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Graph structure-based Heuristics for Optimal Targeting in Social Networks

Massimo Bini, Paolo Frasca, Chiara Ravazzi, Fabrizio Dabbene

Abstract—We consider a dynamic model for competition in a social network, where two strategic agents have fixed beliefs and the non-strategic/regular agents adjust their states according to a distributed consensus protocol. We suppose that one strategic agent must identify k_+ target agents in the network in order to maximally spread her own opinion and alter the average opinion that eventually emerges. In the literature, this problem is cast as the maximization of a set function and, leveraging on the submodular property, is solved in a greedy manner by solving k_+ separate single targeting problems. Our main contribution is to exploit the underlying graph structure to build more refined heuristics. As a first instance, we provide the analytical solution for the optimal targeting problem over Complete Graphs. This result provides a rule to understand whether it is convenient or not to block the opponent’s influence by targeting the same nodes. The argument is then extended to generic graphs leading to more accurate solutions compared to a simple greedy approach. As a second instance, by electrical analogy we provide the analytical solution of the single targeting problem for the Line Graph and derive some useful properties of the objective function for trees. Inspired by these findings, we define a new algorithm which selects the optimal solution on trees in a much faster way with respect to a brute-force approach and works well also over tree-like/sparse graphs. The proposed heuristics are then compared to zero-cost heuristics on different random generated graphs and real social networks. Summarizing, our results suggest a scheme that tells which algorithm is more suitable in terms of accuracy and computational complexity, based on the density of the graphs and its degree distribution.

I. INTRODUCTION

In the course of the last decade, numerous works have considered the problem of optimally allocating resources to influence the outcome of opinion dynamics. The problem has attracted researchers with backgrounds from economics to engineering, who have deployed tools from game theory, optimization and, of course, network science [1]–[6]. A large part of this research assumes a linear model of opinion evolution, as per the influential De Groot model of opinion evolution—see [7], [8] for a contextualization of this model. Under De Groot model, the steady-state opinions satisfy a linear equation defined by a weighted Laplacian matrix associated to the network graph.

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A typical setup considers two strategic agents, holding extreme opinions (say, -1 and $+1$), which compete with the purpose of swaying the average steady-state opinion towards their own. This setup has the mathematical advantage of yielding an objective function that is a linear function of the node opinions.

Actually, two kinds of (closely related) problems have been considered: in one formulation of the problem (*internal* influence), strategic agents have the opportunity to “recruit”, among the regular nodes, influencers that hold their fixed opinions [2]. In another formulation (*external* influence), the strategic agents have the opportunity to create additional edges between themselves and target nodes [5]. Both setups allow for either game theoretic analysis, where the focus is on the interplay between the two strategic players, as well as optimization approaches where one of the players has a fixed strategy and the other is optimizing her strategy by targeting k nodes for internal or external influence.

As both internal and external influence problems are combinatorially hard [6], effective heuristics are needed. Most methods rely on submodularity to advocate greedy heuristics that reduce the general problem of targeting the k_+ best nodes to a sequence of k_+ problems of targeting the best node. Such an approach requires $O(Nk_+)$ evaluations of the equilibrium opinions (where N is the number of network nodes). Each 1-best problem can be easily solved by comparing the N possible solutions: in turn, evaluating each solution requires the solution of a linear system of equations, which can be performed in $O(M)$ operations (where M is the number of network edges [9]). Therefore, this kind of greedy approach typically results in $O(MN)$ cost, with the guarantee of a bounded error. Other heuristic approaches may achieve $O(M)$ cost, though without bounded-error guarantees [10], [11].

In this paper, we concentrate on the external influence problem in which one of the two strategic agents has to optimize the deployment of k_+ additional links between herself and k_+ regular nodes (to which she is not yet connected). On this well studied problem, we provide theoretical results on specific networks. First, we derive a closed-form solution for the Optimal Targeting Problem (OTP) over Complete Graphs leading to a zero-cost rule for the optimal strategy. Then, by electrical analogy, we provide the analytical solution for the Single Targeting Problem (STP) over line graphs and some of the properties of the objective function are extended to the branches of generic tree graphs. These theoretical findings allow us to design new algorithms for general graphs. These heuristics are compared with optimal solution and zero-cost strategies, consisting in targeting with highest degree nodes. To put the results into perspective, we provide a scheme to

identify which is the best heuristic, depending on the cost vs accuracy trade-off and the underlying graph.

a) *Paper outline:* In Section II the model of competition and the OTP are formally introduced. In Section III we derive an explicit solution of the OTP on the Complete Graph and propose a simple heuristic that requires no evaluations of the equilibrium opinions. Section IV presents some analytical results for STP on the Line Graph and on trees. These results lead to a heuristic criterion to accelerate the solution to the 1-best problem by avoiding the evaluation of all N possible solutions (Section V). Section VI collects some concluding remarks. Finally, the Appendix, containing some technical proofs, completes the paper.

b) *Notation:* Throughout this paper, we use the following notation. The set of real numbers is denoted by \mathbb{R} and the set of non-negative integers is denoted by $\mathbb{Z}_{\geq 0}$. We denote column vectors with lower case letters and matrices with upper case letters. The vector of all ones of appropriate dimension is represented by $\mathbb{1}$. We denote the 2-norm of a vector x with the symbol $\|x\|$. Given a matrix A , A^\top denotes its transpose. Moreover, $\text{sr}(A)$ is the spectral radius of the matrix A , and a square matrix A is said to be Schur stable if $\text{sr}(A) < 1$. A matrix A with positive entries is said to be row stochastic if $A\mathbb{1} = \mathbb{1}$, and it is said to be row substochastic if $A\mathbb{1} \leq \mathbb{1}$, where the inequality is entry-wise. We represent the network by a directed graph, a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of nodes, unitary elements of the network, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges or links representing the relationships among such entities. A path in a graph is a sequence of edges which joins a sequence of vertices. A directed graph \mathcal{G} is called strongly connected if there is a path from each vertex in the graph to every other vertex. An undirected graph in which any two vertices are connected by exactly one path is called tree. Given a matrix $W \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ with non-negative entries, the weighted graph associated to W is the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with node set \mathcal{V} , defined by drawing an edge $(i, j) \in \mathcal{E}$ if and only if $W_{ij} > 0$ and putting weights W_{ij} . If W is symmetric, i.e. $W_{ij} = W_{ji}$ for each $i, j \in \mathcal{V}$, the *undirected* edges will be denoted as unordered pairs $\{i, j\}$, corresponding to both the directed links (i, j) and (j, i) . A subset of nodes $\mathcal{U} \subset \mathcal{V}$ is said to be *globally reachable* in \mathcal{G} if for every node $j \in \mathcal{V} \setminus \mathcal{U}$ there exists a path from j to some node $i \in \mathcal{U}$. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph, then the *in-neighborhood* of a node $i \in \mathcal{V}$ is defined as $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. The *in-degree* of a node is defined as $d_i = \sum_{j \in \mathcal{V}} W_{ij}$. We will consider the *normalized weight matrix* $Q = D^{-1}W$ where D is the diagonal matrix with diagonal entries equal to the *in-degree* of node $i \in \mathcal{V}$: $D_{ii} = \sum_{j \in \mathcal{V}} W_{ij}$. We will denote the *Laplacian matrix* by $L = D - W$.

II. OTP AND ITS STATE-OF-THE-ART SOLUTIONS

A. Dynamic model for competition

We consider an influence network described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. Nodes $v \in \mathcal{V}$ represent the agents and \mathcal{E} is the set of edges describing the potential interactions among them. We assume that the set of nodes is partitioned into two disjoint sets: $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$, where \mathcal{R} and \mathcal{S} are the set of

regular and strategic agents, respectively. At an initial stage regular agents have a belief around a certain topic or issue. Then, by interacting with their contacts, each agent updates her opinion by averaging her own with the ones of their neighbors in a distributed manner. Strategic agents, instead, are not updating their opinion, spreading over and over the same idea. These agents can be interpreted as either individuals or media outlets who wish to influence others with their opinion, or as stubborn individuals/opinion leaders, who have a strong influence over some communities and are less likely to change their opinion as time goes on. More formally, the structure of the network is encoded in the adjacency matrix W . We assume that the graph is undirected and we consider the normalized weight matrix $Q = D^{-1}W$. We assume that each agent is endowed with a state $x_s(t) = x_s \in \{-1, 1\}$ for all $s \in \mathcal{S}$ and $x_i(t) \in [-1, +1]$ for all $i \in \mathcal{R}$, representing the opinion/belief at time t . At each time step $t \in \mathbb{Z}_{\geq 0}$ the opinion of a regular agent $i \in \mathcal{R}$ is updated as a response to the interaction with the neighbors, according to the following rule $x_i(t+1) = \sum_{j \in \mathcal{V}} Q_{ij} x_j(t)$ where $Q_{ij} \geq 0$ for all $i \in \mathcal{R}$ and for all $j \in \mathcal{V}$, $Q_{ij} = 0 \iff (i, j) \notin \mathcal{E}$ and $\sum_{j \in \mathcal{V}} Q_{ij} = 1$ for all $i \in \mathcal{R}$.

Assembling opinions of regular and strategic agents in a vector $x(t) = (x^{\mathcal{R}\top}(t), x^{\mathcal{S}\top}(t))^\top = (x^{\mathcal{R}\top}(t), x^{\mathcal{S}\top}(t))^\top$, we can rewrite the dynamics in the following compact form

$$x(t+1) = Qx(t) \quad t = 1, 2, \dots \quad (1)$$

$$Q = \begin{pmatrix} Q^{11} & Q^{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} (D^{11})^{-1}W^{11} & (D^{11})^{-1}W^{12} \\ 0 & (D^{22})^{-1} \end{pmatrix}$$

where matrices Q^{11} , Q^{12} , W^{11} , W^{12} , D^{11} , and D^{12} , are non-negative matrices of appropriate dimensions. Such equation in the social science context is known as the *DeGroot* opinion dynamics model [12], [13] or, more generally, as the *linear averaging dynamics* on \mathcal{G} .

We assume that each strategic agent has at least one link to a regular agent, but no more than one to the same target. Hence, in the paper, we make the standing assumption that the influence matrix $W^{12} \in \{0, 1\}^{\mathcal{R} \times \mathcal{S}}$ is such that $\mathbb{1}^\top W^{12} \geq \mathbb{1}^\top$, and that $\mathcal{G}|_{\mathcal{R}}$, i.e. the graph restricted to nodes in \mathcal{R} , is strongly connected. Under these assumptions it can be shown that the dynamics converges to a final limit profile that corresponds to a disagreement. More precisely, the following proposition holds [14].

Proposition 1. *Let Q^{11} be substochastic and asymptotically stable. Then $I - Q^{11}$ will be invertible with non-negative inverse matrix. Moreover, for every initial state vector $x(0) \in \mathbb{R}^{\mathcal{V}}$, the dynamics in (1) converges to a finite limit profile $\bar{x} = ((\bar{x}^{\mathcal{R}})^\top, (\bar{x}^{\mathcal{S}})^\top)^\top$*

$$\bar{x}^{\mathcal{R}} = \lim_{t \rightarrow +\infty} x^{\mathcal{R}}(t) = (I - Q^{11})^{-1} Q^{12} x^{\mathcal{S}}. \quad (2)$$

Proposition 1 states that the opinions converge asymptotically to a stationary profile that is a combination of the opinions of all strategic agents.

B. Optimal Targeting Problems

The main question addressed in this paper is related to the control of dynamics in (1). The problem is stated as a competition between two strategic agents –of opposite opinion– who try to lead the average asymptotic opinion towards their own by targeting a certain number of regular agents. This competition is set from the perspective of one of the strategic agents, so as to be formalized by the optimization problem of selecting at most k regular agents to connect to in order to maximally shift the average asymptotic opinion of the social network. Formally, we consider the situation where there are N regular agents, indexed by $i \in \mathcal{R} = \{1, \dots, N\}$ and two strategic agents $\mathcal{S} = \{N+1, N+2\} := \{\boxplus, \boxminus\}$ with opinion $x_{\boxplus}(t) = +1$ and the latter with opinion $x_{\boxminus}(t) = -1$ respectively, for all $t \in \mathbb{Z}_{\geq 0}$. We investigate how to identify regular nodes in \mathcal{R} in order to maximize the influence of opinion $+1$ on the final limit profile, assuming that edges of the strategic agent \boxminus are already placed. We use $(\bar{x}_v^{(\mathcal{A})})_{v \in \mathcal{R}}$ to denote the asymptotic opinion profile that emerges from the particular configuration in which the nodes belonging to the set \mathcal{A} are additionally linked to strategic node \boxplus . The OTP is defined as the following optimization problem.

Problem 1 (Optimal Targeting Problem (k_+ OTP)). *Given $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let $\mathcal{A}^- = \{v \in \mathcal{R} : (\boxminus, v) \in \mathcal{E}\} \neq \emptyset$, $\mathcal{A}^{(0)} = \{v \in \mathcal{R} : (\boxplus, v) \in \mathcal{E}\}$. Find the node set \mathcal{A}^+*

$$\mathcal{A}^+ \subseteq \operatorname{argmax}_{\mathcal{A} \subseteq \mathcal{R} \setminus \mathcal{A}^{(0)}: |\mathcal{A}| \leq k_+} F_+(\mathcal{A}), \quad (3)$$

with

$$F_+(\mathcal{A}) = \frac{1}{N} \sum_{v \in \mathcal{R}} \bar{x}_v^{(\mathcal{A})}, \quad \mathcal{A} \subseteq \mathcal{R} \setminus \mathcal{A}^{(0)} \quad (4)$$

where $\bar{x}^{(\mathcal{A})}$ is the limit profile satisfying (2).

In this optimization problem, for any different choice of the set \mathcal{A} , the influence matrices Q^{11} and Q^{12} change and the final limit profile needs to be computed, requiring a new matrix inversion. Then, the complexity of the problem is combinatorial since we need to find the best solution among all $\binom{N}{k_+}$ possible configurations.

Problem 2 (Single Targeting Problem (STP)). *The specific case where $k_+ = 1$, $|\mathcal{A}^-| = 1$, will be referred to as Single Targeting Problem (STP). Then, the OTP reduces to finding the node that maximizes the following objective function: $\max_{v \in \mathcal{R} \setminus \mathcal{A}^{(0)}} F_+(\{v\})$.*

C. Electrical Network Analogy

In the sequel it will be convenient to use the electrical network analogy for the OTP problem. We briefly review the basic notions of such analogy as presented in [10].

We consider a strongly connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where \mathcal{E} is the set of unordered couples $\{i, j\}$. Such graph can be seen as an electrical network $\mathcal{G}_C = (\mathcal{V}, \mathcal{E}, C)$ where the weight matrix W is replaced by the conductance matrix $C \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$, where $C_{ij} = C_{ji}$ is now the conductance between the nodes i and j (notice how the reciprocity assumption must hold). Then, let us define the incidence matrix

$B \in \{0, +1, -1\}^{\mathcal{E} \times \mathcal{V}}$, such that $B\mathbb{1} = 0$ and $B_{ei} \neq 0 \iff i \in e$ with $e \in \mathcal{E}$. It is straightforward to verify that given $e = \{i, j\}$, the e -th row of B has all entries equal to zero except for B_{ei} and B_{ej} : one of them will be $+1$ and the other one -1 . Let $D_C \in \mathbb{R}^{\mathcal{E} \times \mathcal{E}}$ be the diagonal matrix whose entries are $(D_C)_{ee} = C_{ij} = C_{ji}$ with $e = (i, j) \in \mathcal{E}$. It should be noted that $B^\top D_C B = D_{C\mathbb{1}} - C$ where $D_{C\mathbb{1}} = \operatorname{diag}(C\mathbb{1})$. Indeed $D_C B$ associates at each row of B the weight of the corresponding edge multiplied by 1 or -1 , while $B^\top D_C B$ generates the matrix that on each diagonal entry has the sum of all the conductances on such node, while on the ij -th entry it has the conductance value of edge $\{i, j\}$ of negative sign, if present.

Defining $\eta \in \mathbb{R}^{\mathcal{V}}$ as the *input current vector* (positive if ingoing, negative if outgoing), such that $\eta^\top \mathbb{1} = 0$; $V \in \mathbb{R}^{\mathcal{V}}$ as the *voltage vector*, and $\Phi \in \mathbb{R}^{\mathcal{E}}$ as the *current flow vector* (positive if going from i to j on (i, j)), then the usual Kirchoff and Ohm's law can be written as follows

$$L(C)V = \eta \quad (5)$$

where $L(C) := D_{C\mathbb{1}} - C$ is the *Laplacian* of C . Since the graph is strongly connected, $L(C)$ has rank $|\mathcal{V}| - 1$ and $L(C)\mathbb{1} = 0$, making V , up to translations, the unique solution of the system. Also notice that $(L(C)V)_i = 0 \quad \forall i \in \mathcal{V}$ such that $\eta_i = 0$. The Equation (5) resembles the system in Proposition 1, where the asymptotic opinion of regular agents can be interpreted as voltages with 0 input current, while those the strategic nodes as voltages fixed to 1 and -1 with input current different from 0.

From now on, we will exploit the electrical analogy where the agents are nodes in the electrical network and their asymptotic opinions are the associated voltages. In this analogy, the strategic nodes \boxminus and \boxplus are considered voltage sources of value -1 and $+1$ respectively. Thus, the objective function of OTP becomes

$$F_+(\mathcal{A}) = \sum_{i \in \mathcal{R}} V^{(\mathcal{A})}(i)/N$$

where $V^{(\mathcal{A})}(i)$ is the voltage of node i when the set of nodes linked to \boxplus is \mathcal{A} .

In the sequel, we will also make use of two common operations that allow to replace an electrical network by a simpler one without changing certain quantities of interest. Since current never flows between vertices with the same voltage, we can merge vertices having the same voltage into a single one, while keeping all existing edges, voltages and currents are unchanged (*gluing*, [15]). Another useful operation is replacing a portion of the electrical network connecting two nodes h, k by an equivalent resistance, a single resistance denoted as R_{eff} which keeps the difference of voltages $V_h - V_k$ unchanged (*series and parallel laws* [15]).

Remark 1. *It is worth mentioning that considering the case with two strategic nodes is not a restrictive assumption. Indeed, leveraging the electrical network analogy of the problem [10], all the strategic agents of identical opinion can be considered as voltage sources and merged together, without affecting the final results. So, rather than thinking of two competitors as individuals, we can think of them as two groups of people - made up of opinion leaders, media,*

influencers, stubborn individuals - who hold opposing views and try to make their idea prevail by optimally addressing new individuals.

D. Some known heuristics for OTP

The OTP described in Problem 1 is computationally challenging if we are interested in targeting simultaneously $k_+ \geq 2$ nodes in the network. In this section we review some heuristics, practical methods that are not guaranteed to be optimal but useful to obtain an approximation of the solution to the OTP problem.

A heuristic based on the out-degree centrality, defined as the number of outgoing links of the nodes, is a rough but common approach to approximate the OTP solution. This method, summarized in Algorithm 1, consists in selecting the k_+ nodes with highest degree (if there are more subsets with this property, one of them is selected randomly). It should be noticed that this is a zero-cost heuristics, in the sense that it provides a strategy without the burden of the equilibrium opinions' computation. This simple heuristic will be used as a benchmark for the proposed methods.

Algorithm 1 Degree heuristic for k_+ OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, number of available links k_+

Initialization:

\mathcal{D} set of nodes with top- k_+ degree

$\mathcal{A}_{k_+} = \mathcal{D}$

return $\mathcal{A}_{k_+}, F_+(\mathcal{A}_{k_+})$

Another common approach in literature is to solve the optimization problem for one target at a time in a greedy manner, i.e. choosing at each iteration a target which gives the largest gain in the objective function. This approach allows to reduce the complexity and can be applied in large social networks. We review the procedure in Algorithm 2.

Algorithm 2 Greedy algorithm for k_+ OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, number of available links k_+

Initialization:

$\mathcal{A}_0 = \emptyset$

for $i \in \{1, \dots, k_+\}$ **do**

$\mathcal{A}_i = \mathcal{A}_{i-1} \cup \underset{v}{\operatorname{argmax}} \{\Delta(v|\mathcal{A}_{i-1})\}$

end for

return $\mathcal{A}_{k_+}, F_+(\mathcal{A}_{k_+})$

The greedy algorithm starts with the empty set $\mathcal{A}_0 = \emptyset$, and at iteration i it adds a new element maximizing the discrete derivative $\Delta(v|\mathcal{A}_{i-1})$, i.e. $\mathcal{A}_i = \mathcal{A}_{i-1} \cup \underset{v}{\operatorname{argmax}} \{\Delta(v|\mathcal{A}_{i-1})\}$ where $\Delta(v|\mathcal{A}) = F_+(\mathcal{A} \cup \{v\}) - F_+(\mathcal{A})$.

Theorem 1. For any arbitrary instance of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the set function $F_+(\mathcal{A})$ defined in (4) is monotone and submodular.

Submodularity ensures that the greedy procedure in Algorithm 2 provides a good approximation to the optimal solution [16], as stated in the following corollary.

Corollary 1. For $k_+ > 0$ and $\ell < k_+$, it holds that $F_+(\mathcal{A}_\ell) \geq (1 - 1/e^{-\ell/k_+}) F^*$ with $F^* = \max_{|\mathcal{A}| \leq k_+} F_+(\mathcal{A})$.

Algorithm 2 instead of evaluating the objective function for all possible combination of edges, chooses one edge at a time, reducing significantly the complexity from $O(M(N/k_+)^{k_+})$ to $O(MNk_+)$, at the price of a bounded relative error $|F^* - F_+(\mathcal{A}_\ell)|/|F^*| \leq 1/e$.

Using monotonic properties and submodularity of set valued functions to derive a greedy search algorithm with guaranteed performance is quite standard in the set valued optimization. The work by Kempe et al. in [1] is one of the first paper that applies this approach to the field of maximization of the spread of influence in social networks. However, the proof of Theorem 1 is not straightforward. A self-contained proof in our specific setting can be derived following the same lines of [6]. The proof can be also obtained from results in [3]. Leveraging the electrical network analogy, it can be possible to split the strategic node \boxplus , described as a voltage source, into several identical sources, without affecting the electrical network. Then the problem is equivalent to the optimal leader selection problem devised in [3] for which submodularity holds true. We refer the reader to [17] for details of this equivalence.

III. OTP: A BLOCKING APPROACH

In this section, we study the OTP in the Complete Graph. Inspired by this result, we propose a simple heuristic to solve OTP in general graphs that does not require any evaluation of the equilibrium opinions.

A. OTP in Complete Graphs

We consider the situation where there are N regular nodes in the network forming a Complete Graph. In order to compute the objective function in this case, we exploit the anonymity property, i.e. the fact that regular agents share the same neighborhood, except for being connected or not to the strategic agents. Based on the latter, we can distinguish among four kinds of regular nodes, that is, we partition the set \mathcal{R} into

- \mathcal{R}^+ , the set of nodes linked to \boxplus but not to \boxminus and denote $p := |\mathcal{R}^+|$;
- \mathcal{R}^- , the set of nodes linked to \boxminus but not to \boxplus and denote $q := |\mathcal{R}^-|$;
- \mathcal{R}^\pm , the set of nodes linked to both \boxplus and \boxminus and denote $r := |\mathcal{R}^\pm|$; and
- \mathcal{R}^\emptyset , the set of nodes linked to neither of them, so that $N - p - q - r = |\mathcal{R}^\emptyset|$.

Anonymity ensures that the objective function is only a function of p, q, r , that is, we can write $F_+(\mathcal{A}) =: F_+(p, q, r)$. Then, the system of equations in (2) becomes

$$\begin{cases} \bar{x}_{v^+} = \frac{p-1}{N} \bar{x}_{v^+} + \frac{q}{N} \bar{x}_{v^-} + \frac{r}{N} \bar{x}_{v^\pm} + \frac{N-p-q-r}{N} \bar{x}_{v^\emptyset} + \frac{1}{N} \\ \bar{x}_{v^-} = \frac{p}{N} \bar{x}_{v^+} + \frac{q-1}{N} \bar{x}_{v^-} + \frac{r}{N} \bar{x}_{v^\pm} + \frac{N-p-q-r}{N} \bar{x}_{v^\emptyset} - \frac{1}{N} \\ \bar{x}_{v^\pm} = \frac{p}{N+1} \bar{x}_{v^+} + \frac{q}{N+1} \bar{x}_{v^-} + \frac{r-1}{N+1} \bar{x}_{v^\pm} + \frac{N-p-q-r}{N+1} \bar{x}_{v^\emptyset} \\ \bar{x}_{v^\emptyset} = \frac{p}{N-1} \bar{x}_{v^+} + \frac{q}{N-1} \bar{x}_{v^-} + \frac{r}{N-1} \bar{x}_{v^\pm} + \frac{N-p-q-r-1}{N-1} \bar{x}_{v^\emptyset} \end{cases}$$

with $v^+ \in \mathcal{R}^+$, $v^- \in \mathcal{R}^-$, $v^\pm \in \mathcal{R}^\pm$, and $v^\emptyset \in \mathcal{R}^\emptyset$. Solving this system, we find

$$F_+(p, q, r) = \frac{(N+2)(p-q)}{N(N+2)(p+q) + 2N(N+1)r}$$

Notice that if $p = q$, then $F_+(p, q=p, r) = 0$, and that $F_+(p, q, r)$ is decreasing in r . This formula allows us to give an explicit solution to the OTP problem. Let p_0, q_0, r_0 be the number of regular nodes initially linked to strategic agent \boxplus but not to \boxminus , to \boxminus but not to \boxplus , and to both, respectively. Strategic agent \boxplus has k_+ available links to add and define

$$F_+^* := \max_{\mathcal{A} \subseteq \mathcal{R} \setminus \mathcal{A}^{(0)}: |\mathcal{A}| \leq k_+} F_+(\mathcal{A}).$$

Let r_1 and p_1 be the numbers of additional nodes that are targeted by strategic agent \boxplus and initially are, respectively, linked or not to strategic agent \boxminus (with the constraints that $p_1 + r_1 = k_+$, $r_1 \leq q_0$ and $p_1 \leq N - p_0 - q_0 - r_0$). Observe that $F_+(\mathcal{A}) = F_+(p_0 + p_1, q_0 - k_+ + p_1, r_0 + k_+ - p_1)$.

Proposition 2 (k_+ OTP on Complete Graph). *The optimal solution p_1^* satisfies the following properties.*

- If $k_+ < q_0 - p_0$, then $p_1^* = k_+$ and $F_+^* = F_+(p_0 + k_+, q_0, r_0)$;
- If $k_+ = q_0 - p_0$, then $F_+(p, q, r) = 0$ irrespective of p_1 ;
- If $k_+ > q_0 - p_0$, then $p_1^* = \max\{0, k_+ - q_0\}$ and

$$F_+^* = F_+(p_0 + p_1^*, q_0 - k_+ + p_1^*, r_0 + k_+ - p_1^*)$$

The proof is postponed to the Appendix.

Remark 2 (Offensive versus defensive strategy). *The result can be interpreted as follows: in a scenario in which the competing agent \boxminus has a smaller budget of links, in order to maximize the influence of node \boxplus , the optimal strategy is to have an offensive approach and target the same nodes connected to \boxminus . Conversely, if \boxplus is at a disadvantage, the optimal solution is to have a defensive approach, i.e. target nodes that allow to limit the opponent's influence, and then not get overpowered by hitting the same nodes. This result intuitively formalizes a common sense: think of a 10 vs 11 soccer game, where it makes more sense to defend strategically, than to man-mark and leave one free.*

B. Blocking heuristics on general graphs

In Section III-A the OTP has been solved for the Complete Graph. It is worth remarking that, using the greedy method (see Algorithm 2), at each iteration the algorithm would have introduced an error if the condition $p^{(k)} + 1 > q^{(k)}$ was not satisfied, where $p^{(k)}, q^{(k)}$ are the number of nodes connected exclusively to \boxplus and \boxminus at iteration $k = 1, \dots, k_+$, respectively. With this in mind, we design a new heuristic in Algorithm 3. Let us denote by \mathcal{A}^- the set of nodes linked to \boxminus , while by $\mathcal{A}^{(0)}$ the set of initial nodes linked to \boxplus .

Algorithm 3, in practice, compares the $|\mathcal{A}^{(0)} \setminus \mathcal{A}^-| + k_+$ overall edges of agent \boxplus not linked to \boxminus , with the $|\mathcal{A}^- \setminus \mathcal{A}^{(0)}|$ edges not linked to \boxplus of agent \boxminus . If the comparison tells that the former is larger, then extra edges of \boxplus will be used until possible to target the nodes in $\mathcal{A}^- \setminus \mathcal{A}^{(0)}$. Then, if some edges

Algorithm 3 Blocking heuristic for OTP

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, set node \mathcal{A}^- , set node $\mathcal{A}^{(0)}$, number of available links k_+

Initialization:

$$\mathcal{A}_0 = \emptyset, s = 0$$

if $k_+ > |\mathcal{A}^- \setminus \mathcal{A}^{(0)}| - |\mathcal{A}^{(0)} \setminus \mathcal{A}^-|$ **then**

$$\mathcal{A}_s = \mathcal{A}^- \setminus \mathcal{A}^{(0)}, s = |\mathcal{A}^- \setminus \mathcal{A}^{(0)}|$$

end if

if $k_+ > s$ **then**

for $i \in \{s+1, \dots, k_+\}$ **do**

$$\mathcal{A}_i = \mathcal{A}_{i-1} \cup \operatorname{argmax}_{v \in \mathcal{R}} \{\Delta(v | \mathcal{A}_{i-1})\}$$

end for

else

$$\mathcal{A}_{k_+} = \mathcal{A}_s$$

end if

return $\mathcal{A}_{k_+}, F_+(\mathcal{A}_{k_+})$

are still available to agent \boxplus , they will be placed by following the greedy approach (see Algorithm 2). On the other hand, if the first condition is not satisfied, it simply reduces to the Greedy heuristic. It should be noticed that Algorithm 3 has in general a smaller cost than the Greedy heuristic, since it requires to evaluate the equilibrium opinions $\max\{0, k_+ - q\}N$ times.

Experiment 1 (Erdos-Renyi graphs). We compare the proposed algorithms with zero-cost heuristics and greedy approaches. The study is aimed at highlighting the effect of the density of the network on the performance of the algorithms. In particular, we consider random generated Erdos-Renyi graphs [15] with parameters $N = 400$ and $p = a \frac{\log N}{N}$. For each value of $a \in [1.5, 10]$, we generate 50 random graphs and we connect $|\mathcal{A}^-| = 3$ nodes randomly to the strategic agent \boxminus , whereas strategic agent \boxplus has $k_+ = 5$ available budget. The performance of the algorithms are compared in Figure 1 (see dotted curves). It should be noted that the best performance is obtained with the heuristics based on the Blocking approach (Alg. 3, \dots) and, as to be expected, the Degree heuristics (Alg. 1, \dots) performs the worst.

It is worth remarking that the initial choice of the adversary may influence the performance of the proposed heuristics, and it could make sense considering a "strategic" adversary (node \boxminus) by selecting nodes with high centrality, or one maximizing some other measure of influence. The dashed-dotted curves in Fig. 1 show the performance of the heuristics when the node \boxminus chooses three nodes uniformly at random among the 10% of nodes with highest degree. Also in this case the Blocking heuristic outperforms the other heuristics and the gain is higher for sparse networks (for low values of a) and reduces with the density of the network (high values of a). Finally, when the node \boxminus is targeting regular nodes with maximal degree, all heuristics have similar performance (see continuous curves in Fig. 1).

In the following experiment we test the heuristics over graphs with communities and study the performance as function of the overall connectivity.

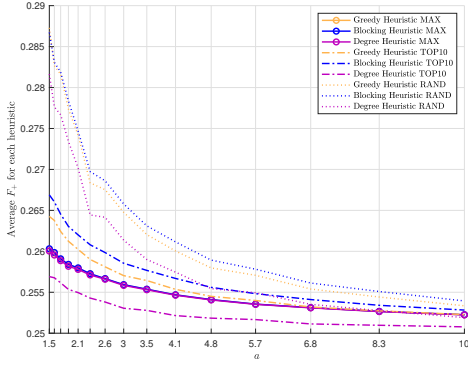


Fig. 1: Erdos-Renyi with $p = a \log(N)/N$: Average F_+ over 50 simulations as a function of connectivity parameter a .

Experiment 2 (Modular graphs). We consider the case with 7 dense communities with total link density $p \in [0, 2, 0.5]$ proportion of links within modules compared to links across equal to 0.99. In Fig. 2 we show single realizations of the networks for different values of p . As p increases, the number of connections among the clusters becomes bigger and bigger.

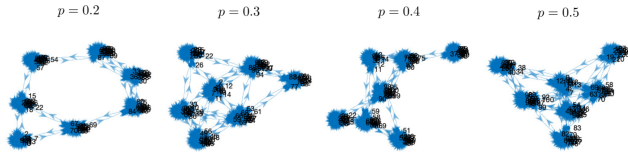


Fig. 2: Single realizations of modular graphs with connectivity parameter $p \in \{0.2, 0.3, 0.4, 0.5\}$.

For each value of total link density, we generate 50 random graphs and we connect $|\mathcal{A}^-| = 3$ nodes randomly to the strategic agent \boxminus , whereas strategic agent \boxplus has $k_+ = 5$ available budget. The performance of the algorithms are compared in Figure 3 (see dotted curves). Also in this case, the Blocking heuristic (Alg. 3, \cdots) outperforms the other heuristics. The Blocking heuristic is effective and robust, even if the strategic agent \boxminus is targeting the nodes using the degree centrality or the Greedy heuristics.

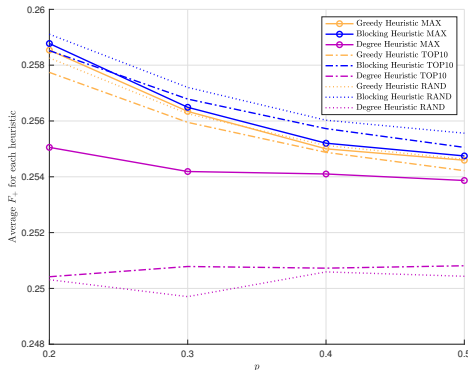


Fig. 3: Modular graphs: Average F_+ over 50 simulations as a function of connectivity parameter $p \in [0.2, 0.5]$.

However, we want to point out that a "smart choice" by the first strategic agent does not automatically imply

	Max	Top-10	Random
Degree	0.25247	0.25013	0.27717
Greedy	0.25201	0.25199	0.27821
Blocking	0.25258	0.25259	0.27424
Blocking (high degree)	0.25258	0.25259	0.27825

TABLE I: Heuristics comparison with different initial scenarios: Values of cost function F_+ , averaged over 50 graphs randomly generated with bimodal degree distribution.

a worse performance of the Blocking algorithm for OTP. On the contrary, in this case, when the adversary chooses highly influential nodes, the blocking strategy becomes particularly advantageous. Conversely, a "dumb" adversary choosing poorly connected nodes may put ourselves in a situation where the blocking strategy partially loses its efficacy. In this case, indeed, a better choice would be to not consider this node and concentrate on highly connected nodes. To show this fact we consider an ensemble of graphs with a bimodal distribution for the nodes' degrees. Table I reports the value of F_+ averaged over 50 graphs with degree distribution which is a mixture of random variable with equal variance, one with mean 10 and mass 0.75 and the other with mean 80 and mass 0.25.

Now, in addition to previous heuristics, we also consider a modification of the Blocking heuristic where the blocking is limited to high degree nodes (*Blocking (high degree)*). We consider each of these, for three different initial targeting strategies of node \boxminus : i) targeting the nodes with max degree (*Max*), ii) targeting the nodes randomly among the ones with degree in the top-10% (*Top-10*), and iii) targeting the nodes randomly (*Random*). In other terms, the Blocking heuristic is strictly dependent on opponent's strategy, but a simple modification can easily bypass potential issues. For the detailed algorithm of this modification, the reader can see [17]. The interplay between strategies based on centrality measures and topology would merit a deeper discussion and will be subject of future research.

IV. STP: ELECTRICAL ANALOGY AND TREES

In this section, we show some analytical results regarding the solution of STP on specific networks: Line Graphs and trees. Before stating and proving our results, we describe a key methodology.

A. Line Graph

We denote by v^+ and v^- the regular nodes that are linked to the strategic nodes \boxplus and \boxminus , respectively.

Proposition 3 (STP on the Line). Assume the strategic node \boxminus is directly connected to a generic node $v^- = \ell$. Then, the objective function reads

$$F_+(k) = \begin{cases} \frac{-k^2 + (N+1)k - (N+1)\ell + \ell^2}{N(k-\ell+2)} & \text{if } k \geq \ell \\ \frac{-k^2 + (N+1)k - (N+1)\ell + \ell^2}{N(\ell-k+2)} & \text{if } k < \ell \end{cases}$$

and attains the maximum value at $k^* = \operatorname{argmax} \{F_+(\lfloor \hat{k} \rfloor), F_+(\lceil \hat{k} \rceil)\}$ with

$$k^* = \begin{cases} \ell - 2 + \sqrt{2N + 6 - 4\ell} & \text{if } \ell < \frac{N+1}{2} \\ \ell + 2 + \sqrt{4\ell + 2 - 2N} & \text{if } \ell \geq \frac{N+1}{2} \end{cases}$$

Proposition 3, whose proof is given in the Appendix, guarantees that there exists an optimal value of $v^+ = k^*$, which is placed on the left or on the right of $v^- = \ell$ depending on the value of ℓ . This fact is quite intuitive: indeed, if v^- is not in the middle, the strategic agent \boxplus is able to influence a larger amount of individuals by targeting an agent on the opposite side of v^- . What is surprising is that, in general, *targeting an agent immediately next to v^- is not an optimal choice*. It is more effective to target an agent slightly on the opposite side, with some nodes of distance. This is due to the impact that also agent \boxminus has, being close to v^+ too. Clearly, if v^- is in the middle, then the optimal choice for \boxplus is to cancel out its effect by targeting the same node. This is what happens in the popular game theoretical Hotelling model [18]. The following remark provide an interpretation of the result using the electrical network analogy.

Remark 3 (A marginal analysis). *Let us consider the following scenario. Suppose agent \boxplus is targeting a node indexed by ℓ (assume ℓ is at least two nodes far from the extremes). One could ask which forces are involved when moving agent \boxplus from node $k = \ell + r$ to $k = \ell + r + 1$ ($r \geq 0$), eventually leading to the equilibrium in the result of Proposition 3.*

To grasp the intuition behind this, one can intuitively think of the usual electrical analogy: when agent \boxplus targets node k , the nodes from k to N (extreme node) are all short-circuited, contributing to the final average opinion with the same voltage $V(k)$. So, moving from node $k = \ell + r$ to $k = \ell + r + 1$, means having one less node contributing in the short-circuited tail. On the other hand, voltage $V(k)$ is intuitively influenced by its vicinity to agent \boxminus : meaning that moving from node $k = \ell + r$ to $k = \ell + r + 1$, the voltage $V(k)$ shared in the short-circuited tail increases, being less influenced by agent \boxminus . Therefore, we can identify these two opposing forces in (i) the number of nodes present in the short-circuited tail, sharing the voltage $V(k)$, and (ii) the amount of such a voltage.

B. Tree Graphs

For the Line Graph we have found an analytical solution for the STP, determining exactly the optimal position of v^+ in order to maximize the influence of \boxplus . In an analogous way, we could think of extending the argument behind the previous section to a generic tree. Indeed, given the position of v^- , for each possible choice of v^+ there exists just one path connecting v^- to v^+ , thus leading to a similar situation as before. By considering the corresponding electrical network it is easy to see that each node that does not belong to this path is short-circuited with one on the path, i.e. the only voltage drops happen along this path.

On the other hand, when considering a generic tree, the computation of $V(i)$ is not straightforward. Indeed, while for the Line Graph each intermediate node between v^- and v^+ produces an identical voltage drop, for the Tree Graph each node belonging to the path between v^- and v^+ contributes to such drop proportionally to the number of nodes of its subtree (see Figure 4 for a better understanding). Because of



Fig. 4: Tree Graph with v^-, v^+

this complication, in order to compute F_+ for a generic Tree Graph, it becomes necessary to have more information about the tree.

More formally, let $\mathcal{T} = (\mathcal{I}, \mathcal{E})$ be a Tree Graph. Then, given a pair of distinct nodes $i, j \in \mathcal{I}$, denote by $\mathcal{I}^{<ij}$ the subtree rooted at node i that does not contain node j , along with the path from i to j (apart from i), i.e.

$$\mathcal{I}^{<ij} = \{h \in \mathcal{I} | \text{path from } h \text{ to } j \text{ goes through } i\}$$

Similarly, denote by c_i the cardinality of the subtree rooted at node $i \in \{\text{path from } v^- \text{ to } v^+\}$ made up by the nodes $j \in \mathcal{I}^{<iv^-} \cap \mathcal{I}^{<iv^+}$. Using this formalism, the objective function F_+ can be written as follows:

$$F_+(k) = \frac{1}{N} \left[|\mathcal{I}^{<v^-v^+}| V(v^-) + |\mathcal{I}^{<v^+v^-}| V(v^+) + \sum_{i \in \{\text{path from } v^- \text{ to } v^+\}} c_i V(i) \right]$$

It is clear from this expression that the optimal node lies on the path from v^- to v^+ and therefore the objective function needs to be evaluated on these nodes only. Actually, we shall now show that the number of evaluations can be reduced further. The proof is presented in the Appendix.

Proposition 4 (Monotonicity over a branch). *Let $\mathcal{T} = (\mathcal{I}, \mathcal{E})$ be a tree. Let us consider $v^- = 1$ as the root and consider a tree branch, a path going from the root to one of the leaves, denoting the nodes in the sequence by $i \in \{1, 2, \dots, L\}$. Then, there exists $k^* \in \{1, \dots, L\}$ such that the objective function $F_+(k)$ is monotone increasing in $k \in [1, k^*] \cap \mathbb{N}$ and monotone decreasing in $k \in [k^*, L] \cap \mathbb{N}$.*

Let us visit one node at a time starting from the root $v^- = 1$. The strategic node \boxplus , currently considering node k , could move to one of its children, looking for a node that increases the objective function F_+ .

Proposition 5 (Exploration of offspring population). *Let $\mathcal{T} = (\mathcal{I}, \mathcal{E})$ be a tree. Let $v^- \in \mathcal{I}$ be the root node, and consider the path from node v^- to a generic node $k \in \mathcal{I}$. Let $\mathcal{O}(k)$ be the offspring of k , denoted as the set of nodes linked to k belonging to the subtree $\mathcal{I}^{<kv^-}$. If there exists $m \in \mathcal{O}(k)$ such that $F(m) > F(k)$, then $F(n) < F(k)$ for each $n \in \mathcal{O}(k) \setminus \{m\}$.*

Proposition 5, whose proof is given in Appendix, guarantees that at most one of its children can increase the objective function value. Then, it is useless to compute the objective function on the other nodes, if an improving node has already been found.

Proposition 4 and Proposition 5 imply that, starting from the root v^- and moving v^+ from v^- to its first neighbors, only one of them will make F_+ increase. This is true also for such improving neighbor and it continuous, as going towards the

leaves, until no improving neighbor is found, thereby identifying the optimal node. This leads to design the following algorithm in order to improve the maximum search algorithm.

Algorithm 4 Tree Graph Single Targeting Algorithm [TGSTA]

Require: $\mathcal{T} = (\mathcal{I}, \mathcal{E})$ tree graph, node v^-

Initialization:

Root node $r = v^-$

Number of visited nodes $s = 0$

Evaluate $F_+(r)$

Flag $f = 0$

while $f = 0$ **do**

$f = 1$

for $\ell \in \mathcal{O}(r)$ **do**

$s = s + 1$

Evaluate $F_+(\ell)$

if $F_+(r) < F_+(\ell)$ **then**

$r = \ell, F_+(r) = F_+(\ell), f = 0$

break for

end if

end for

end while

return $v^+ = r, F_+(r), s$

Theorem 2 (STP over trees). *Let $\mathcal{T} = (\mathcal{I}, \mathcal{E})$ be a tree, Algorithm 4 solves STP.*

Proof. Since $F_+(\cdot)$ admits a maximum on each branch (see Proposition 4) and there is at most one initial node from which a monotonically increasing branch can start (see Proposition 5), we conclude the convergence of the algorithm to the solution of STP. \square

Experiment 3. We consider 50 random trees: we start with a single individual in generation 0 and, for each node, a number of children is generated according to a Poisson distribution with parameter $\lambda \in \{3, 6, 9, 12\}$. The strategic node \boxplus is connected to a node chosen uniformly at random. For each instance, the STP problem is solved using TGSTA (Alg. 4). Figure 5 depicts the average fraction of visited nodes as function of number of regular nodes in the network for different offspring distributions (different curves correspond to different value of λ). It should be noticed that TGSTA allows to reduce largely the computational complexity of maximum search. Moreover, when the size of the tree increases, then the gain becomes larger and just a small fraction of nodes needs to be explored.

V. TREE-LIKE HEURISTICS

We now apply the insights of the previous section and extend Algorithm 4 to graphs that are not trees.

A. STP in Tree-like graphs

We present now an algorithm, referred to *Tree-like Single Targeting Algorithm*, that works as follows. When looking at the root's offspring, it does not stop looking at the first increasing F_+ value found, but it saves each *improving node*

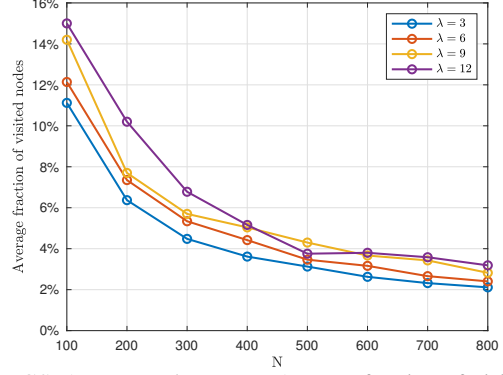


Fig. 5: TGSTA over random trees: Average fraction of visited nodes averaged over 50 experiments for different offspring distributions (Poisson(λ)).

(i.e. a node leading to an increasing value of F_+). In principle, we could use all of them as roots for the next iterations. Indeed, being the graph not a tree, it is possible to have more values in the nearest neighborhood leading to increasing values of F_+ . Then, we decide to use only the node leading to the maximum improvement as the root for the next iteration. The code is summarized in Algorithm 5.

Algorithm 5 Tree-like Single Targeting Algorithm (Tree-like STA)

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph, node v^-

Initialization:

Root node $r = v^-$

Number of visited nodes $s = 0$

while $r \neq \emptyset$ **do**

$v = \emptyset$ empty set of improving nodes

for $\ell \in \mathcal{O}(r)$ **do**

$s = s + 1$

Evaluate $F_+(\ell)$

if $F_+(r) < F_+(\ell)$ **then**

$v = v \cup \{\ell\}$

end if

end for

$r = \operatorname{argmax}_{\hat{v} \in v} F_+(\hat{v})$

end while

return $v^+ = r, F_+(r), s$

It is worth mentioning that the proposed routine is a greedy search algorithm in general graphs: since there are multiple utility-improving moves at each node, the algorithm evaluates all possible transitions at the current node and moves to the one with highest marginal gain. When comparing all the neighbors of a current node, the algorithm is actually looking for the best subsequent target that maximizes the average opinion at that step: this comparison among all the discrete alternatives at each step can be interpreted as a ranking of links based on a sensitivity analysis. Then, a *pruning* of non-optimal alternatives is done at each step, to get a tree where we can exploit the proposed optimal strategies.

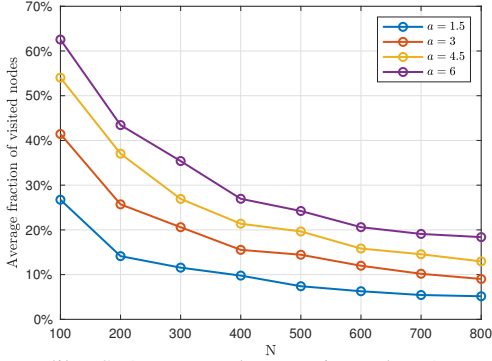


Fig. 6: Tree-like STA over Erdos-Renyi graphs: Average fraction of visited nodes averaged over 50 experiments for different connectivity parameter a .

The main driver for the structural properties in optimal single-targeting on trees is that the equilibrium average opinion can be written as a weighted averaging of opinions of those nodes on the path connecting the two targets (one for each player), where for each node on the path the weight is proportional to the size of the subtree attached to the path at the node. This underlying property of the solution significantly simplifies the single target search on trees but is not true for general graphs. However, our intuition is that the proposed algorithm should work for a class of graph ensembles for which the typical random graph locally behaves like a tree. The formal property of Local Weak Convergence is the key feature that provides this link and formalizes the idea that, for large size of the network, the local structure of the graph near a vertex chosen uniformly at random is approximately a tree. It can be shown that in the sparse regime where the number of edges scales linearly in the size of the networks, the large majority of random models are tree-like. We refer the interested reader to [19] for a deeper discussion on this class of graphs. The core idea is that exact computations supposing to be on a tree provide a good approximation in the large systems limits. This is a common approach applied to a class of problems that arise in combinatorial probability and combinatorial optimization over networks [20].

We corroborate these observations with the following experiment.

Experiment 4 (Erdos-Renyi: sparse versus dense networks). We now consider STP on 50 random generated Erdos-Renyi graphs with connectivity parameter $a \in \{1.5, 3, 4.5, 6\}$, $p = a \log(N)/N$. We solve the STP by using Tree-like-STA (Alg. 5): denote by F^* the optimal value of the objective function and \hat{F} the value of function at node identified by Tree-like STA (Alg. 5). Figure 6 and Table II show the number of visited nodes and the empirical probability of success as a function of the size of the network, where we declare a success when $|F_+^* - \hat{F}_+|/|F_+^*| \leq 1/e$.

Some remarks are in order. The number of visited nodes increases when the graph is less sparse and the probability of success is larger than 0.8 for all networks. Additionally, as the exploration decreases, accuracy gets worse.

	$a = 1.5$	$a = 3$	$a = 4.5$	$a = 6$
$N = 100$	0.900	0.960	0.980	1.000
$N = 200$	0.940	0.960	0.940	0.980
$N = 300$	0.840	0.940	0.960	0.960
$N = 400$	0.840	0.920	0.880	0.940
$N = 500$	0.840	0.960	0.960	0.940
$N = 600$	0.800	0.900	0.940	0.980
$N = 700$	0.920	0.940	0.880	0.960
$N = 800$	0.860	0.860	0.900	0.880

TABLE II: Tree-like STA over Erdos-Renyi graphs: Empirical probability of success for different connectivity parameter a .

In real networks we distinguish two distinct scenarios based on the current target's neighborhood: (i) target's neighborhood is locally tree-like; (ii) target's neighborhood is highly clustered. In the first scenario the algorithm will work for a certain number of steps and simplify the exploration of the graphs. In the second scenario the proposed greedy search will choose at each step the nearest node with highest objective function, while discarding the other options. By repeating this procedure, after some steps, the algorithm will end up exploring zones with locally tree-like structure, where the algorithm is guaranteed to work well. In this regard, the greedy search can be seen as a heuristic that is used locally to exit from these dense zones.

We consider the family of modular graphs in order to test the algorithm in this scenario.

Experiment 5 (Modular graphs). We consider the case with 7 dense communities with total link density $p \in [0, 2, 0.5]$ proportion of links within modules compared to links across equal to 0.99. Figure 7 and Table III show the number of visited nodes and the empirical probability of success as a function of the size of the network, where we declare a success when $|F_+^* - \hat{F}_+|/|F_+^*| \leq 1/e$.

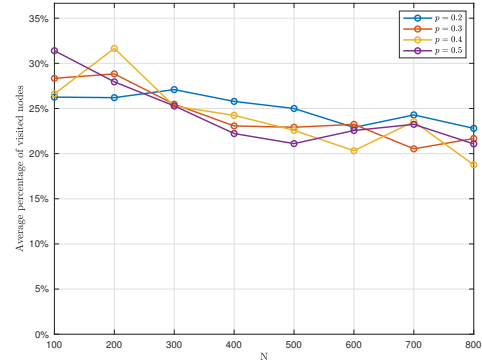


Fig. 7: Tree-like STA over modular graphs: Average fraction of visited nodes averaged over 50 experiments for different connectivity parameter p .

We can see that also in this scenario, the proposed algorithm simplifies the the exploration of the graph and finds a good solution with high probability by visiting only approximately 27% of total nodes.

Experiment 6 (Facebook ego-network). We now test Algorithm 5 on a real large-scale online social network: the Facebook ego-network, retrieved from Stanford Large Network Dataset Collection (<https://snap.stanford.edu/data/>)

	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
$N = 100$	0.840	0.960	0.800	0.860
$N = 200$	0.760	0.780	0.860	0.860
$N = 300$	0.780	0.800	0.820	0.940
$N = 400$	0.800	0.900	0.900	0.880
$N = 500$	0.880	0.860	0.780	0.760
$N = 600$	0.840	0.900	0.880	0.840
$N = 700$	0.920	0.840	0.920	0.880
$N = 800$	0.920	0.860	0.920	0.900

TABLE III: Tree-like STA over modular graphs: Empirical probability of success computed on 50 experiments for different connectivity parameter p .

egonets-Facebook.html). This dataset contains anonymized personal networks of connections between friends and the size of the graph associated is $|\mathcal{V}| = 4039$, while the number of links is equal to $|\mathcal{E}| = 88234$. Such graph is extremely sparse, since the number of nonzero elements $|\mathcal{E}|/|\mathcal{V}|^2 \approx 5 * 10^{-3}$. 10 instances of STP are generated by linking agent \boxplus to a random regular agent. We find that Algorithm 5 reaches the optimum (solving STP by means of the brute-force approach). We obtain that the average fraction of visited nodes is equal to 30%.

When the number of links placed by strategic agent \boxplus is greater than 1, we use a generalized version of the Tree-like heuristic: among the nodes linked to \boxplus , we select as the root v^- from which Algorithm 4 is started the one with smallest degree. The reasoning behind this algorithm, supported by empirical simulations, is that in sparse graphs it is easier to move away from not relevant nodes rather than vice versa. Indeed, if starting the algorithm from high degree nodes, the first steps would generally be more affected by the noise produced by the strong influence of \boxplus .

B. OTP on Tree-like graphs

For the general OTP, when both strategic agents have a number of available or placed nodes greater than 1, we propose an algorithm that is a generalized version of previous Single Targeting algorithms over tree-like graphs. Specifically, it simulates a Greedy heuristic where a sub-optimum is found at each step. Specifically, this is done by selecting at each step a different root node among the ones linked to \boxplus .

Experiment 7. Let us now compare the Tree-like heuristics (Alg. 5) with Greedy heuristics (Alg. 3) on random generated Erdos-Renyi graphs of parameters $N = 200$ and $p = 0.1$. We generate 15 random graphs and we link $|A^-| = 3$ nodes randomly selected to the strategic agent $-$, while $k_+ = 5$. The results are reported below:

	Average F_+	Average fraction of visited nodes at each step
Tree-like heuristic	0.2555	27%
Greedy heuristic	0.2566	100%

We can easily see how the F_+ values are really close to each other, while the average number of computations is almost reduced by one quarter by the Tree-like heuristic.

VI. SUMMARY AND CONCLUDING REMARKS

In this paper we have considered the optimal targeting problem in a social network where two strategic agents are competing. We have both studied the problem for special classes of graphs and proposed heuristics for general graphs. Available heuristics essentially rely on two approaches, namely, the identification of “important” nodes by some general-purpose measure of centrality –the simplest such *centrality measure* is just the node degree–, or a *greedy* approach, in which nodes are targeted one-at-the-time and which is justified by the submodularity of the objective function. We have exemplified these two approaches by Algorithms 1 and 2.

Starting from this background, we have studied the problem in Complete Graphs and on trees: in the former case, we have explicitly found the optimal solution; in the latter, we have identified two key properties that greatly simplify its computation. Complete and tree graphs are extremal examples, as they respectively feature maximal and minimal connectivity (only one path connects any two nodes on a tree). The insights from these two examples are more broadly relevant, as they translate into novel heuristic approaches to the optimal targeting problem. The Complete Graph suggests the approach of *blocking* nodes that have been targeted by the adversary: this approach has zero cost, meaning that it requires no evaluations of the equilibrium opinions (Alg. 3). The tree graphs suggest the approach of *tree-like exploration*, which allows greatly reducing the cost of the greedy approach (Alg. 5).

These four heuristic approaches can –and should– be combined in designing heuristic algorithms for the optimal targeting problem. Note, for instance, that our Algorithm 3 does combine blocking and greedy approach. The most suitable combination shall depend on the known properties of the underlying graph and its choice will require to address the trade-off between cost and accuracy. Let us for instance consider the choice of whether to block or not the opponent’s influence by targeting the same nodes. This is typically a wise choice, at least if one has the possibility of targeting more nodes than one’s opponent. However, a blocking approach is less effective if the graph is very sparse or if the opponent is linked to marginal nodes: the latter issue can be addressed by restricting the blocking procedure to the high degree nodes linked to \boxplus . More generally, if the graph is clearly split between high and low degree nodes, a degree-based approach would be a good choice. Also the choice of applying the Tree-like approach to accelerate the greedy algorithm should mainly be based on the graph structure. Indeed, if the graph is locally tree-like or sparse, a Tree-like approach would perform well and improve the complexity of the heuristic. Further research should concentrate on refining these heuristic guidelines.

APPENDIX

A. Proof of Proposition 2

If k_+ edges are available, then the strategic agent \boxplus can target p_1 nodes not already linked to \boxplus and r_1 nodes in \mathcal{R}^- . Adding these links, we obtain that

$$F_+^* = \max_{p_1, r_1: p_1 + r_1 = k_+} F_+(p_0 + p_1, q_0 - r_1, r_0 + r_1).$$

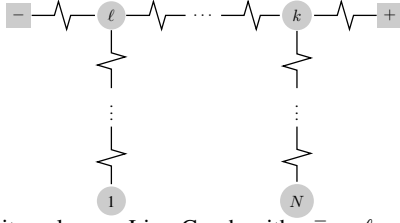


Fig. 8: Circuit analogous Line Graph with $v^- = \ell$, $v^+ = k$, $k \geq \ell$

The objective function F_+ can be increasing or decreasing in p_1 , depending on whether $k_+ \geq q_0 - p_0$. If $k_+ < q_0 - p_0$, then the objective function is negative and the maximizing value p_1^* is obtained with $p_1^* = k_+$ and $r_1^* = 0$. If $k_+ = q_0 - p_0$, then F_+ is always zero. If $k_+ > q_0 - p_0$, then the objective function is positive and the optimum is reached by taking the smallest value of p_1 , that is, the largest value of r_1 . Since the latter is naturally constrained by $r_1 \leq q_0$ and by $p_1 \leq N - p_0 - q_0 - r_0$, the result follows. \square

B. Proof of Proposition 3

Assume the strategic node \boxminus is directly connected to a node ℓ and the strategic node \boxplus will choose a node $k \geq \ell$. By the electrical analogy, we can interpret the strategic nodes \boxminus and \boxplus as voltage sources of value -1 and $+1$ respectively. The nodes $v \in \{1, \dots, \ell - 1\}$ and $v \in \{k + 1, \dots, N\}$ will be short-circuited with the nodes ℓ and k , respectively, i.e. $V(-) = -1$, $V(+)=+1, V(1) = V(2) = \dots = V(\ell)$ and $V(k) = V(k+1) = \dots = V(N)$. We compute the voltage in each node $i = \ell, \ell + 1, \dots, k$ as the voltage drop in the voltage divider (as represented below) where the effective resistances are the summation of the resistances on the left and on the right of node i , that is $R_{-i}^{\text{eff}} = i - \ell + 1$ and $R_{i+}^{\text{eff}} = k - i + 1$ leading to

$$V(i) - V(-) = (V(+)-V(-)) \frac{i - \ell + 1}{i - \ell + 1 + k - i + 1}$$

$$V(i) = 2 \frac{i - \ell + 1}{k - \ell + 2} - 1$$

and $F_+(k) = \frac{1}{N(k-\ell+2)} \left[-k^2 + (N+1)k - (N+1)\ell + \ell^2 \right]$. The maximum value of F_+ is at $\hat{k} = \ell - 2 + \sqrt{2N + 6 - 4\ell}$, when $\ell < \frac{N+1}{2}$. With similar arguments we get the expression for $k < \ell$. \square

C. Proof of Proposition 4

From the electrical analogy we obtain $\forall i = 1, \dots, k$ that $V(i) = 2i/(k+1) - 1$ and $V(j) = V(i)$, $\forall j \in \mathcal{I}^{<i1} \cap \mathcal{I}^{<ik}$. Then, noticing that $\mathcal{I}^{<ik} = \mathcal{I}^{<iL}$, $\forall k : i < k < L$, we have

$$F_+(k) = \frac{1}{N} \left[|\mathcal{I}^{<1k}| V(1) + \sum_{i=2}^{k-1} |\mathcal{I}^{<i1} \cap \mathcal{I}^{<ik}| V(i) + |\mathcal{I}^{<k1}| V(k) \right]$$

$$= \frac{1}{N} \left[|\mathcal{I}^{<1L}| V(1) + \sum_{i=2}^{k-1} |\mathcal{I}^{<i1} \cap \mathcal{I}^{<iL}| V(i) + |\mathcal{I}^{<k1}| V(k) \right]$$

$$= \frac{1}{N} \left[\sum_{i=1}^{k-1} c_i V(i) + V(k) \sum_{j=k}^L c_j \right]$$

where $c_i = |\mathcal{I}^{<i1} \cap \mathcal{I}^{<iL}|$ and $\mathcal{I}^{<1L} = \mathcal{I}^{<11} \cap \mathcal{I}^{<1L}$. Then, putting the expression for $V(i)$, we get

$$F_+(k) = \frac{1}{N} \left[\sum_{i=1}^k c_i \left(\frac{2i}{k+1} - 1 \right) + \left(\frac{2k}{k+1} - 1 \right) \sum_{j=k+1}^L c_j \right].$$

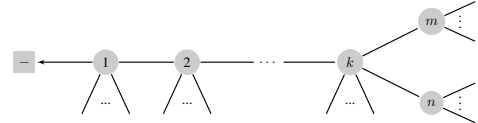


Fig. 9: Generic path from 1 to k on a tree with root $v^- = 1$. Once evaluated $F(k)$, Proposition 5 considers to move to one of the children (here m and n)

Notice that

$$F_+(k+1) - F_+(k)$$

$$= \frac{1}{N} \left[\sum_{i=1}^k c_i \left(\frac{2i}{k+2} - \frac{2i}{k+1} \right) + \left(\frac{2k+2}{k+2} - \frac{2k}{k+1} \right) \sum_{j=k+1}^L c_j \right]$$

$$= \frac{1}{N} \left[\sum_{i=1}^k c_i \left(-\frac{2i}{(k+1)(k+2)} \right) + \frac{2}{(k+1)(k+2)} \sum_{j=k+1}^L c_j \right]$$

$$= \frac{2}{N(k+1)(k+2)} \left(\sum_{j=k+1}^L c_j - \sum_{i=1}^k i c_i \right)$$

Let $G(k)$ be the quantity between brackets: observe that $G(k)$ is decreasing in k and that $G(L-1) \leq c_L - \sum_{i=1}^{L-1} c_i = -(L-1) \leq 0$. We conclude that there exists $k^* \in \{1, \dots, L\}$ such that $F_+(k)$ is decreasing in $k \in \{k^*, \dots, L\}$.

D. Proof of Proposition 5

Let us represent \mathcal{T} as a pseudo-line branch, and let us denote the path from the root node $v^- = 1$ to k , as the length- k path shown in Figure 9. Let us assume that at least one node m in k 's offspring $\mathcal{O}(k)$ is such that $F(m) > F(k)$, where $|\mathcal{O}(k)| \geq 2$, otherwise the proof would be trivial, and denote with n a generic node in such offspring different from m .

Let us consider the unique path from $v^- = 1$ to v^+ and let $c_i^{(v^+)}$ be the cardinality of the subtree generating from each node i in the line. It should be noticed that $c_k^{(k)} = c_k^{(m)} + c_m^{(m)} = c_k^{(n)} + c_n^{(n)}$ and $\xi = |\mathcal{I}^{<k1} \cap \mathcal{I}^{<km} \cap \mathcal{I}^{<kn}| = c_k^{(k)} - c_m^{(m)} - c_n^{(n)}$. By electrical analogy, we have

$$F_+(m) = \frac{1}{N} \left[\sum_{i=1}^{k-1} c_i^{(m)} V^{(m)}(i) + c_k^{(m)} V^{(m)}(k) + c_m^{(m)} V^{(m)}(m) \right]$$

$$= \frac{1}{N} \left[\sum_{i=1}^{k-1} c_i^{(m)} \left(\frac{2i}{k+2} - 1 \right) + \right.$$

$$\left. + (\xi + c_n^{(n)}) \left(\frac{2k}{k+2} - 1 \right) + c_m^{(m)} \left(\frac{2k+2}{k+2} - 1 \right) \right]$$

$$F_+(k)$$

$$= \frac{1}{N} \left[\sum_{i=1}^{k-1} c_i^{(n)} \left(\frac{2i}{k+1} - 1 \right) + (\xi + c_n^{(n)} + c_m^{(m)}) \left(\frac{2k}{k+1} - 1 \right) \right]$$

where $c_i^{(m)} = c_i^{(n)}$. Then

$$\begin{aligned} F_+(m) - F_+(k) &= \frac{1}{N} \left[\sum_{i=1}^{k-1} c_i^{(m)} \left(\frac{2i}{k+2} - \frac{2i}{k+1} \right) + \right. \\ &+ (\xi + c_n^{(n)}) \left(\frac{2k}{k+2} - \frac{2k}{k+1} \right) + c_m^{(m)} \left(\frac{2k+2}{k+2} - \frac{2k}{k+1} \right) \left. \right] \\ &= \frac{2}{N(k+1)(k+2)} \left[- \sum_{i=1}^{k-1} i c_i^{(m)} - k(\xi + c_n^{(n)}) + c_m^{(m)} \right] \\ &= \frac{2}{N(k+1)(k+2)} [-f - k c_n^{(n)} + c_m^{(m)}] \end{aligned}$$

where $f := \sum_{i=1}^{k-1} i c_i^{(m)} + k \xi > 0$.

By hypothesis $F_+(m) - F_+(k) > 0$, then $c_m^{(m)} > k c_n^{(n)} + f > \frac{1}{k} c_n^{(n)} + f$. We thus have

$$\begin{aligned} F_+(n) - F_+(k) &= \frac{2}{N(k+1)(k+2)} (-f - k c_m^{(m)} + c_n^{(n)}) \\ &\leq \frac{2}{N(k+1)(k+2)} [-(k+1)f] \leq 0. \end{aligned}$$

□

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