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SPDEs and Renormalisation

Lorenzo Zambotti

Abstract We review the main ideas of renormalisation of stochastic partial differential equations, as they appear in the theory of regularity structures. We discuss in an informal way noise regularisation, the transformation of canonical-to-renormalised models, the space of models and the role of the continuity of the solution map.

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In the foundational paper [7], Martin Hairer introduced a theory of regularity structures and applied it to solving two important equations, whose well-posedness was an open problem. The first part of [7] is truly a *theory*, in the sense that it can be applied in the same way to a large class of problems: the second part however, which deals with two concrete applications, contains more and more *ad hoc* arguments, which should be adapted if applied in different contexts. Worse, for many other interesting equations the approach proposed in [7] becomes impractical, since the combinatorial complexity can be huge.

Fortunately, the situation has changed recently. The trio of papers

- Martin Hairer (2014),
A theory of regularity structures, Inventiones Math., [7]
- Yvain Bruned, M.H., L.Z. (2016),
Algebraic renormalisation of regularity structures, arXiv, [2]
- Ajay Chandra, M.H. (2016),
An analytic BPHZ theorem for regularity structures, arXiv, [3]

now gives a *completely automatic black box* for obtaining local existence and uniqueness theorems for a wide class of SPDEs. Prominent examples are the following:

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$$\text{(KPZ)} \quad \partial_t u = \Delta u + (\partial_x u)^2 + \xi, \quad x \in \mathbb{R},$$

$$\text{(PAM)} \quad \partial_t u = \Delta u + u \xi, \quad x \in \mathbb{R}^2,$$

$$\text{(\Phi}_3^4) \quad \partial_t u = \Delta u - u^3 + \xi, \quad x \in \mathbb{R}^3,$$

for ξ a space time white noise (only depending in space for PAM).

Each of these equations contains a product between a distribution (generalised function) and another distribution or, in the case of PAM, a function without a sufficient regularity. In order to put the problem in a more general context, note that if $T \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution and $\psi \in \mathcal{S}(\mathbb{R}^d)$, then we can define canonically the product $\psi T = T\psi \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\psi T(\varphi) = T\psi(\varphi) := T(\psi\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The products appearing in the above equations can not be defined in this way since ψ is not expected to belong to $\mathcal{S}(\mathbb{R}^d)$.

Therefore, even with polynomial non-linearities, we do not know how to properly multiply such (random) distributions. The randomness here is in fact not crucial, since the problems are the same if we consider ξ as a deterministic distribution with sufficiently low regularity.

In classical stochastic calculus we have a similar problem in the definition of *stochastic integrals*. If $(B_t)_{t \geq 0}$ is a Brownian motion and $(A_t)_{t \geq 0}$ a smooth process, then we can define canonically by means of a simple integration by parts

$$\int_0^t A_s dB_s := A_t B_t - \int_0^t B_s \dot{A}_s ds, \quad t \geq 0.$$

However we have several possible extensions of this definition to a larger class of non-smooth A (Itô, Stratonovich...). This is related to the fact that, for A a generic process such that the Itô integral is well defined, the map $B \rightarrow \int_0^\bullet A_s dB_s$ is *measurable but not continuous*.

This is the starting point of the Rough Paths theory, initiated by Terry Lyons [9] and later expanded by, among others, Massimiliano Gubinelli, whose ideas on controlled [4] and branched [5] rough paths have much inspired Martin Hairer in the elaboration of Regularity Structures.

Lack of continuity of stochastic integrals with respect to trajectories of the underlying BM has relevant consequences when one looks for approximations by noise-regularisation, as can be already seen in the case of SDEs. Let us consider for example the ODE in \mathbb{R}^d

$$\frac{d}{dt} x_\varepsilon = b(x_\varepsilon) + f(x_\varepsilon) \frac{dB_\varepsilon}{dt} \tag{1}$$

where B_ε is a smooth approximation of a Brownian motion B and b, f are smooth coefficients. Then the Wong-Zakai Theorem [11, 12] states that, as $\varepsilon \rightarrow 0$, we have $x_\varepsilon \rightarrow x$, solution to the Stratonovich SDE

$$dx = b(x) dt + f(x) \circ dB.$$

In order to obtain the Itô SDE in the limit, one has to define rather

$$\frac{d}{dt} \hat{x}_\varepsilon = b(\hat{x}_\varepsilon) - \frac{1}{2} Df(\hat{x}_\varepsilon) f(\hat{x}_\varepsilon) + f(\hat{x}_\varepsilon) \frac{dB_\varepsilon}{dt} \quad (2)$$

and in this case, as $\varepsilon \rightarrow 0$, we have $\hat{x}_\varepsilon \rightarrow \hat{x}$, solution to

$$d\hat{x} = b(\hat{x}) dt + f(\hat{x}) dB. \quad (3)$$

Note that this is already an example of a "renormalised" equation: although there is no diverging quantity, we have to modify the regularised equation (1) if we want to obtain in the limit the Itô SDE (3). In other words, if we want to approximate (3), the correct choice is the less intuitive (2), rather than (1).

The deep reason for that, if any, is that the "product" $f(\hat{x}) dB$ is less intuitive than we would like to think, and it has to be properly interpreted if we want to approximate it with smoother distributions.

Let us go back now to SPDEs. Let $\xi_\varepsilon = \rho_\varepsilon * \xi$ be a regularisation of ξ , with $(\rho_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers, and let u_ε solve

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon)$$

where F is a non-linear function from a certain class, which includes the three equations above. The question is of course: what happens as $\varepsilon \rightarrow 0$? In order to control this limit, we need a topology such that

1. the map $\xi_\varepsilon \mapsto u_\varepsilon$ is continuous
2. $\xi_\varepsilon \rightarrow \xi$ as $\varepsilon \rightarrow 0$.

The first point requires the topology to be "strong", while the second one requires it to be "weak". In fact, no simple solution seems to be available if the regularity of ξ is sufficiently low. The analytic part of the theory of Regularity Structures (RS) gives a framework to solve this problem, by constructing, for a given equation,

- a space of Models \mathcal{M} endowed with a metric d
- a canonical lift of every smooth ξ_ε to a model $\mathbf{X}^\varepsilon \in \mathcal{M}$
- a continuous function $\Phi : \mathcal{M} \rightarrow \mathcal{S}'(\mathbb{R}^d)$ such that $u_\varepsilon = \Phi(\mathbf{X}^\varepsilon)$ solves the regularised equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon).$$

Roughly speaking, the model $\mathbf{X}^\varepsilon \in \mathcal{M}$ contains a finite number of explicit products which are relevant to the given equation. By simplifying a lot, we can say that convergence in (\mathcal{M}, d) means convergence of all these objects as distributions. For instance $\mathbf{X}^\varepsilon \in \mathcal{M}$ can contain

$$\xi_\varepsilon(G * \xi_\varepsilon)$$

(with G the heat kernel). These products can be *ill-defined* in the limit $\varepsilon \rightarrow 0$, e.g.

$$\mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] = \rho_\varepsilon * G * \rho_\varepsilon(0) \rightarrow G(0) = +\infty.$$

Therefore in general \mathbf{X}^ε does *not* converge in (\mathcal{M}, d) as $\varepsilon \rightarrow 0$.

The RS theory identifies a class of equations, called *subcritical*, for which it is enough to *modify a finite number of products* in order to obtain a convergent lift $\hat{\mathbf{X}}^\varepsilon \in \mathcal{M}$ of ξ_ε . For instance

$$\xi_\varepsilon(G * \xi_\varepsilon) \rightarrow \xi_\varepsilon(G * \xi_\varepsilon) - \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)]. \quad (4)$$

The model $\hat{\mathbf{X}}^\varepsilon \in \mathcal{M}$ contains all these modified (*renormalised*) products. Then we define the *renormalised solution* by $\hat{u}_\varepsilon := \Phi(\hat{\mathbf{X}}^\varepsilon)$.

One can summarize the procedure into three steps:

- *Analytic step*: Construction of the space of models (\mathcal{M}, d) and continuity of the solution map $\Phi : \mathcal{M} \rightarrow \mathcal{S}'(\mathbb{R}^d)$, [7]
- *Algebraic step*: Renormalisation of the canonical model $\mathbf{X}^\varepsilon \rightarrow \hat{\mathbf{X}}^\varepsilon \in \mathcal{M}$, [2]
- *Probabilistic step*: Convergence a.s. of the renormalised model $\hat{\mathbf{X}}^\varepsilon$ to $\hat{\mathbf{X}}$ in (\mathcal{M}, d) , [3].

The final result is a *renormalised solution* $\hat{u} := \Phi(\hat{\mathbf{X}})$, which is also the unique solution of a fixed point problem. This works for very general noises, far beyond the Gaussian case.

The Wong-Zakai result for SPDEs is much more subtle than for SDEs [8]: if

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + H(u_\varepsilon) + F(u_\varepsilon) \xi_\varepsilon, \quad x \in \mathbb{R}, \quad (5)$$

then $u_\varepsilon = \Phi(\mathbf{X}^\varepsilon)$ does *not* converge in general and, in order to obtain in the limit the correct Itô SPDE, it is necessary to renormalise the equation and study

$$\partial_t \hat{u}_\varepsilon = \partial_x^2 \hat{u}_\varepsilon + \bar{H}(\hat{u}_\varepsilon) - C_\varepsilon F'(\hat{u}_\varepsilon) F(\hat{u}_\varepsilon) + F(\hat{u}_\varepsilon) \xi_\varepsilon \quad (6)$$

where $C_\varepsilon = \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] = \varepsilon^{-1} c_0$,

$$\bar{H} := H + c_1 F' F^3 + c_2 F'' F' F^2,$$

and $\{c_0, c_1, c_2\}$ are constants depending on the mollifier ρ used in the noise-regularisation. It turns out that $\hat{u}_\varepsilon = \Phi(\hat{\mathbf{X}}^\varepsilon)$, and the limit $\hat{u} := \Phi(\hat{\mathbf{X}})$ solves the Itô SPDE

$$d\hat{u} = (\partial_x^2 \hat{u} + H(\hat{u})) dt + F(\hat{u}) dW_t, \quad (7)$$

where W is the martingale measure associated with ξ , in the spirit of [10].

Therefore, the Wong-Zakai result for SPDEs is that the correct approximation to (7) is not (5) but (6). The Stratonovich integral with respect to W is not well-defined, since the covariation between $F(\hat{u})$ and W is infinite, which explains the

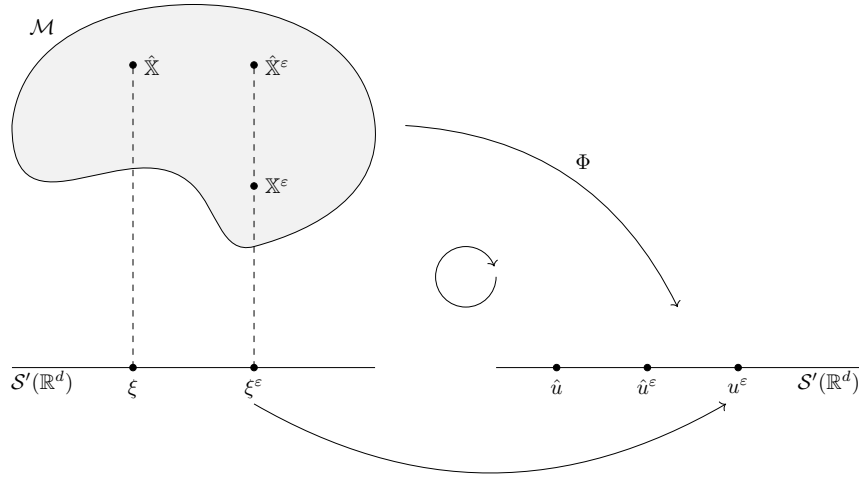


Fig. 1 In this figure we show the factorisation of the map $\xi_\varepsilon \mapsto u_\varepsilon$ into $\xi_\varepsilon \mapsto \mathbf{X}^\varepsilon \mapsto \Phi(\mathbf{X}^\varepsilon) = u_\varepsilon$. We also see that in the space of models \mathcal{M} we have several possible lifts of $\xi_\varepsilon \in \mathcal{S}'(\mathbb{R}^d)$, e.g. the canonical model \mathbf{X}^ε and the renormalised model $\hat{\mathbf{X}}^\varepsilon$; it is the latter that converges to a model $\hat{\mathbf{X}}$, thus providing a lift of ξ . Note that $\hat{u}_\varepsilon = \Phi(\hat{\mathbf{X}}^\varepsilon)$ and $\hat{u} = \Phi(\hat{\mathbf{X}})$.

diverging constant C_ε in (6). The renormalisation procedure should not be seen as a “modification of the equation” but as a *choice of the correct equation*.

It is very interesting to note that there is *nothing singular* in the SPDE (7). The convergence of \hat{u}_ε to \hat{u} is however far from simple and requires the full power of the RS theory [8].

Another important point is the following: the final aim is to renormalise the *unknown* solution $u_\varepsilon = \Phi(\mathbf{X}^\varepsilon)$. One of the main ideas of the RS theory is that, for this, it is enough to renormalise the *finitely many explicit* products defining the model \mathbf{X}^ε , and this can be done following [2]. Then [3] shows that the renormalised model $\hat{\mathbf{X}}^\varepsilon$ converges to $\hat{\mathbf{X}}$ in (\mathcal{M}, d) . Continuity of the solution map $\Phi : \mathcal{M} \rightarrow \mathcal{S}'(\mathbb{R}^d)$ yields convergence of the renormalised solution $\hat{u}_\varepsilon = \Phi(\hat{\mathbf{X}}^\varepsilon)$ to $\hat{u} = \Phi(\hat{\mathbf{X}})$ [7].

For instance, let us consider the regularised version of KPZ:

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi_\varepsilon. \quad (8)$$

The renormalised version is

$$\partial_t \hat{u}_\varepsilon = \partial_x^2 \hat{u}_\varepsilon + (\partial_x \hat{u}_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon \quad (9)$$

with

$$C_\varepsilon = \mathbb{E} \left[(\partial_x G * \xi_\varepsilon)^2 \right] \sim \frac{1}{\varepsilon}.$$

The first mathematical paper on KPZ is [1], where the solution is constructed by the Hopf-Cole transform, namely the simple remark that $z_\varepsilon := \exp(\hat{u}_\varepsilon)$ solves the *linear* equation

$$\partial_t z_\varepsilon = \partial_x^2 z_\varepsilon + z_\varepsilon \xi_\varepsilon. \quad (10)$$

It was not until [6] that a direct approach to (8)-(9) was obtained. The reason why mathematicians have been at loss with this equation for so long, is of course that it is not clear at all how one should handle the term $(\partial_x \hat{u}_\varepsilon)^2 - C_\varepsilon$ as $\varepsilon \rightarrow 0$. Well, now we know that it is enough to handle the convergence as a distribution of the explicit function

$$(\partial_x G * \xi_\varepsilon)^2 - \mathbb{E}[(\partial_x G * \xi_\varepsilon)^2] \quad (11)$$

which is the renormalised version of $(\partial_x G * \xi_\varepsilon)^2$, plus a few other terms which are less evident from the equation, and then the continuity of the solution map Φ does the rest of the job.

The Hopf-Cole transform shows that, as discussed above for (5)-(6), (9) is not merely the renormalisation of (8), it is rather *the correct equation* approximating KPZ. Indeed, $z_\varepsilon = \exp(\hat{u}_\varepsilon)$ converges to the solution z of the linear Itô SPDE

$$dz = \partial_x^2 z dt + z dW_t$$

while $\exp(u_\varepsilon)$ does not converge.

Finally, let us notice that the two examples of renormalised products we have discussed in (4)-(11) are simply given by subtraction of a constant. This might give the impression that the renormalisation procedure reduces to a mere *centering*. Although this is partly true, it should be emphasised that the transformation from \mathbf{X}^ε to $\hat{\mathbf{X}}^\varepsilon$ is described in [2] by the action on \mathcal{M} of a *renormalisation group* which is in general *non-linear and non-commutative*.

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