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# BOUNDED COMPLEXITY, MEAN EQUICONTINUITY AND DISCRETE SPECTRUM

WEN HUANG, JIAN LI, JEAN-PAUL THOUVENOT, LEIYE XU AND XIANGDONG YE

**ABSTRACT.** We study dynamical systems which have bounded complexity with respect to three kinds metrics: the Bowen metric  $d_n$ , the max-mean metric  $\hat{d}_n$  and the mean metric  $\bar{d}_n$ , both in topological dynamics and ergodic theory.

It is shown that a topological dynamical system  $(X, T)$  has bounded complexity with respect to  $d_n$  (resp.  $\hat{d}_n$ ) if and only if it is equicontinuous (resp. equicontinuous in the mean). However, we construct minimal systems which have bounded complexity with respect to  $\bar{d}_n$  but not equicontinuous in the mean.

It turns out that an invariant measure  $\mu$  on  $(X, T)$  has bounded complexity with respect to  $d_n$  if and only if  $(X, T)$  is  $\mu$ -equicontinuous. Meanwhile, it is shown that  $\mu$  has bounded complexity with respect to  $\hat{d}_n$  if and only if  $\mu$  has bounded complexity with respect to  $\bar{d}_n$  if and only if  $(X, T)$  is  $\mu$ -mean equicontinuous if and only if it has discrete spectrum.

## 1. INTRODUCTION

Throughout this paper, by a topological dynamical system (t.d.s. for short) we mean a pair  $(X, T)$ , where  $X$  is a compact metric space with a metric  $d$  and  $T$  is a continuous map from  $X$  to itself. Let  $\mathcal{B}_X$  be the Borel  $\sigma$ -algebra on  $X$  and  $\mu$  be a probability measure on  $(X, \mathcal{B}_X)$ . We say that  $\mu$  is an invariant measure for  $T$  if for every  $B \in \mathcal{B}_X$ ,  $\mu(T^{-1}B) = \mu(B)$ .

Entropy is a very useful invariant to describe the complexity of a dynamical system which measures the rate of the exponential growth of the orbits. For some simple systems (for example dynamical systems with zero entropy) it is useful to consider the complexity function itself. This kind of considerations can be traced back to the work by Morse and Hedlund, who studied the complexity function of a subshift and proved that the boundedness of the function is equivalent to the eventual periodicity of the system (for progress on the high dimensional analogue see [3]). In [10], Ferenczi studied measure-theoretic complexity of ergodic systems using  $\alpha$ -names of a partition and the Hamming distance. He proved that when the measure is ergodic, the complexity function is bounded if and only if the system has discrete spectrum (for the result dealing with non-ergodic case, see [35]). In [21] Katok introduced a notion using the modified notion of spanning sets with respect to an invariant measure  $\mu$  and an error  $\varepsilon$ , which can be used to define the complexity function. In [2], Blanchard, Host and Maass studied topological complexity via the complexity function of an open cover and showed that the complexity function is bounded for any open cover if and only if the system is equicontinuous.

Recently, in the investigation of the Sarnak's conjecture, Huang, Wang and Ye [18] introduced the measure complexity of an invariant measure  $\mu$  similar to the one introduced by Katok [21], by using the mean metric instead of the Bowen metric (for discussion and

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results related to mean metric, see also [27, 34]). They showed that if an invariant measure has discrete spectrum, then the measure complexity with respect to this invariant measure is bounded. An open question was posed as whether the converse statement holds. Motivated by this open question and inspired by the discussions in [10, 21, 2, 17, 25, 12, 13, 16], in this paper, we study topological and measure-theoretic complexity via a sequence of metrics induced by a metric  $d$ , namely the metrics  $d_n$ ,  $\hat{d}_n$  and  $\bar{d}_n$ .

To be precise, for  $n \in \mathbb{N}$ , we define three metrics on  $X$  as follows. For  $x, y \in X$ , let

$$d_n(x, y) = \max\{d(T^i x, T^i y) : 0 \leq i \leq n-1\}, \quad \bar{d}_n(x, y) = \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y)$$

and

$$\hat{d}_n(x, y) = \max\{\bar{d}_k(x, y) : 1 \leq k \leq n\}.$$

It is clear that for all  $x, y \in X$ ,

$$d_n(x, y) \geq \hat{d}_n(x, y) \geq \bar{d}_n(x, y).$$

For  $x \in X$ ,  $\varepsilon > 0$  and a metric  $\rho$  on  $X$ , let  $B_\rho(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}$ . We say that a dynamical system  $(X, T)$  has bounded topological complexity with respect to a sequence of metrics  $\{\rho_n\}$  if for every  $\varepsilon > 0$  there exists a positive integer  $C$  such that for each  $n \in \mathbb{N}$  there are points  $x_1, x_2, \dots, x_m \in X$  with  $m \leq C$  satisfying  $X = \bigcup_{i=1}^m B_{\rho_n}(x_i, \varepsilon)$ . In this paper we will focus on the situation when  $\rho_n = d_n$ ,  $\hat{d}_n$  and  $\bar{d}_n$ .

We also study the measure-theoretic complexity of invariant measures. That is, for a given  $\varepsilon > 0$  and an invariant measure  $\mu$  we consider the measure complexity with respect to  $\{\rho_n\}$  with  $\rho_n = d_n$ ,  $\hat{d}_n$  and  $\bar{d}_n$ , defined by

$$\min\{m \in \mathbb{Z}_+ : \exists x_1, \dots, x_m \in X, \mu(\bigcup_{i=1}^m B_{\rho_n}(x_i, \varepsilon)) > 1 - \varepsilon\}.$$

As expected, the bounded complexity of a topological dynamical system or a measure preserving system is related to various notions of equicontinuity.

It is shown that (see Theorem 3.1 and Theorem 3.5) a topological dynamical system  $(X, T)$  has bounded complexity with respect to  $d_n$  (resp.  $\hat{d}_n$ ) if and only if it is equicontinuous (resp. equicontinuous in the mean). At the same time, we construct minimal systems which have bounded complexity with respect to  $\bar{d}_n$  but not equicontinuous in the mean, which are not uniquely ergodic or uniquely ergodic (see Proposition 3.8 and Proposition 3.9).

It turns out that an invariant measure  $\mu$  on  $(X, T)$  has bounded complexity with respect to  $d_n$  if and only if  $(X, T)$  is  $\mu$ -equicontinuous (see Theorem 4.1). Meanwhile, it is shown that  $\mu$  has bounded complexity with respect to  $\hat{d}_n$  if and only if  $\mu$  has bounded complexity with respect to  $\bar{d}_n$  if and only if  $(X, T)$  is  $\mu$ -mean equicontinuous if and only if  $(X, T)$  is  $\mu$ -equicontinuous in the mean if and only if it has discrete spectrum (see Theorem 4.3, Theorem 4.4 and Theorem 4.7).

The structure of the paper is the following. In Section 2, we recall some basic notions which we will use in the paper. In Section 3, we prove the topological results for systems with bounded complexity with respect to three kinds of metrics. In Section 4, we consider the corresponding results in the measure-theoretical setting. In the Appendix we give some examples.

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## 2. PRELIMINARIES

In this section we recall some notions and aspects of dynamical systems which will be used later.

**2.1. General notions.** In the article, the sets of integers, nonnegative integers and natural numbers are denoted by  $\mathbb{Z}$ ,  $\mathbb{Z}_+$  and  $\mathbb{N}$ , respectively. We use  $\#(A)$  to denote the number of elements of a finite set  $A$ .

A t.d.s.  $(X, T)$  is *transitive* if for each pair of non-empty open subsets  $U$  and  $V$ ,  $N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \emptyset\}$  is infinite; it is *totally transitive* if  $(X, T^n)$  is transitive for each  $n \in \mathbb{N}$ ; and it is *weakly mixing* if  $(X \times X, T \times T)$  is transitive. We say that  $x \in X$  is a *transitive point* if its orbit  $\text{Orb}(x, T) = \{x, Tx, T^2x, \dots\}$  is dense in  $X$ . The set of transitive points is denoted by  $\text{Trans}(X, T)$ . It is well known that if  $(X, T)$  is transitive, then  $\text{Trans}(X, T)$  is a dense  $G_\delta$  subset of  $X$ .

A t.d.s.  $(X, T)$  is *minimal* if  $\text{Trans}(X, T) = X$ , i.e., it contains no proper subsystems. A point  $x \in X$  is called a *minimal point* or *almost periodic point* if  $(\text{Orb}(x, T), T)$  is a minimal subsystem of  $(X, T)$ .

**2.2. Equicontinuity and mean equicontinuity.** A t.d.s.  $(X, T)$  is called *equicontinuous* if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that whenever  $x, y \in X$  with  $d(x, y) < \delta$ ,  $d(T^n x, T^n y) < \varepsilon$  for  $n = 0, 1, 2, \dots$ . It is well known that a t.d.s.  $(X, T)$  with  $T$  being surjective is equicontinuous if and only if there exists a compatible metric  $\rho$  on  $X$  such that  $T$  acts on  $X$  as an isometry, i.e.,  $\rho(Tx, Ty) = \rho(x, y)$  for any  $x, y \in X$ . Moreover, a transitive equicontinuous system is conjugate to a minimal rotation on a compact abelian metric group, and  $(X, T, \mu)$  has discrete spectrum, where  $\mu$  is the unique normalized Haar measure on  $X$ .

When studying dynamical systems with discrete spectrum, Fomin [11] introduced a notion called *stable in the mean in the sense of Lyapunov* or simply *mean-L-stable*. A t.d.s.  $(X, T)$  is *mean-L-stable* if for every  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $d(T^n x, T^n y) < \varepsilon$  for all  $n \in \mathbb{Z}_+$  except a set of upper density less than  $\varepsilon$ . Fomin proved that if a minimal system is mean-L-stable then it is uniquely ergodic. Mean-L-stable systems are also discussed briefly by Oxtoby in [28], and he proved that each transitive mean-L-stable system is uniquely ergodic. Auslander in [1] systematically studied mean-L-stable systems, and provided new examples. See Scarpellini [29] for a related work. It was an open question whether every ergodic invariant measure on a mean-L-stable system has discrete spectrum [29]. This question was answered affirmatively by Li, Tu and Ye in [25].

A t.d.s.  $(X, T)$  is called *mean equicontinuous* (resp. *equicontinuous in the mean*) if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that whenever  $x, y \in X$  with  $d(x, y) < \delta$ ,

$\limsup_{n \rightarrow \infty} \bar{d}_n(x, y) < \varepsilon$  (resp.  $\bar{d}_n(x, y) < \varepsilon$  for each  $n \in \mathbb{N}$ ). It is not hard to show that a dynamical system is mean equicontinuous if and only if it is mean-L-stable. For works related to mean equicontinuity, we refer to [25, 7, 13, 14, 26]. We remark that by the result in [7], a minimal null or tame system is mean equicontinuous. We will show in this paper that a minimal system is mean equicontinuous if and only if it is equicontinuous in the mean (for the proof for the general case, see [31]).

**2.3.  $\mu$ -equicontinuity and  $\mu$ -mean equicontinuity.** When studying the chaotic behaviors of dynamical systems, Huang, Lu and Ye [17] introduced a notion which connects the equicontinuity with respect to a subset or a measure.

Following [17], for a t.d.s.  $(X, T)$ , we say that a subset  $K$  of  $X$  is *equicontinuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $d(T^n x, T^n y) < \varepsilon$  for all  $n \in \mathbb{Z}_+$  and all  $x, y \in K$  with  $d(x, y) < \delta$ . For an invariant measure  $\mu$  on  $(X, T)$ , we say that  $T$  is  *$\mu$ -equicontinuous* if for any  $\tau > 0$  there exists a  $T$ -equicontinuous measurable subset  $K$  of  $X$  with  $\mu(K) > 1 - \tau$ . It was shown in [17] that if  $(X, T)$  is  $\mu$ -equicontinuous and  $\mu$  is ergodic then  $\mu$  has discrete spectrum. We note that  $\mu$ -equicontinuity was studied further in [12].

In the process to study mean equicontinuity, the above notions were generalized to mean equicontinuity with respect to an invariant measure by García-Ramos in [13]. Particularly, he proved that for an ergodic invariant measure  $\mu$ ,  $(X, T)$  is  $\mu$ -mean equicontinuous if and only if  $\mu$  has discrete spectrum. For a different approach, see [24].

**2.4. Hausdorff metric.** Let  $K(X)$  be the hyperspace on  $X$ , i.e., the space of non-empty closed subsets of  $X$  equipped with the Hausdorff metric  $d_H$  defined by

$$d_H(A, B) = \max \left\{ \max_{x \in A} \min_{y \in B} d(x, y), \max_{y \in B} \min_{x \in A} d(x, y) \right\} \text{ for } A, B \in K(X).$$

As  $(X, d)$  is compact,  $(K(X), d_H)$  is also compact. For  $n \in \mathbb{N}$ , it is easy to see that the map  $X^n \rightarrow K(X)$ ,  $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$ , is continuous. Then  $\{A \in K(X) : \#(A) \leq n\}$  is a closed subset of  $K(X)$ .

**2.5. Discrete spectrum.** Let  $(X, T)$  be an invertible t.d.s., that is,  $T$  is a homeomorphism on  $X$ . Let  $\mu$  be an invariant measure on  $(X, T)$  and let  $L^2(\mu) = L^2(X, \mathcal{B}_X, \mu)$  for short. An *eigenfunction* for  $\mu$  is some non-zero function  $f \in L^2(\mu)$  such that  $Uf := f \circ T = \lambda f$  for some  $\lambda \in \mathbb{C}$ . In this case,  $\lambda$  is called the *eigenvalue* corresponding to  $f$ . It is easy to see every eigenvalue has norm one, that is  $|\lambda| = 1$ . If  $f \in L^2(\mu)$  is an eigenfunction, then  $\{U^n f : n \in \mathbb{Z}\}$  is precompact in  $L^2(\mu)$ , that is the closure of  $\{U^n f : n \in \mathbb{Z}\}$  is compact in  $L^2(\mu)$ . Generally, we say that  $f$  is *almost periodic* if  $\{U^n f : n \in \mathbb{Z}\}$  is precompact in  $L^2(\mu)$ . It is well known that the set of all bounded almost periodic functions forms a  $U$ -invariant and conjugation-invariant subalgebra of  $L^2(\mu)$  (denoted by  $A_c$ ). The set of all almost periodic functions is just the closure of  $A_c$  (denoted by  $H_c$ ), and is also spanned by the set of eigenfunctions. The invariant measure  $\mu$  is said to have *discrete spectrum* if  $L^2(\mu)$  is spanned by the set of eigenfunctions, that is  $H_c = L^2(\mu)$ . We remark that when  $\mu$  is not ergodic, the structure of a system  $(X, T, \mu)$  with discrete spectrum can be very complicated, we refer to [23, 8] and the example we provide at the end of Section 4 for details.

### 3. TOPOLOGICAL DYNAMICAL SYSTEMS WITH BOUNDED TOPOLOGICAL COMPLEXITY

In this section we will study the topological complexity of dynamical systems with respect to three kinds of metrics.

**3.1. Topological complexity with respect to  $\{d_n\}$ .** Let  $(X, T)$  be a t.d.s. For  $n \in \mathbb{N}$  and  $x, y \in X$ , define

$$d_n(x, y) = \max\{d(T^i x, T^i y) : i = 0, 1, \dots, n-1\}.$$

It is easy to see that for each  $n \in \mathbb{N}$ ,  $d_n$  is a metric on  $X$  which is topologically equivalent to the metric  $d$ . Let  $x \in X$  and  $\varepsilon > 0$ . The open ball of centre  $x$  and radius  $\varepsilon$  in the metric  $d_n$  is

$$B_{d_n}(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\} = \bigcap_{i=0}^{n-1} T^{-i} B(T^i x, \varepsilon).$$

Let  $K$  be a subset of  $X$ ,  $n \in \mathbb{N}$  and  $\varepsilon > 0$ . A subset  $F$  of  $K$  is said to  $(n, \varepsilon)$ -span  $K$  with respect to  $T$  if for every  $x \in K$  there exists  $y \in F$  with  $d_n(x, y) < \varepsilon$ , that is

$$K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon).$$

Let  $\text{span}_K(n, \varepsilon)$  denote the smallest cardinality of any  $(n, \varepsilon)$ -spanning set for  $K$  with respect to  $K$ , that is

$$\text{span}_K(n, \varepsilon) = \min \left\{ \#(F) : F \subset K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon) \right\}.$$

We say that a subset  $K$  of  $X$  has *bounded topological complexity with respect to  $\{d_n\}$*  if for every  $\varepsilon > 0$  there exists a positive integer  $C = C(\varepsilon)$  such that  $\text{span}_K(n, \varepsilon) \leq C$  for all  $n \geq 1$ . If the whole set  $X$  has bounded topological complexity with respect to  $\{d_n\}$ , we will say that the dynamical system  $(X, T)$  has the property.

We first show that a subset with bounded topological complexity with respect to  $\{d_n\}$  is equivalent to the equicontinuity property.

**Theorem 3.1.** *Let  $(X, T)$  be a t.d.s. and  $K \subset X$  be a compact set. Then  $K$  has bounded topological complexity with respect to  $\{d_n\}$  if and only if it is equicontinuous.*

*Proof.* ( $\Leftarrow$ ) Fix  $\varepsilon > 0$ . By the definition of equicontinuity, there exists  $\delta > 0$  such that  $d(T^n x, T^n y) < \varepsilon$  for all  $n \in \mathbb{Z}_+$  and all  $x, y \in K$  with  $d(x, y) < \delta$ . By the compactness of  $K$ , there exists a finite subset  $F$  of  $K$  such that  $K \subset \bigcup_{x \in F} B(x, \delta)$ . Then  $K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon)$  for all  $n \geq 1$ . So  $K$  has bounded topological complexity with respect to  $\{d_n\}$ .

( $\Rightarrow$ ) Assume the contrary that  $K$  is not equicontinuous. There exists  $\varepsilon > 0$  such that for any  $k \geq 1$  there are  $x_k, y_k \in K$  and  $m_k \in \mathbb{N}$  such that  $d(x_k, y_k) < \frac{1}{k}$  and  $d(T^{m_k} x_k, T^{m_k} y_k) \geq \varepsilon$ . Without loss of generality, we may assume that  $x_k \rightarrow x_0$  as  $k \rightarrow \infty$ . Then we have  $x_0 \in K$  and  $y_k \rightarrow x_0$  as  $k \rightarrow \infty$ . For any  $k \in \mathbb{N}$ , by the triangle inequality, either  $d(T^{m_k} x_k, T^{m_k} x_0) \geq \frac{\varepsilon}{2}$  or  $d(T^{m_k} y_k, T^{m_k} x_0) \geq \frac{\varepsilon}{2}$ . Without loss of generality, we always have  $d(T^{m_k} x_k, T^{m_k} x_0) \geq \frac{\varepsilon}{2}$  for all  $k \in \mathbb{N}$ . Then  $d_{m_k+1}(x_0, x_k) \geq \varepsilon/2$  for all  $k \in \mathbb{N}$ .

As  $K$  has bounded topological complexity with respect to  $\{d_n\}$ , for the constant  $\varepsilon/6$ , there exists  $C > 0$  such that for every  $n \geq 1$  there exists a subset  $F_n$  of  $K$  with  $\#(F_n) \leq C$  such that  $K \subset \bigcup_{x \in F_n} B_{d_n}(x, \varepsilon/6)$ . We view  $\{F_n\}$  as a sequence in the hyperspace  $K(X)$ . By the compactness of  $K(X)$ , there is a subsequence  $F_{n_i} \rightarrow F$  as  $i \rightarrow \infty$  in the Hausdorff metric  $d_H$ . As  $F_n \subset K$  and  $K$  is compact, we have  $F \subset K$ . By the fact  $\{A \in K(X) : \#(A) \leq C\}$  is



closed, we have  $\#(F) \leq C$ . For any  $i \in \mathbb{N}$  and any  $x \in K$ , there exists  $z_{n_i} \in F_{n_i}$  such that  $d_{n_i}(x, z_{n_i}) < \varepsilon/6$ . Without loss of generality, assume that  $z_{n_i} \rightarrow z$  as  $i \rightarrow \infty$ . Then  $z \in F$ . As the sequence  $\{d_n\}$  of metrics is increasing, that is  $d_n(u, v) \leq d_{n+1}(u, v)$  for all  $u, v \in X$  and  $n \in \mathbb{N}$ , we have  $d_{n_i}(x, z_{n_j}) \leq d_{n_j}(x, z_{n_j}) < \varepsilon$  for all  $j \geq i$ . Letting  $j$  go to infinity, we get  $d_{n_i}(x, z) \leq \varepsilon/6$ . This implies that

$$K \subset \bigcup_{z \in F} \{x \in K : d_{n_i}(x, z) \leq \varepsilon/6\}$$

for all  $n_i$ . By the monotonicity of  $\{d_n\}$ , we have

$$K \subset \bigcup_{z \in F} \{x \in K : d_n(x, z) \leq \varepsilon/6\}$$

for all  $n \in \mathbb{N}$ . Enumerate  $F$  as  $\{z_1, \dots, z_m\}$  and let

$$K_j = \bigcap_{n=1}^{\infty} \{x \in K : d_n(x, z_j) \leq \varepsilon/6\}$$

for  $j = 1, \dots, m$ . Then each  $K_j$  is a closed set. By the monotonicity of  $\{d_n\}$ , we have  $K = \bigcup_{j=1}^m K_j$ .

For the sequence  $\{x_k\}$  in  $K$ , passing to a subsequence if necessary we assume that the sequence  $\{x_k\}$  is in the same  $K_j$ . As  $K_j$  is closed,  $x_0$  is also in  $K_j$ . Note that for any  $u, v \in K_j$  and any  $n \geq 1$ ,  $d_n(u, v) \leq d_n(u, z_j) + d_n(z_j, v) \leq \varepsilon/3$ . Particularly, we have  $d_{m_k+1}(x_0, x_k) \leq \varepsilon/3$  for any  $k \in \mathbb{N}$ , which is a contradiction.  $\square$

**Remark 3.2.** In the definition of  $(n, \varepsilon)$ -spanning set  $F$  of  $K$ , we require  $F$  to be a subset of  $K$ . In fact we can define

$$\text{span}'_K(n, \varepsilon) = \min\{\#(F) : F \subset X \text{ and } K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon)\}.$$

It is clear that  $\text{span}_K(n, 2\varepsilon) \leq \text{span}'_K(n, \varepsilon) \leq \text{span}_K(n, \varepsilon)$ . So Proposition 3.1 still holds if in the definition of topological complexity with respect to  $\{d_n\}$  we replace  $\text{span}_K(n, \varepsilon)$  by  $\text{span}'_K(n, \varepsilon)$ .

**Corollary 3.3.** *A dynamical system  $(X, T)$  is equicontinuous if and only if for every  $\varepsilon > 0$  there exists a positive integer  $C$  such that  $\text{span}_X(n, \varepsilon) \leq C$  for all  $n \geq 1$ .*

**Remark 3.4.** It is shown in [2] that the complexity defined by using the open covers is bounded if and only if the system is equicontinuous. In fact, we can prove Corollary 3.3 by using this result and the fact that [33, Theorem 7.7] if  $\alpha$  is an open cover of  $X$  with Lebesgue number  $\delta$  then

$$N(\bigvee_{i=0}^{n-1} T^{-i} \alpha) \leq \text{span}_X(n, \delta/2).$$

**3.2. Topological complexity with respect to  $\{\hat{d}_n\}$ .** For  $n \in \mathbb{N}$  and  $x, y \in X$ , define

$$\hat{d}_n(x, y) = \max\left\{\frac{1}{k} \sum_{i=0}^{k-1} d(T^i x, T^i y) : k = 1, 2, \dots, n\right\}.$$

It is easy to see that for each  $n \in \mathbb{N}$ ,  $\hat{d}_n$  is a metric on  $X$  which is topologically equivalent to the metric  $d$ . For  $x \in X$  and  $\varepsilon > 0$ , let  $B_{\hat{d}_n}(x, \varepsilon) = \{y \in X : \hat{d}_n(x, y) < \varepsilon\}$ . Let  $K$  be a

subset of  $X$ . For  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , define

$$\widehat{\text{span}}_K(n, \varepsilon) = \min \left\{ \#(F) : F \subset K \subset \bigcup_{x \in F} B_{\hat{d}_n}(x, \varepsilon) \right\}.$$

We say that a subset  $K$  of  $X$  has *bounded topological complexity with respect to*  $\{\hat{d}_n\}$  if for every  $\varepsilon > 0$  there exists a positive integer  $C = C(\varepsilon)$  such that  $\widehat{\text{span}}_K(n, \varepsilon) \leq C$  for all  $n \geq 1$ .

As  $\hat{d}_n(x, y) \leq d_n(x, y)$  for all  $n \in \mathbb{N}$  and  $x, y \in X$ , if  $K$  has bounded topological complexity with respect to  $\{d_n\}$  then it is also bounded topological complexity with respect to  $\{\hat{d}_n\}$ . We say that a subset  $K$  of  $X$  is *equicontinuous in the mean* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $\hat{d}_n(x, y) < \varepsilon$  for all  $n \in \mathbb{Z}_+$  and all  $x, y \in K$  with  $d(x, y) < \delta$ .

The following result follows the same lines in Theorem 3.1, just replace the distance  $d_n$  by  $\hat{d}_n$ , as the sequence  $\{\hat{d}_n\}$  of metrics is also increasing.

**Theorem 3.5.** *Let  $(X, T)$  be a t.d.s. and  $K$  be a compact subset of  $X$ . Then  $K$  has bounded topological complexity with respect to  $\hat{d}_n$  if and only if it is equicontinuous in the mean.*

We say that a subset  $K$  of  $X$  is *mean equicontinuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon$$

for all  $x, y \in K$  with  $d(x, y) < \delta$ . If  $X$  is mean equicontinuous then we say that  $(X, T)$  is *mean equicontinuous*. It is clear that if  $K$  is equicontinuous in the mean then it is mean equicontinuous. We can show that for minimal systems they are equivalent.

**Proposition 3.6.** *Let  $(X, T)$  be a minimal t.d.s. Then  $(X, T)$  is mean equicontinuous if and only if equicontinuous in the mean.*

*Proof.* It is clear that equicontinuity in the mean implies mean equicontinuity.

Assume that  $(X, T)$  is mean equicontinuous. For each  $\varepsilon > 0$  there is  $\delta_1 > 0$  such that if  $d(x, y) < \delta_1$  then

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{\varepsilon}{8}.$$

Fix  $z \in X$ . For each  $N \in \mathbb{N}$ , let

$$A_N = \left\{ x \in \overline{B(z, \delta_1/2)} : \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i z) \leq \frac{\varepsilon}{4}, n = N, N+1, \dots \right\}.$$

Then  $A_N$  is closed and  $\overline{B(z, \delta_1/2)} = \bigcup_{N=1}^{\infty} A_N$ . By the Baire Category Theorem, there is  $N_1 \in \mathbb{N}$  such that  $A_{N_1}$  contains an open subset  $U$  of  $X$ . By the minimality we know that there is  $N_2 \in \mathbb{N}$  with  $\bigcup_{i=0}^{N_2-1} T^{-i}U = X$ . Let  $\delta_2$  be the Lebesgue number of the open cover  $\{T^{-i}U : 0 \leq i \leq N_2 - 1\}$  of  $X$ . Let  $N = \max\{N_1, 2N_2\}$ . By the continuity of  $T$ , there exists  $\delta_3 > 0$  such that if  $d(x, y) < \delta_3$  implies that  $d(T^i x, T^i y) < \frac{\varepsilon}{4}$  for any  $0 \leq i \leq N$ . Put  $\delta = \min\{\delta_2, \delta_3\}$ . Let  $x, y \in X$  with  $d(x, y) < \delta$  and  $n \in \mathbb{N}$ . If  $n \leq N$ , then

$$\frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) \leq \frac{1}{n} \cdot n \cdot \frac{\varepsilon}{4} < \varepsilon.$$



If  $n > N$ , there exists  $0 \leq i_0 \leq N_2 - 1$  such that  $x, y \in T^{-i_0}U$ , i.e.,  $T^{i_0}x, T^{i_0}y \in U$ , and then

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) &\leq \frac{1}{n} \sum_{i=0}^{i_0-1} d(T^i x, T^i y) + \frac{1}{n} \sum_{i=0}^{n-1} d(T^i T^{i_0} x, T^i T^{i_0} y) \\ &\leq \frac{\varepsilon}{4} + \frac{1}{n} \sum_{i=0}^{n-1} d(T^i T^{i_0} x, T^i z) + \frac{1}{n} \sum_{i=0}^{n-1} d(T^i T^{i_0} x, T^i z) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

Therefore  $\hat{d}_n(x, y) < \varepsilon$  for all  $n \in \mathbb{Z}_+$ . This implies that  $(X, T)$  is equicontinuous in the mean.  $\square$

**Remark 3.7.** When this paper was finished, we became aware of the work of [31] that Qiu and Zhao can show that in general a t.d.s. is mean equicontinuous if and only if it is equicontinuous in the mean.

**3.3. Topological complexity with respect to  $\{\bar{d}_n\}$ .** For  $n \in \mathbb{N}$  and  $x, y \in X$ , define

$$\bar{d}_n(x, y) = \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y).$$

It is easy to see that for each  $n \in \mathbb{N}$ ,  $\bar{d}_n$  is a metric on  $X$  which is topologically equivalent to the metric  $d$ . For  $x \in X$  and  $\varepsilon > 0$ , let  $B_{\bar{d}_n}(x, \varepsilon) = \{y \in X : \bar{d}_n(x, y) < \varepsilon\}$ . For  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , define

$$\overline{\text{span}}_K(n, \varepsilon) = \min \left\{ \#(F) : F \subset K \subset \bigcup_{x \in F} B_{\bar{d}_n}(x, \varepsilon) \right\}.$$

We say that a subset  $K$  of  $X$  has *bounded topological complexity with respect to  $\{\bar{d}_n\}$*  if for every  $\varepsilon > 0$  there exists a positive integer  $C = C(\varepsilon)$  such that  $\overline{\text{span}}_K(n, \varepsilon) \leq C$  for all  $n \geq 1$ .

As  $\bar{d}_n(x, y) \leq \hat{d}_n(x, y)$  for all  $n \in \mathbb{N}$  and  $x, y \in X$ , if  $K$  has bounded topological complexity with respect to  $\{\hat{d}_n\}$  then it also has bounded topological complexity with respect to  $\{\bar{d}_n\}$ . Intuitively, dynamical systems with bounded topological complexity with respect to  $\{\bar{d}_n\}$  have similar properties of ones with respect to  $\{\hat{d}_n\}$  or  $\{d_n\}$ . But we will see that this is far from being true. The key point is that the sequence  $\{\bar{d}_n\}$  of metrics may be not monotonous. If a dynamical system has bounded topological complexity with respect to  $\{\bar{d}_n\}$ , then by Theorem 4.7 in next section, every invariant measure has discrete spectrum. So it is simple in the measure-theoretic sense. But we have the following proposition which is a surprise in some sense. Since the construction is somewhat long and complicated, we move it to the Appendix.

**Proposition 3.8.** *There is a distal, non-equicontinuous, non-uniquely ergodic, minimal system, which has bounded topological complexity with respect to  $\{\bar{d}_n\}$ .*

We can modify the example in Proposition 3.8 to be uniquely ergodic and also present the construction in the Appendix.

**Proposition 3.9.** *There is a distal, non-equicontinuous, uniquely ergodic, minimal system, which has bounded topological complexity with respect to  $\{\bar{d}_n\}$ .*

**Remark 3.10.** As each distal mean equicontinuous minimal system is equicontinuous, the systems constructed in Propositions 3.8 and 3.9 are not mean equicontinuous.

We have a natural question.

**Question 1.** Is there a non-trivial weakly mixing, even strongly mixing minimal system with bounded topological complexity with respect to  $\{\bar{d}_n\}$ ?

We are just informed by Huang and Xu [19] the above question has an affirmative answer for weakly mixing minimal systems. The question if there is a non-trivial strongly mixing minimal system with bounded topological complexity with respect to  $\{\bar{d}_n\}$  is still open.

#### 4. INVARIANT MEASURES WITH BOUNDED MEASURE-THEORETIC COMPLEXITY

In this section, we will study the measure-theoretic complexity of invariant (Borel probability) measures with respect to three kinds of metrics.

**4.1. Measure-theoretic complexity with respect to  $\{d_n\}$ .** Let  $(X, T)$  be a t.d.s. and  $\mu$  be an invariant measure on  $(X, T)$ . For  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , let

$$\text{span}_\mu(n, \varepsilon) = \min \left\{ \#(F) : F \subset X \text{ and } \mu \left( \bigcup_{x \in F} B_{d_n}(x, \varepsilon) \right) > 1 - \varepsilon \right\}.$$

Recall that this is the same notion defined in [21] by Katok. We say that  $\mu$  has *bounded complexity with respect to  $\{d_n\}$*  if for every  $\varepsilon > 0$  there exists a positive integer  $C = C(\varepsilon)$  such that  $\text{span}_\mu(n, \varepsilon) \leq C$  for all  $n \geq 1$ .

We will show that an invariant measure with bounded complexity with respect to  $\{d_n\}$  is equivalent to the  $\mu$ -equicontinuity property.

**Theorem 4.1.** *Let  $(X, T)$  be a t.d.s. and  $\mu$  be an invariant measure on  $(X, T)$ . Then  $\mu$  has bounded complexity with respect to  $\{d_n\}$  if and only if  $T$  is  $\mu$ -equicontinuous.*

*Proof.* ( $\Leftarrow$ ) First assume that  $(X, T)$  is  $\mu$ -equicontinuous. Fix  $\varepsilon > 0$ . There exists a  $T$ -equicontinuous measurable subset  $K$  of  $X$  with  $\mu(K) > 1 - \varepsilon$ . As the measure  $\mu$  is regular, we can require the set  $K$  to be compact. Now the result follows from Theorem 3.1, as  $\text{span}_\mu(n, \varepsilon) \leq \text{span}_K(n, \varepsilon)$ .

( $\Rightarrow$ ) For any  $\tau > 0$ , we need to find a  $T$ -equicontinuous set  $K$  with  $\mu(K) > 1 - \tau$ . Now fix  $\tau > 0$ . As  $\mu$  has bounded complexity with respect to  $\{d_n\}$ , for any  $M > 0$ , there exists  $C = C_M > 0$  such that for every  $n \geq 1$  there exists a subset  $F_n$  of  $X$  with  $\#(F_n) \leq C$  such that

$$\mu \left( \bigcup_{x \in F_n} B_{d_n} \left( x, \frac{1}{M} \right) \right) > 1 - \frac{\tau}{2^{M+2}}.$$

As the measure  $\mu$  is regular, pick a compact subset  $K_n$  of  $\bigcup_{x \in F_n} B_{d_n} \left( x, \frac{1}{M} \right)$  with  $\mu(K_n) > 1 - \frac{\tau}{2^{M+2}}$ . Without loss of generality, assume that  $F_n \rightarrow F_M, K_n \rightarrow K_M$  as  $n \rightarrow \infty$  in the Hausdorff metric. Then  $\#(F_M) \leq C$ . As  $K_n$  is closed,

$$\mu(K_M) \geq \limsup_{n \rightarrow \infty} \mu(K_n) \geq 1 - \frac{\tau}{2^{M+2}}.$$

For any  $x \in K_M$  and  $n \in \mathbb{N}$ , there exists an  $N > 0$  such that for any  $k > N$  there exists  $x_k \in K_k$  and  $y_k \in F_k$  such that  $d_n(x, x_k) < \frac{1}{M}$  and  $d_k(x_k, y_k) < \frac{1}{M}$ . Without loss of

generality, assume that  $y_k \rightarrow y$  as  $k \rightarrow \infty$ . Then  $y \in F_M$ . By the monotonicity of  $\{d_n\}$ , we have

$$d_n(x, y_k) \leq d_n(x, x_k) + d_n(x_k, y_k) \leq d_n(x, x_k) + d_k(x_k, y_k) \leq \frac{2}{M}.$$

Letting  $k$  go to infinity, one has  $d_n(x, y) \leq \frac{2}{M}$ . Then  $K_M \subset \bigcup_{x \in F_M} B_{d_n}(x, \frac{3}{M})$  and  $\text{span}_{K_M}(n, \frac{3}{M}) \leq \#(F_M) \leq C_M$ .

Let  $K = \bigcap_{M=1}^{\infty} K_M$ . Then  $\mu(K) > 1 - \tau$  and for any  $M \geq 1$ ,

$$\text{span}_K(n, \frac{3}{M}) \leq \text{span}_{K_M}(n, \frac{3}{M}) \leq C_M$$

for all  $n \geq 1$ . Now by Theorem 3.1,  $K$  is  $T$ -equicontinuous. This proves that  $(X, T)$  is  $\mu$ -equicontinuous.  $\square$

**Remark 4.2.** Similar to the observation in Remark 3.4, the open cover version of Theorem 4.1 was proved in [17, Proposition 3.3].

4.2. **Measure-theoretic complexity with respect to  $\{\hat{d}_n\}$ .** For  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , let

$$\widehat{\text{span}}_{\mu}(n, \varepsilon) = \min \left\{ \#(F) : F \subset X \text{ and } \mu \left( \bigcup_{x \in F} B_{\hat{d}_n}(x, \varepsilon) \right) > 1 - \varepsilon \right\}.$$

We say that  $\mu$  has *bounded complexity with respect to  $\{\hat{d}_n\}$*  if for every  $\varepsilon > 0$  there exists a positive integer  $C = C(\varepsilon)$  such that  $\widehat{\text{span}}_{\mu}(n, \varepsilon) \leq C$  for all  $n \geq 1$ .

We will show that an invariant measure with bounded complexity with respect to  $\{\hat{d}_n\}$  is equivalent to the following two kinds of measure-theoretic equicontinuity. We say that  $T$  is  *$\mu$ -equicontinuous in the mean* if for any  $\tau > 0$  there exists a measurable subset  $K$  of  $X$  with  $\mu(K) > 1 - \tau$  which is equicontinuous in the mean, and  *$\mu$ -mean equicontinuous* if for any  $\tau > 0$  there exists a measurable subset  $K$  of  $X$  with  $\mu(K) > 1 - \tau$  which is mean equicontinuous.

**Theorem 4.3.** *Let  $(X, T)$  be a t.d.s. and  $\mu$  be an invariant measure on  $(X, T)$ . Then the following statements are equivalent:*

- (1)  $\mu$  has bounded complexity with respect to  $\hat{d}_n$ ;
- (2)  $T$  is  $\mu$ -equicontinuous in the mean;
- (3)  $T$  is  $\mu$ -mean equicontinuous.

*Proof.* (1)  $\Rightarrow$  (2) Following the proof of Theorem 4.1, we know that for a given  $\tau > 0$ , there is a compact subset  $K$  such that  $\mu(K) \geq 1 - \tau$  and for any  $M \geq 1$ ,  $\widehat{\text{span}}_K(n, \frac{\tau}{M}) \leq C_M$  for all  $n \geq 1$ . By Theorem 3.5,  $K$  is equicontinuous in the mean. This proves that  $(X, T)$  is  $\mu$ -equicontinuous in the mean.

(2)  $\Rightarrow$  (3) is obvious.

(3)  $\Rightarrow$  (1) Now assume that  $(X, T)$  is  $\mu$ -mean equicontinuous. Fix  $\varepsilon > 0$ . Then there is a compact  $K = K(\varepsilon) \subset X$  such that  $\mu(K) > 1 - 2\varepsilon$  and  $K$  is mean equicontinuous. There exists a  $\delta > 0$  such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon/4$$

for all  $x, y \in K$  with  $d(x, y) < \delta$ . As  $K$  is compact, there exists a finite subset  $F$  of  $K$  such that  $K \subset \bigcup_{x \in F} B(x, \delta)$ . Enumerate  $F$  as  $\{x_1, x_2, \dots, x_m\}$ . For  $j = 1, \dots, m$  and  $N \in \mathbb{N}$ , let

$$A_N(x_j) = \left\{ y \in B(x_j, \delta) \cap K : \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x_j, T^i y) < \varepsilon/2, n = N, N+1, \dots \right\}.$$

It is easy to see that for each  $j = 1, \dots, m$ ,  $\{A_N(x_j)\}_{N=1}^\infty$  is an increasing sequence and  $B(x_j, \delta) \cap K = \bigcup_{N=1}^\infty A_N(x_j)$ . Choose  $N_1 \in \mathbb{N}$  and a compact subset  $K_1$  of  $A_{N_1}(x_1)$  such that  $\mu(K_1) > \mu(B(x_1, \delta) \cap K) - \frac{\varepsilon}{2m}$ . Choose  $N_2 \in \mathbb{N}$  and a compact subset  $K_2$  of  $A_{N_2}(x_2)$  such that  $K_1 \cap K_2 = \emptyset$  and  $\mu(K_1 \cup K_2) > \mu((B(x_1, \delta) \cup B(x_2, \delta)) \cap K) - \frac{2\varepsilon}{2m}$ . By induction, we can choose compact subsets  $K_j$  of  $A_{N_j}(x_j)$  for  $j = 1, \dots, m$  with  $\mu(\bigcup_{j=1}^m K_j) > \mu(K) - \frac{\varepsilon}{2} > 1 - \varepsilon$  and  $K_i \cap K_j = \emptyset$  for  $1 \leq i < j \leq m$ .

Let  $K_0 = \bigcup_{j=1}^m K_j$  and  $N_0 = \max\{N_j : j = 1, 2, \dots, m\}$ . There exists  $\delta_1 > 0$  such that for every  $x, y \in K$  with  $d(x, y) < \delta_1$  there exists  $j \in \{1, 2, \dots, m\}$  with  $x, y \in K_j$ . By the continuity of  $T$ , there exists  $\delta_2 > 0$  such that  $d_N(x, y) < \varepsilon$  for every  $x, y \in X$  with  $d(x, y) < \delta_2$ . Let  $\delta_3 = \min\{\delta_1, \delta_2\}$ . By the compactness of  $K_0$ , there exists a finite subset  $H$  of  $K_0$  such that  $K_0 \subset \bigcup_{x \in H} B(x, \delta_3)$ . Fix  $n \geq 1$  and  $y \in K_0$ . There exists  $x \in H$  with  $d(x, y) < \delta_3$ . If  $n < N_0$ , then  $\hat{d}_n(x, y) \leq d_{N_0}(x, y) < \varepsilon$ . If  $n \geq N_0$ , there exists  $j \in \{1, 2, \dots, m\}$  with  $x, y \in K_j \subset A_{N_j}(x_j)$ . By the construction of  $A_{N_j}(x_j)$  and  $n \geq N_j$ ,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) \leq \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i x_j) + \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x_j, T^i y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

For any  $n \geq 1$ , we have  $\hat{d}_n(x, y) < \varepsilon$ . Then

$$K_0 \subset \bigcup_{x \in H} B_{\hat{d}_n}(x, \varepsilon)$$

and

$$\mu\left(\bigcup_{x \in H} B_{\hat{d}_n}(x, \varepsilon)\right) \geq \mu(K_0) > 1 - \varepsilon.$$

This implies that  $\overline{\text{span}}_\mu(n, \varepsilon) \leq \#(H)$  for all  $n \geq 1$ . Then  $\mu$  has bounded complexity with respect to  $\{\hat{d}_n\}$ .  $\square$

**4.3. Measure-theoretic complexity with respect to  $\{\bar{d}_n\}$ .** For  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ , let

$$\overline{\text{span}}_\mu(n, \varepsilon) = \min\left\{\#(F) : F \subset X \text{ and } \mu\left(\bigcup_{x \in F} B_{\bar{d}_n}(x, \varepsilon)\right) > 1 - \varepsilon\right\}.$$

We say that  $\mu$  has *bounded complexity with respect to  $\{\bar{d}_n\}$*  if for every  $\varepsilon > 0$  there exists a positive integer  $C = C(\varepsilon)$  such that  $\overline{\text{span}}_\mu(n, \varepsilon) \leq C$  for all  $n \geq 1$ .

Unlike the topological case, we can prove that bounded measure-theoretic complexity with respect  $\{\bar{d}_n\}$  and  $\{\hat{d}_n\}$  are equivalent.

**Theorem 4.4.** *Let  $(X, T)$  be a t.d.s. and  $\mu$  be an invariant measure on  $(X, T)$ . Then  $\mu$  has bounded complexity with respect to  $\{\bar{d}_n\}$  if and only if it has bounded complexity with respect to  $\{\hat{d}_n\}$ .*

*Proof.* It is clear that if  $\mu$  has bounded complexity with respect to  $\{\hat{d}_n\}$  then by definition it also has bounded complexity with respect to  $\{\bar{d}_n\}$ .

Now assume that  $\mu$  has bounded complexity with respect to  $\{\bar{d}_n\}$ . Let  $\varepsilon > 0$ . There is  $C = C(\varepsilon)$  such that for any  $n \in \mathbb{N}$ , there is  $F_n \in X$  with  $\#(F_n) \leq C$  such that

$$\mu\left(\bigcup_{x \in F_n} B_{\bar{d}_n}(x, \varepsilon/8)\right) > 1 - \varepsilon/8.$$

By the Birkhoff pointwise ergodic theorem for  $\mu \times \mu$  a.e.  $(x, y) \in X^2$

$$\bar{d}_N(x, y) = \frac{1}{N} \sum_{i=0}^{N-1} d(T^i x, T^i y) \rightarrow d^*(x, y).$$

So for a given  $0 < r < \min\{1, \frac{\varepsilon}{2C}\}$ , by the Egorov's theorem there are  $R \subset X^2$  with  $\mu \times \mu(R) > 1 - r^2$  and  $N_0 \in \mathbb{N}$  such that if  $(x, y) \in R$  then

$$|\bar{d}_n(x, y) - \bar{d}_{N_0}(x, y)| < r, \text{ for } n \geq N_0.$$

By the Fubini's theorem there is  $A \subset X$  such that  $\mu(A) > 1 - r$  and for any  $x \in A$ ,  $\mu(R_x) > 1 - r$ , where

$$R_x = \{y \in X : (x, y) \in R\}.$$

Enumerate  $F_{N_0} = \{x_1, x_2, \dots, x_m\}$ . Then  $m \leq C$ . Let  $I = \{1 \leq i \leq m : A \cap B_{\bar{d}_{N_0}}(x_i, \varepsilon/8) \neq \emptyset\}$ . Denote  $\#(I) = m'$ . Then  $1 \leq m' \leq m$ . For each  $i \in I$ , pick  $y_i \in A \cap B_{\bar{d}_{N_0}}(x_i, \varepsilon/8)$ . Then we have  $B_{\bar{d}_{N_0}}(x_i, \varepsilon/8) \subset B_{\bar{d}_{N_0}}(y_i, \varepsilon/4)$  for all  $i \in I$ . As

$$\mu\left(A \cap \bigcap_{i \in I} R_{y_i} \cap \bigcup_{x \in F_{N_0}} B_{\bar{d}_{N_0}}(x, \varepsilon/8)\right) \geq 1 - r - m'r - \varepsilon/8 > 1 - \varepsilon,$$

choose a compact subset

$$K \subset A \cap \bigcap_{i \in I} R_{y_i} \cap \bigcup_{x \in F_{N_0}} B_{\bar{d}_{N_0}}(x, \varepsilon/8)$$

with  $\mu(K) > 1 - \varepsilon$ . If  $x \in K$ , there exists  $i \in I$  such that  $x \in R_{y_i} \cap B_{\bar{d}_{N_0}}(y_i, \varepsilon/4)$ . Then  $(y_i, x) \in R$ . By the construction of  $R$ , for any  $n \geq N_0$ ,

$$\bar{d}_n(x, y_i) = \bar{d}_n(y_i, x) \leq \bar{d}_{N_0}(y_i, x) + r < \varepsilon/4 + r < \varepsilon/2.$$

Let  $\delta_1 > 0$  be a Lebesgue number of the open cover of  $K$  by  $\{K \cap B_{\bar{d}_{N_0}}(y_i, \varepsilon/4) : i \in I\}$ . By the continuity of  $T$ , there exists  $0 < \delta < \delta_1$  such that if  $d(x_1, x_2) < \delta$  then  $d_{N_0}(x_1, x_2) < \varepsilon$ . Let  $x_1, x_2 \in K$  with  $d(x_1, x_2) < \delta$ . There is  $i \in I$  such that  $x_1, x_2 \in A_{y_i} \cap B_{\bar{d}_{N_0}}(y_i, \varepsilon/4)$ . Fix  $n \geq 1$ . If  $n < N_0$ ,  $\bar{d}_n(x_1, x_2) \leq d_{N_0}(x_1, x_2) < \varepsilon$ . If  $n \geq N_0$ ,

$$\bar{d}_n(x_1, x_2) \leq \bar{d}_n(x_1, y_i) + \bar{d}_n(x_2, y_i) < \varepsilon/2 + 2r < \varepsilon.$$

Then  $\hat{d}_n(x_1, x_2) < \varepsilon$  for all  $n \geq 1$ . By the compactness of  $K$ , there exists a finite subset  $H$  of  $K$  such that  $K \subset \bigcup_{x \in H} B(x, \delta)$ . For any  $n \geq 1$ , we have

$$K \subset \bigcup_{x \in H} B_{\hat{d}_n}(x, \varepsilon)$$

and then

$$\mu\left(\bigcup_{x \in H} B_{\hat{d}_n}(x, \varepsilon)\right) \geq \mu(K) > 1 - \varepsilon.$$

This implies that  $\widehat{\text{span}}_\mu(n, \varepsilon) \leq \#(H)$  for all  $n \geq 1$ . Then  $\mu$  has bounded complexity with respect to  $\{\hat{d}_n\}$ .  $\square$

We can restate Proposition 4.1 of [18] as follows.

**Proposition 4.5.** *Let  $(X, T)$  be an invertible t.d.s. and  $\mu$  be an invariant measure on  $(X, T)$ . If  $\mu$  has discrete spectrum, then it has bounded complexity with respect to  $\{\bar{d}_n\}$ .*

It is conjectured in [18] that the converse of Proposition 4.5 is also true. If  $\mu$  is ergodic, by [13, Corollary 39], we know that  $\mu$  has discrete spectrum and if and only if  $\mu$  is mean equicontinuous. So by Theorem 4.3, if an ergodic measure  $\mu$  has bounded complexity with respect to  $\{\bar{d}_n\}$ , then it has discrete spectrum. We will show in Theorem 4.7 that that in general the converse of Proposition 4.5 is also true.

The following result was proved in [24, Theorem 2.7], see also [13, Corollary 39]. Here we provide a different direct proof.

**Proposition 4.6.** *Let  $(X, T)$  be a t.d.s. and  $\mu$  be an ergodic invariant measure on  $(X, T)$ . If  $\mu$  does not have discrete spectrum, then there exists  $\alpha > 0$  such that for  $\mu \times \mu$ -almost every pair  $(x, y) \in X \times X$ ,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \alpha.$$

*Proof.* Let  $\mathcal{B}_\mu$  be the completion of the Borel  $\sigma$ -algebra  $\mathcal{B}_X$  of  $X$  with respect to  $\mu$ . Corresponding to the discrete part of the spectrum of the action of  $T$ , there exists a compact metric abelian group  $(G, +)$  with Haar measure  $\nu$ , an element  $\tau$  of  $G$  such that  $(G, \mathcal{B}_\nu, \nu, S)$  is the Kronecker factor of  $(X, \mathcal{B}_\mu, \mu, T)$  with an associated factor map  $\pi : X \rightarrow G$ , where  $\mathcal{B}_\nu$  be the completion of the Borel  $\sigma$ -algebra of  $G$  with respect to  $\nu$  and  $S$  is the translation by  $\tau$  on  $G$ .

Let  $\mu = \int_G \mu_z d\nu(z)$  be the disintegration of the measure  $\mu$  over  $\nu$ . For  $s \in G$ , let

$$\lambda_s = \int_G \mu_z \times \mu_{z+s} d\nu(z).$$

It is a classical result (see e.g. [15, §4.3.1 Theorem 18]) that there is  $G_0 \subset G$  with  $\nu(G_0) = 1$  such that for every  $s \in G_0$ , the system  $(X \times X, \lambda_s, T \times T)$  is ergodic and

$$\mu \times \mu = \int_G \lambda_s d\nu(s)$$

is the ergodic decomposition  $\mu \times \mu$  under  $T \times T$ .

By the Birkhoff ergodic theorem, the limit

$$\lim_{n \rightarrow +\infty} \bar{d}_n(x, y)$$

exists and equals to

$$\int_{X \times X} d(x_1, x_2) d\lambda_s(x_1, x_2)$$

for some  $s = s(x, y) \in G_0$  for  $\mu \times \mu$ -a.e.  $(x, y) \in X^2$ .

Now it is sufficient to show that if  $(X, \mathcal{B}_\mu, \mu, T)$  does not have discrete spectrum, then there exists  $\alpha > 0$  such that  $\int_{X \times X} d(x_1, x_2) d\lambda_s(x_1, x_2) \geq \alpha$  for all  $s \in G_0$ .



As  $X$  is compact, pick a countable dense subset  $\{y_n : n \in \mathbb{N}\}$  in  $X$ . For  $z \in G$ ,

$$c(z) := \inf_{n \in \mathbb{N}} \int_X d(x, y_n) d\mu_z(x).$$

It is clear that  $c(z) > 0$  if and only if  $\mu_z$  is not a Dirac measure. Moreover,  $c(\cdot)$  is a non-negative measurable function on  $G$ . Put

$$\alpha := \int_G c(z) d\nu(z).$$

Since  $(X, \mathcal{B}_\mu, \mu, T)$  is ergodic and does not have discrete spectrum, by Rohlin's theorem  $\mu_z$  is not a Dirac measure for  $\nu$ -a.e.  $z \in G$ . This means that  $c(z) > 0$  for  $\nu$ -a.e.  $z \in G$  and thus  $\alpha > 0$ . For each  $y \in X$ , there exists a subsequence  $\{n_i\}$  such that  $y_{n_i} \rightarrow y$  as  $i \rightarrow \infty$ . Then for each  $x \in X$ ,  $d(x, y_{n_i}) \rightarrow d(x, y)$  as  $i \rightarrow \infty$ . By the Lebesgue dominated convergence theorem, for each  $z \in G_0$ ,

$$\int_X d(x, y) d\mu_z(x) = \lim_{i \rightarrow \infty} \int_X d(x, y_{n_i}) d\mu_z(x) \geq c(z).$$

Thus, for each  $s \in G_0$ ,

$$\begin{aligned} \int_{X \times X} d(x_1, x_2) d\lambda_s(x_1, x_2) &= \int_G \left( \int_{X \times X} d(x, y) d\mu_z \times \mu_{z+s}(x, y) \right) d\nu(z) \\ &= \int_G \left( \int_X \left( \int_X d(x, y) d\mu_z(x) \right) d\mu_{z+s}(y) \right) d\nu(z) \\ &\geq \int_G \int_X c(z) d\mu_{z+s}(y) d\nu(z) \\ &= \int_G c(z) d\nu(z) = \alpha > 0. \end{aligned}$$

This finishes the proof.  $\square$

Now we are able to show the converse of Proposition 4.5.

**Theorem 4.7.** *Let  $(X, T)$  be an invertible t.d.s. and  $\mu$  be an invariant measure on  $(X, T)$ . If  $\mu$  has bounded complexity with respect to  $\{\bar{d}_n\}$ , then it has discrete spectrum.*

*Proof.* Let  $A$  be the collection of points  $z \in X$  which are generic to some ergodic measure, that is, for each  $z \in A$ ,  $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i z} \rightarrow \mu_z$  as  $n \rightarrow \infty$  and  $\mu_z$  is ergodic. Then  $A$  is measurable and  $\mu(A) = 1$ . We first prove the following Claim.

**Claim 1:**  $\mu_z$  has discrete spectrum for  $\mu$ -a.e.  $z \in A$ .

*Proof of the Claim 1.* Let  $A_1 = \{z \in A : \mu_z \text{ does not have discrete spectrum}\}$ . We need to prove that  $A_1$  is measurable and has zero  $\mu$ -measure. The ergodic decomposition of  $\mu$  can be expressed as  $\mu = \int_A \mu_z d\mu(z)$  (see e.g. [30, Theorem 6.4]). For  $k \in \mathbb{N}$  and  $z \in A$ , put

$$F_k(z) = \mu_z \times \mu_z \left( \left\{ (x, y) \in X \times X : \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \frac{1}{k} \right\} \right).$$

As  $\int_A \mu_z \times \mu_z d\mu(z)$  is an invariant measure on  $(X \times X, T \times T)$ , for each  $k \in \mathbb{N}$ ,  $F_k$  is a measurable function on  $A$ . By Theorem 4.6, we know that  $\mu_z$  does not have discrete spectrum if and only if there exists  $k \in \mathbb{N}$  such that  $F_k(z) = 1$ . Then  $A_1 = \bigcup_{k \in \mathbb{N}} \{z \in A : F_k(z) = 1\}$  and it is measurable. Now it is sufficient to prove  $\mu(G_1) = 0$ . If not,

then  $\mu(A_1) > 0$  and there exists  $k \in \mathbb{N}$  such that  $\mu(\{z \in A: F_k(z) = 1\}) > 0$ . Let  $A_2 = \{z \in A: F_k(z) = 1\}$  and put  $r = \mu(A_2)$ . Then for every  $z \in A_2$  and for  $\mu_z \times \mu_z$ -a.e.  $(x, y) \in X \times X$ ,

$$(1) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \frac{1}{k}.$$

By Theorems 4.3 and 4.4,  $(X, T)$  is  $\mu$ -mean equicontinuous. Then there exists  $M \subset X$  with  $\mu(M) > 1 - \frac{r^2}{4}$  such that  $M$  is mean equicontinuous. By regularity of  $\mu$ , we can assume that  $M$  is compact and  $M \subset A$ . Let  $A_3 = \{z \in A: \mu_z(M) > 1 - \frac{r}{2}\}$ . Then  $A_3$  is measurable, as  $\mu = \int_A \mu_z d\mu(z)$  is the ergodic decomposition of  $\mu$ . We have

$$\begin{aligned} 1 - \frac{r^2}{4} < \mu(M) &= \int_A \mu_z(M) d\mu(z) \leq \int_{A_3} \mu_z(M) d\mu(z) + \int_{A \setminus A_3} \mu_z(M) d\mu(z) \\ &\leq \mu(A_3) + (1 - \mu(A_3))(1 - \frac{r}{2}), \end{aligned}$$

which implies that  $\mu(A_3) > 1 - \frac{r}{2}$ . Then  $\mu(A_2 \cap A_3) > r + (1 - \frac{r}{2}) - 1 = \frac{r}{2} > 0$ . Pick  $z \in A_2 \cap A_3$ . As  $M$  is mean equicontinuous, there exists a  $\delta > 0$  such that for any  $x, y \in M$  with  $d(x, y) < \delta$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{1}{k}.$$

As  $M$  is compact, there exists a finite open cover  $\{U_1, U_2, \dots, U_m\}$  of  $M$  with diameter less than  $\delta$ . Since  $z \in A_3$ ,  $\mu_z(M) > 1 - \frac{r}{2}$ . Then there exists  $i \in \{1, \dots, m\}$  such that  $\mu_z(U_i) > 0$  and also  $\mu_z \times \mu_z(U_i) > 0$ . Note that the diameter of  $U_i$  is less than  $\delta$ , so for any  $x, y \in U_i$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{1}{k},$$

which contradicts to (1). This ends the proof of Claim 1.  $\square$

Let

$$A_0 = \{z \in A: \mu_z \text{ has discrete spectrum}\}.$$

By Claim 1, we have  $\mu(A_0) = 1$ . Let  $f \in C(X)$  be a Lipschitz continuous function on  $X$ . Then there exists  $C > 0$  such that  $|f(x) - f(y)| \leq Cd(x, y)$  for all  $x, y \in X$ .

Recall that the associated operator  $U: L^2(\mu) \rightarrow L^2(\mu)$  is defined by  $Uf = f \circ T$  for all  $f \in L^2(\mu)$ . Inspired by the idea of [29, Theorem 1], we have the following Claim.

**Claim 2:** For any  $\tau > 0$ , there exists  $M^* \in \mathcal{B}$  with  $\mu(M^*) > 1 - \tau$  such that  $f \cdot \mathbf{1}_{M^*}$  is almost periodic, i.e.,  $\{U^n(f \cdot \mathbf{1}_{M^*}): n \in \mathbb{Z}\}$  is precompact in  $L^2(\mu)$ .

*Proof of the Claim 2.* By Theorems 4.3 and 4.4,  $(X, T)$  is  $\mu$ -mean equicontinuous. Fix a constant  $\tau > 0$ . Then there exists  $M \subset X$  with  $\mu(M) > 1 - \tau$  such that  $M$  is mean equicontinuous. Let  $M^* = \bigcup_{n \in \mathbb{Z}} T^{-n}M$ . To show that  $f \cdot \mathbf{1}_{M^*}$  is almost periodic, we only need to prove for any sequence  $\{t_n\}$  in  $\mathbb{Z}$  there exists a subsequence  $\{s_n\}$  of  $\{t_n\}$  such that  $\{U^{s_n}(f \cdot \mathbf{1}_{M^*})\}$  is a Cauchy sequence in  $L^2(\mu)$ .

By regularity of  $\mu$ , we can assume that  $M$  is compact and  $M \subset A_0$ . Choose a countable dense subset  $\{z_m\}$  in  $M$ . As  $\mu_{z_1}$  has discrete spectrum, there exists a subsequence  $\{t_n\}$  of  $\{t_n\}$  such that  $\{U^{t_n} f: n \in \mathbb{N}\}$  is a Cauchy sequence in  $L^2(\mu_{z_1})$ . Inductively assume

that for each  $i \leq m - 1$  we have defined  $\{t_{n,i}\}$  (which is a subsequence of  $\{t_{n,i-1}\}$ ) such that  $\{U^{t_{n,i}} f : n \in \mathbb{N}\}$  is a Cauchy sequence in  $L^2(\mu_{z_i})$ . As  $\mu_{z_m}$  has discrete spectrum, there exists a subsequence  $\{t_{n,m}\}$  of  $\{t_{n,m-1}\}$  such that  $\{U^{t_{n,m}} f : n \in \mathbb{N}\}$  is a Cauchy sequence in  $L^2(\mu_{z_m})$ . Let  $s_n = t_{n,m}$  for  $n \geq 1$ . By the usual diagonal procedure,  $\{U^{s_n} f : n \in \mathbb{N}\}$  is a Cauchy sequence in  $L^2(\mu_{z_m})$  for all  $m \geq 1$ .

Fix  $\varepsilon > 0$ . As  $M$  is mean equicontinuous in  $(X, T)$ , there exists  $\delta > 0$  such that for any  $x, y \in M$  with  $d(x, y) < \delta$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left( d(T^i x, T^i y) \right)^2 < \varepsilon.$$

Fix  $z \in M$ . There exists  $m \in \mathbb{N}$  such that  $d(z, z_m) < \delta$ . For any  $j \neq k \in \mathbb{N}$ ,

$$\begin{aligned} \|U^{s_j} f - U^{s_k} f\|_{L^2(\mu_z)}^2 &= \int_X |U^{s_j} f - U^{s_k} f|^2 d\mu_z = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} |f(T^{s_j+i} z) - f(T^{s_k+i} z)|^2 \\ &\leq C^2 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left( d(T^{s_j+i} z, T^{s_k+i} z) \right)^2 \\ &\leq C^2 \left( \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left( d(T^{s_j+i} z, T^{s_j+i} z_m) \right)^2 \right. \\ &\quad \left. + \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left( d(T^{s_k+i} z, T^{s_k+i} z_m) \right)^2 \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left( d(T^{s_j+i} z_m, T^{s_j+i} z_m) \right)^2 \right) \\ &\leq C^2 \left( 2\varepsilon + \|U^{s_j} f - U^{s_k} f\|_{L^2(\mu_{z_m})}^2 \right). \end{aligned}$$

As  $\{U^{s_n} f : n \in \mathbb{N}\}$  is a Cauchy sequence in  $L^2(\mu_{z_m})$  for all  $m \geq 1$ ,  $\{U^{s_n} f : n \in \mathbb{N}\}$  is also a Cauchy sequence in  $L^2(\mu_z)$ . Then for each  $z \in M$ ,

$$\lim_{N \rightarrow \infty} \sup_{j, k \geq N} \int_X |U^{s_j} f - U^{s_k} f|^2 d\mu_z = 0.$$

For each  $y \in M^*$ , there exists  $n \in \mathbb{Z}$  and  $z \in M$  such that  $T^n z = y$ . Then  $\mu_z = \mu_y$ . For  $z \in M^*$ , put

$$f_N(z) = \sup_{j, k \geq N} \int_X |U^{s_j} f - U^{s_k} f|^2 d\mu_z.$$

By the dominated convergence theorem,

$$(2) \quad \lim_{N \rightarrow \infty} \int_{M^*} f_N(z) d\mu(z) = \int_{M^*} \lim_{N \rightarrow \infty} f_N(x) d\mu(z) = 0.$$

It is easy to see that

$$\begin{aligned} \sup_{j,k \geq N} \int_{M^*} \int_X |U^{sj} f - U^{sk} f|^2 d\mu_z d\mu(z) &\leq \int_{M^*} \left( \sup_{j,k \geq N} \int_X |U^{sj} f - U^{sk} f|^2 d\mu_z \right) d\mu(z) \\ &= \int_{M^*} f_N(z) d\mu(z), \end{aligned}$$

from where we deduce

$$\lim_{N \rightarrow \infty} \left( \sup_{j,k \geq N} \int_{M^*} \int_X |U^{sj} f - U^{sk} f|^2 d\mu_z d\mu(z) \right) = 0 \quad \text{by (2).}$$

As  $\int_{M^*} g d\mu = \int_{M^*} (\int g d\mu_z) d\mu(z)$  for any  $g \in L^2(\mu)$  we have

$$\lim_{N \rightarrow \infty} \left( \sup_{j,k \geq N} \int_{M^*} |U^{sj} f - U^{sk} f|^2 d\mu(z) \right) = 0.$$

Note that  $T(M^*) = M^*$ , so

$$\int_{M^*} |U^{sj} f - U^{sk} f|^2 d\mu(z) = \int |U^{sj}(f \cdot \mathbf{1}_{M^*}) - U^{sk}(f \cdot \mathbf{1}_{M^*})|^2 d\mu.$$

Thus  $\{U^{sn}(f \cdot \mathbf{1}_{M^*})\}$  is a Cauchy sequence in  $L^2(\mu)$ , which ends the proof of Claim 2.  $\square$

Note that The collection of almost periodic functions  $g$  is closed in  $L^2(\mu)$ . As the measure of  $M^*$  in Claim 2 can be arbitrary close to 1,  $f$  is also an almost periodic function in  $L^2(\mu)$ . As the collection of Lipschitz continuous functions is dense in  $C(X)$  (see e.g. [5, Theorem 11.2.4.]) and  $C(X)$  is dense in  $L^2(\mu)$ , then for every function  $g \in L^2(\mu)$  is almost periodic in  $L^2(\mu)$ , that is  $\mu$  has discrete spectrum.  $\square$

**Remark 4.8.** After we have finished this paper, Nhan-Phu Chung informed us that Theorem 4.7 was also proved in [32, Theorem 3.2] with a different method. Note that an invariant measure  $\mu$  has bounded complexity with respect to  $\{\bar{d}_n\}$  in our sense if and only if every  $\varepsilon > 0$ , the scaling sequences with respect to  $\mu$  and  $d$  are bounded as in [32, Definition 3.1]. It should be noticed that in the introduction of [32] it requires a mild condition that the standard (Lebesgue) space  $(X, \mu)$  is non-atomic.

In Theorem 4.7, we show that if an invariant measure  $\mu$  of a t.d.s.  $(X, T)$  has bounded complexity with respect to  $\{\bar{d}_n\}$ , then almost all the ergodic components in the ergodic decomposition of  $\mu$  have discrete spectrum. In the following remark we provide an example which shows that it may happen there are uncountably many pairwise non-isomorphic ergodic components in the ergodic decomposition, and the set of unions of all eigenvalues of the ergodic components are countable.

**Remark 4.9.** The space  $X$  is the product  $\{0, 1\}^{\mathbb{N}} \times (S^1)^{\mathbb{N}}$ . Let  $\{\tau_i : i \in \mathbb{N}\}$  be a family of irrational numbers independent over the rational numbers. The measure  $\mu$  is the product of the Bernoulli measure  $(\frac{1}{2}, \frac{1}{2})$  on  $\{0, 1\}^{\mathbb{N}}$  and the product measure  $\lambda_{\mathbb{N}}$  on  $(S^1)^{\mathbb{N}}$ , where each coordinate is equipped with the Lebesgue measure  $\lambda$ .

The transformation  $T : X \rightarrow X$  is defined in the following way: let  $\omega = (\omega_i)_{i \geq 1} \in \{0, 1\}^{\mathbb{N}}$  and  $w = (w_i)_{i \geq 1} \in (S^1)^{\mathbb{N}}$ . Define  $T(\omega, w) = (\omega, w')$ , where  $(w')_i = w_i$  if  $\omega_i = 0$  and  $(w')_i = T_i w_i$  if  $\omega_i = 1$ , where  $T_i$  is the translation by  $\tau_i$  on  $(S^1)_i$ . It is easy to see that  $\{\omega\} \times (S^1)^{\mathbb{N}}$  is  $T$ -invariant for any  $\omega \in \{0, 1\}^{\mathbb{N}}$ .

Let the distance on  $X$  be the sum of the distances

$$d_1(\omega, \omega') = \sum_{i \geq 1} \frac{1}{2^i} |\omega_i - \omega'_i| \quad \text{and} \quad d_2(s, s') = \sum_{i \geq 1} \frac{1}{2^i} d'(s_i, s'_i),$$

where  $d'$  is the distance on the circle  $S^1$ , so that  $d((\omega, s), (\omega', s')) = d_1(\omega, \omega') + d_2(s, s')$ .

It is not difficult to see that  $T$  has bounded complexity with respect to  $\{\bar{d}_n\}$ . Note that the ergodic components are  $\{\omega\} \times (w', \Pi_{i|\omega_i=1}(S^1)_i)$ , where  $w' \in \Pi_{i|\omega_i=0}(S^1)_i$ .

**Remark 4.10.** Assume that  $(X, T)$  is a minimal system with bounded complexity with respect to  $\{\bar{d}_n\}$  for an invariant measure  $\mu$ . It is interesting to know whether almost all the ergodic measures in the ergodic decomposition of  $\mu$  are isomorphic. After the first draft version of this paper was finished, Cyr and Kra informed us in [4] that there exists an example dose not satisfy the condition, see Proposition B.1 in Appendix B.

#### APPENDIX A. TWO EXAMPLES

The aim of this appendix is to construct two examples announced in Section 3. We remark that the measure complexity for a minimal distal system can be very complicated, see for example [20].

**A.1. The construction of the system in Proposition 3.8.** We view the unit circle  $\mathbb{T}$  as  $\mathbb{R}/\mathbb{Z}$  and also as  $[0, 1) \pmod{1}$ . For  $a \in \mathbb{R}$  we let  $\|a\| = \min\{|a-z| : z \in \mathbb{Z}\}$  which induces a distance on  $\mathbb{T}$ . Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  be an irrational number and  $R_\alpha : \mathbb{T} \rightarrow \mathbb{T}, x \rightarrow x + \alpha$  the rotation on  $\mathbb{T}$  by  $\alpha$ . In this subsection we will construct a skew product map  $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  with  $T(x, y) = (x + \alpha, y + h(x))$  for any  $x, y \in \mathbb{T}$ , where  $h : \mathbb{T} \rightarrow \mathbb{R}$  is continuous and will be defined below.

The general idea to construct  $h$  is the following. First we choose some disjoint intervals (see (3)) in  $\mathbb{T}$  and define  $h_1 : \mathbb{T} \rightarrow \mathbb{R}$  such that  $h_1$  takes positive values in some intervals, negative values in the rest intervals and zero in rest points (see (10) below). This results that  $\sum_{i=0}^{n-1} h_1(R_\alpha^i x)$  is small under some conditions (see Lemmas A.1 and A.2). Then we choose smaller disjoint intervals (see (7)) in  $\mathbb{T}$  and define  $h_2 : \mathbb{T} \rightarrow \mathbb{R}$  such that  $h_2$  takes smaller positive values in some intervals, smaller negative values in the rest intervals and zero in the rest points. We do this for each  $h_k$  and finally we put  $h = \sum_{k=1}^{\infty} h_k$ . Note that for  $n \in \mathbb{N}$  and  $x, y \in \mathbb{T}$ ,  $T^n(x, y) = \left( R_\alpha^n x, y + \sum_{i=0}^{n-1} h(R_\alpha^i x) \right)$ . When calculating  $\sum_{i=0}^{n-1} h(R_\alpha^i x)$ , we only need to take care  $\sum_{i=0}^{n-1} (\sum_{j=1}^k h_j)(R_\alpha^i x)$  when  $k$  large enough (see (15)). By careful choosing  $h_k$  we can show the minimality, non-equicontinuity and non-unique ergodicity. Now let us begin the construction.

Let  $\eta = \frac{1}{100}$ ,  $M_1 = 10$  and  $N_1 = 10M_1$ . As  $\alpha$  is irrational, the two-side orbit  $\{n\alpha : n \in \mathbb{Z}\}$  of 0 under the rotation  $R_\alpha$  are pairwise distinct. Choose  $\delta_1 > 0$  small enough such that the intervals

$$[i\alpha - \delta_1, i\alpha + \delta_1], \quad i = -1, 0, 1, \dots, 2N_1$$

are pairwise disjoint on  $\mathbb{T}$ . Put

$$(3) \quad E_1 = \bigcup_{i=0}^{2N_1-1} [i\alpha - \delta_1, i\alpha + \delta_1],$$

and

$$F_1 = \{i\alpha - \delta_1, i\alpha + \delta_1 : i = 0, 1, \dots, 2N_1 - 1\}.$$

The total length of intervals in  $E_1$  is  $4N_1\delta_1$ . Shrinking  $\delta_1$  if necessary, we can require  $4N_1\delta_1 < \eta/2$ . Put

$$l_1 = \min\{\|x - y\| : x, y \in F_1, x \neq y\}$$

and  $\gamma_0 = 2l_1$ .

For  $k = 2, 3, 4, \dots$ , we will define  $M_k, N_k, \delta_k, E_k, F_k, l_k$  and  $\gamma_{k-1}$  by induction. Assume that  $M_{k-1}, N_{k-1}, \delta_{k-1}, E_{k-1}, F_{k-1}, l_{k-1}$  and  $\gamma_{k-2}$  have been defined such that the total length of intervals in  $E_{k-1}$  is less than  $\frac{\eta}{2^{k-1}}$ . As  $R_\alpha$  is uniquely ergodic on  $\mathbb{T}$ , choose  $M_k > N_{k-1}$  large enough such that for any  $x, y \in \mathbb{T}$ , one has

$$(4) \quad \{0 \leq i \leq M_k - 1 : R_\alpha^i x \in (y, y + l_{k-1})\} \neq \emptyset.$$

and for any  $n \geq M_k$  and any  $x \in \mathbb{T}$ ,

$$(5) \quad \frac{1}{n} \#\{0 \leq i \leq n - 1 : R_\alpha^i x \in E_{k-1}\} < \frac{\eta}{2^{k-1}}.$$

Let  $N_k = 10^k M_k$ . Choose  $\delta_k > 0$  small enough such that

$$\{i\alpha \pm \delta_k : i = 0, 1, 2, \dots, 2N_k - 1\} \cap F_{k-1} = \emptyset,$$

and

$$[i\alpha - \delta_k, i\alpha + \delta_k], \quad i = -1, 0, 1, \dots, 2N_k$$

are pairwise disjoint intervals on  $\mathbb{T}$ . Choose  $0 < \gamma_{k-1} < \delta_{k-1}$  small enough such that

$$(6) \quad [i\alpha - \gamma_{k-1}, i\alpha + \gamma_{k-1}], \quad -2N_k \leq i \leq 2N_k + 2N_{k-1}$$

are pairwise disjoint intervals on  $\mathbb{T}$ . Put

$$(7) \quad E_k = \bigcup_{i=0}^{2N_k-1} [i\alpha - \delta_k, i\alpha + \delta_k]$$

and

$$(8) \quad F_k = F_{k-1} \cup \{i\alpha + \delta_k, i\alpha - \delta_k : i = 0, 1, \dots, 2N_k - 1\}$$

The total length of intervals in  $E_k$  is  $4N_k\delta_k$ . Shrinking  $\delta_k$  if necessary, we can require  $4N_k\delta_k < \frac{\eta}{2^k}$ . Let

$$(9) \quad l_k = \min\left(\left\{\|x - y\| : x, y \in F_k, x \neq y\right\} \cup \left\{\frac{\gamma_i}{2k^2} : i = 1, 2, \dots, k - 1\right\}\right).$$

This finishes the induction.

For each  $k \in \mathbb{N}$ , define  $h_k^*, h_k : \mathbb{R} \rightarrow [-1/2, 1/2)$  such that

$$h_k^*(x) = \begin{cases} \frac{1}{N_k}(1 - |\frac{x-m}{\gamma_k}|), & \text{for } x \in [m - \gamma_k, m + \gamma_k] \text{ with } m \in \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_k(x) = \sum_{i=0}^{N_k-1} h_k^*(x - i\alpha) - \sum_{i=N_k}^{2N_k-1} h_k^*(x - i\alpha).$$



As the intervals in  $E_k$  are pairwise disjoint and  $\gamma_k < \delta_k$ , it is easy to check that for any  $x \in \mathbb{R}$ ,

(10)

$$h_k(x) = \begin{cases} h_k^*(x - i\alpha), & \text{if } x \in [i\alpha - \gamma_k, i\alpha + \gamma_k] \pmod{1}, i = 0, 1, 2, \dots, N_k - 1, \\ -h_k^*(x - i\alpha), & \text{if } x \in [i\alpha - \gamma_k, i\alpha + \gamma_k] \pmod{1}, i = N_k, \dots, 2N_k - 1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular,  $h_k(x) = 0$  for  $x \notin E_k \pmod{1}$ ,

$$h_k(i\alpha) = \begin{cases} \frac{1}{N_k}, & \text{for } i = 0, 1, \dots, N_k - 1, \\ -\frac{1}{N_k}, & \text{for } i = N_k, N_k + 1, \dots, 2N_k - 1, \end{cases}$$

and for any  $z \in [-\delta_x, \delta_x]$ ,

$$(11) \quad \sum_{s=0}^{2N_k-1} h_k(R_\alpha^s z) = 0.$$

It is also easy to see that for any  $x \in \mathbb{R}$ ,

$$(12) \quad |h_k(x)| \leq \frac{1}{N_k} = \frac{1}{10^k M_k} < \frac{1}{10^k},$$

and  $h_k$  is Lipschitz continuous with a Lipschitz constant  $\frac{1}{N_k \gamma_k}$ , that is, for any  $x, y \in \mathbb{R}$ ,

$$(13) \quad |h_k(x) - h_k(y)| \leq \frac{1}{N_k \gamma_k} |x - y|.$$

For any  $x \in \mathbb{R}$ , we have  $h_k(x + 1) = h_k(x)$ , so we can regard  $h_k$  as a function from  $\mathbb{T}$  to  $\mathbb{R}$ . Now, define  $h: \mathbb{T} \rightarrow \mathbb{R}$  as for each  $x \in \mathbb{T}$

$$h(x) = \sum_{k=1}^{\infty} h_k(x).$$

It is easy to see that  $h$  is continuous since

$$\sum_{k=1}^{\infty} |h_k(x)| \leq \sum_{k=1}^{\infty} \frac{1}{10^k} < \frac{1}{2}.$$

For  $k \geq 1$ , we set

$$(14) \quad h_{1,k}(x) = \sum_{i=1}^k h_i(x) \text{ and } h_{k,\infty}(x) = \sum_{i=k}^{\infty} h_i(x).$$

Then

$$h(x) = h_{1,k}(x) + h_{k+1,\infty}(x)$$

and

$$(15) \quad \|h_{k,\infty}(x)\| \leq \sum_{i=k}^{\infty} \|h_i(x)\| \leq \sum_{i=k}^{\infty} \frac{1}{10^i M_i} \leq \frac{1}{M_k} \sum_{i=k}^{\infty} \frac{1}{10^i} = \frac{1}{M_k} \frac{1}{9 \cdot 10^{k-1}} < \frac{1}{9 \cdot 10^{k-1}}.$$

Finally, we define a skew product map as follows:

$$T: \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + h(x)).$$

It is clear that  $T$  is continuous. We will show that the system  $(\mathbb{T}^2, T)$  is as required. By the definition, it is clear that  $(\mathbb{T}^2, T)$  is distal.

For any real function  $g$  on  $\mathbb{T}$  and  $x \in \mathbb{T}$ , we set  $H_0^g \equiv 0$  and

$$H_n^g(x) := \sum_{i=0}^{n-1} g(R_\alpha^i x)$$

for  $n \geq 1$ . Recall that  $h(x) = \sum_{k=1}^{\infty} h_k(x)$ , so

$$H_n^h(x) = \sum_{i=1}^{\infty} H_n^{h_i}(x) = H_n^{h_{1,k}}(x) + H_n^{h_{k+1,\infty}}(x).$$

We choose a compatible metric  $d$  on  $\mathbb{T}^2$  by

$$d((x_1, y_1), (x_2, y_2)) := \|x_1 - x_2\| + \|y_1 - y_2\|,$$

for any  $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$ . We remark that for  $n \in \mathbb{N}$  and  $x, y \in \mathbb{T}$

$$T^n(x, y) = \left( R_\alpha^n x, y + \sum_{i=0}^{n-1} h(R_\alpha^i x) \right) = (R_\alpha^n x, y + H_n^h(x)).$$

To estimate the orbit of  $(x, y)$ , the key point is to control  $H_n^h(x)$ . The following two lemmas will be useful in the estimation.

**Lemma A.1.** *Assume  $x \in \mathbb{T}$ ,  $i \in \mathbb{N}$  and  $m \in \mathbb{N}$ . If  $x, R_\alpha^m x \in E_i^c \cup [-\delta_i, \delta_i]$ , then one has*

$$H_m^{h_i}(x) = 0.$$

*Proof.* Let  $J = \{0 \leq j \leq m-1 : R_\alpha^j x \in [-\delta_i, \delta_i]\}$ . We first claim that

$$\{0 \leq k \leq m-1 : R_\alpha^k x \in E_i\} = \bigcup_{j \in J} \{j+l : 0 \leq l \leq 2N_i - 1\}.$$

To see this equality firstly we note that if  $j \in J$ , then  $R_\alpha^j x \in [-\delta_i, \delta_i]$ . This implies that  $R_\alpha^{j+l} x \in [l\alpha - \delta_i, l\alpha + \delta_i] \subset E_i$  for  $0 \leq l \leq 2N_i - 1$ . Since  $j \leq m-1$  and  $R_\alpha^m x \in E_i^c \cup [\delta_i, \delta_i]$ , one has  $j + 2N_i - 1 \leq m$  and then  $\{j+l : 0 \leq l \leq 2N_i - 1\} \subset \{0, 1, 2, \dots, m-1\}$ . Thus,  $\{0 \leq k \leq m-1 : R_\alpha^k x \in E_i\} \supset \bigcup_{j \in J} \{j+l : 0 \leq l \leq 2N_i - 1\}$ .

Conversely, if  $k \in \{0, 1, 2, \dots, m-1\}$  with  $R_\alpha^k x \in E_i$ . This means that  $R_\alpha^k x \in [s\alpha - \delta_i, s\alpha + \delta_i]$  for some  $0 \leq s \leq 2N_i - 1$ . If  $k < s$ , then  $x \in [(s-k)\alpha - \delta_i, (s-k)\alpha + \delta_i]$ , which contradicts to the assumption  $x \in E_i^c \cup [-\delta_i, \delta_i]$ . This implies  $k \geq s$ . Hence we have  $k-s \in J$  and  $k \in \{(k-s)+l : 0 \leq l \leq 2N_i - 1\}$ . Thus we get  $\{0 \leq k \leq m-1 : R_\alpha^k x \in E_i\} \subset \bigcup_{j \in J} \{j+l : 0 \leq l \leq 2N_i - 1\}$ . This proves the claim.

By the claim, we have

$$\begin{aligned} H_m^{h_i}(x) &= \sum_{\substack{0 \leq k \leq m-1 \\ R_\alpha^k x \notin E_i}} h_i(R_\alpha^k x) + \sum_{\substack{0 \leq k \leq m-1 \\ R_\alpha^k x \in E_i}} h_i(R_\alpha^k x) \\ &= 0 + \sum_{j \in J} \sum_{l=0}^{2N_i-1} h_i(R_\alpha^l(R_\alpha^j x)) = 0. \quad \text{by (11)} \end{aligned}$$

This finishes the proof of Lemma A.1. □

**Lemma A.2.** *Assume  $x \in \mathbb{T}$ ,  $m, k \in \mathbb{N}$  and  $1 \leq j \leq k - 1$ . If  $\|m\alpha\| < l_k$  and  $x, R_\alpha^m x \in [i\alpha - \delta_j, i\alpha + \delta_j]$  for some  $0 \leq i \leq 2N_j - 1$ , then one has*

$$\|H_m^{h_j}(x)\| < \frac{1}{k^2}.$$

*Proof.* First, by (9), one has

$$\begin{aligned} l_k &\leq \min\{\|(i\alpha + \delta_k) - (j\alpha + \delta_k)\| : 0 \leq i < j \leq 2N_k - 1\} \\ &= \min_{0 \leq r \leq 2N_k - 1} \|r\alpha\|. \end{aligned}$$

Thus,  $m \geq 2N_k$  since  $\|m\alpha\| < l_k$ . Next, by the construction of  $E_j$ , one has

$$R_\alpha^{2N_j-i} x, R_\alpha^{m-i} x = R_\alpha^{m-2N_j}(R_\alpha^{2N_j-i} x) \in E_j^c \cup [-\delta_j, \delta_j]$$

since  $x, R_\alpha^m x \in [i\alpha - \delta_j, i\alpha + \delta_j]$ . By Lemma A.1, one has  $H_{m-2N_j}^{h_j}(R_\alpha^{2N_j-i} x) = 0$  and

$$\begin{aligned} H_m^{h_j}(x) &= H_{2N_j-i}^{h_j}(x) + H_{m-2N_j}^{h_j}(R_\alpha^{2N_j-i} x) + H_i^{h_j}(R_\alpha^{m-i} x) \\ &= H_{2N_j-i}^{h_j}(x) + H_i^{h_j}(R_\alpha^{m-i} x) \\ &= (H_{2N_j-i}^{h_j}(x) + H_i^{h_j}(R_\alpha^{-i} x)) + (H_i^{h_j}(R_\alpha^{m-i} x) - H_i^{h_j}(R_\alpha^{-i} x)) \\ &= H_{2N_j}^{h_j}(R_\alpha^{-i} x) + (H_i^{h_j}(R_\alpha^{m-i} x) - H_i^{h_j}(R_\alpha^{-i} x)). \end{aligned}$$

Notice that  $R_\alpha^{-i} x \in [-\delta_j, \delta_j]$ . By (11), one has

$$H_{2N_j}^{h_j}(R_\alpha^{-i} x) = \sum_{s=0}^{2N_j-1} h_j(R_\alpha^s(R_\alpha^{-i} x)) = 0.$$

This implies that

$$\begin{aligned} \|H_m^{h_j}(x)\| &\leq \|H_i^{h_j}(R_\alpha^{m-i} x) - H_i^{h_j}(R_\alpha^{-i} x)\| \\ &\leq \sum_{s=0}^{i-1} \|h_j(R_\alpha^{m-i+s} x) - h_j(R_\alpha^{-i+s} x)\| \\ &\leq i \cdot l_k \cdot \frac{1}{N_j \gamma_j}, \end{aligned}$$

where the last inequality follows from (13) and  $\|R_\alpha^{-i+s} x - R_\alpha^{m-i+s} x\| = \|m\alpha\| < l_k$  for  $s = 0, 1, 2, \dots, i-1$ . Finally, by (9),

$$\|H_m^{h_j}(x)\| \leq 2N_j \cdot \frac{\gamma_j}{2k^2} \cdot \frac{1}{N_j \gamma_j} = \frac{1}{k^2}.$$

This finishes the proof of Lemma A.2.  $\square$

**Proposition A.3.**  $(\mathbb{T}^2, T)$  is minimal.

*Proof.* We need to show every point  $(x, y)$  has a dense orbit. Fix  $(x, y) \in \mathbb{T}^2$ ,  $0 < \epsilon < 1$  and  $k \in \mathbb{N}$ . There exists  $n_1 \in \mathbb{N}$  such that  $R_\alpha^{n_1} x \in [-\epsilon\gamma_k, \epsilon\gamma_k]$ . Let  $(x_1, y_1) = T^{n_1}(x, y)$ . Then  $x_1 = R_\alpha^{n_1} x$  and  $\|x_1\| \leq \epsilon\gamma_k$ .

Now fix  $(x', y') \in \mathbb{T}^2$ . Note that points in  $F_{k-1}$  divide the unite circle into open arcs with length not less than  $l_{k-1}$ . The collection of these arcs is denoted by  $\mathcal{F}_{k-1}$ . There

exists  $(a_1, a_2) \in \mathcal{F}_{k-1}$  such that  $x' \in [a_1, a_2)$ . As  $(a_1, a_2) \cap F_{k-1} = \emptyset$ , either  $(a_1, a_2) \subset E_{k-1}$  or  $(a_1, a_2) \subset E_{k-1}^c$ . If  $(a_1, a_2) \subset E_{k-1}$ , then  $[a_1, a_2) \subset [j\alpha - \delta_{k-1}, j\alpha + \delta_{k-1}]$  for some  $0 \leq j \leq 2N_{k-1} - 1$ , and we take  $a = j\alpha + \delta_{k-1}$ . If  $(a_1, a_2) \subset E_{k-1}^c$ , we take  $a \in [a_1, a_2)$  such that  $x' \in [a, a + l_{k-1}) \subset [a_1, a_2)$  since the length of  $[a_1, a_2)$  is not less than  $l_{k-1}$ . Note that in any case  $(a, a + l_{k-1})$  is a subset of some  $(b_1, b_2) \in \mathcal{F}_{k-1}$ . For any  $1 \leq i \leq k-2$ , as  $F_i \subset F_{k-1}$ ,  $(b_1, b_2) \cap E_i = \emptyset$ . Then  $(b_1, b_2)$  is either a subset of  $[j\alpha - \delta_i, j\alpha + \delta_i]$  for some  $0 \leq j \leq 2N_i - 1$  or a subset of  $E_i^c$ . Summing up the above arguments, one has:

- (i)  $(a, a + l_{k-1}) \subset E_{k-1}^c$  and  $\min\{\|x' - x''\| : x'' \in (a, a + l_{k-1})\} \leq 2\delta_{k-1}$ ;
- (ii) for all  $1 \leq i \leq k-2$ , either  $(a, a + l_{k-1}) \subset [j\alpha - \delta_i, j\alpha + \delta_i]$  for some  $0 \leq j \leq 2N_i - 1$  or  $(a, a + l_{k-1}) \subset E_i^c$ .

By (4), there exists an integer  $n_2 \in [0, M_k)$  such that  $R_\alpha^{n_2}(x_1) \in (a, a + l_{k-1})$ . Suppose

$$y' - (y_1 + H_{n_2}^h(x_1)) = b \pmod{1}.$$

Then  $b \in [0, 1)$ . By (4), there exists an integer  $n_3 \in [[10^k b]M_k, ([10^k b] + 1)M_k)$  such that  $n_3 \geq n_2$  and  $R_\alpha^{n_3}(x_1) \in (a, a + l_{k-1})$ . Note that  $n_3 < 10^k M_k = N_k$  and

$$b - \frac{2}{10^k} \leq \frac{([10^k b] - 1)M_k}{N_k} \leq \frac{n_3 - n_2}{N_k} \leq \frac{([10^k b] + 1)M_k}{N_k} \leq b + \frac{2}{10^k}.$$

By (i) and Lemma A.1, one has  $H_{n_3 - n_2}^{h_{k-1}}(R_\alpha^{n_2} x_1) = 0$ . By (ii) and Lemmas A.1 and A.2, one has

$$\|H_{n_3 - n_2}^{h_i}(R_\alpha^{n_2} x_1)\| < \frac{1}{(k-1)^2}$$

for  $1 \leq i \leq k-2$ . Thus, one has

$$\begin{aligned} \|y' - (y_1 + H_{n_3}^h(x_1))\| &= \|y' - (y_1 + H_{n_2}^h(x_1)) - H_{n_3 - n_2}^h(R_\alpha^{n_2} x_1)\| \\ &= \left\| b - \sum_{i=k}^{\infty} H_{n_3 - n_2}^{h_i}(R_\alpha^{n_2} x_1) - \sum_{i=1}^{k-2} H_{n_3 - n_2}^{h_i}(R_\alpha^{n_2} x_1) \right\| \\ &\leq \|b - H_{n_3 - n_2}^{h_k}(R_\alpha^{n_2} x_1)\| \\ &\quad + \sum_{i=k+1}^{\infty} \|H_{n_3 - n_2}^{h_i}(R_\alpha^{n_2} x_1)\| + \sum_{i=1}^{k-2} \|H_{n_3 - n_2}^{h_i}(R_\alpha^{n_2} x_1)\| \\ &\leq \|b - H_{n_3 - n_2}^{h_k}(n_2\alpha)\| + \|H_{n_3 - n_2}^{h_k}(n_2\alpha) - H_{n_3 - n_2}^{h_k}(n_2\alpha + x_1)\| \\ &\quad + \sum_{i=k+1}^{\infty} \frac{n_3 - n_2}{N_i} + (k-2) \cdot \frac{1}{(k-1)^2} \\ &\leq \left\| b - (n_3 - n_2) \frac{1}{N_k} \right\| + \epsilon \frac{n_3 - n_2}{N_k} + \sum_{i=k+1}^{\infty} \frac{1}{10^i} + \frac{1}{k-1} \\ &\leq \frac{3}{10^k} + 2\epsilon + \frac{1}{k-1}. \end{aligned}$$

We deduces

$$\begin{aligned} d((x', y'), T^{n_3+n_1}(x, y)) &= d((x', y'), T^{n_3}(x_1, y_1)) \\ &= \|x' - R_\alpha^{n_3} x_1\| + \|y' - (y_1 + H_{n_3}^h(x_1))\| \\ &\leq l_{k-1} + 2\delta_{k-1} + \frac{3}{10^k} + 2\epsilon + \frac{1}{k-1}. \end{aligned}$$

This implies that  $(x', y') \in \overline{\text{Orb}((x, y), T)}$  if we let  $k \rightarrow +\infty$  and  $\epsilon \rightarrow 0$ . Hence  $(\mathbb{T}^2, T)$  is minimal.  $\square$

For  $1 \leq j \leq k$ , we let

$$E_{j,k} = \bigcup_{i=j}^k E_i.$$

By (5), for any  $n \geq M_{k+1}$  and  $x \in \mathbb{T}$ ,

$$(16) \quad \frac{1}{n} \#(\{0 \leq i \leq n-1 : R_\alpha^i x \in E_{j,k}\}) < \sum_{i=j}^k \frac{\eta}{2^i} < \frac{\eta}{2^{j-1}} < \frac{1}{2}.$$

**Proposition A.4.**  $(\mathbb{T}^2, T)$  is not equicontinuous.

*Proof.* To show that  $(\mathbb{T}^2, T)$  is not equicontinuous, it is sufficient to show that for any  $\epsilon > 0$ , there exist  $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$  and  $n \in \mathbb{N}$  such that  $d((x_1, y_1), (x_2, y_2)) \leq \epsilon$  and  $d(T^n(x_1, y_1), T^n(x_2, y_2)) \geq \frac{1}{9}$ .

Fix  $\epsilon > 0$ . There exists  $k \in \mathbb{N}$  such that  $l_k + \delta_k < \epsilon$ . Put  $x' = \delta_k + \frac{1}{2}l_k$ . One has  $R_\alpha^i x' \in E_k^c$  and  $h_k(R_\alpha^i x') = 0$  for  $i = 0, 1, \dots, N_k - 1$ . By (16), we can choose integers  $n_1 \in [0, M_k - 1]$  and  $n_2 \in [\frac{1}{2}N_k - M_k, \frac{1}{2}N_k - 1]$  such that

$$R_\alpha^{n_1} 0, R_\alpha^{n_1} x', R_\alpha^{n_2} 0, R_\alpha^{n_2} x' \in E_{1,k-1}^c.$$

By using Lemma A.1 and the fact  $R_\alpha^{n_1} x', R_\alpha^{n_2} x' \in E_k^c$ , we have

$$\begin{aligned} H_{n_2-n_1}^h(R_\alpha^{n_1} 0) &= H_{n_2-n_1}^{h_{1,k-1}}(R_\alpha^{n_1} 0) + H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} 0) + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} 0) \\ &= H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} 0) + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} 0) \\ &= (n_2 - n_1) \frac{1}{N_k} + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} 0) \end{aligned}$$

and

$$\begin{aligned} H_{n_2-n_1}^h(R_\alpha^{n_1} x') &= H_{n_2-n_1}^{h_{1,k-1}}(R_\alpha^{n_1} x') + H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} x') + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} x') \\ &= H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} x'). \end{aligned}$$

Note that  $\frac{1}{2}N_k - 2M_k \leq n_2 - n_1 \leq \frac{1}{2}N_k$  and  $N_k = 10^k M_k$ , so

$$\frac{2}{5} \leq (n_2 - n_1) \frac{1}{N_k} \leq \frac{1}{2}.$$

By (12), we have

$$\begin{aligned} \|H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}0) - H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}x')\| &\leq \sum_{i=k+1}^{\infty} \left( \|H_{n_2-n_1}^{h_i}(R_\alpha^{n_1}0)\| + \|H_{n_2-n_1}^{h_i}(R_\alpha^{n_1}x')\| \right) \\ &\leq \sum_{i=k+1}^{\infty} 2(n_2 - n_1) \frac{1}{N_i} \leq \sum_{i=k+1}^{\infty} \frac{2N_k}{N_i} \\ &\leq 2 \sum_{i=k+1}^{\infty} \frac{1}{10^{i-k}} = \frac{2}{9}. \end{aligned}$$

Thus,

$$\frac{16}{90} \leq \|H_{n_2-n_1}^h(R_\alpha^{n_1}0) - H_{n_2-n_1}^h(R_\alpha^{n_1}x')\| \leq \frac{65}{90},$$

and

$$\begin{aligned} d(T^{n_2-n_1}(R_\alpha^{n_1}0, 0), T^{n_2-n_1}(R_\alpha^{n_1}x', 0)) \\ &= d((R_\alpha^{n_2}0, H_{n_2-n_1}^h(R_\alpha^{n_1}0)), (R_\alpha^{n_2}x', H_{n_2-n_1}^h(R_\alpha^{n_1}x'))) \\ &\geq \|H_{n_2-n_1}^h(R_\alpha^{n_1}0) - H_{n_2-n_1}^h(R_\alpha^{n_1}x')\| \geq \frac{16}{90} \geq \frac{1}{9} \end{aligned}$$

with

$$d((R_\alpha^{n_1}0, 0), (R_\alpha^{n_1}x', 0)) = \|R_\alpha^{n_1}0 - R_\alpha^{n_1}x'\| = \|x'\| = \delta_k + \frac{1}{2}\ell_k \leq \epsilon.$$

This implies that  $(\mathbb{T}^2, T)$  is not equicontinuous.  $\square$

**Proposition A.5.**  $(\mathbb{T}^2, T)$  is not uniquely ergodic.

*Proof.* Let  $m_{\mathbb{T}^2}$  be the unique normalized Haar measure on  $\mathbb{T}^2$ . For any  $m_{\mathbb{T}^2}$ -integrable function  $f(x, y)$ , by the Fubini's theorem, one has

$$\begin{aligned} \int_{\mathbb{T}^2} f \circ T(x, y) dm_{\mathbb{T}^2} &= \int_{\mathbb{T}} \int_{\mathbb{T}} f(R_\alpha x, y + h(x)) dm_{\mathbb{T}}(y) dm_{\mathbb{T}}(x) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} f(R_\alpha x, y) dm_{\mathbb{T}}(y) dm_{\mathbb{T}}(x) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} f(x, y) dm_{\mathbb{T}}(y) dm_{\mathbb{T}}(x) \\ &= \int_{\mathbb{T}^2} f(x, y) dm_{\mathbb{T}^2}. \end{aligned}$$

Therefore  $m_{\mathbb{T}^2}$  is  $T$ -invariant.

If  $(\mathbb{T}^2, T)$  is uniquely ergodic, then  $m_{\mathbb{T}^2}$  is the unique invariant measure. We take a measurable function

$$f(x, y) = \mathbf{1}_{\mathbb{T} \times [0, \frac{1}{2})}(x, y) - \mathbf{1}_{\mathbb{T} \times [\frac{1}{2}, 1)}(x, y).$$

Note that the boundary of  $\mathbb{T} \times [0, \frac{1}{2})$  and  $\mathbb{T} \times [\frac{1}{2}, 1)$  have zero  $m_{\mathbb{T}^2}$ -measure. By unique ergodicity, we have

$$(17) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x, y)) = \int_{\mathbb{T}^2} f dm_{\mathbb{T}^2} = 0$$



for each  $(x, y) \in \mathbb{T}^2$ .

For  $k \geq 1$ , put

$$A_k = \left\{ s \in \left\{ \frac{1}{10}N_k, \frac{1}{10}N_k + 1, \dots, \frac{4}{10}N_k \right\} : R_\alpha^s 0 = s\alpha \in \bigcap_{j=1}^{k-1} E_j^c \right\}.$$

For  $i \in A_k$ , it is clear that  $0, R_\alpha^i 0 \in \bigcap_{j=1}^{k-1} E_j^c \cup [-\delta_j, \delta_j]$ , so we have that by Lemma A.1

$$H_i^h(0) - \frac{i}{N_k} = \sum_{j=1}^{\infty} H_i^{h_j}(0) - \frac{i}{N_k} = \sum_{j=k}^{\infty} H_i^{h_j}(0) - \frac{i}{N_k} = \sum_{j=k+1}^{\infty} H_i^{h_j}(0)$$

where in the last equality we use the fact that  $H_i^{h_k}(0) = \sum_{l=0}^{i-1} h_k(R_\alpha^l 0) = \frac{i}{N_k}$  by (10). Notice that

$$\|H_i^{h_j}(0)\| \leq \sum_{l=0}^{i-1} \|h_j(R_\alpha^l 0)\| \leq \frac{i}{N_j}.$$

Therefore

$$\begin{aligned} \left\| H_i^h(0) - \frac{i}{N_k} \right\| &= \left\| \sum_{j=k+1}^{\infty} H_i^{h_j}(0) \right\| \leq \sum_{j=k+1}^{\infty} \|H_i^{h_j}(0)\| \leq \sum_{j=k+1}^{\infty} \frac{i}{N_j} \\ &\leq \sum_{j=k+1}^{\infty} \frac{i}{10^j M_j} \leq \sum_{j=k+1}^{\infty} \frac{1}{10^j} < \frac{1}{10}. \end{aligned}$$

It is clear that  $\frac{1}{10} \leq \frac{i}{N_k} \leq \frac{4}{10}$ . So  $H_i^h(0) \in [0, \frac{1}{2}]$  and

$$(18) \quad f(T^i(0, 0)) = f((i\alpha, H_i^h(0))) = 1.$$

Put  $S_k = \{0, 1, \dots, \frac{1}{2}N_k - 1\}$  and  $B_k = \{s \in S_k : R_\alpha^s 0 \in E_{1,k-1}\}$ , and by the construction (5)

$$\frac{\#(B_k)}{\frac{1}{2}N_k} \leq \sum_{j=1}^{k-1} \frac{\eta}{2^j} < \eta = \frac{1}{100}.$$

Hence

$$\frac{\#(A_k)}{\frac{1}{2}N_k} \geq \frac{\frac{3}{10}N_k - \#B_k}{\frac{1}{2}N_k} > \frac{59}{100}.$$

Since  $f(T^i(0, 0)) = 1$  for  $i \in A_k$  and  $f(T^i(0, 0)) \in \{-1, 1\}$  for  $i \in S_k \setminus A_k$ , we have

$$\begin{aligned} \frac{1}{\frac{1}{2}N_k} \sum_{i=0}^{\frac{1}{2}N_k-1} f(T^i(0, 0)) &= \frac{1}{\frac{1}{2}N_k} \left( \sum_{i \in A_k} f(T^i(0, 0)) + \sum_{i \in S_k \setminus A_k} f(T^i(0, 0)) \right) \\ &\geq \frac{1}{\frac{1}{2}N_k} (\#(A_k) - \#(S_k \setminus A_k)) \\ &\geq \frac{59}{100} - \frac{41}{100} = \frac{18}{100}. \end{aligned}$$

Thus

$$\limsup_{k \rightarrow \infty} \frac{1}{\frac{1}{2}N_k} \sum_{i=0}^{\frac{1}{2}N_k-1} f(T^i(0,0)) \geq \frac{18}{100} > 0,$$

which contradicts (17). Therefore  $(\mathbb{T}^2, T)$  is not uniquely ergodic. This completes the proof.  $\square$

For any real function  $g$  on  $\mathbb{T}$ ,  $n \in \mathbb{N}$  and  $x, y \in X$ , we set

$$\bar{d}_n^g(x, y) := \frac{1}{n} \sum_{m=0}^{n-1} \|H_m^g(x) - H_m^g(y)\|.$$

Then for any  $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$ , we have

$$\begin{aligned} \bar{d}_n((x_1, y_1), (x_2, y_2)) &= \frac{1}{n} \sum_{m=0}^{n-1} d((R_\alpha^m x_1, y_1 + H_m^h(x_1)), (R_\alpha^m x_2, y_1 + H_m^h(x_2))) \\ &\leq \|x_1 - x_2\| + \|y_1 - y_2\| + \bar{d}_n^h(x_1, x_2). \end{aligned}$$

The main result of this subsection is as follows.

**Proposition A.6.**  $(\mathbb{T}^2, T)$  has bounded topological complexity with respect to  $\{\bar{d}_n\}$ .

*Proof.* It is sufficient to show that for any  $\epsilon \in (0, \frac{1}{100})$ , there exist two constants  $C(\epsilon) > 0$  and  $K(\epsilon) \in \mathbb{N}$  such that  $\overline{\text{span}}(n, 17\epsilon) \leq C(\epsilon)$  for any  $n > K(\epsilon)$ .

First, we choose an integer  $q \in \mathbb{N}$  such that

$$(19) \quad \sum_{i=q+1}^{\infty} \frac{\eta}{2^i} < \epsilon \text{ and } \frac{1}{10^q} < \epsilon.$$

Then there exists  $\delta(\epsilon) > 0$  such that

$$(20) \quad \sum_{i=1}^q \|H_s^{h_i}(x) - H_s^{h_i}(y)\| < \epsilon,$$

for any  $0 \leq s \leq M_{q+1} - 1$  and any  $x, y \in \mathbb{T}$  with  $\|x - y\| < \delta(\epsilon)$ .

Put  $c_\epsilon = \lceil \frac{1}{\epsilon} \rceil$  and  $c_\delta = \lceil \frac{1}{\delta(\epsilon)} \rceil$ . Let

$$C(\epsilon) = 100c_\epsilon^{11}c_\delta \text{ and } K(\epsilon) = 2N_{q+2}.$$

In the following, we are going to show that for any  $n > K(\epsilon)$  there exists a cover  $\mathcal{T}$  of  $\mathbb{T}^2$  (that depends on  $n$ ), such that

$$\#(\mathcal{T}) \leq C(\epsilon) \text{ and } \bar{d}_n((x_1, y_1), (x_2, y_2)) \leq 17\epsilon$$

for any  $(x_1, y_1), (x_2, y_2) \in W \in \mathcal{T}$ . This will imply  $\overline{\text{span}}(n, 17\epsilon) \leq C(\epsilon)$  for any  $n > K(\epsilon)$ .

Now fix an integer  $n > K(\epsilon)$ . There exists a unique integer  $k \geq q + 2$  such that

$$2N_k < n \leq 2N_{k+1}.$$

Recall that

$$\begin{aligned} h &= h_{1,k-1} + h_k + h_{k+1} + h_{k+2,\infty}, \\ \bar{d}_n((x_1, y_1), (x_2, y_2)) &\leq \|x_1 - x_2\| + \|y_1 - y_2\| + \bar{d}_n^h(x_1, x_2), \end{aligned}$$

and

$$\bar{d}_n^h(x_1, x_2) \leq \bar{d}_n^{h_{1,k-1}}(x_1, x_2) + \bar{d}_n^{h_k}(x_1, x_2) + \bar{d}_n^{h_{k+1}}(x_1, x_2) + \bar{d}_n^{h_{k+2,\infty}}(x_1, x_2).$$

We divide the remaining proof into four steps, bounding each term of the sum above.

**Step 1:** We will construct a finite cover  $\mathcal{P}$  of  $\mathbb{T}$  such that

$$\#\mathcal{P} \leq c_\delta c_\epsilon^2 \text{ and } \bar{d}_n^{h_{1,k-1}}(x, y) < 6\epsilon$$

for  $x, y \in P \in \mathcal{P}$ .

Firstly, for any  $x \in \mathbb{T}$  and  $\ell \geq 2$ , we define

$$n_\ell^*(x) = \min\{i \geq 0 : R_\alpha^i x \in E_{1,\ell-1}^c\} \text{ and } x_\ell^* = R_\alpha^{n_\ell^*(x)} x.$$

Clearly,  $n_\ell^*(x) \leq M_\ell - 1$  by (4). By Lemma A.1, if  $R_\alpha^m x \in E_i^c$  for some  $1 \leq i \leq \ell - 1$  and  $m \geq M_\ell$ , one has  $H_{m-n_\ell^*(x)}^{h_i}(x_\ell^*) = 0$  and then

$$(21) \quad H_m^{h_i}(x) = H_{n_\ell^*(x)}^{h_i}(x) + H_{m-n_\ell^*(x)}^{h_i}(x_\ell^*) = H_{n_\ell^*(x)}^{h_i}(x).$$

Next, let

$$\begin{aligned} \mathcal{P}_1 &= \left\{ \left[ \frac{j}{c_\delta}, \frac{j+1}{c_\delta} \right) : 0 \leq j \leq c_\delta - 1 \right\}, \\ \mathcal{P}_2 &= \left\{ \left\{ x \in \mathbb{T} : \sum_{i=1}^q H_{n_{q+1}^*(x)}^{h_i}(x) \in \left[ \frac{j}{c_\epsilon}, \frac{j+1}{c_\epsilon} \right) \right\} : 0 \leq j \leq c_\epsilon - 1 \right\}, \\ \mathcal{P}_3 &= \left\{ \left\{ x \in \mathbb{T} : \sum_{i=q+1}^{k-1} H_{n_k^*(x)}^{h_i}(x) \in \left[ \frac{j}{c_\epsilon}, \frac{j+1}{c_\epsilon} \right) \right\} : 0 \leq j \leq c_\epsilon - 1 \right\}. \end{aligned}$$

Put  $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \mathcal{P}_3$ . It is clear that  $\mathcal{P}$  is a partition of  $\mathbb{T}$  and  $\#\mathcal{P} \leq c_\delta c_\epsilon^2$ .

Fix two points  $x, y$  which are in the same atom of  $\mathcal{P}$ . If there exists  $m \geq M_k$  with  $R_\alpha^m x, R_\alpha^m y \in E_{q+1,k-1}^c$ , then by (21) we have for  $q+1 \leq i \leq k-1$ ,

$$H_m^{h_i}(x) = H_{n_k^*(x)}^{h_i}(x) \text{ and } H_m^{h_i}(y) = H_{n_k^*(y)}^{h_i}(y).$$

Thus,

$$\left\| \sum_{i=q+1}^{k-1} (H_m^{h_i}(x) - H_m^{h_i}(y)) \right\| = \left\| \sum_{i=q+1}^{k-1} (H_{n_k^*(x)}^{h_i}(x) - H_{n_k^*(y)}^{h_i}(y)) \right\| \leq \frac{1}{c_\epsilon} \leq \epsilon,$$

as  $x, y$  are in the same atom in  $\mathcal{P}_3$ .

By (5) for any  $z \in \mathbb{T}$ ,

$$\frac{1}{M_{q+1}} \#\{0 \leq i \leq M_{q+1} - 1 : R_\alpha^i z \in E_{1,q}^c\} \geq 1 - \sum_{i=1}^{\infty} \frac{\eta}{2^i} > \frac{1}{2}.$$

If there exists  $m \geq M_k$  with  $R_\alpha^m x, R_\alpha^m y \in E_{q+1,k-1}^c$ , then we can find an integer  $M \in [m - M_{q+1}, m - 1]$  such that  $R_\alpha^M x \in E_{1,q}^c$  and  $R_\alpha^M y \in E_{1,q}^c$ . Note that  $M \geq m - M_{q+1} > M_{q+1}$ . By (21), for  $1 \leq i \leq q$ ,

$$H_M^{h_i}(x) = H_{n_{q+1}^*(x)}^{h_i}(x) \text{ and } H_M^{h_i}(y) = H_{n_{q+1}^*(y)}^{h_i}(y).$$

Then

$$\left\| \sum_{i=1}^q (H_M^{h_i}(x) - H_M^{h_i}(y)) \right\| = \left\| \sum_{i=1}^q (H_{n_{q+1}^*(x)}^{h_i}(x) - H_{n_{q+1}^*(y)}^{h_i}(y)) \right\| \leq \frac{1}{c_\epsilon} \leq \epsilon,$$

as  $x, y$  are in the same atom in  $\mathcal{P}_2$ .

As  $x, y$  are in the same atom in  $\mathcal{P}_1$ ,  $\|R_\alpha^M x - R_\alpha^M y\| = \|x - y\| \leq \frac{1}{c_\delta} \leq \delta(\epsilon)$ . Note that  $m - M \leq M_{q+1} - 1$ . By (20) we have

$$\left\| \sum_{i=1}^q H_{m-M}^{h_i}(R_\alpha^M x) - \sum_{i=1}^q H_{m-M}^{h_i}(R_\alpha^M y) \right\| < \epsilon.$$

Hence, if there exists  $m \geq M_k$  with  $R_\alpha^m x, R_\alpha^m y \in E_{q+1, k-1}^c$ , then we have

$$\begin{aligned} \left\| H_m^{h_{1, k-1}}(x) - H_m^{h_{1, k-1}}(y) \right\| &\leq \left\| \sum_{i=1}^q (H_m^{h_i}(x) - H_m^{h_i}(y)) \right\| + \left\| \sum_{i=q+1}^{k-1} (H_m^{h_i}(x) - H_m^{h_i}(y)) \right\| \\ &\leq \left\| \sum_{i=1}^q (H_M^{h_i}(x) - H_M^{h_i}(y)) \right\| \\ &\quad + \left\| \sum_{i=1}^q H_{m-M}^{h_i}(R_\alpha^M x) - \sum_{i=1}^q H_{m-M}^{h_i}(R_\alpha^M y) \right\| \\ &\quad + \left\| \sum_{i=q+1}^{k-1} (H_m^{h_i}(x) - H_m^{h_i}(y)) \right\| \\ &\leq 3\epsilon. \end{aligned}$$

Finally,

$$\begin{aligned} \bar{d}_n^{h_{1, k-1}}(x, y) &= \frac{1}{n} \sum_{j=0}^{n-1} \|H_j^{h_{1, k-1}}(x) - H_j^{h_{1, k-1}}(y)\| \\ &\leq \frac{1}{n} \left( \sum_{\substack{M_k \leq j \leq n-1 \\ R_\alpha^j x, R_\alpha^j y \in E_{q+1, k-1}^c}} \|H_j^{h_{1, k-1}}(x) - H_j^{h_{1, k-1}}(y)\| \right. \\ &\quad \left. + \sum_{\substack{M_k \leq j \leq n-1 \\ R_\alpha^j x \in E_{q+1, k-1}}} 1 + \sum_{\substack{M_k \leq j \leq n-1 \\ R_\alpha^j y \in E_{q+1, k-1}}} 1 + \sum_{0 \leq j \leq M_k - 1} 1 \right) \\ &\leq 3\epsilon + \frac{1}{n} \#(\{0 \leq j \leq n-1 : R_\alpha^j x \in E_{q+1, k-1}\}) \\ &\quad + \frac{1}{n-1} \#(\{0 \leq j \leq n-1 : R_\alpha^j y \in E_{q+1, k-1}\}) + \frac{M_k}{n-1} \\ &< 6\epsilon, \end{aligned}$$

where the last inequality follows from (16) and (19).

**Step 2:** We will construct a finite cover  $Q$  of  $\mathbb{T}$  such that

$$\#(Q) \leq 10c_\epsilon^4 \text{ and } \bar{d}_n^{h_k}(x, y) \leq 4\epsilon$$

for any  $x, y \in Q \in \mathcal{Q}$ .

There are two cases. The first case is  $n \leq 2c_\epsilon N_k$ . In this case, we put

$$Q_0 = \mathbb{T} \setminus \left( \bigcup_{-2c_\epsilon N_k \leq i < (2+2c_\epsilon)N_k} [i\alpha - \gamma_k, i\alpha + \gamma_k] \right),$$

and

$$Q_{r,s} = \bigcup_{\frac{rN_k}{c_\epsilon} \leq i < \frac{(r+1)N_k}{c_\epsilon}} \left[ i\alpha + \frac{\gamma_k s}{c_\epsilon^2}, i\alpha + \frac{\gamma_k (s+1)}{c_\epsilon^2} \right].$$

Let

$$\mathcal{Q} = \{Q_0\} \cup \{Q_{r,s} : -2c_\epsilon^2 \leq r \leq (2+2c_\epsilon)c_\epsilon - 1, -c_\epsilon^2 \leq s \leq c_\epsilon^2 - 1\}.$$

It is clear that  $\mathcal{Q}$  is a cover of  $\mathbb{T}$  and  $\#\mathcal{Q} \leq 2c_\epsilon^2 \cdot 5c_\epsilon^2 = \frac{10c_\epsilon^3}{\epsilon} \leq 10c_\epsilon^4$ . For  $x, y \in Q_0$ , one has  $\bar{d}_n^{h_k}(x, y) = 0$  by (10).

Now assume that  $x, y \in Q_{r,s}$  for some  $r$  and  $s$ . There exist integers  $m_1, m_2 \in [\frac{rN_k}{c_\epsilon}, \frac{(r+1)N_k}{c_\epsilon}]$  and  $x_1, y_1 \in [\frac{\gamma_k s}{c_\epsilon^2}, \frac{\gamma_k (s+1)}{c_\epsilon^2}]$  such that

$$x = R_\alpha^{m_1} x_1 \text{ and } y = R_\alpha^{m_2} y_1.$$

Without loss of generality, we can assume that  $m_1 \leq m_2$ . For any  $1 \leq m \leq n$ , one has

$$\begin{aligned} \|H_m^{h_k}(x) - H_m^{h_k}(y)\| &= \left\| \sum_{i=0}^{m-1} (h_k(R_\alpha^i x) - h_k(R_\alpha^i y)) \right\| \\ &\leq \left\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i y_1) \right\| \\ &\leq \left\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i x_1) \right\| \\ &\quad + \left\| \sum_{i=m_2}^{m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \right\| \\ &\leq \sum_{i=m_1}^{m_2-1} \|h_k(R_\alpha^i x_1)\| + \sum_{i=m_1+m}^{m_2+m-1} \|h_k(R_\alpha^i x_1)\| + m \cdot \frac{\gamma_k}{c_\epsilon^2} \cdot \frac{1}{N_k \gamma_k} \\ &\leq 2(m_2 - m_1) \frac{1}{N_k} + m \cdot \frac{\gamma_k}{c_\epsilon^2} \cdot \frac{1}{N_k \gamma_k} \quad \text{by (12) and (13)} \\ &\leq 4\epsilon. \end{aligned}$$

Hence, summing up we obtain

$$\bar{d}_n^{h_k}(x, y) \leq 4\epsilon, \text{ for } x, y \in Q \in \mathcal{Q}.$$

The second case is  $n > 2c_\epsilon N_k$ . In this case, we put

$$Q_0 = \mathbb{T} \setminus \left( \bigcup_{0 \leq i < 2N_k} [i\alpha - \gamma_k, i\alpha + \gamma_k] \right),$$

and

$$Q_{r,s} = \bigcup_{\frac{rN_k}{c_\epsilon} \leq i < \frac{(r+1)N_k}{c_\epsilon}} \left[ i\alpha + \frac{\gamma_k s}{c_\epsilon^2}, i\alpha + \frac{\gamma_k(s+1)}{c_\epsilon^2} \right].$$

Let

$$Q = \{Q_0\} \bigcup \{Q_{r,s} : 0 \leq r \leq 2c_\epsilon - 1, -c_\epsilon^2 \leq s \leq c_\epsilon^2 - 1\}.$$

It is clear that  $Q$  is a cover of  $\mathbb{T}$  and  $\#Q \leq 10c_\epsilon^4$ . Given  $x, y \in Q_0$ , by (6) and (10) one has

$$\#\{0 \leq m \leq n-1 : H_m^{h_k}(x) \neq 0\} \leq 2N_k$$

and

$$\#\{0 \leq m \leq n-1 : H_m^{h_k}(y) \neq 0\} \leq 2N_k.$$

Then by (12)

$$\bar{d}_n^{h_k}(x, y) \leq \frac{1}{n} \cdot 4N_k \leq 2\epsilon.$$

Now assume that  $x, y \in Q_{r,s}$  for some  $r$  and  $s$ . There exist integers  $m_1, m_2 \in [\frac{rN_k}{c_\epsilon}, \frac{(r+1)N_k}{c_\epsilon}]$  and  $x_1, y_1 \in [\frac{\gamma_k s}{c_\epsilon^2}, \frac{\gamma_k(s+1)}{c_\epsilon^2}]$  such that

$$x = R_\alpha^{m_1} x_1 \text{ and } y = R_\alpha^{m_2} y_1.$$

Without loss of generality, we can assume that  $m_1 \leq m_2$ . Recall that  $2N_k < n \leq 2N_{k+1}$ . By (6) and (10) one has

$$(22) \quad h_k(R_\alpha^i x_1) = h_k(R_\alpha^i y_1) = 0$$



for any  $2N_k < i \leq 2N_k + n \leq 2N_k + 2N_{k+1}$ . For any  $1 \leq m \leq n$ , one has

$$\begin{aligned}
\|H_m^{h_k}(x) - H_m^{h_k}(y)\| &= \left\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i y_1) \right\| \\
&\leq \left\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i x_1) \right\| \\
&\quad + \left\| \sum_{i=m_2}^{m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \right\| \\
&\leq \sum_{i=m_1}^{m_2-1} \|h_k(R_\alpha^i x_1)\| + \sum_{i=m_1+m}^{m_2+m-1} \|h_k(R_\alpha^i x_1)\| \\
&\quad + \left\| \sum_{m_2 \leq i \leq m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \right\| \\
&= 2(m_2 - m_1) \frac{1}{N_k} + \left\| \sum_{m_2 \leq i \leq m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \right\| \quad \text{by (12)} \\
&= 2(m_2 - m_1) \frac{1}{N_k} + \left\| \sum_{\substack{m_2 \leq i \leq m_2+m-1 \\ i \leq 2N_k}} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \right\| \quad \text{by (22)} \\
&\leq 2(m_2 - m_1) \frac{1}{N_k} + 2N_k \cdot \frac{\gamma_k}{c_\epsilon^2} \cdot \frac{1}{N_k \gamma_k} \quad \text{by (13)} \\
&\leq 4\epsilon.
\end{aligned}$$

Hence, summing up we get

$$\bar{d}_n^{h_k}(x, y) \leq 4\epsilon, \text{ for } x, y \in Q \in \mathcal{Q}.$$

**Step 3:** We will construct a finite cover  $\mathcal{I}$  of  $\mathbb{T}$  such that

$$\#(\mathcal{I}) \leq 10c_\epsilon^3 \text{ and } \bar{d}_n^{h_{k+1}}(x, y) \leq 4\epsilon$$

for any  $x, y \in I \in \mathcal{I}$ .

Put

$$I_0 = \mathbb{T} \setminus \left( \bigcup_{i=-2N_{k+1}}^{2N_{k+1}} [i\alpha - \gamma_{k+1}, i\alpha + \gamma_{k+1}] \right),$$

and

$$I_{r,s} = \bigcup_{\frac{rN_{k+1}}{c_\epsilon} \leq i < \frac{(r+1)N_{k+1}}{c_\epsilon}} \left[ i\alpha + \frac{\gamma_{k+1}s}{c_\epsilon^2}, i\alpha + \frac{\gamma_{k+1}(s+1)}{c_\epsilon^2} \right].$$

Put

$$\mathcal{I} = \{I_0\} \bigcup \{I_{r,s} : -2c_\epsilon \leq r \leq 2c_\epsilon - 1, -c_\epsilon^2 \leq s \leq c_\epsilon^2 - 1\}.$$

It is clear that  $\mathcal{I}$  is a cover of  $\mathbb{T}$  and  $\#(\mathcal{I}) \leq 10c_\epsilon^3$ . Given  $x, y \in I_0$ , one has  $\bar{d}_n^{h_{k+1}}(x, y) = 0$  by (10).

Now assume that  $x, y \in I_{r,s}$  for some  $r$  and  $s$ . There exist integers  $m_1, m_2 \in [\frac{rN_k}{c_\epsilon}, \frac{(r+1)N_k}{c_\epsilon}]$  and  $x_1, y_1 \in [\frac{\gamma_{k+1}s}{c_\epsilon^2}, \frac{\gamma_{k+1}(s+1)}{c_\epsilon^2}]$  such that

$$x = R_\alpha^{m_1} x_1 \text{ and } y = R_\alpha^{m_2} y_1.$$

Without loss of generality, we can assume that  $m_1 \leq m_2$ . For any  $1 \leq m \leq n$ , one has

$$\begin{aligned} \|H_m^{h_{k+1}}(x) - H_m^{h_{k+1}}(y)\| &= \left\| \sum_{i=0}^{m-1} (h_{k+1}(R_\alpha^i x) - h_{k+1}(R_\alpha^i y)) \right\| \\ &\leq \left\| \sum_{i=m_1}^{m_1+m-1} h_{k+1}(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_{k+1}(R_\alpha^i y_1) \right\| \\ &\leq \left\| \sum_{i=m_1}^{m_1+m-1} h_{k+1}(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_{k+1}(R_\alpha^i x_1) \right\| \\ &\quad + \left\| \sum_{i=m_2}^{m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \right\| \\ &\leq 2(m_2 - m_1) \frac{1}{N_{k+1}} + m \cdot \frac{\gamma_{k+1}}{c_\epsilon^2} \cdot \frac{1}{N_{k+1} \gamma_{k+1}} \quad \text{by (12) and (13)} \\ &\leq 4\epsilon. \end{aligned}$$

Hence, summing up we have

$$\bar{d}_n^{h_{k+1}}(x, y) \leq 4\epsilon, \text{ for } x, y \in Q \in \mathcal{I}.$$

**Step 4:** We will construct a finite cover  $\mathcal{T}$  of  $\mathbb{T}^2$  such that

$$\#(\mathcal{T}) \leq 100c_\epsilon^{11} c_\delta \text{ and } \bar{d}_n((x_1, y_1), (x_2, y_2)) \leq 17\epsilon$$

for any  $(x_1, y_1), (x_2, y_2) \in W \in \mathcal{T}$ .

Note that for any  $x \in \mathbb{T}$ ,

$$\|h_{k+2, \infty}(x)\| \leq \sum_{i=k+2}^{\infty} \frac{1}{N_i} \leq \frac{2}{N_{k+2}}.$$

For any  $x, y \in \mathbb{T}$  and  $1 \leq m \leq n$ , by (19) and  $2N_k < n \leq 2N_{k+1}$ , one has

$$\begin{aligned} \|H_m^{h_{k+2, \infty}}(x) - H_m^{h_{k+2, \infty}}(y)\| &= \left\| \sum_{i=0}^{m-1} (h_{k+2, \infty}(R_\alpha^i x) - h_{k+2, \infty}(R_\alpha^i y)) \right\| \\ &\leq \frac{m}{N_{k+2}} \leq \frac{4N_{k+1}}{N_{k+2}} < \epsilon. \end{aligned}$$

Hence,

$$(23) \quad \bar{d}_n^{h_{k+2, \infty}}(x, y) < \epsilon.$$

Finally, let  $\mathcal{S} = \{[\frac{j}{c_\epsilon}, \frac{j+1}{c_\epsilon}]: j = 0, 1, \dots, c_\epsilon - 1\}$  and put

$$\mathcal{T} = (\mathcal{S} \vee \mathcal{P} \vee \mathcal{Q} \vee \mathcal{I}) \times \mathcal{S}.$$

It is clear that  $\mathcal{T}$  is a finite cover of  $\mathbb{T}^2$  with

$$\#(\mathcal{T}) \leq c_\epsilon \cdot c_\delta c_\epsilon^2 \cdot 10c_\epsilon^4 \cdot 10c_\epsilon^3 \cdot c_\epsilon = 100c_\epsilon^{11} c_\delta = C(\epsilon).$$

Hence, for  $(x_1, y_1), (x_2, y_2) \in W \in \mathcal{T}$ , by Steps 1, 2, 3 and (23), one has

$$\begin{aligned} \bar{d}_n^h(x_1, x_2) &\leq \bar{d}_n^{h_{1,k-1}}(x_1, x_2) + \bar{d}_n^{h_k}(x_1, x_2) + \bar{d}_n^{h_{k+1}}(x_1, x_2) + \bar{d}_n^{h_{k+2,\infty}}(x_1, x_2) \\ &< 6\epsilon + 4\epsilon + 4\epsilon + \epsilon = 15\epsilon. \end{aligned}$$

We deduce that

$$\bar{d}_n((x_1, y_1), (x_2, y_2)) \leq \|x_1 - x_2\| + \|y_1 - y_2\| + \bar{d}_n^h(x_1, x_2) < 17\epsilon.$$

This implies  $\overline{\text{span}}(n, 17\epsilon) \leq C(\epsilon)$  for all  $n > K(\epsilon)$ , which ends the proof.  $\square$

**A.2. The construction of the system in Proposition 3.9.** First we need the following Furstenberg's dichotomy result.

**Proposition A.7** ([9]). *Suppose  $(\Omega_0, \mu_0, T_0)$  is a uniquely ergodic topological dynamical system with  $\mu_0$  being the unique ergodic measure, and  $h : \Omega_0 \rightarrow \mathbb{T}$  is a continuous function. Let  $T : \Omega_0 \times \mathbb{T}$  be defined by  $T(x, y) = (T_0(x), y + h(x))$ . Then exactly one of the following is true:*

- (1)  $T$  is uniquely ergodic and  $\mu_0 \times m_{\mathbb{T}}$  is the unique invariant measure;
- (2) there exists a measurable map  $g : \Omega_0 \rightarrow \mathbb{T}$  and a non-zero integer  $s$  such that  $s \cdot h(x) = g(T_0(x)) - g(x)$  for  $\mu_0$ -almost every  $x \in \Omega_0$ .

Now we modify the example  $(\mathbb{T}^2, T)$  in the previous subsection to be uniquely ergodic. As  $(\mathbb{T}^2, T)$  is not uniquely ergodic, by Furstenberg's dichotomy result there is an  $m_{\mathbb{T}}$ -measurable function  $g(x)$  and a non-zero integer  $s$  such that

$$(24) \quad s \cdot h(x) = g(x + \alpha) - g(x)$$

for  $m_{\mathbb{T}}$ -a.e.  $x \in \mathbb{T}$ . We define

$$\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad (x, y) \mapsto (x, s \cdot y)$$

and

$$\tilde{T} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + s \cdot h(x)).$$

Then  $\tilde{T} \circ \phi = \phi \circ T$ , in other words, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{T} & \mathbb{T}^2 \\ \phi \downarrow & & \phi \downarrow \\ \mathbb{T}^2 & \xrightarrow{\tilde{T}} & \mathbb{T}^2 \end{array}$$

Take an irrational number  $\beta \in \mathbb{R}$  such that  $\alpha$  and  $\beta$  are rationally independent. Then the system defined by

$$T_{\alpha,\beta} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + \beta)$$

is uniquely ergodic and  $m_{\mathbb{T}^2}$  is the unique invariant measure. Finally, we define

$$\tilde{T}_\beta : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + s \cdot h(x) + \beta).$$

We will show that the system  $(\mathbb{T}^2, \tilde{T}_\beta)$  is the one we need. It is clear that  $(\mathbb{T}^2, \tilde{T}_\beta)$  is distal.

**Proposition A.8.**  $(\mathbb{T}^2, \tilde{T}_\beta)$  is uniquely ergodic and minimal.

*Proof.* Let  $K = \{x \in \mathbb{T} : s \cdot h(x) = g(x+\alpha) - g(x)\}$  and  $\pi : \mathbb{T}^2 \rightarrow \mathbb{T}^2, (x, y) \mapsto (x, y - g(x))$ . By (24) one has  $m_{\mathbb{T}}(K) = 1$ . It is easy to see that  $\pi : K \times \mathbb{T} \rightarrow K \times \mathbb{T}$  is an invertible map with  $\pi \circ \tilde{T}_\beta = T_{\alpha, \beta} \circ \pi$ . For each  $\tilde{T}_\beta$ -invariant measure  $\mu$ , we have  $\mu(K \times \mathbb{T}) = 1$  and  $\mu \circ \pi^{-1}$  is  $T_{\alpha, \beta}$ -invariant. We have  $\mu \circ \pi^{-1} = m_{\mathbb{T}^2}$  since  $m_{\mathbb{T}^2}$  is the unique invariant probability measure of  $T_{\alpha, \beta}$ . Thus,  $\mu = m_{\mathbb{T}^2} \circ \pi$ . This implies that  $m_{\mathbb{T}^2} \circ \pi$  is the only invariant measure for  $(\mathbb{T}^2, \tilde{T}_\beta)$ . Moreover,  $(\mathbb{T}^2, \tilde{T}_\beta)$  is minimal since the only invariant measure  $m_{\mathbb{T}^2} \circ \pi$  is of full support.  $\square$

**Proposition A.9.**  $(\mathbb{T}^2, \tilde{T}_\beta)$  is not equicontinuous.

*Proof.* It is sufficient to show for any  $\epsilon > 0$ , there exist  $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$  and a positive integer  $n$  such that  $d((x_1, y_1), (x_2, y_2)) \leq \epsilon$  and  $d(\tilde{T}_\beta^n(x_1, y_1), \tilde{T}_\beta^n(x_2, y_2)) \geq \frac{1}{200}$ .

Assuming that  $10^p \leq |s| < 10^{p+1}$  for some non-negative integer  $p$ . Given  $\epsilon > 0$ , there exists  $k \in \mathbb{N}$  such that  $k > p + 10$  and  $l_k + \delta_k < \epsilon$ . Put  $x' = \delta_k + \frac{1}{2}l_k$ . One has  $R_\alpha^i x' \in E_k^c$  and  $h_k(R_\alpha^i x') = 0$  for  $i = 0, 1, 2, \dots, N_k - 1$ . By (5), for any  $x \in \mathbb{T}$ ,

$$\frac{1}{M_k} \#\{0 \leq i \leq M_k - 1 : R_\alpha^i x \in E_{1, k-1}^c\} \geq 1 - \sum_{i=1}^{\infty} \frac{\eta}{2^i} > \frac{1}{2}.$$

Then there are integers  $n_1 \in [0, M_k - 1]$  and  $n_2 \in [10^{k-p-2}M_k - M_k, 10^{k-p-2}M_k - 1]$  such that  $R_\alpha^{n_1} 0, R_\alpha^{n_1} x', R_\alpha^{n_2} 0, R_\alpha^{n_2} x' \in E_{1, k-1}^c$ . By using Lemma A.1 and the fact  $R_\alpha^{n_1} x', R_\alpha^{n_2} x' \in E_k^c$ , we have

$$\begin{aligned} H_{n_2-n_1}^h(R_\alpha^{n_1} 0) - H_{n_2-n_1}^h(R_\alpha^{n_1} x') &= H_{n_2-n_1}^{h_{1, k-1}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{1, k-1}}(R_\alpha^{n_1} x') + H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} 0) \\ &\quad - H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} x') + H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} x') \\ &= H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} 0) + H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} x') \\ &= (n_2 - n_1) \frac{1}{N_k} + H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} x'). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1, \infty}}(R_\alpha^{n_1} x')\| &\leq \sum_{i=k+1}^{\infty} (|H_{n_2-n_1}^{h_i}(R_\alpha^{n_1} 0)| + |H_{n_2-n_1}^{h_i}(R_\alpha^{n_1} x')|) \\ &\leq \sum_{i=k+1}^{\infty} 2(n_2 - n_1) \frac{1}{10^i M_i} \leq \sum_{i=k+1}^{\infty} \frac{2 \cdot 10^{k-p-2} M_k}{10^i M_i} \\ &\leq 2 \sum_{i=k+1}^{\infty} \frac{10^{-p-2}}{10^{i-k}} = \frac{2}{9} \cdot 10^{-p-2}. \end{aligned}$$

Note that  $10^{k-p-2}M_k - 2M_k \leq n_2 - n_1 \leq 10^{k-p-2}M_k$ . One has

$$|s| \cdot \|H_{n_2-n_1}^h(R_\alpha^{n_1} 0) - H_{n_2-n_1}^h(R_\alpha^{n_1} x')\| \leq |s| \cdot \left(10^{-p-2} + \frac{2}{9} \cdot 10^{-p-2}\right) \leq \frac{2}{10}.$$

and

$$|s| \cdot \|H_{n_2-n_1}^h(R_\alpha^{n_1} 0) - H_{n_2-n_1}^h(R_\alpha^{n_1} x')\| \geq |s| \cdot \left(10^{-p-2} - \frac{2}{10^k} - \frac{2}{9} \cdot 10^{-p-2}\right) \geq \frac{1}{200}.$$

Let  $x_1 = R_\alpha^{n_1} 0$ ,  $x_2 = R_\alpha^{n_1} x'$ ,  $y_1 = y_2 = 0$  and  $n = n_1 - n_2$ . Then

$$d((x_1, y_1), (x_2, y_2)) = \|R_\alpha^{n_1} 0 - R_\alpha^{n_1} x'\| = \|x'\| = \delta_k + \frac{1}{2}\ell_k < \epsilon.$$

and

$$\begin{aligned} d(\tilde{T}_\beta^n(x_1, y_1), \tilde{T}_\beta^n(x_2, y_2)) &= d\left((R_\alpha^{n_2} 0, s \cdot H_{n_2-n_1}^h(R_\alpha^{n_1} 0) + (n_2 - n_1)\beta), \right. \\ &\quad \left. (R_\alpha^{n_2} x', s \cdot H_{n_2-n_1}^h(R_\alpha^{n_1} x') + (n_2 - n_1)\beta)\right) \\ &\geq \|s \cdot (H_{n_2-n_1}^h(R_\alpha^{n_1} 0) - H_{n_2-n_1}^h(R_\alpha^{n_1} x'))\| \geq \frac{1}{200}. \end{aligned}$$

This implies that  $(\mathbb{T}^2, \tilde{T}_\beta)$  is not equicontinuous.  $\square$

**Proposition A.10.**  $(\mathbb{T}^2, \tilde{T}_\beta)$  has bounded topological complexity with respect to  $\{\bar{d}_n\}$ .

*Proof.* For  $\epsilon > 0$ , let  $\mathcal{T}$ ,  $c_\epsilon$  and  $c_\delta$  be defined in Proposition A.6. Then for  $(x_1, y_1), (x_2, y_2) \in W \in \mathcal{T}$ , one has

$$\begin{aligned} \bar{d}_n^{s \cdot h + \beta}(x_1, x_2) &= \frac{1}{n} \sum_{m=0}^{n-1} \|H_m^{sh+\beta}(x) - H_m^{sh+\beta}(y)\| \\ &= \frac{1}{n} \sum_{m=0}^{n-1} \|sH_m^h(x) - sH_m^h(y)\| \\ &\leq \frac{1}{n} \sum_{m=0}^{n-1} |s| \cdot \|H_m^h(x) - H_m^h(y)\| \\ &\leq |s| \cdot \bar{d}_n^h(x_1, x_2) \leq 15|s|\epsilon. \end{aligned}$$

and

$$\bar{d}_n((x_1, y_1), (x_2, y_2)) \leq \|x_1 - x_2\| + \|y_1 - y_2\| + \bar{d}_n^h(x_1, x_2) \leq (15|s| + 2)\epsilon.$$

Hence  $\overline{\text{span}}(n, (15|s| + 2)\epsilon) \leq 100c_\epsilon^{11}c_\delta$ . Thus  $(\mathbb{T}^2, \tilde{T}_\beta)$  has bounded topological complexity with respect to  $\{\bar{d}_n\}$ .  $\square$

## APPENDIX B. AN EXAMPLE BY CYR AND KRA

We first introduce some concepts. Following [6], by an *assignment*, we mean a function  $\Psi$  defined on an abstract metrizable Choquet simplex  $\mathcal{P}$ , whose “value” are measure-theoretic dynamical systems, i.e., for  $p \in \mathcal{P}$ ,  $\Psi(p)$  has the form  $(X_p, \mathcal{B}_p, \mu_p, T_p)$ . Two assignments,  $\Psi$  on a simplex  $\mathcal{P}$  and  $\Psi'$  on a simplex  $\mathcal{P}'$ , are said to be *equivalent* if there exists an affine homeomorphism  $\pi: \mathcal{P} \rightarrow \mathcal{P}'$  of Choquet simplexes such that for every  $p \in \mathcal{P}$  the systems  $\Psi(p)$  and  $\Psi'(\pi(p))$  are isomorphic as measure-theoretic dynamical systems. A topological dynamical system  $(X, T)$  determines an assignment on the simplex of  $T$ -invariant probability measures by the rule  $\mu \rightarrow (X, \mathcal{B}_X, \mu, T)$ , where  $\mathcal{B}_X$  denotes the collection of Borel sets on  $X$ . By [6, Theorem 1] or [22, Theorem 1], we know that if  $Y$  is zero dimensional and  $(Y, S)$  has no periodic points, then the assignment determined by  $(Y, S)$  is equivalent to an assignment determined by some minimal system  $(X, T)$ . If  $(Y, S)$  is invertible, then we can require that  $(X, T)$  is also invertible [22]. Applying [6, Theorem

1] or [22, Theorem 1], there is a minimal system  $(X, T)$  whose assignment is equivalent to that of  $(Y, S)$ .

**Proposition B.1.** *There exists a minimal system with bounded complexity with respect to  $\{\bar{d}_n\}$  for an invariant measure  $\mu$ , for which there exist two non-isomorphic ergodic measures in the ergodic decomposition.*

*Proof.* Pick two Sturmian shifts  $(Y_1, \sigma)$  and  $(Y_2, \sigma)$  in the full shifts  $(\{0, 1\}^{\mathbb{Z}}, \sigma)$  and  $(\{2, 3\}^{\mathbb{Z}}, \sigma)$  respectively. Then  $(Y_1, \sigma)$  and  $(Y_2, \sigma)$  are minimal and uniquely ergodic. Let  $\nu_1$  and  $\nu_2$  be the unique invariant measure of  $(Y_1, \sigma)$  and  $(Y_2, \sigma)$  respectively. Then both  $\nu_1$  and  $\nu_2$  have discrete spectrum. We can require that the spectra of  $\nu_1$  and  $\nu_2$  are different and then  $\nu_1$  and  $\nu_2$  are not isomorphic. Let  $Y = Y_1 \cup Y_2 \subset \{0, 1, 2, 3\}^{\mathbb{Z}}$ . It is clear that  $Y$  is zero dimensional and  $(Y, \sigma)$  has no periodic points.

By [6, Theorem 1] or [22, Theorem 1], there is a minimal system  $(Y, S)$  whose assignment is equivalent to that of  $(Y, \sigma)$ . This means that  $(X, T)$  carries exactly two ergodic measures,  $\mu_1$  and  $\mu_2$ , and  $(X, \mathcal{B}_X, \mu_i, T)$  is isomorphic to  $(Y_i, \mathcal{B}_{Y_i}, \nu_i, \sigma)$ . Let  $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ . As both  $\mu_1$  and  $\mu_2$  have discrete spectrum, so is  $\mu$ . By Proposition 4.5  $\mu$  has bounded complexity with respect to  $\{\bar{d}_n\}$ . But the ergodic measures in the ergodic decomposition of  $\mu$  are  $\mu_1$  and  $\mu_2$ , which are not isomorphic.  $\square$

**Remark B.2.** It should be noticed that the same idea of construction in Proposition B.1 can be used to provide countably many non-isomorphic ergodic measures in the ergodic decomposition, but the uncountably many case is still not clear.

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