section

1.15.2). Furthermore, the current work and that of [2] are complementary, and neither work generalizes the other.

In Chapter 1 of the present paper, we establish a full, quenched thermodynamic formalism for piecewise monotonic random dynamics with general potentials and general driving—the random driving σ can be any invertible ergodic process on Ω. We begin with the random closed dynamics dealt with in [3]: piecewise monotonic interval maps satisfying a random covering condition; we have no Markovian assumptions, our maps may have discontinuities and may lack full branches. The number of branches of our maps need not be uniformly bounded above in ω and our potentials φ_ω need not be uniformly bounded below or above in ω. To this setting we introduce random holes H_ω and formulate sufficient conditions that guarantee a random conformal measure ν_ω and corresponding equivariant measure μ_ω supported on the random survivor set $X_{\omega,\infty}$, and a random ACCIM η_ω supported on H_ω^c. These augment the notion of a random contracting potential [3] with accumulation rates of contiguous "bad" intervals (with zero conformal measure), and extend similar constructions of [53] to the random situation.

To establish the existence of the family of measures $(\nu_\omega)_{\omega\in\Omega}$, we follow the limiting functional approach of [54, 53] by defining a random functional Λ_ω which is a limit of ratios of transfer operators and then showing that Λ_ω may be identified with the open conformal measure ν_ω. This technique improves on the approach of [3], which uses the Schauder-Tichonov Fixed Point Theorem to prove the existence of $\nu_{\omega,0}$, by eliminating the extra steps necessary to show that the family $(\nu_\omega)_{\omega\in\Omega}$ is measurable with respect to m. Several steps are needed to achieve the construction of the conformal random measures. We start in Section 1.4 by giving background material on Birkhoff cone techniques and the construction of our random cones. In Section 1.5 we develop several random Lasota-Yorke type inequalities in terms of the variation and the random functional Λ_ω. Section 1.6 sees the construction of a large measure set of "good" fibers $\Omega_G \subseteq \Omega$ for which we obtain cone invariance at a uniform time step, and in Section 1.7 we show that the remaining "bad" fibers occur infrequently and behave sufficiently well. In Section 1.8 we collect further properties of the random functional Λ, which are then used in Section 1.9 to construct a large measure set of fibers $\Omega_F \subseteq \Omega$ for which we obtain cone contraction with a finite diameter image in a random time step. Using Hilbert metric contraction arguments, Section 1.10 collects together the fruits of Sections 1.6-1.9 to prove our main technical lemma (Lemma 1.10.1), which is then used to (i) obtain the existence of a random density ϕ, (ii) prove the existence of a unique non-atomic random conformal measure ν, and (iii) a random T-invariant measure μ which is absolutely continuous with respect to ν. All these facts are collected in our first result (detailed statements can be found in Corollary 1.10.6, Proposition 1.10.8, Lemma 1.10.9, Lemma 1.10.11, Proposition 1.10.14, Proposition 1.11.5, Lemma 1.11.7, and Theorem 1.13.2).

Theorem A. Given a random open system $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ (see Section 0.1.1) satisfying (T1)-(T3), (LIP), (GP), (A1)-(A2), and (Q1)-(Q3) (see Sections 1.1 and 1.3), the following hold.

(1) There exists a unique random probability measure $\nu \in \mathcal{P}_\Omega(\Omega \times I)$ supported in $\mathcal{X}_\infty$ such that

$$\nu_{\sigma(\omega)}(\mathcal{L}_\omega f) = \lambda_\omega \nu_\omega(f),$$

for each $f \in \mathrm{BV}(I)$, where

$$\lambda_\omega := \nu_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega).$$

Furthermore, we have that $\log \lambda_\omega \in L^1(m)$.

(2) There exists a function $\phi \in \mathrm{BV}_\Omega(I)$ such that $\nu(\phi) = 1$ and for m-a.e. $\omega \in \Omega$ we have

$$\mathcal{L}_\omega \phi_\omega = \lambda_\omega \phi_{\sigma(\omega)}.$$

Moreover, ϕ is unique modulo ν.

(3) The measure $\mu := \phi\nu$ is a T-invariant and ergodic random probability measure supported in $\mathcal{X}_\infty$ and the unique relative equilibrium state for the potential φ satisfying the following variational principle:

$$\mathcal{E}P(\varphi) = h_\mu(T) + \int_{\Omega\times I} \varphi \, d\mu = \sup_{\eta \in \mathcal{P}^H_{T,m}(\Omega\times I)} \left(h_\eta(T) + \int_{\Omega\times I} \varphi \, d\eta \right).$$

Furthermore, for each $\eta \in \mathcal{P}^H_{T,m}(\Omega)$ (the set of all random T-invariant Borel probability measures supported on $\Omega \times I$) different from μ we have that

$$h_\eta(T) + \int_{\Omega\times I} \varphi \, d\eta < h_\mu(T) + \int_{\Omega\times I} \varphi \, d\mu.$$

(4) There exists a random conditionally invariant probability measure η absolutely continuous with respect to ν_0, which is supported on $\cup_{\omega\in\Omega}(\{\omega\} \times I \backslash H_\omega)$, and whose disintegrations are given by

$$\eta_\omega(f) := \frac{\nu_{\omega,0}\left(\mathbb{1}_{H^c_\omega} \phi_\omega f\right)}{\nu_{\omega,0}\left(\mathbb{1}_{H^c_\omega} \phi_\omega\right)}$$

for all $f \in \mathrm{BV}(I)$.

We also show that the operator cocycle is quasi-compact.

Theorem B. With the same hypotheses as Theorem A, for each $f \in \mathrm{BV}(I)$ there exists a measurable function $\Omega \ni \omega \mapsto D(\omega) \in (0, \infty)$ and $\kappa \in (0, 1)$ such that for m-a.e. $\omega \in \Omega$ and all $n \in \mathbb{N}$ we have

$$\| \left(\lambda^n_\omega\right)^{-1} \mathcal{L}^n_\omega f - \nu_\omega(f)\phi_{\sigma^n(\omega)}\|_\infty \leq D(\omega)\|f\|_\infty \kappa^n.$$

Furthermore, for all $A \in \mathscr{B}$ we have

$$\left|\nu_{\omega,0}\left(T^{-n}_\omega(A) \,|\, X_{\omega,n}\right) - \eta_{\sigma^n(\omega)}(A)\right| \leq D(\omega)\kappa^n,$$

and all $f \in \mathrm{BV}$

$$\left| \frac{\eta_\omega\left(f|_{X_{\omega,n}}\right)}{\eta_\omega\left(X_{\omega,n}\right)} - \mu_\omega(f) \right| \leq D(\omega)\|f\|_\infty \kappa^n.$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

For the proof of Theorem B, as well as a more general statement, see Theorem 1.11.2 and Corollary 1.11.8. From quasi-compactness we easily deduce the exponential decay of correlations for the invariant measure μ.

THEOREM C. With the same hypotheses as Theorem A, there exists a measurable function $\Omega \ni \omega \mapsto C(\omega) \in (0, \infty)$ such that for every $h \in \mathrm{BV}(I)$, every $f \in L^1(\mu)$, every $\varkappa \in (\kappa, 1)$, with κ as in Theorem B, every $n \in \mathbb{N}$, and for m-a.e. $\omega \in \Omega$ we have

$$\left|\mu_\omega\left(\left(f_{\sigma^n(\omega)} \circ T_\omega^n\right) h\right) - \mu_{\sigma^n(\omega)}(f_{\sigma^n(\omega)})\mu_\omega(h)\right| \leq C(\omega)\|f_{\sigma^n(\omega)}\|_{L^1(\mu_{\sigma^n(\omega)})}\|h\|_\infty \varkappa^n.$$

Theorem C (stated in more detail in Theorem 1.11.3) is proven in Section 1.11. The presence of holes leads naturally to introduce the notion of fiberwise escape rate $R(\rho_\omega)$ of the measure ρ_ω from the holes; the definition in the random setting in given in 1.12.3. We will show that the escape rate is constant m-almost everywhere and is given in terms of the closed and open expected pressures, denoted with $\mathcal{E}P(\varphi)$ for a given potential φ, which are properly defined in Definition 1.12.1.

THEOREM D. With the same hypotheses as Theorem A, for m-a.e. $\omega \in \Omega$ we have that

$$R(\nu_{\omega,0}) = R(\eta_\omega) = \mathcal{E}P(\varphi_0) - \mathcal{E}P(\varphi) = \int_\Omega \log \frac{\lambda_{\omega,0}}{\lambda_\omega}\, dm(\omega).$$

Theorem D is proven in Section 1.12. The expected pressure function is further developed and used to prove a Bowen's formula type result for the Hausdorff dimension of the survivor set $X_{\omega,\infty}$ for m-a.e. $\omega \in \Omega$ in Section 1.14. This requires us to introduce the bounded distortion property for a given potential, large images of the map T, and large images with respect to the hole H; see Section 1.14 for the full definitions of these terms. We therefore have:

THEOREM E. With the same hypotheses as Theorem A, we additionally suppose that

$$\int_\Omega \log \inf |T'_\omega|\, dm(\omega) > 0$$

and $g_0 = 1/|T'|$ has bounded distortion. Then there exists a unique $h \in [0, 1]$ such that $\mathcal{E}P(t) > 0$ for all $0 \leq t < h$ and $\mathcal{E}P(t) < 0$ for all $h < t \leq 1$.

Furthermore, if T has large images and large images with respect to H, then for m-a.e. $\omega \in \Omega$

$$\mathrm{HD}(X_{\omega,\infty}) = h,$$

where $\mathrm{HD}(A)$ denotes the Hausdorff dimension of the set A.

The proof of Theorem E appears in Section 1.14.

In Section 1.15 we apply our general theory to a large class of random β-transformations with random holes as well as a general random Lasota-Yorke maps with random holes. In fact, our theory applies to all of the finitely-branched examples discussed in [3] (this includes maps which are non-uniformly expanding or have contracting branches which appear infrequently enough that we still maintain on-average expansion) when suitable conditions are put on the holes H_ω. This includes the case where H_ω is composed of finitely many intervals and the number of connected components of H_ω is log-integrable with respect to m.

Summary of Chapter 2. Perturbation formulae for quenched random dynamics with applications to open systems and extreme value theory

The spectral approach to studying deterministic closed dynamical systems $T : X \to X$ on a phase space X, centers on the analysis of a transfer operator $\mathcal{L} : \mathcal{B}(X) \to \mathcal{B}(X)$, given by $\mathcal{L}f(y) = \sum_{x \in T^{-1}y} e^{\varphi(x)} f(x)$, for f in a suitable Banach space $\mathcal{B}(X)$ and for suitable potential function $\varphi : X \to \mathbb{R}$. If the map T is *covering* and the potential function is *contracting* in the sense of [54] (or similarly if $\sup \varphi < P(T)$ as in [31, 30] or if $\sup \varphi - \inf \varphi < h_{top}(T)$ as in [44, 17]), then one obtains the existence of an equilibrium state μ, with associated conformal measure ν, with the topological pressure $P(T)$ and the density of equilibrium state $d\mu/d\nu$ given by the logarithm of the leading (positive) eigenvalue λ of $\mathcal{L}$ and the corresponding positive eigenfunction h, respectively. The map T exhibits an exponential decay of correlations with respect to ν and μ.

Keller and Liverani [49] showed that the leading eigenvalue λ and eigenfunction h of $\mathcal{L}$ vary continuously with respect to certain small perturbations of $\mathcal{L}$. One example of such a perturbation is the introduction of a small hole $H \subset X$. The set of initial conditions of trajectories that never land in H is the survivor set X_∞. For small holes, specialising to Lasota-Yorke maps of the interval, Liverani and Maume-Dechamps [53] apply the perturbation theory of [49] to obtain the existence of a unique conformal measure ν and an absolutely continuous *conditionally* invariant measure μ, with density $h \in \mathrm{BV}([0,1])$. The leading eigenvalue λ is interpretable as an *escape rate*, and the open system displays an exponential decay of correlations with respect to ν and μ.

To obtain finer information on the behaviour of λ with respect to perturbation size, particularly in the situation where the perturbation is not smooth, such as perturbations arising from the introduction of a hole, one requires some additional control on the perturbation. Keller and Liverani [50] develop abstract conditions on $\mathcal{L}$ and its perturbations $\mathcal{L}_\varepsilon$ to ensure good first-order behaviour with respect to the perturbation size. Following [50], several authors [35, 38, 59, 16] have used the Keller-Liverani [49] perturbation theory to obtain similar first-order behaviour of the escape rate with respect to the perturbation size for open systems in various settings.

This "linear response" of λ is exploited in Keller [48] to develop an elegant spectral approach to deriving an exponential extreme value law to describe likelihoods of observing extreme values from evaluating an observation function $h : X \to \mathbb{R}$ along orbits of T. In particular, the $N \to \infty$ limiting law of

$$\nu\left(\left\{x \in X : h(T^j(x)) \le z_N, j = 0, \dots, N-1\right\}\right), \tag{0.0.1}$$

where the thresholds z_N are chosen so that $\lim_{N\to\infty} N\mu(\{x : h(x) > z_N\}) \to t$ for some $t > 0$, is shown to be exponential. The spectral approach of [48] also provides a relatively explicit expression for the limit of (0.0.1), namely

$$\lim_{N\to\infty} \nu\left(\left\{x \in X : h(T^j(x)) \le z_N, j = 0, \dots, N-1\right\}\right) = \exp(-t\theta_0),$$

where θ_0 is the extremal index.

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

In Chapter 2 we begin with sequential composition of linear operators $\mathcal{L}^n_{\omega,0} := \mathcal{L}_{\sigma^{n-1}\omega,0} \circ \cdots \circ \mathcal{L}_{\sigma\omega,0} \circ \mathcal{L}_{\omega,0}$, where $\sigma : \Omega \to \Omega$ is an invertible map on a configuration set Ω. The driving σ could also be an ergodic map on a probability space $(\Omega, \mathcal{F}, m)$. We then consider a family of perturbed cocycles $\mathcal{L}^n_{\omega,\varepsilon} := \mathcal{L}_{\sigma^{n-1}\omega,\varepsilon} \circ \cdots \circ \mathcal{L}_{\sigma\omega,\varepsilon} \circ \mathcal{L}_{\omega,\varepsilon}$, where the size of the perturbation $\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon}$ is quantified by the value $\Delta_{\omega,\varepsilon}$ (Definition 2.1.1). Our first main result is an abstract quenched formula (Theorem 2.1.2) for the Lyapunov multipliers $\lambda_{\omega,0}$ up to first order in the size of the perturbation $\Delta_{\omega,\varepsilon}$ of the operators $\mathcal{L}_{\omega,0}$. This quenched random formula generalizes the main abstract first-order formula in [**50**] stated in the case of a single deterministic operator $\mathcal{L}_0$.

Theorem F. Suppose that assumptions (P1)-(P9) hold (see Section 2.1). If $\Delta_{\omega,\varepsilon} > 0$ for all $\varepsilon > 0$ then for m-a.e. $\omega \in \Omega$:

$$\lim_{\varepsilon\to 0} \frac{\lambda_{\omega,0} - \lambda_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}} = 1 - \sum_{k=0}^{\infty} \hat{q}^{(k)}_{\omega,0} =: \theta_{\omega,0}.$$

The existence of a random quenched equilibrium state, conformal measure, escape rates, and exponential decay of correlations is established in Chapter 1 for relatively large holes, generalizing the large-hole constructions of [**53**] for a single deterministic map T to the random setting with general driving. In contrast, the focus of Chapter 2 is to establish a random quenched analogue of the results of [**53**], [**50**], and [**48**] discussed above. To this end, we let $\mathcal{L}_{\omega,\varepsilon}$ be the transfer operator for the open map T_ω with a hole $H_{\omega,\varepsilon}$ introduced in X, namely $\mathcal{L}_{\omega,\varepsilon}(f) = \mathcal{L}_\omega(\mathbb{1}_{X\setminus H_{\omega,\varepsilon}} f)$. Our second collection of main results is a quenched formula for the derivative of $\lambda_{\omega,\varepsilon}$ with respect to the sample invariant probability measure of the hole $\mu_{\omega,0}(H_{\omega,\varepsilon})$ (Theorem 2.3.6), as well as a quenched formula for the derivative of the fiberwise escape rate $R_\varepsilon(\mu_{\omega,0})$ with respect to the sample invariant probability measure of the hole $\mu_{\omega,0}(H_{\omega,\varepsilon})$ (Corollary 2.3.9).

Corollary G. If $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ is a random open system (see Section 0.1.1) with $\mu_{\omega,0}(H_\omega) > 0$ for all $\varepsilon > 0$ and (C1)–(C8) hold (see Section 2.3), then for m-a.e. $\omega \in \Omega$:

$$\lim_{\varepsilon\to 0} \frac{1 - \lambda_{\omega,\varepsilon}/\lambda_{\omega,0}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} = 1 - \sum_{k=0}^{\infty} \hat{q}^{(k)}_{\omega,0} =: \theta_{\omega,0} \tag{0.0.2}$$

and

$$R_\varepsilon(\mu_{\omega,0}) = \int_\Omega \log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}\, dm(\omega).$$

In addition, if (2.3.12) holds and the $\mu_{\omega,0}$-measures of the random holes scale with ε according to (2.3.13), then for m-a.e. $\omega \in \Omega$

$$\lim_{\varepsilon\to 0} \frac{R_\varepsilon(\mu_{\omega,0})}{\mu_{\omega,0}(H_{\omega,\varepsilon})} = \int_\Omega \theta_{\omega,0}\, dm(\omega).$$

To generalize to the random setting the rescaled distribution of the maxima given by (0.0.1), we now consider the real-valued random observables h_ω defined on the phase space X and construct a process $h_{\sigma^j\omega} \circ T^j_\omega$. We are interested in determining the limiting law of

$$\mu_{\omega,0}\left(\left\{x \in X : h_{\sigma^j\omega}(T^j_\omega(x)) \le z_{\sigma^j\omega,N}, j = 0, \ldots, N-1\right\}\right), \tag{0.0.3}$$

where $\{z_{\sigma^j\omega,N}\}_{0\le j\le N-1}$ is a collection of real-valued thresholds. The sample probability measure $\mu_{\omega,0}$ enjoys the equivariance property $T_\omega^*\mu_{\omega,0}=\mu_{\sigma\omega,0}$, however the process $h_{\sigma^j\omega}\circ T_\omega^j$ is not stationary in the probabilistic sense, which makes the theory slightly more difficult.

The first approach to non-stationary extreme value theory (EVT) was given under convenient assumptions, by Hüsler in [**46, 47**]. He was able to recover the usual extremal behaviour seen for i.i.d. or stationary sequences under Leadbetter's conditions [**51**], namely (i) guaranteed mixing properties for the probability measure governing the process and (ii) that the exceedances should appear scattered through the time period under consideration. Hüsler's results can not be applied in the dynamical systems setting because his uniform bounds on the control of the exceedances are not satisfied for deterministic, random, or sequential compositions of maps. The first contribution dealing explicitly with extreme value theory for random and sequential systems is the paper [**39**]; see also [**37**] for an application to point processes. These works were an adaptation of Leadbetter's conditions and Hüsler's approach: let us call them the probabilistic approach to extreme value theory, to distinguish it from the spectral and perturbative approach used in the current paper.

As in the deterministic case, in order to avoid a degenerate limit distribution, one should conveniently choose the thresholds $z_{\sigma^j\omega,N}$. Hüsler proved convergence to the Gumbel's law if for some $0<t<\infty$ we have convergence of the sum

$$\sum_{j=0}^{N-1}\mu_{\omega,0}\left(h_{\sigma^j\omega}(T_\omega^j(x))>z_{\sigma^j\omega,N}\right)\to t \tag{0.0.4}$$

for m-a.e. ω. In our current framework we will additionally allow the positive number t to be any positive random variable in $L^\infty(m)$. The nonstationary theory developed in [**39**] for quenched random processes, has the further restrictions that the observation function is fixed (ω-independent), and the thresholds z_N (like the scaling t) are just real numbers, and requires the obvious restricted equivalent of (0.0.4). In our framework the observation function h_ω, the scaling t_ω, and the thresholds $z_{\omega,N}$ may all be random (but need not be). We generalize and simplify the requirement (0.0.4) to

$$N\mu_{\omega,0}\left(h_\omega(x)>z_{\omega,N}\right)=t_\omega+\xi_{\omega,N}, \tag{0.0.5}$$

where the scaling t may be a random variable $t\in L^\infty(m)$ and the "errors" $\xi_{\omega,N}$ satisfy (i) $\lim_{N\to\infty}\xi_{\omega,N}=0$ a.e., and (ii) $|\xi_{\omega,N}|\le W<\infty$ for a.e. ω and all sufficiently large N.

We provide a more detailed discussion of the relationship between the conditions (0.0.4) and (0.0.5) at the end of Section 2.4.

In summary, we derive a spectral approach for a quenched random extreme value law, where the dynamics T_ω is random, the observation functions h_ω can be random, the thresholds controlling what is an extreme value can be random, and the scalings of the likelihoods of observing extreme values can be random, all controlled by general invertible ergodic driving. Moreover, we obtain a formula for the explicit form of Gumbel law for the extreme value distribution. This leads to our main extreme value theory result (stated in detail later as Theorem 2.4.5):

Theorem H. For a random open system $(\Omega,m,\sigma,\mathcal{J}_0,T,\mathcal{B},\mathcal{L}_0,\nu_0,\phi_0,H_\varepsilon)$ (see Section 0.1.1), assuming (C1'), (C2), (C3), (C4'), (C5'), (C7'), (C8), and (S) (see Sections 2.3 and

2.4), for m-a.e. $\omega \in \Omega$ one has

$$\begin{aligned}\lim_{N\to\infty} \nu_{\omega,0}\left(x \in X : h_{\sigma^j\omega}(T^j_\omega(x)) \le z_{\sigma^j\omega,N}, j = 0,\dots,N-1\right)\\ = \lim_{N\to\infty} \mu_{\omega,0}\left(x \in X : h_{\sigma^j\omega}(T^j_\omega(x)) \le z_{\sigma^j\omega,N}, j = 0,\dots,N-1\right)\\ = \exp\left(-\int_\Omega t_\omega\theta_{\omega,0}\, dm(\omega)\right),\end{aligned}$$

where $\nu_{\omega,0}$ and $\mu_{\omega,0}$ are the random conformal measure and the random invariant measure, respectively, for our random dynamics, t_ω is a random scaling function, and $\theta_{\omega,0}$ is an ω-local extremal index corresponding to the quantity given in Corollary G.

This result generalizes the spectral approach to extreme value theory in [48], for a single map T, single observation function h, single scaling, and single sequence of thresholds.

Given a family of random holes $\mathcal{H}_{\omega,N} := \{H_{\sigma^j\omega,\varepsilon_N}\}_{j\ge 0}$, one can define the first (random) hitting time to a hole, starting at initial condition x and random configuration ω:

$$\tau_{\omega,\mathcal{H}_{\omega,N}}(x) := \inf\{k \ge 1, T^k_\omega(x) \in H_{\sigma^k\omega,\varepsilon_N}\}.$$

When this family of holes shrink with increasing N according to Condition (S) (see Section 2.4.3), Theorem H provides a description of the statistics of random hitting times, scaled by the measure of the holes (see Theorem 2.4.7).

Corollary I. For a random open system $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ (see Section 0.1.1), assume (C1'), (C2), (C3), (C4'), (C5'), (C7'), (C8), and (S) (see Sections 2.3 and 2.4). For m-a.e. $\omega \in \Omega$ one has

$$\lim_{N\to\infty} \mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega\right) = \exp\left(-\int_\Omega t_\omega\theta_{\omega,0}\, dm(\omega)\right). \tag{0.0.6}$$

By assuming some additional uniformity in ω on the maps T_ω we use a recent random perturbative result [22] to obtain a complete quenched thermodynamic formalism. The following existence result extends Theorem C of [53], which concerned inserting a single small hole into the phase space of a single deterministic map T, to the situation of random map cocycles with small random holes with the random process controlled by general invertible ergodic driving σ.

Theorem J. Suppose that (E1)–(E9) (see Section 2.5) hold for the random open system $(\Omega, m, \sigma, [0,1], T, \mathrm{BV}([0,1]), \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ (see Section 0.1.1). Then for each $\varepsilon > 0$ sufficiently small there exists a unique random T-invariant probability measure $\mu_\varepsilon = \{\mu_{\omega,\varepsilon}\}_{\omega\in\Omega}$ with $\mathrm{supp}(\mu_{\omega,\varepsilon}) \subseteq X_{\omega,\infty,\varepsilon}$. Furthermore, μ_ε is the unique relative equilibrium state for the random open system and satisfies a forward and backward exponential decay of correlations. In addition, there exists a random absolutely continuous (with respect to $\{\nu_{\omega,0}\}_{\omega\in\Omega}$) conditionally invariant probability measure $\varrho_\varepsilon = \{\varrho_{\omega,\varepsilon}\}_{\omega\in\Omega}$ with $\mathrm{supp}(\varrho_{\omega,\varepsilon}) \subseteq [0,1]\backslash H_{\omega,\varepsilon}$ and density function $\psi_{\omega,\varepsilon} \in \mathrm{BV}([0,1])$.

For a more explicit statement of Theorem J as well as the relevant assumptions and definitions see Section 2.5 and Theorem 2.5.12.

In Section 2.6 we prove some quenched limit theorems for closed random dynamics. These limit results are new for more general potentials and their associated equilibrium states. We use two approaches. The first is based on the perturbative technique developed in [**32**], which generalizes the Nagaev-Guivarc'h method to random cocycles. This technique establishes a relation between a suitable twisted operator cocycle and the distribution of the random Birkhoff sums. As a consequence, it is possible to get quenched versions of the large deviation principle, the central limit theorem, and the local central limit theorem. The second approach invokes the martingale techniques previously used in the quenched random setting in [**33**]. We obtain the almost sure invariance principle (ASIP) for the equivariant measure $\mu_{\omega,0}$, which also implies the central limit theorem and the law of iterated logarithms, a general bound for large deviations and a dynamical Borel-Cantelli lemma. In addition, using the Sprindzuk theorem we are able to obtain a quenched shrinking target result.

We conclude in Section 2.7 with several explicit examples of Theorems F–J. We start in Example 2.7.1 with the weight $1/T'_\omega$ for a family of random maps, random scalings, and random observations h_ω with a common extremum location in phase space, which is a common fixed point of the T_ω. The special cases of a fixed map T on the one hand, and a fixed scaling t on the other, are also considered. The same calculations can be extended to observation functions with common extrema on a periodic orbit common to all T_ω. Next in Example 2.7.2 we consider the more difficult case where orbits are distributed according to equilibrium states of a general geometric weight $|DT_\omega|^{-r}$, using random β-maps and random observation functions h_ω with a common extremum at $x = 0$. Example 2.7.3 investigates a fixed map T with random observation functions h_ω with extrema in a shrinking neighbourhood of a fixed point of T, where the neighbourhood lacks the symmetry of Example 2.7.1. In the last example, Example 2.7.4, we again consider random maps T_ω with random observations h_ω, but now the maxima of the observations are not related to fixed or periodic points of T_ω.

Though we apply our results to the setting of random interval maps our results apply equally well to other random settings including random subshifts [**11, 56**], random distance expanding maps [**57**], random polynomial systems [**15**], random transcendental maps [**58**]. In addition, Theorem F (as well as (0.0.2) of Corollary G) applies to sequential and semi-group settings including [**4, 62**]. See Remark 2.5.13 for a sequential version of Theorem J.

0.1. Preliminaries on random open systems

In this section we introduce the general setup of random open systems. We begin with a probability space $(\Omega, \mathscr{F}, m)$ and an ergodic, invertible map $\sigma : \Omega \to \Omega$ which preserves the measure m, i.e.

$$m \circ \sigma^{-1} = m.$$

We will refer to the tuple $(\Omega, \mathscr{F}, m, \sigma)$ as the *base dynamical system*. For each $\omega \in \Omega$, let $\mathcal{J}_{\omega,0}$ be a closed subset of a complete metrisable space X such that the map

$$\Omega \ni \omega \longmapsto \mathcal{J}_{\omega,0}$$

is a closed random set, i.e. $\mathcal{J}_{\omega,0} \subseteq X$ is closed for each $\omega \in \Omega$ and the map $\omega \mapsto \mathcal{J}_{\omega,0}$ is measurable (see [21]), and we consider the maps

$$T_\omega : \mathcal{J}_{\omega,0} \to \mathcal{J}_{\sigma\omega,0}.$$

By $T_\omega^n : \mathcal{J}_{\omega,0} \to \mathcal{J}_{\sigma^n\omega,0}$ we mean the n-fold composition

$$T_{\sigma^n\omega} \circ \cdots \circ T_\omega : \mathcal{J}_{\omega,0} \to \mathcal{J}_{\sigma^n\omega,0}.$$

Given a set $A \subseteq \mathcal{J}_{\sigma^n\omega,0}$ we let

$$T_\omega^{-n}(A) := \{x \in \mathcal{J}_{\omega,0} : T_\omega^n(x) \in A\}$$

denote the inverse image of A under the map T_ω^n for each $\omega \in \Omega$ and $N \geq 1$. Now let

$$\mathcal{J}_0 := \bigcup_{\omega\in\Omega} \{\omega\} \times \mathcal{J}_{\omega,0} \subseteq \Omega \times X,$$

and define the induced skew-product map $T : \mathcal{J}_0 \to \mathcal{J}_0$ by

$$T(\omega, x) = (\sigma\omega, T_\omega(x)).$$

Let $\mathscr{B}$ denote the Borel σ-algebra of X and let $\mathscr{F} \otimes \mathscr{B}$ be the product σ-algebra on $\Omega \times X$. Throughout the text we denote Lebesgue measure by Leb. We suppose the following:

(M1) The map $T : \mathcal{J}_0 \to \mathcal{J}_0$ is measurable with respect to $\mathscr{F} \otimes \mathscr{B}$.

Definition 0.1.1. A measure μ on $\Omega \times X$ with respect to the product σ-algebra $\mathscr{F} \otimes \mathscr{B}$ is said to be *random measure* relative to m if it has marginal m, i.e. if

$$\mu \circ \pi_1^{-1} = m.$$

The disintegrations $\{\mu_\omega\}_{\omega\in\Omega}$ of μ with respect to the partition $(\{\omega\} \times X)_{\omega\in\Omega}$ satisfy the following properties:

(1) For every $B \in \mathscr{B}$, the map $\Omega \ni \omega \longmapsto \mu_\omega(B) \in [0, \infty]$ is measurable,
(2) For m-a.e. $\omega \in \Omega$, the map $\mathscr{B} \ni B \longmapsto \mu_\omega(B) \in [0, \infty]$ is a Borel measure.

We say that the random measure $\mu = \{\mu_\omega\}_{\omega\in\Omega}$ is a *random probability measure* if for m-a.e. $\omega \in \Omega$ the fiber measure μ_ω is a probability measure. Given a set $Y = \cup_{\omega\in\Omega} \{\omega\} \times Y_\omega \subseteq \Omega \times X$, we say that the random measure $\mu = \{\mu_\omega\}_{\omega\in\Omega}$ is supported in Y if $\mathrm{supp}(\mu) \subseteq Y$ and consequently $\mathrm{supp}(\mu_\omega) \subseteq Y_\omega$ for m-a.e. $\omega \in \Omega$. We let $\mathcal{P}_\Omega(Y)$ denote the set of all random probability measures supported in Y. We will frequently denote a random measure μ by $\{\mu_\omega\}_{\omega\in\Omega}$.

The following proposition from Crauel [21], shows that a random probability measure $\{\mu_\omega\}_{\omega\in\Omega}$ on $\mathcal{J}_0$ uniquely identifies a probability measure on $\mathcal{J}_0$.

Proposition 0.1.2 ([21], Propositions 3.3). *If $\{\mu_\omega\}_{\omega\in\Omega} \in \mathcal{P}_\Omega(\mathcal{J}_0)$ is a random probability measure on $\mathcal{J}_0$, then for every bounded measurable function $f : \mathcal{J}_0 \to \mathbb{R}$, the function*

$$\Omega \ni \omega \longmapsto \int_{\mathcal{J}_{\omega,0}} f(\omega, x)\, d\mu_\omega(x)$$

is measurable and

$$\mathscr{F} \otimes \mathscr{B} \ni A \longmapsto \int_\Omega \int_{\mathcal{J}_{\omega,0}} \mathbb{1}_A(\omega, x)\, d\mu_\omega(x)\, dm(\omega)$$

defines a probability measure on $\mathcal{J}_0$.

For functions $f : \mathcal{J}_0 \to \mathbb{R}$ and $F : \Omega \to \mathbb{R}$ we let

$$S_{n,T}(f_\omega) := \sum_{j=0}^{n-1} f_{\sigma^j(\omega)} \circ T_\omega^j \quad \text{and} \quad S_{n,\sigma}(F) := \sum_{j=0}^{n-1} F \circ \sigma^j$$

denote the Birkhoff sums of f and F with respect to T and σ respectively. We will consider a potential of the form $\varphi_0 : \mathcal{J}_0 \to \mathbb{R}$, and for each $n \geq 1$ we consider the weight $g_0^{(n)} : \mathcal{J}_0 \to \mathbb{R}$ whose disintegrations are given by

$$g_{\omega,0}^{(n)} := \exp(S_{n,T}(\varphi_{\omega,0})) = \prod_{j=0}^{n-1} g_{\sigma^j\omega,0}^{(1)} \circ T_\omega^j \tag{0.1.1}$$

for each $\omega \in \Omega$. We will often denote $g_{\omega,0}^{(1)}$ by $g_{\omega,0}$. We assume there exists a family of Banach spaces $\{\mathcal{B}_\omega, \|\cdot\|_{\mathcal{B}_\omega}\}_{\omega\in\Omega}$ of real-valued functions on each $\mathcal{J}_{\omega,0}$ with $g_{\omega,0} \in \mathcal{B}_\omega$ such that the fiberwise (Perron-Frobenius) transfer operator $\mathcal{L}_{\omega,0} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ given by

$$\mathcal{L}_{\omega,0}(f)(x) := \sum_{y\in T_\omega^{-1}(x)} f(y) g_{\omega,0}(y), \quad f \in \mathcal{B}_\omega,\, x \in \mathcal{J}_{\sigma\omega,0} \tag{0.1.2}$$

is well defined. Using induction we see that iterates $\mathcal{L}_{\omega,0}^n : \mathcal{B}_\omega \to \mathcal{B}_{\sigma^n\omega}$ of the transfer operator are given by

$$\mathcal{L}_{\omega,0}^n(f)(x) := \sum_{y\in T_\omega^{-n}(x)} f(y) g_{\omega,0}^{(n)}(y), \quad f \in \mathcal{B}_\omega,\, x \in \mathcal{J}_{\sigma^n\omega,0}.$$

We let $\mathcal{B}$ denote the space of functions $f : \mathcal{J}_0 \to \mathbb{R}$ such that $f_\omega \in \mathcal{B}_\omega$ for each $\omega \in \Omega$ and we define the global transfer operator $\mathcal{L}_0 : \mathcal{B} \to \mathcal{B}$ by

$$(\mathcal{L}_0 f)_\omega(x) := \mathcal{L}_{\sigma^{-1}\omega,0} f_{\sigma^{-1}\omega}(x)$$

for $f \in \mathcal{B}$ and $x \in \mathcal{J}_{\omega,0}$. We assume the following measurability assumption:

(M2) For every measurable function $f \in \mathcal{B}$, the map $(\omega, x) \mapsto (\mathcal{L}_0 f)_\omega(x)$ is measurable.

We suppose the following condition on the existence of a closed conformal measure.

(CCM) There exists a random probability measure $\nu_0 = \{\nu_{\omega,0}\}_{\omega\in\Omega} \in \mathcal{P}_\Omega(\mathcal{J}_0)$ and measurable functions $\lambda_0 : \Omega \to \mathbb{R}\setminus\{0\}$ and $\phi_0 : \mathcal{J}_0 \to (0, \infty)$ with $\phi_0 \in \mathcal{B}$ such that

$$\mathcal{L}_{\omega,0}(\phi_{\omega,0}) = \lambda_{\omega,0}\phi_{\sigma\omega,0} \quad \text{and} \quad \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}(f)) = \lambda_{\omega,0}\nu_{\omega,0}(f)$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

for all $f \in \mathcal{B}_\omega$ where $\phi_{\omega,0}(\cdot) := \phi_0(\omega, \cdot)$. Furthermore, we suppose that the fiber measures $\nu_{\omega,0}$ are non-atomic and that $\lambda_{\omega,0} := \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}\mathbb{1})$ with $\log \lambda_{\omega,0} \in L^1(m)$. We then define the random probability measure μ_0 on $\mathcal{J}_0$ by

$$\mu_{\omega,0}(f) := \int_{\mathcal{J}_{\omega,0}} f\phi_{\omega,0}\, d\nu_{\omega,0}, \qquad f \in L^1(\nu_{\omega,0}). \tag{0.1.3}$$

From the definition, one can easily show that μ_0 is T-invariant, that is,

$$\int_{\mathcal{J}_{\omega,0}} f \circ T_\omega\, d\mu_{\omega,0} = \int_{\mathcal{J}_{\sigma\omega,0}} f\, d\mu_{\sigma\omega,0}, \qquad f \in L^1(\mu_{\sigma\omega,0}). \tag{0.1.4}$$

Remark 0.1.3. Our Assumption (CCM) has been shown to hold in several random settings: random interval maps [3, 2], random subshifts [11, 56], random distance expanding maps [57], random polynomial systems [15], and random transcendental maps [58].

Definition 0.1.4. We will call the collection $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0)$ a *closed random dynamical system* if the assumptions (M1), (M2), and (CCM) are satisfied.

We are now ready to introduce holes into the closed systems.

0.1.1. Random Open Systems. We let $H \subseteq \mathcal{J}_0$ be measurable with respect to the product σ-algebra $\mathscr{F} \otimes \mathscr{B}$ on $\mathcal{J}_0$. For each $\omega \in \Omega$ the sets $H_\omega \subseteq \mathcal{J}_{\omega,0}$ are uniquely determined by the condition that

$$\{\omega\} \times H_\omega = H \cap (\{\omega\} \times \mathcal{J}_{\omega,0}). \tag{0.1.5}$$

Equivalently we have

$$H_\omega = \pi_2(H \cap (\{\omega\} \times \mathcal{J}_{\omega,0})),$$

where $\pi_2 : \mathcal{J}_0 \to \mathcal{J}_{\omega,0}$ is the projection onto the second coordinate. By definition we have that the sets H_ω are $\nu_{\omega,0}$-measurable. Now define

$$\mathcal{J}_\omega := \mathcal{J}_{\omega,0} \backslash H_\omega,$$

and let

$$\mathcal{J} := \bigcup_{\omega\in\Omega} \{\omega\} \times \mathcal{J}_\omega.$$

Throughout the manuscript, in particular Chapter 1, we denote $\mathbb{1}_\omega := \mathbb{1}_{\mathcal{J}_\omega}$. For each $\omega \in \Omega$, $n \geq 0$ we define

$$X_{\omega,n} := \left\{x \in \mathcal{J}_{\omega,0} : T_\omega^j(x) \notin H_{\sigma^j\omega} \text{ for all } 0 \leq j \leq n\right\} = \bigcap_{j=0}^{n} T_\omega^{-j}\left(\mathcal{J}_{\sigma^j\omega}\right) \tag{0.1.6}$$

to be the set of points in $\mathcal{J}_\omega$ which survive, i.e. those points whose trajectories do not land in the holes, for N iterates. We then naturally define

$$X_{\omega,\infty} := \bigcap_{n=0}^{\infty} X_{\omega,n} = \bigcap_{n=0}^{\infty} T_\omega^{-n}(\mathcal{J}_{\sigma^n\omega}) \tag{0.1.7}$$

to be the set of points which will never land in a hole under iteration of the maps T_ω^n for any $n \geq 0$. We call $X_{\omega,\infty}$ the *ω-surviving set*. Note that the sets $X_{\omega,n}$ and $X_{\omega,\infty}$ are forward invariant satisfying the properties

$$T_\omega(X_{\omega,n}) \subseteq X_{\sigma\omega,n-1} \qquad \text{and} \qquad T_\omega(X_{\omega,\infty}) \subseteq X_{\sigma\omega,\infty}. \tag{0.1.8}$$

Now for any $0 \leq \alpha \leq \infty$ we set

$$\hat{X}_{\omega,\alpha} := \mathbb{1}_{X_{\omega,\alpha}} = \prod_{j=0}^{\alpha} \mathbb{1}_{\mathcal{J}_{\sigma^j\omega}} \circ T_\omega^j.$$

The global surviving set is defined as

$$\mathcal{X}_\alpha := \bigcup_{\omega\in\Omega} \{\omega\} \times X_{\omega,\alpha}$$

for each $0 \leq \alpha \leq \infty$. Then $\mathcal{X}_\infty \subseteq \mathcal{J}$ is precisely the set of points that survive under forward iteration of the skew-product map T. We will assume that the fiberwise survivor sets are nonempty:

(X) For m-a.e. $\omega \in \Omega$ we have $X_{\omega,\infty} \neq \emptyset$.

As an immediate consequence of (X) we have that $\mathcal{X}_\infty \neq \emptyset$. In fact, (X) together with the forward invariance of the sets $X_{\omega,\infty}$ imply that $\mathcal{X}_\infty$ is infinite. The following proposition presents a setting in which the survivor set is nonempty for random piecewise continuous open dynamics.

PROPOSITION 0.1.5. *Suppose that for m-a.e. $\omega \in \Omega$ there exist $V_\omega, U_{\omega,1}, \ldots, U_{\omega,k_\omega} \subseteq \mathcal{J}_\omega$ nonempty compact subsets such that for m-a.e. $\omega \in \Omega$*

(1) $T_\omega|_{U_{\omega,j}}$ is continuous for each $1 \leq j \leq k_\omega$,
(2) $T_\omega(U_\omega) \supseteq V_{\sigma\omega}$, where $U_\omega := \cup_{j=1}^{k_\omega} U_{\omega,j} \subseteq V_\omega$ for each ω.

Then $X_{\omega,\infty} \neq \emptyset$ for m-a.e. $\omega \in \Omega$ and consequently $\mathcal{X}_\infty \neq \emptyset$. Furthermore, if

$$m(\{\omega \in \Omega : k_\omega > 1\}) > 0$$

then for m-a.e. $\omega \in \Omega$ the survivor set $X_{\omega,\infty}$ is uncountable.

PROOF. For each $1 \leq j \leq k_\omega$ let $T_{\omega,j} : U_{\omega,j} \to \mathcal{J}_{\sigma\omega,0}$ denote the continuous map $T_\omega|_{U_{\omega,j}}$, and let $T_{\omega,U} : U_\omega \to \mathcal{J}_{\sigma\omega,0}$ denote the map $T_\omega|_{U_\omega}$. Since $V_{\sigma\omega}$ is compact, for each $1 \leq j \leq k_\omega$ we have that $T_{\omega,j}^{-1}(V_{\sigma\omega})$ is a nonempty compact subset of $U_{\omega,j}$. Given $n \geq 1$ let $\gamma = \gamma_0\gamma_1 \ldots \gamma_{n-1}$ be an n-length word with $1 \leq \gamma_j \leq k_{\sigma^j\omega}$ for each $0 \leq j \leq n-1$. Let $\Gamma_{\omega,n}$ denote the finite collection of all such words of length n. Then for each $n \geq 1$ and each $\gamma \in \Gamma_{\omega,n}$

$$T_{\omega,\gamma}^{-n}(V_{\sigma^n\omega}) := T_{\omega,\gamma_0}^{-1} \circ \cdots \circ T_{\sigma^{n-1}\omega,\gamma_{n-1}}^{-1}(V_{\sigma^n\omega}) \subseteq U_{\omega,\gamma_0}$$

is compact. Furthermore, $T_{\omega,\gamma}^{-n}(V_{\sigma^n\omega})$ forms a decreasing sequence in U_{ω,γ_0}. Hence, we see that

$$T_{\omega,U}^{-n}(V_{\sigma^n\omega}) = \bigcup_{\gamma\in\Gamma_{\omega,n}} T_{\omega,\gamma}^{-n}(V_{\sigma^n\omega})$$

forms a decreasing sequence of compact subsets of U_ω. Thus,

$$X_{\omega,\infty} = \bigcap_{n=0}^{\infty} T_\omega^{-n}(V_{\sigma^n\omega}) \supseteq \bigcap_{n=0}^{\infty} T_{\omega,U}^{-n}(V_{\sigma^n\omega}) \neq \emptyset$$

as desired. The final claim follows from the ergodicity of σ, which ensures that for a.e. ω there are infinitely many $j \in \mathbb{N}$ such that $k_{\sigma^j\omega} > 1$, and the usual bijection between a point in $X_{\omega,\infty}$ and an infinite word γ in the fiberwise sequence space $\Sigma_\omega := \{\gamma = \gamma_1\gamma_2\cdots : 1 \leq \gamma_j \leq k_{\sigma^j\omega}\}$. □

Now we define the perturbed operator $\mathcal{L}_\omega : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ by

$$\mathcal{L}_\omega(f) := \mathcal{L}_{\omega,0}\left(f \cdot \mathbb{1}_{\mathcal{J}_\omega}\right) = \sum_{y \in T_\omega^{-1}(x)} f(y)\mathbb{1}_{\mathcal{J}_\omega} g_{\omega,0}(y) = \sum_{y \in T_\omega^{-1}(x)} f(y) g_\omega(y), \quad f \in \mathcal{B}_\omega, \tag{0.1.9}$$

where for each $\omega \in \Omega$ we define $g_\omega := g_{\omega,0}\mathbb{1}_{\mathcal{J}_\omega}$, and, similarly, for each $n \in \mathbb{N}$,

$$g_\omega^{(n)} := \prod_{j=0}^{n-1} g_{\sigma^j\omega} \circ T_\omega^j.$$

Note that measurability of $H \subseteq \mathcal{J}_0$ and condition (M2) imply that for every $f \in \mathcal{B}$ the map $(\omega, x) \mapsto (\mathcal{L}f)_\omega(x)$ is also measurable. Iterates of the perturbed operator $\mathcal{L}_\omega^n : \mathcal{B}_\omega \to \mathcal{B}_{\sigma^n\omega}$ are given by

$$\mathcal{L}_\omega^n := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_\omega,$$

which, using induction, we may write as

$$\mathcal{L}_\omega^n(f) = \mathcal{L}_{\omega,0}^n\left(f \cdot \hat{X}_{\omega,n-1}\right), \qquad f \in \mathcal{B}_\omega. \tag{0.1.10}$$

We denote the normalized transfer operator $\tilde{\mathcal{L}}_\omega : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ by

$$\tilde{\mathcal{L}}_\omega := \lambda_\omega^{-1}\mathcal{L}_\omega.$$

We define the sets $D_{\omega,n}$ to be the support of $\mathcal{L}_{\sigma^{-n}(\omega)}^n \mathbb{1}_{\sigma^{-n}(\omega)}$, that is, we set

$$D_{\omega,n} := \left\{x \in \mathcal{J}_{\sigma^{-n}\omega,0} : \mathcal{L}_{\sigma^{-n}(\omega)}^n \mathbb{1}_{\sigma^{-n}(\omega)}(x) \neq 0\right\}. \tag{0.1.11}$$

Note that, by definition, we have

$$D_{\omega,n+1} \subseteq D_{\omega,n}$$

for each $n \in \mathbb{N}$, and we similarly define

$$D_{\omega,\infty} := \bigcap_{n=0}^{\infty} D_{\omega,n}.$$

From this moment on we will assume that for m-a.e. $\omega \in \Omega$ we have that

$$D_{\omega,\infty} \neq \emptyset. \tag{D}$$

We let

$$\hat{D}_{\omega,\alpha} := \mathbb{1}_{D_{\omega,\alpha}} \tag{0.1.12}$$

for each $0 \leq \alpha \leq \infty$. Since $D_{\omega,n}$ is the support of $\mathcal{L}^n_{\sigma^{-n}(\omega)}\mathbb{1}_{\sigma^{-n}(\omega)}$, using the notation of (0.1.12) we may write

$$\mathcal{L}^n_\omega(f) = \hat{D}_{\sigma^n(\omega),n}\mathcal{L}^n_\omega(f).$$

More generally, we have that, for $k > j$, $D_{\sigma^k(\omega),j}$ is the support of $\mathcal{L}^j_{\sigma^{k-j}(\omega)}\mathbb{1}$, i.e.

$$\mathcal{L}^j_{\sigma^{k-j}(\omega)}(f) = \hat{D}_{\sigma^k(\omega),j}\mathcal{L}^j_{\sigma^{k-j}(\omega)}(f) \tag{0.1.13}$$

for $f \in L^1(\nu_{\sigma^k(\omega),0})$. Note that

$$D_{\omega,n} = T^n_{\sigma^{-n}(\omega)}(X_{\sigma^{-n}(\omega),n-1}). \tag{0.1.14}$$

Finally, we note that since $g^{(n)}_\omega := g^{(n)}_{\omega,0}|_{X_{\omega,n-1}}$, for each $n \in \mathbb{N}$ we have that

$$\inf g^{(n)}_{\omega,0} \leq \inf_{X_{\omega,n-1,\varepsilon}} g^{(n)}_\omega \leq \|g^{(n)}_\omega\|_\infty \leq \|g^{(n)}_{\omega,0}\|_\infty \tag{0.1.15}$$

and

$$\inf g^{(n)}_{\omega,0} \leq \inf_{D_{\sigma^n(\omega),\infty}} \mathcal{L}^n_\omega\mathbb{1}_\omega \leq \|\mathcal{L}^n_\omega\mathbb{1}_\omega\|_\infty \leq \|\mathcal{L}_{\omega,0}\mathbb{1}\|_\infty. \tag{0.1.16}$$

Remark 0.1.6. We note that if $T_\omega(\mathcal{J}_\omega) = \mathcal{J}_{\sigma\omega,0}$ for m-a.e. ω, then

$$D_{\omega,\infty} = \mathcal{J}_{\omega,0}$$

for m-a.e. $\omega \in \Omega$.

Definition 0.1.7. We will call a closed random dynamical system $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0)$ (meaning that (M1), (M2), and (CCM) are satisfied) a *random open system* if assumptions (D) and (X) are also satisfied. We let $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H)$ denote the random open system generated by the random maps $T_\omega : \mathcal{J}_{\omega,0} \to \mathcal{J}_{\sigma\omega,0}$ and random holes $H_\omega \subseteq \mathcal{J}_{\omega,0}$.

CHAPTER 1

Thermodynamic formalism for random interval maps with holes

In this first chapter we consider fixed random holes H_ω and the main objective will be to construct a random conformal measure ν_ω and corresponding equivariant measures μ_ω supported on the random survivor set $X_{\omega,\infty}$. The measures μ_ω will be shown to be the unique relative equilibrium state for the potential φ. We will also get a random absolutely continuous conditionally invariant measure η_ω supported on H_ω^c. Successively we define the escape rate of the closed conformal measure and show that it equals the difference of the expected pressures of the closed and open random systems. Finally we establish a Bowen's like formula for the Hausdorff dimension of the survivor set for a specific potential.

1.1. Preliminaries of random interval maps with holes

We begin with a base dynamical system $(\Omega, \mathscr{F}, m, \sigma)$, i.e. the map $\sigma : \Omega \to \Omega$ is invertible, ergodic, and preserves the measure m. For the remainder of Chapter 1 for each $\omega \in \Omega$ we take $\mathcal{J}_{\omega,0} = I$ to be a compact interval in $\mathbb{R}$, and we consider the map $T_\omega : I \to I$ such that there exists a finite partition $\mathcal{Z}_\omega$ of I such that

(T1) $T_\omega : I \to I$ is surjective,

(T2) $T_\omega(Z)$ is an interval for each $Z \in \mathcal{Z}_\omega$,

(T3) $T_\omega|_Z$ is continuous and strictly monotone for each $Z \in \mathcal{Z}_\omega$.

In addition, we will assume that

(LIP) $\log \#\mathcal{Z}_\omega \in L^1(m)$.

The maps T_ω induce the skew product map $T : \Omega \times I \to \Omega \times I$ given by

$$T(\omega, x) = (\sigma(\omega), T_\omega(x)).$$

For each $n \in \mathbb{N}$ we consider the fiber dynamics of the maps $T_\omega^n : I \to I$ given by the compositions

$$T_\omega^n(x) = T_{\sigma^{n-1}(\omega)} \circ \cdots \circ T_\omega(x).$$

We let $\mathcal{Z}_\omega^{(n)}$, for $n \geq 2$, denote the monotonicity partition of T_ω^n on I which is given by

$$\mathcal{Z}_\omega^{(n)} := \bigvee_{j=0}^{n-1} T_\omega^{-j}\left(\mathcal{Z}_{\sigma^j(\omega)}\right).$$

Given $Z \in \mathcal{Z}_\omega^{(n)}$, we denote by

$$T_{\omega,Z}^{-n} : T_\omega^n(Z) \longrightarrow Z$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

the inverse branch of T_ω^n which takes $T_\omega^n(x)$ to x for each $x \in Z$. We will assume that the partitions $\mathcal{Z}_\omega$ are generating, i.e.

$$\bigvee_{n=1}^{\infty} \mathcal{Z}_\omega^{(n)} = \mathscr{B}, \tag{GP}$$

where $\mathscr{B} = \mathscr{B}(I)$ denotes the Borel σ-algebra of I. Let $\mathrm{B}(I)$ denote the set of all bounded real-valued functions on I and for each $f \in \mathrm{B}(I)$ and each $A \subseteq I$ let

$$\mathrm{var}_A(f) := \sup\left\{\sum_{j=0}^{k-1} |f(x_{j+1}) - f(x_j)| : x_0 < x_1 < \ldots x_k,\, x_j \in A \text{ for all } k \in \mathbb{N}\right\},$$

denote the variation of f over A. If $A = I$ we denote $\mathrm{var}(f) := \mathrm{var}_I(f)$. We let

$$\mathrm{BV}(I) := \{f \in \mathrm{B}(I) : \mathrm{var}(f) < \infty\}$$

denote the set of functions of bounded variation on I. For each $\omega \in \Omega$ we will set the Banach space $\mathcal{B}_\omega = \mathrm{BV}(I)$. Let

$$\|f\|_\infty := \sup(|f|) \qquad \text{and} \qquad \|f\|_{\mathrm{BV}} := \mathrm{var}(f) + \|f\|_\infty$$

be norms on the respective Banach spaces $\mathrm{B}(I)$ and $\mathrm{BV}(I)$. Given a function $f : \Omega \times I \to \mathbb{R}$, by $f_\omega : I \to I$ we mean

$$f_\omega(\cdot) := f(\omega, \cdot).$$

Definition 1.1.1. We say that a function $f : \Omega \times I \to \mathbb{R}$ is *random bounded* if

(i) $f_\omega \in \mathrm{B}(I)$ for each $\omega \in \Omega$,
(ii) for each $x \in I$ the function $\Omega \ni \omega \mapsto f_\omega(x)$ is measurable,
(iii) the function $\Omega \ni \omega \mapsto \|f_\omega\|_\infty$ is measurable.

Let $\mathrm{B}_\Omega(I)$ denote the collection of all random bounded functions on $\Omega \times I$.

Definition 1.1.2. We say that a function $f \in \mathrm{B}_\Omega(I)$ is of *random bounded variation* if $f_\omega \in \mathrm{BV}(I)$ for each $\omega \in \Omega$. We let $\mathrm{BV}_\Omega(I)$, which will take the place of the space $\mathcal{B}$ from Section 0.1, denote the set of all random bounded variation functions.

As in (0.1.2), we define the (closed) transfer operator, $\mathcal{L}_{\omega,0} : \mathrm{B}(I) \to \mathrm{B}(I)$, with respect to the potential $\varphi_0 : \Omega \times I \to \mathbb{R}$ by

$$\mathcal{L}_{\omega,0}(f)(x) := \sum_{y \in T_\omega^{-1}(x)} g_{\omega,0}(y) f(y); \quad f \in \mathrm{B}(I),\ x \in I.$$

For each $\omega \in \Omega$ we let $\mathrm{B}^*(I)$ and $\mathrm{BV}^*(I)$ denote the respective dual spaces of $\mathrm{B}(I)$ and $\mathrm{BV}(I)$. We let $\mathcal{L}^*_{\omega,0} : \mathrm{B}^*(I) \to \mathrm{B}^*(I)$ denote the dual transfer operator.

Definition 1.1.3. We will say that a measurable potential $\varphi_0 : \Omega \times I \to \mathbb{R}$ is *admissible* if for m-a.e. $\omega \in \Omega$ we have

(A1) $\inf \varphi_{\omega,0}, \sup \varphi_{\omega,0} \in L^1(m)$,
(A2) $g_{\omega,0} \in \mathrm{BV}(I)$.

REMARK 1.1.4. Note that if $\varphi_{\omega,0} \in \mathrm{BV}(I)$ for each $\omega \in \Omega$ then (A2) is immediate. Furthermore, as $\varphi_{\omega,0} \in \mathrm{B}(I)$ we also have that $\inf g_{\omega,0}^{(n)} > 0$ for m-a.e. $\omega \in \Omega$ and each $n \in \mathbb{N}$.

As an immediate consequence of (A1) we have that

$$\log \inf g_{\omega,0}, \log \|g_{\omega,0}\|_\infty \in L^1(m). \tag{1.1.1}$$

Note that since we can write

$$g_{\omega,0}^{(n)} := \prod_{j=0}^{n-1} g_{\sigma^j(\omega),0} \circ T_\omega^j.$$

for each $n \in \mathbb{N}$ we must have that $g_{\omega,0}^{(n)} \in \mathrm{BV}(I)$. Clearly we have that the sequence $\|g_{\omega,0}^{(n)}\|_\infty$ is submultiplicative, i.e.

$$\|g_{\omega,0}^{(n+m)}\|_\infty \leq \|g_{\omega,0}^{(n)}\|_\infty \cdot \|g_{\sigma^n(\omega),0}^{(m)}\|_\infty.$$

Similarly we see that the sequence $\inf g_{\omega,0}^{(n)}$ is supermultiplicative. Submultiplicativity and supermultiplicativity of $\|g_{\omega,0}^{(n)}\|_\infty$ and $\inf g_{\omega,0}^{(n)}$ together with (1.1.1) gives that

$$\log \|g_{\omega,0}^{(n)}\|_\infty, \log \inf g_{\omega,0}^{(n)} \in L^1(m) \tag{1.1.2}$$

for each $n \in \mathbb{N}$. Our assumptions (T1) and (LIP) combined with (1.1.2) implies that

$$\log \|\mathcal{L}_{\omega,0}^n \mathbb{1}\|_\infty, \log \inf \mathcal{L}_{\omega,0}^n \mathbb{1} \in L^1(m) \tag{1.1.3}$$

for each $n \in \mathbb{N}$. Note that, in view of (0.1.15)-(0.1.16), (1.1.2) and (1.1.3), imply that

$$\log \inf_{X_{\omega,n-1}} g_\omega^{(n)},\ \log \|g_\omega^{(n)}\|_\infty,\ \log \inf_{D_{\sigma^n(\omega),n}} \mathcal{L}_\omega^n \mathbb{1}_\omega,\ \log \|\mathcal{L}_{\omega,\varepsilon}^n \mathbb{1}_\omega\|_\infty \in L^1(m). \tag{1.1.4}$$

The Birkhoff Ergodic Theorem then implies that the quantities in (0.1.15) and (0.1.16) are tempered, e.g.

$$\lim_{|k|\to\infty} \frac{1}{|k|} \log \inf g_{\sigma^k(\omega)}^{(n)} = 0$$

for m-a.e. $\omega \in \Omega$ and each $n \in \mathbb{N}$.

In addition to the assumptions (T1)-(T3), (A1), (A2), (GP), and (LIP) above, we note that in our current setting assumption (M1) implies that

(M) The map $T : \Omega \times I \to \Omega \times I$ is measurable.

Furthermore, in our current setting our assumption (CCM) translates to the following:

(C) There exists a random probability measure $\nu_0 = \{\nu_{\omega,0}\}_{\omega\in\Omega} \in \mathcal{P}_\Omega(\Omega \times I)$ and measurable functions $\lambda_0 : \Omega \to \mathbb{R}\setminus\{0\}$ and $\phi_0 : \mathcal{J}_0 \to (0,\infty)$ with $\phi_0 \in \mathrm{BV}_\Omega(I)$ such that

$$\mathcal{L}_{\omega,0}(\phi_{\omega,0}) = \lambda_{\omega,0}\phi_{\sigma\omega,0} \quad \text{and} \quad \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}(f)) = \lambda_{\omega,0}\nu_{\omega,0}(f)$$

for all $f \in \mathrm{BV}(I)$. Furthermore, we suppose that the fiber measures $\nu_{\omega,0}$ are non-atomic and that $\lambda_{\omega,0} := \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}\mathbb{1})$ with $\log \lambda_{\omega,0} \in L^1(m)$. The T-invariant random probability measure μ_0 on $\Omega \times I$ is given by

$$\mu_{\omega,0}(f) := \int_I f\phi_{\omega,0}\, d\nu_{\omega,0}, \qquad f \in L^1(\nu_{\omega,0}).$$

Remark 1.1.5. Note that (M) together with (A2) implies that for $f \in \mathrm{BV}_\Omega(I)$ we have $\mathcal{L}_0 f \in \mathrm{BV}_\Omega(I)$.

Remark 1.1.6. Examples which satisfy the conditions (T1)-(T3), (A1), (A2), (GP), (LIP), (M), and (C) can be found in [**3**].

1.1.1. Random Interval Maps with Holes. We now wish to introduce holes into the class of finite branched random weighted covering systems.

Let $H \subseteq \Omega \times I$ be measurable with respect to the product σ-algebra $\mathscr{F} \otimes \mathscr{B}$ on $\Omega \times I$. By definition, i.e. (0.1.5), we have that the sets H_ω are $\nu_{\omega,0}$-measurable. Suppose that $0 < \nu_0(H) < 1$, that is we have

$$0 < \int_\Omega \nu_{\omega,0}(H_\omega)\, dm(\omega) < 1.$$

Now define

$$I_\omega := I \backslash H_\omega.$$

and recall from Section 0.1 that we denote

$$\mathbb{1}_\omega := \mathbb{1}_{I_\omega}.$$

We then let

$$\mathcal{I} := H^c = \bigcup_{\omega\in\Omega} \{\omega\} \times I_\omega.$$

For each $\omega \in \Omega$ and $n \geq 0$ we define the ω-surviving sets $X_{\omega,n}$ and $X_{\omega,\infty}$ as in (0.1.6)-(0.1.7). By definition, for each $n \in \mathbb{N}$, we have that

$$T_\omega(X_{\omega,n}) \subseteq X_{\sigma(\omega),n-1} \quad \text{and} \quad T_\omega(X_{\omega,\infty}) \subseteq X_{\sigma(\omega),\infty}.$$

Note, however, that these survivor sets are, in general, only forward invariant and not backward invariant. For notational convenience for any $0 \leq \alpha \leq \infty$ we set

$$\hat{X}_{\omega,\alpha} := \mathbb{1}_{X_{\omega,\alpha}}.$$

For each $\omega \in \Omega$ we let $\varphi_\omega = \varphi_{\omega,0}|_{I_\omega}$, and thus for each $n \in \mathbb{N}$ this gives

$$g_\omega^{(n)} = g_{\omega,0}^{(n)}|_{X_{\omega,n-1}} = \exp\left(S_{n,T}(\varphi_\omega)\right) = \prod_{j=0}^{n-1} g_{\sigma^j(\omega)} \circ T_\omega^j.$$

Now define the open operator $\mathcal{L}_\omega : L^1(\nu_{\omega,0}) \to L^1(\nu_{\sigma(\omega),0})$, where ν_0 comes from our assumption (C) on the closed system, by

$$\mathcal{L}_\omega(f) := \mathcal{L}_{\omega,0}\left(f \cdot \mathbb{1}_\omega\right), \quad f \in L^1(\nu_{\omega,0}),\ x \in I. \tag{1.1.5}$$

As a consequence of (1.1.5), we have that

$$\mathcal{L}_\omega \mathbb{1} = \mathcal{L}_\omega \mathbb{1}_\omega.$$

Iterates of the open operator $\mathcal{L}_\omega^n : L^1(\nu_{\omega,0}) \to L^1(\nu_{\sigma^n(\omega),0})$ are given by

$$\mathcal{L}_\omega^n := \mathcal{L}_{\sigma^{n-1}(\omega)} \circ \cdots \circ \mathcal{L}_\omega,$$

which, using induction, we may write in terms of the closed operator $\mathcal{L}_{\omega,0}$ as

$$\mathcal{L}_\omega^n(f) = \mathcal{L}_{\omega,0}^n \left(f \cdot \hat{X}_{\omega,n-1} \right), \qquad f \in L^1(\nu_{\omega,0}).$$

1.2. Random conditionally invariant probability measures

In this section we introduce the notion of a random conditionally invariant measure and give suitable conditions for their existence. As in the previous section, we consider random interval maps with holes, however we would like to point out that the definitions, results, and proofs of this section hold in the greater generality of Section 0.1. We begin with the following definition.

Definition 1.2.1. We say that a random probability measure $\eta \in \mathcal{P}_\Omega(\Omega \times I)$ is a random conditionally invariant probability measure (RCIM) if

$$\eta_\omega(T_\omega^{-n}(A) \cap X_{\omega,n}) = \eta_{\sigma^n(\omega)}(A)\eta_\omega(X_{\omega,n}) \tag{1.2.1}$$

for all $n \geq 0$, $\omega \in \Omega$, and all Borel sets $A \subseteq I$. If a RCIM η is absolutely continuous with respect to a random probability measure ζ we call η a random absolutely continuous conditionally invariant probability measure (RACCIM) with respect to ζ.

Straight from the definition of a RCIM we make the following observations.

Observation 1.2.2. Note that if we plug $A = I_\omega = X_{\omega,0}$ into (1.2.1) with $n = 0$, we have that

$$\eta_\omega(I_\omega) = \eta_\omega^2(I_\omega),$$

which immediately implies that $\eta_\omega(I_\omega)$ id either 0 or 1. If $\eta_\omega(H_\omega) = 0$ then we have that η_ω is supported in I_ω.

Observation 1.2.3. Note that since

$$T_\omega^{-n}(X_{\sigma^n(\omega),m}) \cap X_{\omega,n} = X_{\omega,n+m}$$

for each $n, m \in \mathbb{N}$ and $\omega \in \Omega$ we have that if η is a RCIM then

$$\eta_\omega(X_{\omega,n+m}) = \eta_\omega(X_{\omega,n})\eta_{\sigma^n(\omega)}(X_{\sigma^n(\omega),m})$$

for each $n, m \in \mathbb{N}$. In particular, we have that

$$\eta_\omega(X_{\omega,n}) = \prod_{j=0}^{n-1} \eta_{\sigma^j(\omega)}(X_{\sigma^j(\omega),1}).$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

In light of Observation 1.2.3, given a RCIM η, for each $\omega \in \Omega$ we let

$$\alpha_\omega := \eta_\omega(X_{\omega,1}).$$

Thus we have

$$\alpha_\omega^n := \prod_{j=0}^{n-1} \alpha_{\sigma^j(\omega)} = \eta_\omega(X_{\omega,n}). \tag{1.2.2}$$

We now prove a useful identity.

LEMMA 1.2.4. *Given any $f, h \in \mathrm{BV}(I)$, any $\omega \in \Omega$, and $n \in \mathbb{N}$ we have that*

$$\int_{I_{\sigma^n(\omega)}} h \cdot \mathcal{L}_\omega^n f \, d\nu_{\sigma^n(\omega),0} = \lambda_{\omega,0}^n \int_{X_{\omega,n}} f \cdot h \circ T_\omega^n \, d\nu_{\omega,0}.$$

PROOF. To prove the identity we calculate the following:

$$\begin{aligned}
\int_{I_{\sigma^n(\omega)}} h \cdot \mathcal{L}_\omega^n f \, d\nu_{\sigma^n(\omega),0} &= \int_I h \cdot \mathbb{1}_{\sigma^n(\omega)} \cdot \mathcal{L}_{\omega,0}^n \left(f \cdot \hat{X}_{\omega,n-1} \right) d\nu_{\sigma^n(\omega),0} \\
&= \int_I \mathcal{L}_\omega^n \left(f(h \circ T_\omega^n)(\mathbb{1}_{\sigma^n(\omega)} \circ T_\omega^n) \cdot \hat{X}_{\omega,n-1} \right) d\nu_{\sigma^n(\omega),0} \\
&= \lambda_{\omega,0}^n \int_I f(h \circ T_\omega^n) \cdot \hat{X}_{\omega,n} \, d\nu_{\omega,0} \\
&= \lambda_{\omega,0}^n \int_{X_{\omega,n}} f \cdot h \circ T_\omega^n \, d\nu_{\omega,0}.
\end{aligned}$$

□

The following lemma gives a useful characterization of RACCIM (with respect to ν_0) in terms of the transfer operators $\mathcal{L}_\omega$.

LEMMA 1.2.5. *Suppose $\eta = \mathbb{1}_{\mathcal{I}} h \nu_0$ is a random probability measure on $\mathcal{I}$ absolutely continuous with respect to ν_0, whose disintegrations are given by*

$$\eta_\omega = \mathbb{1}_\omega h_\omega \nu_{\omega,0}.$$

Then η is a RACCIM (with respect to ν_0) if and only if there exists $\alpha_\omega > 0$ such that

$$\mathcal{L}_\omega h_\omega = \lambda_{\omega,0} \alpha_\omega h_{\sigma(\omega)} \tag{1.2.3}$$

for each $\omega \in \Omega$.

PROOF. Beginning with the "reverse" direction, we first suppose (1.2.3) holds for all $\omega \in \Omega$. Let $A \in \mathscr{B}$ (Borel σ-algebra). Using Lemma 1.2.4 gives

$$\begin{aligned}
\eta_\omega(T_\omega^{-n}(A) \cap X_{\omega,n}) &= \int_{X_{\omega,n}} (\mathbb{1}_A \circ T_\omega^n) \cdot h_\omega \, d\nu_{\omega,0} \\
&= (\lambda_{\omega,0}^n)^{-1} \int_{I_{\sigma^n(\omega)}} \mathbb{1}_A \cdot \mathcal{L}_\omega^n h_\omega \, d\nu_{\sigma^n(\omega),0} \\
&= \int_{I_{\sigma^n(\omega)}} \mathbb{1}_A \cdot \alpha_\omega^n h_{\sigma^n(\omega)} \, d\nu_{\sigma^n(\omega),0}
\end{aligned}$$

$$= \alpha_\omega^n \eta_{\sigma^n(\omega)}(A). \tag{1.2.4}$$

Inserting $A = I_{\sigma^n(\omega)}$ into (1.2.4) gives

$$\eta_\omega(T_\omega^{-n}(I_{\sigma^n(\omega)}) \cap X_{\omega,n}) = \alpha_\omega^n \eta_{\sigma^n(\omega)}(I_{\sigma^n(\omega)}).$$

Observation 1.2.2 implies that $\eta_{\sigma^n(\omega)}(I_{\sigma^n(\omega)}) = 1$, and thus

$$\alpha_\omega^n = \eta_\omega(T_\omega^{-n}(I_{\sigma^n(\omega)}) \cap X_{\omega,n}) = \eta_\omega(X_{\omega,n}),$$

since $T_\omega^{-n}(I_{\sigma^n(\omega)}) \cap X_{\omega,n} = X_{\omega,n}$. Thus, for $A \in \mathscr{B}$ we have

$$\eta_\omega(T_\omega^{-n}(A) \cap X_{\omega,n}) = \eta_{\sigma^n(\omega)}(A)\eta_\omega(X_{\omega,n})$$

as desired.

Now to prove the opposite direction, suppose $\eta_\omega(\mathbb{1}_\omega h_\omega)$ is a RACCIM. Then by the definition of a RCIM there exists α_ω such that for any $A \in \mathscr{B}$ we have

$$\eta_\omega(T_\omega^{-n}(A) \cap X_{\omega,n}) = \alpha_\omega^n \eta_{\sigma^n(\omega)}(A).$$

So we calculate

$$\begin{aligned}(\lambda_{\omega,0}^n)^{-1} \int_{I_{\sigma^n(\omega)}} \mathbb{1}_A \cdot \mathcal{L}_\omega^n h_\omega \, d\nu_{\sigma^n(\omega),0} &= \int_{X_{\omega,n}} (\mathbb{1}_A \circ T_\omega^n) h_\omega \, d\nu_{\omega,0} = \eta_\omega(T_\omega^{-n}(A) \cap X_{\omega,n}) \\ &= \alpha_\omega^n \eta_{\sigma^n(\omega)}(A) = \alpha_\omega^n \int_{I_{\sigma^n(\omega)}} \mathbb{1}_A \cdot h_{\sigma^n(\omega)} \, d\nu_{\sigma^n(\omega),0}.\end{aligned}$$

So we have

$$\mathcal{L}_\omega^n h_\omega = \lambda_{\omega,0}^n \alpha_\omega^n h_{\sigma^n(\omega)},$$

which completes the proof. □

1.3. Functionals and partitions

In this section we follow [54, 53] and introduce the random functional Λ_ω that we will later show is equivalent to the conformal measure for the open system. We also introduce certain refinements of the partition of monotonicity which are used to define "good" and "bad" intervals and are needed to state our main assumptions on the open system. Following the statement of our main hypotheses, we state our main results.

We begin by defining the functional $\Lambda_\omega : \mathrm{BV}(I) \to \mathbb{R}$ by

$$\Lambda_\omega(f) := \lim_{n\to\infty} \inf_{x \in D_{\sigma^n(\omega),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(\mathbb{1}_\omega)(x)}, \quad f \in \mathrm{BV}(I). \tag{1.3.1}$$

We note that this limit exists as the sequence is bounded and increasing. Indeed, we have that

$$-\|f\|_\infty \le \inf_{x \in D_{\sigma^n(\omega),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(\mathbb{1}_\omega)(x)} \le \|f\|_\infty, \tag{1.3.2}$$

and to see that the ratio is increasing we note that

$$\inf_{x \in D_{\sigma^{n+1}(\omega),n+1}} \frac{\mathcal{L}_\omega^{n+1}(f)(x)}{\mathcal{L}_\omega^{n+1}(\mathbb{1}_\omega)(x)} = \inf_{x \in D_{\sigma^{n+1}(\omega),n+1}} \frac{\mathcal{L}_{\sigma^n(\omega)}\left(\hat{D}_{\sigma^n(\omega),n} \cdot \mathcal{L}_\omega^n(f)\right)(x)}{\mathcal{L}_\omega^{n+1}(\mathbb{1}_\omega)(x)}$$

$$= \inf_{x\in D_{\sigma^{n+1}(\omega),n+1}} \frac{\mathcal{L}_{\sigma^n(\omega)}\left(\hat{D}_{\sigma^n(\omega),n}\cdot \mathcal{L}_\omega^n(\mathbb{1}_\omega)\cdot \frac{\mathcal{L}_\omega^n(f)}{\mathcal{L}_\omega^n(\mathbb{1}_\omega)}\right)(x)}{\mathcal{L}_\omega^{n+1}(\mathbb{1}_\omega)(x)}$$

$$\geq \inf_{x\in D_{\sigma^n(\omega),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(\mathbb{1}_\omega)(x)} \cdot \inf_{x\in D_{\sigma^{n+1}(\omega),n+1}} \frac{\mathcal{L}_{\sigma^n(\omega)}\left(\hat{D}_{\sigma^n(\omega),n}\cdot \mathcal{L}_\omega^n(\mathbb{1}_\omega)\right)(x)}{\mathcal{L}_\omega^{n+1}(\mathbb{1}_\omega)(x)}$$

$$= \inf_{x\in D_{\sigma^n(\omega),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(\mathbb{1}_\omega)(x)}. \tag{1.3.3}$$

In particular, (1.3.2) of the above argument gives that

$$-\|f\|_\infty \leq \inf f \leq \Lambda_\omega(f) \leq \|f\|_\infty. \tag{1.3.4}$$

Observation 1.3.1. One can easily check that the functional Λ_ω has the following properties.

(1) $\Lambda_\omega(\mathbb{1}) = \Lambda_\omega(\mathbb{1}_\omega) = 1$.
(2) Λ_ω is continuous with respect to the supremum norm.
(3) $f \geq h$ implies that $\Lambda_\omega(f) \geq \Lambda_\omega(h)$.
(4) $\Lambda_\omega(cf) = c\Lambda_\omega(f)$.
(5) $\Lambda_\omega(f+h) \geq \Lambda_\omega(f) + \Lambda_\omega(h)$.
(6) $\Lambda_\omega(f+a) = \Lambda_\omega(f) + a$ for all $a \in \mathbb{R}$.
(7) If $A \cap X_{\omega,n} = \emptyset$ for some $n \in \mathbb{N}$ then $\Lambda_\omega(\mathbb{1}_A) = 0$.

Furthermore, we note that the homogeneity (4) and super-additivity (5) imply that Λ_ω is convex. In the sequel, we will show that Λ_ω is in fact linear, and can thus be associated with a unique probability measure on I_ω via the Riesz Representation Theorem.

Remark 1.3.2. Let $f \in \mathrm{BV}(I)$, then for all $x, y \in I_\omega$ we have

$$f(x) \leq f(y) + \mathrm{var}(f).$$

Using property (2) of Λ_ω, together with (1.3.2), implies

$$f(x) \leq \inf f + \mathrm{var}(f) \leq \Lambda_\omega(f) + \mathrm{var}(f) \leq \|f\|_\infty + \mathrm{var}(f). \tag{1.3.5}$$

We set

$$\rho_\omega := \Lambda_{\sigma(\omega)}\left(\mathcal{L}_\omega(\mathbb{1}_\omega)\right). \tag{1.3.6}$$

The following propositions concern various estimates of ρ_ω. We begin by setting

$$\rho_\omega^{(n)} := \inf_{x\in D_{\sigma^{n+1}(\omega),n}} \frac{\mathcal{L}_{\sigma(\omega)}^n(\mathcal{L}_\omega(\mathbb{1}_\omega))(x)}{\mathcal{L}_{\sigma(\omega)}^n(\mathbb{1}_{\sigma(\omega)})(x)}. \tag{1.3.7}$$

Then, by the definition and (1.3.3), we have that

$$\rho_\omega^{(n)} \nearrow \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega(\mathbb{1}_\omega)) = \rho_\omega \tag{1.3.8}$$

as $n \to \infty$.

Remark 1.3.3. Note that (1.3.6) and the definition of $\mathcal{L}_\omega$ together immediately imply that

$$\inf_{I_\omega} g_\omega \leq \rho_\omega \leq \|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty, \tag{1.3.9}$$

and similarly, (1.3.7) implies that

$$\inf_{I_\omega} g_\omega \leq \inf_{D_{\sigma(\omega),1}} \mathcal{L}_\omega(\mathbb{1}_\omega) = \rho_\omega^{(0)} \leq \rho_\omega^{(n)} \leq \|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty \leq \#\mathcal{Z}_\omega \|g_\omega\|_\infty, \tag{1.3.10}$$

for all $\omega \in \Omega$ and $n \geq 0$. Furthermore, (1.3.9), together with (1.1.4), gives that

$$\log \rho_\omega \in L^1(m). \tag{1.3.11}$$

The ergodic theorem then implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \rho_\omega^n = \int_\Omega \log \rho_\omega \, dm(\omega),$$

where

$$\rho_\omega^n := \prod_{j=0}^{n-1} \rho_{\sigma^j(\omega)}.$$

Proposition 1.3.4. *There exists a measurable and finite m-a.e. function $N_\infty : \Omega \to [1, \infty]$ such that*

$$D_{\omega,n} = D_{\omega,\infty}$$

for all $n \geq N_\infty(\omega)$. Furthermore, this implies that

$$\inf_{D_{\omega,\infty}} \mathcal{L}^n_{\sigma^{-n}(\omega)} \mathbb{1}_{\sigma^{-n}(\omega)} > 0 \tag{1.3.12}$$

for all $n \geq N_\infty(\omega)$.

Proof. We proceed via contradiction, assuming that there is a sequence $(n_k)_{k=1}^\infty$ in $\mathbb{N}$ such that

$$D_{\sigma^{n_k+1}(\omega),n_k+1} \subsetneq D_{\sigma^{n_k+1}(\omega),n_k}.$$

Let $x_{n_k} \in D_{\sigma^{n_k+1}(\omega),n_k} \backslash D_{\sigma^{n_k+1}(\omega),n_k+1}$. Then, we have that

$$\rho_\omega^{(n_k)} = \inf_{x \in D_{\sigma^{n_k+1}(\omega),n_k}} \frac{\mathcal{L}^{n_k}_{\sigma(\omega)}(\mathcal{L}_\omega(\mathbb{1}_\omega))(x)}{\mathcal{L}^{n_k}_{\sigma(\omega)}(\mathbb{1}_{\sigma(\omega)})(x)} \leq \frac{\mathcal{L}^{n_k}_{\sigma(\omega)}(\mathcal{L}_\omega(\mathbb{1}_\omega))(x_{n_k})}{\mathcal{L}^{n_k}_{\sigma(\omega)}(\mathbb{1}_{\sigma(\omega)})(x_{n_k})}.$$

By our choice of x_{n_k} and by the definition (0.1.11), we have that the numerator of the quantity on the right is zero, while its denominator is strictly positive. As this holds for each $k \in \mathbb{N}$, this implies that $\rho_\omega = 0$ for m-a.e. $\omega \in \Omega$, which contradicts (1.3.11). Thus, we are done. □

Remark 1.3.5. Note that our assumption (D), that $D_{\omega,\infty} \neq \emptyset$, is satisfied if $T_\omega(I_\omega) \supseteq I_{\sigma(\omega)}$ for m-a.e. $\omega \in \Omega$. Moreover, this also implies that $\rho_\omega \geq \inf_{I_{\sigma(\omega)}} \mathcal{L}_\omega \mathbb{1}_\omega > 0$. This occurs, for example if for m-a.e. $\omega \in \Omega$ there exists a full branch, i.e. there exists $Z \in \mathcal{Z}_\omega$ with $T_\omega(Z) = I$, outside of the hole H_ω, in which case we would have that $D_{\omega,\infty} = I$ for m-a.e. $\omega \in \Omega$.

We now describe various partitions, which depend on the functional Λ_ω, that we will used to obtain a Lasota-Yorke inequality in Section 1.5. Recall that $\mathcal{Z}_\omega^{(n)}$ denotes the

partition of monotonicity of T_ω^n. Now, for each $n \in \mathbb{N}$ and $\omega \in \Omega$ we let $\mathscr{A}_\omega^{(n)}$ be the collection of all finite partitions of I such that

$$\operatorname{var}_{A_i}(g_\omega^{(n)}) \leq 2\|g_\omega^{(n)}\|_\infty \tag{1.3.13}$$

for each $\mathcal{A} = \{A_i\} \in \mathscr{A}_\omega^{(n)}$.

Given $\mathcal{A} \in \mathscr{A}_\omega^{(n)}$, let $\widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A})$ be the coarsest partition amongst all those finer than $\mathcal{A}$ and $\mathcal{Z}_\omega^{(n)}$ such that all elements of $\widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A})$ are either disjoint from $X_{\omega,n-1}$ or contained in $X_{\omega,n-1}$. Now, define the following subcollections:

$$\mathcal{Z}_{\omega,*}^{(n)} := \left\{ Z \in \widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A}) : Z \subseteq X_{\omega,n-1} \right\}, \tag{1.3.14}$$

$$\mathcal{Z}_{\omega,b}^{(n)} := \left\{ Z \in \widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A}) : Z \subseteq X_{\omega,n-1} \text{ and } \Lambda_\omega(\mathbb{1}_Z) = 0 \right\}, \tag{1.3.15}$$

$$\mathcal{Z}_{\omega,g}^{(n)} := \left\{ Z \in \widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A}) : Z \subseteq X_{\omega,n-1} \text{ and } \Lambda_\omega(\mathbb{1}_Z) > 0 \right\}. \tag{1.3.16}$$

Remark 1.3.6. Note that in light of (1.3.1) and (1.3.3), for every $Z \in \mathcal{Z}_{\omega,g}^{(n)}$ we may define the open covering time $M_\omega(Z) \in \mathbb{N}$ to be the least integer such that

$$\inf_{x \in D_{\sigma^{M_\omega(Z)}(\omega), M_\omega(Z)}} \frac{\mathcal{L}_\omega^{M_\omega(Z)} \mathbb{1}_Z(x)}{\mathcal{L}_\omega^{M_\omega(Z)}(\mathbb{1}_\omega)(x)} > 0 \tag{1.3.17}$$

which is finite since the ratio in (1.3.17) increases to $\Lambda_\omega(\mathbb{1}_Z) > 0$. Conversely, given that the ratio in (1.3.17) is increasing by (1.3.3) we see that, for $Z \in \mathcal{Z}_{\omega,*}^{(n)}$, if there exists any $N \in \mathbb{N}$ such that

$$\inf_{x \in D_{\sigma^N(\omega), N}} \frac{\mathcal{L}_\omega^N \mathbb{1}_Z(x)}{\mathcal{L}_\omega^N(\mathbb{1}_\omega)(x)} > 0,$$

then we must have that $\Lambda_\omega(\mathbb{1}_Z) > 0$ and equivalently $Z \in \mathcal{Z}_{\omega,g}^{(n)}$.

We adapt the following definition from [53].

Definition 1.3.7. We say that two elements $W, Z \in \mathcal{Z}_{\omega,*}^{(n)}$ are *contiguous* if either W and Z are contiguous in the usual sense, i.e. they share a boundary point, or if they are separated by a connected component of $\cup_{j=0}^{n-1} T_\omega^{-j}(H_{\sigma^j(\omega)})$.

We will consider random open systems that satisfy the following conditions.

(Q1) For each $\omega \in \Omega$ and $n \in \mathbb{N}$ we let $\xi_\omega^{(n)}$ denote the maximum number of contiguous elements of $\mathcal{Z}_{\omega,b}^{(n)}$. We assume

$$\lim_{n\to\infty} \frac{1}{n} \log \|g_\omega^{(n)}\|_\infty + \limsup_{n\to\infty} \frac{1}{n} \log \xi_\omega^{(n)} < \lim_{n\to\infty} \frac{1}{n} \log \rho_\omega^n = \int_\Omega \log \rho_\omega \, dm(\omega).$$

(Q2) We assume that for each $n \in \mathbb{N}$ we have $\log \xi_\omega^{(n)} \in L^1(m)$.

(Q3) Let

$$\delta_{\omega,n} := \min_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \Lambda_\omega(\mathbb{1}_Z). \tag{1.3.18}$$

We assume that, for each $n \in \mathbb{N}$, $\log \delta_{\omega,n} \in L^1(m)$.

REMARK 1.3.8.

(1) Note that since $\lim_{n\to\infty} \frac{1}{n} \log \xi_\omega^{(n)} \geq 0$, assumption (Q1) implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \|g_\omega^{(n)}\|_\infty < \lim_{n\to\infty} \frac{1}{n} \log \rho_\omega^n.$$

(2) Since $\|g_\omega\|_\infty \leq \|g_{\omega,0}\|_\infty$ and $\inf_{D_{\sigma(\omega),\infty}} \mathcal{L}_\omega \mathbb{1}_\omega \leq \rho_\omega$ to check (Q1) it suffices to have

$$\lim_{n\to\infty} \frac{1}{n} \log \|g_{\omega,0}^{(n)}\|_\infty + \lim_{n\to\infty} \frac{1}{n} \log \xi_\omega^{(n)} < \lim_{n\to\infty} \frac{1}{n} \log \inf_{D_{\sigma^n(\omega),\infty}} \mathcal{L}_\omega^n \mathbb{1}_\omega.$$

(3) One can use the open covering times defined in (1.3.17) to check (Q3). Indeed, note that if

$$N \geq M_{\omega,n} := \max\left\{M_\omega(Z) : Z \in \mathcal{Z}_{\omega,g}^{(n)}\right\}$$

then we have that

$$\begin{aligned}
\delta_{\omega,n} &\geq \min_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \inf_{x\in D_{\sigma^N(\omega),N}} \frac{\mathcal{L}_\omega^N \mathbb{1}_Z(x)}{\mathcal{L}_\omega^N \mathbb{1}_\omega(x)} \\
&\geq \min_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \frac{\inf_{x\in D_{\sigma^N(\omega),N}} \mathcal{L}_\omega^N \mathbb{1}_Z(x)}{\|\mathcal{L}_\omega^N \mathbb{1}_\omega\|_\infty} \\
&\geq \frac{\inf_{X_{\omega,N-1}} g_\omega^{(N)}}{\|\mathcal{L}_\omega^N \mathbb{1}_\omega\|_\infty} \\
&\geq \frac{\inf g_{\omega,0}^{(N)}}{\|\mathcal{L}_{\omega,0}^N \mathbb{1}\|_\infty} > 0.
\end{aligned}$$

Thus (Q3) holds if $\log \inf g_{\omega,0}^{(M_{\omega,n})}, \log \|\mathcal{L}_{\omega,0}^{M_{\omega,n}} \mathbb{1}_\omega\|_\infty \in L^1(m)$ for each $n \in \mathbb{N}$.

REMARK 1.3.9. In Section 1.15 we give several alternate hypotheses to our assumptions (Q1)-(Q3) that are more restrictive, but much simpler to check.

1.4. Random Birkhoff cones and Hilbert metrics

In this section we first recall the theory of convex cones first used by Birkhoff in [10], and then present the random cones on which our operator $\mathcal{L}_\omega$ will act as a contraction. We begin with a definition.

DEFINITION 1.4.1. Given a vector space $\mathcal{V}$, we call a subset $\mathcal{C} \subseteq \mathcal{V}$ a *convex cone* if $\mathcal{C}$ satisfies the following:

(1) $\mathcal{C} \cap -\mathcal{C} = \emptyset$,
(2) for all $\alpha > 0$, $\alpha\mathcal{C} = \mathcal{C}$,
(3) $\mathcal{C}$ is convex,
(4) for all $f, h \in \mathcal{C}$ and all $\alpha_n \in \mathbb{R}$ with $\alpha_n \to \alpha$ as $n \to \infty$, if $h - \alpha_n f \in \mathcal{C}$ for each $n \in \mathbb{N}$, then $h - \alpha f \in \mathcal{C} \cup \{0\}$.

LEMMA 1.4.2 (Lemma 2.1 [54]). *The relation $\leq$ defined on $\mathcal{V}$ by*

$$f \leq h \text{ if and only if } h - f \in \mathcal{C} \cup \{0\}$$

is a partial order satisfying the following:

(i) $f \leq 0 \leq f \implies f = 0$,

(ii) $\lambda > 0$ *and* $f \geq 0 \iff \lambda f \geq 0$,

(iii) $f \leq h \iff 0 \leq h - f$,

(iv) *for all* $\alpha_n \in \mathbb{R}$ *with* $\alpha_n \to \alpha$, $\alpha_n f \leq h \implies \alpha f \leq h$,

(v) $f \geq 0$ *and* $h \geq 0 \implies f + h \geq 0$.

The Hilbert metric on $\mathcal{C}$ is given by the following definition.

DEFINITION 1.4.3. Define a distance $\Theta(f, h)$ by

$$\Theta(f, h) := \log \frac{\beta(f, h)}{\alpha(f, h)},$$

where

$$\alpha(f, h) := \sup \{a > 0 : af \leq h\} \quad \text{and} \quad \beta(f, h) := \inf \{b > 0 : bf \geq h\}.$$

Note that Θ is a pseudo-metric as two elements in the cone may be at an infinite distance from each other. Furthermore, Θ is a projective metric because any two proportional elements must be zero distance from each other. The next theorem, which is due to Birkhoff [10], shows that every positive linear operator that preserves the cone is a contraction provided that the diameter of the image is finite.

THEOREM 1.4.4 ([10]). *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be vector spaces with convex cones $\mathcal{C}_1 \subseteq \mathcal{V}_1$ and $\mathcal{C}_2 \subseteq \mathcal{V}_2$ and a positive linear operator $\mathcal{L} : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\mathcal{L}(\mathcal{C}_1) \subseteq \mathcal{C}_2$. If Θ_i denotes the Hilbert metric on the cone $\mathcal{C}_i$ and if*

$$\Delta = \sup_{f,h \in \mathcal{C}_1} \Theta_2(\mathcal{L}f, \mathcal{L}h),$$

then

$$\Theta_2(\mathcal{L}f, \mathcal{L}h) \leq \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f, h)$$

for all $f, h \in \mathcal{C}_1$.

Note that it is not clear whether $(\mathcal{C}, \Theta)$ is complete. The following lemma of [54] addresses this problem by linking the metric Θ with a suitable norm $\|\cdot\|$ on $\mathcal{V}$.

LEMMA 1.4.5 ([54], Lemma 2.2). *Let $\|\cdot\|$ be a norm on $\mathcal{V}$ such that for all $f, h \in \mathcal{V}$ if $-f \leq h \leq f$, then $\|h\| \leq \|f\|$, and let $\varrho : \mathcal{C} \to (0, \infty)$ be a homogeneous and order-preserving function, which means that for all $f, h \in \mathcal{C}$ with $f \leq h$ and all $\lambda > 0$ we have*

$$\varrho(\lambda f) = \lambda \varrho(f) \qquad \text{and} \qquad \varrho(f) \leq \varrho(h).$$

Then, for all $f, h \in \mathcal{C}$ $\varrho(f) = \varrho(h) > 0$ implies that

$$\|f - h\| \leq \left(e^{\Theta(f,h)} - 1\right) \min\{\|f\|, \|h\|\}.$$

REMARK 1.4.6. Note that the choice $\varrho(\cdot) = \|\cdot\|$ satisfies the hypothesis, however from this moment on we shall make the choice of $\varrho = \Lambda_\omega$.

DEFINITION 1.4.7. For each $a > 0$ and $\omega \in \Omega$ let

$$\mathscr{C}_{\omega,a} := \{f \in \mathrm{BV}(I) : f \geq 0,\ \mathrm{var}(f) \leq a\Lambda_\omega(f)\}. \tag{1.4.1}$$

To see that this cone is non-empty, we note that the function $f + c \in \mathscr{C}_{\omega,a}$ for $f \in \mathrm{BV}(I)$ and $c \geq a^{-1}\mathrm{var}(f) - \inf_{X_\omega} f$. We also define the cone

$$\mathscr{C}_{\omega,+} := \{f \in \mathrm{BV}(I) : f \geq 0\}.$$

Let $\Theta_{\omega,a}$ and $\Theta_{\omega,+}$ denote the Hilbert metrics induced on the respective cones $\mathscr{C}_{\omega,a}$ and $\mathscr{C}_{\omega,+}$. For each $\omega \in \Omega$, $a > 0$, and any set $Y \subseteq \mathscr{C}_{\omega,a}$ we let

$$\mathrm{diam}_{\omega,a}(Y) := \sup_{x,y\in Y} \Theta_{\omega,a}(x,y)$$

and

$$\mathrm{diam}_{\omega,+}(Y) := \sup_{x,y\in Y} \Theta_{\omega,+}(x,y)$$

denote the diameter of Y in the respective cones $\mathscr{C}_{\omega,a}$ and $\mathscr{C}_{\omega,+}$ with respect to the respective metrics $\Theta_{\omega,a}$ and $\Theta_{\omega,+}$. The following lemma collects together the main properties of these metrics.

LEMMA 1.4.8 ([54], Lemmas 4.2, 4.3, 4.5). *For $f, h \in \mathscr{C}_{\omega,+}$ the $\Theta_{\omega,+}$ distance between f, h is given by*

$$\Theta_{\omega,+}(f,h) = \log \sup_{x,y\in X_\omega} \frac{f(y)h(x)}{f(x)h(y)}.$$

If $f, h \in \mathscr{C}_{\omega,a}$, then

$$\Theta_{\omega,+}(f,h) \leq \Theta_{\omega,a}(f,h), \tag{1.4.2}$$

and if $f \in \mathscr{C}_{\omega,\eta a}$, for $\eta \in (0,1)$, we then have

$$\Theta_{\omega,a}(\mathbb{1}, f) \leq \log \frac{\|f\|_\infty + \eta\Lambda_\omega(f)}{\min\{\inf_{X_\omega} f, (1-\eta)\Lambda_\omega(f)\}}.$$

1.5. Lasota-Yorke inequalities

The main goal of this section is to prove a Lasota-Yorke type inequality. We adopt the strategy of [3], where we first prove a less-refined Lasota-Yorke inequality with (random) coefficients that behave in a difficult manner, and then, using the first inequality, prove a second inequality with measurable random coefficients and uniform decay on the variation as in [18].

We now prove a Lasota-Yorke type inequality following the approach of [53] utilizing the "good" and "bad" interval partitions defined in (1.3.14)-(1.3.16).

LEMMA 1.5.1. *For all $\omega \in \Omega$, all $f \in \mathrm{BV}(I)$, and all $n \in \mathbb{N}$ there exist positive, measurable constants $A_\omega^{(n)}$ and $B_\omega^{(n)}$ such that*

$$\mathrm{var}(\mathcal{L}_\omega^n f) \leq A_\omega^{(n)} \mathrm{var}(f) + B_\omega^{(n)} \Lambda_\omega(|f|),$$

where

$$A_\omega^{(n)} := (9 + 16\xi_\omega^{(n)}) \|g_\omega^{(n)}\|_\infty$$

and

$$B_\omega^{(n)} := 8(2\xi_\omega^{(n)} + 1)\|g_\omega^{(n)}\|_\infty \delta_{\omega,n}^{-1}.$$

PROOF. Since $\mathcal{L}_\omega^n(f) = \mathcal{L}_{\omega,0}^n(f \cdot \hat{X}_{\omega,n-1})$, if $Z \in \widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A}) \backslash \mathcal{Z}_{\omega,*}^{(n)}$, then $Z \cap X_{\omega,n-1} = \emptyset$, and thus, we have $\mathcal{L}_\omega^n(f\mathbb{1}_Z) = 0$ for each $f \in \mathrm{BV}(I)$. Thus, considering only intervals Z in $\mathcal{Z}_{\omega,*}^{(n)}$, we are able to write

$$\mathcal{L}_\omega^n f = \sum_{Z \in \mathcal{Z}_{\omega,*}^{(n)}} (\mathbb{1}_Z f g_\omega^{(n)}) \circ T_{\omega,Z}^{-n} \tag{1.5.1}$$

where

$$T_{\omega,Z}^{-n} : T_\omega^n(I_\omega) \to Z$$

is the inverse branch which takes $T_\omega^n(x)$ to x for each $x \in Z$. Now, since

$$\mathbb{1}_Z \circ T_{\omega,Z}^{-n} = \mathbb{1}_{T_\omega^n(Z)},$$

we can rewrite (2.C.3) as

$$\mathcal{L}_\omega^n f = \sum_{Z \in \mathcal{Z}_{\omega,*}^{(n)}} \mathbb{1}_{T_\omega^n(Z)} \left((f g_\omega^{(n)}) \circ T_{\omega,Z}^{-n} \right). \tag{1.5.2}$$

So,

$$\mathrm{var}(\mathcal{L}_\omega^n f) \leq \sum_{Z \in \mathcal{Z}_{\omega,*}^{(n)}} \mathrm{var}\left(\mathbb{1}_{T_\omega^n(Z)} \left((f g_\omega^{(n)}) \circ T_{\omega,Z}^{-n} \right) \right). \tag{1.5.3}$$

Now for each $Z \in \mathcal{Z}_{\omega,*}^{(n)}$, using (2.5.10), we have

$$\begin{aligned}
\mathrm{var}\left(\mathbb{1}_{T_\omega^n(Z)} \left((f g_\omega^{(n)}) \circ T_{\omega,Z}^{-n} \right) \right) &\leq \mathrm{var}_Z(f g_\omega^{(n)}) + 2 \sup_Z \left| f g_\omega^{(n)} \right| \\
&\leq 3\mathrm{var}_Z(f g_\omega^{(n)}) + 2 \inf_Z \left| f g_\omega^{(n)} \right| \\
&\leq 3\|g_\omega^{(n)}\|_\infty \mathrm{var}_Z(f) + 3 \sup_Z |f| \mathrm{var}_Z(g_\omega^{(n)}) + 2\|g_\omega^{(n)}\|_\infty \inf_Z |f| \\
&\leq 3\|g_\omega^{(n)}\|_\infty \mathrm{var}_Z(f) + 6\|g_\omega^{(n)}\|_\infty \sup_Z |f| + 2\|g_\omega^{(n)}\|_\infty \inf_Z |f| \\
&\leq 9\|g_\omega^{(n)}\|_\infty \mathrm{var}_Z(f) + 8\|g_\omega^{(n)}\|_\infty \inf_Z |f|. \qquad (1.5.4)
\end{aligned}$$

Now, using (1.5.4), we may further estimate (1.5.3) as

$$\mathrm{var}(\mathcal{L}_\omega^n f) \leq \sum_{Z \in \mathcal{Z}_{\omega,*}^{(n)}} \left(9\|g_\omega^{(n)}\|_\infty \mathrm{var}_Z(f) + 8\|g_\omega^{(n)}\|_\infty \inf_Z |f| \right)$$

$$\leq 9\|g_\omega^{(n)}\|_\infty \mathrm{var}(f) + 8\|g_\omega^{(n)}\|_\infty \left(\sum_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \inf_Z |f| + \sum_{Z\in\mathcal{Z}_{\omega,b}^{(n)}} \inf_Z |f| \right). \tag{1.5.5}$$

In order to investigate each of the two sums in the line above, we first note that as $\mathcal{Z}_{\omega,g}^{(n)}$ is finite then, by definition, there exists a constant $\delta_{\omega,n} > 0$ (defined by (1.3.18)) such that

$$\inf_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \Lambda_\omega(\mathbb{1}_Z) \geq 2\delta_{\omega,n} > 0.$$

So, we may choose $N_{\omega,n} \in \mathbb{N}$ such that for $x \in D_{\sigma^{N_{\omega,n}}(\omega),N_{\omega,n}}$ we have

$$\inf_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \frac{\mathcal{L}_\omega^{N_{\omega,n}}(\mathbb{1}_Z)(x)}{\mathcal{L}_\omega^{N_{\omega,n}}(\mathbb{1}_\omega)(x)} \geq \delta_{\omega,n}. \tag{1.5.6}$$

Note that since this ratio is increasing we have that (1.5.6) holds for all $\tilde{N} \geq N_{\omega,n}$. Then for each $x \in D_{\sigma^{N_{\omega,n}}(\omega),N_{\omega,n}}$ and $Z \in \mathcal{Z}_{\omega,g}^{(n)}$ we have

$$\mathcal{L}_\omega^{N_{\omega,n}}(|f|\mathbb{1}_Z)(x) \geq \inf_Z |f| \mathcal{L}_\omega^{N_{\omega,n}}(\mathbb{1}_Z)(x) \geq \inf_Z |f| \delta_{\omega,n} \mathcal{L}_\omega^{N_{\omega,n}}(\mathbb{1}_\omega)(x).$$

In particular, for each $x \in D_{\sigma^{N_{\omega,n}}(\omega),N_{\omega,n}}$, we see that

$$\sum_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \inf_Z |f| \leq \delta_{\omega,n}^{-1} \sum_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \frac{\mathcal{L}_\omega^{N_{\omega,n}}(|f|\mathbb{1}_Z)(x)}{\mathcal{L}_\omega^{N_{\omega,n}}(\mathbb{1}_\omega)(x)} \leq \delta_{\omega,n}^{-1} \frac{\mathcal{L}_\omega^{N_{\omega,n}}(|f|)(x)}{\mathcal{L}_\omega^{N_{\omega,n}}(\mathbb{1}_\omega)(x)}. \tag{1.5.7}$$

We are now interested in finding appropriate upper bounds for the sum

$$\sum_{Z\in\mathcal{Z}_{\omega,b}^{(n)}} \inf_Z |f|.$$

However, we must first be able to associate each of the elements of $\mathcal{Z}_{\omega,b}^{(n)}$ with one of the elements of $\mathcal{Z}_{\omega,g}^{(n)}$. To that end, let Z_* and Z^* denote the elements of $\mathcal{Z}_{\omega,*}^{(n)}$ that are the furthest to the left and the right respectively. Now, enumerate each of the elements of $\mathcal{Z}_{\omega,g}^{(n)}$, $Z_1, \ldots, Z_k$ (clearly k depends on ω, n, and $\mathcal{A}$), such that Z_{j+1} is to the right of Z_j for $j = 1, \ldots, k-1$. Given $Z_j \in \mathcal{Z}_{\omega,g}^{(n)}$ $(1 \leq j \leq k)$, with $Z_j \neq Z^*$ let $J_{\omega,+}(Z_j)$ be the union of all contiguous elements $Z \in \mathcal{Z}_{\omega,b}^{(n)}$ which are to the right of Z_j and also to the left of Z_{j+1}. In other words, $J_{\omega,+}(Z_j)$ is the union of all elements of $\mathcal{Z}_{\omega,b}^{(n)}$ between Z_j and Z_{j+1}. Similarly, for $Z_j \neq Z_*$, we define $J_{\omega,-}(Z_j)$ be the union of all contiguous elements $Z \in \mathcal{Z}_{\omega,b}^{(n)}$ which are to the left of Z_j and also to the right of Z_{j-1}. Now, we note that our assumption (Q1) implies that each $J_{\omega,-}(Z)$ and $J_{\omega,+}(Z)$ $(Z \in \mathcal{Z}_{\omega,g}^{(n)})$ is the union of at most ξ_ω^n contiguous elements of $\mathcal{Z}_{\omega,b}^{(n)}$. For $Z \in \mathcal{Z}_{\omega,g}^{(n)}$ let

$$J_{\omega,-}^*(Z) = Z \cup J_{\omega,-}(Z) \qquad \text{and} \qquad J_{\omega,+}^*(Z) = Z \cup J_{\omega,+}(Z).$$

Then for $W \subseteq J_{\omega,-}^*(Z)$ we have

$$\inf_W |f| \leq \inf_Z |f| + \mathrm{var}_{J_{\omega,-}^*(Z)}(f). \tag{1.5.8}$$

We obtain a similar inequality for $W \subseteq J^*_{\omega,+}(Z)$. We now consider the following two cases.

(Case 1:) At least one of the intervals Z_* and Z^* is an element of $\mathcal{Z}^{(n)}_{\omega,g}$.

(Case 2:) Neither of the intervals Z_*, Z^* is an element of $\mathcal{Z}^{(n)}_{\omega,g}$.

If we are in the first case, we assume without loss of generality that $Z_1 = Z_*$, and thus every element $Z \in \mathcal{Z}^{(n)}_{\omega,b}$ is contained in exactly one union $J_{\omega,+}(Z_j)$ for some $Z_j \in \mathcal{Z}^{(n)}_{\omega,g}$ for some $1 \le j \le k$. If $Z_1 \neq Z_*$ and instead we have that $Z_k = Z^*$ we could simply replace $J_{\omega,+}(Z_j)$ with $J_{\omega,-}(Z_j)$ in the previous statement. In view of (1.5.8), Case 1 leads to the conclusion that

$$\sum_{Z\in\mathcal{Z}^{(n)}_{\omega,b}} \inf_Z |f| \le \xi^{(n)}_\omega \left(\sum_{Z\in\mathcal{Z}^{(n)}_{\omega,g}} \inf_Z |f| + \mathrm{var}_{J^*_{\omega,-}(Z)}(f) \right).$$

If we are instead in the second case, then for each $Z \in \mathcal{Z}^{(n)}_{\omega,b}$ to the left of Z_k there is exactly one Z_j, $1 \le j \le k$, such that $Z \subseteq J_{\omega,-}(Z_j)$. This leaves each of the elements $Z \in \mathcal{Z}^{(n)}_{\omega,b}$ to the right of Z_k uniquely contained in the union $J_{\omega,+}(Z_k)$. Thus, Case 2 yields

$$\begin{aligned}\sum_{Z\in\mathcal{Z}^{(n)}_{\omega,b}} \inf_Z |f| &\le \xi^{(n)}_\omega \left(\inf_{Z_k} |f| + \mathrm{var}_{J^*_{\omega,+}(Z_k)}(f) + \sum_{Z\in\mathcal{Z}^{(n)}_{\omega,g}} \inf_Z |f| + \mathrm{var}_{J^*_{\omega,-}(Z)}(f) \right) \\ &\le 2\xi^{(n)}_\omega \left(\mathrm{var}(f) + \sum_{Z\in\mathcal{Z}^{(n)}_{\omega,g}} \inf_Z |f| \right).\end{aligned}$$

Hence, either case gives that

$$\sum_{Z\in\mathcal{Z}^{(n)}_{\omega,b}} \inf_Z |f| \le 2\xi^{(n)}_\omega \left(\mathrm{var}(f) + \sum_{Z\in\mathcal{Z}^{(n)}_{\omega,g}} \inf_Z |f| \right). \tag{1.5.9}$$

Inserting (1.5.7) and (1.5.9) into (1.5.5) gives

$$\begin{aligned}\mathrm{var}(\mathcal{L}^n_\omega f) &\le 9\|g^{(n)}_\omega\|_\infty \mathrm{var}(f) \\ &\quad + 8\|g^{(n)}_\omega\|_\infty \left(2\xi^{(n)}_\omega \left(\mathrm{var}(f) + \sum_{Z\in\mathcal{Z}^{(n)}_{\omega,g}} \inf_Z |f| \right) + \delta^{-1}_{\omega,n} \frac{\mathcal{L}^{N_{\omega,n}}_\omega(|f|)(x)}{\mathcal{L}^{N_{\omega,n}}_\omega(\mathbb{1}_\omega)(x)} \right) \\ &\le (9 + 16\xi^{(n)}_\omega)\|g^{(n)}_\omega\|_\infty \mathrm{var}(f) + 8(2\xi^{(n)}_\omega + 1)\|g^{(n)}_\omega\|_\infty \delta^{-1}_{\omega,n} \frac{\mathcal{L}^{N_{\omega,n}}_\omega(|f|)(x)}{\mathcal{L}^{N_{\omega,n}}_\omega(\mathbb{1}_\omega)(x)}.\end{aligned}$$

In view of (1.3.3), taking the infimum over $x \in D_{\sigma^{N_{\omega,n}}(\omega),N_{\omega,n}}$ allows us to replace the ratio $\frac{\mathcal{L}^{N_{\omega,n}}_\omega(|f|)(x)}{\mathcal{L}^{N_{\omega,n}}_\omega(\mathbb{1}_\omega)(x)}$ with $\Lambda_\omega(|f|)$, that is, we have

$$\mathrm{var}(\mathcal{L}^n_\omega f) \le (9 + 16\xi^{(n)}_\omega)\|g^{(n)}_\omega\|_\infty \mathrm{var}(f) + 8(2\xi^{(n)}_\omega + 1)\|g^{(n)}_\omega\|_\infty \delta^{-1}_{\omega,n} \Lambda_\omega(|f|).$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

Setting

$$A_\omega^{(n)} := (9 + 16\xi_\omega^{(n)})\|g_\omega^{(n)}\|_\infty \qquad \text{and} \qquad B_\omega^{(n)} := 8(2\xi_\omega^{(n)} + 1)\|g_\omega^{(n)}\|_\infty \delta_{\omega,n}^{-1} \tag{1.5.10}$$

finishes the proof. □

REMARK 1.5.2. As a consequence of Lemma 1.5.1 we have that

$$\mathcal{L}_\omega\left(\mathscr{C}_{\omega,+}\right) \subseteq \mathscr{C}_{\sigma(\omega),+}, \tag{1.5.11}$$

and thus $\mathcal{L}_\omega$ is a weak contraction on $\mathscr{C}_{\omega,+}$.

Define the random constants

$$Q_\omega^{(n)} := \frac{A_\omega^{(n)}}{\rho_\omega^n} \quad \text{and} \quad K_\omega^{(n)} := \frac{B_\omega^{(n)}}{\rho_\omega^n}. \tag{1.5.12}$$

In light of our assumption (Q1) on the potential and number of contiguous bad intervals, we see that $Q_\omega^{(n)} \to 0$ exponentially quickly for each $\omega \in \Omega$.

The following proposition now follows from (1.1.4), (1.3.11), and assumptions (Q2)-(Q3).

PROPOSITION 1.5.3. *For each $n \in \mathbb{N}$, $\log^+ Q_\omega^{(n)}, \log K_\omega^{(n)} \in L_m^1(\Omega)$.*

LEMMA 1.5.4. *For each $f \in \mathrm{BV}(I)$ and each $n, k \in \mathbb{N}$ we have*

$$\Lambda_{\sigma^k(\omega)}\left(\mathcal{L}_\omega^k f\right) \geq \Lambda_{\sigma^k(\omega)}\left(\mathcal{L}_\omega^k \mathbb{1}_\omega\right) \cdot \Lambda_\omega(f). \tag{1.5.13}$$

Furthermore, we have that

$$\rho_\omega^n \cdot \Lambda_\omega(f) \leq \Lambda_{\sigma^n(\omega)}(\mathcal{L}_\omega^n f). \tag{1.5.14}$$

In particular, this yields

$$\rho_\omega^n \leq \Lambda_{\sigma^n(\omega)}(\mathcal{L}_\omega^n \mathbb{1}_\omega).$$

PROOF. For each $f \in \mathrm{BV}(I)$ with $f \geq 0$, $k \in \mathbb{N}$, and $x \in D_{\sigma^{n+k}(\omega),n}$ we have

$$\begin{aligned}
\frac{\mathcal{L}_{\sigma^k(\omega)}^n\left(\mathcal{L}_\omega^k f\right)(x)}{\mathcal{L}_{\sigma^k(\omega)}^n\left(\mathbb{1}_{\sigma^k(\omega)}\right)(x)} &= \frac{\mathcal{L}_{\sigma^n(\omega)}^k\left(\mathcal{L}_\omega^n f\right)(x)}{\mathcal{L}_{\sigma^k(\omega)}^n\left(\mathbb{1}_{\sigma^k(\omega)}\right)(x)} \\
&= \frac{\mathcal{L}_{\sigma^n(\omega)}^k\left(\hat{D}_{\sigma^n(\omega),n} \cdot \frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega} \cdot \mathcal{L}_\omega^n \mathbb{1}_\omega\right)(x)}{\mathcal{L}_{\sigma^k(\omega)}^n\left(\mathbb{1}_{\sigma^k(\omega)}\right)(x)} \\
&\geq \frac{\mathcal{L}_{\sigma^k(\omega)}^n\left(\mathcal{L}_\omega^k \mathbb{1}_\omega\right)(x)}{\mathcal{L}_{\sigma^k(\omega)}^n\left(\mathbb{1}_{\sigma^k(\omega)}\right)(x)} \cdot \inf_{D_{\sigma^n(\omega),n}} \frac{\mathcal{L}_\omega^n(f)}{\mathcal{L}_\omega^n(\mathbb{1}_\omega)}.
\end{aligned}$$

Taking the infimum over $x \in D_{\sigma^{n+k}(\omega),n}$ and letting $n \to \infty$ gives

$$\Lambda_{\sigma^k(\omega)}\left(\mathcal{L}_\omega^k f\right) \geq \Lambda_{\sigma^k(\omega)}\left(\mathcal{L}_\omega^k \mathbb{1}_\omega\right) \cdot \Lambda_\omega(f), \tag{1.5.15}$$

proving the first claim. Now to see the second claim we note that as (1.5.15) holds for all $\omega \in \Omega$ with $k = 1$, we must also have

$$\Lambda_{\sigma^{n+1}(\omega)}\left(\mathcal{L}_{\sigma^n(\omega)} f\right) \geq \Lambda_{\sigma^{n+1}(\omega)}\left(\mathcal{L}_{\sigma^n(\omega)} \mathbb{1}_{\sigma^n(\omega)}\right) \cdot \Lambda_{\sigma^n(\omega)}(f) = \rho_{\sigma^n(\omega)} \cdot \Lambda_{\sigma^n(\omega)}(f) \tag{1.5.16}$$

for any $f \in \mathrm{BV}(I)$ and each $n \in \mathbb{N}$. Proceeding via induction, using (1.5.15) as the base case, we now suppose that

$$\Lambda_{\sigma^n(\omega)}\left(\mathcal{L}_\omega^n f\right) \geq \rho_\omega^n \cdot \Lambda_\omega(f) \tag{1.5.17}$$

holds for $n \geq 1$. Using (1.5.16) and (1.5.17), we see

$$\begin{aligned}\Lambda_{\sigma^{n+1}(\omega)}\left(\mathcal{L}_{\sigma^n(\omega)}(\mathcal{L}_\omega^n f)\right) &\geq \rho_{\sigma^n(\omega)} \cdot \Lambda_{\sigma^n(\omega)}(\mathcal{L}_\omega^n f)\\ &\geq \rho_\omega^{n+1} \cdot \Lambda_\omega(f).\end{aligned}$$

Considering $f = \mathbb{1}_\omega$ proves the final claim, and thus we are done. □

Define the normalized operator $\mathcal{L}_\omega : L^1(\nu_{\omega,0}) \to L^1(\nu_{\sigma(\omega),0})$ by

$$\tilde{\mathcal{L}}_\omega f := \rho_\omega^{-1}\mathcal{L}_\omega f; \qquad f \in L^1(\nu_{\omega,0}). \tag{1.5.18}$$

In light of Lemma 1.5.4, for each $\omega \in \Omega$, $n \in \mathbb{N}$, and $f \in \mathrm{BV}(I)$ we have that

$$\Lambda_\omega(f) \leq \Lambda_{\sigma^n(\omega)}\left(\tilde{\mathcal{L}}_\omega^n f\right). \tag{1.5.19}$$

Now, considering the normalized operator, we arrive at the following immediate corollary.

Corollary 1.5.5. *For all $\omega \in \Omega$, all $f \in \mathrm{BV}(I)$, and all $n \in \mathbb{N}$ we have*

$$\mathrm{var}(\tilde{\mathcal{L}}_\omega^n f) \leq Q_\omega^{(n)}\mathrm{var}(f) + K_\omega^{(n)}\Lambda_\omega(|f|).$$

Definition 1.5.6. Since $Q_\omega^{(n)} \to 0$ exponentially fast by our assumption (Q1), we let $N_* \in \mathbb{N}$ be the minimum integer $n \geq 1$ such that

$$-\infty < \int_\Omega \log Q_\omega^{(n)} dm(\omega) < 0, \tag{1.5.20}$$

and we define the number

$$\theta := -\frac{1}{N_*}\int_\Omega \log Q_\omega^{(N_*)} dm(\omega). \tag{1.5.21}$$

Remark 1.5.7. As we are primarily interested in pushing forward in blocks of length N_* we are able to weaken two or our main hypotheses. In particular, we may replace (Q2) and (Q3) with the following:

(Q2') We have $\log \xi_\omega^{(N_*)} \in L^1(m)$.
(Q3') We have $\log \delta_{\omega,N_*} \in L^1(m)$, where $\delta_{\omega,n}$ is defined by (1.3.18).

In light of Corollary 1.5.5 we may now find an appropriate upper bound for the BV norm of the normalized transfer operator.

Lemma 1.5.8. *There exists a measurable function $\omega \mapsto L_\omega \in (0,\infty)$ with $\log L_\omega \in L_m^1(\Omega)$ such that for all $f \in \mathrm{BV}(I)$ and each $1 \leq n \leq N_*$ we have*

$$\|\tilde{\mathcal{L}}_\omega^n f\|_{\mathrm{BV}} \leq L_\omega^n\left(\mathrm{var}(f) + \Lambda_{\sigma^n(\omega)}\left(\tilde{\mathcal{L}}_\omega^n f\right)\right). \tag{1.5.22}$$

where

$$L_\omega^n = L_\omega L_{\sigma(\omega)} \cdots L_{\sigma^{n-1}(\omega)} \geq 6^n.$$

PROOF. Corollary 1.5.5 and (1.5.19) give

$$\begin{aligned}\|\tilde{\mathcal{L}}_\omega^n f\|_{\mathrm{BV}} &= \mathrm{var}(\tilde{\mathcal{L}}_\omega^n f) + \|\tilde{\mathcal{L}}_\omega^n f\|_\infty \le 2\mathrm{var}(\tilde{\mathcal{L}}_\omega^n f) + \Lambda_{\sigma^n(\omega)}\left(\tilde{\mathcal{L}}_\omega^n f\right)\\ &\le 2\left(Q_\omega^{(n)}\mathrm{var}(f) + K_\omega^{(n)}\Lambda_\omega(|f|)\right) + \Lambda_{\sigma^n(\omega)}\left(\tilde{\mathcal{L}}_\omega^n f\right)\\ &\le 2Q_\omega^{(n)}\mathrm{var}(f) + \left(2K_\omega^{(n)}+1\right)\Lambda_{\sigma^n(\omega)}\left(\tilde{\mathcal{L}}_\omega^n f\right).\end{aligned}$$

Now, set

$$\tilde{L}_\omega^{(n)} := \max\left\{6, 2Q_\omega^{(n)}, 2K_\omega^{(n)}+1\right\}.$$

Finally, setting

$$L_\omega := \max\left\{\tilde{L}_\omega^{(j)} : 1 \le j \le N_*\right\} \tag{1.5.23}$$

and

$$L_\omega^n := \prod_{j=0}^{n-1} L_{\sigma^j(\omega)}$$

for all $n \ge 1$ suffices. The log-integrability of L_ω^n follows from Proposition 1.5.3. □

We now define the number $\zeta > 0$ by

$$\zeta := \frac{1}{N_*}\int_\Omega \log L_\omega^{N_*} dm(\omega). \tag{1.5.24}$$

The constants $B_\omega^{(n)}$ and $K_\omega^{(n)}$ in the Lasota-Yorke inequalities from Lemma 1.5.1 and Corollary 1.5.5 grow to infinity with n, making them difficult to use. Furthermore, the rate of decay of the $Q_\omega^{(n)}$ in Corollary 1.5.5 may depend on ω. To remedy these difficulties we prove another, more useful, Lasota-Yorke inequality in the style of Buzzi [18].

PROPOSITION 1.5.9. *For each $\varepsilon > 0$ there exists a measurable, m-a.e. finite function $C_\varepsilon(\omega) > 0$ such that for m-a.e. $\omega \in \Omega$, each $f \in \mathrm{BV}(I)$, and all $n \in \mathbb{N}$ we have*

$$\mathrm{var}(\tilde{\mathcal{L}}_{\sigma^{-n}(\omega)}^n f) \le C_\varepsilon(\omega)e^{-(\theta-\varepsilon)n}\mathrm{var}(f) + C_\varepsilon(\omega)\Lambda_\omega(\tilde{\mathcal{L}}_{\sigma^{-n}(\omega)}^n f).$$

As the proof of Proposition 1.5.9 follows similarly to that of Proposition 4.9 of [3], using (1.5.19) to obtain $\Lambda_\omega(\tilde{\mathcal{L}}_{\sigma^{-n}(\omega)}^n f)$ rather than $\Lambda_{\sigma^{-n}(\omega)}(f)$, we leave it to the dedicated reader.

1.6. Cone invariance on good fibers

In this section we follow Buzzi's approach [18], and describe the good behavior across a large measure set of fibers. In particular, we will show that, for sufficiently many iterates R_*, the normalized transfer operator $\tilde{\mathcal{L}}_\omega^{R_*}$ uniformly contracts the cone $\mathscr{C}_{\omega,a}$ on "good" fibers ω for cone parameters $a > 0$ sufficiently large. Recall that the numbers θ and ζ are given by

$$\theta := -\frac{1}{N_*}\int_\Omega \log Q_\omega^{(N_*)} dm(\omega) > 0 \quad \text{and} \quad \zeta := \frac{1}{N_*}\int_\Omega \log L_\omega^{N_*} dm(\omega) > 0.$$

Note that Lemma 1.5.8 and the ergodic theorem imply that

$$\log 6 \leq \zeta = \lim_{n\to\infty} \frac{1}{nN_*} \sum_{k=0}^{n-1} \log L^{N_*}_{\sigma^{kN_*}(\omega)}. \tag{1.6.1}$$

The following definition is adapted from [**18**, Definition 2.4].

Definition 1.6.1. We will say that ω is a *good fiber* with respect to the numbers ε, a, B_*, and $R_a = q_a N_*$ if the following hold:

(G1) $B_* q_a e^{-\frac{\theta}{2}R_a} \leq \dfrac{1}{3}$,

(G2) $\dfrac{1}{R_a} \displaystyle\sum_{k=0}^{R_a/N_*-1} \log L^{N_*}_{\sigma^{kN_*}(\omega)} \in [\zeta-\varepsilon, \zeta+\varepsilon]$.

Now, we denote

$$\varepsilon_0 := \min\left\{1, \frac{\theta}{2}\right\}. \tag{1.6.2}$$

The following lemma describes the prevalence of the good fibers as well as how to find them.

Lemma 1.6.2. *Given $\varepsilon < \varepsilon_0$ and $a > 0$, there exist parameters B_* and R_a (both of which depend on ε) such that there is a set $\Omega_G \subseteq \Omega$ of good fibers ω with $m(\Omega_G) \geq 1 - \varepsilon/4$.*

Proof. We begin by letting

$$\Omega_1 = \Omega_1(B_*) := \{\omega \in \Omega : C_\varepsilon(\omega) \leq B_*\}, \tag{1.6.3}$$

where $C_\varepsilon(\omega) > 0$ is the m-a.e. finite measurable constant coming from Proposition 1.5.9. Choose B_* sufficiently large such that $m(\Omega_1) \geq 1 - \varepsilon/8$. Noting that $\varepsilon < \theta/2$ by (1.6.2), we set $R_0 = q_0 N_*$ and choose q_0 sufficiently large such that

$$B_* q_0 e^{-(\theta-\varepsilon)R_0} \leq B_* q_0 e^{-\frac{\theta}{2}R_0} \leq \frac{1}{3}.$$

Now let $q_1 \geq q_0$ and define the set

$$\Omega_2 = \Omega_2(q_1) := \{\omega \in \Omega : \text{(G2) holds for the value } R_1 = q_1 N_*\}.$$

Now choose $q_a \geq q_1$ such that $m(\Omega_2(q_a)) \geq 1 - \varepsilon/8$. Set $R_a := q_a N_*$. Set

$$\Omega_G := \Omega_2 \cap \sigma^{-R_a}(\Omega_1). \tag{1.6.4}$$

Then Ω_G is the set of all $\omega \in \Omega$ which are good with respect to the numbers B_* and R_a, and $m(\Omega_G) \geq 1 - \varepsilon/4$. □

In what follows, given a value B_*, we will consider cone parameters

$$a \geq a_0 := 6B_* \tag{1.6.5}$$

and we set

$$q_* = q_{a_0} \quad \text{and} \quad R_* := R_{a_0} = q_* N_*. \tag{1.6.6}$$

Note that (G1) together with Proposition 1.5.9 implies that, for $\varepsilon < \varepsilon_0$ and $\omega \in \Omega_G$, we have

$$\begin{aligned}\operatorname{var}(\tilde{\mathcal{L}}_\omega^{R_*} f) &\le B_* e^{-(\theta-\varepsilon)R_*}\operatorname{var}(f) + B_*\Lambda_{\sigma^{R_*}(\omega)}(\tilde{\mathcal{L}}_\omega^{R_*} f)\\ &\le B_* q_* e^{-\frac{\theta}{2}R_*}\operatorname{var}(f) + B_*\Lambda_{\sigma^{R_*}(\omega)}(\tilde{\mathcal{L}}_\omega^{R_*} f)\\ &\le \frac{1}{3}\operatorname{var}(f) + B_*\Lambda_{\sigma^{R_*}(\omega)}(\tilde{\mathcal{L}}_\omega^{R_*} f). \qquad (1.6.7)\end{aligned}$$

The next lemma shows that the normalized operator is a contraction on the fiber cones $\mathscr{C}_{\omega,a}$ and that the image has finite diameter.

LEMMA 1.6.3. *If ω is good with respect to the numbers ε, a_0, B_*, and R_*, then for each $a \ge a_0$ we have*

$$\tilde{\mathcal{L}}_\omega^{R_*}(\mathscr{C}_{\omega,a}) \subseteq \mathscr{C}_{\sigma^{R_*}(\omega),a/2} \subseteq \mathscr{C}_{\sigma^{R_*}(\omega),a}.$$

PROOF. For ω good and $f \in \mathscr{C}_{\omega,a}$, (1.6.7) and (1.6.5) give

$$\begin{aligned}\operatorname{var}(\tilde{\mathcal{L}}_\omega^{R_*} f) &\le \frac{1}{3}\operatorname{var}(f) + B_*\Lambda_{\sigma^{R_*}(\omega)}(\tilde{\mathcal{L}}_\omega^{R_*} f)\\ &\le \frac{a}{3}\Lambda_\omega(f) + \frac{a}{6}\Lambda_{\sigma^{R_*}(\omega)}(\tilde{\mathcal{L}}_\omega^{R_*} f)\\ &\le \frac{a}{2}\Lambda_{\sigma^{R_*}(\omega)}(\tilde{\mathcal{L}}_\omega^{R_*} f).\end{aligned}$$

Hence we have

$$\tilde{\mathcal{L}}_\omega^{R_*}(\mathscr{C}_{\omega,a}) \subseteq \mathscr{C}_{\sigma^{R_*}(\omega),a/2} \subseteq \mathscr{C}_{\sigma^{R_*}(\omega),a}$$

as desired. □

1.7. Density estimates and cone invariance on bad fibers

In this section we recall the notion of "bad" fibers from [3, 18]. We show that for fibers in the small measure set,

$$\Omega_B := \Omega \backslash \Omega_G,$$

the cone $\mathscr{C}_{\omega,a}$ of positive functions is invariant after sufficiently many iterations for sufficiently large parameters $a > 0$. We accomplish this by introducing the concept of bad blocks (coating intervals), which we then show make up a relatively small portion of an orbit. As the content of this section is adapted from the closed dynamical setting of Section 7 of [3], we do not provide proofs.

Recall that R_* is given by (1.6.6). Following Section 7 of [3], and using the same justifications therein, we define the measurable function $y_* : \Omega \to \mathbb{N}$ so that

$$0 \le y_*(\omega) < R_* \qquad (1.7.1)$$

is the smallest integer such that for either choice of sign $+$ or $-$ we have

$$\lim_{n\to\infty} \frac{1}{n}\#\left\{0 \le k < n : \sigma^{\pm kR_* + y_*(\omega)}(\omega) \in \Omega_G\right\} > 1-\varepsilon, \qquad (1.7.2)$$

$$\lim_{n\to\infty} \frac{1}{n}\#\left\{0 \le k < n : C_\varepsilon\left(\sigma^{\pm kR_* + y_*(\omega)}(\omega)\right) \le B_*\right\} > 1-\varepsilon. \qquad (1.7.3)$$

Clearly, $y_* : \Omega \to \mathbb{N}$ is a measurable function such that

(1.7.4) $y_*(\sigma^{y_*(\omega)}(\omega)) = 0,$

(1.7.5) $y_*(\sigma^{R_*}(\omega)) = y_*(\omega).$

In particular, (1.7.4) and (1.7.5) together imply that

$$y_*(\sigma^{y_*(\omega)+kR_*}(\omega)) = 0 \tag{1.7.6}$$

for all $k \in \mathbb{N}$. Let

$$\Gamma(\omega) := \prod_{k=0}^{q_*-1} L^{N_*}_{\sigma^{kN_*}(\omega)}, \tag{1.7.7}$$

where q_* is given by (1.6.6), and for each $\omega \in \Omega$, given $\varepsilon > 0$, we define the *coating length* $\ell(\omega) = \ell_\varepsilon(\omega)$ as follows:

- if $\omega \in \Omega_G$, then set $\ell(\omega) := 1$,
- if $\omega \in \Omega_B$, then

$$\ell(\omega) := \min\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{0 \le k < n} \left(\mathbb{1}_{\Omega_B} \log \Gamma\right)(\sigma^{kR_*}(\omega)) \le \zeta R_* \sqrt{\varepsilon} \right\}, \tag{1.7.8}$$

where ζ is as in (1.5.24). If the minimum is not attained we set $\ell(\omega) = \infty$.

Since $L^{N_*}_\omega \ge 6^{N_*}$ by Lemma 1.5.8, we must have that

$$\Gamma(\omega) \ge 6^{R_*} \tag{1.7.9}$$

for all $\omega \in \Omega$. It follows from Lemma 1.5.8 that for all $\omega \in \Omega$ we have

$$\operatorname{var}(\tilde{\mathcal{L}}^{R_*}_\omega f) \le \Gamma(\omega)(\operatorname{var}(f) + \Lambda_{\sigma^{R_*}(\omega)}(f)). \tag{1.7.10}$$

Furthermore, if $\omega \in \Omega_G$ it follows from (G2) that

$$R_*(\zeta - \varepsilon) \le \log \Gamma(\omega) \le R_*(\zeta + \varepsilon). \tag{1.7.11}$$

The following proposition collects together some of the key properties of the coating length $\ell(\omega)$.

PROPOSITION 1.7.1. *For all $\varepsilon > 0$ sufficiently small the number $\ell(\omega)$ satisfies the following.*

(i) *For m-a.e. $\omega \in \Omega$ such that $y_*(\omega) = 0$ we have $\ell(\omega) < \infty$,*

(ii) *If $\omega \in \Omega_B$ then $\ell(\omega) \ge 2$.*

REMARK 1.7.2. Given $\omega_0 \in \Omega$, for each $j \ge 0$ let $\omega_{j+1} = \sigma^{\ell(\omega_j)R_*}(\omega_j)$. As a consequence of Proposition 1.7.1 (i) and (1.7.5), we see that for m-a.e. $\omega_0 \in \Omega$ with $y_*(\omega_0) = 0$, we must have that $\ell(\omega_j) < \infty$ for all $j \ge 0$.

DEFINITION 1.7.3. We will call a (finite) sequence $\omega, \sigma(\omega), \dots, \sigma^{\ell(\omega)R_*-1}(\omega)$ of $\ell(\omega)R_*$ fibers a *good block* (originating at ω) if $\omega \in \Omega_G$ (which implies that $\ell(\omega) = 1$). If, on the other hand, $\omega \in \Omega_B$ we call such a sequence a *bad block*, or coating interval, originating at ω.

For $\varepsilon > 0$ sufficiently small we have that $\zeta\sqrt{\varepsilon}/\log 6 < 1$, and so we let $\gamma < 1$ such that

$$\frac{\zeta\sqrt{\varepsilon}}{\log 6} < \gamma < 1. \tag{1.7.12}$$

We now wish to show that the normalized operator $\tilde{\mathcal{L}}_\omega$ is weakly contracting (i.e. non-expanding) on the fiber cones $\mathscr{C}_{\omega,a}$ for sufficiently large values of $a > a_0$. We obtain this cone invariance on blocks of length $\ell(\omega)R_*$, however in order to obtain cone contraction with a finite diameter image we will have to travel along several such blocks. For this reason we introduce the following notation.

Given $\omega \in \Omega$ with $y_*(\omega) = 0$ for each $k \geq 1$ we define the length

$$\Sigma_\omega^{(k)} := \sum_{j=0}^{k-1} \ell(\omega_j)R_*$$

where $\omega_0 := \omega$ and for each $j \geq 1$ we set $\omega_j := \sigma^{\Sigma_\omega^{(j-1)}}(\omega)$. This construction is justified as we recall from Proposition 1.7.1 that for m-a.e. $\omega \in \Omega$ with $y_*(\omega) = 0$ we have that $\ell(\omega) < \infty$. The next lemma was adapted from Lemma 7.5 of [3].

Lemma 1.7.4. *For $\varepsilon > 0$ sufficiently small, each $N \in \mathbb{N}$, and m-a.e. $\omega \in \Omega$ with $y_*(\omega) = 0$ we have that*

$$\operatorname{var}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(N)}} f\right) \leq \left(\frac{1}{3}\right)^{\Sigma_\omega^{(N)}/R_*} \operatorname{var}(f) + \frac{a_*}{6}\Lambda_{\sigma^{\Sigma_\omega^{(N)}}(\omega)}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(N)}} f). \tag{1.7.13}$$

Moreover, we have that

$$\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(N)}}(\mathscr{C}_{\omega,a_*}) \subseteq \mathscr{C}_{\sigma^{\Sigma_\omega^{(N)}}(\omega),a_*/2}, \tag{1.7.14}$$

where

$$a_* = a_*(\varepsilon) := 2a_0 e^{\zeta R_*\sqrt{\varepsilon}} = 12B_* e^{\zeta R_*\sqrt{\varepsilon}}. \tag{1.7.15}$$

Proof. Throughout the proof we will denote $\ell_i = \ell(\omega_i)$ and $L_i = \sum_{k=0}^{i-1}\ell_k$ for each $0 \leq i < N$. Then $\Sigma_\omega^{(N)} = L_N R_*$. Using (1.6.7) on good fibers and (1.7.10) on bad fibers, for any $p \geq 1$ and $f \in \mathscr{C}_{\omega,+}$ we have

(1.7.16)

$$\operatorname{var}(\tilde{\mathcal{L}}_\omega^{pR_*} f) \leq \left(\prod_{j=0}^{p-1} \Phi^{(R_*)}_{\sigma^{jR_*}(\omega)}\right) \operatorname{var}(f) + \sum_{j=0}^{p-1}\left(D^{(R_*)}_{\sigma^{jR_*}(\omega)} \cdot \prod_{k=j+1}^{p-1} \Phi^{(R_*)}_{\sigma^{kR_*}(\omega)}\right)\Lambda_{\sigma^{pR_*}(\omega)}(\tilde{\mathcal{L}}_\omega^{pR_*} f),$$

where

$$\Phi_\tau^{(R_*)} = \begin{cases} B_* e^{-(\theta-\varepsilon)R_*} & \text{for } \tau \in \Omega_G \\ \Gamma(\tau) & \text{for } \tau \in \Omega_B \end{cases} \tag{1.7.17}$$

and

$$D_\tau^{(R_*)} = \begin{cases} B_* & \text{for } \tau \in \Omega_G \\ \Gamma(\tau) & \text{for } \tau \in \Omega_B. \end{cases} \tag{1.7.18}$$

For any $0 \le i < N$ and $0 \le j < \ell_i$ we can write

$$\sum_{0\le k<\ell_i} (\mathbb{1}_{\Omega_B} \log \Gamma)(\sigma^{kR_*}(\omega_i)) = \sum_{0\le k<j} (\mathbb{1}_{\Omega_B} \log \Gamma)(\sigma^{kR_*}(\omega_i)) + \sum_{j\le k<\ell_i} (\mathbb{1}_{\Omega_B} \log \Gamma)(\sigma^{kR_*}(\omega_i)).$$

The definition of $\ell(\omega_i)$, (1.7.8), then implies that

$$\frac{1}{j} \sum_{0\le k<j} (\mathbb{1}_{\Omega_B} \log \Gamma)(\sigma^{kR_*}(\omega_i)) > \zeta R_* \sqrt{\varepsilon},$$

and consequently that

$$\frac{1}{\ell_i - j} \sum_{j\le k<\ell_i} (\mathbb{1}_{\Omega_B} \log \Gamma)(\sigma^{kR_*}(\omega_i)) \le \zeta R_* \sqrt{\varepsilon}. \tag{1.7.19}$$

Now, using (1.7.9), (1.7.19), and (1.7.12) we see that for ε sufficiently small, the proportion of bad blocks is given by

$$\begin{aligned} \frac{1}{\ell_i - j} \#\left\{ j \le k < \ell_i : \sigma^{kR_*}(\omega_i) \in \Omega_B \right\} &= \frac{1}{\ell_i - j} \sum_{j\le k<\ell_i} (\mathbb{1}_{\Omega_B})(\sigma^{kR_*}(\omega_i)) \\ &\le \frac{1}{(\ell_i - j) R_* \log 6} \sum_{j\le k<\ell_i} (\mathbb{1}_{\Omega_B} \log \Gamma)(\sigma^{kR_*}(\omega_i)) \le \gamma. \end{aligned} \tag{1.7.20}$$

In view of (1.7.17), using (1.7.19), (1.7.20), for any $0 \le i < N$ and $0 \le j < \ell_i$ we have

$$\begin{aligned} \prod_{k=j}^{\ell_i - 1} \Phi^{(R_*)}_{\sigma^{kR_*}(\omega_i)} &= \prod_{\substack{j\le k<\ell_i \\ \sigma^{kR_*}(\omega_i)\in\Omega_G}} B_* e^{-(\theta-\varepsilon)R_*} \cdot \prod_{\substack{j\le k<\ell_i \\ \sigma^{kR_*}(\omega_i)\in\Omega_B}} \Gamma(\sigma^{kR_*}(\omega_i)) \\ &\le \left(B_* e^{-(\theta-\varepsilon)R_*}\right)^{(1-\gamma)(\ell_i - j)} \cdot \exp\left(\left(\zeta R_* \sqrt{\varepsilon}\right)(\ell_i - j)\right) \\ &= \left(B_*^{1-\gamma} \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta-\varepsilon)(1-\gamma)\right) R_*\right)\right)^{\ell_i - j} \\ &< \left(B_* \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta-\varepsilon)(1-\gamma)\right) R_*\right)\right)^{\ell_i - j}. \end{aligned} \tag{1.7.21}$$

Now for any $0 \le j < L_N$ there must exist some $0 \le i_0 < N$ and some $0 \le j_0 < \ell_{i_0+1}$ such that $L_{i_0-1} + j_0 = j < L_{i_0}$. Thus, using (1.7.21) we can write

$$\begin{aligned} \prod_{k=j}^{L_N - 1} \Phi^{(R_*)}_{\sigma^{kR_*}(\omega)} &= \prod_{k=j_0}^{\ell_{i_0+1}-1} \Phi^{(R_*)}_{\sigma^{kR_*}(\omega_{i_0})} \cdot \prod_{i=i_0+1}^{N-1} \prod_{k=0}^{\ell_i - 1} \Phi^{(R_*)}_{\sigma^{kR_*}(\omega_i)} \\ &< \left(B_* \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta-\varepsilon)(1-\gamma)\right) R_*\right)\right)^{\ell_{i_0+1} - j_0} \cdot \\ &\quad \cdot \prod_{i=i_0+1}^{N-1} \left(B_* \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta-\varepsilon)(1-\gamma)\right) R_*\right)\right)^{\ell_i} \\ &= \left(B_* \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta-\varepsilon)(1-\gamma)\right) R_*\right)\right)^{L_N - j}. \end{aligned} \tag{1.7.22}$$

Now, since $B_*, \Gamma(\omega) \geq 1$ for all $\omega \in \Omega$, using (1.7.18) and (1.7.19), we have that for $0 \leq i < N$ and $0 \leq j < \ell_i$

$$D^{(R_*)}_{\sigma^{jR_*}(\omega_i)} \leq B_* \Gamma(\sigma^{jR_*}(\omega_i)) \leq B_* \cdot \prod_{\substack{j \leq k < \ell_i \\ \sigma^{kR_*}(\omega_i) \in \Omega_B}} \Gamma(\sigma^{kR_*}(\omega_i)) \leq B_* \left(e^{\zeta R_* \sqrt{\varepsilon}}\right)^{(\ell_i - j)}. \tag{1.7.23}$$

Similarly to the reasoning used to obtain (1.7.22), for any $0 \leq j < L_N$ we see that we can improve (1.7.23) so that we have

$$D^{(R_*)}_{\sigma^{jR_*}(\omega_i)} \leq B_* \left(e^{\zeta R_* \sqrt{\varepsilon}}\right)^{L_N - j}. \tag{1.7.24}$$

Thus, inserting (1.7.22) and (1.7.24) into (1.7.16) (with $p = L_N$) we see that

$$\begin{aligned}
\operatorname{var}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) &\leq \left(\prod_{j=0}^{L_N-1} \Phi^{(R_*)}_{\sigma^{jR_*}(\omega)}\right) \operatorname{var}(f) + \sum_{j=0}^{L_N-1} \left(D^{(R_*)}_{\sigma^{jR_*}(\omega)} \cdot \prod_{k=j+1}^{L_N-1} \Phi^{(R_*)}_{\sigma^{kR_*}(\omega)}\right) \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right), \\
&\leq \left(B_* \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)\right) R_*\right)\right)^{L_N} \operatorname{var}(f) \\
&\quad + \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) \sum_{j=0}^{L_N-1} B_* \left(e^{\zeta R_* \sqrt{\varepsilon}}\right)^{(L_N - j)} \cdot \left(B_* \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)\right) R_*\right)\right)^{L_N - j - 1} \\
&= \left(B_* \exp\left(\left(\zeta\sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)\right) R_*\right)\right)^{L_N} \operatorname{var}(f) \\
&\quad + B_* e^{\zeta R_* \sqrt{\varepsilon}} \cdot \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) \sum_{j=0}^{L_N-1} \left(B_* \exp\left(\left(2\zeta\sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)\right) R_*\right)\right)^{L_N - j - 1}.
\end{aligned}$$

Therefore, taking $\varepsilon > 0$ sufficiently small[1] in conjunction with (G1), we have that

$$\begin{aligned}
\operatorname{var}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) &\leq \left(B_* e^{-\frac{\theta}{2} R_*}\right)^{L_N} \operatorname{var}(f) + B_* e^{\zeta R_* \sqrt{\varepsilon}} \cdot \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) \sum_{j=0}^{L_N-1} \left(B_* e^{-\frac{\theta}{2} R_*}\right)^{L_N - j - 1} \\
&\leq \left(\frac{1}{3}\right)^{L_N} \operatorname{var}(f) + B_* e^{\zeta R_* \sqrt{\varepsilon}} \cdot \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) \sum_{j=0}^{L_N-1} \left(\frac{1}{3}\right)^{L_N - j - 1},
\end{aligned}$$

and so we must have that

$$\begin{aligned}
\operatorname{var}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) &\leq \left(\frac{1}{3}\right)^{L_N} \operatorname{var}(f) + 2B_* e^{\zeta R_* \sqrt{\varepsilon}} \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right) \\
&= \left(\frac{1}{3}\right)^{L_N} \operatorname{var}(f) + \frac{a_*}{6} \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right),
\end{aligned}$$

which proves the first claim. Thus, for any $f \in \mathscr{C}_{\omega, a_*}$ we have that

$$\operatorname{var}(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f) \leq \left(\frac{1}{3}\right)^{L_N} \Lambda_\omega(f) + \frac{a_*}{6} \Lambda_{\sigma^{\Sigma^{(N)}_\omega}(\omega)}\left(\tilde{\mathcal{L}}^{\Sigma^{(N)}_\omega}_\omega f\right)$$

[1] Any $\varepsilon < \min\left\{\left(\frac{\log 55}{4\zeta}\right)^2, \left(\frac{\theta}{8\zeta}\right)^2\right\}$ such that $\frac{\sqrt{\varepsilon}\zeta}{2\log 55} \leq \gamma$, which implies $-\frac{\theta}{2} > 2\sqrt{\varepsilon}\zeta - (\theta - \varepsilon)(1-\gamma) > \sqrt{\varepsilon}\zeta - (\theta - \varepsilon)(1-\gamma)$, will suffice; see Observation 7.4 of [3] for details.

$$\leq \frac{a_*}{3}\Lambda_{\sigma^{\Sigma_\omega^{(N)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(N)}} f\right) + \frac{a_*}{6}\Lambda_{\sigma^{\Sigma_\omega^{(N)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(N)}} f\right),$$

where we have used the fact that $a_* > 1$, and consequently we have

$$\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(N)}}(\mathscr{C}_{\omega,a_*}) \subseteq \mathscr{C}_{\sigma^{\Sigma_\omega^{(N)}}(\omega),a_*/2} \subseteq \mathscr{C}_{\sigma^{\Sigma_\omega^{(N)}}(\omega),a_*}$$

as desired. □

The next lemma shows that the total length of the bad blocks take up only a small proportion of an orbit, however before stating the result we establish the following notation. For each $n \in \mathbb{N}$ we let $K_n \geq 0$ be the integer such that

$$n = K_n R_* + h(n) \tag{1.7.25}$$

where $0 \leq h(n) < R_*$ is a remainder term. Given $\omega_0 \in \Omega$, let

$$\omega_j = \sigma^{\ell(\omega_{j-1})R_*}(\omega_{j-1}) \tag{1.7.26}$$

for each $j \geq 1$. Then for each $n \in \mathbb{N}$ we can break the n-length σ-orbit of ω_0 in Ω into $k_{\omega_0}(n) + 1$ blocks of length $\ell(\omega_j)R_*$ (for $0 \leq j \leq k_{\omega_0}(n)$) plus some remaining block of length $r_{\omega_0}(n)R_*$ where $0 \leq r_{\omega_0}(n) < \ell(\omega_{k_{\omega_0}(n)+1})$ plus a remainder segment of length $h(n)$, i.e. we can write

$$n = \sum_{0\leq j\leq k_{\omega_0}(n)} \ell(\omega_j)R_* + r_{\omega_0}(n)R_* + h(n); \tag{1.7.27}$$

see Figure 1. We also note that (1.7.25) and (1.7.27) imply that

$$K_n = \sum_{0\leq j\leq k_{\omega_0}(n)} \ell(\omega_j) + r_{\omega_0}(n). \tag{1.7.28}$$

The proof of the following lemma is nearly identical to that of Lemma 7.6 of [3], and

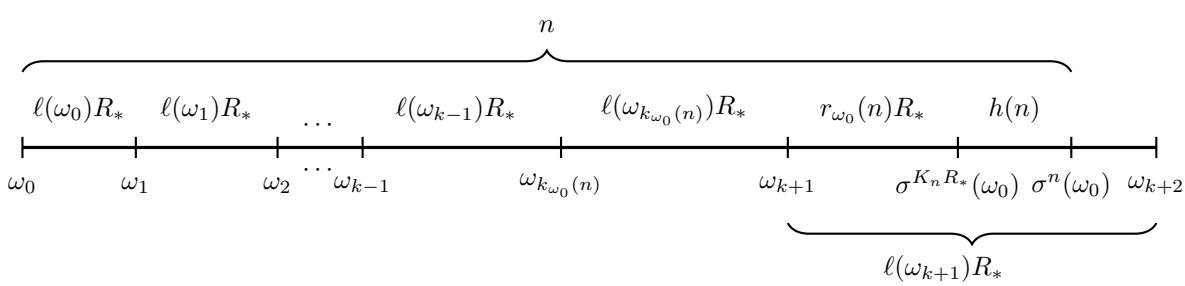

FIGURE 1. The decomposition of $n = \sum_{0\leq j\leq k_{\omega_0}(n)} \ell(\omega_j)R_* + r_{\omega_0}(n)R_* + h(n)$ and the fibers ω_j.

therefore it shall be omitted.

LEMMA 1.7.5. *There exists a measurable function $N_0 : \Omega \to \mathbb{N}$ such that for all $n \geq N_0(\omega_0)$ and for m-a.e. $\omega_0 \in \Omega$ with $y_*(\omega_0) = 0$ we have*

$$E_{\omega_0}(n) := \sum_{\substack{0\leq j\leq k_{\omega_0}(n)\\ \omega_j\in\Omega_B}} \ell(\omega_j) + r_{\omega_0}(n) < Y \cdot \varepsilon K_n \leq \frac{Y}{R_*}\varepsilon n$$

where

$$Y = Y_\varepsilon := \frac{2(2+\zeta)R_*}{\zeta R_* \sqrt{\varepsilon}}, \tag{1.7.29}$$

and where K_n is as in (1.7.25), ω_j is as in (1.7.26), and $k_{\omega_0}(n)$ and $r_{\omega_0}(n)$ are as in (1.7.27).

To end this section we note that

$$\varepsilon \cdot Y_\varepsilon \to 0 \tag{1.7.30}$$

as $\varepsilon \to 0$. For the remainder of the document we will assume that $\varepsilon > 0$ is always taken sufficiently small such that the results of Section 1.7 apply.

1.8. Further properties of Λ_ω

In this section we prove some additional properties of the functional Λ_ω that will be necessary in Section 1.9 to obtain cone contraction with finite diameter. In particular, in the main result of this section, which is a version of Lemma 3.11 of [**53**] and dates back to [**52**, Lemma 3.2], we show that for a function $f \in \mathscr{C}_{\omega,a_*}$ there exists a partition element on which the function f takes values at least as large as $\Lambda_\omega(f)/4$.

Now we prove the following upper and lower bounds for $\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} f\right)$.

Lemma 1.8.1. *For m-a.e. $\omega \in \Omega$ such that $y_*(\omega) = 0$, and each $k \in \mathbb{N}$ we have that*

$$\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega\right)\Lambda_\omega(f) \le \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} f\right) \le a_* \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega\right)\Lambda_\omega(f).$$

Proof. From Lemma 1.5.4 we already see that the first inequality holds. Now, fix $\omega \in \Omega$ (with $y_*(\omega) = 0$) and $k \in \mathbb{N}$. To see the other inequality we first let $n, N \in \mathbb{N}$ and $x \in D_{\sigma^N(\omega),N+n}$, then

$$\begin{aligned}
\frac{\tilde{\mathcal{L}}_\omega^{N+n}(f)(x)}{\tilde{\mathcal{L}}_{\sigma^N(\omega)}^n(\mathbb{1}_{\sigma^N(\omega)})(x)} &= \frac{\tilde{\mathcal{L}}_\omega^{N+n}(f)(x)}{\tilde{\mathcal{L}}_\omega^{N+n}(\mathbb{1}_\omega)(x)} \cdot \frac{\tilde{\mathcal{L}}_\omega^{N+n}(\mathbb{1}_\omega)(x)}{\tilde{\mathcal{L}}_{\sigma^N(\omega)}^n(\mathbb{1}_{\sigma^N(\omega)})(x)} \\
&= \frac{\tilde{\mathcal{L}}_\omega^{N+n}(f)(x)}{\tilde{\mathcal{L}}_\omega^{N+n}(\mathbb{1}_\omega)(x)} \cdot \frac{\tilde{\mathcal{L}}_{\sigma^N(\omega)}^n\left(\mathcal{L}_\omega^N(\mathbb{1}_\omega)\cdot \mathbb{1}_{\sigma^N(\omega)}\right)(x)}{\tilde{\mathcal{L}}_{\sigma^N(\omega)}^n(\mathbb{1}_{\sigma^N(\omega)})(x)} \\
&\le \frac{\tilde{\mathcal{L}}_\omega^{N+n}(f)(x)}{\tilde{\mathcal{L}}_\omega^{N+n}(\mathbb{1}_\omega)(x)} \cdot \|\tilde{\mathcal{L}}_\omega^N \mathbb{1}_\omega\|_\infty.
\end{aligned}$$

Now taking the infimum over $x \in D_{\sigma^N(\omega),N+n}$ and letting $n \to \infty$ gives

$$\Lambda_{\sigma^N(\omega)}(\tilde{\mathcal{L}}_\omega^N f) \le \|\tilde{\mathcal{L}}_\omega^N \mathbb{1}_\omega\|_\infty \Lambda_\omega(f). \tag{1.8.1}$$

Now, set $N = \Sigma_\omega^{(k)}$. Since $\mathbb{1}_\omega \in \mathscr{C}_{\omega,a_*}$, (1.7.14) from Lemma 1.7.4 implies that

$$\begin{aligned}
\|\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega\|_\infty &\le \operatorname{var}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_\omega)) + \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_\omega)) \\
&\le \left(\frac{a_*}{2} + 1\right)\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_\omega))
\end{aligned}$$

$$\leq a_* \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_\omega)), \tag{1.8.2}$$

where we have used the fact that $a_* > 2$ which follows from (1.7.15). Combining (1.8.2) with (1.8.1), we see that

$$\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} f) \leq a_* \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega)\Lambda_\omega(f)$$

completing the proof. □

LEMMA 1.8.2. *For each $\delta > 0$ and each $\omega \in \Omega$ there exists $N_{\omega,\delta}$ such that for each $n \geq N_{\omega,\delta}$, $\mathcal{Z}_\omega^{(n)}$ has the property that*

$$\sup_{Z \in \mathcal{Z}_\omega^{(n)}} \Lambda_\omega(\mathbb{1}_Z) \leq \delta.$$

PROOF. Choose $N_{\omega,\delta} \in \mathbb{N}$ such that

$$\frac{\|g_\omega^{(n)}\|_\infty}{\rho_\omega^n} \leq \delta$$

for each $n \geq N_{\omega,\delta}$. Now, fix some $n \geq N_{\omega,\delta}$ and let $m \in \mathbb{N}$. Then, for $Z \in \mathcal{Z}_\omega^{(n)}$ we have

$$\mathcal{L}_\omega^n \mathbb{1}_Z(x) \leq \|g_\omega^{(n)}\|_\infty \leq \delta \rho_\omega^n.$$

For each $x \in D_{\sigma^{n+m}(\omega),n+m} \subseteq D_{\sigma^{n+m}(\omega),m}$ we have

$$\begin{aligned}\frac{\mathcal{L}_\omega^{n+m}\mathbb{1}_Z(x)}{\mathcal{L}_\omega^{n+m}\mathbb{1}_\omega(x)} &\leq \frac{\|\mathcal{L}_\omega^n \mathbb{1}_Z\|_\infty \mathcal{L}_{\sigma^n(\omega)}^m \mathbb{1}_{\sigma^n(\omega)}(x)}{\mathcal{L}_\omega^{n+m}\mathbb{1}_\omega(x)} \leq \frac{\delta\rho_\omega^n \mathcal{L}_{\sigma^n(\omega)}^m \mathbb{1}_{\sigma^n(\omega)}(x)}{\mathcal{L}_{\sigma^n(\omega)}^m\left(\mathcal{L}_\omega^n \mathbb{1}_\omega\right)(x)} \\ &\leq \delta\rho_\omega^n \frac{1}{\inf_{y \in D_{\sigma^{n+m}(\omega),m}} \frac{\mathcal{L}_{\sigma^n(\omega)}^m(\mathcal{L}_\omega^n \mathbb{1}_\omega)(y)}{\mathcal{L}_{\sigma^n(\omega)}^m \mathbb{1}_{\sigma^n(\omega)}(y)}}.\end{aligned}$$

In view of Lemma 1.5.4, taking the infimum over $x \in D_{\sigma^{n+m}(\omega),n+m}$ and letting $m \to \infty$ gives

$$\Lambda_\omega(\mathbb{1}_Z) \leq \delta\rho_\omega^n \cdot \frac{1}{\Lambda_{\sigma^n(\omega)}(\mathcal{L}_\omega^n \mathbb{1}_\omega)} \leq \delta\rho_\omega^n(\rho_\omega^n)^{-1} = \delta.$$

□

We are now ready to prove the main result of this section, a random version of Lemma 3.11 in [53]. Let

$$\delta_0 := \frac{1}{8a_*^3}. \tag{1.8.3}$$

LEMMA 1.8.3. *For m-a.e. $\omega \in \Omega$ with $y_*(\omega) = 0$, for all $\delta < \delta_0$, all $n \geq N_{\omega,\delta}$ (where $N_{\omega,\delta}$ is as in Lemma 1.8.2), and all $f \in \mathscr{C}_{\omega,a_*}$ there exists $Z_f \in \mathcal{Z}_{\omega,g}^{(n)}$ such that*

$$\inf_{Z_f} f \geq \frac{1}{4}\Lambda_\omega(f).$$

PROOF. We shall the prove the lemma via contradiction. To that end suppose that the conclusion is false, that is we suppose that

$$\inf_Z f < \frac{\Lambda_\omega(f)}{4} \tag{1.8.4}$$

for all $Z \in \mathcal{Z}_{\omega,g}^{(n)}$. Then, for each $n \geq N_{\omega,\delta}$ and each $k \in \mathbb{N}$ such that $n < \Sigma_\omega^{(k)}$, using (1.8.4) we can write

$$\begin{aligned}
\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} f &= \sum_{Z \in \mathcal{Z}_\omega^{(n)}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(f\mathbb{1}_Z) = \sum_{Z \in \mathcal{Z}_{\omega,*}^{(n)}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(f\mathbb{1}_Z) \\
&= \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(f\mathbb{1}_Z) + \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(f\mathbb{1}_Z) \\
&\leq \frac{\Lambda_\omega(f)}{4} \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z) + \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\mathrm{var}_Z(f) + \|f\|_\infty \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z).
\end{aligned} \tag{1.8.5}$$

Now for $Z \in \mathcal{Z}_{\omega,b}^{(n)}$, Lemma 1.8.1 implies that

$$\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) \leq a_* \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}\mathbb{1}_\omega\right)\Lambda_\omega(\mathbb{1}_Z) = 0. \tag{1.8.6}$$

Thus, for $Z \in \mathcal{Z}_{\omega,b}^{(n)}$, using (1.8.6) and (1.7.13), applied along the blocks $\Sigma_\omega^{(k)}$, we have that

$$\begin{aligned}
\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z) &\leq \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) + \mathrm{var}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) \\
&= \mathrm{var}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) \\
&\leq 2\left(\frac{1}{3}\right)^{\Sigma_\omega^{(k)}/R_*} + a_*\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) = 2\left(\frac{1}{3}\right)^{\Sigma_\omega^{(k)}/R_*}.
\end{aligned} \tag{1.8.7}$$

Note that the right-hand side above goes to zero as $k \to \infty$. On the other hand, for $Z \in \mathcal{Z}_{\omega,g}^{(n)}$ we again use (1.7.13) in conjunction with Lemma 1.8.1 to get that

$$\begin{aligned}
\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z) &\leq \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) + \mathrm{var}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) \\
&\leq 2\left(\frac{1}{3}\right)^{\Sigma_\omega^{(k)}/R_*} + a_*\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}(\mathbb{1}_Z)\right) \\
&\leq 2\left(\frac{1}{3}\right)^{\Sigma_\omega^{(k)}/R_*} + a_*^2\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}\mathbb{1}_\omega\right)\Lambda_\omega(\mathbb{1}_Z).
\end{aligned} \tag{1.8.8}$$

Substituting (1.8.8) and (1.8.7) into (1.8.5), applying the functional $\Lambda_{\sigma^{\Sigma_\omega^{(k)}}}$ to both sides yields

$$\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} f\right) \leq \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}}\mathbb{1}_\omega\right) \cdot \frac{\Lambda_\omega(f)}{4}$$

$$
\begin{aligned}
&+ \sum_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \left(\left(a_*\Lambda_\omega(\mathbb{1}_Z) + 2\left(\frac{1}{3}\right)^{\Sigma_\omega^{(k)}/R_*} + a_*^2\Lambda_\omega(\mathbb{1}_Z) \right) \mathrm{var}_Z(f) \right) \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)} \left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega \right) \\
&+ \sum_{Z\in\mathcal{Z}_{\omega,b}^{(n)}} 2\left(\frac{1}{3}\right)^{\Sigma_\omega^{(k)}/R_*} \|f\|_\infty \Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)} \left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega \right).
\end{aligned} \tag{1.8.9}
$$

Dividing (1.8.9) on both sides by $\Lambda_{\sigma^{\Sigma_\omega^{(k)}}(\omega)} \left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega \right)$, letting $k \to \infty$, and using Lemmas 1.8.1, 1.8.2, and 1.7.4 gives us

$$
\begin{aligned}
\Lambda_\omega(f) &\le \frac{\Lambda_\omega(f)}{4} + \sum_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} (a_* + a_*^2)\Lambda_\omega(\mathbb{1}_Z)\mathrm{var}_Z(f) \\
&\le \frac{\Lambda_\omega(f)}{4} + (a_* + a_*^2)\mathrm{var}(f) \sup_{Z\in\mathcal{Z}_{\omega,g}^{(n)}} \Lambda_\omega(\mathbb{1}_Z) \\
&\le \left(\frac{1}{4} + 2a_*^3\delta \right) \Lambda_\omega(f).
\end{aligned}
$$

Given our choice (1.8.3) of $\delta < \delta_0$ we arrive at the contradiction

$$
\Lambda_\omega(f) \le \frac{1}{2}\Lambda_\omega(f),
$$

and thus we are done. □

1.9. Finding finite diameter images

We now find a large measure set of fibers $\omega \in \Omega_F \subseteq \Omega$ for which the image of the cone $\mathscr{C}_{\omega,a_*}$ has a finite diameter image after sufficiently many iterates of the normalized operator $\tilde{\mathcal{L}}_\omega$. Towards accomplishing this task we first recall that for each $\omega \in \Omega$ with $y_*(\omega) = 0$ and each $k \ge 0$

$$
\Sigma_\omega^{(k)} := \sum_{j=0}^{k-1} \ell(\omega_j) R_*
$$

where $\omega_0 := \omega$ and for each $j \ge 1$ we set $\omega_j := \sigma^{\Sigma_\omega^{(j-1)}}(\omega)$. For each $\omega \in \Omega$ with $y_*(\omega) = 0$, we define the number

$$
\Sigma_\omega := \min \left\{ \Sigma_\omega^{(k)} : \inf_{x\in D_{\sigma^{\Sigma_\omega^{(k)}}(\omega),\Sigma_\omega^{(k)}}} \frac{\mathcal{L}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_Z(x)}{\mathcal{L}_\omega^{\Sigma_\omega^{(k)}} \mathbb{1}_\omega(x)} \ge \frac{\Lambda_\omega(\mathbb{1}_Z)}{2} \text{ for all } Z \in \mathcal{Z}_{\omega,g}^{(N_{\omega,\delta_0})} \right\}. \tag{1.9.1}
$$

Note that by definition we must have that $\Sigma_\omega \ge N_{\omega,\delta_0}$. Recall from the proof of Lemma 1.6.2 that the set $\Omega_1 = \Omega_1(B_*)$ is given by

$$
\Omega_1 := \{\omega \in \Omega : C_\varepsilon(\omega) \le B_*\}, \tag{1.9.2}
$$

where $C_\varepsilon(\omega)$ comes from Proposition 1.5.9 and B_* was chosen sufficiently large such that $m(\Omega_1) \geq 1 - \varepsilon/8$. For $\alpha_* > 0$ and $C_* \geq 1$ we consider the following

(F1) $\Lambda_{\sigma^{\Sigma_\omega}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_Z\right) \geq \alpha_*$ for all $Z \in \mathcal{Z}_{\omega,g}^{(N_{\omega,\delta_0})}$,

(F2) $C_*^{-1} \leq \inf\limits_{D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_\omega \leq \|\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_\omega\|_\infty \leq C_*$.

Now, we define the set Ω_3, depending on parameters $S_* = kR_*$ for some $k \in \mathbb{N}$, $\alpha_* > 0$, and $C_* \geq 1$, by

$$\Omega_3 = \Omega_3(S_*, \alpha_*, C_*) := \{\omega \in \Omega : \Sigma_\omega \leq S_*, \text{ and (F1)} - \text{(F2) hold }\}, \tag{1.9.3}$$

and choose $S_* = kR_*$, $\alpha_* > 0$, and $C_* \geq 1$ such that $m(\Omega_3) \geq 1 - \varepsilon/8$. Finally, we define

$$\Omega_F := \Omega_1 \cap \Omega_3, \tag{1.9.4}$$

which must of course have measure $m(\Omega_F) \geq 1 - \varepsilon$. Furthermore, in light of the definition of Ω_G from (1.6.4), we have that

$$\sigma^{-R_*}(\Omega_F) \subseteq \Omega_G.$$

Lemma 1.9.1. *For all $\omega \in \Omega_F$ such that $y_*(\omega) = 0$ we have that*

$$\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathscr{C}_{\omega,a_*} \subseteq \mathscr{C}_{\sigma^{\Sigma_\omega}(\omega),a_*/2} \subseteq \mathscr{C}_{\sigma^{\Sigma_\omega}(\omega),a_*}$$

with

$$\operatorname{diam}_{\sigma^{\Sigma_\omega}(\omega),a_*}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathscr{C}_{\omega,a_*}\right) \leq \Delta := 2\log\frac{8C_*^2 a_*(3+a_*)}{\alpha_*} < \infty. \tag{1.9.5}$$

Proof. The invariance follows from Lemma 1.7.4. To show that the diameter is finite we first note that for $0 \not\equiv f \in \mathscr{C}_{\omega,a_*}$ we must have that $\Lambda_\omega(f) > 0$ by definition. Now, Lemma 1.4.8 implies that for $f \in \mathscr{C}_{\omega,a_*}$ we have

$$\Theta_{\sigma^{\Sigma_\omega}(\omega),a_*}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f, \mathbb{1}_{\sigma^{\Sigma_\omega}(\omega)}) \leq \log\frac{\|\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f\|_\infty + \frac{1}{2}\Lambda_{\sigma^{\Sigma_\omega}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f\right)}{\min\left\{\inf_{D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}}\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f, \frac{1}{2}\Lambda_{\sigma^{\Sigma_\omega}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f\right)\right\}}. \tag{1.9.6}$$

Using Lemmas 1.7.4 and 1.8.1 and (F2) we bound the numerator by

$$\begin{aligned}\|\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f\|_\infty + \frac{1}{2}\Lambda_{\sigma^{\Sigma_\omega}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f\right) &\leq \operatorname{var}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f\right) + \frac{3}{2}\Lambda_{\sigma^{\Sigma_\omega}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}f\right)\\ &\leq \frac{3a_* + a_*^2}{2}\Lambda_{\sigma^{\Sigma_\omega}(\omega)}\left(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_\omega\right)\Lambda_\omega(f)\\ &\leq \frac{C_* a_*(3+a_*)}{2}\Lambda_\omega(f).\end{aligned} \tag{1.9.7}$$

To find a lower bound for the denominator we first note that for each $f \in \mathscr{C}_{\omega,a_*}$, by Lemma 1.8.3 there exists $Z_f \in \mathcal{Z}_{\omega,g}^{(N_{\omega,\delta_0})}$ such that

$$\inf f|_{Z_f} \geq \frac{\Lambda_\omega(f)}{4}. \tag{1.9.8}$$

Thus, using (1.9.8), for each $x \in D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}$ we have that

$$\begin{aligned}\inf_{x\in D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega} f(x) &\geq \inf_{x\in D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega}(f\mathbb{1}_{Z_f})(x)\\ &\geq \inf_{Z_f} f \cdot \inf_{x\in D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_{Z_f}(x)\\ &\geq \frac{\Lambda_\omega(f)}{4} \inf_{x\in D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_{Z_f}(x)\\ &\geq \frac{\Lambda_\omega(f)}{4} \inf_{y\in D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \frac{\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_{Z_f}(y)}{\tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_\omega(y)} \inf_{z\in D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega}\mathbb{1}_\omega(z).\end{aligned}$$

In light of conditions (F1)-(F2) we in fact have that

$$\inf_{D_{\sigma^{\Sigma_\omega}(\omega),\Sigma_\omega}} \tilde{\mathcal{L}}_\omega^{\Sigma_\omega} f \geq \frac{\alpha_*\Lambda_\omega(f)}{8C_*} > 0. \tag{1.9.9}$$

Combining the estimates (1.9.7) and (1.9.9) with (1.9.6) gives

$$\Theta_{\sigma^{\Sigma_\omega}(\omega),a_*}(\tilde{\mathcal{L}}_\omega^{\Sigma_\omega} f, \mathbb{1}_{\sigma^{\Sigma_\omega}(\omega)}) \leq \log \frac{8C_*^2 a_*(3+a_*)}{\alpha_*} < \infty.$$

Taking the supremum over all functions $f \in \mathscr{C}_{\omega,a_*}$, and applying the triangle inequality finishes the proof. □

To end this section we recall from Section 1.7 that $0 \leq y_*(\omega) < R_*$ is chosen to be the smallest integer such that for either choice of sign $+$ or $-$ we have

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq k < n : \sigma^{\pm kR_* + y_*(\omega)}(\omega) \in \Omega_G\right\} > 1-\varepsilon, \tag{1.9.10}$$

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq k < n : C_\varepsilon\left(\sigma^{\pm kR_* + y_*(\omega)}(\omega)\right) \leq B_*\right\} > 1-\varepsilon. \tag{1.9.11}$$

In light of the definition of Ω_F (1.9.4) and using the same reasoning as in Section 1.7 for the existence of y_* (see Section 7 of [3]), for each $\omega \in \Omega$, we now let $0 \leq v_*(\omega) < R_*$ be the least integer such that for either choice of sign $+$ or $-$ we have that the following hold:

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq k < n : \sigma^{\pm kR_* + v_*(\omega)}(\omega) \in \Omega_G\right\} > 1-\varepsilon, \tag{1.9.12}$$

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq k < n : \sigma^{\pm kR_* + v_*(\omega)}(\omega) \in \Omega_F\right\} > 1-\varepsilon. \tag{1.9.13}$$

Two significant properties of v_* are the following:

$$v_*(\sigma^{v_*(\omega)}(\omega)) = 0, \tag{1.9.14}$$

$$\text{if } v_*(\omega) = 0, \text{ then } y_*(\omega) = 0. \tag{1.9.15}$$

1.10. Conformal and invariant measures

We are now ready to bring together all of the results from Sections 1.5-1.9 to establish the existence of conformal and invariant measures supported in the survivor set $X_{\omega,\infty}$. We follow the methods of [3] and [53], and we begin with the following technical lemma from which the rest of our results will follow.

LEMMA 1.10.1. *Let $f, h \in \mathrm{BV}_\Omega(I)$, let $\varepsilon > 0$ sufficiently small such that the results of Section 1.7 apply, and let $V : \Omega \to (0, \infty)$ be a measurable function. Suppose that for each $n \in \mathbb{N}$, each $|p| \le n$, each $l \ge 0$, and for m-a.e. $\omega \in \Omega$ we have $f_{\sigma^p(\omega)} \in \mathscr{C}_{\sigma^p(\omega),+}$ with $\mathrm{var}(f_{\sigma^p(\omega)}) \le e^{\varepsilon n} V(\omega)$ and $h_{\sigma^{p-l}(\omega)} \in \mathscr{C}_{\sigma^{p-l}(\omega),+}$ with $\mathrm{var}(h_{\sigma^{p-l}(\omega)}) \le e^{\varepsilon(n+l)} V(\omega)$. Then there exists $\vartheta \in (0,1)$ and a measurable function $N_3 : \Omega \to \mathbb{N}$ such that for all $n \ge N_3(\omega)$, all $l \ge 0$, and all $|p| \le n$ we have*

$$\Theta_{\sigma^{n+p}(\omega),+}\left(\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)}, \tilde{\mathcal{L}}^{n+l}_{\sigma^{p-l}(\omega)} h_{\sigma^{p-l}(\omega)}\right) \le \Delta \vartheta^n. \tag{1.10.1}$$

Furthermore, Δ, defined in (1.9.5), and ϑ do not depend on V.

PROOF. We begin by noting that by (1.5.11) for each $l \ge 0$ we have that $\tilde{\mathcal{L}}^l_{\sigma^{p-l}(\omega)} h_{\sigma^{p-l}(\omega)} \in \mathscr{C}_{\sigma^p(\omega),+}$ for each $h_{\sigma^{p-l}(\omega)} \in \mathscr{C}_{\sigma^{p-l}(\omega),+}$, and let

$$h_l = \tilde{\mathcal{L}}^l_{\sigma^{p-l}(\omega)} h_{\sigma^{p-l}(\omega)}.$$

Set $v_* = v_*(\sigma^p(\omega))$ (defined in Section 1.7) and let $d_* = d_*(\sigma^p(\omega)) \ge 0$ be the smallest integer that satisfies

$$v_* + d_* R_* \ge \frac{\varepsilon n + \log V(\omega)}{\theta - \varepsilon}, \tag{1.10.2}$$

$$\sigma^{p+v_*+d_*R_*}(\omega) \in \Omega_F. \tag{1.10.3}$$

where θ was defined in (1.5.21). Choose

$$N_1(\omega) \ge \frac{\log V(\omega)}{\varepsilon} \tag{1.10.4}$$

and let $n \ge N_1(\omega)$. Now using (1.10.4) to write

$$\frac{4\varepsilon n}{\theta} = \frac{\varepsilon n + \varepsilon n}{\theta/2} \ge \frac{\varepsilon n + \log V(\omega)}{\theta/2}$$

and then using (1.6.2) and (1.7.1), we see that (1.10.2) is satisfied for any $d_* R_* \ge 4\varepsilon n/\theta$. Using (1.9.13), the construction of v_*, and the ergodic decomposition of σ^{R_*} following (1.9.13), we have for m-a.e. $\omega \in \Omega$ there is an infinite, increasing sequence of integers $d_j \ge 0$ satisfying (1.10.3). Furthermore, (1.9.13) implies that

$$\lim_{n\to\infty} \frac{1}{n/R_*} \# \left\{ 0 \le k < \frac{n}{R_*} : \sigma^{\pm k R_* + v_*(\omega)}(\omega) \notin \Omega_F \right\} < \varepsilon,$$

and thus for $n \in \mathbb{N}$ sufficiently large (depending measurably on ω), say $n \ge N_2(\omega) \ge N_1(\omega)$, we have that

$$\# \left\{ 0 \le k < \frac{n}{R_*} : \sigma^{\pm k R_* + v_*(\omega)}(\omega) \notin \Omega_F \right\} < \frac{\varepsilon n}{R_*}.$$

Thus the smallest integer d_* satisfying (1.10.2) and (1.10.3) also satisfies

$$d_* R_* \le \frac{4\varepsilon n}{\theta} + \varepsilon n = \left(\frac{4+\theta}{\theta}\right) \varepsilon n. \tag{1.10.5}$$

Let

$$\hat{v}_* = v_* + d_* R_*. \tag{1.10.6}$$

Now, we wish to examine the iteration of our operator cocycle along a collection ΣR_* of

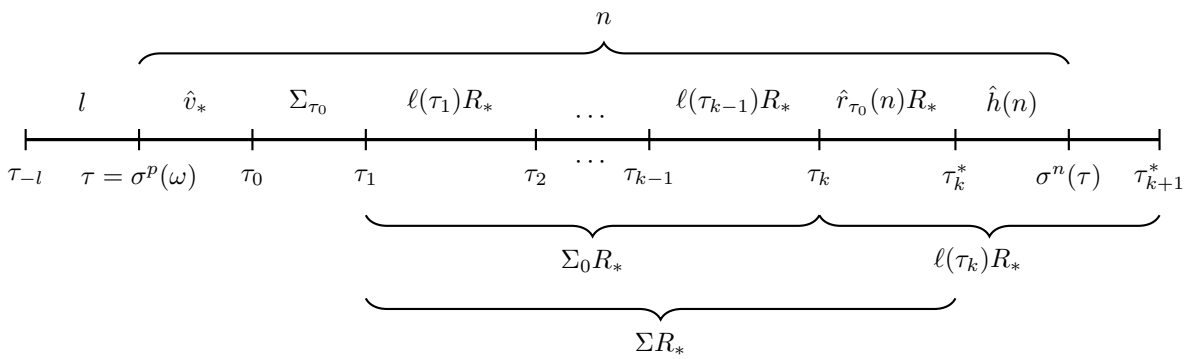

FIGURE 2. The fibers τ_j and the decomposition of $n = \hat{v}_* + \Sigma_{\tau_0} + \Sigma R_* + \hat{h}(n)$.

blocks, each of length $\ell(\omega)R_*$, so that the images of $\tilde{\mathcal{L}}_\omega^{\ell(\omega)R_*}$ are contained in $\mathscr{C}_{\sigma^{\ell(\omega)R_*}(\omega),a_*/2}$ as in Lemma 1.7.4; see Figure 2.

We begin by establishing some simplifying notation. To that end, set $\tau = \sigma^p(\omega)$, $\tau_{-l} = \sigma^{p-l}(\omega)$, and $\tau_0 = \sigma^{p+\hat{v}_*}(\omega)$; see Figure 2. Note that in light of (1.9.14), (1.9.15), and (1.10.6) we have that

$$v_*(\tau_0) = y_*(\tau_0) = 0. \tag{1.10.7}$$

Now, by our choice of d_*, we have that if $f_\tau \in \mathscr{C}_{\tau,+}$ with $\mathrm{var}(f_\tau) \le e^{\varepsilon n} V(\omega)$, then

$$\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau \in \mathscr{C}_{\tau_0,a_*}. \tag{1.10.8}$$

Indeed, applying Proposition 1.5.9, (1.10.2), (1.10.3), and the definition of Ω_F (1.9.4), we have

$$\begin{aligned}
\mathrm{var}\left(\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau\right) &\le C_\varepsilon(\sigma^{\hat{v}_*}(\tau)) e^{-(\theta-\varepsilon)\hat{v}_*} \mathrm{var}(f_\tau) + C_\varepsilon(\sigma^{\hat{v}_*}(\tau)) \Lambda_{\sigma^{\hat{v}_*}(\tau)}\left(\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau\right) \\
&\le B_* e^{-(\theta-\varepsilon)\hat{v}_*} \mathrm{var}(f_\tau) + B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)}\left(\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau\right) \\
&\le B_* \frac{\mathrm{var}(f_\tau)}{e^{\varepsilon n} V(\omega)} + B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)}\left(\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau\right) \\
&\le B_* + B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)}\left(\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau\right) \\
&\le 2B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)}\left(\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau\right) \le \frac{a_*}{6} \Lambda_{\sigma^{\hat{v}_*}(\tau)}\left(\tilde{\mathcal{L}}_\tau^{\hat{v}_*} f_\tau\right),
\end{aligned}$$

where we recall that $a_* > 12B_*$ is defined in (1.7.15). A similar calculation yields that if $h_{\tau_{-l}} \in \mathscr{C}_{\tau_{-l},+}$ with $\mathrm{var}(h_{\tau_{-l}}) \le e^{\varepsilon(n+l)} V(\omega)$, then $\tilde{\mathcal{L}}_{\tau_{-l}}^{l+\hat{v}_*} h_{\tau_{-l}} \in \mathscr{C}_{\tau_0,a_*}$.

We now set $\tau_1 = \sigma^{\Sigma_{\tau_0}}(\tau_0)$ and for each $j \ge 2$ let $\tau_j = \sigma^{\ell(\tau_{j-1})R_*}(\tau_{j-1})$. Note that since $\tau_0 \in \Omega_F$, we have that $\Sigma_{\tau_0} \le S_*$.

As there are only finitely many blocks (good and bad) that will occur within an orbit of length n, let $k \geq 1$ be the integer such that

$$\hat{v}_* + \Sigma_{\tau_0} + \sum_{j=1}^{k-1} \ell(\tau_j) R_* \leq n < \hat{v}_* + \Sigma_{\tau_0} + \sum_{j=1}^{k} \ell(\tau_j) R_*,$$

and let

$$\Sigma_0 := \sum_{j=1}^{k-1} \ell(\tau_j) \qquad \text{and} \qquad \hat{r}_{\tau_0}(n) := r_{\tau_0}(n - \hat{v}_*)$$

where $r_{\tau_0}(n - \hat{v}_*)$ is the number defined in (1.7.27). Finally setting

$$\Sigma = \Sigma_0 + \hat{r}_{\tau_0}(n), \qquad \hat{h}(n) := n - \hat{v}_* - \Sigma_{\tau_0} - \Sigma R_*, \qquad \text{and} \qquad \tau_k^* := \sigma^{\hat{r}_{\tau_0}(n)}(\tau_k),$$

we have the right decomposition of our orbit length n into blocks which do not expand distances in the fiber cones $\mathscr{C}_{\omega,a_*}$ and $\mathscr{C}_{\omega,+}$. Now let

$$n \geq N_3(\omega) := \max\left\{N_2(\omega), \frac{R_*}{\varepsilon}, \frac{S_*}{\varepsilon}\right\}. \tag{1.10.9}$$

Since $v_*, \hat{h}(n) \leq R_*$, by (1.10.9), (1.10.5), and for

$$\varepsilon < \frac{\theta}{8(1+\theta)} \tag{1.10.10}$$

sufficiently small, we must have that

$$\begin{aligned} \Sigma R_* &= n - \hat{v}_* - \Sigma_{\tau_0} - \hat{h}(n) = n - v_* - d_* R_* - \Sigma_{\tau_0} - \hat{h}(n) \\ &\geq n - \left(\frac{4+\theta}{\theta}\right)\varepsilon n - 2R_* - S_* \geq n - \left(\frac{4+\theta}{\theta}\right)\varepsilon n - 3\varepsilon n \\ &\geq n\left(1 - 4\varepsilon\left(\frac{1+\theta}{\theta}\right)\right) > \frac{n}{2}. \end{aligned} \tag{1.10.11}$$

Now we note that since $\tilde{\mathcal{L}}_{\tau_k^*}^{\hat{h}(n)}(\mathscr{C}_{\tau_k^*,+}) \subseteq \mathscr{C}_{\sigma^{n+p}(\omega),+}$ we have that $\tilde{\mathcal{L}}_{\tau_k^*}^{\hat{h}(n)}$ is a weak contraction, and hence, we have

$$\Theta_{\sigma^{n+p}(\omega),+}\left(\tilde{\mathcal{L}}_{\tau_k^*}^{\hat{h}(n)} f', \tilde{\mathcal{L}}_{\tau_k^*}^{\hat{h}(n)} h'\right) \leq \Theta_{\tau_k^*,+}(f', h'), \qquad f', h' \in \Theta_{\tau_k^*,+}. \tag{1.10.12}$$

Recall that $E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0})$, defined in Lemma 1.7.5, is the total length of the bad blocks of the $n - \hat{v}_*$ length orbit starting at τ_0, i.e.

$$E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) = \sum_{\substack{1 \leq j < k \\ \tau_j \in \Omega_B}} \ell(\tau_j) + r_{\tau_0}(n - \hat{v}_*).$$

Lemma 1.7.5 then gives that

$$E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) < Y\varepsilon\Sigma. \tag{1.10.13}$$

We are now poised to calculate (1.10.1), but first we note that we can write

$$n = \hat{v}_* + \Sigma_{\tau_0} + \Sigma R_* + \hat{h}(n)$$

$$= \hat{v}_* + \Sigma_{\tau_0} + \Sigma_0 R_* + \hat{r}_{\tau_0}(n) + \hat{h}(n) \tag{1.10.14}$$

and that the number of good blocks contained in the orbit of length $n - \hat{v}_* - \Sigma_{\tau_0}$ is given by

$$\Sigma_G := \#\left\{1 \le j \le k : \tau_j \in \Omega_G\right\} = \Sigma - E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) \le \Sigma_0. \tag{1.10.15}$$

Now, using (1.10.14) we combine (in order) (1.10.12), (1.4.2), and Theorem 1.4.4 (repeatedly) in conjunction with the fact that $\tau_0 \in \Omega_F$ to see that

$$\begin{aligned}
&\Theta_{\sigma^{n+p}(\omega),+}\left(\tilde{\mathcal{L}}_\tau^n(f_\tau), \tilde{\mathcal{L}}_{\tau_{-l}}^{n+l}(h_{\tau_{-l}})\right) \\
&\quad = \Theta_{\sigma^{n+p}(\omega),+}\left(\tilde{\mathcal{L}}_{\tau_k}^{\hat{h}(n)} \circ \tilde{\mathcal{L}}_{\tau_1}^{\Sigma R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(f_\tau), \tilde{\mathcal{L}}_{\tau_k}^{\hat{h}(n)} \circ \tilde{\mathcal{L}}_{\tau_1}^{\Sigma R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*} \circ \tilde{\mathcal{L}}_{\tau_{-l}}^{l}(h_{\tau_{-l}})\right) \\
&\quad \le \Theta_{\tau_k^*,+}\left(\tilde{\mathcal{L}}_{\tau_1}^{\Sigma R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(f_\tau), \tilde{\mathcal{L}}_{\tau_1}^{\Sigma R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(h_l)\right) \\
&\quad \le \Theta_{\tau_k^*,a_*}\left(\tilde{\mathcal{L}}_{\tau_k}^{\hat{r}_{\tau_0}(n)} \circ \tilde{\mathcal{L}}_{\tau_1}^{\Sigma_0 R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(f_\tau), \tilde{\mathcal{L}}_{\tau_k}^{\hat{r}_{\tau_0}(n)} \circ \tilde{\mathcal{L}}_{\tau_1}^{\Sigma_0 R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(h_l)\right) \\
&\quad \le \Theta_{\tau_k,a_*}\left(\tilde{\mathcal{L}}_{\tau_1}^{\Sigma_0 R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(f_\tau), \tilde{\mathcal{L}}_{\tau_1}^{\Sigma_0 R_*} \circ \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(h_l)\right) \\
&\quad \le \left(\tanh\left(\frac{\Delta}{4}\right)\right)^{\Sigma_G} \Theta_{\tau_1,a_*}\left(\tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(f_\tau), \tilde{\mathcal{L}}_{\tau_0}^{\Sigma_{\tau_0}} \circ \tilde{\mathcal{L}}_\tau^{\hat{v}_*}(h_l)\right).
\end{aligned} \tag{1.10.16}$$

Now since $\tau_0 \in \Omega_F$ and in light of (1.10.8), applying Lemma 1.9.1 allows us to estimate the Θ_{τ_1,a_*} term in the right hand side of (1.10.16) to give

$$\Theta_{\sigma^{n+p}(\omega),+}\left(\tilde{\mathcal{L}}_\tau^n(f_\tau), \tilde{\mathcal{L}}_{\tau_{-l}}^{n+l}(h_{\tau_{-l}})\right) \le \left(\tanh\left(\frac{\Delta}{4}\right)\right)^{\Sigma_G} \Delta. \tag{1.10.17}$$

Using (1.10.15), the fact that $E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) \ge 1$, (1.10.13), and (1.10.11), we see that

$$\begin{aligned}
\Sigma_G &= \Sigma - E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) \\
&\ge \Sigma - Y\varepsilon\Sigma \\
&= \Sigma\left(1 - Y\varepsilon\right) \\
&\ge \frac{\left(1 - Y\varepsilon\right) n}{2R_*}.
\end{aligned} \tag{1.10.18}$$

In light of (1.7.30), for all $\varepsilon > 0$ sufficiently small we have that $1 - Y\varepsilon > 0$. Finally, inserting (1.10.17) and (1.10.18) into (1.10.16) gives

$$\Theta_{\sigma^{n+p}(\omega),+}\left(\tilde{\mathcal{L}}_\tau^n(f_\tau), \tilde{\mathcal{L}}_{\tau_{-l}}^{n+l}(h_{\tau_{-l}})\right) \le \Delta\vartheta^n,$$

where

$$\vartheta := \left(\tanh\left(\frac{\Delta}{4}\right)\right)^{\frac{(1-Y\varepsilon)}{2R_*}} < 1,$$

which completes the proof. □

Combining Lemma 1.10.1 together with Lemma 1.4.5 gives the following immediate corollary.

COROLLARY 1.10.2. *Suppose $\varepsilon > 0$, $V : \Omega \to (0, \infty)$, $f_{\sigma^p(\omega)} \in \mathscr{C}_{\sigma^p(\omega),+}$, and $h_{\sigma^{p-l}(\omega)} \in \mathscr{C}_{\sigma^{p-l}(\omega),+}$ all satisfy the hypotheses of Lemma 1.10.1. Then there exists $\kappa \in (0, 1)$ such that for m-a.e. $\omega \in \Omega$, all $n \geq N_3(\omega)$, all $l \geq 0$, and all $|p| \leq n$ we have*

$$\|\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)} - \tilde{\mathcal{L}}^{n+l}_{\sigma^{p-l}(\omega)} h_{\sigma^{p-l}(\omega)}\|_\infty \leq \|\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)}\|_\infty \left(e^{\Delta\vartheta^n} - 1\right).$$

Notice that if we wish to apply Lemma 1.10.1 (or Corollary 1.10.2) repeatedly iterating in the forward direction, i.e. taking $p = 0$ so that we push forward starting from the ω fiber, then we only need that $f \in \mathscr{C}_{\omega,+}$ and do not need to be concerned with the assumption on the variation. Indeed, as $p = 0$ is fixed, then we will have $\mathrm{var}(f) \leq \mathrm{var}(f) \cdot e^{\varepsilon n}$ for any $n \geq 1$. However, if we wish to apply Lemma 1.10.1 repeatedly with $p = -n$ for n increasing to ∞, then we will need to consider special functions f.

DEFINITION 1.10.3. We let the set $\mathcal{D}$ denote the set of functions $f \in \mathrm{BV}_\Omega(I)$ such that for each $\varepsilon > 0$ there exists a measurable function $V_{f,\varepsilon} : \Omega \to (0, \infty)$ such that the following hold for all $n \in \mathbb{Z}$ with $|n|$ sufficiently large:

($\mathcal{D}$1) $\mathrm{var}(f_{\sigma^n(\omega)}) \leq V_{f,\varepsilon}(\omega) e^{\varepsilon|n|}$,

($\mathcal{D}$2) $\Lambda_{\sigma^n(\omega)}(|f_{\sigma^n(\omega)}|) \geq V^{-1}_{f,\varepsilon}(\omega) e^{-\varepsilon|n|}$.

Let $\mathcal{D}^+ \subseteq \mathcal{D}$ denote the collection of all functions $f \in \mathcal{D}$ such that $f_\omega \geq 0$ for each $\omega \in \Omega$.

REMARK 1.10.4. Note that the space $\mathcal{D}$ is nonempty. In particular, $\mathcal{D}$ contains any function $f : \Omega \times I \to \mathbb{R}$ such that f_ω is equal to some fixed function $f \in \mathrm{BV}(I)$ with $0 < \inf |f|$. More generally, $\mathcal{D}$ contains any functions $f \in \mathrm{BV}_\Omega(I)$ such that $\log \mathrm{var}(f_\omega)$, $\log \Lambda_\omega(|f_\omega|) \in L^1(m)$.

REMARK 1.10.5. Note that if $f \in \mathcal{D}$ then taking $V_f(\omega) = V^2_{f,\varepsilon}(\omega)$ measurable and $\varepsilon' = \varepsilon/2$ we have that

$$\frac{\mathrm{var}(f_{\sigma^{-n}(\omega)})}{\Lambda_{\sigma^{-n}(\omega)}(f_{\sigma^{-n}(\omega)})} \leq V_f(\omega) \frac{\mathrm{var}(f_\omega)}{\Lambda_\omega(f_\omega)} e^{\varepsilon' n}.$$

In the following corollary we establish the existence of an invariant density.

COROLLARY 1.10.6. *There exists a function $\phi \in \mathrm{BV}_\Omega(I)$ and a measurable function $\lambda : \Omega \to \mathbb{R}^+$ such that for m-a.e. $\omega \in \Omega$*

$$\mathcal{L}_\omega \phi_\omega = \lambda_\omega \phi_{\sigma(\omega)} \qquad \text{and} \qquad \Lambda_\omega(\phi_\omega) = 1. \tag{1.10.19}$$

Furthermore, we have that $\log \lambda_\omega \in L^1(m)$ and for m-a.e. $\omega \in \Omega$, $\lambda_\omega \geq \rho_\omega$.

PROOF. First we note that for any $f \in \mathcal{D}_+$, Lemma 1.5.4 and Remark 1.10.5 give that

$$\mathrm{var}\left(f_{\omega,n}\right) = \frac{\rho^n_{\sigma^{-n}(\omega)}}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)} \mathrm{var}\left(f_{\sigma^{-n}(\omega)}\right) \leq \frac{\mathrm{var}(f_{\sigma^{-n}(\omega)})}{\Lambda_{\sigma^{-n}(\omega)}(f_{\sigma^{-n}(\omega)})} \leq V_f(\omega) \frac{\mathrm{var}(f_\omega)}{\Lambda_\omega(f_\omega)} e^{\varepsilon n}$$

for all $n \in \mathbb{N}$ sufficiently large, say for $n \geq N_4(\omega)$, and some measurable $V_f : \Omega \to (0, \infty)$, where

$$f_{\omega,n} := \frac{f_{\sigma^{-n}(\omega)} \rho^n_{\sigma^{-n}(\omega)}}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)} \in \mathscr{C}_{\sigma^{-n}(\omega),+}.$$

Thus, Corollary 1.10.2 (with $p = -n$ and $V(\omega) = V_f(\omega)\mathrm{var}(f_\omega)/\Lambda_\omega(f_\omega)$) gives that

$$(\tilde{\mathcal{L}}^n_{\sigma^{-n}(\omega)} f_{\omega,n})_{n\in\mathbb{N}} = \left(\frac{\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f\right)} \right)_{n\in\mathbb{N}}$$

forms a Cauchy sequence in $\mathscr{C}_{\omega,+}$, and therefore there must exist some $\phi_{\omega,f} \in \mathscr{C}_{\omega,+}$ with

$$\phi_{\omega,f} := \lim_{n\to\infty} \frac{\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)}. \tag{1.10.20}$$

By construction we have that $\Lambda_\omega(\phi_{\omega,f}) = 1$. Now, in view of calculating $\mathcal{L}_\omega \phi_{\omega,f}$, we note that (1.8.1) (with $N = 1$ and $f = \mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}$) gives that

$$\frac{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n+1}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)} \le \frac{\|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty \Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)} = \|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty. \tag{1.10.21}$$

Lemma 1.5.4 (with $k = 1$ and $f = \mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}$) implies that

$$\frac{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n+1}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)} \ge \frac{\rho_\omega \Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)} = \rho_\omega,$$

and thus, together with (1.10.21), we have

$$\frac{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n+1}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)}{\Lambda_\omega\left(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}\right)} \in [\rho_\omega, \|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty]. \tag{1.10.22}$$

Thus there must exist a sequence $(n_k)_{k\in\mathbb{N}}$ along which this ratio converges to some value $\lambda_{\omega,f}$, that is

$$\lambda_{\omega,f} := \lim_{k\to\infty} \frac{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n_k+1}_{\sigma^{-n_k}(\omega)} f\right)}{\Lambda_\omega\left(\mathcal{L}^{n_k}_{\sigma^{-n_k}(\omega)} f\right)}.$$

Hence we have

$$\begin{aligned} \mathcal{L}_\omega \phi_{\omega,f} &= \lim_{k\to\infty} \frac{\mathcal{L}^{n_k+1}_{\sigma^{-n_k}(\omega)} f}{\Lambda_\omega\left(\mathcal{L}^{n_k}_{\sigma^{-n_k}(\omega)} f\right)} \\ &= \lim_{k\to\infty} \frac{\mathcal{L}^{n_k+1}_{\sigma^{-n_k}(\omega)} f}{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n_k+1}_{\sigma^{-n_k}(\omega)} f\right)} \cdot \frac{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n_k+1}_{\sigma^{-n_k}(\omega)} f\right)}{\Lambda_\omega\left(\mathcal{L}^{n_k}_{\sigma^{-n_k}(\omega)} f\right)} = \lambda_{\omega,f}\phi_{\sigma(\omega),f}. \end{aligned} \tag{1.10.23}$$

From (1.10.23) it follows that $\lambda_{\omega,f}$ does not depend on the sequence $(n_k)_{k\in\mathbb{N}}$, and in fact we have

$$\lambda_{\omega,f} = \lim_{n\to\infty} \frac{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n+1}_{\sigma^{-n}(\omega)} f\right)}{\Lambda_\omega\left(\mathcal{L}^{n}_{\sigma^{-n}(\omega)} f\right)},$$

and thus,

$$\mathcal{L}_\omega \phi_{\omega,f} = \lambda_{\omega,f}\phi_{\sigma(\omega),f}. \tag{1.10.24}$$

To see that $\phi_{\omega,f}$ and $\lambda_{\omega,f}$ do not depend on f, we apply Lemma 1.10.1 (with $p=-n$, $l=0$, and $V(\omega) = \max\{V_f(\omega)\mathrm{var}(f_\omega)/\Lambda_\omega(f_\omega), V_h(\omega)\mathrm{var}(h_\omega)/\Lambda_\omega(h_\omega)\}$) to functions $f,h\in\mathcal{D}_+$ to get that

$$\Theta_{\omega,+}\left(\phi_{\omega,f},\phi_{\omega,h}\right) \le \Theta_{\omega,+}\left(\phi_{\omega,f},f_{\omega,n}\right) + \Theta_{\omega,+}\left(f_{\omega,n},h_{\omega,n}\right) + \Theta_{\omega,+}\left(\phi_{\omega,h},h_{\omega,n}\right) \le 3\Delta\vartheta^n \tag{1.10.25}$$

for each $n\ge N_3(\omega)$. Thus, inserting (1.10.25) into Lemma 1.4.5 yields

$$\|\phi_{\omega,f}-\phi_{\omega,h}\|_\infty \le \|\phi_{\omega,f}\|_\infty\left(e^{\left(\Theta_{\omega,+}(\phi_{\omega,f},\phi_{\omega,h})\right)}-1\right) \le \|\phi_{\omega,f}\|_\infty\left(e^{3\Delta\vartheta^n}-1\right),$$

which converges to zero exponentially fast as n tends towards infinity. Thus we must in fact have that $\phi_{\omega,f}=\phi_{\omega,h}$ for all f,h. Moreover, in light of (1.10.24), this implies that $\lambda_{\omega,f}=\lambda_{\omega,h}$. We denote the common values by ϕ_ω and λ_ω respectively. It follows from (1.3.11) and (1.10.22) that

$$0<\rho_\omega\le\lambda_\omega\le\|\mathcal{L}_\omega\mathbb{1}_\omega\|_\infty. \tag{1.10.26}$$

Measurability of the map $\omega\mapsto\lambda_\omega$ follows from the measurability of the sequence

$$\left(\frac{\Lambda_{\sigma(\omega)}\left(\mathcal{L}^{n+1}_{\sigma^{-n}(\omega)}\mathbb{1}_{\sigma^{-n}(\omega)}\right)}{\Lambda_\omega\left(\mathcal{L}^{n}_{\sigma^{-n}(\omega)}\mathbb{1}_{\sigma^{-n}(\omega)}\right)}\right)_{n\in\mathbb{N}}.$$

The log-integrability of λ_ω follows from the log-integrability of ρ_ω and (1.10.26). Finally, measurability of the maps $\omega\mapsto\inf\phi_\omega$ and $\omega\mapsto\|\phi_\omega\|_\infty$ follows from the fact that we have

$$\phi_\omega = \lim_{n\to\infty}\frac{\mathcal{L}^n_{\sigma^{-n}(\omega)}\mathbb{1}_{\sigma^{-n}(\omega)}}{\Lambda_\omega(\mathcal{L}^n_{\sigma^{-n}(\omega)}\mathbb{1}_{\sigma^{-n}(\omega)})},$$

which is a limit of measurable functions, and thus finishes the proof.

□

Remark 1.10.7. For each $k\in\mathbb{N}$, inducting on (1.10.24) for any $f\in\mathcal{D}_+$ yields

$$\begin{aligned}\mathcal{L}^k_\omega(\phi_\omega) &= \lim_{n\to\infty}\frac{\mathcal{L}^{n+k}_{\sigma^{-n}(\omega)}f_{\sigma^{-n}(\omega)}}{\Lambda_\omega(\mathcal{L}^n_{\sigma^{-n}(\omega)}f_{\sigma^{-n}(\omega)})}\\ &= \lim_{n\to\infty}\frac{\mathcal{L}^{n+k}_{\sigma^{-n}(\omega)}f_{\sigma^{-n}(\omega)}}{\Lambda_{\sigma^k(\omega)}(\mathcal{L}^{n+k}_{\sigma^{-n}(\omega)}f_{\sigma^{-n}(\omega)})}\cdot\frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}^{n+k}_{\sigma^{-n}(\omega)}f_{\sigma^{-n}(\omega)})}{\Lambda_\omega(\mathcal{L}^n_{\sigma^{-n}(\omega)}f_{\sigma^{-n}(\omega)})}\end{aligned}$$

$$= \phi_{\sigma^k(\omega)} \cdot \lim_{n\to\infty} \frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}^{n+k}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})}{\Lambda_\omega(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})}. \tag{1.10.27}$$

The final limit in (1.10.27) telescopes to give us

$$\begin{aligned}\lim_{n\to\infty} \frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}^{n+k}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})}{\Lambda_\omega(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})} &= \lim_{n\to\infty} \frac{\Lambda_{\sigma(\omega)}(\mathcal{L}^{n+1}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})}{\Lambda_\omega(\mathcal{L}^n_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})} \cdots \frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}^{n+k}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})}{\Lambda_{\sigma^{k-1}(\omega)}(\mathcal{L}^{n+k-1}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})} \\ &= \lambda_\omega \lambda_{\sigma(\omega)} \cdots \lambda_{\sigma^{k-1}(\omega)},\end{aligned}$$

For each $k \geq 1$ we denote

$$\lambda^k_\omega := \lambda_\omega \lambda_{\sigma(\omega)} \cdots \lambda_{\sigma^{k-1}(\omega)}. \tag{1.10.28}$$

Rewriting (1.10.27) gives

$$\mathcal{L}^k_\omega \phi_\omega = \lambda^k_\omega \phi_{\sigma^k(\omega)}.$$

The following proposition shows that the density ϕ_ω coming from Corollary 1.10.6 is in fact supported on the set $D_{\omega,\infty}$.

PROPOSITION 1.10.8. *For m-a.e. $\omega \in \Omega$ we have that*

$$\inf_{D_{\omega,\infty}} \phi_\omega > 0.$$

PROOF. First we note that since $\Lambda_\omega(\phi_\omega) = 1 > 0$ for m-a.e. $\omega \in \Omega$, using the definition of Λ_ω (1.3.1), we must in fact have that

$$\inf_{D_{\sigma^n(\omega),n}} \mathcal{L}^n_\omega(\phi_\omega) > 0$$

for $n \in \mathbb{N}$ sufficiently large, which, in turn implies that

$$\inf_{X_{\omega,n-1}} \phi_\omega > 0 \tag{1.10.29}$$

for all $n \in \mathbb{N}$ sufficiently large. Next, for m-a.e. $\omega \in \Omega$ and all $n \in \mathbb{N}$ we use (1.10.19) to see that

$$\begin{aligned}\inf_{D_{\omega,\infty}} \phi_\omega &= \left(\lambda^n_{\sigma^{-n}(\omega)}\right)^{-1} \inf_{D_{\omega,\infty}} \mathcal{L}^n_{\sigma^{-n}(\omega)} \phi_{\sigma^{-n}(\omega)} \\ &\geq \inf_{X_{\sigma^{-n}(\omega),n-1}} \phi_{\sigma^{-n}(\omega)} \left(\lambda^n_{\sigma^{-n}(\omega)}\right)^{-1} \inf_{D_{\omega,\infty}} \mathcal{L}^n_{\sigma^{-n}(\omega)} \mathbb{1}_{\sigma^{-n}(\omega)}.\end{aligned} \tag{1.10.30}$$

As the right hand side is strictly positive for all $n \in \mathbb{N}$ sufficiently large by (1.10.29), (1.10.26), and (1.3.12), we are finished. □

LEMMA 1.10.9. *For each $\omega \in \Omega$ the functional Λ_ω is linear, positive, and enjoys the property that*

$$\Lambda_{\sigma(\omega)}(\mathcal{L}_\omega f) = \lambda_\omega \Lambda_\omega(f) \tag{1.10.31}$$

for each $f \in \mathrm{BV}(I)$. Furthermore, for each $\omega \in \Omega$ we have that

$$\lambda_\omega = \rho_\omega = \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega). \tag{1.10.32}$$

PROOF. Positivity of Λ_ω follows from the initial properties of Λ_ω shown in Observation 1.3.1. To prove the remaining claims we first prove a more robust limit characterization of Λ_ω than the one given by its definition, (1.3.1). Now, for any two sequences of points $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ with $x_n, y_n \in D_{\sigma^n(\omega),n}$ we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega}(x_n) - \frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega}(y_n)\right| &= \lim_{n\to\infty}\left|\frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega}(y_n)\right| \cdot \left|\frac{\mathcal{L}_\omega^n f(x_n)}{\mathcal{L}_\omega^n \mathbb{1}_\omega(x_n)} \cdot \frac{\mathcal{L}_\omega^n \mathbb{1}_\omega(y_n)}{\mathcal{L}_\omega^n f(y_n)} - 1\right| \\ &\leq \|f\|_\infty \limsup_{n\to\infty}\left|\exp\left(\Theta_{\sigma^n(\omega),+}\left(\tilde{\mathcal{L}}_\omega^n f, \tilde{\mathcal{L}}_\omega^n \mathbb{1}_\omega\right)\right) - 1\right| = 0. \end{aligned} \tag{1.10.33}$$

Thus, we have shown that we may remove the infimum from (1.3.1), which defines the functional Λ_ω, that is now we may write

$$\Lambda_\omega(f) = \lim_{n\to\infty} \frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega}(x_n) \tag{1.10.34}$$

for all $f \in \mathscr{C}_{\omega,+}$ and all $x_n \in D_{\sigma^n(\omega),n}$. Moreover, this identity also shows that the functional Λ_ω is linear. To extend (1.10.34) to all of $\mathrm{BV}(I)$, we simply write $f = f_+ - f_-$ so that $f_+, f_- \in \mathscr{C}_{\omega,+}$ for each $f \in \mathrm{BV}(I)$ so that we have

$$\Lambda_\omega(f) = \Lambda_\omega(f_+) - \Lambda_\omega(f_-) = \lim_{n\to\infty} \frac{\mathcal{L}_\omega^n f_+}{\mathcal{L}_\omega^n \mathbb{1}_\omega} - \lim_{n\to\infty} \frac{\mathcal{L}_\omega^n f_-}{\mathcal{L}_\omega^n \mathbb{1}_\omega} = \lim_{n\to\infty} \frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega}. \tag{1.10.35}$$

To prove (1.10.31) and (1.10.32) we use (1.10.35) to note that

$$\begin{aligned}\Lambda_{\sigma(\omega)}(\mathcal{L}_\omega f) &= \lim_{n\to\infty} \frac{\mathcal{L}_\omega^{n+1} f}{\mathcal{L}_{\sigma(\omega)}^n \mathbb{1}_{\sigma(\omega)}}(x_{n+1}) \\ &= \lim_{n\to\infty} \frac{\mathcal{L}_\omega^{n+1} f}{\mathcal{L}_\omega^{n+1} \mathbb{1}_\omega}(x_{n+1}) \cdot \frac{\mathcal{L}_\omega^{n+1} \mathbb{1}_\omega}{\mathcal{L}_{\sigma(\omega)}^n \mathbb{1}_{\sigma(\omega)}}(x_{n+1}) \\ &= \Lambda_\omega(f) \cdot \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega). \end{aligned} \tag{1.10.36}$$

Considering the case where $f = \phi_\omega$ in (1.10.36) in conjunction with the fact that $\Lambda_\omega(\phi_\omega) = 1$ and $\mathcal{L}_\omega \phi_\omega = \lambda_\omega \phi_{\sigma(\omega)}$ gives

(1.10.37)
$$\rho_\omega := \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega) = \Lambda_\omega(\phi_\omega)\Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega) = \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \phi_\omega) = \Lambda_{\sigma(\omega)}(\lambda_\omega \phi_{\sigma(\omega)}) = \lambda_\omega,$$

which finishes the proof. □

REMARK 1.10.10. In light of the fact that $\log \rho_\omega \in L^1(m)$ by (1.3.11), Lemma 1.10.9 implies that

$$\log \lambda_\omega \in L^1(m). \tag{1.10.38}$$

In the next lemma we are finally able to show that the functional Λ_ω can be thought of as Borel probability measure for the random open system.

LEMMA 1.10.11. *There exists a non-atomic Borel probability measure ν_ω on I_ω such that*

$$\Lambda_\omega(f) = \int_{I_\omega} f \, d\nu_\omega$$

for all $f \in \mathrm{BV}(I)$. *Consequently, we have that*

$$\nu_{\sigma(\omega)}(\mathcal{L}_\omega f) = \lambda_\omega \nu_\omega(f) \tag{1.10.39}$$

for all $f \in \mathrm{BV}(I)$. *Furthermore, we have that* $\mathrm{supp}(\nu_\omega) \subseteq X_{\omega,\infty}$.

Proof. The proof that the functional Λ_ω can be equated to a non-atomic Borel probability measure ν_ω goes exactly like the proof of Lemma 4.3 in [53]. Thus, we have only to prove that $\mathrm{supp}(\nu_\omega) \subseteq X_{\omega,\infty}$. To that end, suppose $f \in L^1(\nu_{\omega,0})$ with $f \equiv 0$ on $X_{\omega,n-1}$. Then

$$\int_I f\, d\nu_\omega = (\lambda_\omega^n)^{-1} \int_I \mathcal{L}_\omega^n(f)\, d\nu_{\sigma^n(\omega)} = (\lambda_\omega^n)^{-1} \int_I \mathcal{L}_{\omega,0}^n(\hat{X}_{\omega,n-1} \cdot f)\, d\nu_{\sigma^n(\omega)} = 0.$$

As $0 < \lambda_\omega^n < \infty$ for each $n \in \mathbb{N}$, we must have that $\mathrm{supp}(\nu_\omega) \subseteq X_{\omega,\infty}$. □

Remark 1.10.12. We can immediately see, cf. [31, 5], that the conformality of the family $(\nu_\omega)_{\omega\in\Omega}$ produced in Lemma 1.10.11 enjoys the property that for each $n \geq 1$ and each set A on which $T_\omega^n|_A$ is one-to-one we have

$$\nu_{\sigma^n(\omega)}(T_\omega^n(A)) = \lambda_\omega^n \int_A e^{-S_{n,T}(\varphi_\omega)}\, d\nu_\omega.$$

In particular, this gives that for each $n \geq 1$ and each $Z \in \mathcal{Z}_\omega^{(n)}$ we have

$$\nu_{\sigma^n(\omega)}(T_\omega^n(Z)) = \lambda_\omega^n \int_Z e^{-S_{n,T}(\varphi_\omega)}\, d\nu_\omega.$$

Remark 1.10.13. In light of Lemmas 1.10.9 and 1.10.11, the normalized operator $\tilde{\mathcal{L}}_\omega$ is given by $\tilde{\mathcal{L}}_\omega(\cdot) := \rho_\omega^{-1}\mathcal{L}_\omega(\cdot) = \lambda_\omega^{-1}\mathcal{L}_\omega(\cdot)$. Furthermore, $\tilde{\mathcal{L}}_\omega$ enjoys the properties

$$\tilde{\mathcal{L}}_\omega \phi_\omega = \phi_{\sigma(\omega)} \qquad \text{and} \qquad \nu_{\sigma(\omega)}\left(\tilde{\mathcal{L}}_\omega(f)\right) = \nu_\omega(f)$$

for all $f \in \mathrm{BV}(I)$.

For each $\omega \in \Omega$ we may now define the measure $\mu_\omega \in \mathcal{P}(I)$ by

$$\mu_\omega(f) := \int_{X_{\omega,\infty}} f\phi_\omega\, d\nu_\omega, \qquad f \in L^1(\nu_\omega). \tag{1.10.40}$$

Lemma 1.10.11 and Proposition 1.10.8 together show that, for m-a.e. $\omega \in \Omega$, μ_ω is a non-atomic Borel probability measure with $\mathrm{supp}(\mu_\omega) \subseteq X_{\omega,\infty}$, which is absolutely continuous with respect to ν_ω. Furthermore, in view of Proposition 1.10.8, for m-a.e. $\omega \in \Omega$, we may now define the fully normalized transfer operator $\mathscr{L}_\omega : \mathrm{BV}(I) \to \mathrm{BV}(I)$ by

$$\mathscr{L}_\omega f := \frac{1}{\phi_{\sigma(\omega)}} \tilde{\mathcal{L}}_\omega(f\phi_\omega) = \frac{1}{\lambda_\omega \phi_{\sigma(\omega)}} \mathcal{L}_\omega(f\phi_\omega), \qquad f \in \mathrm{BV}(I). \tag{1.10.41}$$

As an immediate consequence of Remark 1.10.13 and (1.10.41), we get that

$$\mathscr{L}_\omega \mathbb{1}_\omega = \mathbb{1}_{\sigma(\omega)}. \tag{1.10.42}$$

We end this section with the following proposition which shows that the family $(\mu_\omega)_{\omega\in\Omega}$ of measures is T-invariant.

PROPOSITION 1.10.14. *The family* $(\mu_\omega)_{\omega\in\Omega}$ *defined by* (1.10.40) *is* T*-invariant in the sense that*

$$\int_{X_{\omega,\infty}} f\circ T_\omega\, d\mu_\omega = \int_{X_{\sigma(\omega),\infty}} f\, d\mu_{\sigma(\omega)} \tag{1.10.43}$$

for $f\in L^1(\mu_{\sigma(\omega)}) = L^1(\nu_{\sigma(\omega)})$.

The proof of Proposition 1.10.14 goes just like the proof of Proposition 8.11 of [3], and has thus been omitted.

1.11. Decay of correlations

We are now ready to show that images under the normalized transfer operator $\tilde{\mathcal{L}}_\omega$ converge exponentially fast to the invariant density as well as the fact that the invariant measure μ_ω established in Section 1.10 satisfies an exponential decay of correlations. Furthermore, we show that the families $(\nu_\omega)_{\omega\in\Omega}$ and $(\mu_\omega)_{\omega\in\Omega}$ are in fact random measures and then introduce the RACCIM η supported on $\mathcal{I}$.

To begin this section we state a lemma which shows that the BV norm of the invariant density ϕ_ω does not grow too much along a σ-orbit of fibers by providing a measurable upper bound. In fact, we show that the BV norm of ϕ_ω is tempered. As the proof of the following lemma is the same as the proof of Lemma 8.5 in the closed dynamical setting of [3], its proof is omitted.

LEMMA 1.11.1. *For all* $\delta>0$ *there exists a measurable random constant* $C(\omega,\delta)>0$ *such that for all* $k\in\mathbb{Z}$ *and* m*-a.e.* $\omega\in\Omega$ *we have*

$$\|\phi_{\sigma^k(\omega)}\|_{\mathrm{BV}} = \|\phi_{\sigma^k(\omega)}\|_\infty + \mathrm{var}(\phi_{\sigma^k(\omega)}) \le C(\omega,\delta)e^{\delta|k|}.$$

Consequently, we have that $q\in\mathcal{D}$.

We are now able to prove the following theorem which completes the proof of Theorem B.

THEOREM 1.11.2. *There exists a measurable,* m*-a.e. finite function* $D:\Omega\to\mathbb{R}$ *and* $\kappa<1$ *such that for each* $f\in\mathcal{D}$*, each* $n\in\mathbb{N}$*, and each* $|p|\le n$ *we have*

$$\|\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)} - \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\phi_{\sigma^{p+n}(\omega)}\|_\infty \le D(\omega)\|f_{\sigma^p(\omega)}\|_\infty \kappa^n \tag{1.11.1}$$

and

$$\|\mathscr{L}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)} - \mu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\mathbb{1}_{\sigma^{p+n}(\omega)}\|_\infty \le D(\omega)\|f_{\sigma^p(\omega)}\|_\infty \kappa^n. \tag{1.11.2}$$

PROOF. We first note that for m-a.e. $\omega\in\Omega$, all $n\ge N_3(\omega)$, all $|p|\le n$, and all $f\in\mathcal{D}_+$ we may use Lemma 1.10.1 to get that

$$\begin{aligned}\Theta_{\sigma^{p+n}(\omega),+}&\left(\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)}, \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\phi_{\sigma^{p+n}(\omega)}\right)\\ &= \Theta_{\sigma^{p+n}(\omega),+}\left(\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)}, \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\tilde{\mathcal{L}}^n_{\sigma^p(\omega)}\phi_{\sigma^p(\omega)}\right) \le \Delta\vartheta^n.\end{aligned}$$

Applying Lemma 1.4.5 with $\varrho=\nu_{\sigma^{p+n}(\omega)}$ and $\|\cdot\|=\|\cdot\|_\infty$, together with Lemma 1.10.1 then gives

$$\|\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)} - \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\phi_{\sigma^{p+n}(\omega)}\|_\infty$$

$$\begin{aligned}
&\leq \left(\exp\left(\Theta_{\sigma^{p+n}(\omega),+}\left(\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)}, \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\phi_{\sigma^{p+n}(\omega)}\right)\right) - 1\right) \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\|\phi_{\sigma^{p+n}(\omega)}\|_\infty \\
&\leq \|f_{\sigma^p(\omega)}\|_\infty \|\phi_{\sigma^{p+n}(\omega)}\|_\infty \left(e^{\Delta\vartheta^n} - 1\right) \\
&\leq \|f_{\sigma^p(\omega)}\|_\infty \|\phi_{\sigma^{p+n}(\omega)}\|_\infty \tilde{\kappa}^n
\end{aligned} \tag{1.11.3}$$

for some $\tilde{\kappa} \in (0,1)$[2]. Since $\|\phi_\omega\|_\infty$ is tempered, as a consequence of Lemma 1.11.1, for each $n \in \mathbb{N}$ and each $\delta > 0$ we can find a tempered function $A_{n,\delta} : \Omega \to \mathbb{R}$ such that

$$\|\phi_{\sigma^{p+n}(\omega)}\|_\infty \leq A_{n,\delta}(\omega) e^{(p+n)\delta} \|\phi_\omega\|_\infty \leq A_{n,\delta}(\omega) e^{2n\delta} \|\phi_\omega\|_\infty, \tag{1.11.4}$$

where we have used the fact that $|p| \leq n$. For each $\delta > 0$ we let

$$B_\delta(\omega) := \max_{1 \leq n \leq N_3(\omega)} \{A_{n,\delta}(\omega)\} \tilde{\kappa}^{-N_3(\omega)}. \tag{1.11.5}$$

Combining (1.11.3) - (1.11.5), for any $n \in \mathbb{N}$ we see that

$$\begin{aligned}
\|\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)} - \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\phi_{\sigma^{p+n}(\omega)}\|_\infty &\leq 2B_\delta(\omega) e^{2n\delta} \|f_{\sigma^p(\omega)}\|_\infty \|\phi_\omega\|_\infty \tilde{\kappa}^n \\
&\leq 2B(\omega) \|f_{\sigma^p(\omega)}\|_\infty \|\phi_\omega\|_\infty \kappa^n,
\end{aligned} \tag{1.11.6}$$

where here we have fixed $\delta > 0$ sufficiently small such that

$$e^{2\delta}\tilde{\kappa} =: \kappa < 1,$$

and we have set $B(\omega) = B_\delta(\omega)$

Now, to extend (1.11.6) to all of $\mathcal{D}$ we write a function $f \in \mathcal{D}_+$ as $f = f_+ - f_-$, where $f_+, f_- \in \mathcal{D}_+$. Applying the triangle inequality and using (1.11.6) twice gives

$$\|\tilde{\mathcal{L}}^n_{\sigma^p(\omega)} f_{\sigma^p(\omega)} - \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\phi_{\sigma^{p+n}(\omega)}\|_\infty \leq 4B(\omega) \|f_{\sigma^p(\omega)}\|_\infty \|\phi_\omega\|_\infty \kappa^n.$$

Setting $D(\omega) := 4B(\omega)\|\phi_\omega\|_\infty$ finishes the proof of (1.11.1). To prove the second claim concerning $\mathscr{L}$ follows easily from the first claim in a similar fashion as in the proof of Theorem 10.4 of [3]. □

From the previous result we easily deduce that the invariant measure μ satisfies an exponential decay of correlations. The following theorem, whose proof is exactly the same as Theorem 11.1 of [3], completes the proof of Theorem C.

THEOREM 1.11.3. *For m-a.e. every $\omega \in \Omega$, every $n \in \mathbb{N}$, every $|p| \leq n$, every $f \in L^1(\mu)$, and every $h \in \mathcal{D}$ we have*

$$\left|\mu_\tau\left(\left(f_{\sigma^n(\tau)} \circ T^n_\tau\right) h_\tau\right) - \mu_{\sigma^n(\tau)}(f_{\sigma^n(\tau)})\mu_\tau(h_\tau)\right| \leq D(\omega) \|f_{\sigma^n(\tau)}\|_{L^1(\mu_{\sigma^n(\tau)})} \|h_\tau\|_\infty \kappa^n,$$

where $\tau = \sigma^p(\omega)$.

REMARK 1.11.4. Note that Theorem 1.11.2 implies a stronger limit characterization of the measure ν_ω than what is concluded in (1.10.34). Indeed, Theorem 1.11.2 implies that

$$\nu_\omega(f) = \lim_{n\to\infty} \frac{\mathcal{L}^n_\omega f(x_n)}{\mathcal{L}^n_\omega \mathbb{1}_\omega(y_n)}$$

[2]Any $\tilde{\kappa} > \vartheta$ will work for n sufficiently large.

for any pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with $x_n, y_n \in D_{\sigma^n(\omega),\infty}$, which further implies that

$$\nu_\omega(f) = \lim_{n\to\infty} \frac{\|\mathcal{L}_\omega^n f\|_\infty}{\inf \mathcal{L}_\omega^n \mathbb{1}_\omega} = \lim_{n\to\infty} \frac{\inf \mathcal{L}_\omega^n f}{\|\mathcal{L}_\omega^n \mathbb{1}_\omega\|_\infty}. \tag{1.11.7}$$

Furthermore, the same holds for $\nu_{\omega,0}$; see Lemma 9.2 and Proposition 10.4 of [3].

We now address the uniqueness of the families of measures $\nu = (\nu_\omega)_{\omega\in\Omega}$ and $\mu = (\mu_\omega)_{\omega\in\Omega}$ as well as the invariant density ϕ.

Proposition 1.11.5.

(1) The family $\nu = (\nu_\omega)_{\omega\in\Omega}$ is a random probability measure which is uniquely determined by (1.10.39).
(2) The global invariant density $q \in \mathcal{D}$ produced in Corollary 1.10.6 is the unique element of $L^1(\nu)$ (modulo ν) such that

$$\tilde{\mathcal{L}}_\omega \phi_\omega = \phi_{\sigma(\omega)}.$$

(3) The family $\mu = (\mu_\omega)_{\omega\in\Omega}$ is a unique random T-invariant probability measure which is absolutely continuous with respect to ν.

Proof. The fact that the family $(\nu_\omega)_{\omega\in\Omega}$ is a random measure as in Definition 0.1.1 follows from the limit characterization given in (1.11.7), as we have that ν_ω is a limit of measurable functions. Indeed, for every interval $J \subset I$, the measurability of the function $\omega \mapsto \nu_\omega(J)$ follows from the fact that it is given by the limit of measurable functions by (1.11.7) applied to the characteristic function $f_\omega = \mathbb{1}_J$. Since $\mathscr{B}$ is generated by intervals, $\omega \mapsto \nu_\omega(B)$ is measurable for every $B \in \mathscr{B}$. Furthermore, ν_ω is a Borel probability measure for m-a.e. $\omega \in \Omega$ from Proposition 1.10.11.

The remainder of the proof Proposition 1.11.5 follows along exactly like the proofs of Propositions 9.4 and 10.4 of [3], and is therefore left to the reader. □

The proof of the following proposition is the same as the proof of Proposition 4.7 of [57], and so it is omitted.

Proposition 1.11.6. *The random T-invariant probability measure μ defined in (1.10.40) is ergodic.*

In the following lemma we establish the existence of the unique random absolutely continuous conditionally invariant probability measure η with density in BV.

Lemma 1.11.7. *The random measure $\eta \in \mathcal{P}_\Omega(\Omega \times I)$, whose disintegrations are given by*

$$\eta_\omega(f) := \frac{\nu_{\omega,0}(f \cdot \mathbb{1}_\omega \cdot \phi_\omega)}{\nu_{\omega,0}(\mathbb{1}_\omega \cdot \phi_\omega)},$$

is the unique random absolutely continuous (with respect to ν_0) conditionally invariant probability measure supported on $\mathcal{I}$ with fiberwise density of bounded variation.

Proof. The fact that η is an RACCIM follows from Lemma 1.2.5 and uniqueness follows from Proposition 1.11.5. □

As a corollary of Theorem 1.11.2, the following results gives the exponential convergence of the closed conformal measure $\nu_{\omega,0}$ conditioned on the survivor set to the RACCIM η_ω.

COROLLARY 1.11.8. *For m-a.e. every $\omega \in \Omega$, every $n \in \mathbb{N}$, every $|p| \leq n$, every $A \in \mathscr{B}$, and every $f \in \mathcal{D}$ we have*

$$\left|\nu_{\sigma^p(\omega),0}\left(T^{-n}_{\sigma^p(\omega)}(A) \mid X_{\sigma^p(\omega),n}\right) - \eta_{\sigma^{p+n}(\omega)}(A)\right| \leq \frac{D(\omega)}{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\phi_{\sigma^{p+n}(\omega)}\right)}\kappa^n$$

and

$$\left|\frac{\eta_{\sigma^p(\omega)}\left(f_{\sigma^p(\omega)}|_{X_{\sigma^p(\omega),n}}\right)}{\eta_{\sigma^p(\omega)}\left(X_{\sigma^p(\omega),n}\right)} - \mu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\right| \leq D(\omega)\|f_{\sigma^p(\omega)}\|_\infty \kappa^n$$

for m-a.e. $\omega \in \Omega$.

PROOF. For $A \in \mathscr{B}$, Lemma 1.2.4 allows us to write

$$\begin{aligned}
\nu_{\sigma^p(\omega),0}\left(T^{-n}_{\sigma^p(\omega)}(A) \cap X_{\sigma^p(\omega),n}\right) &= \int_{X_{\sigma^p(\omega),n}} \mathbb{1}_{\sigma^p(\omega)}\mathbb{1}_A \circ T^n_{\sigma^p(\omega)}\, d\nu_{\sigma^p(\omega),0} \\
&= \left(\lambda^n_{\sigma^p(\omega),0}\right)^{-1} \int_{I_{\sigma^{p+n}(\omega)}} \mathbb{1}_A \mathcal{L}^n_{\sigma^p(\omega)}\mathbb{1}_{\sigma^p(\omega)}\, d\nu_{\sigma^{p+n}(\omega),0} \\
&= \nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_A \mathbb{1}_{\sigma^{p+n}(\omega)}\tilde{\mathcal{L}}^n_{\sigma^p(\omega)}(\mathbb{1}_{\sigma^p(\omega)})\right).
\end{aligned}$$

So, if $A = I$, then we have

$$\nu_{\sigma^p(\omega),0}\left(X_{\sigma^p(\omega),n}\right) = \nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\tilde{\mathcal{L}}^n_{\sigma^p(\omega)}(\mathbb{1}_{\sigma^p(\omega)})\right).$$

Thus, we apply (1.11.1) of Theorem 1.11.2 together with elementary calculation to get that

$$\begin{aligned}
&\left|\nu_{\sigma^p(\omega),0}\left(T^{-n}_{\sigma^p(\omega)}(A) \mid X_{\sigma^p(\omega),n}\right) - \eta_{\sigma^{p+n}(\omega)}(A)\right| \\
&\quad = \left|\frac{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_A \mathbb{1}_{\sigma^{p+n}(\omega)}\tilde{\mathcal{L}}^n_{\sigma^p(\omega)}(\mathbb{1}_{\sigma^p(\omega)})\right)}{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\tilde{\mathcal{L}}^n_{\sigma^p(\omega)}(\mathbb{1}_{\sigma^p(\omega)})\right)} - \frac{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_A \mathbb{1}_{\sigma^{p+n}(\omega)}\phi_{\sigma^{p+n}(\omega)}\right)}{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\phi_{\sigma^{p+n}(\omega)}\right)}\right| \\
&\quad \leq \frac{D(\omega)}{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\phi_{\sigma^{p+n}(\omega)}\right)}\kappa^n.
\end{aligned}$$

To see the second claim we note that for $f \in \mathcal{D}$

$$\begin{aligned}
\frac{\eta_{\sigma^p(\omega)}\left(f_{\sigma^p(\omega)}|_{X_{\sigma^p(\omega),n}}\right)}{\eta_{\sigma^p(\omega)}\left(X_{\sigma^p(\omega),n}\right)} &= \frac{\nu_{\sigma^p(\omega),0}\left(f_{\sigma^p(\omega)}\hat{X}_{\sigma^p(\omega),n}\mathbb{1}_{\sigma^p(\omega)}\phi_{\sigma^p(\omega)}\right)}{\nu_{\sigma^p(\omega),0}\left(\hat{X}_{\sigma^p(\omega),n}\mathbb{1}_{\sigma^p(\omega)}\phi_{\sigma^p(\omega)}\right)} \\
&= \frac{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\tilde{\mathcal{L}}^n_{\sigma^p(\omega)}(f_{\sigma^p(\omega)}\phi_{\sigma^p(\omega)})\right)}{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\tilde{\mathcal{L}}^n_{\sigma^p(\omega)}(\phi_{\sigma^p(\omega)})\right)}
\end{aligned}$$

$$= \frac{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\mathscr{L}^n_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\phi_{\sigma^{p+n}(\omega)}\right)}{\nu_{\sigma^{p+n}(\omega),0}\left(\mathbb{1}_{\sigma^{p+n}(\omega)}\phi_{\sigma^{p+n}(\omega)}\right)}$$
$$= \eta_{\sigma^{p+n}(\omega)}\left(\mathscr{L}^n_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\right).$$

Thus, applying (1.11.2) of Theorem 1.11.2, we have

$$\left|\frac{\eta_{\sigma^p(\omega)}\left(f_{\sigma^p(\omega)}|_{X_{\sigma^p(\omega),n}}\right)}{\eta_{\sigma^p(\omega)}\left(X_{\sigma^p(\omega),n}\right)} - \mu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\right| = \left|\eta_{\sigma^{p+n}(\omega)}\left(\mathscr{L}^n_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\right) - \mu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\right|$$
$$\leq D(\omega)\|f_{\sigma^p(\omega)}\|_\infty \kappa^n,$$

which finishes the proof. □

Remark 1.11.9. Note that the second claim of Corollary 1.11.8 differs from the third claim of Theorem A of [53] where the substitute the function $f \in \mathcal{D}$ (or $f \in \mathrm{BV}(I)$) with the function $\mathbb{1}_A$ for A a Borel set. As stated this is not true as this result can not hold for general $A \in \mathscr{B}$; taking $A = X_\infty$ produces a counterexample. However, if A is taken as the union of finitely many intervals, then $\mathbb{1}_A \in \mathrm{BV}(I)$ and the third claim of Theorem A of [53] holds.

1.12. Expected pressures and escape rates

We now establish the rate at which mass escapes through the hole with respect to the closed conformal measure $\nu_{\omega,0}$ and the RACCIM η_ω in terms of the open and closed expected pressures.

Definition 1.12.1. Given a potential φ_0 on the closed system we define the expected pressure of the closed and open systems respectively by

$$\mathcal{E}P(\varphi_0) := \int_\Omega \log \lambda_{\omega,0}\, dm(\omega) \quad \text{and} \quad \mathcal{E}P(\varphi) := \int_\Omega \log \lambda_\omega\, dm(\omega).$$

In light of (1.10.38) (and (C)), we see that the definition of the expected pressures $\mathcal{E}P(\varphi_0)$, $\mathcal{E}P(\varphi) \in \mathbb{R}$, are well defined. Since $\log \lambda_\omega, \log \lambda_{\omega,0} \in L^1(m)$, Birkhoff's Ergodic Theorem gives that

$$\mathcal{E}P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \lambda^n_\omega = \lim_{n\to\infty} \frac{1}{n} \log \lambda^n_{\sigma^{-n}(\omega)} \tag{1.12.1}$$

and

$$\mathcal{E}P(\varphi_0) = \lim_{n\to\infty} \frac{1}{n} \log \lambda^n_{\omega,0} = \lim_{n\to\infty} \frac{1}{n} \log \lambda^n_{\sigma^{-n}(\omega),0}. \tag{1.12.2}$$

The following lemma, which is the open analogue of Lemma 10.1 of [3], gives an alternate method for calculating the expected pressure.

Lemma 1.12.2. *For m-a.e. $\omega \in \Omega$ we have that*

$$\lim_{n\to\infty} \|\frac{1}{n} \log \mathcal{L}^n_{\sigma^{-n}(\omega)} \mathbb{1}_{\sigma^{-n}(\omega)} - \frac{1}{n} \log \lambda^n_{\sigma^{-n}(\omega)}\|_\infty = 0 \tag{1.12.3}$$

and

$$\lim_{n\to\infty} \|\frac{1}{n}\log \mathcal{L}_\omega^n \mathbb{1}_\omega - \frac{1}{n}\log \lambda_\omega^n\|_\infty = 0. \tag{1.12.4}$$

Furthermore, for m-a.e. $\omega \in \Omega$ we have that

$$\lim_{n\to\infty} \frac{1}{n}\log \inf_{D_{\sigma^n(\omega),\infty}} \phi_{\sigma^n(\omega)} = \lim_{n\to\infty} \frac{1}{n}\log \|\phi_{\sigma^n(\omega)}\|_\infty = 0.$$

As the proof of the previous lemma is exactly the same as the proof of Lemma 10.1 of [3] where infima are taken over the appropriate $D_{\omega,\infty}$ sets, the proof is left to the reader. Now, in view of the fact that

$$\mathcal{L}_\omega \mathbb{1}_\omega \le \mathcal{L}_{\omega,0}\mathbb{1},$$

Lemma 1.12.2 and Lemma 10.1 of [3], together with (1.12.1) and (1.12.2) imply that

$$\mathcal{E}P(\varphi) \le \mathcal{E}P(\varphi_0). \tag{1.12.5}$$

We now define the fiberwise escape rates of a random measure.

Definition 1.12.3. Given a random probability measure ϱ on $\mathcal{I}$, for each $\omega \in \Omega$, we define the lower and upper fiberwise escape rates respectively by the following:

$$\underline{R}(\varrho_\omega) := -\limsup_{n\to\infty} \frac{1}{n}\log \varrho_\omega(X_{\omega,n}) \quad \text{and} \quad \overline{R}(\varrho_\omega) := -\liminf_{n\to\infty} \frac{1}{n}\log \varrho_\omega(X_{\omega,n}).$$

If $\underline{R}(\varrho_\omega) = \overline{R}(\varrho_\omega)$, we say the escape rate exists and denote the common value by $R(\varrho_\omega)$.

The previous results allow us to calculate the following escape rates, thus proving Theorem D.

Proposition 1.12.4. *For m-a.e. $\omega \in \Omega$ we have that*

$$R(\nu_{\omega,0}) = R(\eta_\omega) = \mathcal{E}P(\varphi_0) - \mathcal{E}P(\varphi).$$

Proof. We begin by noting that

$$\begin{aligned}\nu_{\omega,0}(X_{\omega,n-1}) &= \left(\lambda_{\omega,0}^n\right)^{-1} \nu_{\sigma^n(\omega),0}\left(\mathcal{L}_{\omega,0}^n(\hat{X}_{\omega,n-1})\right) = \frac{\lambda_\omega^n}{\lambda_{\omega,0}^n}\nu_{\sigma^n(\omega),0}\left(\tilde{\mathcal{L}}_\omega^n \mathbb{1}_\omega\right)\\ &= \frac{\lambda_\omega^n}{\lambda_{\omega,0}^n}\left(\nu_{\sigma^n(\omega),0}(\phi_{\sigma^n(\omega)}) - \nu_{\sigma^n(\omega),0}\left(\tilde{\mathcal{L}}_\omega^n \mathbb{1}_\omega - \phi_{\sigma^n(\omega)}\right)\right).\end{aligned}$$

Using Theorem 1.11.2 gives that

$$-R(\nu_{\omega,0}) = \lim_{n\to\infty}\frac{1}{n}\log\frac{\lambda_\omega^n}{\lambda_{\omega,0}^n} + \lim_{n\to\infty}\frac{1}{n}\log \nu_{\sigma^n(\omega),0}(\phi_{\sigma^n(\omega)}).$$

Thus, the temperedness of $\inf_{D_{\omega,\infty}} \phi_\omega$ and $\|\phi_\omega\|_\infty$, coming from Lemma 1.12.2, imply that

$$0 = \lim_{n\to\infty}\frac{1}{n}\log \inf_{D_{\sigma^n(\omega),\infty}} \phi_{\sigma^n(\omega)} \le \lim_{n\to\infty}\frac{1}{n}\log \nu_{\sigma^n(\omega),0}(\phi_{\sigma^n(\omega)}) \le \lim_{n\to\infty}\frac{1}{n}\log \|q_{\sigma^n(\omega)}\|_\infty = 0,$$

which, when combined with (1.12.1) and (1.12.2), completes the proof of the first claim. As the second equality follows similarly to the first, we are done. □

REMARK 1.12.5. If there exists a T-invariant measure μ_0 on the closed system which is absolutely continuous with respect to ν_0 then the proof of Proposition 1.12.4, with minor adjustments, also shows that for m-a.e. $\omega \in \Omega$ we have that

$$R(\nu_{\omega,0}) = R(\mu_{\omega,0}) = \mathcal{E}P(\varphi_0) - \mathcal{E}P(\varphi).$$

1.13. Equilibrium states

In this short section we show that the invariant measures $\mu = \{\mu_\omega\}_{\omega\in\Omega}$ constructed in Section 1.10 are in fact the unique relative equilibrium states for the random open system.

DEFINITION 1.13.1. Let $\mathcal{P}^H_{T,m}(I)$ denote the collection of T-invariant random probability measures ζ on $\Omega \times I$, such that the disintegration $\{\zeta_\omega\}$ satisfy $\zeta_\omega(H_\omega) = 0$ for m a.e. $\omega \in \Omega$. We say that a measure $\zeta \in \mathcal{P}^H_{T,m}(\Omega)$ is a *relative equilibrium state* for the random open system (open) potential φ if

$$\mathcal{E}P(\varphi) = h_\zeta(T) + \int_{\Omega\times I} \varphi \, d\zeta,$$

where $h_\zeta(T)$ denotes the entropy of T with respect to ζ.

The proof of the next result follows similarly to the proof of Theorem 2.23 in [3] (see also Remark 2.24, Lemma 12.2 and Lemma 12.3).

THEOREM 1.13.2. *The random measure $\mu \in \mathcal{P}^H_{T,m}(\Omega)$ with disintegration $\{\mu_\omega\}_{\omega\in\Omega}$ produced in (1.10.40) is the unique relative equilibrium state for the potential φ. It satisfies the following variational principle:*

$$\mathcal{E}P(\varphi) = h_\mu(T) + \int_{\Omega\times I} \varphi \, d\mu = \sup_{\zeta\in\mathcal{P}^H_{T,m}(\Omega\times I)} \left(h_\zeta(T) + \int_{\Omega\times I} \varphi \, d\zeta \right).$$

Furthermore, for each $\zeta \in \mathcal{P}^H_{T,m}(\Omega)$ different from μ we have that

$$h_\zeta(T) + \int_{\Omega\times I} \varphi \, d\zeta < h_\mu(T) + \int_{\Omega\times I} \varphi \, d\mu$$

1.14. Bowen's formula

This section is devoted to proving a formula for the Hausdorff dimension of the survivor set in terms of the expected pressure function, which was first proven by Bowen in [13] in the setting quasi-Fuchsian groups. In this section we will consider geometric potentials of the form $\varphi_{0,t}(\omega, x) = -t \log |T'_\omega|(x)$ ($t \in [0, 1]$) for maps T_ω which are expanding on average. We denote the expected pressure of $\varphi_{0,t}$ by $\mathcal{E}P_0(t)$ and the expected pressure of the open potential φ_t by $\mathcal{E}P(t)$. In the case that $t = 1$, the fiberwise closed conformal measures $\nu_{\omega,0,1}$ are equal to Lebesgue measure and $\lambda_{\omega,0,1} = 1$. Furthermore, we note that for any $t \geq 0$ we have that

$$g^{(n)}_{\omega,0,t} = \left(g^{(n)}_{\omega,0,1}\right)^t = \frac{1}{|(T^n_\omega)'|^t}. \tag{1.14.1}$$

DEFINITION 1.14.1. We will say that the weight function $g_{\omega,0}$ has the *Bounded Distortion Property* if for m-a.e. $\omega \in \Omega$ there exists $K_\omega \geq 1$ such that for all $n \in \mathbb{N}$, all $Z \in \mathcal{Z}_\omega^{(n)}$, and all $x, y \in Z$ we have that

$$\frac{g_{\omega,0}^{(n)}(x)}{g_{\omega,0}^{(n)}(y)} \leq K_\omega.$$

We now adapt the following definitions from [53] to the random setting.

DEFINITION 1.14.2. We will say that the map T has *large images* if for m-a.e. $\omega \in \Omega$ we have

$$\inf_{n\in\mathbb{N}} \inf_{Z\in\mathcal{Z}_\omega^{(n)}} \nu_{\sigma^n(\omega),0}\left(T_\omega^n(Z)\right) > 0.$$

T is said to have *large images with respect to* H if for m-a.e. $\omega \in \Omega$, each $n \in \mathbb{N}$, and each $Z \in \mathcal{Z}_\omega^{(n)}$ with $Z \cap X_{\omega,\infty} \neq \emptyset$ we have

$$T_\omega^n(Z \cap X_{\omega,n-1}) \supseteq X_{\sigma^n(\omega),\infty}.$$

REMARK 1.14.3. If T has large images with respect to H then it follows from Remark 1.10.12 that $\operatorname{supp}(\nu_{\omega,t}) = X_{\omega,\infty}$ for any $t \in [0,1]$.

We now prove a formula for the Hausdorff dimension of the surviving set, à la Bowen, proving Theorem E.

THEOREM 1.14.4. *Suppose that* $\int_\Omega \log\inf|T'_\omega|\, dm(\omega) > 0$ *and that* $g_0 = 1/|T'|$ *satisfies the bounded distortion property, then there exists a unique* $h \in [0,1]$ *such that* $\mathcal{E}P(t) > 0$ *for all* $0 \leq t < h$ *and* $\mathcal{E}P(t) < 0$ *for all* $h < t \leq 1$. *Furthermore, if* T *has large images and large images with respect to* H, *then for* m*-a.e.* $\omega \in \Omega$

$$\mathrm{HD}(X_{\omega,\infty}) = h.$$

PROOF. We will prove this theorem in a series of lemmas.

LEMMA 1.14.5. *The function* $\mathcal{E}P(t)$ *is strictly decreasing and there exists* $h \in [0,1]$ *such that* $\mathcal{E}P(t) > 0$ *for all* $0 \leq t < h$ *and* $\mathcal{E}P(t) < 0$ *for all* $h < t \leq 1$.

PROOF. We first note that, using (1.14.1), for any $n \in \mathbb{N}$ and $s < t \in [0,1]$ we can write

$$\mathcal{L}_{\omega,t}^n \mathbb{1}_\omega \leq \|g_{\omega,1}^{(n)}\|_\infty^{t-s} \mathcal{L}_{\omega,s}^n \mathbb{1}_\omega.$$

This immediately implies that

$$\mathcal{E}P(t) < \mathcal{E}P(s)$$

since for m-a.e. $\omega \in \Omega$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|g_{\omega,1}^{(n)}\|_\infty < 0.$$

Now since $\mathcal{E}P(0) \geq 0$ and $\mathcal{E}P(1) \leq \mathcal{E}P_0(1) = 0$, there must exist some $h \in [0,1]$ such that for all $s < h < t$ we have

$$\mathcal{E}P(t) < 0 < \mathcal{E}P(s).$$

□

To prove the remaining claim of Theorem 1.14.4, we now suppose that T has large images and large images with respect to H.

LEMMA 1.14.6. *If T has large images and large images with respect to H, and $\nu_{\omega,t}(Z) > 0$ for all $t \in [0,1]$, all $n \in \mathbb{N}$, and all $Z \in \mathcal{Z}_\omega^{(n)}$, then for all $x \in Z$ we have*

$$K_\omega^{-1} \leq \frac{\left(g_{\omega,0,1}^{(n)}\right)^t(x)}{\lambda_{\omega,t}^n \nu_{\omega,t}(Z)} \leq K_\omega \quad \text{and} \quad K_\omega^{-1} \leq \frac{g_{\omega,0,1}^{(n)}(x)}{\mathrm{Leb}(Z)} \leq K_\omega.$$

PROOF. In light of Remark 1.14.3, $\mathrm{supp}(\nu_{\omega,t}) = X_{\omega,\infty}$, and thus for any $Z \in \mathcal{Z}_\omega^{(n)}$ for any $n \geq 1$, $\nu_{\omega,t}(Z) > 0$ if and only if $Z \cap X_{\omega,\infty} \neq \emptyset$. Furthermore, since T has large images with respect to H, we have that

$$\nu_{\sigma^n(\omega),t}(T_\omega^n(Z)) = 1 \tag{1.14.2}$$

for any $Z \in \mathcal{Z}_\omega^{(n)}$ with $Z \cap X_{\omega,\infty} \neq \emptyset$. Thus, we may write

$$\begin{aligned}
\nu_{\omega,t}(Z) &= \int_{X_{\omega,\infty}} \mathbb{1}_Z \, d\nu_{\omega,t} = \left(\lambda_{\omega,t}^n\right)^{-1} \int_{X_{\sigma^n(\omega),\infty}} \mathcal{L}_{\omega,t}^n \mathbb{1}_Z \, d\nu_{\sigma^n(\omega),t} \\
&= \left(\lambda_{\omega,t}^n\right)^{-1} \int_{X_{\sigma^n(\omega),\infty}} \mathcal{L}_{\omega,0,t}^n \left(\mathbb{1}_Z \hat{X}_{\omega,n-1}\right) \, d\nu_{\sigma^n(\omega),t} \\
&= \left(\lambda_{\omega,t}^n\right)^{-1} \int_{T_\omega^n(Z)} \left(\left(g_{\omega,0,1}^{(n)}\right)^t \hat{X}_{\omega,n-1}\right) \circ T_{\omega,Z}^{-n} \, d\nu_{\sigma^n(\omega),t} \\
&= \left(\lambda_{\omega,t}^n\right)^{-1} \int_{T_\omega^n(Z)} \left(g_{\omega,0,1}^{(n)}\right)^t \circ T_{\omega,Z}^{-n} \, d\nu_{\sigma^n(\omega),t}. \qquad (1.14.3)
\end{aligned}$$

The Bounded Distortion Property implies that for $x \in Z$ we have

$$\begin{aligned}
K_\omega^{-1} \nu_{\sigma^n(\omega),t}(T_\omega^n(Z)) \left(g_{\omega,0,1}^{(n)}\right)^t(x) &\leq \int_{T_\omega^n(Z)} \left(g_{\omega,0,1}^{(n)}\right)^t \circ T_{\omega,Z}^{-n} \, d\nu_{\sigma^n(\omega),t} \\
&\leq K_\omega \nu_{\sigma^n(\omega),t}(T_\omega^n(Z)) \left(g_{\omega,0,1}^{(n)}\right)^t(x).
\end{aligned}$$

Thus

$$K_\omega^{-1} \frac{\left(g_{\omega,0,1}^{(n)}\right)^t(x)}{\lambda_{\omega,t}^n \nu_{\omega,t}(Z)} \leq \frac{1}{\nu_{\sigma^n(\omega),t}(T_\omega^n(Z))} \leq K_\omega \frac{\left(g_{\omega,0,1}^{(n)}\right)^t(x)}{\lambda_{\omega,t}^n \nu_{\omega,t}(Z)},$$

which then implies that

$$K_\omega^{-1} \frac{1}{\nu_{\sigma^n(\omega),t}(T_\omega^n(Z))} \leq \frac{\left(g_{\omega,0,1}^{(n)}\right)^t(x)}{\lambda_{\omega,t}^n \nu_{\omega,t}(Z)} \leq K_\omega \frac{1}{\nu_{\sigma^n(\omega),t}(T_\omega^n(Z))}.$$

The first claim follows from (1.14.2). The proof of the second claim involving the Lebesgue measure follows similarly noting that $\nu_{\omega,0,1} = \mathrm{Leb}$ and $\lambda_{\omega,0,1} = 1$. □

Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $\operatorname{diam}(Z) < \varepsilon$ for all $Z \in \mathcal{Z}_\omega^{(n)}$. Denote

$$\mathcal{F}_\omega := \left\{ Z \in \mathcal{Z}_\omega^{(n)} : Z \cap X_{\omega,\infty} \neq \emptyset \right\},$$

which is a cover of $X_{\omega,\infty}$ by sets of diameter less than ε. Then, letting x_Z be any element of Z and using Lemma 1.14.6 twice (first with respect to Leb and then with respect to $\nu_{\omega,t}$), we have

$$\begin{aligned}\sum_{Z\in\mathcal{F}_\omega} \operatorname{diam}^t(Z) &\leq \sum_{Z\in\mathcal{F}_\omega} \operatorname{Leb}^t(Z) \leq K_\omega^t \sum_{Z\in\mathcal{F}_\omega} \left(g_{\omega,0,1}^{(n)}\right)^t (x_Z) \\ &\leq K_\omega^{2t} \sum_{Z\in\mathcal{F}_\omega} \lambda_{\omega,t}^n \nu_{\omega,t}(Z) = K_\omega^{2t} \lambda_{\omega,t}^n \nu_{\omega,t}(X_{\omega,\infty}) = K_\omega^{2t} \lambda_{\omega,t}^n. \end{aligned} \tag{1.14.4}$$

Now, if $t > h$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_{\omega,t}^n = \mathcal{E}P(t) < 0,$$

and thus, for $\delta > 0$ sufficiently small and all $n \in \mathbb{N}$ sufficiently large,

$$\lambda_{\omega,t}^n \leq e^{-n\delta}.$$

Consequently, we see that the right-hand side of (1.14.4) must go to zero, and thus we must have that $\operatorname{HD}(X_{\omega,\infty}) \leq h$.

For the lower bound we turn to the following result of Young.

Proposition 1.14.7 (Proposition, [66]). *Let X be a metric space and $Z \subseteq X$. Assume there exists a probability measure μ such that $\mu(Z) > 0$. For any $x \in Z$ we define*

$$\underline{d_\mu}(x) := \liminf_{\varepsilon\to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}.$$

If $\underline{d_\mu}(x) \geq d$ for each $x \in Z$, then $\operatorname{HD}(Z) \geq d$.

We will use this result to prove a lower bound for the dimension, thus completing the proof of Theorem 1.14.4. Let $x \in X_{\omega,\infty}$, let $\varepsilon > 0$, and in light of Lemma 1.14.6, let $n_{\omega,0}+1$ be the least positive integer such that there exists $y_0 \in B(x,\varepsilon)$ such that

$$g_{\omega,0}^{(n_{\omega,0}+1)}(y_0) \leq 2\varepsilon K_\omega.$$

Note that as $\varepsilon \to 0$ we must have that $n_{\omega,0} \to \infty$. So we must have

$$g_{\omega,0}^{(n_{\omega,0})}(y_0) g_{\sigma^{n_{\omega,0}}(\omega),0}(T_\omega^{n_{\omega,0}}(y_0)) = g_{\omega,0}^{(n_{\omega,0}+1)}(y_0) \leq 2\varepsilon K_\omega. \tag{1.14.5}$$

Thus, using (1.14.5) and the definition of $n_{\omega,0}$ we have that

$$2\varepsilon K_\omega < g_{\omega,0}^{(n_{\omega,0})}(y_0) \leq \frac{2\varepsilon K_\omega}{\inf g_{\sigma^{n_{\omega,0}}(\omega),0}}.$$

Now let $Z_0 \in \mathcal{Z}_\omega^{(n_{\omega,0})}$ be the partition element containing y_0. Then Lemma 1.14.6 gives that

$$\operatorname{diam}(Z_0) \leq K_\omega g_{\omega,0}^{(n_{\omega,0})}(y_0) \leq \frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,0}}(\omega),0}}, \tag{1.14.6}$$

and

$$\operatorname{diam}(Z_0) \geq K_\omega^{-1} g_{\omega,0}^{(n_{\omega,0})}(y_0) > 2\varepsilon. \tag{1.14.7}$$

Combining (1.14.6) and (1.14.7) gives

$$2\varepsilon < \operatorname{diam}(Z_0) \leq \frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,0}}(\omega),0}}. \tag{1.14.8}$$

Now, we define

$$B_{\omega,1} := B(x,\varepsilon)\backslash Z_0,$$

which may be empty. If $B_{\omega,1} \neq \emptyset$, then we let $n_{\omega,1}+1$ be the least positive integer such that there exists $y_1 \in B_{\omega,1}$ such that

$$g_{\omega,0}^{(n_{\omega,1}+1)}(y_1) \leq 2\varepsilon K_\omega.$$

Following the same line of reasoning to derive (1.14.8), we see that

$$2\varepsilon < \operatorname{diam}(Z_1) \leq \frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,1}}(\omega),0}}, \tag{1.14.9}$$

where $Z_1 \in \mathcal{Z}_\omega^{(n_{\omega,1})}$ is the partition element containing y_1. Note that by definition we have that $n_{\omega,1} \geq n_{\omega,0}$ and $Z_0 \cap Z_1 = \emptyset$. This immediately implies that

$$B(x,\varepsilon) \subseteq Z_0 \cup Z_1,$$

as otherwise using the same construction we could find some $y_2 \in B_{\omega,1}\backslash Z_1$, some $n_{\omega,2} \geq n_{\omega,1}$ and a partition element $Z_2 \in \mathcal{Z}_\omega^{(n_{\omega,2})}$ containing y_2 with diameter greater than 2ε. But this would produce three disjoint intervals each with diameter greater than 2ε all of which intersect $B(x,\varepsilon)$, which would obviously be a contradiction.

Now, using (1.14.3) and Lemma 1.14.6 gives that

$$\nu_{\omega,t}(Z_j) = \left(\lambda_{\omega,t}^{n_{\omega,j}}\right)^{-1} \int_{T_\omega^{n_{\omega,j}}(Z)} \left(g_{\omega,0,1}^{(n_{\omega,j})}\right)^t \circ T_{\omega,Z}^{-n_{\omega,j}} \, d\nu_{\sigma^{n_{\omega,j}}(\omega),t} \leq K_\omega^t \left(\lambda_{\omega,t}^{n_{\omega,j}}\right)^{-1} \operatorname{diam}^t(Z_j)$$

for $j \in \{0,1\}$. Using this we see that

$$\begin{aligned}
&\log \nu_{\omega,t}(B(x,\varepsilon)) \leq \log\left(\nu_{\omega,t}(Z_0) + \nu_{\omega,t}(Z_1)\right) \\
&\leq t\log K_\omega + \log\left(\left(\lambda_{\omega,t}^{n_{\omega,0}}\right)^{-1} \operatorname{diam}^t(Z_0) + \left(\lambda_{\omega,t}^{n_{\omega,1}}\right)^{-1} \operatorname{diam}^t(Z_1)\right) \\
&\leq t\log K_\omega + \log\left(\left(\lambda_{\omega,t}^{n_{\omega,0}}\right)^{-1} \left(\frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,0}}(\omega),0}}\right)^t + \left(\lambda_{\omega,t}^{n_{\omega,1}}\right)^{-1} \left(\frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,1}}(\omega),0}}\right)^t\right) \\
&= t\log K_\omega + t\log 2\varepsilon K_\omega^2 + \log\left(\left(\lambda_{\omega,t}^{n_{\omega,0}}\right)^{-1} \left(\inf g_{\sigma^{n_{\omega,0}}(\omega),0}\right)^{-t} + \left(\lambda_{\omega,t}^{n_{\omega,1}}\right)^{-1} \left(\inf g_{\sigma^{n_{\omega,1}}(\omega),0}\right)^{-t}\right).
\end{aligned} \tag{1.14.10}$$

Now since $\log\inf g_{\omega,0} \in L^1(m)$, $\inf g_{\omega,0}$ is tempered and thus for each $\delta > 0$ and all $n \in \mathbb{N}$ sufficiently large we have that

$$e^{-nt\delta} \leq \inf\left(g_{\sigma^n(\omega),0}\right)^t. \tag{1.14.11}$$

From (1.12.1) we get that there for all $n \in \mathbb{N}$ sufficiently large

$$\lambda_{\omega,t}^n \geq e^{n(\mathcal{E}P(t)-\delta)}. \tag{1.14.12}$$

Thus combining (1.14.11) and (1.14.12) with (1.14.10) gives

$$\begin{aligned}\log \nu_{\omega,t}(B(x,\varepsilon)) &\leq t\log K_\omega + t\log 2\varepsilon K_\omega^2 + \log\left(e^{n_{\omega,0}\delta(t+1)-n_{\omega,0}\mathcal{E}P(t)} + e^{n_{\omega,1}\delta(t+1)-n_{\omega,1}\mathcal{E}P(t)}\right)\\ &\leq t\log K_\omega + t\log 2\varepsilon K_\omega^2 + \log\left(e^{2n_{\omega,0}\delta-n_{\omega,0}\mathcal{E}P(t)} + e^{2n_{\omega,1}\delta-n_{\omega,1}\mathcal{E}P(t)}\right),\end{aligned} \tag{1.14.13}$$

where we have used the fact that $t \in [0,1]$. Then for $\delta > 0$ sufficiently small and $n_{\omega,0}$ and $n_{\omega,1}$ sufficiently large (which requires $\varepsilon > 0$ to be sufficiently small) we have that

$$\log\left(e^{2n_{\omega,0}\delta-n_{\omega,0}\mathcal{E}P(t)} + e^{2n_{\omega,1}\delta-n_{\omega,1}\mathcal{E}P(t)}\right) < 0, \tag{1.14.14}$$

since for all $t < h$ we have that $\mathcal{E}P(t) > 0$. Dividing both sides of (1.14.13) by $\log\varepsilon < 0$ and using (1.14.14) yields

$$\begin{aligned}\frac{\log \nu_{\omega,t}(B(x,\varepsilon))}{\log\varepsilon} &\geq t\frac{\log K_\omega}{\log\varepsilon} + t\frac{\log 2\varepsilon K_\omega^2}{\log\varepsilon} + \frac{\log\left(e^{2n_{\omega,0}\delta-n_{\omega,0}\mathcal{E}P(t)} + e^{2n_{\omega,1}\delta-n_{\omega,1}\mathcal{E}P(t)}\right)}{\log\varepsilon}\\ &\geq t\frac{\log K_\omega}{\log\varepsilon} + t\frac{\log 2K_\omega^2}{\log\varepsilon} + t.\end{aligned} \tag{1.14.15}$$

Taking a liminf of (1.14.15) as ε goes to 0 gives that $\underline{d}_{\nu_{\omega,t}}(x) \geq t$ for all $x \in X_{\omega,\infty}$. As this holds for all $t < h$, we must in fact have that $\underline{d}_{\nu_{\omega,t}}(x) \geq h$. In light of Proposition 1.14.7, we have proven Theorem 1.14.4. □

1.15. Examples

In this final section we present several examples of our general theory of open random systems. In particular, we show that our results hold for a large class of random β-transformations with random holes which have uniform covering times as well as a large class of random Lasota-Yorke maps with random holes. However, we note that in principle any of the finitely branched classes of maps treated in [3] will satisfy our assumptions given a suitable choice of hole. This includes random systems where we allow non-uniformly expanding maps, or even maps with contracting branches to appear with positive probability. We also note that the examples we present allow for both random maps and random hole, which, to the authors' knowledge, has not appeared in literature until now. Before presenting our examples, we first give alternate hypotheses (to our assumptions (Q1)-(Q3)) that are more restrictive but simpler to check.

We begin by recalling the definitions of the various partitions constructed in Section 1.3 which are used in producing our main Lasota-Yorke inequality (Lemma 1.5.1) and are implicitly a part of our main assumptions (Q1)-(Q3). Recall that $\mathcal{Z}_\omega^{(n)}$ denotes the partition of monotonicity of T_ω^n, and $\mathscr{A}_\omega^{(n)}$ denotes the collection of all finite partitions of I such that

$$\mathrm{var}_{A_i}(g_\omega^{(n)}) \leq 2\|g_\omega^{(n)}\|_\infty \tag{1.15.1}$$

for each $\mathcal{A} = \{A_i\} \in \mathscr{A}_\omega^{(n)}$. Given $\mathcal{A} \in \mathscr{A}_\omega^{(n)}$, $\widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A})$ denotes the coarsest partition amongst all those finer than $\mathcal{A}$ and $\mathcal{Z}_\omega^{(n)}$ such that all elements of $\widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{A})$ are either

disjoint from $X_{\omega,n-1}$ or contained in $X_{\omega,n-1}$. From $\widehat{\mathcal{Z}}_\omega^{(n)}$ we recall the subcollections $\mathcal{Z}_{\omega,*}^{(n)}$, $\mathcal{Z}_{\omega,b}^{(n)}$, and $\mathcal{Z}_{\omega,g}^{(n)}$ defined in (1.3.14)-(1.3.16).

For the purposes of showing that examples easily satisfy our conditions, we take the more general approach to partitions found in Section 2.2 of [3], and instead now set, for $\hat{\alpha} \geq 0$, $\overline{\mathscr{A}}_\omega^{(n)}(\hat{\alpha})$ to be the collection of all finite partitions of I such that

$$\mathrm{var}_{A_i}(g_\omega^{(n)}) \leq \hat{\alpha}\|g_\omega^{(n)}\|_\infty \tag{1.15.2}$$

for each $\mathcal{A} = \{A_i\} \in \overline{\mathscr{A}}_\omega^{(n)}(\hat{\alpha})$. Note that for some $\hat{\alpha} \leq 1$ the collection $\overline{\mathscr{A}}_\omega^{(n)}(\hat{\alpha})$ may be empty, but such partitions always exist for any $\hat{\alpha} > 1$, and may exist even with $\hat{\alpha} = 0$ if the weight function g_ω is constant; see [60] Lemma 6. We now suppose that we can find $\hat{\alpha} \geq 0$ sufficiently large such that

(Z) $\mathcal{Z}_\omega^{(n)} \in \overline{\mathscr{A}}_\omega^{(n)}(\hat{\alpha})$ for each $n \in \mathbb{N}$ and each $\omega \in \Omega$.

Now we set $\overline{\mathcal{Z}}_\omega^{(n)}$ be the coarsest partition such that all elements of $\mathcal{Z}_\omega^{(n)}$ are either disjoint from $X_{\omega,n-1}$ or contained in $X_{\omega,n-1}$. Note that $\overline{\mathcal{Z}}_\omega^{(n)} = \widehat{\mathcal{Z}}_\omega^{(n)}(\mathcal{Z}_\omega^{(n)})$. Now, define the following subcollections:

$$\begin{aligned}
\overline{\mathcal{Z}}_{\omega,*}^{(n)} &:= \left\{Z \in \overline{\mathcal{Z}}_\omega^{(n)} : Z \subseteq X_{\omega,n-1}\right\},\\
\overline{\mathcal{Z}}_{\omega,b}^{(n)} &:= \left\{Z \in \overline{\mathcal{Z}}_\omega^{(n)} : Z \subseteq X_{\omega,n-1} \text{ and } \Lambda_\omega(\mathbb{1}_Z) = 0\right\},\\
\overline{\mathcal{Z}}_{\omega,g}^{(n)} &:= \left\{Z \in \overline{\mathcal{Z}}_\omega^{(n)} : Z \subseteq X_{\omega,n-1} \text{ and } \Lambda_\omega(\mathbb{1}_Z) > 0\right\}.
\end{aligned}$$

Consider the collection $\overline{\mathcal{Z}}_{\omega,F}^{(n)} \subseteq \overline{\mathcal{Z}}_{\omega,*}^{(n)}$ such that for $Z \in \overline{\mathcal{Z}}_{\omega,F}^{(n)}$ we have $T_\omega^n(Z) = I$. We will elements of $\overline{\mathcal{Z}}_{\omega,F}^{(n)}$ "full intervals". We let $\overline{\mathcal{Z}}_{\omega,U}^{(n)} := \overline{\mathcal{Z}}_{\omega,*}^{(n)} \backslash \overline{\mathcal{Z}}_{\omega,F}^{(n)}$. Since for any $Z \in \overline{\mathcal{Z}}_{\omega,F}^{(n)}$ we have that $T_\omega^n(Z) = I$, and hence

$$\Lambda_\omega(\mathbb{1}_Z) \geq \frac{\inf_{D_{\sigma^n(\omega),n}} \mathcal{L}_\omega^n \mathbb{1}_Z}{\sup_{D_{\sigma^n(\omega),n}} \mathcal{L}_\omega^n \mathbb{1}_\omega} \geq \frac{\inf g_{\omega,0}^{(n)}}{\|\mathcal{L}_{\omega,0}^n \mathbb{1}\|_\infty} > 0. \tag{1.15.3}$$

Consequently, we have that $\overline{\mathcal{Z}}_{\omega,F}^{(n)} \subseteq \overline{\mathcal{Z}}_{\omega,g}^{(n)}$, and thus

$$\overline{\mathcal{Z}}_{\omega,b}^{(n)} \subseteq \overline{\mathcal{Z}}_{\omega,U}^{(n)}. \tag{1.15.4}$$

We let $\zeta_\omega^{(n)} \geq 0$ denote the maximum number of contiguous non-full intervals for T_ω^n in $\overline{\mathcal{Z}}_{\omega,*}^{(n)}$ for each $\omega \in \Omega$ and $n \in \mathbb{N}$. Note that $\zeta_\omega^{(1)}$ may be equal to 0, but $\zeta_\omega^{(n)} \geq 1$ for all $n \geq 2$, and so it follows from (1.15.4) that

$$0 \leq \log \xi_\omega^{(n)} \leq \log \zeta_\omega^{(n)} \tag{1.15.5}$$

for all $n \geq 2$. In the interest of having assumptions that are easier to check than (Q1)-(Q3) we introduce the following simpler assumptions which use the collections $\overline{\mathcal{Z}}_{\omega,F}^{(n)}$ and $\overline{\mathcal{Z}}_{\omega,U}^{(n)}$ rather than $\mathcal{Z}_{\omega,g}^{(n)}$ and $\mathcal{Z}_{\omega,b}^{(n)}$. We assume the following:

($\widehat{\text{Q0}}$) $\overline{\mathcal{Z}}_{\omega,F}^{(1)} \neq \emptyset$ for m-a.e. $\omega \in \Omega$,

($\widehat{\text{Q1}}$)

$$\lim_{n\to\infty}\frac{1}{n}\log\|g_\omega^{(n)}\|_\infty+\limsup_{n\to\infty}\frac{1}{n}\log\zeta_\omega^{(n)}<\lim_{n\to\infty}\frac{1}{n}\log\rho_\omega^n=\int_\Omega\log\rho_\omega\,dm(\omega),$$

($\widehat{\text{Q2}}$) for each $n\in\mathbb{N}$ we have $\log^+\zeta_\omega^{(n)}\in L^1(m)$,

($\widehat{\text{Q3}}$) for each $n\in\mathbb{N}$, $\log\hat{\delta}_{\omega,n}\in L^1(m)$, where

$$\hat{\delta}_{\omega,n}:=\min_{Z\in\overline{\mathcal{Z}}_{\omega,F}^{(n)}}\Lambda_\omega(\mathbb{1}_Z). \tag{1.15.6}$$

Our assumptions (Q1)-(Q3) are used exclusively in Section 1.5, and primarily in Lemma 1.5.1. In the proof of Lemma 1.5.1, the good and bad interval collections $\mathcal{Z}_{\omega,g}^{(n)}$ and $\mathcal{Z}_{\omega,b}^{(n)}$ are used only to estimate the variation of a function and can easily be replaced by the collections $\overline{\mathcal{Z}}_{\omega,F}^{(n)}$ and $\overline{\mathcal{Z}}_{\omega,U}^{(n)}$. Therefore, we can easily replace the assumptions (Q1)-(Q3) with ($\widehat{\text{Q0}}$)-($\widehat{\text{Q3}}$) without any changes. In particular, we are still able to construct the number N_* which is defined in (1.5.20). Note that by replacing the 2 in (1.15.1) with the $\hat{\alpha}\geq 0$ that appears in (1.15.2), the constant coefficients which appear in the definitions of $A_\omega^{(n)}$ and $B_\omega^{(n)}$ in (1.5.10) at the end of Lemma 1.5.1 may be different, consequently changing the value of N_*. This ultimately does not affect our general theory as we only care that such a value N_* satisfying (1.5.20) exists.

Remark 1.15.1. As in Remark 1.5.7, we again note that checking (Z), ($\widehat{\text{Q2}}$), and ($\widehat{\text{Q3}}$) for all $n\in\mathbb{N}$ could be difficult and that it suffices to instead check these conditions only for $n=N_*$. Thus we may replace (Z), ($\widehat{\text{Q2}}$), and ($\widehat{\text{Q3}}$) with the following:

(Z') there exists $\hat{\alpha}\geq 0$ such that $\mathcal{Z}_\omega^{(N_*)}\in\overline{\mathscr{A}}_\omega^{(N_*)}(\hat{\alpha})$ for each $\omega\in\Omega$,

($\widehat{\text{Q2}}'$) $\log^+\zeta_\omega^{(N_*)}\in L^1(m)$,

($\widehat{\text{Q3}}'$) $\log\hat{\delta}_{\omega,N_*}\in L^1(m)$.

The following proposition gives that assumption ($\widehat{\text{Q3}}$) is always satisfied, and thus that we really only need to assume ($\widehat{\text{Q1}}$) and ($\widehat{\text{Q2}}$).

Proposition 1.15.2. *Assumption* ($\widehat{\text{Q3}}$) *is trivially satisfied.*

Proof. As the right hand side of (1.15.3) is log-integrable by (1.1.2) and (1.1.3), we must also have $\log\hat{\delta}_{\omega,n}\in L^1(m)$. □

Recall that two elements $W,Z\in\mathcal{Z}_{\omega,*}^{(n)}$ are said to be contiguous if either W and Z are contiguous in the usual sense, i.e. they share a boundary point, or if they are separated by a connected component of $\cup_{j=0}^{n-1}T_\omega^{-j}(H_{\sigma^j(\omega)})$. The following proposition gives an upper bound for the exponential growth of the number $\zeta_\omega^{(n)}$ which will be useful in checking our assumption ($\widehat{\text{Q1}}$), which implies (Q1).

PROPOSITION 1.15.3. *The following inequality holds for $\zeta_\omega^{(n)}$, the largest number of contiguous non-full intervals for T_ω^n:*

$$\zeta_\omega^{(n)} \leq n \prod_{j=0}^{n-1} (\zeta_{\sigma^j(\omega)}^{(1)} + 2). \tag{1.15.7}$$

Consequently, using (1.15.5) *and the ergodic theorem, we have that*

$$\lim_{n\to\infty} \frac{1}{n} \log \xi_\omega^{(n)} \leq \lim_{n\to\infty} \frac{1}{n} \log \zeta_\omega^{(n)} \leq \int_\Omega \log(\zeta_\omega^{(1)} + 2)\, dm(\omega). \tag{1.15.8}$$

PROOF. This is a random version of [**53**, Lemma 6.3]. We sketch the argument here. To upper bound $\zeta_\omega^{(n+1)}$, we observe that the largest number of contiguous non-full intervals for T_ω^{n+1} is given by

$$\zeta_\omega^{(n+1)} \leq \zeta_\omega^{(1)}(\zeta_{\sigma(\omega)}^{(n)} + 2) + 2\zeta_{\sigma(\omega)}^{(n)}. \tag{1.15.9}$$

Indeed, the first term on the right hand side accounts for the (worst case) scenario that all non-full branches of $T_{\sigma(\omega)}^n$ are pulled back inside contiguous non-full intervals for T_ω. For each non-full interval of T_ω, there at most $\zeta_{\sigma(\omega)}^{(n)} + 2$ contiguous non-full intervals for T_ω^{n+1}, as in addition to the $\zeta_{\sigma(\omega)}^{(n)}$ non-full intervals pulled back from $T_{\sigma(\omega)}^n$, there may be full branches of $T_{\sigma(\omega)}^n$ to the left and right of these which are only partially pulled back inside the corresponding branch of T_ω. The second term in (1.15.9) accounts for an extra (at most) $\zeta_{\sigma(\omega)}^{(n)}$ non full branches of $T_{\sigma(\omega)}^n$ pulled back inside the full branches of T_ω neighboring the cluster of $\zeta_\omega^{(1)}$ non-full branches.

Rearranging (1.15.9) yields $\zeta_\omega^{(n+1)} \leq \zeta_{\sigma(\omega)}^{(n)}(\zeta_\omega^{(1)} + 2) + 2\zeta_\omega^{(1)}$. The claim follows directly by induction. □

Let $\hbar_\omega$ denote the number of connected components of H_ω. The following lemma shows that the conditions

LEMMA 1.15.4. *If assumption* (Z') *holds as well as*

$$\log \hbar_\omega \in L^1(m), \tag{CCH}$$

then $\log \zeta_\omega^{(1)} \in L^1(m)$. *Consequently, we have that* (Q2') *and* ($\widehat{\text{Q2}}$) *hold.*

PROOF. Since condition (Z') holds, we see that (LIP) in conjunction with (CCH) gives that $\log \#\overline{\mathcal{Z}}_{\omega,*}^{(1)} \in L^1(m)$, and thus we must have that $\log \zeta_\omega^{(1)} \in L^1(m)$. In light of (1.15.7) we see that (Q2') and ($\widehat{\text{Q2}}$) hold if

$$\log(\zeta_\omega^{(1)} + 2) \in L^1(m),$$

and thus we are done. □

For each $n \in \mathbb{N}$ define

$$F_\omega^{(n)} := \min_{y\in[0,1]} \#\{T_\omega^{-n}(y)\}. \tag{1.15.10}$$

Since the sequences $\{\omega \mapsto \|g_\omega^{(n)}\|_\infty\}_{n\in\mathbb{N}}$ and $\{\omega \mapsto \inf \mathcal{L}_\omega^n \mathbb{1}_\omega\}_{n\in\mathbb{N}}$ are submultiplicative and supermultiplicative, respectively, the subadditive ergodic theorem implies that the assumption that

$$\lim_{n\to\infty} \frac{1}{n} \log \|g_\omega^{(n)}\|_\infty < \lim_{n\to\infty} \frac{1}{n} \log \inf_{D_{\sigma^n(\omega),n}} \mathcal{L}_\omega^n \mathbb{1}_\omega$$

is equivalent to the assumption that there exist $N \in \mathbb{N}$ such that

$$\int_\Omega \log \|g_\omega^{(N)}\|_\infty \, dm(\omega) < \int_\Omega \log \inf_{D_{\sigma^N(\omega),N}} \mathcal{L}_\omega^N \mathbb{1}_\omega \, dm(\omega). \tag{1.15.11}$$

A useful lower bound for the right hand side is the following:

$$\inf_{D_{\sigma^N(\omega),N}} \mathcal{L}_\omega^N \mathbb{1}_\omega \geq \inf_{X_{\omega,N-1}} g_\omega^{(N)} F_\omega^{(N)} \geq \inf g_{\omega,0}^{(N)} F_\omega^{(N)}. \tag{1.15.12}$$

The next lemma, which offers a sufficient condition to check assumptions (Q1) and ($\widehat{\text{Q1}}$), follows from (1.15.8), (1.15.11), (1.15.12), and the calculations from the proof of Lemma 13.18 in [3].

LEMMA 1.15.5. *If there exists $N \in \mathbb{N}$ such that*

$$\frac{1}{N} \int_\Omega \sup S_{N,T}(\varphi_{\omega,0}) - \inf S_{N,T}(\varphi_{\omega,0}) + N \log(\zeta_\omega^{(1)} + 2) \, dm(\omega) < \frac{1}{N} \int_\Omega \log F_\omega^{(N)} \, dm(\omega),$$

then ($\widehat{\text{Q1}}$) *(and thus* (Q1)*) holds.*

The following definition will be useful in checking our measurability assumptions for examples.

DEFINITION 1.15.6. We say that a function $f : \Omega \to \mathbb{R}_+$ is m-continuous function if there is a partition of $\Omega \pmod m$ into at most countably many Borel sets $\Omega_1, \Omega_2, \ldots$ such that f is constant on each Ω_j, say $f|_{\Omega_j} = f_j$.

We now give specific classes of random maps with holes which meet our assumptions. In principle, any of the classes of finitely branched maps discussed in Section 13 of [3] (including random non-uniformly expanding maps) will fit our current assumptions given a suitable hole H.

1.15.1. Random β-Transformations With Holes. For this first example we consider the class of maps described in Section 13.2 in [3]. These are β-transformations for which the last (non-full) branch is not too small so that each branch in the random closed system has a uniform covering time. In particular we assume there is some $\delta > 0$ such that for m-a.e. $\omega \in \Omega$ we have

$$\beta_\omega \in \bigcup_{k=1}^\infty [k+\delta, k+1].$$

Further suppose that the map $\omega \mapsto \beta_\omega$ is m-continuous. We consider the random β-transformation $T_\omega : [0,1] \to [0,1]$ given by

$$T_\omega(x) = \beta_\omega x \pmod 1$$

and the potential

$$\varphi_{\omega,0} = -t \log |T'_\omega| = -t \log \beta_\omega$$

for $t \geq 0$. In addition, we assume that

$$\int_\Omega \log \lfloor \beta_\omega \rfloor \, dm(\omega) > \log 3 \tag{1.15.13}$$

and

$$\int_\Omega \log \lceil \beta_\omega \rceil \, dm(\omega) < \infty. \tag{1.15.14}$$

Note that we allow β_ω arbitrarily large. It follows from Lemma 13.6 of [3] that our assumptions (T1)-(T3), (LIP), (GP), (A1)-(A2), (M), (C), and (Z) are satisfied.

To check the remainder of our assumptions we must now describe the choice of hole H_ω. For our holes H_ω we will consider intervals of length at most $1/\beta_\omega$ so that H_ω may not intersect more than two monotonicity partition elements. To ensure that ($\widehat{\text{Q0}}$) is satisfied we assume there is a full-branched element $Z \in \mathcal{Z}^{(1)}_{\omega,F}$ such that $Z \cap H_\omega = \emptyset$ for each $\omega \in \Omega$, and thus, in light of Remark 1.3.5, we also have that assumption (D) is satisfied with $D_{\omega,\infty} = I$ for each $\omega \in \Omega$.

Now, we note that since (1.15.14) implies our assumption (LIP), Lemma 1.15.4 implies that assumption ($\widehat{\text{Q2}}$) is satisfied. Thus, we have only to check the condition ($\widehat{\text{Q1}}$). Depending on H_ω we may have that

$$\inf \mathcal{L}_\omega \mathbb{1}_\omega = \frac{\lfloor \beta_\omega \rfloor - 1}{\beta^t_\omega},$$

for example if H_ω is the last full branch. To ensure that ($\widehat{\text{Q1}}$) holds, note that (1.15.11) holds with $N = 1$, and thus it suffices to have

$$\int_\Omega \log(\lfloor \beta_\omega \rfloor - 1) \, dm(\omega) > \int_\Omega \log(\zeta^{(1)}_\omega + 2) \, dm(\omega),$$

since

$$\int_\Omega \log \inf \mathcal{L}_\omega \mathbb{1}_\omega - \log \|g_\omega\|_\infty \, dm(\omega) \geq \int_\Omega \log \frac{\lfloor \beta_\omega \rfloor - 1}{\beta^t_\omega} + \log \beta^t_\omega \, dm(\omega).$$

Depending on the placement of H_ω we may have $\zeta^{(1)}_\omega = i$ for any $i \in \{0, 1, 2, 3\}$. Thus, we obtain the following lemma assuming the worst case scenario, i.e. assuming $\zeta^{(1)}_\omega = 3$ for m-a.e. $\omega \in \Omega$.

LEMMA 1.15.7. *If $H_\omega \subseteq I$ is such that $\zeta^{(1)}_\omega \leq 3$ for m-a.e. $\omega \in \Omega$, then Theorems A-D if*

$$\int_\Omega \log(\lfloor \beta_\omega \rfloor - 1) \, dm(\omega) > \log 5.$$

On the other hand, if we have that H_ω is equal to the monotonicity partition element which contains 1, then $\zeta^{(1)}_\omega = 0$ and

$$\inf \mathcal{L}_\omega \mathbb{1}_\omega = \frac{\lfloor \beta_\omega \rfloor}{\beta^t_\omega}.$$

Furthermore, the additional hypotheses necessary for Theorem E are satisfied. In particular, the fact that T has large images follows from the fact that these maps have a uniform covering time; see Lemma 13.5 of [3]. Thus, we thus have the following lemma.

LEMMA 1.15.8. *If $H_\omega = Z_1$, where $1 \in Z_1 \in \mathcal{Z}_\omega$, for m-a.e. $\omega \in \Omega$ then Theorems A-E hold.*

More generally, we can consider general potentials, non-linear maps, and holes which are unions of finitely many intervals so that condition (CCH) holds.

1.15.2. Random Open Lasota-Yorke Maps. We now present an example of a large class of random Lasota-Yorke maps with holes. The following lemma summarizes the closed setting for this particular class of random maps was treated in Section 13.6 of [3].

LEMMA 1.15.9. *Let $\varphi_0 : \Omega \to \mathrm{BV}(I)$ be an m-continuous function, and let $\varphi_0 : \Omega \times I \to \mathbb{R}$ be given by $\varphi_{\omega,0} := -t \log |T'_\omega| = \varphi_0(\omega)$ for $t \geq 0$. Then $g_{\omega,0} = e^{\varphi_{\omega,0}} = 1/|T'_\omega|^t \in \mathrm{BV}(I)$ for m a.e. $\omega \in \Omega$. We further suppose the system satisfies the following:*

(1) $\log \#\mathcal{Z}_\omega \in L^1(m)$,
(2) there exists $M(n) \in \mathbb{N}$ such that for any $\omega \in \Omega$ and any $Z \in \mathcal{Z}_\omega^{(n)}$ we have that $T_\omega^{M(n)}(Z) = I$,
(3) for each $\omega \in \Omega$, $Z \in \mathcal{Z}_\omega$, and $x \in Z$
  (a) $T_\omega|_Z \in C^2$,
  (b) there exists $K \geq 1$ such that
$$\frac{|T''_\omega(x)|}{|T'_\omega(x)|} \leq K,$$
(4) there exist $1 < \lambda \leq \Lambda < \infty$ and $n_0 \in \mathbb{N}$ such that
  (a) $|T'_\omega| \leq \Lambda$ for m-a.e. $\omega \in \Omega$,
  (b) $|(T_\omega^{n_0})'| \geq \lambda^{n_0}$ for m-a.e. $\omega \in \Omega$
  (c) $\frac{1}{n_0} \int_\Omega \log F_\omega^{(n_0)} \, dm(\omega) > t \log \frac{\Lambda}{\lambda}$,
(5) for each $n \in \mathbb{N}$ there exists
$$\varepsilon_n := \inf_{\omega \in \Omega} \min_{Z \in \mathcal{Z}_\omega^{(n)}} \mathrm{diam}(Z) > 0.$$

Then Theorems 2.19-2.23 of [3] hold, and in particular, our assumptions (T1)-(T3), (LIP), (GP), (A1)-(A2), (M), and (C) hold.

The following lemma gives a large class of random Lasota-Yorke maps with holes for which our results apply. In particular, we allow our hole to be composed of finitely many intervals which may change depending on the fiber ω, provided the number of connected components of the hole is log-integrable over Ω (CCH).

LEMMA 1.15.10. *Let $\varphi_{\omega,0} = -t \log |T'_\omega|$ and suppose the hypotheses of Lemma 1.15.9 hold. Additionally we suppose that $H \subseteq \Omega \times I$ such that (CCH) holds as well as the following:*

(6) for m-a.e. $\omega \in \Omega$ there exists $Z \in \mathcal{Z}_\omega$ with $Z \cap H_\omega = \emptyset$ such that $T_\omega(Z) = I$,
(7) $\frac{1}{n_0} \int_\Omega \log F_\omega^{(n_0)} \, dm(\omega) > t \log \frac{\Lambda}{\lambda} + \int_\Omega \log(\zeta_\omega^{(1)} + 2) \, dm(\omega)$.

Then the hypotheses of Theorems A-D hold. If in addition we have that

(8) *there exists* $M : \mathbb{N} \to \mathbb{N}$ *such that* $T_\omega^{M(n)}(Z) = I$ *for* m*-a.e.* $\omega \in \Omega$ *and each* $Z \in \mathcal{Z}_\omega^{(n)}$*, i.e. there is a uniform covering time,*

(9) *for* m*-a.e.* $\omega \in \Omega$ *there exists* $Z_1, \dots, Z_k \in \mathcal{Z}_\omega$ *such that* $H_\omega = \cup_{j=1}^k Z_j$ *and* $T_\omega(Z) = I$ *for all* $Z \in \mathcal{Z}_\omega$ *with* $Z \cap H_\omega = \emptyset$*,*

then the hypotheses of Theorem E also hold.

Proof. The conclusion of Lemma 1.15.9 leaves only to check assumptions (D) and ($\widehat{\text{Q0}}$)-($\widehat{\text{Q3}}$). But in light of Proposition 1.15.2 we see that ($\widehat{\text{Q3}}$) holds, and hypothesis (6) implies (D) (by Remark 1.3.5) and ($\widehat{\text{Q0}}$) hold.

To check our remaining hypotheses on the open system we first show that (Z') holds. To see this we note that equation (13.20) of [3], together with the fact that hypothesis (2) of Lemma 1.15.9 implies that $\Lambda^{-kn_0t} \le g_{\omega,0}^{(kn_0)} \le \lambda^{-kn_0t} < 1$, gives that for any $\omega \in \Omega$ and $Z \in \mathcal{Z}_\omega^{(kn_0)}$ we have

$$\begin{aligned}
\operatorname{var}_Z(g_{\omega,0}^{(kn_0)}) &\le 2\|g_{\omega,0}^{(kn_0)}\|_\infty + \frac{tK}{\Lambda - 1} \cdot \left(\frac{\Lambda}{\lambda^t}\right)^{kn_0} \\
&\le 2\|g_{\omega,0}^{(kn_0)}\|_\infty + \frac{tK}{\Lambda - 1} \cdot \left(\frac{\Lambda}{\lambda^t}\right)^{kn_0} \cdot \Lambda^{kn_0t}\|g_{\omega,0}^{(kn_0)}\|_\infty \\
&\le \hat{\alpha}_k \|g_{\omega,0}^{(kn_0)}\|_\infty,
\end{aligned}$$

where

$$\hat{\alpha}_k := 2 + \frac{tK}{\Lambda - 1} \cdot \left(\frac{\Lambda^{2t}}{\lambda^t}\right)^{kn_0}.$$

Taking k_* so large that

$$\int_\Omega \log Q_\omega^{(k_* n_0)}\, dm(\omega) < 0,$$

where $Q_\omega^{(k_* n_0)}$ is defined as in (1.5.12), and setting $N_* = k_* n_0$, we then see that (Z') holds, that is we have that $\mathcal{Z}_\omega^{(N_*)} \in \overline{\mathscr{A}}_\omega^{(N_*)}(\hat{\alpha}_{k_*})$ for each $\omega \in \Omega$. Thus, Lemma 1.15.4 together with (CCH) ensures that ($\widehat{\text{Q2}}$) holds. Now taking (7) in conjunction with Lemma 1.15.5 implies assumption ($\widehat{\text{Q1}}$), and thus the hypotheses of Theorems A-D hold.

The hypotheses of Theorem E hold since the assumptions (8) and (9) together imply that T has large images and large images with respect to H, and assumptions (3) and (4)(b) give the bounded distortion condition for $g_{\omega,0}$.

□

Remark 1.15.11. If one wishes to work with general potentials rather than the geometric potentials in Lemmas 1.15.9 and 1.15.10 then one could replace (4) of Lemma 1.15.9 with (1.15.11) and (7) of Lemma 1.15.10 with Lemma 1.15.5.

CHAPTER 2

Perturbation formulae for quenched random dynamics with applications to open systems and extreme value theory

In this second chapterwe first develop a perturbation theory for the quasi-compact linear operators cocycle $\mathcal{L}_\omega^n$ and its perturbed version $\mathcal{L}_{\omega,\varepsilon}^n$ By defining $\lambda_{\omega,\varepsilon}$ as the leading Lyapunov multiplier of $\mathcal{L}_{\omega,\varepsilon}$, we will get a first order formula for $\lambda_{\omega,\varepsilon}$ in terms of $\lambda_{\omega,0}$ and in the size of the perturbation $\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon}$. Whenever $\mathcal{L}_{\omega,\varepsilon}$ is a transfer operator cocycle for a random map cocycle T_ω^n, it will be defined by the introduction of small random holes $H_{\omega,\varepsilon}$. The first-order perturbation for the Lyapunov multiplier will therefore been used to obtain a quenched extreme value theory, by using a suitable spectral approach. By pursuing with the perturbative scheme, we will establish the existence of equilibrium states and conditionally invariant measures and we finally prove quenched limit theorems for equilibrium states arising from contracting potentials.

2.1. Sequential perturbation theorem

In this section we briefly depart from the setting of random dynamical systems to prove a general perturbation result for sequential operators acting on sequential Banach spaces. In particular, we will not require any measurability or notion of randomness in this section.

Suppose that Ω is a set and that the map $\sigma : \Omega \to \Omega$ is invertible. Furthermore, we suppose that there is a family of (fiberwise) normed vector spaces $\{\mathcal{B}_\omega, \|\cdot\|_{\mathcal{B}_\omega}\}_{\omega\in\Omega}$ and dual spaces $\left\{\mathcal{B}_\omega^*, \|\cdot\|_{\mathcal{B}_\omega^*}\right\}_{\omega\in\Omega}$ such that for each $\omega \in \Omega$ and each $0 \le \varepsilon \le \varepsilon_0$ there are operators $\mathcal{L}_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ such that the following hold.

(P1) There exists a function $C_1 : \Omega \to \mathbb{R}_+$ such that for $f \in \mathcal{B}_\omega$ we have

$$\sup_{\varepsilon\ge 0} \|\mathcal{L}_{\omega,\varepsilon}(f)\|_{\mathcal{B}_{\sigma\omega}} \le C_1(\omega)\|f\|_{\mathcal{B}_\omega}.$$

(P2) For each $\omega \in \Omega$ and $\varepsilon \ge 0$ there is a functional $\nu_{\omega,\varepsilon} \in \mathcal{B}_\omega^*$, the dual space of $\mathcal{B}_\omega$, $\lambda_{\omega,\varepsilon} \in \mathbb{C}\setminus\{0\}$, and $\phi_{\omega,\varepsilon} \in \mathcal{B}_\omega$ such that

$$\mathcal{L}_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = \lambda_{\omega,\varepsilon}\phi_{\sigma\omega,\varepsilon} \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(\mathcal{L}_{\omega,\varepsilon}(f)) = \lambda_{\omega,\varepsilon}\nu_{\omega,\varepsilon}(f)$$

for all $f \in \mathcal{B}_\omega$. Furthermore we assume that

$$\nu_{\omega,0}(\phi_{\omega,\varepsilon}) = 1.$$

(P3) There is an operator $Q_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ such that for each $f \in \mathcal{B}_\omega$ we have

$$\lambda_{\omega,\varepsilon}^{-1}\mathcal{L}_{\omega,\varepsilon}(f) = \nu_{\omega,\varepsilon}(f)\cdot\phi_{\sigma\omega,\varepsilon} + Q_{\omega,\varepsilon}(f).$$

Furthermore, we have

$$Q_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = 0 \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(Q_{\omega,\varepsilon}(f)) = 0.$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

Note that assumptions (P2) and (P3) together imply that

$$\nu_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = 1.$$

(P4) There exists a function $C_2 : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\varepsilon \geq 0} \|\phi_{\omega,\varepsilon}\|_{\mathcal{B}_\omega} = C_2(\omega) < \infty.$$

For each $\omega \in \Omega$ and $\varepsilon \geq 0$ we define the quantities

$$\Delta_{\omega,\varepsilon} := \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,0})\right) \tag{2.1.1}$$

and

$$\eta_{\omega,\varepsilon} := \|\nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})\|_{\mathcal{B}^*_\omega}. \tag{2.1.2}$$

(P5) For each $\omega \in \Omega$ we have

$$\lim_{\varepsilon \to 0} \eta_{\omega,\varepsilon} = 0.$$

(P6) For each $\omega \in \Omega$ such that $\Delta_{\omega,\varepsilon} > 0$ for every $\varepsilon > 0$, we have that there exists a function $C_3 : \Omega \to \mathbb{R}_+$ such that

$$\limsup_{\varepsilon \to 0} \frac{\eta_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}} := C_3(\omega) < \infty.$$

Given $\omega \in \Omega$, if there is $\varepsilon_0 > 0$ such that for each $\varepsilon \leq \varepsilon_0$ we have that $\Delta_{\omega,\varepsilon} = 0$ then we also have that $\eta_{\omega,\varepsilon} = 0$ for each $\varepsilon \leq \varepsilon_0$.

(P7) For each $\omega \in \Omega$ we have

$$\lim_{\varepsilon \to 0} \nu_{\omega,\varepsilon}(\phi_{\omega,0}) = 1.$$

(P8) For each $\omega \in \Omega$ with $\Delta_{\omega,\varepsilon} > 0$ for all $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \limsup_{\varepsilon \to 0} \Delta^{-1}_{\omega,\varepsilon} \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})\left(Q^n_{\sigma^{-n}\omega,\varepsilon}\phi_{\sigma^{-n}\omega,0}\right)\right) = 0.$$

(P9) For each $\omega \in \Omega$ with $\Delta_{\omega,\varepsilon} > 0$ for all $\varepsilon > 0$ we have the limit

$$q^{(k)}_{\omega,0} := \lim_{\varepsilon \to 0} \frac{\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\mathcal{L}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0} - \mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)}{\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,0})\right)}$$

exists for each $k \geq 0$.

Consider the identity

$$\begin{aligned} \lambda_{\omega,0} - \lambda_{\omega,\varepsilon} &= \lambda_{\omega,0}\nu_{\omega,0}(\phi_{\omega,\varepsilon}) - \nu_{\sigma\omega,0}(\lambda_{\omega,\varepsilon}\phi_{\sigma\omega,\varepsilon}) \\ &= \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}(\phi_{\omega,\varepsilon})) - \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,\varepsilon}(\phi_{\omega,\varepsilon})) \\ &= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,\varepsilon})\right). \end{aligned} \tag{2.1.3}$$

It then follows from (2.1.3), together with (2.1.2) and assumption (P4), that

$$|\lambda_{\omega,0} - \lambda_{\omega,\varepsilon}| \leq C_2(\omega)\eta_{\omega,\varepsilon}. \tag{2.1.4}$$

In particular, given assumption (P5), (2.1.4) implies

$$\lim_{\varepsilon \to 0} \lambda_{\omega,\varepsilon} = \lambda_{\omega,0} \tag{2.1.5}$$

for each $\omega \in \Omega$.

REMARK 2.1.1. Note that (P6) and (2.1.4) imply that

$$\limsup_{\varepsilon \to 0} \frac{|\lambda_{\omega,0} - \lambda_{\omega,\varepsilon}|}{\Delta_{\omega,\varepsilon}} \leq C_2(\omega)C_3(\omega) < \infty.$$

For $n \geq 1$ we define the normalized operator $\tilde{\mathcal{L}}^n_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma^n\omega}$ by

$$\tilde{\mathcal{L}}^n_{\omega,\varepsilon} := (\lambda^n_{\omega,\varepsilon})^{-1}\mathcal{L}^n_{\omega,\varepsilon}$$

where

$$\lambda^n_{\omega,\varepsilon} := \lambda_{\omega,\varepsilon} \cdot \dots \cdot \lambda_{\sigma^{n-1}\omega,\varepsilon}.$$

In view of assumption (P3), induction gives

$$\tilde{\mathcal{L}}^n_{\omega,\varepsilon}(f) = \nu_{\omega,\varepsilon}(f) \cdot \phi_{\sigma^n\omega,\varepsilon} + Q^n_{\omega,\varepsilon}(f)$$

for each $n \geq 1$ and all $f \in \mathcal{B}_\omega$. Similarly with (2.1.3), we have that

$$\lambda^n_{\omega,0} - \lambda^n_{\omega,\varepsilon} = \nu_{\sigma^n\omega,0}\left(\left(\mathcal{L}^n_{\omega,0} - \mathcal{L}^n_{\omega,\varepsilon}\right)(\phi_{\omega,\varepsilon})\right).$$

We now arrive at the main result of this section. We prove a differentiability result for the perturbed quantities $\lambda_{\omega,\varepsilon}$ as $\varepsilon \to 0$ in the spirit of Keller and Liverani [50].

THEOREM 2.1.2. *Suppose that assumptions* (P1)-(P8) *hold. If there is some* $\varepsilon_0 > 0$ *such that* $\Delta_{\omega,\varepsilon} = 0$ *for* $\varepsilon \leq \varepsilon_0$ *then*

$$\lambda_{\omega,0} = \lambda_{\omega,\varepsilon},$$

or if (P9) *holds then*

$$\lim_{\varepsilon \to 0} \frac{\lambda_{\omega,0} - \lambda_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}} = 1 - \sum_{k=0}^{\infty} (\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0})^{-1} q^{(k)}_{\omega,0}.$$

PROOF. Fix $\omega \in \Omega$. First, we note that assumption (P6) implies that if there is some ε_0 such that $\Delta_{\omega,\varepsilon} = 0$ for all $\varepsilon \leq \varepsilon_0$ (which implies, by assumption, that $\eta_{\omega,\varepsilon} = 0$), then (2.1.4) immediately implies that

$$\lambda_{\omega,0} = \lambda_{\omega,\varepsilon}.$$

Now we suppose that $\Delta_{\omega,\varepsilon} > 0$ for all $\varepsilon > 0$. Using (P2), (P3), and (2.1.3), for each $n \geq 1$ and all $\omega \in \Omega$ we have

$$\begin{aligned}
\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})(\lambda_{\omega,0} - \lambda_{\omega,\varepsilon}) &= \nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,\varepsilon})\right) \\
&= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}) \cdot \phi_{\omega,\varepsilon})\right) \\
&= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}) \cdot \phi_{\omega,\varepsilon} + Q^n_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}) - Q^n_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))\right) \\
&= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,\varepsilon} - Q^n_{\sigma^{-n}\omega,\varepsilon})(\phi_{\sigma^{-n}\omega,0})\right) \\
&= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,0} - \tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,0} + \tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,\varepsilon} - Q^n_{\sigma^{-n}\omega,\varepsilon})(\phi_{\sigma^{-n}\omega,0})\right) \\
&= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,0})\right)
\end{aligned}$$

$$
\begin{aligned}
&- \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,0}-\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,\varepsilon})(\phi_{\sigma^{-n}\omega,0})\right)\\
&\qquad - \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^n_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))\right)
\end{aligned}
$$

(2.1.6)

$$
= \Delta_{\omega,\varepsilon} - \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,0}-\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,\varepsilon})(\phi_{\sigma^{-n}\omega,0})\right)
$$

(2.1.7)

$$
- \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^n_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))\right).
$$

Since

$$
(\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,0}-\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,\varepsilon})(\phi_{\sigma^{-n}\omega,0}) = \phi_{\omega,0} - \tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}),
$$

using a telescoping argument, the second term of (2.1.6)

$$
D_2 := -\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,0}-\tilde{\mathcal{L}}^n_{\sigma^{-n}\omega,\varepsilon})(\phi_{\sigma^{-n}\omega,0})\right)
$$

can be rewritten as

$$
\begin{aligned}
D_2 &= -\sum_{k=0}^{n-1}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\tilde{\mathcal{L}}_{\sigma^{-(k+1)}\omega,0}-\tilde{\mathcal{L}}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)\\
&= -\sum_{k=0}^{n-1}\nu_{\sigma\omega,0}\Big((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\tilde{\mathcal{L}}_{\sigma^{-(k+1)}\omega,0}\\
&\qquad -\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon}+\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon}-\tilde{\mathcal{L}}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{k=0}^{n-1}\nu_{\sigma\omega,0}\Big((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\tilde{\mathcal{L}}_{\sigma^{-(k+1)}\omega,0}\\
&\qquad -\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon}+\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\Big) \qquad (2.1.8)\\
&\qquad + \sum_{k=0}^{n-1}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\tilde{\mathcal{L}}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right). \qquad (2.1.9)
\end{aligned}
$$

Reindexing, the second summand of the above calculation, and multiplying by 1, (2.1.9), can be rewritten as

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^{k+1}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)\\
&\qquad = \sum_{k=1}^{n}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right)\\
&\qquad = \sum_{k=1}^{n}\lambda^{-1}_{\sigma^{-k}\omega,0}\lambda_{\sigma^{-k}\omega,0}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right). \qquad (2.1.10)
\end{aligned}
$$

And (2.1.8) from above can again be broken into two parts and then rewritten as

$$
\begin{aligned}
&-\sum_{k=0}^{n-1}\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0}-\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)\\
&-\sum_{k=1}^{n}\lambda^{-1}_{\sigma^{-k}\omega,0}\lambda_{\sigma^{-k}\omega,\varepsilon}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right),
\end{aligned}
\tag{2.1.11}
$$

where in the second sum we have used the fact that $\mathcal{L}_{\sigma^{-k}\omega,\varepsilon}=\lambda_{\sigma^{-k}\omega,\varepsilon}\tilde{\mathcal{L}}_{\sigma^{-k}\omega,\varepsilon}$. Altogether using (2.1.10) and (2.1.11), D_2 we can be written as

$$
\begin{aligned}
D_2 &= -\sum_{k=0}^{n-1}\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0}-\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)\\
&\quad-\sum_{k=1}^{n}\lambda^{-1}_{\sigma^{-k}\omega,0}\lambda_{\sigma^{-k}\omega,\varepsilon}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right)\\
&\quad+\sum_{k=1}^{n}\lambda^{-1}_{\sigma^{-k}\omega,0}\lambda_{\sigma^{-k}\omega,0}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right)\\
&= -\sum_{k=0}^{n-1}\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0}-\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)\\
&\quad=+\sum_{k=1}^{n}\lambda^{-1}_{\sigma^{-k}\omega,0}\left(\lambda_{\sigma^{-k}\omega,0}-\lambda_{\sigma^{-k}\omega,\varepsilon}\right)\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right).
\end{aligned}
\tag{2.1.12}
$$

Now for each $k\geq 0$ we let

$$
q^{(k)}_{\omega,\varepsilon}:=\frac{\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\mathcal{L}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0}-\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)}{\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,0})\right)}.
\tag{2.1.13}
$$

Using (2.1.12) we can continue our rephrasing of $\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})(\lambda_{\omega,0}-\lambda_{\omega,\varepsilon})$ from (2.1.6) to get

$$
\begin{aligned}
&\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})(\lambda_{\omega,0}-\lambda_{\omega,\varepsilon})\\
&\quad=\Delta_{\omega,\varepsilon}-\sum_{k=0}^{n-1}\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0}-\mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)\\
&\qquad+\sum_{k=1}^{n}\lambda^{-1}_{\sigma^{-k}\omega,0}\left(\lambda_{\sigma^{-k}\omega,0}-\lambda_{\sigma^{-k}\omega,\varepsilon}\right)\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right)\\
&\qquad-\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^n_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))\right)\\
&\quad=\Delta_{\omega,\varepsilon}\left(1-\sum_{k=0}^{n-1}\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}(\lambda^k_{\sigma^{-k}\omega,\varepsilon})^{-1}q^{(k)}_{\omega,\varepsilon}\right)
\end{aligned}
$$

$$+\sum_{k=1}^{n}\lambda_{\sigma^{-k}\omega,0}^{-1}\left(\lambda_{\sigma^{-k}\omega,0}-\lambda_{\sigma^{-k}\omega,\varepsilon}\right)\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^{k}_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right)$$

(2.1.14)

$$-\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^{n}_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))\right).$$

Dividing the calculation culminating in (2.1.14) by $\Delta_{\omega,\varepsilon}$ on both sides gives

$$\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})\frac{\lambda_{\omega,0}-\lambda_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}} \tag{2.1.15}$$

$$=1-\sum_{k=0}^{n-1}\lambda_{\sigma^{-(k+1)}\omega,0}^{-1}(\lambda^{k}_{\sigma^{-k}\omega,\varepsilon})^{-1}q^{(k)}_{\omega,\varepsilon}$$

$$+\Delta_{\omega,\varepsilon}^{-1}\sum_{k=1}^{n}\lambda_{\sigma^{-k}\omega,0}^{-1}\left(\lambda_{\sigma^{-k}\omega,0}-\lambda_{\sigma^{-k}\omega,\varepsilon}\right)\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^{k}_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})\right) \tag{2.1.16}$$

$$-\Delta_{\omega,\varepsilon}^{-1}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^{n}_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))\right). \tag{2.1.17}$$

Assumption (P8) ensures that (2.1.17) goes to zero as $\varepsilon\to 0$ and $n\to\infty$. Now, using (2.1.2), (2.1.4), (P1), and (P4) we bound (2.1.16) by

$$\Delta_{\omega,\varepsilon}^{-1}\sum_{k=1}^{n}\left|\lambda_{\sigma^{-k}\omega,0}\right|^{-1}\left|\lambda_{\sigma^{-k}\omega,0}-\lambda_{\sigma^{-k}\omega,\varepsilon}\right|\eta_{\omega,\varepsilon}\|\tilde{\mathcal{L}}^{k}_{\sigma^{-k}\omega,\varepsilon}(\phi_{\sigma^{-k}\omega,0})\|_{\mathcal{B}_\omega}$$

$$\leq\frac{\eta_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}}\sum_{k=1}^{n}C_2(\sigma^{-k}\omega)\left|\lambda_{\sigma^{-k}\omega,0}\right|^{-1}\eta_{\sigma^{-k}\omega,\varepsilon}\left|\lambda^{k}_{\sigma^{-k}\omega,\varepsilon}\right|^{-1}\|\mathcal{L}^{k}_{\sigma^{-k}\omega,\varepsilon}(\phi_{\sigma^{-k}\omega,0})\|_{\mathcal{B}_\omega}$$

(2.1.18)

$$\leq\frac{\eta_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}}\sum_{k=1}^{n}(C_2(\sigma^{-k}\omega))^2C_1^{k}(\sigma^{-k}\omega)\left|\lambda_{\sigma^{-k}\omega,0}\right|^{-1}\eta_{\sigma^{-k}\omega,\varepsilon}\left|\lambda^{k}_{\sigma^{-k}\omega,\varepsilon}\right|^{-1}.$$

In view of (P5), (P6), and (2.1.5), for fixed n, we may continue from (2.1.18) and let $\varepsilon\to 0$ to see that

$$\lim_{\varepsilon\to 0}\frac{\eta_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}}\sum_{k=1}^{n}(C_2(\sigma^{-k}\omega))^2C_1^{k}(\sigma^{-k}\omega)\left|\lambda_{\sigma^{-k}\omega,0}\right|^{-1}\eta_{\sigma^{-k}\omega,\varepsilon}\left|\lambda^{k}_{\sigma^{-k}\omega,\varepsilon}\right|^{-1}=0. \tag{2.1.19}$$

In light of (P6), (P7), (2.1.19), and (2.1.15)–(2.1.17) together with (P8) and (P9), we see that first letting $\varepsilon\to 0$ and then $n\to\infty$ gives

$$\lim_{\varepsilon\to 0}\frac{\lambda_{\omega,0}-\lambda_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}}=1-\sum_{k=0}^{\infty}(\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0})^{-1}q^{(k)}_{\omega,0}$$

as desired. □

In the sequel we will refer to the quantity on the right hand side of the last equation in the proof of the previous theorem by $\theta_{\omega,0}$, i.e. we set

$$\theta_{\omega,0}:=1-\sum_{k=0}^{\infty}(\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0})^{-1}q^{(k)}_{\omega,0}. \tag{2.1.20}$$

2.2. Random open systems

We now return to the general random setting of Section 0.1. Suppose that $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0)$ is a closed random dynamical system as in Definition 0.1.4. That is we have a base dynamical system $(\Omega, \mathscr{F}, m, \sigma)$, complete metrisable spaces $\mathcal{J}_{\omega,0}$ such that the map $\Omega \ni \omega \mapsto \mathcal{J}_{\omega,0}$ is a closed random set, and maps $T_\omega : \mathcal{J}_{\omega,0} \to \mathcal{J}_{\sigma\omega,0}$. In addition, we assume that conditions (M1), (M2), and (CCM) hold, and that the transfer operator $\mathcal{L}_{\omega,0}$ acting on the family of Banach spaces $\{\mathcal{B}_\omega, \|\cdot\|_{\mathcal{B}_\omega}\}_{\omega\in\Omega}$ is given by

$$\mathcal{L}_{\omega,0}(f)(x) := \sum_{y\in T_\omega^{-1}(x)} f(y)g_{\omega,0}(y), \quad f \in \mathcal{B}_\omega,\, x \in \mathcal{J}_{\sigma\omega,0},$$

where $g_{\omega,0}(x) = e^{\varphi_{\omega,0}(x)}$ for a suitably chosen random potential $\varphi_{\omega,0}$. We will also assume that the fiberwise Banach spaces $\mathcal{B}_\omega \subseteq L^\infty(\nu_{\omega,0})$, where $\nu_0 = (\nu_{\omega,0})_{\omega\in\Omega}$ is the random probability measure given by (CCM), and that there exists a measurable m-a.e. finite function $K : \Omega \to [1, \infty)$ such that

(B) $\quad \|f\|_{\infty,\omega} \le K_\omega \|f\|_{\mathcal{B}_\omega}$

for all $f \in \mathcal{B}_\omega$ and each $\omega \in \Omega$, where $\|\cdot\|_{\infty,\omega}$ denotes the supremum norm with respect to $\nu_{\omega,0}$. Now for each $\varepsilon > 0$ we let $H_\varepsilon \subseteq \mathcal{J}_0$ be measurable with respect to the product σ-algebra $\mathscr{F} \otimes \mathscr{B}$ on $\mathcal{J}_0$ such that

(A) $\quad H'_\varepsilon \subseteq H_\varepsilon$ for each $0 < \varepsilon' \le \varepsilon$.

Then the sets $H_{\omega,\varepsilon} \subseteq \mathcal{J}_{\omega,0}$ are uniquely determined by the condition that

$$\{\omega\} \times H_{\omega,\varepsilon} = H_\varepsilon \cap (\{\omega\} \times \mathcal{J}_{\omega,0}),$$

or equivalently that

$$H_{\omega,\varepsilon} = \pi_2(H_\varepsilon \cap (\{\omega\} \times \mathcal{J}_{\omega,0})),$$

where $\pi_2 : \mathcal{J}_0 \to \mathcal{J}_{\omega,0}$ is the projection onto the second coordinate. The sets $H_{\omega,\varepsilon}$ are then $\nu_{\omega,0}$-measurable, and (A) implies that

(A') $\quad H_{\omega,\varepsilon'} \subseteq H_{\omega,\varepsilon}$ for each $\varepsilon' \le \varepsilon$ and each $\omega \in \Omega$.

For each $\varepsilon > 0$ we set

$$\Omega_{+,\varepsilon} := \{\omega \in \Omega : \mu_{\omega,0}(H_{\omega,\varepsilon}) > 0\}$$

and then define

$$\Omega_+ := \bigcap_{\varepsilon>0} \Omega_{+,\varepsilon}. \tag{2.2.1}$$

Remark 2.2.1. Note that the set Ω_+ is measurable as it is the intersection of a decreasing family of measurable sets and it is not necessarily σ-invariant.

For each ω and each $\varepsilon > 0$ we define the fibers $\mathcal{J}_{\omega,\varepsilon} := \mathcal{J}_{\omega,0} \backslash H_{\omega,\varepsilon}$ and

$$\mathcal{J}_\varepsilon := \mathcal{J}_0 \backslash H_\varepsilon = \bigcup_{\omega \in \Omega} \{\omega\} \times \mathcal{J}_{\omega,\varepsilon}.$$

We define the surviving sets $X_{\omega,n,\varepsilon}$, $X_{\omega,\infty,\varepsilon}$, $X_{n,\varepsilon}$, and $X_{\infty,\varepsilon}$ as in Section 0.1.

Remark 2.2.2. Note that since (A) implies that $X_{\omega,\infty,\varepsilon} \subseteq X_{\omega,\infty,\varepsilon'}$ for all $\varepsilon' < \varepsilon$, (X) holds, i.e. $X_{\omega,\infty,\varepsilon} \neq \emptyset$ for m-a.e. $\omega \in \Omega$, if there exists $\varepsilon > 0$ such that $X_{\omega,\infty,\varepsilon} \neq \emptyset$ for m-a.e. $\omega \in \Omega$. Furthermore, since $T_\omega(X_{\omega,\infty,\varepsilon}) \subseteq X_{\sigma\omega,\infty,\varepsilon}$, if $X_{\omega,\infty,\varepsilon} \neq \emptyset$ then $X_{\sigma^N\omega,\infty,\varepsilon'} \neq \emptyset$ for each $N \geq 1$ and $\varepsilon' \leq \varepsilon$. As $\mathcal{X}_{\infty,\varepsilon}$ is forward invariant we have that $X_{\omega,\infty,\varepsilon} \neq \emptyset$ not only implies that $\mathcal{X}_{\infty,\varepsilon} \neq \emptyset$, but also that $\mathcal{X}_{\infty,\varepsilon}$ is infinite.

Now for each $\varepsilon > 0$ define the open transfer operator $\mathcal{L}_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ by

$$\mathcal{L}_{\omega,\varepsilon}(f) := \mathcal{L}_{\omega,0}(\mathbb{1}_{\mathcal{J}_{\omega,\varepsilon}} f), \qquad f \in \mathcal{B}_\omega.$$

Iterates of the perturbed operator $\mathcal{L}^n_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma^n\omega}$ are given by

$$\mathcal{L}^n_{\omega,\varepsilon} := \mathcal{L}_{\sigma^{n-1}\omega,\varepsilon} \circ \cdots \circ \mathcal{L}_{\omega,\varepsilon},$$

which, using induction, we may write as

$$\mathcal{L}^n_{\omega,\varepsilon}(f) = \mathcal{L}^n_{\omega,0}\left(f \cdot \hat{X}_{\omega,n-1,\varepsilon}\right), \qquad f \in \mathcal{B}_\omega. \tag{2.2.2}$$

For every $\varepsilon \geq 0$ we let

$$\tilde{\mathcal{L}}_{\omega,\varepsilon} := \lambda^{-1}_{\omega,\varepsilon}\mathcal{L}_{\omega,\varepsilon}.$$

For the remainder of the manuscript we will suppose that $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0)$ is a closed random dynamical system as Definition 0.1.4, for each $\varepsilon > 0$ $H_\varepsilon \subseteq \mathcal{J}_0$ such that condition (A) holds, and that for each $\varepsilon > 0$, $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ is a random open system as in Definition 0.1.7. In summary, we assume that condition (A) holds in addition to the assumptions (M1), (M2), (CCM), (B), and (X) from Section 0.1.

2.2.1. Some of the Terms from Sequential Perturbation Theorem. In this short section we develop a more thorough understanding of some of the vital terms in the general sequential perturbation theorem of Section 2.1 in the setting of random open systems. We begin by calculating the quantities $\eta_{\omega,\varepsilon}$ and $\Delta_{\omega,\varepsilon}$ from Section 2.1. In particular, we have that

$$\begin{aligned}
\Delta_{\omega,\varepsilon} &:= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,0})\right) \\
&= \nu_{\sigma\omega,0}\left(\mathcal{L}_{\omega,0}(\phi_{\omega,0} \cdot \mathbb{1}_{H_{\omega,\varepsilon}})\right) \\
&= \lambda_{\omega,0} \cdot \nu_{\omega,0}(\phi_{\omega,0} \cdot \mathbb{1}_{H_{\omega,\varepsilon}}) \\
&= \lambda_{\omega,0} \cdot \mu_{\omega,0}(H_{\omega,\varepsilon})
\end{aligned} \tag{2.2.3}$$

and

$$\begin{aligned}
\eta_{\omega,\varepsilon} :&= \|\nu_{\sigma\omega,0}\left(\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon}\right)\|_{\mathcal{B}_\omega} \\
&= \sup_{\|\psi\|_{\mathcal{B}_\omega} \leq 1} \nu_{\sigma\omega,0}\left(\mathcal{L}_{\omega,0}(\psi \cdot \mathbb{1}_{H_{\omega,\varepsilon}})\right)
\end{aligned}$$

$$= \lambda_{\omega,0} \cdot \sup_{\|\psi\|_{\mathcal{B}_\omega} \le 1} \nu_{\omega,0}\left(\psi \cdot \mathbb{1}_{H_{\omega,\varepsilon}}\right)$$

$$\le K_\omega \lambda_{\omega,0} \cdot \nu_{\omega,0}\left(H_{\omega,\varepsilon}\right), \tag{2.2.4}$$

where the last line follows from (B).

For $\varepsilon > 0$ and $\mu_{\omega,0}(H_{\omega,\varepsilon}) > 0$, calculating $q^{(k)}_{\omega,\varepsilon}$ in this setting gives

$$q^{(k)}_{\omega,\varepsilon} := \frac{\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\mathcal{L}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0} - \mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0})\right)}{\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,0})\right)}$$

$$= \frac{\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0} \cdot \mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon}) \cap \left(\bigcap_{j=1}^k T^{-(k+1)+j}_{\sigma^{-(k+1)}\omega}(H^c_{\sigma^{-j}\omega,\varepsilon})\right) \cap H_{\sigma^{-(k+1)}\omega,\varepsilon}\right)}{\mu_{\omega,0}(H_{\omega,\varepsilon})}$$

$$= \frac{\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0} \cdot \mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon}) \cap \left(\bigcap_{j=1}^k T^{-(k+1)+j}_{\sigma^{-(k+1)}\omega}(H^c_{\sigma^{-j}\omega,\varepsilon})\right) \cap H_{\sigma^{-(k+1)}\omega,\varepsilon}\right)}{\mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon})\right)}.$$

For notational convenience we define the quantity $\hat{q}^{(k)}_{\omega,\varepsilon}$ by

$$\hat{q}^{(k)}_{\omega,\varepsilon} := \frac{\mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon}) \cap \left(\bigcap_{j=1}^k T^{-(k+1)+j}_{\sigma^{-(k+1)}\omega}(H^c_{\sigma^{-j}\omega,\varepsilon})\right) \cap H_{\sigma^{-(k+1)}\omega,\varepsilon}\right)}{\mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon})\right)}, \tag{2.2.5}$$

and thus we have that

$$\hat{q}^{(k)}_{\omega,\varepsilon} = \left(\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0}\right)^{-1} q^{(k)}_{\omega,\varepsilon}.$$

In light of (2.2.5), one can think of $\hat{q}^{(k)}_{\omega,\varepsilon}$ as the conditional probability (on the fiber $\sigma^{-(k+1)}\omega$) of a point starting in the hole $H_{\sigma^{-(k+1)}\omega,\varepsilon}$, leaving and avoiding holes for k steps, and finally landing in the hole $H_{\omega,\varepsilon}$ after exactly $k+1$ steps conditioned on the trajectory of the point landing in $H_{\omega,\varepsilon}$. Similarly, for $\omega \in \Omega_+$, we set

$$\hat{q}^{(k)}_{\omega,0} := \left(\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0}\right)^{-1} q^{(k)}_{\omega,0}.$$

2.3. Quenched perturbation theorem and escape rate asymptotics for random open systems

In this section we introduce versions of the assumptions (P1)–(P9) tailored to random open systems. Under these assumptions we then prove a derivative result akin to Theorem 2.1.2 for random open systems as well as a similar derivative result for the escape rate.

Suppose $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ is a random open system. We assume the following conditions hold:

(C1) There exists a measurable m-a.e. finite function $C_1 : \Omega \to \mathbb{R}_+$ such that for $f \in \mathcal{B}_\omega$ we have

$$\sup_{\varepsilon \ge 0} \|\mathcal{L}_{\omega,\varepsilon}(f)\|_{\mathcal{B}_{\sigma\omega}} \le C_1(\omega)\|f\|_{\mathcal{B}_\omega}.$$

(C2) For each $\varepsilon \geq 0$ there is a random measure $\{\nu_{\omega,\varepsilon}\}_{\omega\in\Omega}$ supported in $\mathcal{J}_0$ and measurable functions $\lambda_\varepsilon : \Omega \to (0,\infty)$ with $\log \lambda_{\omega,\varepsilon} \in L^1(m)$ and $\phi_\varepsilon : \mathcal{J}_0 \to \mathbb{R}$ such that

$$\mathcal{L}_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = \lambda_{\omega,\varepsilon}\phi_{\sigma\omega,\varepsilon} \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(\mathcal{L}_{\omega,\varepsilon}(f)) = \lambda_{\omega,\varepsilon}\nu_{\omega,\varepsilon}(f)$$

for all $f \in \mathcal{B}_\omega$. Furthermore we assume that for m-a.e. $\omega \in \Omega$

$$\nu_{\omega,0}(\phi_{\omega,\varepsilon}) = 1 \quad \text{and} \quad \nu_{\omega,0}(\mathbb{1}) = 1.$$

(C3) There is an operator $Q_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ such that for m-a.e. $\omega \in \Omega$ and each $f \in \mathcal{B}_\omega$ we have

$$\lambda_{\omega,\varepsilon}^{-1}\mathcal{L}_{\omega,\varepsilon}(f) = \nu_{\omega,\varepsilon}(f)\cdot\phi_{\sigma\omega,\varepsilon} + Q_{\omega,\varepsilon}(f).$$

Furthermore, for m-a.e. $\omega \in \Omega$ we have

$$Q_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = 0 \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(Q_{\omega,\varepsilon}(f)) = 0.$$

(C4) For each $f \in \mathcal{B}$ there exist measurable functions $C_f : \Omega \to (0,\infty)$ and $\alpha : \Omega\times\mathbb{N} \to (0,\infty)$ with $\alpha_\omega(N) \to 0$ as $N \to \infty$ such that for m-a.e. $\omega \in \Omega$ and all $N \in \mathbb{N}$

$$\sup_{\varepsilon\geq 0} \|Q^N_{\omega,\varepsilon} f_\omega\|_{\infty,\sigma^N\omega} \leq C_f(\omega)\alpha_\omega(N)\|f_\omega\|_{\mathcal{B}_\omega},$$

$$\sup_{\varepsilon\geq 0} \|Q^N_{\sigma^{-N}\omega,\varepsilon} f_{\sigma^{-N}\omega}\|_{\infty,\omega} \leq C_f(\omega)\alpha_\omega(N)\|f_{\sigma^{-N}\omega}\|_{\mathcal{B}_{\sigma^{-N}\omega}},$$

and $\|\phi_{\sigma^N\omega,0}\|_{\infty,\sigma^N\omega}\alpha_\omega(N) \to 0$, $\|\phi_{\sigma^{-N}\omega,0}\|_{\infty,\sigma^{-N}\omega}\alpha_\omega(N) \to 0$ as $N \to \infty$.

(C5) There exists a measurable m-a.e. finite function $C_2 : \Omega \to [1,\infty)$ such that

$$\sup_{\varepsilon\geq 0} \|\phi_{\omega,\varepsilon}\|_{\infty,\omega} \leq C_2(\omega) \quad \text{and} \quad \|\phi_{\omega,0}\|_{\mathcal{B}_\omega} \leq C_2(\omega).$$

(C6) For m-a.e. $\omega \in \Omega$ we have

$$\lim_{\varepsilon\to 0} \nu_{\omega,0}(H_{\omega,\varepsilon}) = 0.$$

(C7) There exists a measurable m-a.e. finite function $C_3 : \Omega \to [1,\infty)$ such that for all $\varepsilon > 0$ sufficiently small we have

$$\inf \phi_{\omega,0} \geq C_3^{-1}(\omega) > 0 \qquad \text{and} \qquad \operatorname*{ess\,inf}_\omega \inf \phi_{\omega,\varepsilon} \geq 0.$$

(C8) For m-a.e. $\omega \in \Omega_+$ we have that the limit $\hat{q}^{(k)}_{\omega,0} := \lim_{\varepsilon\to 0} \hat{q}^{(k)}_{\omega,\varepsilon}$ exists for each $k \geq 0$, where $\hat{q}^{(k)}_{\omega,\varepsilon}$ is as in (2.2.5).

Remark 2.3.1. To obtain the scaling required in (C2) and (C3), in particular to obtain the assumption that $\nu_{\omega,0}(\phi_{\omega,\varepsilon}) = 1$, suppose that $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ is a random open system satisfying the following properties:

(O1) For each $\varepsilon \geq 0$ there is a random measure $\left\{\nu'_{\omega,\varepsilon}\right\}_{\omega\in\Omega}$ with $\nu'_{\omega,\varepsilon} \in \mathcal{B}^*_\omega$, the dual space of $\mathcal{B}_\omega$, $\lambda'_{\omega,\varepsilon} \in \mathbb{C}\setminus\{0\}$, and $\phi'_{\omega,\varepsilon} \in \mathcal{B}_\omega$ such that

$$\mathcal{L}_{\omega,\varepsilon}(\phi'_{\omega,\varepsilon}) = \lambda'_{\omega,\varepsilon}\phi'_{\sigma\omega,\varepsilon} \quad \text{and} \quad \nu'_{\sigma\omega,\varepsilon}(\mathcal{L}_{\omega,\varepsilon}(f)) = \lambda'_{\omega,\varepsilon}\nu'_{\omega,\varepsilon}(f)$$

for all $f \in \mathcal{B}_\omega$.

(O2) There is an operator $Q'_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ such that for m-a.e. $\omega \in \Omega$ and each $f \in \mathcal{B}_\omega$ we have

$$(\lambda'_{\omega,\varepsilon})^{-1}\mathcal{L}_{\omega,\varepsilon}(f) = \nu'_{\omega,\varepsilon}(f) \cdot \phi'_{\sigma\omega,\varepsilon} + Q'_{\omega,\varepsilon}(f).$$

Furthermore, for m-a.e. $\omega \in \Omega$ we have

$$Q'_{\omega,\varepsilon}(\phi'_{\omega,\varepsilon}) = 0 \quad \text{and} \quad \nu'_{\sigma\omega,\varepsilon}(Q'_{\omega,\varepsilon}(f)) = 0.$$

Then, for every $\varepsilon > 0$ and each $f \in \mathcal{B}_\omega$, by setting

$$\begin{aligned}
\phi_{\omega,0} &:= \phi'_{\omega,0}, & \phi_{\omega,\varepsilon} &:= \frac{1}{\nu_{\omega,0}(\phi'_{\omega,\varepsilon})} \cdot \phi'_{\omega,\varepsilon},\\
\nu_{\omega,0}(f) &:= \nu'_{\omega,0}(f), & \nu_{\omega,\varepsilon}(f) &:= \nu_{\omega,0}(\phi'_{\omega,\varepsilon})\nu'_{\omega,\varepsilon}(f),\\
\lambda_{\omega,0} &:= \lambda'_{\omega,0}, & \lambda_{\omega,\varepsilon} &:= \frac{\nu_{\sigma\omega,0}(\phi'_{\sigma\omega,\varepsilon})}{\nu_{\omega,0}(\phi'_{\omega,\varepsilon})}\lambda'_{\omega,\varepsilon},\\
Q_{\omega,0}(f) &:= Q'_{\omega,0}(f), & Q_{\omega,\varepsilon}(f) &:= \frac{\nu_{\omega,0}(\phi'_{\omega,\varepsilon})}{\nu_{\sigma\omega,0}(\phi'_{\sigma\omega,\varepsilon})}Q'_{\omega,\varepsilon}(f)
\end{aligned}$$

we see that (C2) and (C3) hold, and in particular, $\nu_{\omega,0}(\phi_{\omega,\varepsilon}) = 1$. Furthermore, (O1) and (O2) together imply that $\nu'_{\omega,\varepsilon}(\phi'_{\omega,\varepsilon}) = \nu_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = 1$ for m-a.e. $\omega \in \Omega$. Note that the measures $\nu_{\omega,\varepsilon}$ (for $\varepsilon > 0$) described in (C2) are not necessarily probability measures and should not be confused with the conformal measures for the open systems.

Now given $N \in \mathbb{N}$ and $\omega \in \Omega$, for $\psi_\omega \in \mathcal{B}_\omega$ such that $\nu_{\omega,0}(\psi_\omega) = 1$ we have

$$\begin{aligned}
\int_{X_{\omega,N-1,\varepsilon}} \psi_\omega \, d\nu_{\omega,0} &= \int_{\mathcal{J}_{\omega,0}} \psi_\omega \cdot \prod_{j=0}^{N-1} \mathbb{1}_{\mathcal{J}_{\sigma^{j-N}\omega,\varepsilon}} \circ T^j_\omega \, d\nu_{\omega,0}\\
&= \left(\lambda^N_{\omega,0}\right)^{-1} \int_{\mathcal{J}_{\sigma^N\omega,0}} \mathcal{L}^N_{\omega,0}\left(\psi_\omega \cdot \prod_{j=0}^{N-1} \mathbb{1}_{\mathcal{J}_{\sigma^{j-N}\omega,\varepsilon}} \circ T^j_\omega\right) d\nu_{\sigma^N\omega,0}\\
&= \frac{\lambda^N_{\omega,\varepsilon}}{\lambda^N_{\omega,0}} \int_{\mathcal{J}_{\sigma^N\omega,0}} \tilde{\mathcal{L}}^N_{\omega,\varepsilon}(\psi_\omega) \, d\nu_{\sigma^N\omega,0}\\
&= \frac{\lambda^N_{\omega,\varepsilon}}{\lambda^N_{\omega,0}} \int_{\mathcal{J}_{\sigma^N\omega,0}} \nu_{\omega,\varepsilon}(\psi_\omega) \cdot \phi_{\sigma^N\omega,\varepsilon} \, d\nu_{\sigma^N\omega,0} + \frac{\lambda^N_{\omega,\varepsilon}}{\lambda^N_{\omega,0}} \int_{\mathcal{J}_{\sigma^N\omega,0}} Q^N_{\omega,\varepsilon}(\psi_\omega) \, d\nu_{\sigma^N\omega,0}.
\end{aligned}$$

Thus if $\psi_\omega = \mathbb{1}$ we have

$$\nu_{\omega,0}(X_{\omega,N-1,\varepsilon}) = \frac{\lambda^N_{\omega,\varepsilon}}{\lambda^N_{\omega,0}}\nu_{\omega,\varepsilon}(\mathbb{1}) + \frac{\lambda^N_{\omega,\varepsilon}}{\lambda^N_{\omega,0}} \int_{\mathcal{J}_{\sigma^N\omega,0}} Q^N_{\omega,\varepsilon}(\mathbb{1}) \, d\nu_{\sigma^N\omega,0}, \tag{2.3.1}$$

and if $\psi_\omega = \phi_{\omega,0}$, then we have

$$\mu_{\omega,0}(X_{\omega,N-1,\varepsilon}) = \frac{\lambda^N_{\omega,\varepsilon}}{\lambda^N_{\omega,0}}\nu_{\omega,\varepsilon}(\phi_{\omega,0}) + \frac{\lambda^N_{\omega,\varepsilon}}{\lambda^N_{\omega,0}} \int_{\mathcal{J}_{\sigma^N\omega,0}} Q^N_{\omega,\varepsilon}(\phi_{\omega,0}) \, d\nu_{\sigma^N\omega,0}. \tag{2.3.2}$$

REMARK 2.3.2. Using (2.1.3) and (C7) we see that

$$\begin{aligned}\lambda_{\omega,0} - \lambda_{\omega,\varepsilon} &= \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,\varepsilon})\right) = \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon}}\phi_{\omega,\varepsilon})\right)\\ &= \lambda_{\omega,0}\nu_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon}}\phi_{\omega,\varepsilon}) \geq \lambda_{\omega,0}\nu_{\omega,0}(H_{\omega,\varepsilon})\inf\phi_{\omega,\varepsilon} \geq 0.\end{aligned}$$

Thus, we have that $\lambda_{\omega,0} \geq \lambda_{\omega,\varepsilon}$. Note that if $\nu_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon}}\phi_{\omega,\varepsilon}) > 0$ then we have that $\lambda_{\omega,0} > \lambda_{\omega,\varepsilon}$. Consequently, if $\nu_{\omega,0}(H_{\omega,\varepsilon}) = 0$ then we have that $\lambda_{\omega,0} = \lambda_{\omega,\varepsilon}$.

Recall from Definition 1.12.3 that the upper and lower fiberwise escape rate of a random probability measure ζ on $\mathcal{J}_0$, for each $\varepsilon > 0$, is given by the following:

$$\underline{R}_\varepsilon(\zeta_\omega) := -\limsup_{N\to\infty}\frac{1}{N}\log\zeta_\omega(X_{\omega,N,\varepsilon}) \quad \text{and} \quad \overline{R}_\varepsilon(\zeta_\omega) := -\liminf_{N\to\infty}\frac{1}{N}\log\zeta_\omega(X_{\omega,N,\varepsilon}).$$

If $\underline{R}_\varepsilon(\zeta_\omega) = \overline{R}_\varepsilon(\zeta_\omega)$ then the fiberwise escape rate exist is denoted the common value by $R_\varepsilon(\zeta_\omega)$. As an immediate consequence of (2.3.1), (2.3.2), and assumptions (C4), (C5), and (C7) for each (fixed) $\varepsilon > 0$ we have that

(2.3.3)

$$\lim_{N\to\infty}\frac{1}{N}\log\nu_{\omega,0}(X_{\omega,N,\varepsilon}) = \lim_{N\to\infty}\frac{1}{N}\log\mu_{\omega,0}(X_{\omega,N,\varepsilon}) = \lim_{N\to\infty}\frac{1}{N}\log\lambda_{\omega,\varepsilon}^N - \lim_{N\to\infty}\frac{1}{N}\log\lambda_{\omega,0}^N.$$

Since $\log\lambda_{\omega,\varepsilon} \in L^1(m)$ for all $\varepsilon \geq 0$ by (C2), the following proposition follows directly from Birkhoff's Ergodic Theorem.

PROPOSITION 2.3.3. *Given a random open system $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ satisfying conditions (C1)-(C7), for m-a.e. $\omega \in \Omega$ we have that*

$$R_\varepsilon(\nu_{\omega,0}) = R_\varepsilon(\mu_{\omega,0}) = \int_\Omega \log\lambda_{\omega,0}\,dm(\omega) - \int_\Omega \log\lambda_{\omega,\varepsilon}\,dm(\omega). \tag{2.3.4}$$

REMARK 2.3.4. We remark that if we were to replace the supremum norm (with respect to $\nu_{\omega,0}$) $\|\cdot\|_{\infty,\omega}$ everywhere in our assumption (C4) with the norm $\|\cdot\|_\infty := \sup(|\cdot|)$, then, using an argument similar to that of Example 7.4 of [2], we would be able to prove a stronger result than is given in Proposition 2.3.3. Namely we could show that for every $0 \leq \varepsilon' < \varepsilon$ and m-a.e. $\omega \in \Omega$ we would have

$$R_\varepsilon(\nu_{\omega,\varepsilon'}) = R_\varepsilon(\mu_{\omega,\varepsilon'}) = \int_\Omega \log\lambda_{\omega,\varepsilon'}\,dm(\omega) - \int_\Omega \log\lambda_{\omega,\varepsilon}\,dm(\omega).$$

We now begin to work toward an application of Theorem 2.1.2 to random open systems. The following implications are immediate: (C1) $\Longrightarrow$ (P1), (C2) $\Longrightarrow$ (P2), (C3) $\Longrightarrow$ (P3), (C5) $\Longrightarrow$ (P4), and in light of (2.2.3) and (2.2.4) we also have that (C6) $\Longrightarrow$ (P5) and (C7) $\Longrightarrow$ (P6). Thus, in order to ensure that Theorem 2.1.2 applies for the random open dynamical setting we need only to check assumptions (P7) and (P8). We now prove the following lemma showing that (P7) and (P8) follow from assumptions (C1)-(C7).

Recall that the set Ω_+, defined in (2.2.1), is the set of all fibers ω such that $\mu_{\omega,0}(H_{\omega,\varepsilon}) > 0$ for all $\varepsilon > 0$.

LEMMA 2.3.5. *Given a random open system $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ satisfying conditions (C1)-(C7), for m-a.e. $\omega \in \Omega_+$ we have that*

$$\lim_{\varepsilon\to 0}\nu_{\omega,\varepsilon}(\phi_{\omega,0}) = 1 \tag{2.3.5}$$

and

$$\lim_{N\to\infty}\limsup_{\varepsilon\to 0}\Delta_{\omega,\varepsilon}^{-1}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^N_{\sigma^{-N}\omega,\varepsilon}\phi_{\sigma^{-N}\omega,0})\right)=0. \tag{2.3.6}$$

Proof. First, using (2.2.3), we note that if $\mu_{\omega,0}(H_{\omega,\varepsilon})>0$ then so is $\Delta_{\omega,\varepsilon}$. To prove (2.3.5), we note that for fixed $N\in\mathbb{N}$ we have

$$\lim_{\varepsilon\to 0}\frac{\lambda^N_{\omega,0}}{\lambda^N_{\omega,\varepsilon}}\mu_{\omega,0}(X_{\omega,N-1,\varepsilon})=1 \tag{2.3.7}$$

since $\lambda^N_{\omega,0}/\lambda^N_{\omega,\varepsilon}\to 1$ (by (2.1.4), (2.2.4), (C5), and (C6)) and non-atomicity of $\nu_{\omega,0}$ (CCM) together with (C3) imply that $\mu_{\omega,0}(X_{\omega,N-1,\varepsilon})\to 1$ as $\varepsilon\to 0$. Using (2.3.2) we can write

$$\nu_{\omega,\varepsilon}(\phi_{\omega,0})=\frac{\lambda^N_{\omega,0}}{\lambda^N_{\omega,\varepsilon}}\mu_{\omega,0}(X_{\omega,N-1,\varepsilon})-\nu_{\sigma^N\omega,0}(Q^N_{\omega,\varepsilon}(\phi_{\omega,0})), \tag{2.3.8}$$

and thus using (2.3.7) and (C4), for each $\omega\in\Omega$ and each $N\in\mathbb{N}$ we can write

$$\begin{aligned}\lim_{\varepsilon\to 0}|1-\nu_{\omega,\varepsilon}(\phi_{\omega,0})| &\le \lim_{\varepsilon\to 0}\left|1-\frac{\lambda^N_{\omega,0}}{\lambda^N_{\omega,\varepsilon}}\mu_{\omega,0}(X_{\omega,N-1,\varepsilon})\right|+\|Q^N_{\omega,\varepsilon}(\phi_{\omega,0})\|_{\infty,\sigma^N\omega}\\ &\le C_{\phi_0}(\omega)\alpha_\omega(N)\|\phi_{\omega,0}\|_{\mathcal{B}_\omega}.\end{aligned}$$

As this holds for each $N\in\mathbb{N}$ and as the right-hand side of the previous equation goes to zero as $N\to\infty$, we must in fact have that

$$\lim_{\varepsilon\to 0}|1-\nu_{\omega,\varepsilon}(\phi_{\omega,0})|=0,$$

which yields the first claim.

Now, for the second claim, using (2.2.3), we note that (C7) implies

$$\begin{aligned}\Delta_{\omega,\varepsilon}^{-1}\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^N_{\sigma^{-N}\omega,\varepsilon}\phi_{\sigma^{-N}\omega})\right) &= \frac{\nu_{\omega,0}\left(\mathbb{1}_{H_{\omega,\varepsilon}}\cdot Q^N_{\sigma^{-N}\omega,\varepsilon}(\phi_{\sigma^{-N}\omega,0})\right)}{\mu_{\omega,0}(H_{\omega,\varepsilon})}\\ &\le \frac{\nu_{\omega,0}(H_{\omega,\varepsilon})}{\mu_{\omega,0}(H_{\omega,\varepsilon})}\|Q^N_{\sigma^{-N}\omega,\varepsilon}(\phi_{\sigma^{-N}\omega,0})\|_{\infty,\omega}\\ &\le C_3(\omega)\|Q^N_{\sigma^{-N}\omega,\varepsilon}(\phi_{\sigma^{-N}\omega,0})\|_{\infty,\omega}.\end{aligned}$$

Thus, letting $\varepsilon\to 0$ first and then $N\to\infty$, the second claim follows from (C4). □

Now recall from (2.1.20) that if $\mu_{\omega,0}(H_{\omega,\varepsilon})>0$ for each $\varepsilon>0$, then $\theta_{\omega,0}$ is given by

$$\theta_{\omega,0}:=1-\sum_{k=0}^{\infty}(\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0})^{-1}q^{(k)}_{\omega,0}=1-\sum_{k=0}^{\infty}\hat{q}^{(k)}_{\omega,0}.$$

In light of Lemma 2.3.5, we see that (C1)-(C8) imply (P1)-(P9), and thus we have the following Theorem and first main result of this section.

Theorem 2.3.6. *Suppose that* (C1)-(C7) *hold for a random open system* $(\Omega,m,\sigma,\mathcal{J}_0,T,\mathcal{B},\mathcal{L}_0,\nu_0,\phi_0,H_\varepsilon)$. *For* m*-a.e.* $\omega\in\Omega$ *if there is some* $\varepsilon_0>0$ *such that* $\mu_{\omega,0}(H_{\omega,\varepsilon})=0$ *for* $\varepsilon\le\varepsilon_0$ *then*

$$\lambda_{\omega,0}=\lambda_{\omega,\varepsilon},$$

or if (C8) *holds, then*

$$\lim_{\varepsilon\to 0}\frac{\lambda_{\omega,0}-\lambda_{\omega,\varepsilon}}{\lambda_{\omega,0}\mu_{\omega,0}(H_{\omega,\varepsilon})}=\theta_{\omega,0}.$$

Furthermore, the map $\Omega_+\ni\omega\mapsto\theta_{\omega,0}$ *is measurable.*

PROOF. All statements follow directly from Theorem 2.1.2, except measurability of $\theta_{\omega,0}$, which follows from its construction as a limit of measurable objects. □

REMARK 2.3.7. As Theorem 2.1.2 does not require any measurability, one could restate Theorem 2.3.6 to hold for sequential open systems satisfying the sequential versions of hypotheses (C1)-(C8).

The following lemma will be useful for bounding $\theta_{\omega,0}$.

LEMMA 2.3.8. *Suppose* $m(\Omega\backslash\Omega_+)=0$. *Then for each* $\varepsilon>0$ *and* m*-a.e.* $\omega\in\Omega$ *we have*

$$\sum_{k=0}^{\infty}(\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0})^{-1}q^{(k)}_{\omega,\varepsilon}=\sum_{k=0}^{\infty}\hat{q}^{(k)}_{\omega,\varepsilon}=1.$$

PROOF. Recall from Section 0.1.1 that the hole $H_\varepsilon\subseteq\mathcal{J}_0$ is given by

$$H_\varepsilon:=\bigcup_{\omega\in\Omega}\{\omega\}\times H_{\omega,\varepsilon},$$

and the skew map $T:\mathcal{J}_0\to\mathcal{J}_0$ is given by

$$T(\omega,x)=(\sigma\omega,T_\omega(x)).$$

Let $\tau_{H_\varepsilon}(\omega,x)$ denote the first return time of the point $(\omega,x)\in H_\varepsilon$ into H_ε. For each $k\geq 0$ let $B_k:=\{(\omega,x)\in H_\varepsilon:\tau_{H_\varepsilon}(\omega,x)=k+1\}$ be the set of points (ω,x) that remain outside of the hole H_ε for exactly k iterates, and for each $\omega\in\Omega$ we set $B_{k,\omega}:=\pi_2(B_k\cap\{\omega\}\times\mathcal{J}_{\omega,0})$. Then $B_{k,\omega}$ is precisely the set

$$B_{k,\omega}=\left\{x\in H_{\omega,\varepsilon}:T^{k+1}_\omega(x)\in H_{\sigma^{k+1}\omega,\varepsilon}\text{ and }T^j_\omega(x)\notin H_{\sigma^j\omega,\varepsilon}\text{ for all }1\leq j\leq k\right\}.$$

For each $k\geq 0$ we can disintegrate μ_0 as

$$\mu_0\left(B_k\right)=\int_\Omega\mu_{\omega,0}(B_{k,\omega})\,dm(\omega),$$

where the last equality follows from the σ-invariance m. In particular, for fixed $\omega\in\Omega_+$ and each $k\geq 0$ we have

(2.3.9)

$$\mu_{\sigma^{-(k+1)}\omega,0}(B_{k,\sigma^{-(k+1)}\omega})=\mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon})\cap\left(\bigcap_{j=1}^{k}T^{-(k+1)+j}_{\sigma^{-(k+1)}\omega}(H^c_{\sigma^{-j}\omega,\varepsilon})\right)\cap H_{\sigma^{-(k+1)}\omega,\varepsilon}\right).$$

Recall from (2.2.5) that $\hat{q}^{(k)}_{\omega,\varepsilon}$ is given by

$$\hat{q}^{(k)}_{\omega,\varepsilon}:=\frac{\mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon})\cap\left(\bigcap_{j=1}^{k}T^{-(k+1)+j}_{\sigma^{-(k+1)}\omega}(H^c_{\sigma^{-j}\omega,\varepsilon})\right)\cap H_{\sigma^{-(k+1)}\omega,\varepsilon}\right)}{\mu_{\sigma^{-(k+1)}\omega,0}\left(T^{-(k+1)}_{\sigma^{-(k+1)}\omega}(H_{\omega,\varepsilon})\right)},$$

which is well defined since $\omega \in \Omega_+$ by assumption. Note that since $\sum_{k=0}^{\infty} \mu_{\sigma^{-(k+1)}\omega,0}(B_{k,\sigma^{-(k+1)}\omega}) \leq \mu_{\omega,0}(H_{\omega,\varepsilon})$ we must have that $\sum_{k=0}^{\infty} \hat{q}_{\omega,\varepsilon}^{(k)} \leq 1$. Now, as the measure μ_0 is T-invariant, the right-hand side of (2.3.9) is equal to $\hat{q}_{\omega,\varepsilon}^{(k)}\mu_{\omega,0}(H_{\omega,\varepsilon})$, and therefore

$$\int_\Omega \hat{q}_{\omega,\varepsilon}^{(k)}\ \mu_{\omega,0}(H_{\omega,\varepsilon})\ dm(\omega) = \int_\Omega \mu_{\omega,0}(B_{k,\omega})\ dm(\omega) = \mu_0(B_k). \tag{2.3.10}$$

By the Poincaré recurrence theorem we have

$$\sum_{k=0}^{\infty} \mu_0(B_k) = \mu_0(H_\varepsilon) = \int_\Omega \mu_{\omega,0}(H_{\omega,\varepsilon})\, dm(\omega).$$

By interchanging the sum with the integral above (possible by Tonelli's Theorem) and using (2.3.10), we have

$$\int_\Omega \left(\sum_{k=0}^{\infty} \hat{q}_{\omega,\varepsilon}^{(k)}\ \mu_{\omega,0}(H_{\omega,\varepsilon}) \right)\ dm(\omega) = \int_\Omega \mu_{\omega,0}(H_{\omega,\varepsilon})\ dm(\omega),$$

which implies

$$\int_\Omega \left(\mu_{\omega,0}(H_{\omega,\varepsilon}) \left(\sum_{k=0}^{\infty} \hat{q}_{\omega,\varepsilon}^{(k)} - 1 \right) \right) dm(\omega) = 0.$$

Since we already have that $\sum_{k=0}^{\infty} \hat{q}_{\omega,\varepsilon}^{(k)} \leq 1$ for m-a.e. $\omega \in \Omega$, we must in fact have $\sum_{k=0}^{\infty} \hat{q}_{\omega,\varepsilon}^{(k)} = 1$, which completes the proof. □

By definition, we have that $\hat{q}_{\omega,\varepsilon}^{(k)} \in [0,1]$ and thus (C8) implies that $\hat{q}_{\omega,0}^{(k)} \in [0,1]$ for each $k \geq 0$ as well. In light of the previous lemma, dominated convergence implies that

$$\theta_{\omega,0} \in [0,1]. \tag{2.3.11}$$

2.3.1. Escape Rate Asymptotics. If we have some additional ω-control on the size of the holes we obtain the following corollary of Theorem 2.3.6, which provides a formula for the escape rate asymptotics for small random holes. The scaling of the holes (2.3.13) takes a similar form to the scaling we will shortly use in the next section for our quenched extreme value theory.

COROLLARY 2.3.9. *Suppose that* (C1)–(C8) *hold for a random open system* $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ *and there exists* $A \in L^1(m)$ *such that for* m*-a.e.* $\omega \in \Omega$ *and all* $\varepsilon > 0$ *sufficiently small we have*

$$\frac{\log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} \leq A(\omega). \tag{2.3.12}$$

Further suppose that there is some $\kappa(\varepsilon)$ *with* $\kappa(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$, $t \in L^\infty(m)$ *with* $t_\omega > 0$, *and* $\xi_{\omega,\varepsilon} \in L^\infty(m)$ *with* $\xi_{\omega,\varepsilon} \to 0$ *as* $\varepsilon \to 0$ *for* m*-a.e.* $\omega \in \Omega$ *such that*

$$\mu_{\omega,0}(H_{\omega,\varepsilon}) = \frac{t_\omega + \xi_{\omega,\varepsilon}}{\kappa(\varepsilon)} \tag{2.3.13}$$

and $|\xi_{\omega,\varepsilon}| < Ct_\omega$ *for all* $\varepsilon > 0$ *sufficiently small for some* $C > 0$. *Then for* m*-a.e.* $\omega \in \Omega$ *we have*

$$\lim_{\varepsilon\to 0} \frac{R_\varepsilon(\mu_{\omega,0})}{\mu_{\omega,0}(H_{\omega,\varepsilon})} = \frac{\int_\Omega \theta_{\omega,0} t_\omega \, dm(\omega)}{t_\omega}.$$

In particular, if t_ω *is constant* m*-a.e. then*

$$\lim_{\varepsilon\to 0} \frac{R_\varepsilon(\mu_{\omega,0})}{\mu_{\omega,0}(H_{\omega,\varepsilon})} = \int_\Omega \theta_{\omega,0} \, dm(\omega).$$

Proof. First, we note that (2.3.13) implies that $\mu_{\omega,0}(H_{\omega,\varepsilon}) > 0$ for m-a.e. $\omega \in \Omega$ and all $\varepsilon > 0$ sufficiently small, i.e. $m(\Omega \backslash \Omega_+) = 0$. Using Theorem 2.3.6 we have that

$$\lim_{\varepsilon\to 0} \frac{\log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} = \theta_{\omega,0} \tag{2.3.14}$$

since

$$\frac{\log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} = \frac{\log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}}{\lambda_{\omega,0} - \lambda_{\omega,\varepsilon}} \cdot \frac{\lambda_{\omega,0} - \lambda_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} \longrightarrow \frac{1}{\lambda_{\omega,0}} \cdot \lambda_{\omega,0}\theta_{\omega,0} = \theta_{\omega,0}$$

as $\varepsilon \to 0$. In light of (2.3.3) and (2.3.4), we use (2.3.13) to get that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{R_\varepsilon(\mu_{\omega,0})}{\mu_{\omega,0}(H_{\omega,\varepsilon})} &= \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{N} \frac{\log \lambda^N_{\omega,0} - \log \lambda^N_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} \\
&= \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{\log \lambda_{\sigma^j\omega,0} - \log \lambda_{\sigma^j\omega,\varepsilon}}{\mu_{\sigma^j\omega,0}(H_{\sigma^j\omega,\varepsilon})} \cdot \frac{\mu_{\sigma^j\omega,0}(H_{\sigma^j\omega,\varepsilon})}{\mu_{\omega,0}(H_{\omega,\varepsilon})} \\
&= \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{\log \lambda_{\sigma^j\omega,0} - \log \lambda_{\sigma^j\omega,\varepsilon}}{\mu_{\sigma^j\omega,0}(H_{\sigma^j\omega,\varepsilon})} \cdot \frac{t_{\sigma^j\omega} + \xi_{\sigma^j\omega,\varepsilon}}{t_\omega + \xi_{\omega,\varepsilon}} \\
&= \lim_{\varepsilon\to 0} \frac{1}{t_\omega + \xi_{\omega,\varepsilon}} \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{\log \lambda_{\sigma^j\omega,0} - \log \lambda_{\sigma^j\omega,\varepsilon}}{\mu_{\sigma^j\omega,0}(H_{\sigma^j\omega,\varepsilon})} \cdot (t_{\sigma^j\omega} + \xi_{\sigma^j\omega,\varepsilon}) \\
&= \lim_{\varepsilon\to 0} \frac{1}{t_\omega + \xi_{\omega,\varepsilon}} \int_\Omega \frac{\log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} \cdot (t_\omega + \xi_{\omega,\varepsilon}) \, dm(\omega),
\end{aligned}$$

where the last line follows from Birkhoff (which is applicable thanks to (2.3.12) and the fact that $t_\omega, \xi_{\omega,\varepsilon} \in L^\infty(m)$). As $\xi_{\omega,\varepsilon} \to 0$ by assumption, (2.3.12) allows us to apply Dominated Convergence which, in view of (2.3.14), implies

$$\lim_{\varepsilon\to 0} \frac{R_\varepsilon(\mu_{\omega,0})}{\mu_{\omega,0}(H_{\omega,\varepsilon})} = \frac{1}{t_\omega} \lim_{\varepsilon\to 0} \int_\Omega \frac{\log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} \cdot (t_\omega + \xi_{\omega,\varepsilon}) \, dm(\omega) = \frac{\int_\Omega \theta_{\omega,0} t_\omega \, dm(\omega)}{t_\omega},$$

completing the proof. □

In the next section we will present easily checkable assumptions which will imply the hypotheses of Corollary 2.3.9.

2.4. Quenched extreme value law

2.4.1. Gumbel's law in the quenched regime. Suppose that $h_\omega : \mathcal{J}_{\omega,0} \to \mathbb{R}$ is a continuous function for each $\omega \in \Omega$. In our quenched random extreme value theory, h_ω is an observation function that is allowed to depend on ω. For each $\omega \in \Omega$ let $\overline{z}_\omega$ be the essential supremum of h_ω with respect to $\nu_{\omega,0}$, that is

$$\overline{z}_\omega := \sup \left\{ z \in \mathbb{R} \ : \nu_{\omega,0}(\{x \in \mathcal{J}_{\omega,0} : h_\omega(x) \geq z\}) > 0 \right\}.$$

Similarly we define $\underline{z}_\omega$ to be the essential infimum of h_ω with respect to $\nu_{\omega,0}$. Suppose $\underline{z}_\omega < \overline{z}_\omega$ for m-a.e. $\omega \in \Omega$, and for each $z \in [\underline{z}_\omega, \overline{z}_\omega]$ we define the set

$$V_{\omega,z_\omega} := \{x \in \mathcal{J}_{\omega,0} : h_\omega(x) - z_\omega > 0\},$$

which represents points x in our phase space where the observation h_ω exceeds a random threshold z_ω at base configuration ω. Our theory allows for random thresholds z_ω so that we may consider both "anomalous" and "absolute" exceedances. For example in a real-world application, $h_\omega(x)$ may represent the surface ocean temperature at a spatial location x for an ocean system configuration ω. Random temperature exceedances above z_ω allow one to describe extreme value statistics for temperature anomalies, e.g. those above a climatological seasonal mean (on average the surface ocean is warmer in summer and colder in winter). On the other hand, non-random absolute temperature exceedances above z are more relevant for marine life. We suppose that

$$\nu_{\omega,0}(V_{\omega,\overline{z}_\omega}) = 0.$$

As is standard in extreme value theory, to develop an exponential law we will consider an increasing sequence of thresholds $z_{\omega,0} < z_{\omega,1} < \cdots$. For each $N \geq 0$ we take $z_{\omega,N} \in [\underline{z}_\omega, \overline{z}_\omega]$ and the choice will be made explicit in a moment. For each $k, N \in \mathbb{N}$ define $G^{(k)}_{\omega,N} : \mathcal{J}_{\omega,0} \to \mathbb{R}$ by

$$G^{(k)}_{\omega,N}(x) := h_{\sigma^N\omega}(T^N_\omega(x)) - z_{\sigma^N\omega,k}.$$

If $G^{(k)}_{\omega,N} > 0$ then $h_{\sigma^N\omega}(T^N_\omega(x)) > z_{\sigma^N\omega,k}$. Our extreme value law concerns the large N limit of the likelihood of continued threshold non-exceedances:

$$\nu_{\omega,0}\left(\left\{x \in \mathcal{J}_{\omega,0} : \max\left(G^{(N)}_{\omega,0}(x), \ldots, G^{(N)}_{\omega,N-1}(x)\right) \leq 0\right\}\right). \tag{2.4.1}$$

We may easily transform (2.4.1) into the language of random open systems: one immediately has that $G^{(k)}_{\omega,N} > 0$ is equivalent to $T^N_\omega(x) \in V_{\sigma^N\omega,z_{\sigma^N\omega,k}}$. Therefore, for each $N \geq 0$ we have

$$(2.4.1) = \nu_{\omega,0}\left(\left\{x \in \mathcal{J}_{\omega,0} : T^j_\omega(x) \notin V_{\sigma^j\omega,z_{\sigma^j\omega,N}} \text{ for } j = 0, \ldots, N-1\right\}\right) = \nu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right), \tag{2.4.2}$$

where to obtain the second equality we identify the sets $V_{\omega,z_{\omega,N}}$ with holes $H_{\omega,\varepsilon_N} \subset \mathcal{J}_{\omega,0}$ for each $N \in \mathbb{N}$ [1]. Now using (2.3.1) and (2.4.2) we may convert (2.4.1) into the spectral

[1] In Section 2.4 we consider a decreasing sequence of holes, whereas in previous sections we considered a decreasing family of holes $H_{\omega,\varepsilon}$ parameterised by $\varepsilon > 0$. For the sake of notational continuity, in this section, and in the sequel, we denote a decreasing sequence of holes by H_{ω,ε_N}. Note that ε_N here is just a parameter and should not be thought of as a real number, but instead as an index and that H_{ω,ε_N} serves

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

expression:

$$(2.4.1) = \frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} \left(\nu_{\omega,\varepsilon_N}(\mathbb{1}) + \nu_{\sigma^N\omega,0}\left(Q^N_{\omega,\varepsilon_N}(\mathbb{1})\right)\right). \tag{2.4.3}$$

Similarly, using (2.3.2), we can write

$$\begin{aligned}\mu_{\omega,0}\left(\left\{x \in \mathcal{J}_{\omega,0} : T^j_\omega(x) \notin V_{\sigma^j\omega, z_{\sigma^j\omega,N}} \text{ for } j = 0,\dots,N-1\right\}\right) &= \mu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right)\\ &= \frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} \left(\nu_{\omega,\varepsilon_N}(\phi_{\omega,0}) + \nu_{\sigma^N\omega,0}\left(Q^N_{\omega,\varepsilon_N}(\phi_{\omega,0})\right)\right). \qquad (2.4.4)\end{aligned}$$

Before we state our main result concerning the $N \to \infty$ limit, in addition to (C2), (C3), and (C8), we make the following uniform adjustments to some of the assumptions in Section 2.3 as well as an assumption on the choice of the sequence of thresholds, condition (S). At the end of this section, we will compare it with the Hüsler condition, which is the usual prescription for non stationary processes, as we anticipated in the Introduction.

(S) For any fixed random scaling function $t \in L^\infty(m)$ with $t > 0$, we may find sequences of functions $z_N, \xi_N \in L^\infty(m)$ and a constant $W < \infty$ satisfying

$$\mu_{\omega,0}(\{h_\omega(x) - z_{\omega,N} > 0\}) = (t_\omega + \xi_{\omega,N})/N, \text{ for a.e. } \omega \text{ and each } N \geq 1$$

where:
(i) $\lim_{N\to\infty} \xi_{\omega,N} = 0$ for a.e. ω and
(ii) $|\xi_{\omega,N}| \leq W$ for a.e. ω and all $N \geq 1$.

(C1') There exists $C_1 \geq 1$ such that for m-e.a. $\omega \in \Omega$ we have

$$C_1^{-1} \leq \mathcal{L}_{\omega,0}\mathbb{1} \leq C_1.$$

(C4') For each $f \in \mathcal{B}$ and each $N \in \mathbb{N}$ there exists $C_f > 0$ and $\alpha(N) > 0$ (independent of ω) with $\alpha := \sum_{N=1}^\infty \alpha(N) < \infty$ such that for m-a.e. $\omega \in \Omega$, all $N \in \mathbb{N}$

$$\sup_{\varepsilon \geq 0} \|Q^N_{\omega,\varepsilon} f_\omega\|_{\infty,\sigma^N\omega} \leq C_f \alpha(N) \|f_\omega\|_{\mathcal{B}_\omega}.$$

(C5') There exists $C_2 \geq 1$ such that

$$\sup_{\varepsilon \geq 0} \|\phi_{\omega,\varepsilon}\|_{\infty,\omega} \leq C_2 \quad \text{and} \quad \|\phi_{\omega,0}\|_{\mathcal{B}_\omega} \leq C_2$$

for m-a.e. $\omega \in \Omega$.

(C7') There exists $C_3 \geq 1$ such that for all $\varepsilon > 0$ sufficiently small we have

$$\operatorname*{ess\,inf}_\omega \inf \phi_{\omega,0} \geq C_3^{-1} > 0 \qquad \text{and} \qquad \operatorname*{ess\,inf}_\omega \inf \phi_{\omega,\varepsilon} \geq 0.$$

REMARK 2.4.1. Note that (S) implies that $\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > 0$ for each $N \in \mathbb{N}$ since $t_\omega > 0$, and in particular we have $m(\Omega \backslash \Omega_+) = 0$.

REMARK 2.4.2. Note that since $\lambda_{\omega,0} = \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}\mathbb{1})$, (C1') implies that

$$C_1^{-1} \leq \lambda_{\omega,0} \leq C_1$$

for m-a.e. ω.

as an alternative notation to $H_{\omega,N}$. Furthermore, note that the measure of the hole depends on the fiber ω with $\mu_{\omega,0}(H_{\omega,\varepsilon_N}) \to 0$ as $N \to \infty$.

REMARK 2.4.3. Note that conditions (S), (C5'), and (C7') together imply (C6), thus Theorem 2.3.6 applies. Furthermore, these same conditions along with Remark 2.4.2 imply that there exists $A \geq 1$ such that

$$\frac{\log \lambda_{\omega,0} - \log \lambda_{\omega,\varepsilon}}{\mu_{\omega,0}(H_{\omega,\varepsilon})} \leq A$$

for m-a.e. $\omega \in \Omega$ and all $\varepsilon > 0$ sufficiently small, and thus (2.3.12) and (2.3.13) hold, meaning that Corollary 2.3.9 applies as well.

The following lemma shows that $\nu_{\omega,\varepsilon_N}(\phi_{\omega,0})$ converges to 1 uniformly in ω under our assumptions (C1'), (C4'), (C5'), (C7'), and (S).

LEMMA 2.4.4. *For each $N \in \mathbb{N}$ there exists $C_{\varepsilon_N} \geq 1$ with $C_{\varepsilon_N} \to 1$ as $N \to \infty$ such that*

$$C_{\varepsilon_N}^{-1} \leq \nu_{\omega,\varepsilon_N}(\mathbb{1}),\ \nu_{\omega,\varepsilon_N}(\phi_{\omega,0}) \leq C_{\varepsilon_N} \tag{2.4.5}$$

for m-a.e. $\omega \in \Omega$.

PROOF. Note that

$$\nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon_N})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon_N}\phi_{\sigma^{-k}\omega,0})\right) = \lambda_{\omega,0}\nu_{\omega,0}\left(\mathbb{1}_{H_{\omega,\varepsilon_N}}\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon_N}\phi_{\sigma^{-k}\omega,0}\right). \tag{2.4.6}$$

Further, by (C7') we have

$$\nu_{\omega,0}(H_{\omega,\varepsilon_N}) \leq C_3\mu_{\omega,0}(H_{\omega,\varepsilon_N}) = \frac{C_3(t_\omega + \xi_{\omega,N})}{N} \tag{2.4.7}$$

for m-a.e. $\omega \in \Omega$. Following the same derivation of (2.1.3), and using (C2) gives that

$$\lambda_{\omega,0} - \lambda_{\omega,\varepsilon_N} = \nu_{\sigma\omega,0}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon_N})(\phi_{\omega,\varepsilon_N})\right) = \lambda_{\omega,0}\nu_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon_N}}\phi_{\omega,\varepsilon_N}). \tag{2.4.8}$$

Thus it follows from Remark 2.3.2, (2.4.8), (C5'), and (2.4.7) that

$$0 \leq \lambda_{\omega,0} - \lambda_{\omega,\varepsilon_N} \leq \frac{C_2C_3\lambda_{\omega,0}(t_\omega + \xi_{\omega,N})}{N}. \tag{2.4.9}$$

Using Remark 2.4.2 and (2.4.9), for all N sufficiently large, we see that

$$1 \leq \frac{\lambda_{\omega,0}}{\lambda_{\omega,\varepsilon_N}} = \frac{\lambda_{\omega,0}}{\lambda_{\omega,0} - \lambda_{\omega,0}\nu_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon_N}}\phi_{\omega,\varepsilon_N})} \leq \frac{1}{1 - C_2\nu_{\omega,0}(H_{\omega,\varepsilon_N})} \leq \frac{1}{1 - E_N}, \tag{2.4.10}$$

where

$$E_N := \frac{C_2C_3(|t|_\infty + W)}{N} \to 0$$

as $N \to \infty$.

CLAIM 2.4.4.1. *For every n, ε_N, and m-a.e. $\omega \in \Omega$ we have*

$$|1 - \nu_{\omega,\varepsilon_N}(\phi_{\omega,0})| \leq n\left(\frac{1}{1 - E_N}\right)^n \cdot \left(C_1E_N + \frac{C_1C_2E_N}{1 - E_N}\right) + C_2C_{\phi_0}\alpha(n).$$

PROOF. Using (C2) and a telescoping argument, we can write

$$\begin{aligned}
|1-\nu_{\omega,\varepsilon_N}(\phi_{\omega,0})| &= |\nu_{\sigma^n\omega,0}(\phi_{\sigma^n\omega,0}) - \nu_{\omega,\varepsilon_N}(\phi_{\omega,0})| \\
&= \left|\nu_{\sigma^n\omega,0}(\phi_{\sigma^n\omega,0}) - \nu_{\sigma^n\omega,0}\big(\nu_{\omega,\varepsilon_N}(\phi_{\omega,0})\cdot\phi_{\sigma^n\omega,\varepsilon_N}\big)\right| \\
&= \left|\nu_{\sigma^n\omega,0}\left(\phi_{\sigma^n\omega,0} - \tilde{\mathcal{L}}^n_{\omega,\varepsilon_N}(\phi_{\omega,0}) + Q^n_{\omega,\varepsilon_N}(\phi_{\omega,0})\right)\right| \\
&\le \sum_{k=0}^{n-1}\left|\nu_{\sigma^n\omega,0}\left(\tilde{\mathcal{L}}^k_{\sigma^{n-k}\omega,\varepsilon_N}(\phi_{\sigma^{n-k}\omega,0}) - \tilde{\mathcal{L}}^{k+1}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right| \\
&\qquad + \left|\nu_{\sigma^n\omega,0}\left(Q^n_{\omega,\varepsilon_N}(\phi_{\omega,0})\right)\right| \\
&= \sum_{k=0}^{n-1}\left|\nu_{\sigma^n\omega,0}\left(\left(\tilde{\mathcal{L}}^k_{\sigma^{n-k}\omega,\varepsilon_N}\right)\left(\tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,0} - \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right| \qquad (2.4.11) \\
&\qquad + \left|\nu_{\sigma^n\omega,0}\left(Q^n_{\omega,\varepsilon_N}(\phi_{\omega,0})\right)\right|. \qquad (2.4.12)
\end{aligned}$$

First we note that we can estimate (2.4.12) as

$$\left|\nu_{\sigma^n\omega,0}\left(Q^n_{\omega,\varepsilon_N}(\phi_{\omega,0})\right)\right| \le \|Q^n_{\omega,\varepsilon_N}(\phi_{\omega,0})\|_{\infty,\sigma^n\omega} \le C_2C_{\phi_0}\alpha(n). \qquad (2.4.13)$$

Now recall that

$$\hat{X}_{\sigma^{-k}\omega,k,\varepsilon_N} := \mathbb{1}_{\bigcap_{j=0}^k T^{-j}_{\sigma^{-k}\omega}(\mathcal{J}_{\sigma^{j-k}\omega,\varepsilon_N})}.$$

Using (C2), we can write (2.4.11) as

$$\begin{aligned}
&\sum_{k=0}^{n-1}\left|\nu_{\sigma^n\omega,0}\left(\left(\tilde{\mathcal{L}}^k_{\sigma^{n-k}\omega,\varepsilon_N}\right)\left(\tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,0} - \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right| \\
&\quad = \sum_{k=0}^{n-1}\left|\nu_{\sigma^n\omega,0}\left(\left(\lambda^k_{\sigma^{n-k}\omega,\varepsilon_N}\right)^{-1}\mathcal{L}^k_{\sigma^{n-k}\omega,0}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N}\cdot\left(\tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,0} - \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right)\right| \\
&\quad = \sum_{k=0}^{n-1}\frac{\lambda^k_{\sigma^{n-k}\omega,0}}{\lambda^k_{\sigma^{n-k}\omega,\varepsilon_N}}\left|\nu_{\sigma^{n-k}\omega,0}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N}\cdot\left(\tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,0} - \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right|. \qquad (2.4.14)
\end{aligned}$$

Now, since $\lambda_{\omega,0}/\lambda_{\omega,\varepsilon_N} \ge 1$ (by Remark 2.3.2) for m a.e. ω and $\varepsilon \ge 0$, and thus $\lambda^k_{\sigma^{n-k}\omega,0}/\lambda^k_{\sigma^{n-k}\omega,\varepsilon_N} \ge 1$ for each $k \ge 1$, we can write

$$\begin{aligned}
(2.4.14) &\le \frac{\lambda^n_{\omega,0}}{\lambda^n_{\omega,\varepsilon_N}}\sum_{k=0}^{n-1}\left|\nu_{\sigma^{n-k}\omega,0}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N}\cdot\left(\tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,0} - \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right| \\
&\le \left(\frac{1}{1-E_N}\right)^n\cdot\sum_{k=0}^{n-1}\left|\nu_{\sigma^{n-k}\omega,0}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N}\cdot\left(\tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,0} - \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right|, \qquad (2.4.15)
\end{aligned}$$

Using

$$\tilde{\mathcal{L}}_{\omega,0} - \tilde{\mathcal{L}}_{\omega,\varepsilon} = \tilde{\mathcal{L}}_{\omega,0} - \lambda^{-1}_{\omega,0}\mathcal{L}_{\omega,\varepsilon} + \lambda^{-1}_{\omega,0}\mathcal{L}_{\omega,\varepsilon} - \tilde{\mathcal{L}}_{\omega,\varepsilon}$$

$$
\begin{aligned}
&= \lambda_{\omega,0}^{-1}\left(\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon}\right) + (\lambda_{\omega,0}^{-1} - \lambda_{\omega,\varepsilon}^{-1})\mathcal{L}_{\omega,\varepsilon} \\
&= \lambda_{\omega,0}^{-1}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon}) + \lambda_{\omega,0}\lambda_{\omega,\varepsilon}(\lambda_{\omega,0}^{-1} - \lambda_{\omega,\varepsilon}^{-1})\tilde{\mathcal{L}}_{\omega,\varepsilon}\right) \\
&= \lambda_{\omega,0}^{-1}\left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon}) + (\lambda_{\omega,\varepsilon} - \lambda_{\omega,0})\tilde{\mathcal{L}}_{\omega,\varepsilon}\right),
\end{aligned}
$$

the fact that

$$
\mathcal{L}_{\omega,\varepsilon_N}\left((f_{\sigma\omega} \circ T_\omega) \cdot h_\omega\right) = f_{\sigma\omega} \cdot \mathcal{L}_{\omega,\varepsilon_N}(h_\omega),
$$

for all $\varepsilon_N \geq 0$ and $\omega \in \Omega$, and Remark 2.4.2, we may estimate the sum in (2.4.15) by

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \left|\nu_{\sigma^{n-k}\omega,0}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \cdot \left(\tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,0} - \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right| \\
&\quad \leq \sum_{k=0}^{n-1} \lambda_{\sigma^{n-(k+1)}\omega,0}^{-1} \left|\nu_{\sigma^{n-k}\omega,0}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \cdot \left(\mathcal{L}_{\sigma^{n-(k+1)}\omega,0} - \mathcal{L}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right| \\
&\qquad + \sum_{k=0}^{n-1} \lambda_{\sigma^{n-(k+1)}\omega,0}^{-1} \left|(\lambda_{\sigma^{n-(k+1)}\omega,\varepsilon_N} - \lambda_{\sigma^{n-(k+1)}\omega,0}) \cdot \nu_{\sigma^{n-k}\omega,0}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \cdot \tilde{\mathcal{L}}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}(\phi_{\sigma^{n-(k+1)}\omega,0})\right)\right|
\end{aligned}
$$

(2.4.16)

$$
\begin{aligned}
&\quad = \sum_{k=0}^{n-1} \lambda_{\sigma^{n-(k+1)}\omega,0}^{-1} \left|\nu_{\sigma^{n-k}\omega,0}\left(\left(\mathcal{L}_{\sigma^{n-(k+1)}\omega,0} - \mathcal{L}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)\left(\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \circ T_{\sigma^{n-(k+1)}\omega}\right) \cdot \phi_{\sigma^{n-(k+1)}\omega,0}\right)\right)\right| \\
&\qquad + \sum_{k=0}^{n-1} \left(\lambda_{\sigma^{n-(k+1)}\omega,0}^{-1} \left|1 - \frac{\lambda_{\sigma^{n-(k+1)}\omega,0}}{\lambda_{\sigma^{n-(k+1)}\omega,\varepsilon_N}}\right|\right.
\end{aligned}
$$

(2.4.17)

$$
\left. \cdot \left|\nu_{\sigma^{n-k}\omega,0}\left(\mathcal{L}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\left(\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \circ T_{\sigma^{n-(k+1)}\omega}\right) \cdot \phi_{\sigma^{n-(k+1)}\omega,0}\right)\right)\right|\right).
$$

Using (2.4.8), (2.4.9), and Remark 2.4.2 we can estimate (2.4.16) to get

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \lambda_{\sigma^{n-(k+1)}\omega,0}^{-1} \left|\nu_{\sigma^{n-k}\omega,0}\left(\left(\mathcal{L}_{\sigma^{n-(k+1)}\omega,0} - \mathcal{L}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\right)\left(\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \circ T_{\sigma^{n-(k+1)}\omega}\right) \cdot \phi_{\sigma^{n-(k+1)}\omega,0}\right)\right)\right| \\
&\quad = \sum_{k=0}^{n-1} \left|\nu_{\sigma^{n-(k+1)}\omega,0}\left(\mathbb{1}_{H_{\sigma^{n-(k+1)}\omega,\varepsilon_N}} \cdot \left(\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \circ T_{\sigma^{n-(k+1)}\omega}\right) \cdot \phi_{\sigma^{n-(k+1)}\omega,0}\right)\right)\right|
\end{aligned}
$$

(2.4.18)

$$
\leq \sum_{k=0}^{n-1} \nu_{\sigma^{n-(k+1)}\omega,0}\left(\mathbb{1}_{H_{\sigma^{n-(k+1)}\omega,\varepsilon_N}}\right) \|\phi_{\sigma^{n-(k+1)}\omega,0}\|_{\infty,\sigma^{n-(k+1)}\omega} \leq \frac{C_2 C_3(t_\omega + \xi_{\omega,N}) \cdot n}{N} \leq nE_N.
$$

Since $\mathcal{L}_{\omega,\varepsilon}(f) = \mathcal{L}_{\omega,0}(\hat{X}_{\omega,0,\varepsilon} f)$ we can rewrite the second product in the sum in (2.4.17) so that we have

$$
\nu_{\sigma^{n-k}\omega,0}\left(\mathcal{L}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\left(\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N} \circ T_{\sigma^{n-(k+1)}\omega}\right) \cdot \phi_{\sigma^{n-(k+1)}\omega,0}\right)\right)
$$

$$
\begin{aligned}
&= \nu_{\sigma^{n-k}\omega,0}\left(\mathcal{L}_{\sigma^{n-(k+1)}\omega,0}\left(\hat{X}_{\sigma^{n-(k+1)}\omega,0,\varepsilon_N}\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N}\circ T_{\sigma^{n-(k+1)}\omega}\right)\cdot\phi_{\sigma^{n-(k+1)}\omega,0}\right)\right)\\
&\le \|\mathcal{L}_{\sigma^{n-(k+1)}\omega,0}\mathbb{1}\|_{\infty,\sigma^{n-k}\omega}\|\phi_{\sigma^{n-(k+1)}\omega,0}\|_{\infty,\sigma^{n-(k+1)}\omega}\le C_1C_2.
\end{aligned}
\tag{2.4.19}
$$

Inserting (2.4.19) into (2.4.17) and using (2.4.10) yields

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\left|1-\frac{\lambda_{\sigma^{n-(k+1)}\omega,0}}{\lambda_{\sigma^{n-(k+1)}\omega,\varepsilon_N}}\right|\cdot\left|\nu_{\sigma^{n-k}\omega,0}\left(\mathcal{L}_{\sigma^{n-(k+1)}\omega,\varepsilon_N}\left(\left(\hat{X}_{\sigma^{n-k}\omega,k,\varepsilon_N}\circ T_{\sigma^{n-(k+1)}\omega}\right)\cdot\phi_{\sigma^{n-(k+1)}\omega,0}\right)\right)\right|\\
&\le\sum_{k=0}^{n-1}\frac{C_1C_2E_N}{1-E_N}=n\cdot\frac{C_1C_2E_N}{1-E_N}.
\end{aligned}
\tag{2.4.20}
$$

Thus, collecting the estimates (2.4.16)-(2.4.20) together with (2.4.13) and inserting into (2.4.15) yields

$$
|1-\nu_{\omega,\varepsilon_N}(\phi_{\omega,0})|\le\left(\frac{1}{1-E_N}\right)^n\cdot\left(nE_N+\frac{C_1C_2E_N\cdot n}{1-E_N}\right)+C_2C_{\phi_0}\alpha(n),
\tag{2.4.21}
$$

which finishes the proof of the claim. ■

To finish the proof of Lemma 2.4.4, we note that (2.4.21) holds for m-a.e. ω, every N sufficiently large, and each $n\ge 1$. Given a $\delta>0$, choose and fix n so that $C_2C_{\phi_0}\alpha(n)<\delta/2$. Because $\lim_{N\to\infty}E_N=0$, we may choose N large enough so that first summand in (2.4.21) is also smaller than $\delta/2$. Thus, $\lim_{N\to\infty}\nu_{\omega,\varepsilon_N}(\phi_{\omega,0})=1$, uniformly in ω. This proves (2.4.5) for $\nu_{\omega,\varepsilon_N}(\phi_{\omega,0})$; we immediately obtain the other inequality using (C5') and (C7'), and thus the proof of Lemma 2.4.4 is complete.

□

We now obtain a formula for the explicit form of Gumbel law for the extreme value distribution.

THEOREM 2.4.5. *Given a random open system* $(\Omega,m,\sigma,\mathcal{J}_0,T,\mathcal{B},\mathcal{L}_0,\nu_0,\phi_0,H_\varepsilon)$ *satisfying* (C1'), (C2), (C3), (C4'), (C5'), (C7'), (C8), *and* (S), *for almost every* $\omega\in\Omega$ *one has*

$$
\lim_{N\to\infty}\nu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right)=\lim_{N\to\infty}\mu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right)=\lim_{N\to\infty}\frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}}=\exp\left(-\int_\Omega t_\omega\theta_{\omega,0}\,dm(\omega)\right).
\tag{2.4.22}
$$

PROOF.

Step 1: Estimating $\lambda_{\omega,\varepsilon_N}/\lambda_{\omega,0}$. To work towards constructing an estimate for $\lambda^N_{\omega,\varepsilon_N}/\lambda^N_{\omega,0}$, we first estimate $\lambda_{\omega,\varepsilon_N}/\lambda_{\omega,0}$. For brevity, in this step we drop the N subscripts on ε. Following the proof of Theorem 2.1.2 up to equation (2.1.15), which uses assumptions (C2) and (C3), we have

$$
\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})\frac{\lambda_{\omega,0}-\lambda_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}}
$$

$$
\begin{aligned}
(2.4.23) \quad &= \underbrace{1-\sum_{k=0}^{n-1}\lambda^{-1}_{\sigma^{-(k+1)}\omega,0}(\lambda^{k}_{\sigma^{-k}\omega,\varepsilon})^{-1}q^{(k)}_{\omega,\varepsilon}}_{=:\theta_{\omega,\varepsilon,n}} \\
(2.4.24) \quad &+ \Delta^{-1}_{\omega,\varepsilon}\underbrace{\sum_{k=1}^{n}\lambda^{-1}_{\sigma^{-k}\omega,0}(\lambda_{\sigma^{-k}\omega,0}-\lambda_{\sigma^{-k}\omega,\varepsilon})\nu_{\sigma\omega,0}((\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^{k}_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0}))}_{=:\theta'_{\omega,\varepsilon,n}} \\
(2.4.25) \quad &+ \Delta^{-1}_{\omega,\varepsilon}\underbrace{\nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}-\mathcal{L}_{\omega,\varepsilon})(Q^{n}_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))}_{=:\theta''_{\omega,\varepsilon,n}}.
\end{aligned}
$$

By first rearranging to solve for $\lambda_{\omega,\varepsilon}$ we have

$$
\lambda_{\omega,\varepsilon} = \lambda_{\omega,0}-\frac{\theta_{\omega,\varepsilon,n}\Delta_{\omega,\varepsilon}+\theta'_{\omega,\varepsilon,n}+\theta''_{\omega,\varepsilon,n}}{\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})},
$$

and thus

$$
(2.4.26) \quad \frac{\lambda_{\omega,\varepsilon}}{\lambda_{\omega,0}} = 1-\underbrace{\frac{\theta_{\omega,\varepsilon,n}\Delta_{\omega,\varepsilon}}{\lambda_{\omega,0}\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})}}_{=:Y^{(1)}_{\omega,\varepsilon,n}}-\underbrace{\frac{\theta'_{\omega,\varepsilon,n}}{\lambda_{\omega,0}\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})}}_{Y^{(2)}_{\omega,\varepsilon,n}}-\underbrace{\frac{\theta''_{\omega,\varepsilon,n}}{\lambda_{\omega,0}\nu_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0})}}_{=:Y^{(3)}_{\omega,\varepsilon,n}}.
$$

Setting $Y_{\omega,\varepsilon,n} := Y^{(1)}_{\omega,\varepsilon,n}+Y^{(2)}_{\omega,\varepsilon,n}+Y^{(3)}_{\omega,\varepsilon,n}$ applying Taylor to $\log(1-\cdot)$, we obtain

$$
(2.4.27) \quad \frac{\lambda_{\omega,\varepsilon}}{\lambda_{\omega,0}} = \exp\left(-Y_{\omega,\varepsilon,n}-\frac{Y^2_{\omega,\varepsilon,n}}{2(1-y)^2}\right),
$$

where $0 \le y \le Y_{\omega,\varepsilon,n}$. Setting $\varepsilon = \varepsilon_N$ in (2.4.27) we obtain

$$
\begin{aligned}
(2.4.28) \quad \frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} &= \exp\left(-\sum_{i=0}^{N-1}Y_{\sigma^i\omega,\varepsilon_N,n}-\sum_{i=0}^{N-1}\frac{Y^2_{\sigma^i\omega,\varepsilon_N,n}}{2(1-y)^2}\right) \\
(2.4.29) \quad &= \exp\left(-\frac{1}{N}\sum_{i=0}^{N-1}\left(g^{(1)}_{N,n}(\sigma^i\omega)+g^{(2)}_{N,n}(\sigma^i\omega)+g^{(3)}_{N,n}(\sigma^i\omega)\right)-\sum_{i=0}^{N-1}\frac{Y^2_{\sigma^i\omega,\varepsilon_N,n}}{2(1-y)^2}\right),
\end{aligned}
$$

where $g^{(j)}_{N,n}(\omega) = NY^{(j)}_{\omega,\varepsilon_N,n}$ for $j=1,2,3$ and $0 \le y \le Y_{\omega,\varepsilon_N,n}$.

Step 2: A non-standard ergodic lemma. In preparation for estimating the products along orbits contained in $\lambda^N_{\omega,\varepsilon_N}/\lambda^N_{\omega,0}$, we state and prove a non-standard ergodic lemma.

LEMMA 2.4.6. *For $N \ge 0$, let $g_N \in L^1(m)$. Suppose that as $N \to \infty$, $g_N \to g$ m-almost everywhere for some $g \in L^1(m)$ and that $\lim_{N\to\infty}\int_\Omega |g_N - g|\,dm = 0$. Then for m-a.e. $\omega \in \Omega$, $\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} g_N(\sigma^i\omega)$ exists and equals $\mathbb{E}(g) = \int_\Omega g\,dm$.*

PROOF. We write

$$
\left|\frac{1}{N}\sum_{i=0}^{N-1}g_N(\sigma^i\omega)-\mathbb{E}(g)\right| \le \frac{1}{N}\sum_{i=0}^{N-1}|g_N(\sigma^i\omega)-g(\sigma^i\omega)|+\left|\frac{1}{N}\sum_{i=0}^{N-1}g(\sigma^i\omega)-\mathbb{E}(g)\right|.
$$

First, we note that the Birkhoff Ergodic Theorem implies that

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{i=0}^{N-1} g(\sigma^i\omega)-\mathbb{E}(g)\right|=0.$$

Now, to deal with the remaining term, since $g_N \to g$ almost everywhere, given $\delta>0$, we let $N_\delta\in\mathbb{N}$ be sufficiently large such that

$$m\left(\Omega_\delta := \{\omega\in\Omega : |g(\omega)-g_N(\omega)|<\delta \text{ for all } N\geq N_\delta\}\right)>1-\delta.$$

Birkhoff applied to $\mathbb{1}_{\Omega_\delta}$ then gives that for each $\delta>0$ and all $N\geq N_\delta$ sufficiently large we have that

$$\frac{1}{N}\#\left\{0\leq k<N : \sigma^k\omega\in\Omega_\delta\right\}>1-\delta. \tag{2.4.30}$$

Since $\lim_{N\to\infty}\int_\Omega |g_N-g|\,dm(\omega)=0$, for each $\delta>0$ there exists $N'_\delta>0$ such that $\int_\Omega |g_N-g|\,dm<\delta$ for all $N\geq N'_\delta$. We apply the Birkhoff Ergodic Theorem to the functions $|g_N-g|\cdot\mathbb{1}_{\Omega_\delta}$ and $|g_N-g|\cdot\mathbb{1}_{\Omega_\delta^c}$. Using (2.4.30) gives that for m-a.e. $\omega\in\Omega$ and all $N\geq\max(N_\delta,N'_\delta)$ sufficiently large (so that the Birkhoff error is less than δ, noting that N depends on ω) we have

$$\begin{aligned}
&\frac{1}{N}\sum_{i=0}^{N-1}|g_N(\sigma^i\omega)-g(\sigma^i\omega)|\\
&\quad=\frac{1}{N}\sum_{i=0}^{N-1}|g_N(\sigma^i\omega)-g(\sigma^i\omega)|\mathbb{1}_{\Omega_\delta}(\sigma^i\omega)+\frac{1}{N}\sum_{i=0}^{N-1}|g_N(\sigma^i\omega)-g(\sigma^i\omega)|\mathbb{1}_{\Omega_\delta^c}(\sigma^i\omega)\\
&\quad<\delta+\int_{\Omega_\delta^c}|g_N-g|\,dm+\delta\leq 2\delta+\int_\Omega|g_N-g|\,dm<3\delta.
\end{aligned}$$

As this holds for every $\delta>0$, we must in fact have that

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{i=0}^{N-1} g_N(\sigma^i\omega)-\mathbb{E}(g)\right|=0$$

as desired. □

Step 3: Estimating $g^{(1)}$.

In this step we construct estimates of $g^{(1)}$ that are required to apply Lemma 2.4.6.

In preparation for the first use of Lemma 2.4.6 we recall that

$$g^{(1)}_{N,n}(\omega):=\frac{N\theta_{\omega,\varepsilon_N,n}\Delta_{\omega,\varepsilon_N}}{\lambda_{\omega,0}\nu_{\sigma^{-n}\omega,\varepsilon_N}(\phi_{\sigma^{-n}\omega,0})}=\frac{N\theta_{\omega,\varepsilon_N,n}\mu_{\omega,0}(H_{\omega,\varepsilon_N})}{\nu_{\sigma^{-n}\omega,\varepsilon_N}(\phi_{\sigma^{-n}\omega,0})}=\frac{\theta_{\omega,\varepsilon_N,n}(t_\omega+\xi_{\omega,N})}{\nu_{\sigma^{-n}\omega,\varepsilon_N}(\phi_{\sigma^{-n}\omega,0})},$$

where $\lim_{N\to\infty}\xi_{\omega,N}=0$ for a.e. ω and $|\xi_{\omega,N}|\leq W$ by (S). We also set $g^{(1)}_n(\omega):=\theta_{\omega,0,n}t_\omega$, where

$$\theta_{\omega,0,n}:=1-\sum_{k=0}^{n-1}\left(\lambda^{k+1}_{\sigma^{-(k+1)}\omega,0}\right)^{-1}q^{(k)}_{\omega,0}=1-\sum_{k=0}^{n-1}\hat{q}^{(k)}_{\omega,0}. \tag{2.4.31}$$

By Lemma 2.3.8 and (C8) we see that $0 \le \hat{q}^{(k)}_{\omega,0}, \hat{q}^{(k)}_{\omega,\varepsilon_N} \le 1$ for each k, N, and m-a.e. ω. Thus, (C8) and the Dominated Convergence Theorem imply that $\lim_{N\to\infty} \|\hat{q}^{(k)}_{\omega,\varepsilon_N} - \hat{q}^{(k)}_{\omega,0}\|_1 = 0$. From (2.4.23) we have that

$$\theta_{\omega,\varepsilon_N,n} := 1 - \sum_{k=0}^{n-1} \lambda^{-1}_{\sigma^{-(k+1)}\omega,0} (\lambda^k_{\sigma^{-k}\omega,\varepsilon_N})^{-1} q^{(k)}_{\omega,\varepsilon_N} = 1 - \sum_{k=0}^{n-1} \frac{\lambda^k_{\sigma^{-k}\omega,0}}{\lambda^k_{\sigma^{-k}\omega,\varepsilon_N}} \hat{q}^{(k)}_{\omega,\varepsilon_N}, \tag{2.4.32}$$

and thus that $\theta_{\omega,0,N} \in [0,1]$ and $\theta_{\omega,\varepsilon_N,n} \le 1$ for each n and m-a.e. ω. Using (2.4.10) and the fact that $\lambda_{\omega,0}/\lambda_{\omega,\varepsilon_N} \ge 1$ (by Remark 2.3.2), we have that

$$\theta_{\omega,\varepsilon_N,n} \ge 1 - \frac{\lambda^{n-1}_{\sigma^{-(n-1)}\omega,0}}{\lambda^{n-1}_{\sigma^{-(n-1)}\omega,\varepsilon_N}} \sum_{k=0}^{n-1} \hat{q}^{(k)}_{\omega,\varepsilon_N} \ge 1 - \left(1 + \frac{C_1 E_N}{C_1 - E_N}\right)^n.$$

For fixed n, again we apply Dominated Convergence to get that

$$\lim_{N\to\infty} \|\theta_{\omega,\varepsilon_N,n} - \theta_{\omega,0,n}\|_1 = 0. \tag{2.4.33}$$

Using Lemma 2.4.4 and (S) we see that $g^{(1)}_{N,n}, g^{(1)}_n \in L^1(m)$ for each N, n.

Referring to the first term in the Y-sum in (2.4.28), we may now apply Lemma 2.4.6 to conclude that

$$\sum_{i=0}^{N-1} \frac{\theta_{\sigma^i\omega,\varepsilon_N,n}\Delta_{\sigma^i\omega,\varepsilon_N}}{\lambda_{\sigma^i\omega,0}\nu_{\sigma^{-n+i}\omega,\varepsilon_N}(\phi_{\sigma^{-n+i}\omega,0})} = \frac{1}{N}\sum_{i=0}^{N-1} g^{(1)}_{N,n}(\sigma^i\omega) \to \int_\Omega g^{(1)}_n(\omega)\, dm(\omega) = \int_\Omega \theta_{\omega,0,n} t_\omega\, dm(\omega),$$

as $N \to \infty$ for each n and m-a.e. ω.

Step 4: Estimating $g^{(2)}$. We now perform a similar analysis to the previous step to control the terms in the sum (2.4.28) corresponding to θ'. Again in preparation for applying Lemma 2.4.6, recall that

$$g^{(2)}_{N,n}(\omega) = \frac{N\theta'_{\omega,\varepsilon_N,n}}{\lambda_{\omega,0}\nu_{\sigma^{-n}\omega,\varepsilon_N}(\phi_{\sigma^{-n}\omega,0})}$$

and set $g^{(2)}_n(\omega) \equiv 0$ for each n. Using (2.4.24), (2.4.6), (2.4.9), and (C7'), for sufficiently large N we have

$$\begin{aligned}
\theta'_{\omega,\varepsilon_N,n} &:= \sum_{k=1}^{n} \lambda^{-1}_{\sigma^{-k}\omega,0}(\lambda_{\sigma^{-k}\omega,0} - \lambda_{\sigma^{-k}\omega,\varepsilon})\nu_{\sigma\omega,0}((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon})(\phi_{\sigma^{-k}\omega,0})) \\
&\le \frac{C_2C_3\lambda_{\omega,0}}{N}\sum_{k=1}^{n} \lambda^{-1}_{\sigma^{-k}\omega,0}\left(\lambda_{\sigma^{-k}\omega,0}(t_{\sigma^{-k}\omega} + \xi_{\sigma^{-k}\omega,N})\right)\nu_{\omega,0}\left(\mathbb{1}_{H_{\omega,\varepsilon_N}}\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon_N}\phi_{\sigma^{-k}\omega,0}\right) \\
&\le \frac{C_2C_3\lambda_{\omega,0}}{N}\sum_{k=1}^{n}(t_{\sigma^{-k}\omega} + \xi_{\sigma^{-k}\omega,N})\nu_{\omega,0}(H_{\omega,\varepsilon_N})\|\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon_N}\phi_{\sigma^{-k}\omega,0}\|_{\infty,\omega} \\
&\le \frac{C_2C_3\lambda_{\omega,0}}{N}\cdot\frac{C_3(t_\omega + \xi_{\omega,N})}{N}\sum_{k=1}^{n}(t_{\sigma^{-k}\omega} + \xi_{\sigma^{-k}\omega,N})\|\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon_N}\phi_{\sigma^{-k}\omega,0}\|_{\infty,\omega}.
\end{aligned}$$

Using Lemma 2.4.4, (C5'), and (C4') we note that

$$\|\tilde{\mathcal{L}}^k_{\sigma^{-k}\omega,\varepsilon_N}\phi_{\sigma^{-k}\omega,0}\|_{\infty,\omega} = \|\nu_{\sigma^{-k}\omega,\varepsilon_N}(\phi_{\sigma^{-k}\omega,0})\phi_{\omega,\varepsilon_N} + Q^k_{\sigma^{-k}\omega,\varepsilon_N}\phi_{\sigma^{-k}\omega,0}\|_{\infty,\omega}$$

$$\begin{aligned} &\leq \|C_{\varepsilon_N} C_2 + C_{\phi_0} C_2 \alpha(k)\|_{\infty,\omega} \\ &\leq C_2 C_{\varepsilon_N} + C_2 C_{\phi_0} \alpha. \end{aligned} \tag{2.4.34}$$

Finally, we have for N sufficiently large and by (S)

$$g^{(2)}_{N,n}(\omega) \leq \frac{N C_2 C_3 \lambda_{\omega,0}(C_2 C_{\varepsilon_N} + C_2 C_{\phi_0}\alpha)}{N \lambda_{\omega,0} \nu_{\sigma^{-n}\omega,\varepsilon_N}(\phi_{\sigma^{-n}\omega,0})} \cdot \frac{C_3(t_\omega + \xi_{\omega,N})}{N} \sum_{k=1}^{n} (t_{\sigma^{-k}\omega} + \xi_{\sigma^{-k}\omega,N}).$$

Integrability of $g^{(2)}_{N,n}$ follows from the the fact that $t \in L^\infty(m)$ and $|\xi_{\omega,N}| \leq W$, for almost all ω. Moreover by (S), for each n we have that $g^{(2)}_{N,n} \to 0$ almost everywhere as $N \to \infty$. By dominated convergence we have again that $\lim_{N\to\infty} \|g^{(2)}_{N,n} - g^{(2)}_n\|_1 \to 0$ for each n.

Referring to $Y^{(2)}_{\omega,\varepsilon_N,n}$ in (2.4.26), we may now apply Lemma 2.4.6 to conclude that

$$\sum_{i=0}^{N-1} \frac{\theta'_{\sigma^i\omega,\varepsilon_N,n}}{\lambda_{\sigma^i\omega,0}\nu_{\sigma^{-n+i}\omega,\varepsilon_N}(\phi_{\sigma^{-n+i}\omega,0})} = \frac{1}{N}\sum_{i=0}^{N-1} g^{(2)}_{N,n}(\sigma^i\omega) \to \int_\Omega g^{(2)}_n(\omega)\, dm(\omega) = 0,$$

as $N \to \infty$ for each n and a.e. ω.

Step 5: Estimating $g^{(3)}$. We repeat a similar analysis to control the terms in the sum (2.4.28) corresponding to θ''. Again in preparation for applying Lemma 2.4.6, recall that

$$g^{(3)}_{N,n}(\omega) = \frac{N\theta''_{\omega,\varepsilon_N,n}}{\lambda_{\omega,0}\nu_{\sigma^{-n}\omega,\varepsilon_N}(\phi_{\sigma^{-n}\omega,0})}.$$

We begin developing an upper bound for $|g^{(3)}_{N,n}|$. Using (2.4.25), (C2), (C4'), and (C7') we have for sufficiently large N that

$$\begin{aligned} |\theta''_{\omega,\varepsilon_N,n}| &:= \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon_N}} Q^n_{\sigma^{-n}\omega,\varepsilon}(\phi_{\sigma^{-n}\omega,0}))) \\ &= \lambda_{\omega,0}\nu_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon_N}} Q^n_{\sigma^{-n}\omega,\varepsilon_N}\phi_{\sigma^{-n}\omega,0}) \\ &\leq \lambda_{\omega,0}\nu_{\omega,0}(H_{\omega,\varepsilon_N})\|Q^n_{\sigma^{-n}\omega,\varepsilon_N}\phi_{\sigma^{-n}\omega,0}\|_{\infty,\omega} \\ &\leq \lambda_{\omega,0}\nu_{\omega,0}(H_{\omega,\varepsilon_N})C_{\phi_0}\|\phi_{\sigma^{-n}\omega,0}\|_{\mathcal{B}_{\sigma^{-n}\omega}}\alpha(n) \\ &\leq \lambda_{\omega,0}C_{\phi_0}C_2C_3\alpha(n)\mu_{\omega,0}(H_{\omega,\varepsilon_N}) \\ &= \frac{\lambda_{\omega,0}C_{\phi_0}C_2C_3\alpha(n)(t_\omega + \xi_{\omega,N})}{N}. \end{aligned}$$

Therefore, using Lemma 2.4.4

$$|g^{(3)}_{N,n}(\omega)| \leq \frac{N\lambda_{\omega,0}C_{\phi_0}C_2C_3\alpha(n)(t_\omega+\xi_{\omega,N})}{N\lambda_{\omega,0}\nu_{\sigma^{-n}\omega,\varepsilon_N}(\phi_{\sigma^{-n}\omega,0}))} \leq C_{\varepsilon_N}C_{\phi_0}C_2C_3\alpha(n)(t_\omega+\xi_{\omega,N}) =: \tilde{g}^{(3)}_{N,n}(\omega).$$

We set $\tilde{g}^{(3)}_n(\omega) = C_{\phi_0}C_2C_3\alpha(n)t_\omega$. Integrability of $\tilde{g}^{(3)}_{N,n}$ and $\tilde{g}^{(3)}_n$ follows from (S) and the fact $t \in L^\infty(m)$ and $|\xi_{\omega,N}| \leq W$, for almost all ω. Similarly, (recalling that $C_{\varepsilon_N} \to 1$ as $N \to \infty$ by Lemma 2.4.4) for each n, $\tilde{g}^{(3)}_{N,n} \to \tilde{g}^{(3)}_n$ almost everywhere as $N \to \infty$. For the same reasons by (S) and dominated convergence we also have that $\lim_{N\to\infty}\|\tilde{g}^{(3)}_{N,n} - \tilde{g}^{(3)}_n\|_1 \to 0$ for each n.

Referring to $Y^{(3)}_{\omega,\varepsilon_N,n}$ in (2.4.26), we may now apply Lemma 2.4.6 to $\tilde{g}^{(3)}_{N,n}$ and $\tilde{g}^{(3)}_n$ to conclude that

$$\frac{1}{N}\sum_{i=0}^{N-1} g^{(3)}_{N,n}(\sigma^i\omega) \le \frac{1}{N}\sum_{i=0}^{N-1} \tilde{g}^{(3)}_{N,n}(\sigma^i\omega) \to \int_\Omega \tilde{g}^{(3)}_n(\omega)\, dm(\omega) = C_{\phi_0}C_2C_3\alpha(n)\int_\Omega t_\omega\, dm(\omega)$$

as $N\to\infty$ for each n and a.e. ω.

Step 6: Finishing up. Recall from (2.4.3) that

$$(2.4.1) = \frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}}\left(\nu_{\omega,\varepsilon_N}(\mathbb{1}) + \nu_{\sigma^N\omega,0}\left(Q^N_{\omega,\varepsilon_N}(\mathbb{1})\right)\right).$$

Using (C4') and Lemma 2.4.4 we see that $\lim_{N\to\infty}\left(\nu_{\omega,\varepsilon_N}(\mathbb{1}) + \nu_{\sigma^N\omega,0}\left(Q^N_{\omega,\varepsilon_N}(\mathbb{1})\right)\right) = 1$ for m-a.e. $\omega\in\Omega$. By (2.4.29) and Steps 3, 4, and 5 we see that for any n

$$\begin{aligned}(2.4.35)\quad &\lim_{N\to\infty}\exp\left(-\frac{1}{N}\sum_{i=0}^{N-1}\left(g^{(1)}_{N,n}(\sigma^i\omega)+g^{(2)}_{N,n}(\sigma^i\omega)+g^{(3)}_{N,n}(\sigma^i\omega)\right)\right)\exp\left(-\sum_{i=0}^{N-1}\frac{Y^2_{\sigma^i\omega,\varepsilon_N,n}}{2(1-y)^2}\right)\\ &\le \lim_{N\to\infty}\frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,\varepsilon_N}}\\ &\le \lim_{N\to\infty}\exp\left(-\frac{1}{N}\sum_{i=0}^{N-1}\left(g^{(1)}_{N,n}(\sigma^i\omega)+g^{(2)}_{N,n}(\sigma^i\omega)-\tilde{g}^{(3)}_{N,n}(\sigma^i\omega)\right)\right)\exp\left(-\sum_{i=0}^{N-1}\frac{Y^2_{\sigma^i\omega,\varepsilon_N,n}}{2(1-y)^2}\right),\end{aligned}$$

where $0\le y\le Y_{\sigma^i\omega,\varepsilon_N,n}$. We now treat the Taylor remainder terms. From Steps 3, 4, and 5 for all N sufficiently large we have and for almost all ω:

$$(2.4.36)\qquad |g^{(1)}_{N,n}| \le C_{\varepsilon_N}(|t|_\infty + W),$$

$$(2.4.37)\qquad |g^{(2)}_{N,n}| \le C_{\varepsilon_N}C_2C_3(C_2C_{\varepsilon_N}+C_2C_{\phi_0}\alpha)\cdot\frac{C_3n(|t|_\infty+W)^2}{N},$$

$$(2.4.38)\qquad |g^{(3)}_{N,n}| \le \tilde{g}^{(3)}_{N,n} := C_{\phi_0}C_2C_3\alpha(|t|_\infty+W).$$

Further,

$$(2.4.39)\qquad \sum_{i=0}^{N-1}\frac{Y^2_{\sigma^i\omega,\varepsilon_N,n}}{2(1-y)^2} \le \sum_{i=0}^{N-1}\frac{Y^2_{\sigma^i\omega,\varepsilon_N,n}}{2(1-Y_{\sigma^i\omega,\varepsilon_N,n})^2} = \sum_{i=0}^{N-1}\frac{(G_{\sigma^i\omega,\varepsilon_N,n}/N)^2}{2(1-(G_{\sigma^i\omega,\varepsilon_N,n}/N))^2},$$

where $G_{\omega,\varepsilon_N,n} := g^{(1)}_{N,n}+g^{(2)}_{N,n}+g^{(3)}_{N,n}$. Using the bounds (2.4.36)–(2.4.38) we see that (2.4.39) approaches 0 for almost all ω for each n as $N\to\infty$. Therefore, combining the expressions developed in Steps 3, 4, and 5 for the $N\to\infty$ limits with (2.4.35) we have

$$\exp\left(-\int_\Omega \theta_{\omega,0,n}t_\omega\, dm(\omega)\right) \le \lim_{N\to\infty}\frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} \le \exp\left(-\int_\Omega \theta_{\omega,0,n}t_\omega\, dm(\omega) + C_2C_3C_{\phi_0}\alpha(n)\int_\Omega t_\omega\, dm(\omega)\right).$$

Recalling the definitions of $\theta_{\omega,0,n}$ (2.4.31) and $\theta_{\omega,0}$ (2.1.20) and the fact that $0\le\theta_{\omega,0,n}\le 1$ (by (2.3.11)), we may use dominated convergence to take the $n\to\infty$ limit to obtain

$$\exp\left(-\int_\Omega \theta_{\omega,0}t_\omega\, dm(\omega)\right) \le \lim_{N\to\infty}\frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} \le \exp\left(-\int_\Omega \theta_{\omega,0}t_\omega\, dm(\omega)\right),$$

thus completing the proof that

$$\lim_{N\to\infty} \nu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right) = \lim_{N\to\infty} \frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} = \exp\left(-\int_\Omega t_\omega \theta_{\omega,0}\, dm(\omega)\right).$$

To see that $\lim_{N\to\infty} \mu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right)$ is also equal to this value, we simply recall that (2.4.4) gives that

$$\mu_{\omega,0}(X_{\omega,N-1,\varepsilon_N}) = \frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} \left(\nu_{\omega,\varepsilon_N}(\phi_{\omega,0}) + \nu_{\sigma^N\omega,0}\left(Q^N_{\omega,\varepsilon_N}(\phi_{\omega,0})\right)\right).$$

Now since (C4') and Lemma 2.4.4 together give that

$$\lim_{N\to\infty} \left(\nu_{\omega,\varepsilon_N}(\phi_{\omega,0}) + \nu_{\sigma^N\omega,0}\left(Q^N_{\omega,\varepsilon_N}(\phi_{\omega,0})\right)\right) = 1$$

for m-a.e. $\omega \in \Omega$, we must in fact have that

$$\lim_{N\to\infty} \mu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right) = \lim_{N\to\infty} \frac{\lambda^N_{\omega,\varepsilon_N}}{\lambda^N_{\omega,0}} = \exp\left(-\int_\Omega t_\omega \theta_{\omega,0}\, dm(\omega)\right),$$

which completes the proof of Theorem 2.4.5.

□

2.4.2. The relationship between condition (0.0.5) and the Hüsler condition. We now return to the discussion initiated in the Introduction to compare our assumption (0.0.5) for the thresholds $z_{\omega,N}$ with the Hüsler type condition (0.0.4). We show that in the more general situation considered in our paper with random boundary level t_ω, the limit (0.0.4) will follow from the simpler assumption (0.0.5), provided we replace t in (0.0.4) with the expectation of t_ω.

Recall our assumption for the choice of the thresholds (S) is $\mu_{\omega,0}(h_\omega(x) > z_{\omega,N}) = (t_\omega + \xi_{\omega,N})/N$, where $\xi_{\omega,N}$ goes to zero almost surely when $N \to \infty$ and $\xi_{\omega,N} \le W$ for a.e. ω. It is immediate to see by dominated convergence that:

$$\lim_{N\to\infty} \int_\Omega |N\mu_{\omega,0}(h_\omega(x) > z_{\omega,N}) - t_\omega|\, dm(\omega) = 0. \tag{2.4.40}$$

Applying our non-standard ergodic Lemma 2.4.6 with $g_N(\omega) := N\mu_{\omega,0}(h_\omega(x) > z_{\omega,N})$ and $g(\omega) := t_\omega$, one may transform the sum in (0.0.4) as follows:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} N\mu_{\omega,0}(h_{\sigma^j\omega}(T^j_\omega(x)) > z_{\sigma^j\omega,N}) = \int_\Omega t_\omega\, dm(\omega),$$

which is the condition (0.0.4) with t replaced by the expectation of t_ω.

2.4.3. Hitting time statistics. It is well known that in the deterministic setting there is a close relationship between extreme value theory and the statistics of first hitting time, see for instance [**36, 55**]. We now show how our Theorem 2.4.5, with a slight modification, can be interpreted in terms of a suitable definition of (quenched) first hitting time distribution. Let us consider as in the previous sections, a sequence of small random holes $\mathcal{H}_{\omega,N} := \{H_{\sigma^j\omega,\varepsilon_N}\}_{j\ge 0}$, and define the first random hitting time as

$$\tau_{\omega,\mathcal{H}_{\omega,N}}(x) = \inf\{k \ge 1, T^k_\omega(x) \in H_{\sigma^k\omega,\varepsilon_N}\}.$$

We recall that the usual statistics of hitting times is written in the form $\mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}} > t\right)$, for nonnegative values of t. Since the sets $H_{\sigma^j\omega,\varepsilon_N}$ have measure tending to zero when $N \to \infty$, and therefore the first hitting times could eventually grow to infinity, one needs a rescaling in order to get a meaningful limit distribution. This is achieved in the next Proposition. In our current setting, condition (S) reads: $\mu_{\omega,0}(H_{\omega,\varepsilon_N}) = \frac{t_\omega+\xi_{\omega,N}}{N}$, with $\lim_{N\to\infty}\xi_{\omega,N} = 0$ for a.e. ω and $|\xi_{\omega,N}| \le W$ for a.e. ω and all $N \ge 1$.

PROPOSITION 2.4.7. *If our random open system $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ satisfies the assumptions of Theorem 2.4.5 with the sequence H_{ω,ε_N} verifying condition (S), then the first random hitting time satisfies the limit, for ω m-a.e.*

$$\lim_{N\to\infty} \mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega\right) = \exp\left(-\int_\Omega t_\omega\theta_{\omega,0}dm\right). \tag{2.4.41}$$

PROOF. For $N \ge 1$ the event,

$$\{\tau_{\omega,\mathcal{H}_{\omega,N}} > N\} = \{x \in I; T_\omega(x) \in H^c_{\sigma\omega,\varepsilon_N}, \dots, T^N_\omega(x) \in H^c_{\sigma^N\omega,\varepsilon_N}\} \tag{2.4.42}$$

is also equal to

$$T^{-1}_\omega\left(x \in I; x \in H^c_{\sigma\omega,\varepsilon_N}, T_{\sigma\omega}(x) \in H^c_{\sigma^2\omega,\varepsilon_N}, \dots, T^{N-1}_{\sigma\omega}(x) \in H^c_{\sigma^N\omega,\varepsilon_N}\right).$$

Then, by equivariance of μ_0 we obtain the link between the statistics of hitting time and extreme value theory:

$$\begin{aligned}
\mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}} > N\right) &= \mu_{\omega,0}\left(T^{-1}_\omega\left(x \in I : x \in H^c_{\sigma\omega,\varepsilon_N}, T_{\sigma\omega}(x) \in H^c_{\sigma^2\omega,\varepsilon_N}, \dots, T^{N-1}_{\sigma\omega}(x) \in H^c_{\sigma^N\omega,\varepsilon_N}\right)\right) \qquad (2.4.43)\\
&= \mu_{\sigma\omega,0}\left(x \in H^c_{\sigma\omega,\varepsilon_N}, T_{\sigma\omega}(x) \in H^c_{\sigma^2\omega,\varepsilon_N}, \dots, T^{N-1}_{\sigma\omega}(x) \in H^c_{\sigma^N\omega,\varepsilon_N}\right)\\
&= \mu_{\sigma\omega,0}(X_{\sigma\omega,N-1,\varepsilon_N}). \qquad (2.4.44)
\end{aligned}$$

In order to rescale the eventually growing first random hitting times, we invoke the condition (S); by substituting $N = (t_\omega + \xi_{\omega,N})/\mu_{\omega,0}(H_{\omega,\varepsilon_N})$ in the LHS of (2.4.43) we have

$$\mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}} > N\right) = \mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega + \xi_{\omega,N}\right). \tag{2.4.45}$$

Our final preparation before applying Theorem 2.4.5 is to show that

$$|\mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega + \xi_{\omega,N}\right) - \mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega\right)| \to 0,\ N \to \infty. \tag{2.4.46}$$

Since by a standard trick, see for instance eq. 5.3.6 in [55],

$$\{\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega\}\backslash\{\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega + \xi_{\omega,N}\} \subset \bigcup_{j=\left\lceil\frac{t_\omega}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}\right\rceil}^{\left\lceil\frac{t_\omega+\xi_{\omega,n}}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}\right\rceil} T^{-j}_\omega(H_{\sigma^j\omega,\varepsilon_N})$$

we have by equivariance

$$|\mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega + \xi_{\omega,N}\right) - \mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega\right)| \le \sum_{j=\left\lceil\frac{t_\omega}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}\right\rceil}^{\left\lceil\frac{t_\omega+\xi_{\omega,n}}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}\right\rceil} \mu_{\sigma^j\omega,0}(H_{\sigma^j\omega,N}).$$

For N large enough:

$$\sum_{j=\left\lceil \frac{t_\omega}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}\right\rceil}^{\left\lceil \frac{t_\omega+\xi_{\omega,n}}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}\right\rceil} \mu_{\sigma^j\omega,0}(H_{\sigma^j\omega,N}) \leq \left\lceil \frac{|\xi_{\omega,N}|}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}\right\rceil \frac{|t|_\infty + W}{N} \leq \left\lceil \frac{|\xi_{\omega,N}|\ N}{t_\omega - |\xi_{\omega,N}|}\right\rceil \frac{|t|_\infty + W}{N},$$

which goes to zero by (S). Recalling $t_\omega > 0$ for a.e. ω, the final expression above goes to zero as $N \to \infty$ for ω m-a.e.

Using (2.4.44)–(2.4.46) and noting that $\lim_{N\to\infty} \mu_{\sigma\omega,0}(X_{\sigma\omega,N-1,\varepsilon_N})$ is nonrandom, applying Theorem 2.4.5 yields

$$\lim_{N\to\infty} \mu_{\omega,0}\left(\tau_{\omega,\mathcal{H}_{\omega,N}}\mu_{\omega,0}(H_{\omega,\varepsilon_N}) > t_\omega\right) = \exp\left(-\int_\Omega t_\omega \theta_{\omega,0}\ dm\right). \tag{2.4.47}$$

□

2.5. Quenched thermodynamic formalism for random open interval maps via perturbation

In this section we present an explicit class of random piecewise-monotonic interval maps for which our Theorem 2.3.6, Corollary 2.3.9, and Theorem 2.4.5 apply. Using a perturbative approach, we introduce a family of small random holes parameterised by $\varepsilon > 0$ into a random closed dynamical system, and for every small ε we prove (i) the existence of a unique random conformal measure $\{\nu_{\omega,\varepsilon}\}_{\omega\in\Omega}$ with fiberwise support in $X_{\omega,\infty,\varepsilon}$ and (ii) a unique random absolutely continuous invariant measure $\{\mu_{\omega,\varepsilon}\}_{\omega\in\Omega}$ which satisfies an exponential decay of correlations and is the unique relative equilibirum state for the random open system $(\Omega, m, \sigma, \mathcal{J}_0, T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$. In addition, we prove the existence of a random absolutely continuous (with respect to $\nu_{\omega,0}$) conditionally invariant probability measure $\{\varrho_{\omega,\varepsilon}\}_{\omega\in\Omega}$ with fiberwise support in $[0,1]\backslash H_{\omega,\varepsilon}$.

We now suppose that the spaces $\mathcal{J}_{\omega,0} = [0,1]$ for each $\omega \in \Omega$ and the maps $T_\omega : [0,1] \to [0,1]$ are surjective, finitely-branched, piecewise monotone, nonsingular (with respect to Lebesgue), and that there exists $C \geq 1$ such that

$$\operatorname*{ess\,sup}_\omega |T'_\omega| \leq C \qquad \text{and} \qquad \operatorname*{ess\,sup}_\omega D(T_\omega) \leq C, \tag{E1}$$

where $D(T_\omega) := \sup_{y\in[0,1]} \#T_\omega^{-1}(y)$. We let $\mathcal{Z}_{\omega,0}$ denote the (finite) monotonicity partition of T_ω and for each $n \geq 2$ we let $\mathcal{Z}_{\omega,0}^{(n)}$ denote the partition of monotonicity of T_ω^n.

(MC) The map $\sigma : \Omega \to \Omega$ is a homeomorphism, the skew-product map $T : \Omega \times [0,1] \to \Omega \times [0,1]$ is measurable, and $\omega \mapsto T_\omega$ has countable range.

Remark 2.5.1. Under assumption (MC), the family of transfer operator cocycles $\{\mathcal{L}_{\omega,\varepsilon}\}_{\varepsilon\geq 0}$ satisfies the conditions of Theorem 17 [40] (m-continuity and σ a homeomorphism). Note that condition (MC) implies that T satisfies (M1) and the cocycle generated by $\mathcal{L}_0$ satisfies condition (M2).

Recall that the *variation* of $f : [0,1] \to \mathbb{R}_+$ on $Z \subset [0,1]$ be

$$\mathrm{var}_Z(f) = \sup_{x_0<\cdots<x_k,\, x_j \in Z} \sum_{j=0}^{k-1} |f(x_{j+1}) - f(x_j)|\,,$$

and $\mathrm{var}(f) := \mathrm{var}_{[0,1]}(f)$. We let $\mathrm{BV} = \mathrm{BV}([0,1])$ denote the set of functions on $[0,1]$ that have bounded variation. Given a non-atomic and fully supported measure ν (i.e. for any nondegenerate interval $J \subseteq [0,1]$ we have $\nu(J) > 0$) we let $\mathrm{BV}_\nu \subseteq L^\infty(\nu)$ be the set of (equivalence classes of) functions of *bounded variation* on $[0,1]$, with norm given by

$$\|f\|_{\mathrm{BV}_\nu} := \inf_{\tilde f = f \ \nu \text{ a.e.}} \mathrm{var}(\tilde f) + \nu(|f|).$$

If we require to emphasise that elements of BV_ν are equivalence classes, we denote these by $[f]_\nu$ (resp. $[f]_1$). Note that if $f \in \mathrm{BV}$ is a function of bounded variation, then it is always possible to choose a representative of minimal variation from the equivalence class $[f]_\nu$. We define $\mathrm{BV}_1 \subseteq L^\infty(\mathrm{Leb})$ and $\|\cdot\|_{\mathrm{BV}_1}$ similarly, with the measure ν replaced with Lebesgue measure. We denote the supremum norm on $L^\infty(\mathrm{Leb})$ by $\|\cdot\|_{\infty,1}$. It follows from Rychlik [60] that BV_ν and BV_1 are Banach spaces. The following proposition gives the equivalence of the norms $\|\cdot\|_{\mathrm{BV}_\nu}$ and $\|\cdot\|_{\mathrm{BV}_1}$.

PROPOSITION 2.5.2. *Given a fully supported and non-atomic measure ν on $[0,1]$ and $f \in \mathrm{BV}$ we have that*

$$(1/2)\|f\|_{\mathrm{BV}_1} \le \|f\|_{\mathrm{BV}_\nu} \le 2\|f\|_{\mathrm{BV}_1}.$$

PROOF. We first show that for $f \in \mathrm{BV}$ we have

$$[f]_\nu \cap \mathrm{BV} = [f]_1 \cap \mathrm{BV}. \tag{2.5.1}$$

To see this let $\tilde f \in [f]_\nu \cap \mathrm{BV}$. As ν is a fully supported and non-atomic measure, we must have that the set $\{x : f(x) \neq \tilde f(x)\}$ is countable. Thus $\tilde f \in [f]_1 \cap \mathrm{BV}$. As Leb is also fully supported and non-atomic the same reasoning implies that the reverse inclusion also holds, proving (2.5.1). As a direct consequence of (2.5.1) we have that

$$\inf_{\tilde f = f \ \nu \text{ a.e.}} \mathrm{var}(\tilde f) = \inf_{\tilde f = f \ \mathrm{Leb} \text{ a.e.}} \mathrm{var}(\tilde f). \tag{2.5.2}$$

Since f is continuous everywhere except on a set of at most countably many points, letting $\mathcal{C}$ denote the set of intervals of continuity for f, we have

$$\mathrm{Leb}\text{-}\operatorname*{ess\,inf}_{[0,1]} f = \inf_{J\in\mathcal{C}} \mathrm{Leb}\text{-}\operatorname*{ess\,inf}_{J} f = \inf_{J\in\mathcal{C}} \inf_{J} f = \inf_{J\in\mathcal{C}} \nu\text{-}\operatorname*{ess\,inf}_{J} f = \nu\text{-}\operatorname*{ess\,inf}_{[0,1]} f. \tag{2.5.3}$$

Using similar reasoning we must also have

$$\mathrm{Leb}\text{-}\operatorname{ess\,sup} f = \nu\text{-}\operatorname{ess\,sup} f. \tag{2.5.4}$$

Combining (2.5.2) and (2.5.3) we have

$$\begin{aligned}\|f\|_{\mathrm{BV}_1} = \inf_{\tilde f = f \ \mathrm{Leb} \ a.e.} \mathrm{var}(\tilde f) + \mathrm{Leb}(|f|) &\le 2 \inf_{\tilde f = f \ \mathrm{Leb} \ a.e.} \mathrm{var}(\tilde f) + \mathrm{Leb}\text{-}\operatorname{ess\,inf} |f| \\ &= 2 \inf_{\tilde f = f \ \nu \ a.e.} \mathrm{var}(\tilde f) + \nu\text{-}\operatorname{ess\,inf} |f|\end{aligned}$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

$$\leq 2 \inf_{\tilde{f}=f \ \nu \ a.e.} \operatorname{var}(\tilde{f}) + \nu(|f|) \leq 2\|f\|_{\mathrm{BV}_\nu}.$$

Similarly, using (2.5.2) and (2.5.4) we have $\|f\|_{\mathrm{BV}_\nu} \leq 2\|f\|_{\mathrm{BV}_1}$, and thus the proof is complete.

□

Proposition 2.5.2 will be used later to provide (non-random) equivalence of $\|\cdot\|_{\mathrm{BV}_{\nu_{\omega,0}}}$ and $\|\cdot\|_{\mathrm{BV}_1}$ for each $\omega \in \Omega$. We set $J_\omega := |T'_\omega|$ and define the random Perron–Frobenius operator, acting on functions in BV

$$P_\omega(f)(x) := \sum_{y \in T_\omega^{-1}(x)} \frac{f(y)}{J_\omega(y)}.$$

The operator P satisfies the well-known property that

$$\int_{[0,1]} P_\omega(f)\, d\mathrm{Leb} = \int_{[0,1]} f\, d\mathrm{Leb} \tag{2.5.5}$$

for m-a.e. $\omega \in \Omega$ and all $f \in \mathrm{BV}$. Recall from Section 2.2 that $g_0 = \{g_{\omega,0}\}_{\omega\in\Omega}$ and that

$$\mathcal{L}_{\omega,0}(f)(x) := \sum_{y \in T_\omega^{-1}(x)} g_{\omega,0}(y) f(y), \quad f \in \mathrm{BV}.$$

We assume that the weight function $g_{\omega,0}$ lies in BV for each $\omega \in \Omega$ and satisfies

$$\operatorname{ess\,sup}_\omega \|g_{\omega,0}\|_{\infty,1} < \infty, \tag{E2}$$

and

$$\operatorname{ess\,inf}_\omega \inf g_{\omega,0} > 0. \tag{E3}$$

Note that (E1) and (E2) together imply

$$\operatorname{ess\,sup}_\omega \|\mathcal{L}_{\omega,0}\mathbb{1}\|_{\infty,1} \leq \operatorname{ess\,sup}_\omega D(T_\omega)\|g_{\omega,0}\|_{\infty,1} < \infty \tag{2.5.6}$$

and

$$\operatorname{ess\,sup}_\omega \|g_{\omega,0} J_\omega\|_{\infty,1} < \infty. \tag{2.5.7}$$

We also assume a uniform covering condition[2] :

(E4) For every subinterval $J \subset [0,1]$ there is a $k = k(J)$ such that for a.e. ω one has $T_\omega^k(J) = [0,1]$.

Concerning the open system we assume that the holes $H_{\omega,\varepsilon} \subseteq [0,1]$ are chosen so that assumption (A) holds. We also assume for each $\omega \in \Omega$ and each $\varepsilon > 0$ that $H_{\omega,\varepsilon}$ is composed of a finite union of intervals such that

(E5) There is a uniform-in-ε and uniform-in-ω upper bound on the number of connected components of $H_{\omega,\varepsilon}$,

[2]We could replace the covering condition with the assumption of a strongly contracting potential. See [2] for details.

and

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{\omega} \operatorname{Leb}(H_{\omega,\varepsilon}) = 0, \tag{E6}$$

and

(EX) There exists an $\varepsilon > 0$ and an open neighborhood $\tilde{H}_{\omega,\varepsilon} \supseteq H_{\omega,\varepsilon}$ such that $T_\omega(U_\omega) \supseteq \tilde{H}^c_{\sigma\omega,\varepsilon}$, where $U_\omega := \cup_{Z\in\mathcal{Z}_{\omega,0}} \overline{Z}_\varepsilon$ and $\overline{Z_\varepsilon}$ denotes the closure of $Z_\varepsilon \in A_{\omega,\varepsilon} := \{Z \cap \tilde{H}^c_{\omega,\varepsilon} : Z \in \mathcal{Z}_{\omega,0}\}$ with $m(\{\omega \in \Omega : \#A_{\omega,\varepsilon} \geq 2\}) > 0$.

Remark 2.5.3. Assumption (EX) is satisfied for any random open system such that each map contains at least two intervals of monotonicity, the holes are contained in the interior of exactly one interval of monotonicity, and the image of the complement of the hole is the full interval, i.e. $T_\omega(H^c_{\omega,\varepsilon}) = [0,1]$. In particular, (EX) is satisfied if there exists a full branch outside of the hole.

Recall that condition (X) states

(X) For m-a.e. $\omega \in \Omega$ we have $X_{\omega,\infty,\varepsilon} \neq \emptyset$.

Remark 2.5.4. Note that since (A) (i.e. $H_{\varepsilon'} \subseteq H_\varepsilon$ for $\varepsilon' < \varepsilon$) implies that $X_{\omega,\infty,\varepsilon'} \supseteq X_{\omega,\infty,\varepsilon}$ for all $\varepsilon' < \varepsilon$, (X) holds if there exists $\varepsilon > 0$ such that $X_{\omega,\infty,\varepsilon} \neq \emptyset$ for m-a.e. $\omega \in \Omega$. Furthermore, since $T_\omega(X_{\omega,\infty,\varepsilon}) \subseteq X_{\sigma\omega,\infty,\varepsilon}$, if $X_{\omega,\infty,\varepsilon} \neq \emptyset$ then $X_{\sigma^N\omega,\infty,\varepsilon'} \neq \emptyset$ for each $N \geq 1$ and $\varepsilon' \leq \varepsilon$. As $\mathcal{X}_{\infty,\varepsilon}$ is forward invariant we have that $X_{\omega,\infty,\varepsilon} \neq \emptyset$ not only implies that $\mathcal{X}_{\infty,\varepsilon} \neq \emptyset$, but also that $\mathcal{X}_{\infty,\varepsilon}$ is infinite.

The following proposition ensures that condition (X) holds.

Proposition 2.5.5. *The assumption* (EX) *implies* (X).

Proof. In light of Remark 2.5.4, to show that (X) is satisfied, it suffices to show that there is some $\varepsilon > 0$ such that $X_{\omega,\infty,\varepsilon} \neq \emptyset$ for m-a.e. $\omega \in \Omega$.

Let $\tilde{T}_{\omega,Z}$ denote the continuous extension of T_ω onto $\overline{Z}_\varepsilon$ for each $Z_\varepsilon \in A_{\omega,\varepsilon}$, and let $\tilde{X}_{\omega,\infty,\varepsilon}$ denote the survivor set for the open system consisting of the maps $\tilde{T}_\omega$ and holes $\tilde{H}_{\omega,\varepsilon}$. By Proposition 0.1.5 (taking $V_\omega = [0,1]\backslash\tilde{H}_{\omega,\varepsilon}$ and $U_{\omega,j} = \overline{Z}_\varepsilon$ for each $1 \leq j \leq \#A_{\omega,\varepsilon}$) we see that $\tilde{X}_{\omega,\infty,\varepsilon}$ is uncountable. Let

$$\mathcal{D}_\omega := \bigcup_{j\geq 0} \tilde{T}^{-j}_\omega(\cup_{Z_\varepsilon\in A_{\sigma^j\omega,\varepsilon}} \overline{Z}_\varepsilon \backslash Z_\varepsilon).$$

Since the survivor set for the original (unmodified) open system $X_{\omega,\infty,\varepsilon} \subseteq \tilde{X}_{\omega,\infty,\varepsilon}\backslash\mathcal{D}_\omega$, and since $\mathcal{D}_\omega$ is at most countable, we must in fact have that $X_{\omega,\infty,\varepsilon} \neq \emptyset$, thus satisfying (X). □

Further, we suppose that for m-a.e. $\omega \in \Omega$ and all $\varepsilon > 0$ sufficiently small

$$T_\omega(\mathcal{J}_{\omega,\varepsilon}) = [0,1] \tag{E7}$$

and there exists $n' \geq 1$ and $\varepsilon_0 > 0$ such that [3]

$$9 \cdot \operatorname*{ess\,sup}_{\omega} \|g^{(n')}_{\omega,0}\|_{\infty,1} < \operatorname*{ess\,inf}_{\omega} \inf_{0\leq\varepsilon\leq\varepsilon_0} \inf \mathcal{L}^{n'}_{\omega,\varepsilon}\mathbb{1}, \tag{E8}$$

[3]Note that the 9 appearing in (E8) is not optimal. See Section 1.15 and [3] for how this assumption may be improved.

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

where

$$\mathcal{L}_{\omega,\varepsilon}(f)(x) := \mathcal{L}_{\omega,0}(\mathbb{1}_{H^c_{\omega,\varepsilon}} f)(x) = \sum_{y\in T_\omega^{-1}(x)} g_{\omega,\varepsilon}(y)f(y), \quad f \in \mathrm{BV}$$

and $g_{\omega,\varepsilon} := g_{\omega,0}\mathbb{1}_{H^c_{\omega,\varepsilon}}$ as in Section 0.1.1.

Note that (E7) and (E3) together imply that $\mathcal{L}_{\omega,\varepsilon}\mathbb{1}(x) > 0$ for all $x \in [0,1]$:

$$\operatorname*{ess\,inf}_{\omega} \inf_{\varepsilon\le\varepsilon_0} \inf \mathcal{L}_{\omega,\varepsilon}\mathbb{1} > \operatorname*{ess\,inf}_{\omega} \inf g_{\omega,0} > 0, \tag{2.5.8}$$

and since $\|g_{\omega,\varepsilon}\|_{\infty,1} \le \|g_{\omega,0}\|_{\infty,1}$ for all $\varepsilon > 0$, (E8) is equivalent to the following

$$9 \cdot \sup_{0\le\varepsilon\le\varepsilon_0} \operatorname*{ess\,sup}_{\omega} \|g^{(n')}_{\omega,\varepsilon}\|_{\infty,1} < \operatorname*{ess\,inf}_{\omega} \inf_{0\le\varepsilon\le\varepsilon_0} \inf \mathcal{L}^{n'}_{\omega,\varepsilon}\mathbb{1}. \tag{2.5.9}$$

Remark 2.5.6. Note that the assumption (E7) is equivalent to there existing $N' \ge 1$ such that for m-a.e. $\omega \in \Omega$ and all $\varepsilon \ge 0$ sufficiently small

$$T^{N'}_\omega(X_{\omega,N'-1,\varepsilon}) = [0,1].$$

Indeed, since the surviving sets are forward invariant (0.1.8), we have that $T^{N'-1}_\omega(X_{\omega,N'-1,\varepsilon}) \subseteq X_{\sigma^{N'-1}\omega,0,\varepsilon} = \mathcal{J}_{\sigma^{N'-1}\omega,\varepsilon}$, and thus,

$$[0,1] = T^{N'}_\omega(X_{\omega,N'-1,\varepsilon}) \supseteq T_{\sigma^{N'-1}\omega}(\mathcal{J}_{\sigma^{N'-1}\omega,\varepsilon}).$$

For each $n \in \mathbb{N}$ and $\omega \in \Omega$ we let $\mathscr{A}^{(n)}_{\omega,0}$ be the collection of all finite partitions of $[0,1]$ such that

$$\mathrm{var}_{A_i}(g^{(n)}_{\omega,0}) \le 2\|g^{(n)}_{\omega,0}\|_{\infty,1} \tag{2.5.10}$$

for each $\mathcal{A} = \{A_i\} \in \mathscr{A}^{(n)}_{\omega,0}$. Given $\mathcal{A} \in \mathscr{A}^{(n)}_{\omega,0}$, let $\widehat{\mathcal{Z}}^{(n)}_{\omega,\varepsilon}(\mathcal{A})$ be the coarsest partition amongst all those finer than $\mathcal{A}$ and $\mathcal{Z}^{(n)}_{\omega,0}$ such that all elements of $\widehat{\mathcal{Z}}^{(n)}_{\omega,\varepsilon}(\mathcal{A})$ are either disjoint from $X_{\omega,n-1,\varepsilon}$ or contained in $X_{\omega,n-1,\varepsilon}$.

Remark 2.5.7. Note that if $\mathrm{var}_Z(g_{\omega,0}) \le 2\|g^{(n)}_{\omega,0}\|_{\infty,1}$ for each $Z \in \mathcal{Z}^{(n)}_{\omega,0}$ then we can take the partition $\mathcal{A} = \mathcal{Z}^{(n)}_{\omega,0}$. Furthermore, the 2 above can be replaced by some $\hat{\alpha} \ge 0$ (depending on $g_{\omega,0}$) following the techniques of Section 1.15 and [3].

Define the subcollection

$$\mathcal{Z}^{(n)}_{\omega,*,\varepsilon} := \{Z \in \widehat{\mathcal{Z}}^{(n)}_{\omega,\varepsilon}(\mathcal{A}) : Z \subseteq X_{\omega,n-1,\varepsilon}\}. \tag{2.5.11}$$

Recalling that $g^{(n)}_{\omega,\varepsilon} := g^{(n)}_{\omega,0}\mathbb{1}_{X_{\omega,n-1,\varepsilon}}$, (2.5.10) implies that

$$\mathrm{var}_Z(g^{(n)}_{\omega,\varepsilon}) \le 2\|g^{(n)}_{\omega,0}\|_{\infty,1} \tag{2.5.12}$$

for each $Z \in \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}$. We assume the following covering condition for the open system

(E9) There exists $k_o(n') \in \mathbb{N}$ such that for m-a.e. $\omega \in \Omega$, all $\varepsilon > 0$ sufficiently small, and all $Z \in \mathcal{Z}^{(n')}_{\omega,*,\varepsilon}$ we have $T^{k_o(n')}_\omega(Z) = [0,1]$, where n' is the number coming from (E8).

Remark 2.5.8. Note that the uniform open covering time assumption (E9) clearly holds if (E4) holds and if there are only finitely many maps T_ω. In Remark 2.C.2 we present an alternative assumption to (E9).

The following lemma extends several results in [33] from the specific weight $g_{\omega,0} = 1/|T'_\omega|$ to general weights satisfying the conditions just outlined.

LEMMA 2.5.9. *Assume that a family of random piecewise-monotonic interval maps $\{T_\omega\}$ satisfies (E2), (E3), and (E4), as well as (E8) and (MC) for $\varepsilon = 0$. Then (C1') and the $\varepsilon = 0$ parts of (C2), (C3), (C4'), (C5'), and (C7') as well as (CCM) hold. Further, $\nu_{\omega,0}$ is fully supported, condition (C4') holds with $C_f = K$, for some $K < \infty$, and with $\alpha(N) = \gamma^N$ for some $\gamma < 1$.*

PROOF. See Appendix 2.A. □

In what follows we consider transfer operators acting on the Banach spaces $\mathcal{B}_\omega = \mathrm{BV}_{\nu_{\omega,0}}$ for a.e. $\omega \in \Omega$. The norm we will use is $\|\cdot\|_{\mathcal{B}_\omega} = \|\cdot\|_{\mathrm{BV}_{\nu_{\omega,0}}} := \mathrm{var}(\cdot) + \nu_{\omega,0}(|\cdot|)$. As $\nu_{\omega,0}$ is fully supported and non-atomic (Lemma 2.5.9), Proposition 2.5.2 implies that

$$(1/2)\|f\|_{\mathrm{BV}_1} \le \|f\|_{\mathcal{B}_\omega} \le 2\|f\|_{\mathrm{BV}_1} \tag{2.5.13}$$

for m-a.e. $\omega \in \Omega$ and $f \in \mathrm{BV}$. Furthermore, applying (2.5.13) twice, we see that

$$(1/4)\|f\|_{\mathcal{B}_\omega} \le \|f\|_{\mathcal{B}_{\sigma^n\omega}} \le 4\|f\|_{\mathcal{B}_\omega} \tag{2.5.14}$$

for m-a.e. $\omega \in \Omega$ and all $n \in \mathbb{Z}$. It follows from the proof of Proposition 2.5.2 that

$$\|f\|_{\infty,\omega} = \|f\|_{\infty,1} \tag{2.5.15}$$

for all $f \in \mathrm{BV}$ and m-a.e. $\omega \in \Omega$, where $\|\cdot\|_{\infty,\omega}$ denotes the supremum norm with respect to $\nu_{\omega,0}$. From (2.5.13) we see that (B) is clearly satisfied.

From Lemma 2.5.9 we have that $\lambda_{\omega,0} := \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}\mathbb{1})$ and thus we may update (2.5.8) to get

$$\operatorname*{ess\,inf}_\omega \lambda^{n'}_{\omega,0} \ge \operatorname*{ess\,inf}_\omega \inf_{\varepsilon\le\varepsilon_0} \inf \mathcal{L}^{n'}_{\omega,\varepsilon}\mathbb{1} \ge \operatorname*{ess\,inf}_\omega \inf g^{(n')}_{\omega,0} > 0. \tag{2.5.16}$$

Note that since the conditions (B), (X), and (CCM) have been verified and we have assumed (A) and (MC), we see that $(\Omega, m, \sigma, [0,1], T, \mathrm{BV}, \mathcal{L}_0, \phi_0, H_\varepsilon)$ forms a random open system as defined in Section 0.1.1 for all $\varepsilon > 0$ sufficiently small. We now use hyperbolicity of the $\varepsilon = 0$ transfer operator cocycle to guarantee that we have hyperbolic cocycles for small $\varepsilon > 0$, which will yield (C2), (C3), (C4'), (C5'), (C6), and (C7') for small positive ε.

LEMMA 2.5.10. *Assume that the conditions (E1)–(E9) hold for the random open system $(\Omega, m, \sigma, [0,1], T, \mathrm{BV}_1, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$. Then for sufficiently small $\varepsilon > 0$, conditions (C2), (C3), (C4'), (C5'), (C6), and (C7') hold. Furthermore, the functionals $\nu_{\omega,\varepsilon} \in \mathrm{BV}_1^*$ can be identified with non-atomic Borel measures.*

PROOF. For each ω and $\varepsilon > 0$ we define $\hat{\mathcal{L}}_{\omega,\varepsilon} := \lambda^{-1}_{\omega,0}\mathcal{L}_{\omega,\varepsilon}$; note that $\hat{\mathcal{L}}_{\omega,0} = \tilde{\mathcal{L}}_{\omega,0}$. Our strategy is to apply Theorem 4.8 [22], to conclude that for small ε the cocycles $\{\hat{\mathcal{L}}_{\omega,\varepsilon}\}$ are uniformly hyperbolic when considered as cocycles on the Banach space $(\mathrm{BV}_1, \|\cdot\|_{\mathrm{BV}_1})$. Because of (2.5.13), we will conclude the existence of a uniformly hyperbolic splitting in $\|\cdot\|_{\mathcal{B}_\omega}$ for a.e. ω.

First, we note that Theorem 4.8 [22] assumes that the Banach space on which the transfer operator cocycle acts is separable. A careful check of the proof of Theorem 4.8 [22] shows that it holds for the Banach space $(\mathrm{BV}_1, \|\cdot\|_{\mathrm{BV}_1})$ under the alternative condition (MC) (see Appendix 2.B). To apply Theorem A [22] we require, in our notation, that:

(1) $\hat{\mathcal{L}}_{\omega,0}$ is a hyperbolic transfer operator cocycle on BV_1 with norm $\|\cdot\|_{\mathrm{BV}_1}$ and a one-dimensional leading Oseledets space (see Definition 3.1 [22]), and slow and fast growth rates $0 < \gamma < \Gamma$, respectively. We will construct γ and Γ shortly.
(2) The family of cocycles $\{\hat{\mathcal{L}}_{\omega,\varepsilon}\}_{0\le\varepsilon\le\varepsilon_0}$ satisfy a uniform Lasota–Yorke inequality

$$\|\hat{\mathcal{L}}^k_{\omega,\varepsilon} f\|_{\mathrm{BV}_1} \le A\alpha^k \|f\|_{\mathrm{BV}_1} + B^k \|f\|_1$$

for a.e. ω and $0 \le \varepsilon \le \varepsilon_0$, where $\alpha \le \gamma < \Gamma \le B$.
(3) $\lim_{\varepsilon\to 0} \operatorname{ess\,sup}_\omega \left|\!\left|\!\left| \hat{\mathcal{L}}_{\omega,0} - \hat{\mathcal{L}}_{\omega,\varepsilon} \right|\!\right|\!\right| = 0$, where $|||\cdot|||$ is the $\mathrm{BV} - L^1(\mathrm{Leb})$ triple norm.

By Lemma 2.5.9 we obtain a unique measurable family of equivariant functions $\{\phi_{\omega,0}\}$ satisfying (C7') and (C5') for $\varepsilon = 0$. We have the equivariant splitting $\operatorname{span}\{\phi_{\omega,0}\} \oplus V_\omega$, where $V_\omega = \{f \in \mathrm{BV}_1 : \nu_{\omega,0}(f) = 0\}$. We claim that this splitting is hyperbolic in the sense of Definition 3.1 [22]; this will yield item (1) above. To show this, we verify conditions (H1)–(H3) in [22]. In our setting, Condition (H1) [22] requires the norm of the projection onto the top space spanned by $\phi_{\omega,0}$, along the annihilator of $\nu_{\omega,0}$, to be uniformly bounded in ω. This is true because this projection acting on $f \in \mathrm{BV}_1$ is $\nu_{\omega,0}(f)\phi_{\omega,0}$ and therefore

$$\|\nu_{\omega,0}(f)\phi_{\omega,0}\|_{\mathrm{BV}_1} \le \operatorname*{ess\,sup}_\omega \|\phi_{\omega,0}\|_{\mathrm{BV}_1} \cdot \nu_{\omega,0}(f) \le 2 \operatorname*{ess\,sup}_\omega \|\phi_{\omega,0}\|_{\mathrm{BV}_1} \cdot \|f\|_{\mathrm{BV}_1},$$

using (C5') and equivalence of $\|\cdot\|_{\mathrm{BV}_1}$ and $\|\cdot\|_{\mathcal{B}_\omega}$ (2.5.13). Next, we define

$$\alpha^{n'} := \frac{9 \operatorname{ess\,sup}_\omega \|g^{(n')}_{\omega,0}\|_{\infty,1}}{\operatorname{ess\,inf}_\omega \inf_{\varepsilon\ge 0} \inf \mathcal{L}^{n'}_{\omega,0}\mathbb{1}} < 1,$$

which is possible by (E8). Condition (H2) requires $\|\hat{\mathcal{L}}^n_{\omega,0}\phi_{\omega,0}\|_{\mathrm{BV}_1} \ge C\Gamma^n \|\phi_{\omega,0}\|_{\mathrm{BV}_1}$ for some $C > 0$, $\Gamma > 0$, all n and a.e. ω. By (C7') one has

$$\|\hat{\mathcal{L}}^n_{\omega,0}\phi_{\omega,0}\|_{\mathrm{BV}_1} = \|\phi_{\sigma^n\omega,0}\|_{\mathrm{BV}_1} \ge \operatorname*{ess\,inf}_\omega \inf \phi_{\sigma^n\omega,0} > 0,$$

and thus we obtain (H2) with $C = \Gamma = 1$. Condition (H3) requires $\|\hat{\mathcal{L}}^n_{\omega,0}|_{V_\omega}\|_{\mathrm{BV}_1} \le K\gamma^n$ for some $K < \infty$, $\alpha \le \gamma < 1$, all n and a.e. ω. This is provided by the $\varepsilon = 0$ part of (C4')—specifically the stronger exponential version guaranteed by Lemma 2.5.9—and the equivalence of $\|\cdot\|_{\mathrm{BV}_1}$ and $\|\cdot\|_{\mathcal{B}_{\sigma^n\omega}}$.

For item (2) we begin with the Lasota–Yorke inequality for $\operatorname{var}(\cdot)$ and $\nu_{\omega,0}(|\cdot|)$ provided by the final line of the proof of Lemma 2.C.1 (equation (2.C.7)). Dividing through by $\lambda^{n'}_{\omega,0}$ we obtain

$$\begin{aligned}\operatorname{var}(\hat{\mathcal{L}}^{n'}_{\omega,\varepsilon} f) &\le \frac{9\|g^{(n')}_{\omega,\varepsilon}\|_{\infty,1}}{\lambda^{n'}_{\omega,0}} \operatorname{var}(f) + \frac{8\|g^{(n')}_{\omega,\varepsilon}\|_{\infty,1}}{\lambda^{n'}_{\omega,0} \min_{Z\in\mathcal{Z}^{(n')}_{\omega,*,\varepsilon}(\mathcal{A})} \nu_{\omega,0}(Z)} \nu_{\omega,0}(|f|) \\ &\le \alpha^{n'} \operatorname{var}(f) + \frac{\alpha^{n'}}{\min_{Z\in\mathcal{Z}^{(n')}_{\omega,*,\varepsilon}(\mathcal{A})} \nu_{\omega,0}(Z)} \nu_{\omega,0}(|f|), \qquad (2.5.17)\end{aligned}$$

noting that $\operatorname{ess\,inf}_\omega \min_{Z\in\mathcal{Z}^{(n')}_{\omega,*,\varepsilon}(\mathcal{A})} \nu_{\omega,0}(Z) > 0$ since the uniform open covering assumption (E9) together with (E3) and (2.5.16) imply that and equivariance of the backward adjoint

cocycle together imply that for $Z \in \mathcal{Z}^{(n')}_{\omega,*,\varepsilon}$ we have

$$\nu_{\omega,0}(Z) = \nu_{\sigma^{k_o(n')}\omega,0}\left(\left(\lambda^{k_o(n')}_{\omega,0}\right)^{-1}\mathcal{L}^{k_o(n')}_{\omega,0}\mathbb{1}_Z\right) \geq \frac{\inf g^{k_o(n')}_{\omega,0}}{\lambda^{k_o(n')}_{\omega,0}} > 0. \tag{2.5.18}$$

As the holes $H_{\omega,\varepsilon}$ are composed of finite unions of disjoint intervals, assumption (E6) implies that the radii of each of these intervals must go to zero as $\varepsilon \to 0$. Thus, using (E6) together with the fact that $\nu_{\omega,0}$ is fully supported and non-atomic, we see that (C6) must hold.

We construct a uniform Lasota–Yorke inequality for all n in the usual way by using blocks of length jn'; we write this as

$$\mathrm{var}(\hat{\mathcal{L}}^n_{\omega,\varepsilon}f) \leq A_1\alpha^n\mathrm{var}(f) + A_2^n\nu_{\omega,0}(|f|) \tag{2.5.19}$$

for some $A_2 > \alpha$. We now wish to convert this to an inequality

$$\mathrm{var}(\hat{\mathcal{L}}^n_{\omega,\varepsilon}f) \leq A_1'\alpha^n\mathrm{var}(f) + (A_2')^n\|f\|_1. \tag{2.5.20}$$

In light of (E2), (2.5.16), (E1), and (2.5.5), we see that there is a constant B so that for m-a.e. $\omega \in \Omega$ we have

$$\begin{aligned}
\|\hat{\mathcal{L}}^{n'}_{\omega,\varepsilon}f\|_1 &= \left(\lambda^{n'}_{\omega,0}\right)^{-1}\int_{[0,1]}\left|\sum_{y\in T_\omega^{-n'}x} g^{(n')}_{\omega,0}(y)\hat{X}_{\omega,n'-1,\varepsilon}(y)f(y)\right| d\mathrm{Leb}(x)\\
&\leq \frac{\|g^{(n')}_{\omega,0}J^{(n')}_\omega\|_{\infty,1}}{\lambda^{n'}_{\omega,0}}\int_{[0,1]}\left|\sum_{y\in T_\omega^{-n'}(x)}\frac{f(y)}{J^{(n')}_\omega(y)}\right| d\mathrm{Leb}(x)\\
&= \frac{\|g^{(n')}_{\omega,0}J^{(n')}_\omega\|_{\infty,1}}{\lambda^{n'}_{\omega,0}}\int_{[0,1]}\left|P^{n'}_\omega(f)\right| d\mathrm{Leb}(x)\\
&\leq \frac{\|g^{(n')}_{\omega,0}J^{(n')}_\omega\|_{\infty,1}}{\lambda^{n'}_{\omega,0}}\|f\|_1 \leq B^{n'}\|f\|_1.
\end{aligned}$$

Using the non-atomicicty of $\nu_{\omega,0}$ from (CCM) (shown in (2.5.9)) and the fact that $\mathrm{var}(|f|) \leq \mathrm{var}(f)$, we may apply Lemma 5.2 [12] to $\nu_{\omega,0}$ to obtain that for each $\zeta > 0$, there is a $B_\zeta < \infty$ such that $\nu_{\omega,0}(|f|) \leq \zeta\mathrm{var}(f) + B_\zeta\|f\|_1$. Now using (2.5.19) we see that

$$\begin{aligned}
\|\hat{\mathcal{L}}^n_{\omega,\varepsilon}f\|_{\mathrm{BV}_1} &= \mathrm{var}(\hat{\mathcal{L}}^n_{\omega,\varepsilon}f) + \|\hat{\mathcal{L}}^n_{\omega,\varepsilon}(f)\|_1\\
&\leq A_1\alpha^n\mathrm{var}(f) + A_2^n\nu_{\omega,0}(|f|) + B^n\|f\|_1\\
&\leq (A_1\alpha^n + A_2^n\zeta)\mathrm{var}(f) + (B^n + A_2^nB_\zeta)\|f\|_1\\
&\leq (A_1\alpha^n + A_2^n\zeta)\|f\|_{\mathrm{BV}_1} + (B^n + A_2^n(B_\zeta - \zeta) - A_1\alpha^n))\|f\|_1.
\end{aligned}$$

Selecting ζ sufficiently small and n'' sufficiently large so that $C(\alpha^{n''} + \zeta) < 1$ we again (by proceeding in blocks of n'') arrive at a uniform Lasota–Yorke inequality of the form (2.5.20) for all $n \geq 0$.

For item (3) we note that

$$\left|\left|\left|\hat{\mathcal{L}}_{\omega,0} - \hat{\mathcal{L}}_{\omega,\varepsilon}\right|\right|\right| := \sup_{\|f\|_{\mathrm{BV}_1}=1}\|(\hat{\mathcal{L}}_{\omega,0} - \hat{\mathcal{L}}_{\omega,\varepsilon})f\|_1 = \sup_{\|f\|_{\mathrm{BV}_1}=1}\|\hat{\mathcal{L}}_{\omega,0}(f\mathbb{1}_{H_{\omega,\varepsilon}})\|_1 \leq \|\hat{\mathcal{L}}_{\omega,0}(\mathbb{1}_{H_{\omega,\varepsilon}})\|_1$$

$$
\begin{aligned}
&= \lambda_{\omega,0}^{-1} \int_{[0,1]} \left| \sum_{y \in T_\omega^{-1}(x)} g_{\omega,0}(y) \mathbb{1}_{H_{\omega,\varepsilon}}(y) \right| d\mathrm{Leb}(x) \\
&\leq \frac{\|g_{\omega,0} J_\omega\|_{\infty,1}}{\lambda_{\omega,0}} \int_{[0,1]} \left| \sum_{y \in T_\omega^{-1}(x)} \frac{\mathbb{1}_{H_{\omega,\varepsilon}}(y)}{J_\omega(y)} \right| d\mathrm{Leb}(x) \\
&= \frac{\|g_{\omega,0} J_\omega\|_{\infty,1}}{\lambda_{\omega,0}} \int_{[0,1]} \left| P_\omega(\mathbb{1}_{H_{\omega,\varepsilon}}) \right| d\mathrm{Leb}(x) \\
&\leq \operatorname*{ess\,sup}_\omega \frac{\|g_{\omega,0} J_\omega\|_{\infty,1}}{\lambda_{\omega,0}} \cdot \operatorname*{ess\,sup}_\omega \mathrm{Leb}(H_{\omega,\varepsilon}).
\end{aligned}
$$

Because (E2), (E1) and (2.5.16) imply $\operatorname{ess\,sup}_\omega {}^{\|g_{\omega,0} J_\omega\|_{\infty,1}}/\lambda_{\omega,0} < \infty$, and since (E6) implies that $\lim_{\varepsilon \to 0} \operatorname{ess\,sup}_\omega \mathrm{Leb}(H_{\omega,\varepsilon}) = 0$, we obtain item (3).

We may now apply Theorem 4.8 [22] to conclude that given $\delta > 0$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the cocycle generated by $\hat{\mathcal{L}}_\varepsilon$ is hyperbolic, with

(i) the existence of an equivariant family $\hat{\phi}_{\omega,\varepsilon} \in \mathrm{BV}$ with $\operatorname{ess\,sup}_\omega \|\hat{\phi}_{\omega,\varepsilon} - \phi_{\omega,0}\|_{\mathrm{BV}_1} < \delta$,

(ii) existence of corresponding Lyapunov multipliers $\hat{\lambda}_{\omega,\varepsilon}$ satisfying $|\hat{\lambda}_{\omega,\varepsilon} - 1| < \delta$,

(iii) operators $\hat{Q}_{\omega,\varepsilon}$ satisfying $\|(\hat{Q}_{\omega,\varepsilon})^n\|_{\mathrm{BV}_1} \leq K'(\gamma + \delta)^n$, where γ is the decay rate for $Q_{\omega,0}$ from the proof of Lemma 2.5.9.

To obtain an $\hat{\mathcal{L}}^*_{\omega,\varepsilon}$-equivariant family of linear functionals $\hat{\nu}_{\omega,\varepsilon} \in \mathrm{BV}_1^*$ we apply Corollary 2.5 [32]. Using the one-dimensionality of the leading Oseledets space for the forward cocycle, this result shows that the leading Oseledets space for the backward adjoint cocycle is also one-dimensional. This leading Oseledets space is spanned by some $\hat{\nu}_{\omega,\varepsilon} \in \mathrm{BV}_1^*$, satisfying $\hat{\nu}_{\sigma\omega,\varepsilon}(\hat{\mathcal{L}}_{\omega,\varepsilon}(f)) = \hat{\vartheta}_{\omega,\varepsilon} \hat{\nu}_{\omega,\varepsilon}(f)$, for Lyapunov multipliers $\hat{\vartheta}_{\omega,\varepsilon}$. By Lemma 2.6 [32], we may scale the $\hat{\nu}_{\omega,\varepsilon}$ so that $\hat{\nu}_{\omega,\varepsilon}(\hat{\phi}_{\omega,\varepsilon}) = 1$ for a.e. ω. We show that in fact $\hat{\vartheta}_{\omega,\varepsilon} = \hat{\lambda}_{\omega,\varepsilon}$ m-a.e. Indeed,

$$
1 = \hat{\nu}_{\sigma\omega,\varepsilon}(\hat{\phi}_{\sigma\omega,\varepsilon}) = \hat{\nu}_{\sigma\omega,\varepsilon}(\hat{\mathcal{L}}_{\omega,\varepsilon} \hat{\phi}_{\omega,\varepsilon} / \hat{\lambda}_{\omega,\varepsilon}) = (\hat{\vartheta}_{\omega,\varepsilon} / \hat{\lambda}_{\omega,\varepsilon}) \hat{\nu}_{\omega,\varepsilon}(\hat{\phi}_{\omega,\varepsilon}) = \hat{\vartheta}_{\omega,\varepsilon} / \hat{\lambda}_{\omega,\varepsilon}.
$$

Note that $\mathcal{L}_{\omega,\varepsilon} = \lambda_{\omega,0} \hat{\mathcal{L}}_{\omega,\varepsilon}$, and we now define $\phi_{\omega,\varepsilon}, \lambda_{\omega,\varepsilon}, Q_{\omega,\varepsilon}$, and $\nu_{\omega,\varepsilon}$ by the following:

$$
\phi_{\omega,\varepsilon} := \frac{1}{\nu_{\omega,0}(\hat{\phi}_{\omega,\varepsilon})} \cdot \hat{\phi}_{\omega,\varepsilon}, \qquad \nu_{\omega,\varepsilon}(f) := \nu_{\omega,0}(\hat{\phi}_{\omega,\varepsilon}) \hat{\nu}_{\omega,\varepsilon}(f),
$$

$$
\lambda_{\omega,\varepsilon} := \lambda_{\omega,0} \frac{\nu_{\sigma\omega,0}(\hat{\phi}_{\sigma\omega,\varepsilon})}{\nu_{\omega,0}(\hat{\phi}_{\omega,\varepsilon})} \hat{\lambda}_{\omega,\varepsilon}, \qquad Q_{\omega,\varepsilon}(f) := \frac{\nu_{\omega,0}(\hat{\phi}_{\omega,\varepsilon})}{\nu_{\sigma\omega,0}(\hat{\phi}_{\sigma\omega,\varepsilon})} \hat{Q}_{\omega,\varepsilon}(f)
$$

Clearly all of the properties of (C2) and (C3) are now satisfied except for the log-integrability of $\lambda_{\omega,\varepsilon}$ in (C2). To demonstrate this last point, we note that by uniform hyperbolicity of the perturbed cocycles, $\lambda'_{\omega,\varepsilon}$ are uniformly bounded below and are therefore log-integrable. Since

$$
|1 - \nu_{\omega,0}(\hat{\phi}_{\omega,\varepsilon})| = |\nu_{\omega,0}(\phi_{\omega,0}) - \nu_{\omega,0}(\hat{\phi}_{\omega,\varepsilon})| \leq \|\hat{\phi}_{\omega,\varepsilon} - \phi_{\omega,0}\|_{\mathrm{BV}_1} < \delta,
$$

the $\lambda_{\omega,\varepsilon}$ are uniformly small perturbations of the $\hat{\lambda}_{\omega,\varepsilon}$, and since $\lambda_{\omega,0}$ is log-integrable by (E2) and (2.5.16), we must therefore have that the log integrability condition on $\lambda_{\omega,\varepsilon}$ in

(C2) is satisfied. Point (i) above, combined with the $\varepsilon = 0$ part of (C5') (resp. (C7')) and the uniform estimate for $|1 - \nu_{\omega,0}(\hat{\phi}_{\omega,\varepsilon})|$, immediately yields the $\varepsilon > 0$ part of (C5') (resp. (C7')). Point (iii) above combined with the same estimates also ensures that the norm of $\|Q^n_{\omega,\varepsilon}\|_{\infty,1}$ decays exponentially fast, uniformly in ω and ε, satisfying the stronger exponential version of (C4'). In fact point (iii) implies the stronger statement that $\|Q^n_{\omega,\varepsilon}\|_{\mathrm{BV}_1}$ decays exponentially fast, uniformly in ω and ε.

Finally we show that $\nu_{\omega,\varepsilon} : C^0([0,1]) \to \mathbb{C}$ is a positive linear functional with $\nu_{\omega,\varepsilon}(f) \in \mathbb{R}$ if f is real. From this fact it will follow by Riesz-Markov (e.g. Theorem A.3.11 [**65**]) that $\nu_{\omega,\varepsilon}$ can be identified with a real finite Borel measure on $[0,1]$. By linearity we may consider the two cases: (i) $f = \phi_{\omega,\varepsilon} > 0$ (the generator of the leading Oseledets space) and (ii) $f \in F_{\omega,\varepsilon}$, where $F_{\omega,\varepsilon}$ is the Oseledets space complementary to $\mathrm{span}\{\phi_{\omega,\varepsilon}\}$. In case (i) $\nu_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = 1 > 0$. In case (ii), Lemma 2.6 [**32**] implies $\nu_{\omega,\varepsilon}(f) = 0$. In summary we see that $\nu_{\omega,\varepsilon}$ is positive. □

Remark 2.5.11. As we have just shown that assumptions (C1'), (C2), (C3), (C4'), (C5'), (C6), (C7') (Lemmas 2.5.9 and 2.5.10), we see that Proposition 2.3.3 holds as well as Theorem 2.3.6 for the random open system $(\Omega, m, \sigma, [0,1], T, \mathrm{BV}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ under the additional assumption of (C8). In light of Remark 2.4.3, if we assume (S) in addition to (C8) then both Corollary 2.3.9 and Theorem 2.4.5 apply.

The following theorem is the main result of this section and elaborates on the dynamical significance of the perturbed objects produced in Lemma 2.5.10.

Theorem 2.5.12. *Suppose $(\Omega, m, \sigma, [0,1], T, \mathcal{B}, \mathcal{L}_0, \nu_0, \phi_0, H_\varepsilon)$ is a random open system and that the assumptions of Lemma 2.5.10 hold. Then there exists $\varepsilon_0 > 0$ sufficiently small such that for every $0 \le \varepsilon < \varepsilon_0$ we have the following:*

(1) There exists a unique random probability measure $\zeta_\varepsilon = \{\zeta_{\omega,\varepsilon}\}_{\omega\in\Omega}$ on $[0,1]$ such that, for $\varepsilon > 0$, $\zeta_{\omega,\varepsilon}$ is supported in $X_{\omega,\infty,\varepsilon}$ and

$$\zeta_{\sigma\omega,\varepsilon}(\mathcal{L}_{\omega,\varepsilon} f) = \rho_{\omega,\varepsilon}\zeta_{\omega,\varepsilon}(f),$$

for m-a.e. $\omega \in \Omega$ and each $f \in \mathrm{BV}$, where

$$\rho_{\omega,\varepsilon} := \zeta_{\sigma\omega,\varepsilon}(\mathcal{L}_{\omega,\varepsilon}\mathbb{1}).$$

Furthermore, for $\varepsilon = 0$ we have $\zeta_{\omega,0} = \nu_{\omega,0}$ and $\rho_{\omega,0} = \lambda_{\omega,0}$ and for $\varepsilon > 0$ we have that $C_1^{-1} \le \rho_{\omega,\varepsilon} \le C_1$ for m-a.e. $\omega \in \Omega$.

(2) There exists a measurable function $\psi_\varepsilon : \Omega \times [0,1] \to (0,\infty)$ such that $\zeta_{\omega,\varepsilon}(\psi_{\omega,\varepsilon}) = 1$ and

$$\mathcal{L}_{\omega,\varepsilon}\psi_{\omega,\varepsilon} = \rho_{\omega,\varepsilon}\psi_{\sigma\omega,\varepsilon}$$

for m-a.e. $\omega \in \Omega$. Moreover, ψ_ε is unique modulo ζ_ε, and there exists $C \ge 1$ such that $C^{-1} \le \psi_{\omega,\varepsilon} \le C$ for m-a.e. $\omega \in \Omega$. Furthermore, for m-a.e. $\omega \in \Omega$ we have that $\rho_{\omega,\varepsilon} \to \rho_{\omega,0} = \lambda_{\omega,0}$ and $\psi_{\omega,\varepsilon} \to \psi_{\omega,0} = \phi_{\omega,0}$ (in $\mathcal{B}_\omega$) as $\varepsilon \to 0$, where $\phi_{\omega,0}$ and $\lambda_{\omega,0}$ are defined in Lemma 2.5.9.

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

(3) *The random measure* $\mu_\varepsilon = \{\mu_{\omega,\varepsilon} := \psi_{\omega,\varepsilon}\zeta_{\omega,\varepsilon}\}_{\omega\in\Omega}$ *is a* T*-invariant and ergodic random probability measure whose fiberwise support, for* $\varepsilon > 0$*, is contained in* $X_{\omega,\infty,\varepsilon}$*. Furthermore,* μ_ε *is the unique relative equilibrium state, i.e.*

$$\int_\Omega \log \rho_{\omega,\varepsilon}\, dm(\omega) =: \mathcal{E}P_\varepsilon(\log g_0) = h_{\mu_\varepsilon}(T) + \int_{\mathcal{J}_0} \log g_0\, d\mu_\varepsilon = \sup_{\eta_\varepsilon \in \mathcal{P}^{H_\varepsilon}_{T,m}(\mathcal{J}_0)} \left(h_{\eta_\varepsilon}(T) + \int_{\mathcal{J}_0} \log g_0\, d\eta_\varepsilon \right),$$

where $h_{\eta_\varepsilon}(T)$ *is the entropy of the measure* η_ε*,* $\mathcal{E}P_\varepsilon(\log g_0)$ *is the expected pressure of the weight function* $g_0 = \{g_{\omega,0}\}_{\omega\in\Omega}$*, and* $\mathcal{P}^{H_\varepsilon}_{T,m}(\mathcal{J}_0)$ *denotes the collection of* T*-invariant random probability measures* η_ε *on* $\mathcal{J}_0$ *whose disintegration* $\{\eta_{\omega,\varepsilon}\}_{\omega\in\Omega}$ *satisfies* $\eta_{\omega,\varepsilon}(H_{\omega,\varepsilon}) = 0$ *for* m*-a.e.* $\omega \in \Omega$*. Furthermore,* $\lim_{\varepsilon\to 0} \mathcal{E}P_\varepsilon(\log g_0) = \mathcal{E}P_0(\log g_0) := \int_\Omega \log \lambda_{\omega,0}\, dm$*.*

(4) *For* $\varepsilon > 0$*, let* $\varrho_\varepsilon = \{\varrho_{\omega,\varepsilon}\}_{\omega\in\Omega}$ *be the random probability measure with fiberwise support in* $[0,1]\backslash H_{\omega,\varepsilon}$ *whose disintegrations are given by*

$$\varrho_{\omega,\varepsilon}(f) := \frac{\nu_{\omega,0}\left(\mathbb{1}_{H^c_{\omega,\varepsilon}}\psi_{\omega,\varepsilon} f\right)}{\nu_{\omega,0}\left(\mathbb{1}_{H^c_{\omega,\varepsilon}}\psi_{\omega,\varepsilon}\right)}$$

for all $f \in \mathrm{BV}$*.* $\varrho_{\omega,\varepsilon}$ *is the unique random conditionally invariant probability measure that is absolutely continuous (with respect to* $\{\nu_{\omega,0}\}_{\omega\in\Omega}$*) with density of bounded variation.*

(5) *For each* $f \in \mathrm{BV}$ *there exists* $D > 0$ *and* $\kappa_\varepsilon \in (0,1)$ *such that for* m*-a.e.* $\omega \in \Omega$ *and all* $n \in \mathbb{N}$ *we have*

$$\| \left(\rho^n_{\omega,\varepsilon}\right)^{-1} \mathcal{L}^n_{\omega,\varepsilon} f - \zeta_{\omega,\varepsilon}(f)\psi_{\sigma^n\omega,\varepsilon}\|_{\mathcal{B}_{\sigma^n\omega}} \le D\|f\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon.$$

Furthermore, for all $A \in \mathscr{B}$ *and* $f \in \mathrm{BV}$ *we have*

$$\left|\nu_{\omega,0}\left(T^{-n}_\omega(A) \,|\, X_{\omega,n,\varepsilon}\right) - \varrho_{\sigma^n\omega,\varepsilon}(A)\right| \le D\kappa^n_\varepsilon,$$

and

$$\left| \frac{\varrho_{\omega,\varepsilon}\left(f|_{X_{\omega,n,\varepsilon}}\right)}{\varrho_{\omega,\varepsilon}(X_{\omega,n,\varepsilon})} - \mu_{\omega,\varepsilon}(f) \right| \le D\|f\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon.$$

In addition, we have $\lim_{\varepsilon\to 0}\kappa_\varepsilon = \kappa_0$*, where* κ_0 *is defined in Lemma 2.5.9.*

(6) *There exists* $C > 0$ *such that for every* $f, h \in \mathrm{BV}$*, every* $n \in \mathbb{N}$ *sufficiently large, and for* m*-a.e.* $\omega \in \Omega$ *we have*

$$|\mu_{\omega,\varepsilon}\left((f \circ T^n_\omega)\, h\right) - \mu_{\sigma^n\omega,\varepsilon}(f)\mu_{\omega,\varepsilon}(h)| \le C\|f\|_{\infty,\omega}\|h\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon.$$

Proof. First we note that the claims of items (1) – (3) and (5) – (6) above for $\varepsilon = 0$ follow immediately from Lemma 2.5.9. Now we are left to prove each of the claims for $\varepsilon > 0$.

Claims (1) – (3) follow from Lemma 2.5.10 with the scaling:

$$\psi_{\omega,\varepsilon} := \nu_{\omega,\varepsilon}(\phi_{\omega,\varepsilon})\phi_{\omega,\varepsilon}, \qquad \zeta_{\omega,\varepsilon}(f) := \frac{\nu_{\omega,\varepsilon}(f)}{\nu_{\omega,\varepsilon}(\mathbb{1})}, \qquad \rho_{\omega,\varepsilon} := \frac{\nu_{\omega,\varepsilon}(\mathbb{1})}{\nu_{\sigma\omega,\varepsilon}(\mathbb{1})}\lambda_{\omega,\varepsilon}.$$

The uniform boundedness on $\rho_{\omega,\varepsilon}$ and $\psi_{\omega,\varepsilon}$ follows from the uniform boundedness on $\lambda_{\omega,\varepsilon}$ and $\phi_{\omega,\varepsilon}$ coming from Lemma 2.5.10. The fact that μ_ε is the unique relative equilibrium state follows similarly to the proof of Theorem 2.23 in [3] (see also Remark 2.24, Lemma 12.2 and Lemma 12.3). The claim that $\mathrm{supp}(\zeta_{\omega,\varepsilon}) \subseteq X_{\omega,\infty,\varepsilon}$ follows similarly to Lemma 1.10.11. Noting that $\|f\|_{\infty,\omega} = \|f\|_{\infty,1}$ for $f \in \mathrm{BV}$ by the proof of Proposition 2.5.2, we now proof Claims (4) – (6).

Claim (4) follows from Lemma 1.2.5 and the uniqueness of the density $\psi_{\omega,\varepsilon} \in \mathrm{BV}$.

The first item of Claim (5) follows from Lemma 2.5.10 and the remaining items are proved similarly to Corollary 1.11.8.

Claim (6) follows from Claim (5) and is proven in Appendix 2.D. □

Remark 2.5.13. If one considers a two-sided (bi-infinite) sequential analogue of random open systems, then because Theorem 4.8 [22] also applies to two-sided sequential systems, one could prove similar results to items (1), (2), (4), (5), and (6) of Theorem 2.5.12.

2.6. Limit theorems

In this section we prove a few limit theorems for the *closed* systems discussed in Section 2.5 $(\Omega, m, \sigma, [0,1], T, \mathrm{BV}, \mathcal{L}_0, \nu_0, \phi_0)$. We will in fact show that such systems are *admissible* in the sense of [32]. This will allow us to adapt to our setting the spectral approach *à la Nagaev-Guivarc'h* developed in [32] and get a quenched central limit theorem, a quenched large deviation theorem and a quenched local central limit theorem. We will also present an alternative approach based on martingale techniques [33, 1, 34], which produces an almost sure invariance principle (ASIP) for the random measure $\mu_0 = \{\mu_{\omega,0}\}_{\omega\in\Omega}$. Moreover, the ASIP implies that μ_0 satisfies the central limit theorem as well as the law of the iterated logarithm. The martingale approach will also give an upper bound for any (large) deviation from the expected value and a Borel-Cantelli dynamical lemma. At the moment we could not extend the previous limit theorems to the *open* systems investigated in Section 2.5. There are a few reasons for that which concern the Banach space $\mathcal{B}_{\omega,\varepsilon}$ associated to those systems and defined by the norm: $||\cdot||_{\mathcal{B}_{\omega,\varepsilon}} = \mathrm{var}(\cdot) + \zeta_{\omega,\varepsilon}(|\cdot|)$. First of all, we do not know if the random cocycle $\mathcal{R}_\varepsilon = (\Omega, m, \sigma, \mathcal{B}_{\omega,\varepsilon}, \tilde{\mathcal{L}}_{\omega,\varepsilon})$ is quasi-compact which is an essential requirement for admissibility. Second, the results of Theorem 2.5.12 are not particularly compatible with the Banach spaces $\mathcal{B}_{\omega,\varepsilon}$, $\varepsilon > 0$, since the inequalities of items (5) and (6) are in terms of the norms $\|\cdot\|_{\infty,\omega}$ and $\|\cdot\|_{\mathcal{B}_\omega}$ which are defined modulo $\nu_{\omega,0}$ and the $\mathcal{B}_{\omega,\varepsilon}$ norm is defined via $\zeta_{\omega,\varepsilon}$, a measure which is supported on a $\nu_{\omega,0}$-null set.

2.6.1. The Nagaev-Guivarc'h approach. The paper [32] developed a general scheme to adapt the Nagaev-Guivarc'h approach to random quenched dynamical systems, allowing one to prove limit theorems by exploring the connection between a *twisted* operator cocycle and the distribution of the Birkhoff sums. The results in [32] were confined to the geometric potential $|\det(DT_\omega)|^{-1}$ and the associated conformal measure, Lebesgue measure. We now show how to extend those results to the systems verifying the assumptions stated in Section 2.5 and the results of Theorem 2.5.12, whenever $\varepsilon = 0$, that is we will consider random closed systems for a larger class of potentials.

The starting point is to replace the linear operator $\mathcal{L}_{\omega,0}$ associated to the geometric potential and the (conformal) Lebesgue measure introduced in [32], with our operator $\mathcal{L}_{\omega,0}$

and the associated conformal measures $\nu_{\omega,0}$. In particular, if we work with the normalized operator $\tilde{\mathcal{L}}_{\omega,0} := \lambda_{\omega,0}^{-1}\mathcal{L}_{\omega,0}$ the results in [32] are reproducible almost verbatim with a few precautions which we are going to explain. As before, let $\mathcal{B}_\omega$ be the Banach space defined by $||\cdot||_{\mathcal{B}_\omega} = \mathrm{var}(\cdot) + \nu_{\omega,0}(|\cdot|)$, where the variation is defined using equivalence classes mod-$\nu_{\omega,0}$. In order to apply the theory in [32] we must show that our random cocycle is *admissible*. This reduces to check two sets of properties which were listed in [32] respectively as conditions (V1) to (V9) and conditions (C0) up to (C4). The first set of conditions reproduces the classical properties of the total variation of a function and its relationship with the $L^1(\mathrm{Leb})$ norm. We should emphasize that in our case the variation is defined using equivalence classes mod-$\nu_{\omega,0}$. It is easy to check that properties (V1, V2, V3, V5, V8, V9) hold with respect to this variation. In particular (V3) asserts that for any $f \in \mathcal{B}_\omega$ we have $||f||_{L^\infty(\nu_{\omega,0})} \leq ||f||_{\mathcal{B}_\omega}$; we will refer to it in the following just as the (V3) property. *Notation*: recall that given an element $f \in \mathcal{B}_\omega$, the L^∞ norm of f with respect to the measure $\nu_{\omega,0}$ is denoted by $||f||_{\infty,\omega}$ in the rest of this section.

Property (V7) is not used in the current paper; (V6) is a general density embedding result proved in Hofbauer and Keller (Lemma 5 [45]). We elaborate on property (V4). To obtain (V4) in our situation one needs to prove that the unit ball of $\mathcal{B}_\omega$ is compactly injected into $L^1(\nu_{\omega,0})$. As we will see, this is used to get the quasi-compactness of the random cocycle. The result follows easily by adapting Proposition 2.3.4 in [14] to our conformal measure $\nu_{\omega,0}$, which is fully supported and non-atomic. We now rename the other set of properties (C0)–(C4) in [32] as ($\mathcal{C}0$) to ($\mathcal{C}4$) to distinguish them from our (C) properties stated earlier in Sections 2.3 and 2.4.

- Assumption ($\mathcal{C}0$) coincides with our condition (MC).
- Condition ($\mathcal{C}1$) requires us to prove in our case that

$$||\tilde{\mathcal{L}}_{\omega,0}f||_{\mathcal{B}_{\sigma\omega}} \leq K||f||_{\mathcal{B}_\omega}, \tag{2.6.1}$$

  for every $f \in \mathcal{B}_\omega$ and for m-a.e. ω, with ω-independent K.
- Condition ($\mathcal{C}2$) asks that there exists $N \in \mathbb{N}$ and measurable $\tilde{\alpha}^N, \tilde{\beta}^N : \Omega \to (0,\infty)$, with $\int_\Omega \log \tilde{\alpha}^N(\omega)dm(\omega) < 0$, such that for every $f \in \mathcal{B}_\omega$ and m-a.e. $\omega \in \Omega$,

$$||\tilde{\mathcal{L}}^N_{\omega,0}f||_{\mathcal{B}_{\sigma^N\omega}} \leq \tilde{\alpha}^N(\omega)||f||_{\mathcal{B}_\omega} + \tilde{\beta}^N(\omega)||f||_{L^1(\nu_{\omega,0})}. \tag{2.6.2}$$

- Condition ($\mathcal{C}3$) is the content of the first display equation in item 5 in the statement of Theorem 2.5.12.
- Condition ($\mathcal{C}4$) is only used to obtain Lemma 2.11 in [32]. There are three results in that Lemma which we now compare with our situation. The third result is the decay of correlations stated in item 6 of Theorem 2.5.12. The second result is the almost-sure strictly positive lower bound for the density $\phi_{\omega,0}$ stated in item 2 of Theorem 2.5.12. The first result requires that $\operatorname{ess\,sup}_{\omega\in\Omega} ||\phi_{\omega,0}||_{\mathcal{B}_\omega} < \infty$. This follows by checking that the proof of Proposition 1 in [33] works in our current setting with the obvious modifications; we note that Proposition 1 [33] only assumes conditions ($\mathcal{C}1$) and ($\mathcal{C}3$).

We are thus left with showing conditions ($\mathcal{C}1$) and ($\mathcal{C}2$) in our setting. We will get both at the same time as a consequence of the following argument, which consists in adapting to our current situation the final part of Lemma 2.C.1. Our starting point will be the inequality

(2.C.6). Since we are working with the normalized operator cocycle $\tilde{\mathcal{L}}^N_{\omega,0}$, we have to divide the expression in (2.C.6) by $\lambda^{n'}_{\omega,0}$; moreover we have to replace the measure $\nu_{\omega,0}$ with the equivalent conformal probability measure $\nu_{\omega,0}$. Therefore (2.C.6) now becomes

$$(\lambda^n_{\omega,0})^{-1}\mathrm{var}\left(\mathbb{1}_{T^n_\omega(Z)}\left((fg^{(n)}_{\omega,0})\circ T^{-n}_{\omega,Z}\right)\right) \le 9\frac{\|g^{(n)}_{\omega,0}\|_{\infty,\omega}}{\lambda^n_{\omega,0}}\mathrm{var}_Z(f)+\frac{8\|g^{(n)}_{\omega,0}\|_{\infty,\omega}}{\lambda^n_{\omega,0}\nu_{\omega,0}(Z)}\nu_{\omega,0}(|f|_Z), \tag{2.6.3}$$

where Z is an element of $\mathcal{Z}^{(n)}_{\omega,0}$. Since each element of the partition $\mathcal{Z}^{(n)}_{\omega,0}$ has nonempty interior and the measure $\nu_{\omega,0}$ charges open intervals, if we set

$$\overline{\nu}^{(n)}_{\omega,0} := \min_{Z\in\mathcal{Z}^{(n)}_{\omega,0}} \nu_{\omega,0}(Z) > 0,$$

and we take the sum over the $Z \in \mathcal{Z}^{(n)}_{\omega,0}$ we finally get

$$\mathrm{var}(\tilde{\mathcal{L}}^n_{\omega,0}f) \le 9\frac{\|g^{(n)}_{\omega,0}\|_{\infty,\omega}}{\lambda^n_{\omega,0}}\mathrm{var}(f) + 8\frac{\|g^{(n)}_{\omega,0}\|_{\infty,\omega}\nu_{\omega,0}(|f|)}{\lambda^n_{\omega,0}\overline{\nu}^{(n)}_{\omega,0}}. \tag{2.6.4}$$

Notice that condition $(\mathcal{C}2)$ requires that $\tilde{\alpha}^N < 1$, which in our case becomes

$$9\frac{\operatorname{ess\,sup}_\omega \|g^{(n)}_{\omega,0}\|_{\infty,\omega}}{\operatorname{ess\,inf}_\omega \lambda^n_{\omega,0}} < 1, \tag{2.6.5}$$

or equivalently

$$9\frac{\operatorname{ess\,sup}_\omega \|g^{(n)}_{\omega,0}\|_{\infty,\omega}}{\operatorname{ess\,inf}_\omega \inf \mathcal{L}^n_{\omega,0}\mathbb{1}} < 1,$$

which is guaranteed by (E8). We now move on to check condition $(\mathcal{C}1)$. From (2.6.4), setting $n = 1$ we see that a sufficient condition for $(\mathcal{C}1)$ is

$$\frac{\operatorname{ess\,sup}_\omega \|g^{(1)}_{\omega,0}\|_{\infty,\omega}}{\operatorname{ess\,inf}_\omega \lambda^1_{\omega,0}\overline{\nu}_{\omega,0}} < \infty. \tag{2.6.6}$$

or equivalently

$$\frac{\operatorname{ess\,sup}_\omega \|g^{(1)}_{\omega,0}\|_{\infty,\omega}}{\operatorname{ess\,inf}_\omega \inf \mathcal{L}_{\omega,0}\mathbb{1}\overline{\nu}_{\omega,0}} < \infty \tag{2.6.7}$$

Condition (2.6.7) is in principle checkable ($n = 1$) and we could *assume* it as a part of the admissibility condition for our random cocycle $\mathcal{R} = (\Omega, m, \sigma, \mathcal{B}_\omega, \tilde{\mathcal{L}}_{\omega,0})$.

The admissibility conditions stated in [32], in particular $(\mathcal{C}2)$ and $(\mathcal{C}3)$, were sufficient to prove the quasi-compactness of the random cocycle introduced in [32], but they rely on another assumption, which was a part of the Banach space construction, namely that BV_1 was compactly injected into $L^1(\mathrm{Leb})$. We saw above that the same result holds for our Banach space $\mathcal{B}_\omega$ and our measure $\nu_{\omega,0}$. In order to prove quasi-compactness following Lemma 2.1 in [32], we use condition $(\mathcal{C}2)$ with the almost-sure bound $\tilde{\alpha}^N(\omega) < 1$. We must additionally prove that the top Lyapunov exponent $\Lambda(\mathcal{R})$ defined by the limit $\Lambda(\mathcal{R}) = \lim_{n\to\infty}\frac{1}{n}\log||\tilde{\mathcal{L}}^n_{\omega,0}||_{\mathcal{B}_\omega}$ is not less than zero. This will follow by applying item

5 and the bound on the density $\phi_{\omega,0}$ in item 2, both in Theorem 2.5.12. Since the density $\phi_{\omega,0}$ is for m-a.e. $\omega \in \Omega$ bounded from below by the constant C^{-1}, we have that $\limsup_{n\to\infty} \frac{1}{n} \log \|\phi_{\sigma^n\omega,0}\|_{\mathcal{B}_{\sigma^n\omega}} \geq \limsup_{n\to\infty} \frac{1}{n} \log C^{-1} = 0$. Now we continue as in the proof of Theorem 3.2 in [32]. Let N be the integer for which $\tilde{\alpha}^N(\omega) < 1$ ω a.s. We then consider the cocycle $\mathcal{R}_N$ generated by the map $\omega \to \mathcal{L}^N_{\omega,0}$. It is easy to verify that $\Lambda(\mathcal{R}_N) = N\Lambda(\mathcal{R})$ and the index of compactness κ (see section 2.1 in [32] for the definition) of the two cocycles also satisfies $\kappa(\mathcal{R}_N) = N\kappa(\mathcal{R})$. Because $\int \log \tilde{\alpha}^N(\omega)\ dm < 0 \leq \Lambda(\mathcal{R}_N)$, Lemma 2.1 in [32] guarantees that $\kappa(\mathcal{R}_N) \leq \int \log \tilde{\alpha}^N(\omega)\ dm$, proving quasicompactness of $\mathcal{R}$.

As we said at the beginning of this section, we could now follow almost verbatim the proofs in [32] by using our operators $\mathcal{L}_{\omega,0}$ and by replacing the Lebesgue measure with the conformal measures $\nu_{\omega,0}$. We briefly sketch the main steps of the approach; we first define the *observable* v as a measurable map $v : \Omega \times [0,1] \to \mathbb{R}$ with the additional properties that $||v(\omega,x)||_{L^1(\nu_0)} < \infty$ and $\operatorname{ess\,sup}_\omega ||v_\omega||_{\infty,\omega} < \infty$, where we set $v_\omega(x) = v(\omega,x)$. Moreover we assume v_ω is fibrewise centered: $\int v_\omega(x) d\mu_{\omega,0}(x) = 0$, for m-a.e. $\omega \in \Omega$. We then define the twisted (normalized) transfer operator cocycle as:

$$\tilde{\mathcal{L}}^\theta_{\omega,0}(f) = \tilde{\mathcal{L}}_{\omega,0}(e^{\theta v_\omega(\cdot)} f),\ \ f \in \mathcal{B}_\omega.$$

The link with the limit theorems is provided by the following equality which follows easily by the duality property of the operator:

$$\int \tilde{\mathcal{L}}^{\theta,n}_{\omega,0}(f) d\nu_{\sigma^n\omega,0} = \int e^{\theta S_n v_\omega(\cdot)} f d\nu_{\omega,0},$$

where $\tilde{\mathcal{L}}^{\theta,n}_{\omega,0} = \tilde{\mathcal{L}}^\theta_{\sigma^{n-1}\omega,0} \circ \cdots \circ \tilde{\mathcal{L}}^\theta_{\omega,0}$. The adaptation of Theorem 3.12 in [32] to our case, shows that the twisted operator is quasi-compact (in $\mathcal{B}_\omega$) for θ close to zero. Moreover, by denoting with

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \log ||\tilde{\mathcal{L}}^{\theta,n}_{\omega,0}||_{\mathcal{B}_\omega}$$

the top Lyapunov exponent of the cocycle, the map $\theta \to \Lambda(\theta)$ is of class C^2 and strictly convex in a neighborhood of zero. We have also the analog of Lemma 4.3 in [32], linking the asymptotic behavior of characteristic functions associated to Birkhoff sums with $\Lambda(\theta)$:

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \int e^{\theta S_n v_\omega(x)} d\mu_{\omega,0} \right| = \Lambda(\theta).$$

We are now ready to collect our results on a few limit theorems; we first define the variance

$$\Sigma^2 = \int v_\omega(x)^2\ d\mu_{\omega,0}\ dm + 2\sum_{n=1}^{\infty} \int v_\omega(x) v(T^n_\omega(x))\ d\mu_{\omega,0}\ dm.$$

We also define the *aperiodicity condition* by asking that for m-a.e. $\omega \in \Omega$ and for every compact interval $J \subset \mathbb{R} \setminus \{0\}$, there exists $C(\omega) > 0$ and $\rho \in (0,1)$ such that $||\tilde{\mathcal{L}}^{it,n}_{\omega,0}||_{\mathcal{B}_\omega} \leq C(\omega)\rho^n$, for $t \in J$ and $n \geq 0$.

THEOREM 2.6.1. *Suppose that our random cocycle $\mathcal{R} = (\Omega, m, \sigma, \mathcal{B}_\omega, \tilde{\mathcal{L}}_{\omega,0})$ is admissible and take the centered observable v verifying $\operatorname{ess\,sup}_\omega ||v_\omega||_{\infty,\omega} < \infty$. Then:*

- *(**Large deviations**). There exists $\varkappa_0 > 0$ and a non-random function $c : (-\varkappa_0, \varkappa_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_{\omega,0}(S_n v_\omega(\cdot) > n\varkappa) = -c(\varkappa), \text{ for } 0 < \varkappa < \varkappa_0, \text{ and } m - a.e.\ \omega \in \Omega.$$

- *(**Central limit theorem**). Assume that $\Sigma^2 > 0$. Then for every bounded and continuous function $\phi : \mathbb{R} \to \mathbb{R}$ and m-a.e. $\omega \in \Omega$, we have*

$$\lim_{n\to\infty} \int \phi\left(\frac{S_n v_\omega(x)}{\sqrt{n}}\right) d\mu_{\omega,0} = \int \phi \, d\mathcal{N}(0, \Sigma^2).$$

- *(**Local central limit theorem**). Suppose the aperiodicity condition holds. Then for m-a.e. $\omega \in \Omega$ and every bounded interval $J \subset \mathbb{R}$, we have*

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_{\omega,0}(s + S_n v_\omega(\cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0.$$

2.6.2. The martingale approach. The deviation result quoted above allows to control, asymptotically, deviations of order $\varkappa$, for $\varkappa$ in a sufficiently small bounded interval around 0. We now show how to extend that result to any $\varkappa$ by getting an exponential bound on the deviation of the distribution function instead of an asymptotic expansion; in particular our bound shows that deviation probability stays small for finite n,which is a typical concentration property. We now derive this result since it will provide us with the martingale that is used to obtain the ASIP. Recall that the equivariant measure $\mu_{\omega,0}$ is equivalent to $\nu_{\omega,0}$. We consider again the fibrewise centered observable v from the previous section, and we wish to estimate

$$\mu_{\omega,0}\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} v_{\sigma^k\omega} \circ T_\omega^k\right| > \varkappa\right).$$

We will use the following result (Azuma, [7]): Let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence of martingale differences. If there is $a > 0$ such that $||M_i||_\infty < a$ for all i, then we have for all $b \in \mathbb{R}$:

$$\mu_{\omega,0}\left(\sum_{i=1}^{n} M_i \geq nb\right) \leq e^{-n\frac{b^2}{2a^2}}.$$

If we denote $\mathcal{F}_\omega^k := (T_\omega^k)^{-1}(\mathcal{F})$, we can easily prove that for a measurable map $\phi : [0,1] \to \mathbb{R}$, we have (the expectations $\mathbb{E}_\omega$ will be taken with respect to $\mu_{\omega,0}$):

$$\mathbb{E}_\omega(\phi \circ T^l | \mathcal{F}_\omega^n) = \left(\frac{\tilde{\mathcal{L}}_{\sigma^l\omega,0}^{n-l}(\lambda_{\sigma^l\omega,0}\phi)}{\lambda_{\sigma^n\omega,0}}\right) \circ T_\omega^n \tag{2.6.8}$$

We now set:

$$M_n := v_{\sigma^n\omega} + G_n - G_{n+1} \circ T_{\sigma^n\omega},$$

with $G_0 = 0$ and

$$G_{n+1} = \frac{\tilde{\mathcal{L}}_{\sigma^n\omega,0}(v_{\sigma^n\omega}\lambda_{\sigma^n\omega,0} + G_n\lambda_{\sigma^n\omega,0})}{\lambda_{\sigma^{k+1}\omega,0}} \tag{2.6.9}$$

It is easy to check that

$$\mathbb{E}_\omega(M_n \circ T_\omega^n | \mathcal{F}_\omega^{n+1}) = 0,$$

which means that the sequence $(M_n \circ T_\omega^n)$ is a reversed martingale difference with respect to the filtration $\mathcal{F}_\omega^n$. By iterating (2.6.9) we get

$$G_n = \frac{1}{\lambda_{\sigma^n\omega,0}} \sum_{j=0}^{n-1} \tilde{\mathcal{L}}_{\sigma^j\omega,0}^{(n-j)}(v_{\sigma^j\omega}\lambda_{\sigma^j\omega,0}), \tag{2.6.10}$$

and by a telescopic trick we have

$$\sum_{k=0}^{n-1} M_k \circ T_\omega^k = \sum_{k=0}^{n-1} v_{\sigma^k\omega} T_\omega^n - G_n \circ T_\omega^n.$$

Suppose for the moment we could bound G_n uniformly in n, but not necessarily in ω, by $||G_n||_{\infty,\omega} \leq C_1(\omega)$. Since by assumption there exists a constant C_2 such that $\operatorname{ess\,sup}_\omega ||v_\omega||_{\infty,\omega} \leq C_2$, we have $||M_n||_{\infty,\omega} \leq C_2 + 2C_1(\omega)$, and by Azuma:

$$\mu_{\omega,0}\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} M_k \circ T_\omega^k\right| > \frac{\varkappa}{2}\right) \leq 2\exp\left\{-\frac{\varkappa^2}{8(C_2+2C_1(\omega))^2}n\right\}.$$

Therefore

$$\begin{aligned}
\mu_{\omega,0}\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} v_{\sigma^k\omega} \circ T_\omega^k\right| > \varkappa\right) &\leq \mu_{\omega,0}\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} M_k \circ T_\omega^k\right| + \frac{1}{n}C_1(\omega) > \varkappa\right) \\
&\leq \mu_{\omega,0}\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} M_k \circ T_\omega^k\right| > \frac{\varkappa}{2}\right) \\
&\leq 2\exp\left\{-\frac{\varkappa^2}{8(C_2+2C_1(\omega))^2}n\right\},
\end{aligned}$$

provided $n > n_0$, where n_0 verifies $\frac{1}{n_0}C_1(\omega) \leq \frac{\varkappa}{2}$.
In order to estimate C_1, we proceed in the following manner. We have

$$||G_n||_\infty \leq \frac{1}{\lambda_{\sigma^n\omega,0}} \sum_{j=0}^{n-1} ||\tilde{\mathcal{L}}_{\sigma^j\omega,0}^{(n-j)}(v_{\sigma^j\omega}\lambda_{\sigma^j\omega,0})||_{\infty,\omega}. \tag{2.6.11}$$

The multiplier $\lambda_{\omega,0}$ is bounded from above and from below ω a.s. respectively by, say, U and $1/U$ by conditions ($\mathcal{C}1$) and item 1 Theorem 2.5.12, respectively. Then we use item 5 of Theorem 2.5.12 and property (V3) to bound the term into the sum. In conclusion we get:

$$||G_n||_{\infty,\omega} \leq U^2 D C_2 \sum_{j=0}^{\infty} \kappa^{n-j} := C_1.$$

We summarize this result in the following

THEOREM 2.6.2. *(**Large deviations bound**)*
Suppose that our random cocycle $\mathcal{R} = (\Omega, m, \sigma, \mathcal{B}_\omega, \tilde{\mathcal{L}}_{\omega,0})$ is admissible and take the fibrewise centered observable v satisfying $\operatorname{ess\,sup}_\omega ||v_\omega||_{\infty,\omega} = C_2 < \infty$. Then there exists $C_1 > 0$ such that for m-a.e. $\omega \in \Omega$ and for any $\varkappa > 0$:

$$\mu_{\omega,0}\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} v_{\sigma^k\omega} \circ T_\omega^k\right| > \varkappa\right) \leq 2\exp\left\{-\frac{\varkappa^2}{8(C_2+2C_1)^2}n\right\},$$

provided $n > n_0$, where n_0 is any integer number satisfying $\frac{1}{n_0}C_1 \leq \frac{\varkappa}{2}$.

We point out that whenever our random cocycle $\mathcal{R} = (\Omega, m, \sigma, \mathcal{B}_\omega, \tilde{\mathcal{L}}_{\omega,0})$ is admissible, we satisfy all the assumptions in the paper [**33**], which allows us to get the almost sure invariance principle. We should simply replace the (conformal) Lebesgue measure with our random conformal measure $\nu_{\omega,0}$, and use our operators $\tilde{\mathcal{L}}_{\omega,0}$, as we already did in Section 2.6.1 for the Nagaev-Guivarc'h approach. It is worthwhile to observe that [**33**] relies on the construction of a suitable martingale, which in our case is precisely the reversed martingale difference $M_n \circ T_\omega^n$ obtained above in the proof of the large deviation bound. We denote with Σ_n^2 the variance $\Sigma_n^2 = \mathbb{E}_\omega\left(\sum_{k=0}^\infty v_{\sigma^k\omega} \circ T_\omega^k\right)^2$, where v_ω is a centered observable, and with Σ^2 the quantity introduced above, in the statement of the central limit theorem. We have the following:

THEOREM 2.6.3. *(**Almost sure invariance principle**)*
Suppose that our random cocycle $\mathcal{R} = (\Omega, m, \sigma, \mathcal{B}_\omega, \tilde{\mathcal{L}}_{\omega,0})$ is admissible and consider a centered observable v_ω. Then $\lim_{n\to\infty}\frac{1}{n}\Sigma_n^2 = \Sigma^2$. Moreover one of the following cases hold:
(i) $\Sigma = 0$, and this is equivalent to the existence of $\psi \in L^2(\mu_0)$

$$v = \psi - \psi \circ T,$$

where T is the induced skew product map and μ its invariant measure, see Definition 0.1.1.
(ii) $\Sigma^2 > 0$ and in this case for m a.e. $\omega \in \Omega$, $\forall \varrho > \frac{5}{4}$, by enlarging the probability space $(X, \mathcal{F}, \mu_{\omega,0})$ if necessary, it is possible to find a sequence $(B_k)_k$ of independent centered Gaussian random variables such that

$$\sup_{1\leq k\leq n}\left|\sum_{i=1}^k (v_{\sigma^i\omega} \circ T_\omega^i) - \sum_{i=1}^k B_i\right| = O(n^{1/4}\log^\varrho(n)),\ \ \mu_{\omega,0}\ \ a.s..$$

We show in Section 2.4.47 that the distribution of the first hitting random time in a decreasing sequence of holes follows an exponential law, see Proposition 2.4.7. We now prove a recurrence result by giving a quantitative estimate of the *asymptotic number of entry times in a descending sequence of balls*. This is known as the *shrinking target problem*.

PROPOSITION 2.6.4. *Suppose that our random cocycle $\mathcal{R} = (\Omega, m, \sigma, \mathcal{B}_\omega, \tilde{\mathcal{L}}_{\omega,0})$ is admissible. For each ω let $B_{\omega,\xi_{k,\omega}}(p)$ be a descending sequence of balls centered at the point p and with radii $\xi_{k+1,\omega} < \xi_{k,\omega}$, such that*

$$E_{\omega,n} := \sum_{k=0}^{n-1} \mu_{\sigma_\omega^k,0}(B_{\sigma^k\omega,\xi_{k,\sigma^k\omega}}(p)) \to \infty \qquad \text{for } m\text{-a.e. } \omega.$$

Then for m-a.e. ω and $\mu_{\omega,0}$-almost all x, $T^k_\omega(x) \in B_{\sigma^k\omega,\xi_{k,\sigma^k\omega}}(p)$ for infinitely many k and

$$\frac{1}{E_{\omega,n}}\#\{0 \le k < n : T^k_\omega(x) \in B_{\sigma^k\omega,\xi_{k,\sigma^k\omega}}(p)\} \to 1.$$

The proposition is a simple consequence of the following *Borel-Cantelli* like property, whenever we put the observable v in the theorem below as $v_{k,\sigma^k\omega} = \mathbb{1}_{B_{\sigma^k\omega,\xi_{k,\sigma^k\omega}}(p)}$.

THEOREM 2.6.5. *Suppose that our random cocycle $\mathcal{R} = (\Omega, m, \sigma, \mathcal{B}_\omega, \tilde{\mathcal{L}}_{\omega,0})$ is admissible. Take a sequence of nonnegative observables v_n, satisfying $\sup_n \operatorname{ess\,sup}_\omega ||v_{n,\omega}||_{\mathcal{B}_\omega} \le M$. Suppose that for m a.e. $\omega \in \Omega$*

$$E_{\omega,n} := \sum_{k=0}^{n-1} \int v_{k,\sigma^k\omega}(x) d\mu_{\sigma^k_\omega,0}(x) \to \infty,\ n \to \infty.$$

Then for m a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{1}{E_{\omega,n}} \sum_{k=0}^{n-1} v_{k,\sigma^k\omega}(T^k_\omega x) = 1,$$

for $\mu_{\omega,0}$-almost all x.

PROOF. Let us write

$$S_\omega := \int \left(\sum_{m<k\le n} \left(v_{k,\sigma^k\omega}(T^k_\omega y) - \int v_{k,\sigma^k\omega} d\mu_{\sigma^k_\omega,0} \right) \right)^2 d\mu_{\omega,0}(y).$$

If we could prove that $S_\omega \le \text{constant} \sum_{m<k\le n} \int v_{k,\sigma^k\omega} d\mu_{\sigma^k\omega,0}$, then the result will follow by applying Sprindzuk theorem [61][4], where we identify $\int v_{k,\sigma^k\omega} d\mu_{\sigma^k\omega,0}$ with the functions g_k and h_k in the previous footnote. So we have

$$\begin{aligned} S_\omega &= \int \left[\sum_{m<k\le n} v_{k,\sigma^k\omega}(T^k_\omega y) \right]^2 d\mu_{\omega,0}(y) - \left[\sum_{m<k\le n} \int v_{k,\sigma^k\omega} d\mu_{\sigma^k_\omega,0} \right]^2 \\ &= \sum_{m<k\le n} \int v_{k,\sigma^k\omega}(T^k_\omega y)^2 d\mu_{\omega,0} + 2 \sum_{m<i<j\le n} \int v_{i,\sigma^i\omega}(T^i_\omega y) v_{j,\sigma^j\omega}(T^j_\omega y) d\mu_{\omega,0} \end{aligned}$$

[4]We recall here the Sprindzuk theorem. Let $(\Omega, \mathcal{B}, \mu_0)$ be a probability space and let $(f_k)_k$ be a sequence of nonnegative and measurable functions on Ω. Moreover, let $(g_k)_k$ and $(h_k)_k$ be bounded sequences of real numbers such that $0 \le g_k \le h_k$. Assume that there exists $C > 0$ such that

$$\int \left(\sum_{m<k\le n} (f_k(x) - g_k) \right)^2 d\mu_0(x) \le C \sum_{m<k\le n} h_k$$

for $m < n$. Then, for every $\varepsilon > 0$ we have

$$\sum_{1\le k\le n} f_k(x) = \sum_{1\le k\le n} g_k + O(\Theta^{1/2}(n) \log^{3/2+\varepsilon} \Theta(n)),$$

for μ a.e. $x \in \Omega$, where $\Theta(n) = \sum_{1\le k\le n} h_k$.

$$- \sum_{m<k\le n} \left[\int v_{k,\sigma^k\omega} d\mu_{\sigma^k\omega,0} \right]^2 - 2 \sum_{m<i<j\le n} \int v_{i,\sigma^i\omega} d\mu_{\sigma^i\omega,0} \int v_{j,\sigma^j\omega} d\mu_{\sigma^j\omega,0}.$$

We now discard the third negative piece and bound the first by Hölder inequality as

$$\sum_{m<k\le n} \int v_{k,\sigma^k\omega}(T^k_\omega y)^2 d\mu_{\omega,0} \le \sum_{m<k\le n} M \int v_{k,\sigma^k\omega} d\mu_{\sigma^k\omega,0}.$$

Then for the remaining two pieces we use the decay of correlations given in Theorem 2.21 of our paper [3], where the observables are taken in $\mathcal{B}_\omega$ and in $L^1(\mu_{\omega,0})$. We can apply it to our case since the two observables coincide with v_ω, and the measures $\mu_{\omega,0}$ and $\nu_{\omega,0}$ are equivalent. We point out that this result improves the decay bound given in item (6) of Theorem 2.5.12 where the presence of holes for $\varepsilon > 0$ forced us to use $L^\infty(\nu_{\omega,\varepsilon})$ instead of $L^1(\nu_{\omega,\varepsilon})$. Hence:

$$\left| \int v_{i,\sigma^i\omega}(T^i_\omega y) v_{j,\sigma^j\omega}(T^j_\omega y) d\mu_{\omega,0} - \int v_{i,\sigma^i\omega} d\mu_{\sigma^i\omega,0} \int v_{j,\sigma^j\omega} d\mu_{\sigma^j\omega,0} \right| \le CM\kappa^{j-i} \int v_{j,\sigma^j\omega} d\mu_{\sigma^j\omega,0}.$$

In conclusion, we get

$$\begin{aligned} S_\omega &\le \sum_{m<k\le n} M \int v_{k,\sigma^k\omega} d\mu_{\sigma^k\omega,0} + CM \sum_{m<i<j\le n} \kappa^{j-i} \int v_{j,\sigma^j\omega} d\mu_{\sigma^j\omega,0} \\ &\le M \left(\frac{C}{1-\kappa} + 1 \right) \sum_{m<k\le n} \int v_{k,\sigma^k\omega} d\mu_{\sigma^k\omega,0}, \end{aligned}$$

which satisfies Sprindzuk. □

2.7. Examples

In this section we present explicit examples of random dynamical systems to illustrate the quenched extremal index formula in Theorem 2.4.5. In all cases, Theorem 2.5.12 can be applied to guarantee the existence of all objects therein for the perturbed random open system (in brief, the thermodynamic formalism for the closed random system implies a thermodynamic formalism for the perturbed random open system). In Examples 2.7.1–2.7.3 we derive explicit expressions for the quenched extreme value laws.

Our examples are piecewise-monotonic interval map cocycles whose transfer operators act on $\mathcal{B}_\omega = \mathrm{BV} := \mathrm{BV}([0,1])$ for a.e. ω. The norm we will use is $\|\cdot\|_{\mathcal{B}_\omega} = \|\cdot\|_{\mathrm{BV}_{\nu_{\omega,0}}} := \mathrm{var}(\cdot) + \nu_{\omega,0}(|\cdot|)$. In Section 2.5 we noted that (B) is automatically satisfied. Lemma 2.5.9 shows that (CCM) holds under assumptions (E2), (E3), (E4), and (E8). To obtain a random open system, in addition to (B) and (CCM) we need (MC) (implying (M1) and (M2)), (A), and (X). Therefore, in each example we verify conditions (A), (X) (Section 0.1.1) and (MC) and (E1)–(E9) (Section 2.5); and where required (S).

Example 2.7.1. **Random maps and random holes centered on a non-random fixed point (random observations with non-random maximum)**

We show that nontrivial quenched extremal indices occur even for holes centered on a non-random fixed point x_0. To define a random (open) map, let (Ω, m) be a complete probability space, and $\sigma : (\Omega, m) \to (\Omega, m)$ be an ergodic, invertible, probability-preserving

transformation. For example, one could consider an irrational rotation on the circle. Let $\Omega = \cup_{j=1}^{\infty}\Omega_j$ be a countable partition of Ω into measurable sets on which $\omega \mapsto T_\omega$ is constant. This ensures that (MC) is satisfied. For each $\omega \in \Omega$ let $\mathcal{J}_{\omega,0} = [0,1]$ and let $T_\omega : \mathcal{J}_{\omega,0} \to \mathcal{J}_{\sigma\omega,0}$ be random, with all maps fixing $x_0 \in [0,1]$. The observation functions $h_\omega : \mathcal{J}_{\omega,0} \to \mathbb{R}$ have a unique maximum at x_0 for a.e. ω.

To make all of the objects and calculations as simple as possible and illustrate some of the underlying mechanisms for nontrivial extremal indices we use the following specific family of maps $\{T_\omega\}$:

$$T_\omega(x) = \begin{cases} 1 - 2x/(1-1/s_\omega), & 0 \le x \le (1-1/s_\omega)/2; \\ s_\omega x - (s_\omega - 1)/2, & (1-1/s_\omega)/2 \le x \le (1+1/s_\omega)/2; \\ 1 - (2x - (1+1/s_\omega))/(1-1/s_\omega) & (1+1/s_\omega)/2 \le x \le 1, \end{cases} \tag{2.7.1}$$

where $1 < s \le s_\omega \le S < \infty$ and $s|_{\Omega_j}$ is constant for each $j \ge 1$; thus (M1) holds. These maps have three full linear branches, and therefore Lebesgue measure is preserved by all maps T_ω; i.e. $\mu_{\omega,0} = \text{Leb}$ for a.e. ω. The central branch has slope s_ω and passes through the fixed point $x_0 = 1/2$, which lies at the center of the central branch; see Figure 1. A

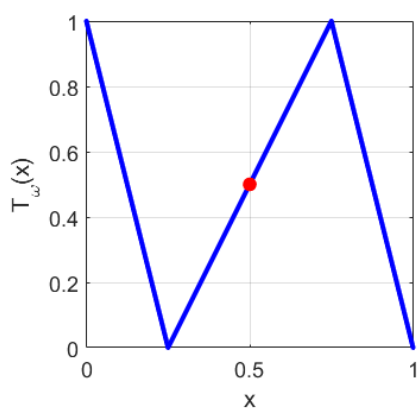

FIGURE 1. Graph of map T_ω, with $s_\omega = 2$.

specific random driving could be $\sigma : S^1 \to S^1$ given by $\sigma(\omega) = \omega + \alpha$ for some $\alpha \notin \mathbb{Q}$ and $s_\omega = s^0 + s^1 \cdot \omega$ for $1 < s^0 < \infty$ and $0 < s^1 < \infty$, but only the ergodicity of σ will be important for us. Setting $g_{\omega,0} = 1/|T'_\omega|$ it is immediate that (E1), (E2), (E3), and (E4) hold.

We select a measurable family of observation functions $h : \Omega \times [0,1] \to \mathbb{R}$. For a.e. $\omega \in \Omega$, h_ω is C^1, has a unique maximum at $x_0 = 1/2$, and is a locally even function about x_0. For small ε_N we will then have that $H_{\omega,\varepsilon_N} = \{x \in [0,1] : h_\omega(x) > z_{\omega,N}\}$ is a small neighbourhood of x_0, satisfying (E5) and (A). In light of Remark 2.5.3 and Proposition 2.5.5 we see that (X) holds. The corresponding cocycle of open operators $\mathcal{L}_{\omega,\varepsilon}$ satisfy (MC). Given an essentially bounded function t, the $z_{\omega,N}$ are chosen to satisfy (S). Because the h_ω are C^1 with unique maxima and $\mu_{\omega,0}$ is Lebesgue for all ω we can make such a choice of $z_{\omega,N}$ with[5] $\xi_{\omega,N} \equiv 0$. Furthermore, as $\mu_{\omega,0}$ is Lebesgue we have that (S) implies that assumption

[5]In the situation where $z_{\omega,N}$ is constant a.e. for each $N \ge 1$, in general we cannot satisfy (S) for a given fixed scaling function t. In the case where t is also constant a.e. if h_ω is random but sufficiently

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

(E6) is automatically satisfied. With the above choices, (E7) is clearly satisfied for $n' = 1$. To show condition (E8) with $n' = 1$ we note that $\|g_{\omega,0}\|_\infty = (1 - 1/s_\omega)/2 < (1 - 1/S)/2$, while $\inf \mathcal{L}_{\omega,\varepsilon}\mathbb{1} \geq 2 \cdot (1 - 1/s_\omega)/2 \geq 1 - 1/s$. Therefore we require $(1 - 1/S)/2 < 1 - 1/s$, which by elementary rearrangement is always true for $0 < s < S < \infty$. Because there is a full branch outside the branch with the hole, (E9) is satisfied with $n' = 1$ and $k_o(n') = 1$.

At this point we have checked all of the hypotheses of Theorem 2.5.12 and we obtain that for sufficiently small holes as defined above, the corresponding random open dynamical system has a quenched thermodynamic formalism and quenched decay of correlations. We note that this result does not require (S); the holes H_{ε_N} do not need to scale in any particular way with N, they simply need to be sufficiently small. In order to next apply Theorem 2.4.5 we do require (S) and additionally (C8).

To finish the example, we verify (C8). We claim that for each fixed $k > 0$, one has $\hat{q}^{(k)}_{\omega,\varepsilon_N} = 0$ for sufficiently sufficiently large N (sufficiently small ε_N). If this were not the case, there must be a positive $\mu_{\omega,0}$-measure set of points that (i) lie in $H_{\sigma^{-k}\omega,\varepsilon_N}$, (ii) land outside the sequence of holes $H_{\sigma^{-(k-1)}\omega,\varepsilon_N}, \ldots, H_{\sigma^{-1}\omega,\varepsilon_N}$ for the next $k-1$ iterations, and then (iii) land in H_{ω,ε_N} on the kth iteration. In this example, because all H_{ω,ε_N} are neighbourhoods of x_0 of diameter smaller than $(|t|_\infty + \gamma_N)/N$, and the maps are locally expanding about x_0, for fixed k one can find a large enough N so that it is impossible to leave a small neighbourhood of x_0 and return after k iterations. For $k = 0$, by definition

$$\hat{q}^{(0)}_{\omega,\varepsilon_N} = \frac{\mu_{\sigma^{-1}\omega,0}(T^{-1}_{\sigma^{-1}\omega}H_{\omega,\varepsilon_N} \cap H_{\sigma^{-1}\omega,\varepsilon_N})}{\mu_{\sigma^{-1}\omega,0}(T^{-1}_{\sigma^{-1}\omega}H_{\omega,\varepsilon_N})} = \frac{\mu_{\sigma^{-1}\omega,0}(T^{-1}_{\sigma^{-1}\omega}H_{\omega,\varepsilon_N} \cap H_{\sigma^{-1}\omega,\varepsilon_N})}{\mu_{\omega,0}(H_{\omega,\varepsilon_N})}. \tag{2.7.2}$$

There are two cases to consider:

Case 1: $H_{\sigma^{-1}\omega,\varepsilon_N}$ is larger than the local preimage of H_{ω,ε_N}; that is, $T^{-1}_{\sigma^{-1}\omega}H_{\omega,\varepsilon_N} \cap H_{\omega,\varepsilon_N} \subset H_{\sigma^{-1}\omega,\varepsilon_N}$. Because of the linearity of the branch containing x_0, one has $\hat{q}^{(0)}_{\omega,\varepsilon_N} = 1/T'_{\sigma^{-1}\omega}(x_0)$.

Case 2: $H_{\sigma^{-1}\omega,\varepsilon_N}$ is smaller than the local preimage of H_{ω,ε_N}; that is, $T^{-1}_{\sigma^{-1}\omega}H_{\omega,\varepsilon_N} \cap H_{\omega,\varepsilon_N} \supset H_{\sigma^{-1}\omega,\varepsilon_N}$. By (2.7.2) and (S) we then have

$$\frac{t_{\sigma^{-1}\omega} - \gamma_N}{t_\omega + \gamma_N} \leq \hat{q}^{(0)}_{\omega,\varepsilon_N} \leq \frac{t_{\sigma^{-1}\omega} + \gamma_N}{t_\omega - \gamma_N}$$

and thus for such an ω, $\lim_{N\to\infty} \hat{q}^{(0)}_{\omega,\varepsilon_N} = t_{\sigma^{-1}\omega}/t_\omega$. Thus, combining the two cases,

$$\hat{q}^{(0)}_{\omega,0} := \lim_{N\to\infty} \hat{q}^{(0)}_{\omega,\varepsilon_N} = \min\left\{\frac{t_{\sigma^{-1}\omega}}{t_\omega}, \frac{1}{|T'_{\sigma^{-1}\omega}(x_0)|}\right\}$$

exists for a.e. ω, verifying (C8).

Recalling that $\theta_{\omega,0} = 1 - \sum_{k=0}^\infty \hat{q}^{(k)}_{\omega,0} = 1 - \hat{q}^{(0)}_{\omega,0}$, we may now apply Theorem 2.4.5 to obtain the quenched extreme value law:

$$\lim_{N\to\infty} \nu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right) = \exp\left(-\int_\Omega t_\omega \theta_{\omega,0}\, dm(\omega)\right)$$

close to a non-random observation h for each ω, we can satisfy (S) by incorporating the error term $\xi_{\omega,N}$. This situation corresponds to a decreasing family of holes whose length fluctuates slightly in ω with the fluctuation decreasing to zero sufficiently fast. A similar situation can occur if the scaling function is a general measurable function. These situations motivate the use of the error term $\xi_{\omega,N}$ in condition (S).

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

$$= \exp\left(-\int_\Omega t_\omega \left(1-\min\left\{\frac{t_{\sigma^{-1}\omega}}{t_\omega}, \frac{1}{|T'_{\sigma^{-1}\omega}(x_0)|}\right\}\right) dm(\omega)\right) \tag{2.7.3}$$

This formula is a generalization of the formula in Remark 8 [48], where we may create nontrivial laws from either the random dynamics T_ω, or the random scalings t_ω, or both. The following two special cases consider these mechanisms separately.

(1) *Random maps, non-random scaling (t_ω takes a constant value $t > 0$):* Since $|T'_\omega| > 1$, in this case (2.7.3) becomes

$$\lim_{N\to\infty} \nu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right) = \exp\left(-t\left[1-\int_\Omega \frac{1}{|T'_\omega(x_0)|}\, dm(\omega)\right]\right), \tag{2.7.4}$$

and we see that we can interpret $\theta = 1 - \int_\Omega \frac{1}{|T'_\omega(x_0)|}\, dm(\omega)$ as an extremal index.

(2) *Fixed map, random scaling:* Suppose $T_\omega \equiv T$; then we may replace $T'_\omega(x_0)$ with $T'(x_0)$ in (2.7.3), and we see we the extremal index depends on the choice of random scalings t_ω alone; of course the thresholds $z_{\omega,N}$ depend on the chosen t_ω.

Similar results could be obtained with the T_ω possessing nonlinear branches. The arguments above can also be extended to the case where x_0 is a periodic point of prime period p for all maps; the formula (2.7.3) now includes $(T^p_\omega)'(x_0)$ and $t_\omega, t_{\sigma^1\omega}, \ldots, t_{\sigma^{-(p-1)}\omega}$. If the scaling $t_\omega = t$ is non-random, one would simply replace $T'_\omega(x_0)$ in (2.7.4) with $(T^p_\omega)'(x_0)$.

We recall that for deterministic T (including non-uniformly hyperbolic maps), the extremal index enjoys a dichotomy, in the sense that it is equal to 1 when a single hole H_{ε_N} shrinks to an aperiodic point, and it is strictly smaller than 1 when the hole shrinks to a periodic point. In the latter case the extremal index can be expressed in terms of the period; see [55] for a general account of the above facts. Example 2.7.1 shows that in a simple random setting there are many more ways to obtain nontrivial exponential limit laws, e.g. by random scaling, or by the existence of periodic orbits for only a positive measure set of ω.

Example 2.7.2. **Random β-maps, random holes containing a non-random fixed point (random observations with non-random maximum), general geometric potential**

We show how a nontrivial extremal index can arise from random β-maps where statistics are generated by an equilibrium state of a general geometric potential. Consider the "no short branches" random β-map example of Section 13.2 [3], where $\beta_\omega \in \{2\} \cup \bigcup_{2\le k\le K}[k+\delta, k+1]$ for a.e. ω, and some $\delta > 0$ and $K < \infty$. The measurability of $\omega \mapsto \beta_\omega$ yields (M1). We use the weight $g_\omega = 1/|T'_\omega|^r$, $r \ge 0$. To obtain (MC) the base dynamics is driven by an ergodic homeomorphism σ on a Borel subset Ω of a separable, complete metric space, assuming that $\omega \mapsto \mathcal{L}_{\omega,0}$ has countable range, as in Example 2.7.1. Our random observation function $h : \Omega \times [0,1] \to \mathbb{R}$ is measurable and for a.e. ω, h_ω is C^1 with a unique maximum at $x = 0$. We select a measurable scaling function t_ω; either or both of h_ω and t_ω could be ω-independent. By assigning thresholds $z_{\omega,N}$ (which could also be ω-independent) we obtain a decreasing family of holes H_{ω,ε_N}, which are decreasing intervals with left endpoint at $x = 0$.

Clearly (E2) and (E3) hold and Lemma 13.5 [3] provides (E4). Conditions (E5) and (E6), and (A) are immediate, as is (E7) with $n' = 1$. Condition (EX) holds because there

is at least one full branch outside the branch with the hole. Regarding (E8), $\|g_{\omega,\varepsilon}\|_\infty \le \beta_\omega^{-r}$ and arguing as in the previous example, since $\inf \mathcal{L}_{\omega,\varepsilon}\mathbb{1} \ge \lfloor \beta_\omega \rfloor / \beta_\omega^r$, we see (E8) holds with $n' = 1$ because $\beta_\omega \ge 2$. Because there is at least one full branch outside the branch with the hole, (E9) holds with $n' = 1$ and $k_o(n') = 1$. We have now checked all of the hypotheses of Theorem 2.5.12. By this theorem, for sufficiently small holes H_ε, the corresponding random open dynamical system has a quenched thermodynamic formalism and quenched decay of correlations. As in the previous example, we emphasise that this result does not require (S).

Because $\phi_{\omega,0}$ is uniformly bounded above and below, as long as $z_{\omega,N}$ is random, we may adjust $z_{\omega,N}$ to obtain (S). We now verify (C8) for our holes, which are of the form $H_{\omega,\varepsilon_N} = [0, r_{\omega,N}]$, in preparation to apply Theorem 2.4.5. The same arguments from Case 1 and Case 2 of the previous example apply. Case 2 is unchanged. In Case 1 we have

$$\hat{q}^{(0)}_{\omega,\varepsilon_N} = \frac{\mu_{\sigma^{-1}\omega,0}(T^{-1}_{\sigma^{-1}\omega} H_{\omega,\varepsilon_N} \cap H_{\sigma^{-1}\omega,\varepsilon_N})}{\mu_{\sigma^{-1}\omega,0}(T^{-1}_{\sigma^{-1}\omega} H_{\omega,\varepsilon_N})}.$$

Because $T^{-1}_{\sigma^{-1}\omega} H_{\omega,\varepsilon_N}$ is a finite union of left-closed intervals, using the fact that $\phi_{\sigma^{-1}\omega,0} \in \mathrm{BV}$, and therefore has left- and right-hand limits everywhere, we may redefine $\phi_{\sigma^{-1}\omega,0}(y)$ at the finite collection of points $y \in T^{-1}_{\sigma^{-1}\omega}(0)$ so that

$$\lim_{N\to\infty} \frac{\mu_{\sigma^{-1}\omega,0}(T^{-1}_{\sigma^{-1}\omega} H_{\omega,\varepsilon_N} \cap H_{\sigma^{-1}\omega,\varepsilon_N})}{\mu_{\sigma^{-1}\omega,0}(T^{-1}_{\sigma^{-1}\omega} H_{\omega,\varepsilon_N})} = \frac{\phi_{\sigma^{-1}\omega,0}(0)}{\sum_{y\in T^{-1}_{\sigma^{-1}\omega}(0)} \phi_{\sigma^{-1}\omega,0}(y)},$$

recalling that in Case 1, $T^{-1}_{\sigma^{-1}\omega} H_{\omega,\varepsilon_N} \subset H_{\sigma^{-1}\omega,\varepsilon_N}$. Taking the minimum of Cases 1 and 2 as in Example 1, we see that $\hat{q}^{(0)}_{\omega,0} = \lim_{N\to\infty} \hat{q}^{(0)}_{\omega,\varepsilon_N}$ exists, which verifies (C8). Finally,

(2.7.5)
$$\lim_{N\to\infty} \nu_{\omega,0}\left(X_{\omega,N-1,\varepsilon_N}\right) = \exp\left(-\int_\Omega t_\omega \left(1 - \min\left\{\frac{t_{\sigma^{-1}\omega}}{t_\omega}, \frac{\phi_{\sigma^{-1}\omega,0}(0)}{\sum_{y\in T^{-1}_{\sigma^{-1}\omega}(0)} \phi_{\sigma^{-1}\omega,0}(y)}\right\}\right) dm(\omega)\right)$$

EXAMPLE 2.7.3. **A fixed map with random holes containing a fixed point (random observations with a non-random maximum)**

Let us now consider more closely the case of a fixed map and holes moving randomly around a fixed point z, a situation previously considered in [9] in the annealed framework.

We take a piecewise uniformly expanding map T of the unit interval I of class C^2, and such that T is surjective on the domains of local injectivity. Moreover T preserves a mixing measure μ equivalent to the Lebesgue measure Leb, with strictly positive density ρ. We moreover assume that T and ρ are continuous at a fixed point x_0. The observation functions h_ω have a common maximum at x_0, leading to holes H_{ω,ε_N} that are closed intervals containing the point x_0 for any N. In Example 2.7.1 the holes were centered on x_0 but could vary dramatically in diameter between ω-fibers. In this example the holes need not be centered on x_0 but must become more identical as they shrink. Specifically, we assume

$$\sup_k \frac{\mathrm{Leb}(H_{\omega,\varepsilon_N} \Delta H_{\sigma^{-k}\omega,\varepsilon_N})}{\mathrm{Leb}(H_{\omega,\varepsilon_N})} \to 0, \ N \to \infty, \tag{2.7.6}$$

where the use of Lebesgue is for simplicity.

Since the sample measures $\mu_{\omega,0}$ coincide with μ, we may easily verify condition (S) by choosing conveniently the size of the hole H_{ω,ε_N}. Moreover, by choosing the holes to be contained entirely within exactly one interval of monotonicity, we see that (EX) holds, and thus (X) holds via Remark 2.5.3 and Proposition 2.5.5. Since the weights g_ω are nonrandom and equal to $1/|T'|$, conditions (E1) to (E7) clearly hold. Since T is continuous in x_0,, the holes will belong to one branch of T. Therefore if $D(T)$ will denote the number of branches of T and $\lambda_m := \min_I |T'|$, $\lambda_M := \max_I |T'|$, it will be enough to have $D(T) - 1 > \frac{\lambda_M}{\lambda_m}$ in order to satisfy (E9) with $n' = 1$. Moreover, still keeping $n' = 1$ and since we have finitely many branches, condition (E9a) in Remark 2.C.2 is satisfied with $k_o(n') = 1$. We may now apply Theorem 2.5.12 to obtain a quenched thermodynamic formalism for our fixed map with sufficiently small random holes.

Whenever the point x_0 is not periodic, one obtains that all the $\hat{q}^{(k)}_{\omega,\varepsilon_N}$ are zero by repeating the argument given in Example 2.7.1 for $k \geq 0$ and using the continuity of T at x_0. We now take x_0 as a periodic point of minimal period p and we assume that T^p and ρ are continuous at x_0. We now begin to evaluate

$$\hat{q}^{(p-1)}_{\omega,\varepsilon_N} = \frac{\mu(T^{-p}H_{\omega,\varepsilon_N} \cap H_{\sigma^{-p}\omega,\varepsilon_N})}{\mu(H_{\omega,\varepsilon_N})}. \tag{2.7.7}$$

Since T^p is continuous and expanding in the neighborhood of x_0, by taking N large enough, the set $T^{-p}H_{\omega,\varepsilon_N} \cap H_{\sigma^{-p}\omega,\varepsilon_N}$ has only one connected component. Denote the local branch of T through x_0 by T_{x_0}. Therefore by a local change of variable we have for the upper bound of the numerator

$$\begin{aligned} \mu(T^{-p}H_{\omega,\varepsilon_N} \cap H_{\sigma^{-p}\omega,\varepsilon_N}) &= \int_{T^p(T^{-p}H_{\omega,\varepsilon_N} \cap H_{\sigma^{-p}\omega,\varepsilon_N})} \rho(T_{x_0}^{-p}y)|DT^p(T_{x_0}^{-p}y)|^{-1} d\mathrm{Leb}(y) \\ &\leq \sup_{H_{\omega,\varepsilon_N}} |DT^p|^{-1} \sup_{H_{\omega,\varepsilon_N}} \rho \ \mathrm{Leb}(H_{\omega,\varepsilon_N}). \end{aligned} \tag{2.7.8}$$

For the lower bound of the numerator, since T^p is locally expanding, $T^p(H_{\sigma^{-p}\omega,\varepsilon_N}) \supset H_{\sigma^{-p}\omega,\varepsilon_N}$, and by (2.7.8)

$$\begin{aligned} \mu(T^{-p}H_{\omega,\varepsilon_N} \cap H_{\sigma^{-p}\omega,\varepsilon_N}) &\geq \int_{H_{\omega,\varepsilon_N} \cap H_{\sigma^{-p}\omega,\varepsilon_N}} \rho(T_{x_0}^{-p}y)|DT^p(T_{x_0}^{-p}y))|^{-1} d\mathrm{Leb}(y) \\ &\geq \inf_{H_{\omega,\varepsilon_N}} |DT^p|^{-1} \inf_{H_{\omega,\varepsilon_N}} \rho \ \mathrm{Leb}(H_{\omega,\varepsilon_N}) - \int_{H_{\omega,\varepsilon_N} \setminus H_{\sigma^{-p}\omega,\varepsilon_N}} \rho(T_{x_0}^{-p}y)|DT^p(T_{x_0}^{-p}y)|^{-1} d\mathrm{Leb}(y) \end{aligned}$$

We can bound the second negative term as

$$\int_{H_{\omega,\varepsilon_N} \setminus H_{\sigma^{-p}\omega,\varepsilon_N}} \rho(T_{x_0}^{-p}y)|DT^p(T_{x_0}^{-p}y)|^{-1}(y) d\mathrm{Leb}(y) \leq \sup_{H_{\omega,\varepsilon_N}} \rho \sup_{H_{\omega,\varepsilon_N}} |DT^p|^{-1} \ \mathrm{Leb}(H_{\omega,\varepsilon_N} \Delta H_{\sigma^{-p}\omega,\varepsilon_N}).$$

We are now in a position to verify the existence of the limit $\lim_{\varepsilon_N \to 0} \hat{q}^{(p-1)}_{\omega,\varepsilon_N}$. Using the above bounds we have

$$\frac{\inf_{H_{\omega,\varepsilon_N}} |DT^p|^{-1} \inf_{H_{\omega,\varepsilon_N}} \rho \ \mathrm{Leb}(H_{\omega,\varepsilon_N})}{\sup_{H_{\omega,\varepsilon_N}} \rho \mathrm{Leb}(H_{\omega,\varepsilon_N})} - \frac{\sup_{H_{\omega,\varepsilon_N}} |DT^p|^{-1} \ \mathrm{Leb}(H_{\omega,\varepsilon_N} \Delta H_{\sigma^{-p}\omega,\varepsilon_N})}{\mathrm{Leb}(H_{\omega,\varepsilon_N})}$$

$$\leq \hat{q}^{(p-1)}_{\omega,\varepsilon_N} \leq \frac{\sup_{H_{\omega,\varepsilon_N}} |DT^p|^{-1} \sup_{H_{\omega,\varepsilon_N}} \rho \text{ Leb}(H_{\omega,\varepsilon_N})}{\inf_{H_{\omega,\varepsilon_N}} \rho \text{Leb}(H_{\omega,\varepsilon_N})}.$$

Since the map T^p and the density ρ are continuous at x_0, and by using the assumption (2.7.6), we will finally obtain the ω-independent value

$$\theta_{\omega,0} = 1 - \frac{1}{|DT^p(x_0)|}.$$

Theorem 2.4.5 may now be applied to obtain the quenched extreme value law.

Example 2.7.4. **Random maps with random holes**

We saw in Examples 2.7.1 and 2.7.3 that an extremal index less than one could be obtained for observables reaching their maximum around a point which was periodic for all the random maps, or, for a fixed map, when the holes shrink around the periodic point. We now produce an example where periodicity is not responsible for getting an extremal index less than one. This example is the quenched version of the annealed cases investigated in sections 4.1.2 and 4.2.1 of the paper [**19**].

Let $\Omega = \{0, \dots, l-1\}^{\mathbb{Z}}$, with σ the bilateral shift map, and m an invariant ergodic measure. To each letter $j = 0, \dots, l-1$ we associate a point $v(j)$ in the unit circle S^1 and we consider the well-known observable in the extreme value theory literature:

$$h_\omega(x) = -\log|x - v(\omega_0)|, \ x \in S$$

where ω_0 denotes the 0-th coordinate of $\omega \in \Omega$.

In this case the hole H_{ω,ε_N} will be the ball $B(v(\omega_0), e^{-z_N(\omega)})$, of center $v(\omega_0)$ and radius $e^{-z_N(\omega)}$;. For each $\omega \in \Omega$ we associate a map T_{ω_0}, where $T_0, \dots, T_{l-1}$ are maps of the circle which we will take as β-maps of the form[6] $T_i(x) = \beta_i x + r_i \pmod 1$, with $\beta_i \in \mathbb{N}$, $\beta_i \geq 3$, and $0 \leq r_i < 1$. Since the range of $\omega \mapsto \mathcal{L}_{\omega,0}$ is finite and the shift is a homeomorphism on Ω with respect to the usual metrics for Ω, assumption (MC) is verified. Since the potential is equal to $1/|T'_\omega|$, conditions (E1) to (E7) clearly hold. Condition (E8) holds as in Example 2.7.1, which uses similar piecewise linear expanding maps. Condition (E9) is a consequence of the fact that we have finitely many maps each of which is full branches; it will be therefore enough to invoke (E9a) with $n' = k_o(n') = 1$. As we have chosen $\beta_i \geq 3$ we have that (EX) holds and thus (X) follows via Proposition 2.5.5.

At this point we may apply Theorem 2.5.12 to obtain a quenched thermodynamic formalism for sufficiently small holes.

[6]The reason for this choice is that, in order to compute the quantities $\hat{q}^{(k)}_{\omega,0}$, we have to follow the itinerary of the points $v(i)$ under the composition of the maps T_i and compare with their predecessors. As it will be clear in the computation below for $k = 0$ and 1, the task will be relatively easy and generalizable to any $k > 0$, if all the maps are at least C^1 which in particular means that the image of the point $T_i(v_j), i, j = 0, \dots, l-1$ is not a discontinuous point of the $T_i, i = 0, \dots, l-1$,. It will also be true that all the maps $T_i, 0 = 1, \dots, l-1$ are differentiable with bounded derivative in small neighbourhoods of any $v_i, i = 0, \dots, l-1$. If these conditions are relaxed, it could be that the limit defining the $\hat{q}^{(k)}_{\omega,0}$ for some k is not immediately computable, we defer to section 3.3 and Proposition 3.4 in [**6**] for a detailed discussion of the computation of the extremal index in presence of discontinuities. Another advantage of our choice is that, as in Example 2.7.1, all the sample measure $\mu_{\omega,0}$ are equivalent to Lebesgue, Leb.

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

As a more concrete example, we will consider an alphabet of four letters $\mathcal{A} := \{0, 1, 2, 3\}$ and we set the associations

$$i \to v_i := v(i),\ i = 0, 1, 2, 3,$$

where the points $v_i \in (0, 1)$, are chosen on the unit interval according to the following prescriptions:

$$T_1(v_1) = T_2(v_2) = v_0;\ T_0(v_0) = v_3;\ T_3(v_i) \neq v_3, i = 0, 1, 2, 3. \tag{2.7.9}$$

Since the sample measures coincide with the Lebesgue measure, condition (S) reduces to $2e^{-z_N(\omega)} = \frac{t_\omega + \xi_{\omega,N}}{N}$ and we can solve for $z_N(\omega)$ by setting $\xi_{\omega,N} \equiv 0$. The prescription (2.7.9) clearly avoids any sort of periodicity when we take the first iteration of the random maps needed to compute $\hat{q}^{(0)}_{\omega,0}$; nevertheless we will show that $\hat{q}^{(0)}_{\omega,0} > 0$ and this is sufficient to conclude that $\theta_{\omega,0}$ is smaller than 1. Of course we need to prove that the limits defining all the other $\hat{q}^{(k)}_{\omega,0}$ for any $k > 0$ exist. We will show it for $k = 1$ and the same arguments could be generalized to any $k > 1$.

Using T-invariance of Lebesgue, we have (for simplicity we write z_N instead of $z_N(\omega)$):

$$\hat{q}^{(0)}_{\omega,0} = \lim_{N\to\infty} \frac{\mathrm{Leb}\left(T^{-1}_{\sigma^{-1}\omega} B(v(\omega_0), e^{-z_N}) \cap B(v((\sigma^{-1}\omega)_0), e^{-z_N})\right)}{\mathrm{Leb}(B(v(\omega_0), e^{-z_N}))},$$

provided the limit exists, which we are going to establish. Consider first a point ω in the cylinder $[\omega_{-1} = 0, \omega_0 = 3]$; the quantity we have to compute is therefore

$$\hat{q}^{(0)}_{\omega,0} = \frac{\mathrm{Leb}\left(T_0^{-1} B(v_3, e^{-z_N}) \cap B(v_0, e^{-z_N})\right)}{\mathrm{Leb}(B(v_3, e^{-z_N}))} \tag{2.7.10}$$

If N is large enough, by the prescription (2.7.9), $T_0(v_0) = v_3$, we see that the local preimage of $B(v_3, e^{-z_N})$ under T_0^{-1} will be strictly included into $B(v_0, e^{-z_N})$ and its length will be contracted by a factor β_0^{-1}. Therefore the ratio (2.7.10) will be simply β_0^{-1}. The same happens with the cylinders $[\omega_{-1} = 1, \omega_0 = 0]$ and $[\omega_{-1} = 2, \omega_0 = 0]$, producing respectively the quantities β_1^{-1} and β_2^{-1}. For all the other cylinders the ratio will be zero since the sets entering the numerator of (2.7.10) will be disjoint for N large enough. In conclusion we have

$$\int_\Omega t_\omega \hat{q}^{(0)}_{\omega,0} dm = \int_{[\omega_{-1}=0,\omega_0=3]} t_\omega \beta_0^{-1} dm + \int_{[\omega_{-1}=1,\omega_0=0]} t_\omega \beta_1^{-1} dm + \int_{[\omega_{-1}=2,\omega_0=0]} t_\omega \beta_2^{-1} dm.$$

If we now take $t_\omega = t$, ω-independent, and we choose m as the Bernoulli measure with equal weights $1/4$, the preceding expression assumes the simpler form

$$\int_\Omega t \hat{q}^{(0)}_{\omega,0} dm = \frac{t}{16}(\beta_0^{-1} + \beta_1^{-1} + \beta_2^{-1}).$$

To compute $\hat{q}^{(1)}_{\omega,0}$ we have to split the integral over cylinders of length three. As a concrete example let us consider the cylinder $[\omega_{-2} = i, \omega_{-1} = j, \omega_0 = k]$, where $i, j, k \in \mathcal{A}$. By using the preceding notations, we have to control the set

$$T_i^{-1} T_j^{-1} B(v_k, e^{-z_N}) \cap T_i^{-1} B^c(v_j, e^{-z_N}) \cap B(v_i, e^{-z_N}) =$$
$$T_i^{-1}\left(T_j^{-1} B(v_k, e^{-z_N}) \cap B^c(v_j, e^{-z_N})\right) \cap B(v_i, e^{-z_N}).$$

Let us first consider the intersection $T_j^{-1}B(v_k, e^{-z_N}) \cap B^c(v_j, e^{-z_N})$. Call u one of the preimages of $T_j^{-1}(v_k)$ and $T_{j,u}^{-1}$ the inverse branch giving $T_{j,u}^{-1}(v_k) = u$. If $u = v_j$ then the intersection $T_{j,u}^{-1}B(v_k, e^{-z_N}) \cap B^c(v_j, e^{-z_N})$ is empty. Otherwise, by taking N large enough, the set $T_{j,u}^{-1}B(v_k, e^{-z_N})$ will be completely included in $B^c(v_j, e^{-z_N})$ and moreover we have, by the linearity of the maps, $T_{j,u}^{-1}B(v_k, e^{-z_N}) = B(u, \beta_j^{-1}e^{-z_N})$. We are therefore left with the computation of $T_i^{-1}B(u, \beta_j^{-1}e^{-z_N}) \cap B(v_i, e^{-z_N})$. If $u \in \mathcal{A}$ we proceed as in the computation of $\hat{q}_{\omega,0}^{(0)}$ by using the prescriptions (2.7.9); otherwise such an intersection will be empty for N large enough. For instance, if, in the example we are considering, we take $k = 0, j = 2, i = 0$ and we suppose that among the β_2 preimages of v_0 there is, besides v_0, v_3 too, namely $T_2(v_3) = v_0$, and no other element of $\mathcal{A}$, [7] then we get the contribution for $\hat{q}_{\omega,0}^{(1)}$:

$$\int_{[\omega_{-2}=0,\omega_{-1}=2,\omega_0=0]} t_\omega \beta_2^{-1}\beta_0^{-1} dm.$$

From the above it follows that $\theta_{\omega,0}$ exists for a.e. ω, and that we may apply Theorem 2.4.5.

It is not difficult to give an example where all the $\hat{q}_{\omega,0}^{(k)}$ can be explicitly computed. Let us take our beta maps $T_i(x) = \beta_i x + r_i$-mod 1 in the particular case where all the r_i are equal to the irrational number r. Then take a sequence of random balls $B(v((\sigma^k\omega)_0), e^{-z_N}), k \geq 0$ with the centers $v((\sigma^k\omega)_0), k \geq 0$ which are rational numbers. From what we discussed above, it follows that a necessary condition to get a $\hat{q}_{\omega,0}^{(k)} \neq 0$ is that the center $v((\sigma^{-(k+1)}\omega)_0)$ will be sent to the center $v(\omega_0)$. Let z one of this rational centers; the iterate $T_\omega^n(z)$ has the form $T_\omega^n(z) = \beta_{\omega_{n-1}} \cdots \beta_{\omega_0} z + k_n r$,-mod1, where k_n is an integer number. Therefore such an iterate will be never a rational number which shows that all the $\hat{q}_{\omega,0}^{(k)} = 0$ for any $k \geq 0$ and ω, and therefore $\theta_{\omega,0} = 1$.

[7] This condition does not intervene to compute $\hat{q}_{\omega,0}^{(0)}$ and shows that it is strictly less than one.

Chapter 2 Appendices

2.A. A version of [33] with general weights

In this appendix we outline how to extend relevant results from [33] from the Perron–Frobenius weight $g_{\omega,0} = 1/|T'_\omega|$ to the general class of weights $g_{\omega,0}$ in Section 2.5. To begin, we note that there is a unique measurable equivariant family of functions $\{\phi_{\omega,0}\}_{\omega\in\Omega}$ guaranteed by Theorem 2.19 [3] (called q_ω there). We wish to obtain uniform control on the essential infimum and essential supremum of $\phi_{\omega,0}$ for a suitable class of maps.

In [33] we work with the space $\mathrm{BV}_1 = \{h \in L^1(\mathrm{Leb}) : \mathrm{var}(h) < \infty\}$ and use the norm $\|\cdot\|_{\mathrm{BV}_1} = \mathrm{var}(\cdot) + \|\cdot\|_1$. Here we have a measurable family of random conformal probability measures $\nu_{\omega,0}$ (guaranteed by Theorem 2.19 [3]) and we work with the random spaces $\mathcal{B}_\omega = \mathrm{BV}_{\nu_{\omega,0}} = \{h \in L^1(\nu_{\omega,0}) : \mathrm{var}(h) < \infty\}$ and the random norms $\|\cdot\|_{\mathcal{B}_\omega} = \mathrm{var}(\cdot) + \|\cdot\|_{L^1(\nu_{\omega,0})}$. We also work with the normalised transfer operator $\tilde{\mathcal{L}}_{\omega,0}(f) := \mathcal{L}_{\omega,0}(f)/\nu_{\omega,0}(\mathcal{L}_{\omega,0}\mathbb{1})$. All of the variation axioms (V1)–(V8) in [33] hold with the obvious replacements.

Proof of Lemma 2.5.9.

C1': Since $\operatorname{ess\,inf}_\omega \inf \mathcal{L}_{\omega,0}\mathbb{1} \geq \operatorname{ess\,inf}_\omega \inf g_{\omega,0}$, (E3) and (2.5.6) together imply that (C1') holds.

C7' ($\varepsilon = 0$): We show $\operatorname{ess\,inf}_\omega \inf \phi_{\omega,0} > 0$; this will give us (C7'). The statement $\operatorname{ess\,inf}_\omega \inf \phi_{\omega,0} > 0$ is a generalized version of Lemmas 1 and 5 [33] and we follow the strategy in [33]. The result follows from Lemma 2.A.2 below, which in turn depends on (2.A.3) and Lemma 2.A.1.

C4' ($\varepsilon = 0$): It will be sufficient to show that there is a $K < \infty$ and $0 < \gamma < 1$ such that for all $f \in \mathcal{B}_\omega$ with $\nu_{\omega,0}(f) = 0$ and a.e. ω, one has

$$\|\tilde{\mathcal{L}}^n_{\omega,0} f\|_{\mathcal{B}_{\sigma^n\omega}} \leq K\gamma^n \|f\|_{\mathcal{B}_\omega} \text{ for all } n \geq 0. \tag{2.A.1}$$

This is a generalized version of Lemma 2 [33], which has an identical proof, making the replacements outlined in the proof of Lemma 2.A.1 below, and using Lemmas 2.A.1–2.A.5 and (2.A.3). We also use the non-random equivalence (2.5.13) of the $\mathcal{B}_\omega$ norm to the usual BV_1 norm.

C2 ($\varepsilon = 0$), C3 ($\varepsilon = 0$), C5' ($\varepsilon = 0$): We wish to show that there is a unique measurable, nonnegative family $\phi_{\omega,0}$ with the property that $\phi_{\omega,0} \in \mathcal{B}_\omega$, $\int \phi_{\omega,0}\, d\nu_{\omega,0} = 1$, $\tilde{\mathcal{L}}_{\omega,0}\phi_{\omega,0} = \phi_{\sigma\omega,0}$ for a.e. ω, $\operatorname{ess\,sup}_\omega \|\phi_{\omega,0}\|_{\mathcal{B}_\omega} < \infty$, and

$$\operatorname*{ess\,sup}_\omega \|\phi_{\omega,0}\|_{\mathcal{B}_\omega}. \tag{2.A.2}$$

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

We note that again the norm equivalence (2.5.13). This is a generalized version of Proposition 1 [33]. To obtain this generalization, in the proof of Proposition 1 [33], one modifies the space Y to become

$$Y = \{v : \Omega \times X \to \mathbb{R} : v \text{ measurable}, v_\omega := v(\omega, \cdot) \in \mathcal{B}_\omega \text{ and } \operatorname*{ess\,sup}_\omega \|v_\omega\|_{\mathcal{B}_\omega} < \infty\}.$$

All of the arguments go through as per [33] with the appropriate substitutions. Our modified proof of Proposition 1 [33] will also use the modified Lemmas 2.A.1–2.A.5, and inequality (2.A.3) below.

CCM: Finally, we note that (CCM) follows from (C2) ($\varepsilon = 0$) together with non-atomicity of $\nu_{\omega,0}$. But non-atomicity of $\nu_{\omega,0}$ follows from the random covering assumption as in the proof of Proposition 3.1 [3]

Full Support: Following the proof of Claim 3.1.1 of [3], we are able to show that $\nu_{\omega,0}$ is fully supported on $[0,1]$, i.e. $\nu_{\omega,0}(J) > 0$ for any non-degenerate interval $J \subseteq [0,1]$. □

We note that by (2.5.19) with $\varepsilon = 0$, we have our uniform Lasota–Yorke equality.

$$\operatorname{var}(\tilde{\mathcal{L}}^n_{\omega,0}\psi) \le A\alpha^n \operatorname{var}(\psi) + B\nu_{\omega,0}(|\psi|), \tag{2.A.3}$$

for all $n \ge 1$ and a.e. ω. This immediately provides a suitable general version of (H2) [33], which is that there is a $C < \infty$ such that

$$\|\tilde{\mathcal{L}}_{\omega,0}\psi\|_{\mathcal{B}_{\sigma\omega}} \le C\|\psi\|_{\mathcal{B}_\omega} \text{ for a.e. } \omega. \tag{2.A.4}$$

Define random cones $\mathcal{C}_{a,\omega} = \{\psi \in \mathcal{B}_\omega : \psi \ge 0, \operatorname{var}\psi \le a \int \psi \, d\nu_{\omega,0}\}$.

Lemma 2.A.1 (General weight version of Lemma A.1 [33]). *For sufficiently large $a > 0$ we have that $\tilde{\mathcal{L}}^{RN}_{\omega,0}\mathcal{C}_{a,\omega} \subset \mathcal{C}_{a/2,\sigma^{RN}\omega}$ for sufficiently large R and a.e. ω.*

Proof. Identical to [33], substituting $\mathcal{C}_{a,\omega}$ for $\mathcal{C}_a$, $\tilde{\mathcal{L}}_{\omega,0}$ for $\mathcal{L}_{\omega,0}$, and $\nu_{\omega,0}$ for Lebesgue. □

Lemma 2.A.2 (General weight version of Lemma 1 [33]). *If one has uniform covering in the sense of (11) [33], then there is an N such that for each $a > 0$ and sufficiently large n, there exists $c > 0$ such that*

$$\operatorname*{ess\,inf}_\omega \tilde{\mathcal{L}}^{Nn}_\omega h \ge (c/2)\nu_{\omega,0}(|h|) \text{ for every } h \in \mathcal{C}_{\omega,a} \text{ and a.e. } \omega. \tag{2.A.5}$$

Proof. Making all of the obvious substitutions, as per Lemma 2.A.1 and its proof, we subdivide the unit interval into an equipartition according to $\nu_{\omega,0}$ mass. This is possible because $\nu_{\omega,0}$ is non-atomic (Proposition 3.1 [3]). We conclude, as in the proof of Lemma 1 [33], that there is an interval J of $\nu_{\omega,0}$-measure $1/n$ such that for each $f \in \mathcal{C}_{\omega,a}$, one has $\inf_J f \ge (1/2)\nu_{\omega,0}(f)$. Then using uniform covering and the facts that $\operatorname{ess\,inf}_\omega \inf g_{\omega,0} > 0$, $\operatorname{ess\,sup}_\omega g_{\omega,0} < \infty$, and $\operatorname{ess\,sup}_\omega D(T_\omega) < \infty$, we obtain $\operatorname{ess\,inf}_\omega \inf \tilde{\mathcal{L}}^k_{\omega,0} f \ge \alpha_0^* > 0$, where k is the uniform covering time for the interval J. The rest of the proof follows as in [33]. □

Lemma 2.A.3 (General weight version of Lemma A.2 [33]). *Assume that $\psi, \psi' \in \mathcal{C}_{a,\omega}$ and $\int \psi \, d\nu_{\omega,0} = \int \psi' \, d\nu_{\omega,0} = 1$. Then $\|\psi - \psi'\|_{\mathcal{B}_\omega} \le 2(1+a)\Theta_{a,\omega}(\psi, \psi')$.*

Proof. Identical to [33], substituting $\nu_{\omega,0}$ for Lebesgue. The randomness of the Hilbert metric $\Theta_{a,\omega}$ only appears because the functions lie in $\mathcal{C}_{a,\omega}$. □

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

LEMMA 2.A.4 (General weight version of Lemma A.3 [33]). *For any $a \geq 2\mathrm{var}(\mathbb{1}_X)$, we have that $\tilde{\mathcal{L}}^{RN}_{\omega,0}$ is a contraction on $\mathcal{C}_{\omega,a}$ for any sufficiently large R and a.e. $\omega \in \Omega$.*

PROOF. The proof in [33] may be followed, making the substitutions as in Lemma 2.A.1 and its proof. The first inequality reads

$$\operatorname*{ess\,sup}_{\omega} \tilde{\mathcal{L}}^{RN}_{\omega,0} f \leq \nu_{\sigma^{RN}\omega}(|\tilde{\mathcal{L}}^{RN}_{\omega,0} f|) + C_{var}(\tilde{\mathcal{L}}^{RN}_{\omega,0} f) \leq (1 + C_{var}a/2)\nu_{\omega,0}(|f|) = 1 + C_{var}a/2,$$

where we have used axiom (V3) [33] and the weak contracting property of $\tilde{\mathcal{L}}^{RN}_{\omega,0}$ in the $\nu_{\omega,0}$ norm. The rest of the proof follows as in [33]. □

Let $\mathcal{B}_{\omega,0} = \{\psi \in \mathcal{B}_\omega : \int \psi \, d\nu_{\omega,0} = 0\}$.

LEMMA 2.A.5 (General weight version of Lemma A.4 [33]).

PROOF. Identical to [33], substituting as per Lemma 2.A.1 and Lemma 2.A.3 and their proofs above, and using (2.A.4). □

LEMMA 2.A.6 (General weight version of Lemma 5 [33]). $\operatorname{ess\,inf}_\omega \phi_{\omega,0} \geq c/2$ *for a.e. ω.*

PROOF. Identical to [33] with the appropriate substitutions. □

2.B. A summary of checks that relevant results from [22] can be applied to BV_1

The stability result Theorem 4.8 [22] assumes that the underlying Banach space is separable, however this separability assumption is only used to obtain measurability of various objects (and in fact Theorem 3.9 [22] may be applied to sequential dynamics). We use these results for the non-separable space BV_1 in the proof of Lemma 2.5.10, and we therefore need to check that all relevant results in [22] hold for BV_1, under the m-continuity assumption on $\omega \mapsto \mathcal{L}_{\omega,0}$; the latter will provide the required measurability. All section, theorem, proposition, and lemma numbers below refer to numbering in [22].

There are no issues of measurability in Section 3, including Theorem 3.9, until Section 4, so we begin our justifications from Section 4.

Theorem 4.8: This is the main stability theorem. We would substitute "separable strongly measurable random dynamical system" with m-continuous random dynamical system. This theorem relies on Propositions 4.15 and 4.16, Lemma 4.21, Proposition 4.22, and Lemma 4.23.

Proposition 4.15: There is no measurability involved.

Proposition 4.16: Uses Lemmas 4.17 and 4.19. We note that the bounds (95)–(97) in the proof are simpler in our application of this result as our top space is one-dimensional.

Lemma 4.17: This may be replaced by Theorem 17 [40], which treats the m-continuous setting. This removes any use of Lemma B.1 and Lemma B.7.

Lemma 4.18: This uses Lemma 4.17 and the fact that compositions of m-continuous maps are m-continuous. The latter replaces the use of Lemma A.5 [43], which is used in several results. This replacement will not be mentioned further.

Lemma 4.19: We will assume that $\omega \mapsto \Pi_\omega$ is m-continuous in the statement of the lemma. At the start of the proof we would now instead have $\omega \mapsto \Pi_\omega(X)$ is m-continuous by the definition of m-continuity (see e.g. (4) in [40]); this removes the use of Lemma B.2 in the proof. Lian's thesis is quoted regarding measurable maps/bases connected with a measurable space $\Pi_\omega(X)$. In our application, Π_ω has rank 1 and therefore stating that there is a (in our case m-continuous) map $e : \Omega \to \Pi_\omega(X)$ is trivial. One proceeds similarly for the dual basis.

Lemma 4.21: This concerns measurability and integrability of $\omega \mapsto \det(\mathcal{L}_\omega^n|E_{i,\omega})$ and $\omega \mapsto \log \|\mathcal{L}_\omega^n|E_{i,\omega}\|$ where $E_{i,\omega}$ is an Oseledets space. m-continuity of $\omega \mapsto E_{i,\omega}$ is provided by Theorem 17 [40], removing the need for Lemma B.2. The m-continuity of $\omega \mapsto \log \|\mathcal{L}_\omega^n|E_{i,\omega}\|$ follows from Lemma 7 [40]. Because in our application setting we only require one-dimensional $E_{i,\omega}$, the determinants are given by norms and there is nothing more to do concerning determinants. This removes the need for Proposition B.8. Lemma B.16 [43] may be replaced with Lemma 7 [40] to cover the m-continuous setting. Proposition B.6 is not required in the P-continuity setting.

Proposition 4.22: There is no measurability involved.

Lemma 4.23: There is no measurability involved.

2.C. A var–$\nu_{\omega,0}(|\cdot|)$ Lasota–Yorke inequality

Recall from Section 2.5 that $\mathcal{Z}_{\omega,0}^{(n)}$ denotes the partition of monotonicity of T_ω^n and that $\mathscr{A}_{\omega,0}^{(n)}$ is the collection of all finite partitions of $[0,1]$ such that

$$\mathrm{var}_{A_i}(g_{\omega,0}^{(n)}) \le 2\|g_{\omega,0}^{(n)}\|_\infty \tag{2.C.1}$$

for each $\mathcal{A} = \{A_i\} \in \mathscr{A}_{\omega,0}^{(n)}$. Given $\mathcal{A} \in \mathscr{A}_{\omega,0}^{(n)}$, we set $\mathcal{Z}_{\omega,*,\varepsilon}^{(n)} := \left\{ Z \in \widehat{\mathcal{Z}}_{\omega,\varepsilon}^{(n)}(\mathcal{A}) : Z \subseteq X_{\omega,n-1,\varepsilon} \right\}$ where $\widehat{\mathcal{Z}}_{\omega,\varepsilon}^{(n)}(\mathcal{A})$ is the coarsest partition amongst all those finer than $\mathcal{A}$ and $\mathcal{Z}_{\omega,0}^{(n)}$ such that all elements of $\widehat{\mathcal{Z}}_{\omega,\varepsilon}^{(n)}(\mathcal{A})$ are either disjoint from $X_{\omega,n-1,\varepsilon}$ or contained in $X_{\omega,n-1,\varepsilon}$. Then (2.C.1) implies that

$$\mathrm{var}_Z(g_{\omega,\varepsilon}^{(n)}) \le 2\|g_{\omega,0}^{(n)}\|_\infty \tag{2.C.2}$$

for each $Z \in \mathcal{Z}_{\omega,*,\varepsilon}^{(n)}$. We now prove a Lasota–Yorke inequality inspired by Lemma 1.5.1.

Lemma 2.C.1. *For any $f \in \mathrm{BV}_{\nu_{\omega,0}}$ we have*

$$\mathrm{var}(\mathcal{L}_{\omega,\varepsilon}^n(f)) \le 9\|g_{\omega,\varepsilon}^{(n)}\|_\infty \mathrm{var}(f) + \frac{8\|g_{\omega,\varepsilon}^{(n)}\|_\infty}{\min_{Z\in\mathcal{Z}_{\omega,*,\varepsilon}^{(n)}(A)} \nu_{\omega,0}(Z)} \nu_{\omega,0}(|f|).$$

PROOF. Since $\mathcal{L}^n_{\omega,\varepsilon}(f) = \mathcal{L}^n_{\omega,0}(f \cdot \hat{X}_{\omega,n-1,\varepsilon})$, if $Z \in \widehat{\mathcal{Z}}^{(n)}_{\omega,\varepsilon}(\mathcal{A}) \backslash \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}$, then $Z \cap X_{\omega,n-1,\varepsilon} = \emptyset$, and thus, we have $\mathcal{L}^n_{\omega,\varepsilon}(f\mathbb{1}_Z) = 0$ for each $f \in \mathrm{BV}_{\nu_{\omega,0}}$. Thus, considering only on intervals Z in $\mathcal{Z}^{(n)}_{\omega,*,\varepsilon}$, we are able to write

$$\mathcal{L}^n_{\omega,\varepsilon} f = \sum_{Z \in \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}} (\mathbb{1}_Z f g^{(n)}_{\omega,\varepsilon}) \circ T^{-n}_{\omega,Z} \tag{2.C.3}$$

where

$$T^{-n}_{\omega,Z} : T^n_\omega(I_{\omega,\varepsilon}) \to Z$$

is the inverse branch which takes $T^n_\omega(x)$ to x for each $x \in Z$. Now, since

$$\mathbb{1}_Z \circ T^{-n}_{\omega,Z} = \mathbb{1}_{T^n_\omega(Z)},$$

we can rewrite (2.C.3) as

$$\mathcal{L}^n_{\omega,\varepsilon} f = \sum_{Z \in \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}} \mathbb{1}_{T^n_\omega(Z)} \left((f g^{(n)}_{\omega,\varepsilon}) \circ T^{-n}_{\omega,Z} \right). \tag{2.C.4}$$

So,

$$\mathrm{var}(\mathcal{L}^n_{\omega,\varepsilon} f) \le \sum_{Z \in \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}} \mathrm{var}\left(\mathbb{1}_{T^n_\omega(Z)} \left((f g^{(n)}_{\omega,\varepsilon}) \circ T^{-n}_{\omega,Z} \right) \right). \tag{2.C.5}$$

Now for each $Z \in \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}$ we have

$$\begin{aligned}
\mathrm{var}\left(\mathbb{1}_{T^n_\omega(Z)} \left((f g^{(n)}_{\omega,\varepsilon}) \circ T^{-n}_{\omega,Z} \right) \right) &\le \mathrm{var}_Z(f g^{(n)}_{\omega,\varepsilon}) + 2 \sup_Z \left| f g^{(n)}_{\omega,\varepsilon} \right| \\
&\le 3\mathrm{var}_Z(f g^{(n)}_{\omega,\varepsilon}) + 2 \inf_Z \left| f g^{(n)}_{\omega,\varepsilon} \right| \\
&\le 3\|g^{(n)}_{\omega,\varepsilon}\|_\infty \mathrm{var}_Z(f) + 3 \sup_Z |f| \mathrm{var}_Z(g^{(n)}_{\omega,\varepsilon}) + 2\|g^{(n)}_{\omega,\varepsilon}\|_\infty \inf_Z |f| \\
&\le 3\|g^{(n)}_{\omega,\varepsilon}\|_\infty \mathrm{var}_Z(f) + 6\|g^{(n)}_{\omega,\varepsilon}\|_\infty \sup_Z |f| + 2\|g^{(n)}_{\omega,\varepsilon}\|_\infty \inf_Z |f| \\
&\le 9\|g^{(n)}_{\omega,\varepsilon}\|_\infty \mathrm{var}_Z(f) + 8\|g^{(n)}_{\omega,\varepsilon}\|_\infty \inf_Z |f| \\
&\le 9\|g^{(n)}_{\omega,\varepsilon}\|_\infty \mathrm{var}_Z(f) + 8\|g^{(n)}_{\omega,\varepsilon}\|_\infty \frac{\nu_{\omega,0}(|f|_Z|)}{\nu_{\omega,0}(Z)}. \qquad (2.C.6)
\end{aligned}$$

Using (2.C.6), we may further estimate (2.C.5) as

$$\begin{aligned}
\mathrm{var}(\mathcal{L}^n_{\omega,\varepsilon} f) &\le \sum_{Z \in \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}} \left(9\|g^{(n)}_{\omega,\varepsilon}\|_\infty \mathrm{var}_Z(f) + 8\|g^{(n)}_{\omega,\varepsilon}\|_\infty \frac{\nu_{\omega,0}(|f|_Z|)}{\nu_{\omega,0}(Z)} \right) \\
&\le 9\|g^{(n)}_{\omega,\varepsilon}\|_\infty \mathrm{var}(f) + \frac{8\|g^{(n)}_{\omega,\varepsilon}\|_\infty}{\min_{Z \in \mathcal{Z}^{(n)}_{\omega,*,\varepsilon}(A)} \nu_{\omega,0}(Z)} \nu_{\omega,0}(|f|), \qquad (2.C.7)
\end{aligned}$$

and thus we are done. □

REMARK 2.C.2. Note that we could have used Leb or any probability measure in (2.C.6) rather than $\nu_{\omega,0}$. Furthermore, Lemma 2.C.1 could be applied to Section 2.5 with a measure other than $\nu_{\omega,0}$ if the appropriate changes are made to the assumption (E9) so that a uniform-in-ω lower bound similar to (2.5.18) may be calculated.

In particular if we replace (E9) with the following:

(E9a) There exists $k_o(n') \in \mathbb{N}$ and $\delta > 0$ such that for m-a.e. $\omega \in \Omega$, all $\varepsilon > 0$ sufficiently small, we have $\mathrm{Leb}(Z) > \delta$ for all $Z \in \mathcal{Z}^{(n')}_{\omega,*,\varepsilon}(\mathcal{A})$,

(E9b) There exists $c > 0$ such that $\operatorname{ess\,inf}_\omega |T'_\omega| > c$,

then the claims of Section 2.5 hold with $\nu_{\omega,0}$ in (2.C.6) replaced with Leb.

Indeed, to obtain a replacement for (2.5.18) one could use (E9a) and (E9b) to get

$$\begin{aligned}\mathrm{Leb}(Z) = \mathrm{Leb}(\tilde{\mathcal{L}}^{k_o(n')}_{\omega,0}\mathbb{1}_Z) &\geq \frac{\inf g^{(k_o(n'))}_{\omega,0} \inf J^{(k_o(n'))}_\omega}{\lambda^{k_o(n')}_{\omega,0}} \mathrm{Leb}(P^{k_o(n')}_\omega \mathbb{1}_Z) \\ &\geq \operatorname*{ess\,inf}_\omega \frac{\inf g^{(k_o(n'))}_{\omega,0} \inf J^{(k_o(n'))}_\omega}{\lambda^{k_o(n')}_{\omega,0}} \mathrm{Leb}(Z) > 0\end{aligned}$$

for all $Z \in \mathcal{Z}^{(n')}_{\omega,*,\varepsilon}(\mathcal{A})$, where we have also used (E3) for the final inequality. As (E9) is only used to prove (2.5.18), the remainder of Section 2.5 can be carried out with the appropriate notational changes. In fact, the proof of Lemma 2.5.10 can be simplified by replacing $\nu_{\omega,0}$ with Leb as Lemma 5.2 of [12] would no longer be needed.

REMARK 2.C.3. Note that the 2 appearing in (2.C.1), and thus the 9 and 8 appearing in (2.C.7), are not optimal. See [3] and Section 1.15 for how these estimates can be improved.

2.D. Proof of Claim (6) of Theorem 2.5.12

PROOF OF CLAIM (6) OF THEOREM 2.5.12: Set $\mathcal{B}_\omega = \mathrm{BV}_{\nu_{\omega,0}}$. Define the fully normalized operator $\mathscr{L}_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ given by

$$\mathscr{L}_{\omega,\varepsilon}(f) := \frac{1}{\rho_{\omega,\varepsilon}\psi_{\sigma\omega,\varepsilon}} \mathcal{L}_{\omega,\varepsilon}(f \cdot \psi_{\omega,\varepsilon}).$$

Then we have that $\mathscr{L}_{\omega,\varepsilon}\mathbb{1} = \mathbb{1}$ and $\mu_{\sigma\omega,\varepsilon}(\mathscr{L}_{\omega,\varepsilon} f) = \mu_{\omega,\varepsilon}(f)$. Following the proof of Theorem 1.11.2, we can prove the following similar statement to Claim (5) of Theorem 2.5.12: For each $h \in \mathcal{B}_\omega$, m-a.e. $\omega \in \Omega$ and all $n \in \mathbb{N}$ we have

$$\|\mathscr{L}^n_{\omega,\varepsilon} h - \mu_{\omega,\varepsilon}(h)\mathbb{1}\|_{\mathcal{B}_{\sigma^n\omega}} = \|\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\|_{\mathcal{B}_{\sigma^n\omega}} \leq D\|h\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon, \tag{2.D.1}$$

where $\hat{h} := h - \mu_{\omega,\varepsilon}(h)$ and D, κ_ε are as in Claim (5) of Theorem 2.5.12. Using standard arguments (see Theorem 11.1 [3]) we have that

$$|\mu_{\omega,\varepsilon}((f \circ T^n_\omega)\, h) - \mu_{\sigma^n\omega,\varepsilon}(f)\mu_{\omega,\varepsilon}(h)| = \mu_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\right).$$

Note that at this stage we are unable to apply (2.D.1) as the $\|\cdot\|_{\mathcal{B}_\omega}$ norm and the measure $\mu_{\omega,\varepsilon}$ are incompatible. Now from the third statement of Claim (5) of Theorem 2.5.12 we

have that

$$\left|\mu_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\right)-\frac{\varrho_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\hat{X}_{\sigma^n\omega,n,\varepsilon}\right)}{\varrho_{\sigma^n\omega,\varepsilon}\left(X_{\sigma^n\omega,n,\varepsilon}\right)}\right|\leq D\|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\|_{\mathcal{B}_{\sigma^n\omega}}\kappa^n_\varepsilon,$$

and thus we must have that

$$\mu_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\right)\leq D\|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\|_{\mathcal{B}_{\sigma^n\omega}}\kappa^n_\varepsilon+\frac{\varrho_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\hat{X}_{\sigma^n\omega,n,\varepsilon}\right)}{\varrho_{\sigma^n\omega,\varepsilon}(X_{\sigma^n\omega,n,\varepsilon})}. \tag{2.D.2}$$

Using (2.D.1) and (2.5.14), we have that

(2.D.3)

$$\frac{\varrho_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\hat{X}_{\sigma^n\omega,n,\varepsilon}\right)}{\varrho_{\sigma^n\omega,\varepsilon}(X_{\sigma^n\omega,n,\varepsilon})}\leq\|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\|_{\sigma^n\omega,\infty}\leq\|f\|_{\sigma^n\omega,\infty}\|\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\|_{\mathcal{B}_{\sigma^n\omega}}\leq D\|f\|_{\infty,\omega}\|h\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon.$$

Combining (2.D.2) and (2.D.3) and using (2.D.1) again we see that

$$\begin{aligned}
|\mu_{\omega,\varepsilon}\left((f\circ T^n_\omega)\,h\right)-\mu_{\sigma^n\omega,\varepsilon}(f)\mu_{\omega,\varepsilon}(h)|&\leq\mu_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\right)\\
&\leq D\|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\|_{\mathcal{B}_{\sigma^n\omega}}\kappa^n_\varepsilon+\frac{\varrho_{\sigma^n\omega,\varepsilon}\left(\left|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\right|\hat{X}_{\sigma^n\omega,n,\varepsilon}\right)}{\varrho_{\sigma^n\omega,\varepsilon}(X_{\sigma^n\omega,n,\varepsilon})}\\
&\leq D\|f\mathscr{L}^n_{\omega,\varepsilon}\hat{h}\|_{\mathcal{B}_{\sigma^n\omega}}\kappa^n_\varepsilon+D\|f\|_{\infty,\omega}\|h\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon\\
&\leq D^2\|f\|_{\mathcal{B}_\omega}\|h\|_{\mathcal{B}_\omega}\kappa^{2n}_\varepsilon+D\|f\|_{\infty,\omega}\|h\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon\\
&\leq\tilde{D}\|f\|_{\infty,\omega}\|h\|_{\mathcal{B}_\omega}\kappa^n_\varepsilon
\end{aligned}$$

for all n sufficiently large, and thus the proof of Claim (6) of Theorem 2.5.12 is complete. □

Acknowledgments

The authors thank Harry Crimmins for helpful discussions. JA, GF, and CG-T thank the Centro de Giorgi in Pisa and CIRM in Luminy for their support and hospitality. JA is supported by the ARC Discovery projects DP180101223 and DP210100357, and thanks the School of Mathematics and Physics at the University of Queensland for their hospitality. GF, CG-T, and SV are partially supported by the ARC Discovery Project DP180101223. The research of SV was supported by the project *Dynamics and Information Research Institute* within the agreement between UniCredit Bank R&D group for financial support through the "Dynamics and Information Theory Institute" at the Scuola Normale Superiore di Pisa and by the Laboratoire International Associé LIA LYSM, of the French CNRS and INdAM (Italy).

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

Bibliography

[1] Jason Atnip. An Almost Sure Invariance Principle for Several Classes of Random Dynamical Systems. 2017. Preprint: arXiv: 1702.07691. 122

[2] Jason Atnip, Gary Froyland, Cecilia González-Tokman, and Sandro Vaienti. Equilibrium states for non-transitive random open and closed dynamical systems. *arXiv preprint arXiv:2107.03776*, 2021. 6, 7, 17, 93, 113

[3] Jason Atnip, Gary Froyland, Cecilia González-Tokman, and Sandro Vaienti. Thermodynamic Formalism for Random Weighted Covering Systems. *Communications in Mathematical Physics*, July 2021. 5, 6, 7, 9, 17, 24, 33, 39, 41, 43, 45, 46, 52, 63, 64, 65, 67, 68, 69, 74, 75, 78, 79, 80, 81, 114, 115, 122, 130, 133, 139, 140, 144

[4] Jason Atnip, Hiroki Sumi, and Mariusz Urbański. The Dynamics and Geometry of Semi-Hyperbolic Rational Semigroups. *arxiv preprint arXiv:2004.13990*, 2020. 14

[5] Jason Atnip and Mariusz Urbański. Critically finite random maps of an interval. *Discrete & Continuous Dynamical Systems - A*, 40(8):4839, 2020. 6, 62

[6] H. Aytac, J.M. Freitas, and S. Vaienti. Laws of rare events for deterministic and random dynamical systems. *Trans. Amer Math. Soc.*, 367:8229–8278, 2015. 136

[7] Kazuopki Azuma. Weighted sums of certain dependent random variables. *Tôhoku Math. Journ.*, 19(3):357–367, 1967. 126

[8] Wael Bahsoun, Jörg Schmeling, and Sandro Vaienti. On transfer operators and maps with random holes. *Nonlinearity*, 28(3):713–727, March 2015. 6

[9] Wael Bahsoun and Sandro Vaienti. Escape rates formulae and metastablilty for randomly perturbed maps. *Nonlinearity*, 26(5):1415–1438, May 2013. 6, 134

[10] Garrett Birkhoff. *Lattice Theory*. American Mathematical Society, New York, 1940. 31, 32

[11] Thomas Bogenschütz and Volker Matthias Gundlach. Ruelle's Transfer Operator for Random Subshifts of Finite Type. *Ergod. Th. & Dynam. Sys*, 15:413–447, 1995. 14, 17

[12] Christopher Bose, Gary Froyland, Cecilia González-Tokman, and Rua Murray. Ulam's method for Lasota–Yorke maps with holes. *SIAM Journal on Applied Dynamical Systems*, 13(2):1010–1032, 2014. 118, 144

[13] Rufus Bowen. Hausdorff dimension of quasi-circles. *Publications Mathématiques de l'IHÉS*, 50:11–25, 1979. 6, 69

[14] A. Boyarsky and P. Gòra. *Laws of Chaos*. Birkhäuser, 1997. 123

[15] Rainer Brück and Matthias Büger. Generalized iteration. *Computational Methods and Function Theory*, 3(1-2, [On table of contents: 2004]):201–252, 2003. 14, 17

[16] Henk Bruin, Mark F. Demers, and Mike Todd. Hitting and escaping statistics: mixing, targets and holes. *Advances in Mathematics*, 328:1263–1298, April 2018. 10

[17] Henk Bruin and Mike Todd. Equilibrium States for Interval Maps: Potentials with $\sup\varphi - \inf\varphi < h_{\text{top}}(f)$. *Communications in Mathematical Physics*, 283(3):579–611, November 2008. 10

[18] Jérôme Buzzi. Exponential Decay of Correlations for Random Lasota-Yorke Maps. *Communications in Mathematical Physics*, 208(1):25–54, 1999. 33, 39, 40, 41

[19] Théophile Caby, Davide Faranda, Sandro Vaienti, and Pascal Yiou. On the computation of the extremal index for time series. *Journal of Statistical Physics*, 179:1666–1697, 2020. 136

[20] Pierre Collet, Servet Martínez, and Véronique Maume-Deschamps. On the existence of conditionally invariant probability measures in dynamical systems. *Nonlinearity*, 13(4):1263–1274, June 2000. 5

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

[21] Hans Crauel. *Random probability measures on Polish spaces*, volume 11 of *Stochastics Monographs*. Taylor & Francis, London, 2002. 15

[22] Harry Crimmins. Stability of hyperbolic Oseledets splittings for quasi-compact operator cocycles. *arXiv:1912.03008 [math]*, December 2019. arXiv: 1912.03008. 3, 13, 116, 117, 119, 122, 141

[23] Mark Demers, Paul Wright, and Lai-Sang Young. Escape Rates and Physically Relevant Measures for Billiards with Small Holes. *Communications in Mathematical Physics*, 294(2):353–388, March 2010. 6

[24] Mark F. Demers. Escape rates and physical measures for the infinite horizon Lorentz gas with holes. *Dynamical Systems*, 28(3):393–422, September 2013. 6

[25] Mark F. Demers and Bastien Fernandez. Escape rates and singular limiting distributions for intermittent maps with holes. *Transactions of the American Mathematical Society*, 368(7):4907–4932, December 2015. 6

[26] Mark F. Demers and Mike Todd. Equilibrium states, pressure and escape for multimodal maps with holes. *Israel Journal of Mathematics*, 221(1):367–424, September 2017. 6

[27] Mark F. Demers and Mike Todd. Slow and Fast Escape for Open Intermittent Maps. *Communications in Mathematical Physics*, 351(2):775–835, April 2017. 6

[28] Mark F. Demers and Mike Todd. Asymptotic escape rates and limiting distributions for multimodal maps. *Ergodic Theory and Dynamical Systems*, pages 1–50, March 2020. 6

[29] Mark F. Demers, Paul Wright, and Lai-Sang Young. Entropy, Lyapunov exponents and escape rates in open systems. *Ergodic Theory and Dynamical Systems*, 32(4):1270–1301, August 2012. 6

[30] Manfred Denker, Gerhard Keller, and Mariusz Urbański. On the uniqueness of equilibrium states for piecewise monotone mappings. *Studia Mathematica*, 97(1):27–36, 1990. 10

[31] Manfred Denker and Mariusz Urbański. On the existence of conformal measures. *Transactions of the American Mathematical Society*, 328(2):563–587, 1991. 10, 62

[32] D. Dragičević, G. Froyland, C. González-Tokman, and S. Vaienti. A Spectral Approach for Quenched Limit Theorems for Random Expanding Dynamical Systems. *Communications in Mathematical Physics*, 360(3):1121–1187, June 2018. 14, 119, 120, 122, 123, 124, 125

[33] Davor Dragičević, Gary Froyland, Cecilia González-Tokman, and Sandro Vaienti. Almost sure invariance principle for random piecewise expanding maps. *Nonlinearity*, 31(5):2252–2280, May 2018. 3, 14, 116, 122, 123, 128, 139, 140, 141

[34] Davor Dragičević and Yeor Hafouta. Almost Sure Invariance Principle for Random Distance Expanding Maps with a Nonuniform Decay of Correlations. 2020. Preprint: arXiv: 2005.01114. 122

[35] Andrew Ferguson and Mark Pollicott. Escape rates for Gibbs measures. *Ergodic Theory and Dynamical Systems*, 32(03):961–988, June 2012. 10

[36] A.C.M. Freitas, J.M. Freitas, and M. Todd. Hitting time statistics and extreme value theory. *Probab. Theory Related Fields*, 147(3):675–710, 2010. 109

[37] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, Mario Magalhaes, and Sandro Vaienti. Point processes of non stationary sequences generated by sequential and random dynamical systems. *Journal of Statistical Physics*, 181:1365–1409, 2020. 12

[38] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, and Mike Todd. Speed of convergence for laws of rare events and escape rates. *Stochastic Process. Appl.*, 125(4):1653–1687, 2015. 10

[39] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, and Sandro Vaienti. Extreme value laws for non stationary processes generated by sequential and random dynamical systems. *Annales de l'Institut Henri Poincaré*, 53(3):1341–1370, 2017. 12

[40] Gary Froyland, Simon Lloyd, and Anthony Quas. A semi-invertible Oseledets Theorem with applications to transfer operator cocycles. *Discrete and Continuous Dynamical Systems*, 33(9):3835–3860, March 2013. 111, 141, 142

[41] Gary Froyland and Ognjen Stancevic. Metastability, lyapunov exponents, escape rates, and topological entropy in random dynamical systems. *Stochastics and Dynamics*, 13(04):1350004, December 2013. Publisher: World Scientific Publishing Co. 6

[42] Brett Geiger and William Ott. Nonstationary open dynamical systems. *arXiv:1808.05315 [math]*, May 2020. arXiv: 1808.05315. 6

[43] Cecilia González-Tokman and Anthony Quas. A semi-invertible operator Oseledets theorem. *Ergodic Theory and Dynamical Systems*, 34(4):1230–1272, 2014. 142

[44] Franz Hofbauer and Gerhard Keller. Equilibrium states for piecewise monotonic transformations. *Ergodic Theory and Dynamical Systems*, 2(1):23–43, March 1982. 10

[45] Franz Hofbauer and Gerhard Keller. Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.*, 180:119–140, 1982. 123

[46] J. Hüsler. Asymptotic approximation of crossing probabilities of random sequences. *Z. Wahrsch. Verw. Gebiete*, 63(2):257–270, 1983. 12

[47] J. Hüsler. Extreme values of nonstationary random sequences. *J. Appl. Probab.*, 23(4):937–950, 1986. 12

[48] Gerhard Keller. Rare events, exponential hitting times and extremal indices via spectral perturbation†. *Dynamical Systems*, 27(1):11–27, 2012. 10, 11, 13, 133

[49] Gerhard Keller and Carlangelo Liverani. Stability of the spectrum for transfer operators. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 28(1):141–152, 1999. Publisher: Scuola normale superiore. 10

[50] Gerhard Keller and Carlangelo Liverani. Rare Events, Escape Rates and Quasistationarity: Some Exact Formulae. *Journal of Statistical Physics*, 135(3):519–534, 2009. 10, 11, 84

[51] M.R. Leadbetter, G. Lindgren, and H. Rootzen. Extremes and related properties of random sequences and series. 1983. 12

[52] Carlangelo Liverani. Decay of correlations for piecewise expanding maps. *Journal of Statistical Physics*, 78(3):1111–1129, February 1995. 47

[53] Carlangelo Liverani and Véronique Maume-Deschamps. Lasota–Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 39(3):385–412, May 2003. 5, 7, 10, 11, 13, 27, 30, 33, 47, 48, 52, 62, 67, 70, 77

[54] Carlangelo Liverani, Benoit Saussol, and Sandro Vaienti. Conformal measure and decay of correlation for covering weighted systems. *Ergodic Theory and Dynamical Systems*, 18(6):1399–1420, December 1998. 5, 7, 10, 27, 32, 33

[55] Valerio Lucarini, Davide Faranda, Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, Mark Holland, Tobias Kuna, Matthew Nicol, Mike Todd, and Sandro Vaienti. Extremes and recurrence in dynamical systems. 2016. 109, 110, 133

[56] Volker Mayer and Mariusz Urbański. Countable alphabet random subhifts of finite type with weakly positive transfer operator. *Journal of Statistical Physics*, 160(5):1405–1431, 2015. 14, 17

[57] Volker Mayer, Mariusz Urbański, and Bartlomiej Skorulski. *Distance Expanding Random Mappings, Thermodynamical Formalism, Gibbs Measures and Fractal Geometry*, volume 2036 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011. 14, 17, 65

[58] Volker Mayer and Mariusz Urbański. Random dynamics of transcendental functions. *Journal d'Analyse Mathématique*, 134(1):201–235, February 2018. 14, 17

[59] Mark Pollicott and Mariusz Urbanski. *Open conformal systems and perturbations of transfer operators*. Springer Berlin Heidelberg, New York, NY, 1st edition edition, 2017. 10

[60] Marek Rychlik. Bounded variation and invariant measures. *Studia Mathematica*, 76:69–80, 1983. 75, 112

[61] V.G. Sprindzuk. *Metric theory of Diophantine approximations*. V. H. Winston and Sons, Washington D.C., 1979. Translated from the Russian and edited by Richard A. Silverman, With a foreword by Donald J. Newman, Scripta Series in Mathematics. 129

[62] Manuel Stadlbauer, Paulo Varandas, and Xuan Zhang. Quenched and annealed equilibrium states for random Ruelle expanding maps and applications. *arxiv preprint arXiv: 2004.04763*, 2020. 14

[63] Mariusz Urbański. On Hausdorff dimension of invariant sets for expanding maps of a circle. *Ergodic Theory and Dynamical Systems*, 6(2):295–309, June 1986. 6

[64] H. Van Den Bedem and N. Chernov. Expanding maps of an interval with holes. *Ergodic Theory and Dynamical Systems*, 22(03), June 2002. 5

[65] Marcelo Viana and Krerley Oliveira. *Foundations of ergodic theory*. Number 151. Cambridge University Press, 2016. 120

[66] Lai-Sang Young. Dimension, entropy and Lyapunov exponents. *Ergodic Theory and Dynamical Systems*, 2(1):109–124, March 1982. 72

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

Glossary of Assumptions

A:

For each $\varepsilon > 0$ $H_\varepsilon \subseteq \mathcal{J}_0$ is measurable with respect to the product σ-algebra $\mathscr{F} \otimes \mathscr{B}$ on $\mathcal{J}_0$ such that $H'_\varepsilon \subseteq H_\varepsilon$ for each $0 < \varepsilon' \leq \varepsilon$ ..... *see page* 88

A':

$H_{\omega,\varepsilon'} \subseteq H_{\omega,\varepsilon}$ for each $\varepsilon' \leq \varepsilon$ and each $\omega \in \Omega$ ..... *see page* 88

B:

For each ω $\mathcal{B}_\omega \subseteq L^\infty(\nu_{\omega,0})$, where $\nu_0 = (\nu_{\omega,0})_{\omega\in\Omega}$ is the random probability measure given by (CCM), and there exists a measurable m-a.e. finite function $K : \Omega \to [1,\infty)$ such that $\|f\|_{\infty,\omega} \leq K_\omega \|f\|_{\mathcal{B}_\omega}$ for all $f \in \mathcal{B}_\omega$ and each $\omega \in \Omega$, where $\|\cdot\|_{\infty,\omega}$ denotes the supremum norm with respect to $\nu_{\omega,0}$. ..... *see page* 88

C:

There exists a random probability measure $\nu_0 = \{\nu_{\omega,0}\}_{\omega\in\Omega} \in \mathcal{P}_\Omega(\Omega \times I)$ and measurable functions $\lambda_0 : \Omega \to \mathbb{R}\setminus\{0\}$ and $\phi_0 : \mathcal{J}_0 \to (0,\infty)$ with $\phi_0 \in \mathrm{BV}_\Omega(I)$ such that

$$\mathcal{L}_{\omega,0}(\phi_{\omega,0}) = \lambda_{\omega,0}\phi_{\sigma\omega,0} \quad \text{and} \quad \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}(f)) = \lambda_{\omega,0}\nu_{\omega,0}(f)$$

for all $f \in \mathrm{BV}(I)$. Furthermore, we suppose that the fiber measures $\nu_{\omega,0}$ are non-atomic and that $\lambda_{\omega,0} := \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}\mathbb{1})$ with $\log\lambda_{\omega,0} \in L^1(m)$. The T-invariant random probability measure μ_0 on $\Omega \times I$ is given by

$$\mu_{\omega,0}(f) := \int_I f\phi_{\omega,0}\, d\nu_{\omega,0}, \qquad f \in L^1(\nu_{\omega,0}).$$

..... *see page* 23

C1:

There exists a measurable m-a.e. finite function $C_1 : \Omega \to \mathbb{R}_+$ such that for $f \in \mathcal{B}_\omega$ we have

$$\sup_{\varepsilon\geq 0} \|\mathcal{L}_{\omega,\varepsilon}(f)\|_{\mathcal{B}_{\sigma\omega}} \leq C_1(\omega)\|f\|_{\mathcal{B}_\omega}$$

..... *see page* 90

C1':

There exists $C_1 \geq 1$ such that for m-e.a. $\omega \in \Omega$ we have

$$C_1^{-1} \leq \mathcal{L}_{\omega,0}\mathbb{1} \leq C_1$$

..... *see page* 99

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

C2:

For each $\varepsilon \geq 0$ there is a random measure $\{\nu_{\omega,\varepsilon}\}_{\omega\in\Omega}$ supported in $\mathcal{J}_0$ and measurable functions $\lambda_\varepsilon : \Omega \to (0,\infty)$ with $\log \lambda_{\omega,\varepsilon} \in L^1(m)$ and $\phi_\varepsilon : \mathcal{J}_0 \to \mathbb{R}$ such that

$$\mathcal{L}_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = \lambda_{\omega,\varepsilon}\phi_{\sigma\omega,\varepsilon} \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(\mathcal{L}_{\omega,\varepsilon}(f)) = \lambda_{\omega,\varepsilon}\nu_{\omega,\varepsilon}(f)$$

for all $f \in \mathcal{B}_\omega$. Furthermore, for m-a.e. $\omega \in \Omega$

$$\nu_{\omega,0}(\phi_{\omega,\varepsilon}) = 1 \quad \text{and} \quad \nu_{\omega,0}(\mathbb{1}) = 1$$

. . . . . *see page* 91

C3:

There is an operator $Q_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ such that for m-a.e. $\omega \in \Omega$ and each $f \in \mathcal{B}_\omega$ we have

$$\lambda_{\omega,\varepsilon}^{-1}\mathcal{L}_{\omega,\varepsilon}(f) = \nu_{\omega,\varepsilon}(f) \cdot \phi_{\sigma\omega,\varepsilon} + Q_{\omega,\varepsilon}(f).$$

Furthermore, for m-a.e. $\omega \in \Omega$ we have

$$Q_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = 0 \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(Q_{\omega,\varepsilon}(f)) = 0$$

. . . . . *see page* 91

C4:

For each $f \in \mathcal{B}$ there exist measurable functions $C_f : \Omega \to (0,\infty)$ and $\alpha : \Omega \times \mathbb{N} \to (0,\infty)$ with $\alpha_\omega(N) \to 0$ as $N \to \infty$ such that for m-a.e. $\omega \in \Omega$ and all $N \in \mathbb{N}$

$$\sup_{\varepsilon\geq 0} \|Q^N_{\omega,\varepsilon} f_\omega\|_{\infty,\sigma^N\omega} \leq C_f(\omega)\alpha_\omega(N)\|f_\omega\|_{\mathcal{B}_\omega},$$

$$\sup_{\varepsilon\geq 0} \|Q^N_{\sigma^{-N}\omega,\varepsilon} f_{\sigma^{-N}\omega}\|_{\infty,\omega} \leq C_f(\omega)\alpha_\omega(N)\|f_{\sigma^{-N}\omega}\|_{\mathcal{B}_{\sigma^{-N}\omega}},$$

and $\|\phi_{\sigma^N\omega,0}\|_{\infty,\sigma^N\omega}\alpha_\omega(N) \to 0$, $\|\phi_{\sigma^{-N}\omega,0}\|_{\infty,\sigma^{-N}\omega}\alpha_\omega(N) \to 0$ as $N \to \infty$ . . . . *see page* 91

C4':

For each $f \in \mathcal{B}$ and each $N \in \mathbb{N}$ there exists $C_f > 0$ and $\alpha(N) > 0$ (independent of ω) with $\alpha := \sum_{N=1}^\infty \alpha(N) < \infty$ such that for m-a.e. $\omega \in \Omega$, all $N \in \mathbb{N}$

$$\sup_{\varepsilon\geq 0} \|Q^N_{\omega,\varepsilon} f_\omega\|_{\infty,\sigma^N\omega} \leq C_f\alpha(N)\|f_\omega\|_{\mathcal{B}_\omega}$$

. . . . . *see page* 99

C5:

There exists a measurable m-a.e. finite function $C_2 : \Omega \to [1,\infty)$ such that

$$\sup_{\varepsilon\geq 0} \|\phi_{\omega,\varepsilon}\|_{\infty,\omega} \leq C_2(\omega) \quad \text{and} \quad \|\phi_{\omega,0}\|_{\mathcal{B}_\omega} \leq C_2(\omega)$$

. . . . . *see page* 91

C5':

There exists $C_2 \geq 1$ such that

$$\sup_{\varepsilon\geq 0} \|\phi_{\omega,\varepsilon}\|_{\infty,\omega} \leq C_2 \quad \text{and} \quad \|\phi_{\omega,0}\|_{\mathcal{B}_\omega} \leq C_2$$

for m-a.e. $\omega \in \Omega$ ..... *see page* 99

C6:

For m-a.e. $\omega \in \Omega$ we have

$$\lim_{\varepsilon \to 0} \nu_{\omega,0}(H_{\omega,\varepsilon}) = 0$$

..... *see page* 91

C7:

There exists a measurable m-a.e. finite function $C_3 : \Omega \to [1, \infty)$ such that for all $\varepsilon > 0$ sufficiently small we have

$$\inf \phi_{\omega,0} \geq C_3^{-1}(\omega) > 0 \qquad \text{and} \qquad \operatorname*{ess\,inf}_{\omega} \inf \phi_{\omega,\varepsilon} \geq 0$$

..... *see page* 91

C7':

There exists $C_3 \geq 1$ such that for all $\varepsilon > 0$ sufficiently small we have

$$\operatorname*{ess\,inf}_{\omega} \inf \phi_{\omega,0} \geq C_3^{-1} > 0 \qquad \text{and} \qquad \operatorname*{ess\,inf}_{\omega} \inf \phi_{\omega,\varepsilon} \geq 0$$

..... *see page* 99

C8:

For m-a.e. $\omega \in \Omega_+$ we have that the limit $\hat{q}^{(k)}_{\omega,0} := \lim_{\varepsilon \to 0} \hat{q}^{(k)}_{\omega,\varepsilon}$ exists for each $k \geq 0$, where $\hat{q}^{(k)}_{\omega,\varepsilon}$ is as in (2.2.5) ..... *see page* 91

CCM:

There exists a random probability measure $\nu_0 = \{\nu_{\omega,0}\}_{\omega \in \Omega} \in \mathcal{P}_\Omega(\mathcal{J}_0)$ and measurable functions $\lambda_0 : \Omega \to \mathbb{R} \setminus \{0\}$ and $\phi_0 : \mathcal{J}_0 \to (0, \infty)$ with $\phi_0 \in \mathcal{B}$ such that

$$\mathcal{L}_{\omega,0}(\phi_{\omega,0}) = \lambda_{\omega,0} \phi_{\sigma\omega,0} \quad \text{and} \quad \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}(f)) = \lambda_{\omega,0} \nu_{\omega,0}(f)$$

for all $f \in \mathcal{B}_\omega$ where $\phi_{\omega,0}(\cdot) := \phi_0(\omega, \cdot)$. Furthermore, the fiber measures $\nu_{\omega,0}$ are non-atomic and that $\lambda_{\omega,0} := \nu_{\sigma\omega,0}(\mathcal{L}_{\omega,0}\mathbb{1})$ with $\log \lambda_{\omega,0} \in L^1(m)$. The T-invariant random probability measure μ_0 on $\mathcal{J}_0$ is then given by

$$\mu_{\omega,0}(f) := \int_{\mathcal{J}_{\omega,0}} f \phi_{\omega,0} \, d\nu_{\omega,0}, \qquad f \in L^1(\nu_{\omega,0}).$$

..... *see page* 16

E1:

There exists $C \geq 1$ such that

$$\operatorname*{ess\,sup}_{\omega} |T'_\omega| \leq C \qquad \text{and} \qquad \operatorname*{ess\,sup}_{\omega} D(T_\omega) \leq C,$$

where $D(T_\omega) := \sup_{y \in [0,1]} \# T_\omega^{-1}(y)$ ..... *see page* 111

E2:

The weight function $g_{\omega,0}$ lies in BV for each $\omega \in \Omega$ and satisfies

$$\operatorname*{ess\,sup}_{\omega} \|g_{\omega,0}\|_{\infty,1} < \infty$$

. . . . . *see page* 113

E3:

$$\operatorname*{ess\,inf}_{\omega} \inf g_{\omega,0} > 0$$

. . . . . *see page* 113

E4:

For every subinterval $J \subset [0,1]$ there is a $k = k(J)$ such that for a.e. ω one has $T_\omega^k(J) = [0,1]$ . . . . . *see page* 113

E5:

There is a uniform-in-ε and uniform-in-ω upper bound on the number of connected components of $H_{\omega,\varepsilon}$ . . . . . *see page* 113

E6:

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{\omega} \operatorname{Leb}(H_{\omega,\varepsilon}) = 0$$

. . . . . *see page* 114

E7:

For m-a.e. $\omega \in \Omega$ and all $\varepsilon > 0$ sufficiently small

$$T_\omega(\mathcal{J}_{\omega,\varepsilon}) = [0,1]$$

. . . . . *see page* 114

E8:

There exists $n' \geq 1$ and $\varepsilon_0 > 0$ such that

$$9 \cdot \operatorname*{ess\,sup}_{\omega} \|g_{\omega,0}^{(n')}\|_{\infty,1} < \operatorname*{ess\,inf}_{\omega} \inf_{0\leq\varepsilon\leq\varepsilon_0} \inf \mathcal{L}_{\omega,\varepsilon}^{n'} \mathbb{1}$$

. . . . . *see page* 114

E9:

There exists $k_o(n') \in \mathbb{N}$ such that for m-a.e. $\omega \in \Omega$, all $\varepsilon > 0$ sufficiently small, and all $Z \in \mathcal{Z}_{\omega,*,\varepsilon}^{(n')}$ we have $T_\omega^{k_o(n')}(Z) = [0,1]$, where n' is the number coming from (E8) . . . . . *see page* 115

E9a:

There exists $k_o(n') \in \mathbb{N}$ and $\delta > 0$ such that for m-a.e. $\omega \in \Omega$, all $\varepsilon > 0$ sufficiently small, we have $\operatorname{Leb}(Z) > \delta$ for all $Z \in \mathcal{Z}_{\omega,*,\varepsilon}^{(n')}(\mathcal{A})$ . . . . . *see page* 144

E9b:

There exists $c > 0$ such that $\operatorname{ess\,inf}_\omega |T'_\omega| > c$ . . . . . *see page* 144

EX:

There exists an $\varepsilon > 0$ and an open neighborhood $\tilde{H}_{\omega,\varepsilon} \supseteq H_{\omega,\varepsilon}$ such that $T_\omega(U_\omega) \supseteq \tilde{H}_{\sigma\omega,\varepsilon}^c$, where $U_\omega := \cup_{Z\in\mathcal{Z}_{\omega,0}} \overline{Z}_\varepsilon$ and $\overline{Z_\varepsilon}$ denotes the closure of $Z_\varepsilon \in A_{\omega,\varepsilon} := \{Z \cap \tilde{H}_{\omega,\varepsilon}^c : Z \in \mathcal{Z}_{\omega,0}\}$ with $m(\{\omega \in \Omega : \#A_{\omega,\varepsilon} \geq 2\}) > 0$ . . . . *see page* 114

M:

The map $T : \Omega \times I \to \Omega \times I$ is measurable ..... *see page* 23

M1:

The map $T : \mathcal{J}_0 \to \mathcal{J}_0$ is measurable with respect to $\mathscr{F} \otimes \mathscr{B}$ ..... *see page* 15

M2:

For every measurable function $f \in \mathcal{B}$, the map $(\omega, x) \mapsto (\mathcal{L}_0 f)_\omega(x)$ is measurable ..... *see page* 16

MC:

The map $\sigma : \Omega \to \Omega$ is a homeomorphism, the skew-product map $T : \Omega \times [0,1] \to \Omega \times [0,1]$ is measurable, and $\omega \mapsto T_\omega$ has countable range .... *see page* 111

P1:

There exists a function $C_1 : \Omega \to \mathbb{R}_+$ such that for $f \in \mathcal{B}_\omega$ we have

$$\sup_{\varepsilon \geq 0} \|\mathcal{L}_{\omega,\varepsilon}(f)\|_{\mathcal{B}_{\sigma\omega}} \leq C_1(\omega)\|f\|_{\mathcal{B}_\omega}$$

..... *see page* 82

P2:

For each $\omega \in \Omega$ and $\varepsilon \geq 0$ there is a functional $\nu_{\omega,\varepsilon} \in \mathcal{B}_\omega^*$, the dual space of $\mathcal{B}_\omega$, $\lambda_{\omega,\varepsilon} \in \mathbb{C} \setminus \{0\}$, and $\phi_{\omega,\varepsilon} \in \mathcal{B}_\omega$ such that

$$\mathcal{L}_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = \lambda_{\omega,\varepsilon}\phi_{\sigma\omega,\varepsilon} \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(\mathcal{L}_{\omega,\varepsilon}(f)) = \lambda_{\omega,\varepsilon}\nu_{\omega,\varepsilon}(f)$$

for all $f \in \mathcal{B}_\omega$. Furthermore,

$$\nu_{\omega,0}(\phi_{\omega,\varepsilon}) = 1.$$

..... *see page* 82

P3:

There is an operator $Q_{\omega,\varepsilon} : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega}$ such that for each $f \in \mathcal{B}_\omega$ we have

$$\lambda_{\omega,\varepsilon}^{-1}\mathcal{L}_{\omega,\varepsilon}(f) = \nu_{\omega,\varepsilon}(f) \cdot \phi_{\sigma\omega,\varepsilon} + Q_{\omega,\varepsilon}(f).$$

Furthermore, we have

$$Q_{\omega,\varepsilon}(\phi_{\omega,\varepsilon}) = 0 \quad \text{and} \quad \nu_{\sigma\omega,\varepsilon}(Q_{\omega,\varepsilon}(f)) = 0$$

..... *see page* 82

P4:

There exists a function $C_2 : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\varepsilon \geq 0} \|\phi_{\omega,\varepsilon}\|_{\mathcal{B}_\omega} = C_2(\omega) < \infty$$

..... *see page* 83

P5:

For each $\omega \in \Omega$ we have

$$\lim_{\varepsilon \to 0} \eta_{\omega,\varepsilon} = 0$$

..... *see page* 83

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

P6:

For each $\omega \in \Omega$ such that $\Delta_{\omega,\varepsilon} > 0$ for every $\varepsilon > 0$, there exists a function $C_3 : \Omega \to \mathbb{R}_+$ such that

$$\limsup_{\varepsilon \to 0} \frac{\eta_{\omega,\varepsilon}}{\Delta_{\omega,\varepsilon}} := C_3(\omega) < \infty.$$

Given $\omega \in \Omega$, if there is $\varepsilon_0 > 0$ such that for each $\varepsilon \leq \varepsilon_0$ we have that $\Delta_{\omega,\varepsilon} = 0$ then we also have that $\eta_{\omega,\varepsilon} = 0$ for each $\varepsilon \leq \varepsilon_0$ ..... *see page* 83

P7:

For each $\omega \in \Omega$

$$\lim_{\varepsilon \to 0} \nu_{\omega,\varepsilon}(\phi_{\omega,0}) = 1$$

..... *see page* 83

P8:

For each $\omega \in \Omega$ with $\Delta_{\omega,\varepsilon} > 0$ for all $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \limsup_{\varepsilon \to 0} \Delta_{\omega,\varepsilon}^{-1} \nu_{\sigma\omega,0} \left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon}) \left(Q^n_{\sigma^{-n}\omega,\varepsilon} \phi_{\sigma^{-n}\omega,0} \right) \right) = 0.$$

..... *see page* 83

P9:

For each $\omega \in \Omega$ with $\Delta_{\omega,\varepsilon} > 0$ for all $\varepsilon > 0$ the limit

$$q^{(k)}_{\omega,0} := \lim_{\varepsilon \to 0} \frac{\nu_{\sigma\omega,0} \left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\mathcal{L}^k_{\sigma^{-k}\omega,\varepsilon})(\mathcal{L}_{\sigma^{-(k+1)}\omega,0} - \mathcal{L}_{\sigma^{-(k+1)}\omega,\varepsilon})(\phi_{\sigma^{-(k+1)}\omega,0}) \right)}{\nu_{\sigma\omega,0} \left((\mathcal{L}_{\omega,0} - \mathcal{L}_{\omega,\varepsilon})(\phi_{\omega,0}) \right)}$$

exists for each $k \geq 0$ ..... *see page* 83

Q1:

$$\lim_{n \to \infty} \frac{1}{n} \log \|g^{(n)}_\omega\|_\infty + \limsup_{n \to \infty} \frac{1}{n} \log \xi^{(n)}_\omega < \lim_{n \to \infty} \frac{1}{n} \log \rho^n_\omega = \int_\Omega \log \rho_\omega \, dm(\omega).$$

..... *see page* 30

Q2:

For each $n \in \mathbb{N}$ we have $\log \xi^{(n)}_\omega \in L^1(m)$. ..... *see page* 30

Q2':

$\log \xi^{(N_*)}_\omega \in L^1(m)$. ..... *see page* 38

Q3:

For each $n \in \mathbb{N}$, $\log \delta_{\omega,n} \in L^1(m)$, where

$$\delta_{\omega,n} := \min_{Z \in \mathcal{Z}^{(n)}_{\omega,g}} \Lambda_\omega(\mathbb{1}_Z).$$

..... *see page* 30

Q3':

$\log \delta_{\omega,N_*} \in L^1(m)$, where $\delta_{\omega,n}$ is defined by (1.3.18) ..... *see page* 38

$\widehat{\mathbf{Q0}}$:

$\overline{\mathcal{Z}}^{(1)}_{\omega,F} \neq \emptyset$ for m-a.e. $\omega \in \Omega$ ..... *see page* 75

$\widehat{\mathbf{Q1}}$:

$$\lim_{n\to\infty} \frac{1}{n} \log \|g^{(n)}_\omega\|_\infty + \limsup_{n\to\infty} \frac{1}{n} \log \zeta^{(n)}_\omega < \lim_{n\to\infty} \frac{1}{n} \log \rho^n_\omega = \int_\Omega \log \rho_\omega \, dm(\omega),$$

..... *see page* 76

$\widehat{\mathbf{Q2}}$:

For each $n \in \mathbb{N}$ we have $\log^+ \zeta^{(n)}_\omega \in L^1(m)$ ..... *see page* 76

$\widehat{\mathbf{Q2}}$':

$\log^+ \zeta^{(N_*)}_\omega \in L^1(m)$ ..... *see page* 76

$\widehat{\mathbf{Q3}}$:

For each $n \in \mathbb{N}$, $\log \hat{\delta}_{\omega,n} \in L^1(m)$, where

$$\hat{\delta}_{\omega,n} := \min_{Z \in \overline{\mathcal{Z}}^{(n)}_{\omega,F}} \Lambda_\omega(\mathbb{1}_Z).$$

..... *see page* 76

$\widehat{\mathbf{Q3}}$':

$\log \hat{\delta}_{\omega,N_*} \in L^1(m)$ ..... *see page* 76

S:

For any fixed random scaling function $t \in L^\infty(m)$ with $t > 0$, there are sequences of functions $z_N, \xi_N \in L^\infty(m)$ and a constant $W < \infty$ satisfying

$$\mu_{\omega,0}(\{h_\omega(x) - z_{\omega,N} > 0\}) = (t_\omega + \xi_{\omega,N})/N, \text{ for a.e. } \omega \text{ and each } N \geq 1$$

where:
(i) $\lim_{N\to\infty} \xi_{\omega,N} = 0$ for a.e. ω and
(ii) $|\xi_{\omega,N}| \leq W$ for a.e. ω and all $N \geq 1$ ..... *see page* 99

X:

For m-a.e. $\omega \in \Omega$ we have $X_{\omega,\infty} \neq \emptyset$ ..... *see page* 18

Z:

There exists $\hat{\alpha} \geq 0$ sufficiently large such that $\mathcal{Z}^{(n)}_\omega \in \overline{\mathscr{A}}^{(n)}_\omega(\hat{\alpha})$ for each $n \in \mathbb{N}$ and each $\omega \in \Omega$. ..... *see page* 75

Z':

There exists $\hat{\alpha} \geq 0$ such that $\mathcal{Z}^{(N_*)}_\omega \in \overline{\mathscr{A}}^{(N_*)}_\omega(\hat{\alpha})$ for each $\omega \in \Omega$ .... *see page* 76

Index

$A_\omega^{(n)}$, 34
B_*, 40
$B_\omega^{(n)}$, 34
$C(\omega,\delta)$, 63
$C_\varepsilon(\omega)$, 39
$D_{\omega,\infty}$, 19
$D_{\omega,n}$, 19
$G_{\omega,N}^{(k)}$, 98
H_{ω,ε_N}, 98
I_ω, 24
J_ω, 113
$J_{\omega,+}(Z)$, 35
$J_{\omega,-}(Z)$, 35
$K_\omega^{(n)}$, 37
L_ω, 38
$M_\omega(Z)$, 30
N_*, 38
P_ω, 113
$Q_\omega^{(n)}$, 37
$R(\varrho_\omega)$, 68
R_*, 40
$R_\varepsilon(\zeta_\omega)$, 93
R_a, 40
$S_{n,T}$, 16
$S_{n,\sigma}$, 16
$T:\mathcal{J}_0\to\mathcal{J}_0$, 15
$T_\omega^n:\mathcal{J}_{\omega,0}\to\mathcal{J}_{\sigma^n\omega,0}$, 15
V_{ω,z_ω}, 98
$V_{f,\varepsilon}$, 57
$X_{\omega,\infty,\varepsilon}$, 89
$X_{\omega,\infty}$, 18
$X_{\omega,n,\varepsilon}$, 89
$X_{\omega,n}$, 17
$\mathrm{BV}(I)$, 22
BV_1, 112
$\mathrm{BV}_\Omega(I)$, 22
BV_ν, 112
$\mathrm{B}(I)$, 22
$\mathrm{B}_\Omega(I)$, 22
$\Delta_{\omega,\varepsilon}$, 83, 89
$\Gamma(\omega)$, 42
$\Lambda(\mathcal{R})$, 124
Leb, 15
$\Lambda_\omega(f)$, 27
Ω, 14
Ω_+, 88
Ω_B, 41
Ω_F, 51
Ω_G, 40
Σ_ω, 50
$\Sigma_\omega^{(k)}$, 43
$\Theta_{\omega,+}$, 33
$\Theta_{\omega,a}$, 33
$\underline{R}(\varrho_\omega)$, 68
$\underline{z}_\omega$, 98
$\underline{R}_\varepsilon(\zeta_\omega)$, 93
$\mathcal{A}$, 30
$\mathcal{B}$, 16
$\mathcal{B}_\omega$, 16
$\mathcal{D}$, 57
$\mathcal{D}^+$, 57
$\mathcal{E}P(\varphi)$, 67
$\mathcal{E}P(\varphi_0)$, 67
$\mathcal{I}$, 24
$\mathcal{J}$, 17
$\mathcal{J}_0$, 15
$\mathcal{J}_\omega$, 17
$\mathcal{J}_{\omega,0}$, 14
$\mathcal{L}_\omega$, 19
$\mathcal{L}_{\omega,0}$, 16

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque

$\mathcal{L}_{\omega,\varepsilon}$, 82, 89
$\mathcal{P}_\Omega(Y)$, 15
$\mathcal{P}^H_{T,m}(I)$, 69
$\mathcal{X}_{\infty,\varepsilon}$, 89
$\mathcal{X}_\infty$, 18
$\mathcal{X}_{n,\varepsilon}$, 89
$\mathcal{X}_n$, 18
$\mathcal{Z}_\omega$, 21
$\mathcal{Z}^{(n)}_\omega$, 21
$\mathcal{Z}^{(n)}_{\omega,*,\varepsilon}$, 115
$\mathcal{Z}^{(n)}_{\omega,*}$, 30
$\mathcal{Z}^{(n)}_{\omega,b}$, 30
$\mathcal{Z}^{(n)}_{\omega,g}$, 30
$\mathrm{diam}_{\omega,+}(Y)$, 33
$\mathrm{diam}_{\omega,a}(Y)$, 33
$\delta_{\omega,n}$, 31
$\ell(\omega)$, 42
$\eta_{\omega,\varepsilon}$, 83, 90
$\hat{q}^{(k)}_{\omega,0}$, 90
$\hat{q}^{(k)}_{\omega,\varepsilon}$, 90
$\hat{\mathcal{L}}_{\omega,0}$, 116
$\hat{\mathcal{L}}_{\omega,\varepsilon}$, 116
$\hat{D}_{\omega,\infty}$, 20
$\hat{D}_{\omega,n}$, 20
$\hat{X}_{\omega,\infty}$, 18
$\hat{X}_{\omega,n}$, 18
$\mathbb{1}_\omega$, 17, 24
$\|f\|_{\mathrm{BV}}$, 22
$\|f\|_\infty$, 22
$\overline{R}(\varrho_\omega)$, 68
$\overline{z}_\omega$, 98
$\overline{\mathcal{Z}}^{(n)}_\omega$, 75
$\overline{\mathcal{Z}}^{(n)}_{\omega,*}$, 75
$\overline{\mathcal{Z}}^{(n)}_{\omega,F}$, 75
$\overline{\mathcal{Z}}^{(n)}_{\omega,U}$, 75
$\overline{\mathcal{Z}}^{(n)}_{\omega,b}$, 75
$\overline{\mathcal{Z}}^{(n)}_{\omega,g}$, 75
$\overline{R}_\varepsilon(\zeta_\omega)$, 93
$\mathscr{h}_\omega$, 77
ρ_ω, 28
$\mathscr{A}^{(n)}_\omega$, 30
$\mathscr{B}$, 15
$\mathscr{C}_{\omega,+}$, 33
$\mathscr{C}_{\omega,a}$, 33
$\mathscr{F}$, 14
$\mathscr{F}\otimes\mathscr{B}$, 15
$\mathscr{L}_\omega$, 62
$\sigma:\Omega\to\Omega$, 14
θ, 38
$\theta_{\omega,0,n}$, 105
$\theta_{\omega,0}$, 88, 94
$\theta_{\omega,\varepsilon_N,n}$, 106
$\tau_{\omega,\mathcal{H}_{\omega,N}}$, 110
$\mathrm{var}(f)$, 22
$\mathrm{var}_A(f)$, 22
$\varphi_0:\mathcal{J}_0\to\mathbb{R}$, 16
$\widetilde{\mathcal{Z}}^{(n)}_\omega(\mathcal{A})$, 30
$\xi_{\omega,N}$, 99
$\xi^{(n)}_\omega$, 30
ζ, 39
$\tilde{\mathcal{L}}_\omega$, 19
$\tilde{\mathcal{L}}_{\omega,\varepsilon}$, 89
a_0, 40
$g_{\omega,0}$, 16
$g_{\omega,\varepsilon}$, 115
m, 14
q_*, 40
$q^{(k)}_{\omega,0}$, 83
$q^{(k)}_{\omega,\varepsilon}$, 86
t_ω, 99
$y_*:\Omega\to\mathbb{N}$, 42
$z_{\omega,N}$, 98

admissible potential, 22

bad block, 42
base dynamical system, 14

closed random dynamical system, 17
closed random set, 15
contiguous, 30
convex cone, 31
covering, 113

expected pressure, 67

fiberwise escape rate, 68, 93

good block, 42
good fiber, 40

Lyapunov exponent, 124

open covering time, 30

potential, 16

random absolutely continuous conditionally invariant probability measure (RACCIM), 25
random bounded, 22
random bounded variation, 22
random conditionally invariant probability measure (RCIM), 25
random measure, 15
random probability measure, 15
relative equilibrium state, 69

surviving set, 18

weight, 16

18 Oct 2022 02:26:55 PDT
221018-Vaienti Version 1 - Submitted to Asterisque