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# A NEW STOCHASTIC MODEL FOR CARSHARING SUITED TO FREE-FLOATING

CHRISTINE FRICKER, HANENE MOHAMED, ALESSIA RIGONAT, AND MARTIN TRÉPANIER

**ABSTRACT.** Free-floating has an increasing popularity in carsharing but imbalance problem shows the need of stochastic modelling and analysis. In this paper, a new stochastic model suited to free-floating is proposed, taking into account the sharing of public space between private and free-floating cars. The capacity seen by free-floating cars in such a model turns out to be random. We show that unlike station-based car-sharing systems, it is not bounded. Moreover, a stochastic averaging principle governs the free-floating car behavior. We exhibit a phase transition between a non-saturated regime where free-floating cars can always be parked and a saturated one where free-floating cars do not find an available parking space with positive probability. This probability is completely determined by the environment which implies that the operator cannot act on the proportion of saturated areas, i.e. without parking spaces. We solve the dimensioning problem; the more free-floating cars in the system, the more satisfied users are.

**Keywords.** free-floating carsharing, stochastic model, mean-field approach, averaging principle, dimensioning problem.

## 1. INTRODUCTION

Over the last few decades, carsharing has emerged as an alternative mode of transportation that is ecological, economical and adapted to the architecture of large cities.

Carsharing exists in two forms. The first one is round-trip where the user needs to return the car at the departure point. It involves station-based systems, with stations with a fixed number of parking spaces (station capacities). The second one proposes one-way trips. The user picks up a car at a departure point, makes a trip and returns the car at destination. It can be either station-based or free-floating systems. They are more convenient and their popularity increases, especially free-floating ones.

In the round-trip systems, there is no interaction between stations since the parking space after departure remains reserved during the trip. Thus the only problem for the user is to find an available car. The one-way systems are facing the imbalance problem, with zones with high demand and no car and zones with low demand and many cars. Moreover, in contrast to round-trip systems, the users have to find both an available car and an available parking space. This issue is not only due to heterogeneity on demand. Even in a homogeneous system, the users should face this situation due to the randomness of their behavior. Managing such systems is not easy. Much work has been done to study of station-based carsharing systems in order to understand their behavior and improve their reliability. An issue is the dimensioning problem: how many cars would be needed per station on average to minimize the proportion of empty stations (with no available cars) or full stations (with no available parking spaces)?

To address these issues, stochastic models have been proposed to take into account the randomness of the system. The intuitive result is that, in the case of a simple homogeneous model for bike-sharing (or carsharing without reservation) systems, the stations should be roughly half full. Indeed, [4] proves that the optimal fleet size per station in this case is half the station capacity plus an additional term which is the *load* of the system, i.e. the product of the arrival rate of user demand per station by the average trip time. Other refined stochastic models that take into

account the heterogeneity of the system in for example [5], the impact of the reservation (of the car, the parking space or both of them) in [6] or the presence of bikes and e-bikes in [1] have been proposed and analyzed by probabilistic or mean field techniques. Moreover the bikesharing model motivates other analytical studies. In [12], others metrics motivated by bikesharing are studied. [2] deals with trip times under a phase-type distribution.

Another more recent form of carsharing without physical stations gains in popularity. Recall that, in such free-floating systems, cars are parked in public areas. The user picks up the car of his choice, generally closest to his geographical position, makes his trip and parks the car in an available public space around his destination. Free-floating offers more flexibility to the user and brings an answer to the problem of saturation at the destination. For [10], because of these structural differences with station-based carsharing systems, free-floating has the main part of journeys while station-based systems behaves as a complement.

But despite these differences, free-floating has always been analyzed in the literature as a station-based system where the service area is divided into small zones, from  $0.25 \text{ km}^2$  to  $1 \text{ km}^2$ , acting as stations [11].

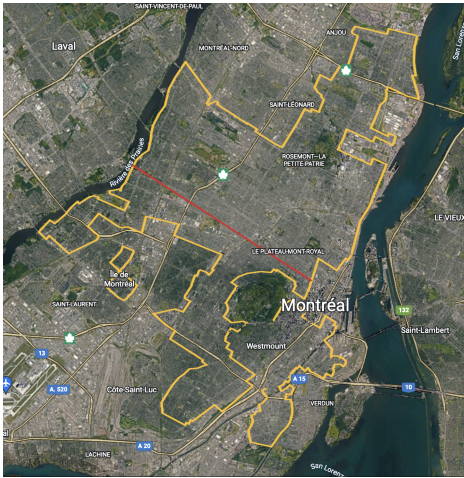


FIGURE 1. Commu-  
nauto service area,  
Montreal

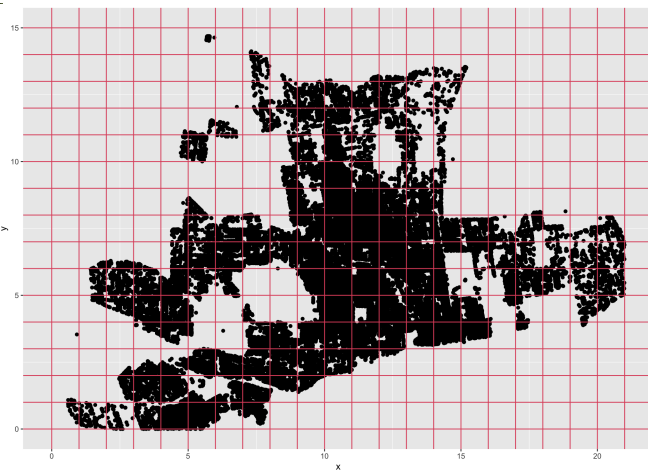


FIGURE 2. Recon-  
struction of Commu-  
nauto service area

As an example, for our case study on free-floating in Montreal, Figures 1 and 2 show the Communauto free-floating service area in Montreal. Figure 1 shows a capture of this zone with St. Laurent Boulevard in red as a landmark. In Figure 2, a reconstruction of this zone is made through GPS data of the positions of Communauto's free-floating cars. A rotation is made in order to make the Saint Laurent Boulevard coincide with the horizontal axis and a mesh is applied to the whole service area.

Such an approach allows to use a framework where results have been obtained. Nevertheless, this approach has some drawbacks. The parameters of such station-based model depend on the mesh which is used. In particular, the capacity of each zone can be defined as the number of available parking spaces added to the number of the free-floating cars parked in this zone. Thus, the choice of a fixed capacity as in a station-based system does no longer make sense. Indeed, for free-floating, the cars of the system share available places in public areas with the private cars. The number of private cars being random, this makes the residual capacity per zone seen by the free-floating cars as random. This key feature is specific to free-floating and cannot be captured

by the classical modeling of a free-floating system as a station-based system. Note that, for free-floating, the capacity can not be obtained by data analysis without data on public parking spaces in the service area.

The paper deals with free-floating carsharing. In this paper, our first objective is to propose a new model for carsharing systems suited to free-floating which take into account that the free-floating cars share public space with much more numerous private cars. Within this framework, our second objective is to measure the dissatisfaction of the user, i.e. the difficulty for the user both to find a car at departure and to return his car at destination. The third one is to solve in this framework the dimensioning problem, which is the optimal number of free-floating cars the operator must put to optimize the satisfaction of the user. Then the last objective is to present simulations which validate the results.

## 2. MATERIAL AND METHODS

**2.1. Modelling.** The first step is to consider an homogeneous framework. Let us clarify the main features of our new model suited to free-floating. The service area is divided into  $N$  zones, each of which has a fairly large overall parking capacity, say of the same order as  $N$ . The set of  $N$  zones is completely symmetric in terms of free-floating car demand (a random user demand with the same rate for each zone), destination choice (uniform among the  $N$  zones), and random reservation and trip time. The heterogeneity of the trip times of the users is modelled by a common distribution. Indeed it is now widely accepted now that in these models the moving free-floating cars are *indistinguishable*. carsharing cars coo-exist with a large number of private cars and share public parking space with them. Private cars arrive in each zone according to a Poisson process of parameter of the same order as  $N$ . This choice reflects the numerical imbalance between the private and free-floating cars. The interarrival times and parking times of private cars are assumed independent respectively identically distributed with exponential distribution. In conclusion, the parameters of the model do not depend on the specific zone.

This modeling aims to take into account the randomness of the residual capacity available to the carsharing system in the public space. The homogeneity is natural for a mean field approach. In the following, thanks to our methodology, the model could be modified with various behaviors. We will keep in our model that the motionless free-floating cars are either available or reserved. This shows how other more refined variants can also be modeled. Moreover, the model is robust enough to be generalized to a spatially heterogeneous setting, making all parameters dependent on the zone considered. Modeling via clusters remains suitable for a mean field approach, such as what was done for bike sharing systems by [3]. As for the temporal heterogeneity (peak and off-peak hours), it is sufficient to set the model parameters corresponding to the chosen time window. In this way, we obtain an approximation of the model behavior specific to this time window.

As far as we know, this is the first model of a free-floating carsharing system that does not reduce it to a station-based system. This point of view is original and seems much more relevant for the system than the classical approaches used up to now to study free-floating carsharing systems. This model takes into account the interactions between private and free-floating cars and highlights that the residual capacity (the parking spaces not occupied by private cars) *ignore* the free-floating cars.

**2.2. Mean field.** For carsharing, the mean-field approach applied to a set of  $N$  zones means that, when  $N$  becomes large, the states of two zones become independent and identically distributed. This classical result allows us to reduce the state of the whole system to that of a single zone, when the system is large enough. A nice queueing interpretation of the limiting state process of a zone (when  $N$  becomes large) allows us to obtain explicitly the stationary distributions of the number of available parking spaces, the number of available free-floating cars and the number of reserved cars. This simple expression of the steady-state behavior of the free-floating system is used for

the dimensioning problem. This means characterizing the optimal number of free-floating cars per zone, allowing to minimize the number of zones without both available cars and parking spaces.

**2.3. Averaging.** This new stochastic model takes into account the interaction between private and carsharing cars sharing public space. The number of carsharing cars is negligible compared to the total number of private ones that could share the service area (850 for Communauto Vs 800 000 private cars in Montreal). Thus intuitively, the private car behavior ignores the presence of free-floating cars. Moreover since the number of private cars is large, their evolution is fast and free-floating cars, arriving at a much smaller rate, see the number of private cars in average. Averaging has been extensively investigated in stochastic networks since [9], the first famous framework given by loss networks in [8]. Because of this, loss networks are always a key reference or conversely, due to the universality of dynamics of loss networks, averaging is present in a huge number of models. It is the case here. Mathematically, the study deals with the fast process of private cars compared to the slow process of carsharing vehicles. It allows to find the behavior of the free-floating cars in a large system although the evolution equation has discontinuities, due to the presence of private cars which condition available parking spaces. To understand the averaging principle, we study the processes at different time scales. Told in a different way, ignoring the sparse presence of free-floating cars, the private cars left a residual number of available parking spaces.

**2.4. Phase transition.** There are two regimes: One with a large number of available parking spaces (of the order of  $N$ ) and the free-floating cars always be parked. It is called *the Montreal regime*. The other one where private cars saturate the parking spaces of the free-floating service area and there is a positive probability that the free-floating cars can not be parked in the target zone and have to park in another zone. This regime is called *the Paris regime*. Note that in the latter case, the probability that the user has to accept to turn around his destination and park slightly away is an important parameter for the operator. This probability allows him to measure the discomfort suffered by the user in case of saturation.

### 3. RESULTS

**3.1. The model.** The service area is divided into  $N$  zones, each of them with a global capacity of order  $N$ , say equal to  $cN$  where  $c > 0$  is a fixed constant depending on the urban structure of the service area. Let  $M$  be the total number of free-floating cars in the system. The number  $M/N$  of free-floating cars per zone is denoted by  $s_N$  in the following. The dynamics of the model are as follows:

- Users of free-floating cars arrive at zone  $i$  ( $1 \leq i \leq N$ ) according to a Poisson process of rate  $\lambda$ . Each user reserves a free-floating car, if there is at least one available car. The reservation duration is random with an exponential distribution with parameter  $\eta$ . Otherwise, if no car is available in the zone  $i$ , he leaves the system looking for another means of transportation.
- After reservation, he starts a trip whose duration time has an exponential distribution with parameter  $\mu$ .
- At the end of the trip, the user chooses zone  $j \in \{1, \dots, N\}$  with probability  $1/N$ .
- If the user does not find a parking space in zone  $j$ , he starts a new trip with exponential distribution with the same parameter  $\mu$ . Notice that the destination choice is again made uniformly over the  $N$  zones.
- Private cars arrive at zone  $i$  according to a Poisson process of parameter  $\alpha N$ . This choice is consistent with the assumption that private cars outnumber the free-floating ones. The arrival rate of private cars in a zone is thus of order  $N$ , larger than that of free-floating cars, of order 1.
- Each private car stays in a zone for a random time with exponential distribution of parameter  $\beta$ .

- If there is no available parking space in the zone, the arriving private car leaves the system.

**3.2. The Markov process for the state of zones.** For a given zone  $i$  in  $\{1, \dots, N\}$ , we consider  $(X_i^N(t), m_i^N(t), V_i^N(t), R_i^N(t))$  a four component stochastic process such that, at time  $t$ ,

- $X_i^N(t)$  is the number of private cars parked in this zone,
- $m_i^N(t)$  is the number of available spaces in this zone,
- $V_i^N(t)$  is the number of *available* free-floating cars parked in this zone,
- $R_i^N(t)$  is the number of *reserved* free-floating cars parked in this zone.

Note that a distinction is made between available parked free-floating cars and those already reserved in order to obtain a Markovian framework. Moreover it is obvious that, at any time  $t \geq 0$  and for any zone  $i$ ,

$$X_i^N(t) + m_i^N(t) + V_i^N(t) + R_i^N(t) = cN.$$

Thus the state of a given zone  $i$  is reduced to the three component stochastic process  $(m_i^N(t), V_i^N(t), R_i^N(t))$  since  $X_i^N(t) = cN - m_i^N(t) - V_i^N(t) - R_i^N(t)$ .

The model presented here is as simple as possible. Variants can be studied. For example, we can consider that, if private cars do not find a parking space in the target zone, they could look for a parking space in another zone. And also we can imagine that, if free-floating users do not find an available car in the target zone, a proportion of them could look for one in another zone. Note also that, for sake of simplicity, we consider that all users make a reservation.

The state of the  $N$  zones is given by  $(m_i^N(t), V_i^N(t), R_i^N(t), 1 \leq i \leq N)$  which is a Markov process on state space

$$\left\{ (m_i, v_i, r_i)_{1 \leq i \leq N} \in \mathbb{N}^{3N}, \forall i \in \{1, \dots, N\}, m_i + v_i + r_i \leq cN, \sum_{i=1}^N v_i + r_i \leq M \right\}.$$

The transitions of this Markov process modify only the state of one zone. The transitions from  $((m_i, v_i, r_i), 1 \leq i \leq N)$  which change the state of zone  $i$  move  $(m_i, v_i, r_i)$  to

$$\begin{cases} (m_i + 1, v_i, r_i) & \beta(cN - m_i - v_i - r_i) \\ (m_i - 1, v_i + 1, r_i) & \frac{\mu}{N} \left( M - \sum_{l=1}^N (v_l + r_l) \right) \mathbf{1}_{m_i > 0} \\ (m_i, v_i - 1, r_i + 1) & \lambda \mathbf{1}_{v_i > 0} \\ (m_i + 1, v_i, r_i - 1) & \eta r_i \\ (m_i - 1, v_i, r_i) & \alpha N \mathbf{1}_{m_i > 0}. \end{cases}$$

and let the remaining  $((m_j, v_j, r_j), 1 \leq j \leq N, j \neq i)$  unchanged. The first transition occurs when a private car leaves zone  $i$  making a new parking space available. It happens at rate  $\beta x_i$ , i.e. at rate  $\beta(cN - m_i - v_i - r_i)$ . The second corresponds to the arrival of a free-floating car being available for users and taking a parking place. Note that there are  $M - v_1 - r_1 - \dots - v_N - r_N$  free-floating cars in move for a trip with exponential distribution with parameter  $\mu$ . For the moving free-floating cars, the probability to park in zone  $i$  is  $1/N$ . Therefore we get the rate associated to this transition. The third one happens when a user reserves an available free-floating car in zone  $i$ , it occurs at rate  $\lambda$ . This car remains parked but reserved at the same parking space. The fourth corresponds to the departure of a reserved free-floating car making a new parking space available in the zone. The last one corresponds to the arrival of a private car taking a parking space in zone  $i$ .

**3.3. Two regimes.** Intuitively, due to the homogeneity of the system, all zones have the same limit behavior. First we focus on the evolution of the number of available parking spaces  $(m_i^N(t))_{t \geq 0}$  in a given zone  $i$  in  $\{1, \dots, N\}$ . If the state of zone  $i$  is  $(m_i, v_i, r_i)$  at time  $t$ , the number of available parking spaces  $m_i$  increases by 1 at rate  $\beta(cN - m_i - v_i - r_i) + \eta r_i$  and decreases by 1 if  $m_i > 0$  at rate  $\alpha N + \mu(M - v_1 - r_1 - \dots - v_N - r_N)/N$ . Roughly speaking  $v_i$  and  $r_i$  are negligible compared

to  $cN$  since the sum  $v_1 + r_1 + \dots + v_N + r_N$  is less or equal to  $M$  of order  $N$  (recall that  $M/N$  tends to a constant  $s$ , the fleet size per station, called *the fleet size parameter*). Thus, for  $N$  large enough, the process  $(m_i(t))$  behaves like a birth and death process with birth rate  $\beta cN$  and death rate  $\alpha N$ . It is well known that such a birth-and-death process is ergodic and admits a unique invariant measure which is geometric with parameter  $\beta c/\alpha$  if  $\beta c/\alpha < 1$ , and is transient and tends to  $+\infty$  if  $\beta c/\alpha > 1$ . It is remarkable that the limiting number of available parking spaces does not depend on the carsharing parameters. Intuitively, it is due to the orders of magnitude of random variables, recalling that  $m_i^N(t) = cN - X_i^N(t) - V_i^N(t) - R_i^N(t)$ .

Thus we highlight two regimes with a phase transition when  $\beta c/\alpha$  is equal to 1 clearly identified in term of parameters related to the *environment*, i.e. the private cars and the urban characteristic of the service area (parameters  $\alpha$ ,  $\beta$  and  $c$ ). Intuitively,

- *The saturated regime:* When  $\beta c/\alpha < 1$ , the private cars occupy *all* the parking spaces in the zone which means a number equivalent to  $cN$  when  $N$  is large. In other words if the arrival rate of the private cars, i.e.  $\alpha N$ , exceeds the parking completion rate  $\beta cN$ , the zone will not have time to free a parking space before the arrival of a new private car. Indeed, as seen previously, the number of parking spaces is of order 1, with geometric distribution with parameter  $\beta c/\alpha$ . With probability  $1 - \beta c/\alpha$ , a free-floating car arriving in a zone can not be parked in this zone.
- *The unsaturated regime:* When  $\beta c/\alpha > 1$ , the private cars leave a proportion of order  $N$  of the parking spaces in the zone. As the number of free-floating cars are of order 1 in a zone, free-floating cars can be parked without any restriction. They always find an available parking space.

3.3.1. *The saturated regime.* It holds when  $\beta c/\alpha < 1$ . We present briefly the behavior.

**The mean-field approximation.** In this case, the free-floating cars face some positive probability  $1 - \beta c/\alpha > 0$  of not finding an available parking space (and thus looking for a space in another zone). This measures the difficulty (and not impossibility) to park at destination for a user of the free-floating carsharing system. This quantity depends only on the private car parameters (the *environment* for the free-floating system) and could be computed, provided by data on parking spaces in the service area and on the movement of private cars in this area.

Recall that the behavior of the large  $N$ -zone system (approximated by the mean-field limit) can be summarized by the behavior of a single zone, two zone behaviors becoming independent. The limiting free-floating state process of a zone is the limiting number of available free-floating cars and the limiting number of reserved cars  $(V(t), R(t))$ . It behaves like an inhomogeneous Markov process with transitions, from  $(v, r)$  at time  $t$ ,

$$(v, r) \rightarrow \begin{cases} (v-1, r+1) & \lambda \mathbf{1}_{v>0} \\ (v, r-1) & \eta r \\ (v+1, r) & \frac{\beta c}{\alpha} \mu (s - \mathbb{E}(V(t) + R(t))) \end{cases}$$

that is easy to understand intuitively. The couple  $(V(t), R(t))$  can be interpreted (see Figure ??) as two queues in tandem with

- the first one, a one-server queue with arrival rate  $(\beta c/\alpha) \mu (s - \mathbb{E}(V(t) + R(t)))$  at time  $t$  and service time  $\lambda$ ,
- and the second one, an infinite-server queue with service rate  $\eta$ .

Note that the  $\beta c/\alpha$  factor in the arrival rate of the first queue (the available free-floating cars) is interpreted as an acceptance probability since the free-floating cars have a probability  $1 - \beta c/\alpha$  of not finding a parking space in a given zone (rejection probability). It is as if private cars is thinning the return of free-floating cars with probability  $\beta c/\alpha$ . This replaces the role of the finite station capacity of the model usually proposed for station-based carsharing systems.

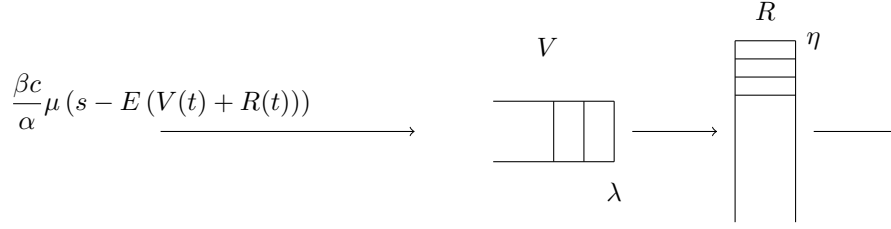


FIGURE 3. Dynamics of  $(V(t), R(t))$  as a tandem of two queues. The *horizontal* is a  $M/M/1$  queue, the *vertical* queue is  $M/M/\infty$  queue.

It can be summarized like this.

**Proposition 1** (Behavior of a zone in the saturated case). *When  $\beta c/\alpha < 1$ , when  $N$  gets large, after some time  $t_o$ , the number of available parking spaces at time  $t$  is stationary with geometric distribution with parameter  $\beta c/\alpha$ . The free-floating cars which return to a zone find no parking space with probability  $\beta c/\alpha$ . The joint numbers of available and reserved cars  $(V(t), R(t))$  behave after  $t_0$  behave like the inhomogeneous tandem of queues of Figure 3.*

**Steady-state behavior.** Using the product-form of the invariant measure of the underlying two-dimension Markov process (i.e. with queueing interpretation, the tandem of two queues), the steady-state of the state of a zone  $(V, R)$  is given by two independent random variables with respective distributions: a geometric <sup>1</sup> distribution with parameter

$$(1) \quad \rho = \frac{\beta c \mu}{\alpha \lambda} (s - \mathbb{E}(V + R))$$

and a Poisson distribution with parameter  $\lambda \rho/\eta$ . It is important to mention that the residual capacity  $m$  available for the free-floating cars, previously considered as a zone-specific constant and arbitrarily fixed, turns out to be unbounded in our modeling, random with a geometric distribution depending only on the environment parameters (private cars and characteristics of the zone).

Equation (1) is a fixed point equation on  $\rho$ . Indeed, using that  $V$  and  $R$  have respective geometric and Poisson distributions,  $\mathbb{E}(V) = \rho/(1 - \rho)$  and  $\mathbb{E}(R) = \lambda \rho/\eta$ . Thus, equation (1) can be rewritten as follows

$$\rho = \frac{\beta c \mu}{\alpha \lambda} \left( s - \frac{\rho}{1 - \rho} - \frac{\lambda \rho}{\eta} \right)$$

or equivalently

$$(2) \quad s = \frac{\rho}{1 - \rho} + \left( \frac{\lambda}{\eta} + \frac{\alpha \lambda}{\beta c \mu} \right) \rho.$$

It is clear that the right-hand side of equation (2) is a function of  $\rho$  defined on  $[0, +\infty[\setminus\{1\}]$ , strictly increasing on both intervals  $[0, 1[$  and  $]1, +\infty[$ . More precisely,  $s$  is strictly increasing from  $[0, 1[$  to  $[0, +\infty[$ , and from  $]1, +\infty[$  to  $] -\infty, +\infty[$ . In conclusion, for a fixed  $s$ , there is a unique solution  $\rho \in ]0, 1[$ . Moreover  $\rho$  is given explicitly. See Proposition 2. Indeed, equation (2) is a polynomial equation of second order in  $\rho$ , with two real solutions, and it can be straightforwardly checked that it has exactly one solution in  $]0, 1[$ .

This is summarized in the following result.

**Proposition 2** (Stationary mean-field limit). *When  $\beta c/\alpha < 1$ , the limit as  $N$  becomes large of the joint numbers of available and reserved free-floating cars  $(V(t), R(t))$  in a given zone has the*

<sup>1</sup>a probability measure  $\pi$  on  $\mathbb{N}$  such that,  $\pi(k) = (1 - \rho) \rho^k$ ,  $k \in \mathbb{N}$  with  $\rho \in ]0, 1[$ .



following stationary distribution

$$(3) \quad \text{geom}(\rho) \otimes \text{Poisson} \left( \frac{\lambda\rho}{\eta} \right)$$

where  $\rho$  is given by

$$(4) \quad \rho = \frac{A + s + 1 - \sqrt{(A + s + 1)^2 - 4sA}}{2A} \quad \text{with} \quad A = \lambda \left( \frac{1}{\mu} \frac{\alpha}{\beta c} + \frac{1}{\eta} \right).$$

It means that the limiting number of available and reserved free-floating cars as  $N$  is large at equilibrium are independent and with simple distributions with explicit parameters.

These two limits in  $N$  and  $t$  are not so easy to handle. For example, it is hard, even the possible limit is obtained, to prove the long term convergence of the mean-filed limit to its *equilibrium point*  $(V, R)$ . To obtain the long-time behavior of the state process of a zone, we use changes of time-scales. It is the aim of the following section.

**The different time-scales.** What follows is introduced in order to obtain the long-time behavior of the system when it gets large, by results on convergence of the processes. Roughly speaking, to investigate the stationary state as  $N$  gets large, it is convenient to accelerate time in  $Nt$ . Moreover to understand the behavior, the difference between the order of the rate of transitions concerning the numbers of private cars  $X^N(t)$  and free-floating  $V^N(t)$  lead us to investigate different timescales (standard in  $t$ , accelerated in  $Nt$  and slowed down in  $t/N$ ). The proofs are left to the paper [7].

The accelerated-time gives the simplest point of view. It allows to obtain the steady-state for large  $N$ .

**Proposition 3** (Accelerated time-scale behavior). *If  $\beta c/\alpha < 1$  and  $\mu < \lambda$ , for the convergence in distribution,*

- (1) *If  $X^N(0)/N \xrightarrow[N \rightarrow \infty]{d} x_0$  then  $(X^N(Nt)/N) \rightarrow_{N \rightarrow \infty} (c)$ .*
- (2) *For each  $t > 0$  fixed, as  $N \rightarrow \infty$ ,  $m^N(t) \rightarrow Y$  where  $Y$  is random variable on  $\mathbb{N}$  with geometric distribution of parameter  $\beta c/\alpha$ .*
- (3) *For each  $t > 0$  fixed, the random variables  $(V_1^N(Nt), R_1^N(Nt))$  converges as  $N$  gets large with a limiting distribution  $\text{geom} \otimes \text{Poisson}$  as previously explained.*

Considering the normal time scale, we capture that saturation by private holds after some time  $t_0$ . Indeed, in the standard time-scale ( $t \mapsto t$ ), the two processes  $(X^N(t))$  and  $(m^N(t))$  are both moving fast (at order  $N$ ) such that  $(m^N(t))$ , when  $N$  tends to  $+\infty$ , roughly speaking becomes of order 1 after some finite time  $t_0$ . They reach instantaneously (at fixed  $t > t_0$ ) an equilibrium for a fixed value  $v$  of the slow process  $(V^N(t))$  at  $t$ , moving at order 1. Recall that, when  $\beta c/\alpha < 1$ , the equilibrium distribution of the number of available places is geometric with parameter  $\beta c/\alpha$ . The following result holds for the saturated case.

**Proposition 4** (Standard time-scale behavior). *If  $\beta c/\alpha < 1$  and  $\mu < \lambda$ , for the convergence in distribution,*

- (1) *If  $X^N(0)/N \xrightarrow[N \rightarrow \infty]{d} x_0$  then process  $(X^N(t)/N)$  converges to a deterministic process given by*

$$\left( \frac{X^N(t)}{N} \right) \rightarrow_{N \rightarrow \infty} (x(t)) = \left( \frac{\alpha}{\beta} + \left( x_0 - \frac{\alpha}{\beta} \right) e^{-\beta t} \right) \wedge c.$$

- (2) *There exists  $t_0 > 0$  such that, for  $t > t_0$  fixed, as  $N \rightarrow \infty$ ,  $m^N(t) \rightarrow Y$  where  $Y$  is a random variable on  $\mathbb{N}$  with geometric distribution of parameter  $\beta c/\alpha$ .*
- (3)  *$(V_1^N(t), R_1^N(t))$  converges after  $t_0$  to the inhomogeneous process described by the tandem of queues.*

Considering a slow time-scale ( $t \mapsto t/N$ ), we only capture the dynamics of the fast process (private cars), the slow process (free-floating cars) being static in time.

**Theorem 1** (Slow time-scale behavior). *Assume  $\beta c/\alpha < 1$ . Given  $\lim_{N \rightarrow \infty} (m^N(0), V^N(0)) = (m_0, v_0)$ , the sequence of processes  $(m^N(t/N), V^N(t/N))$  converges in distribution to the process  $(v_0, L_m(t))$ , where  $L_m(t)$  is the number of customers in a  $M/M/1$  queue with arrival rate  $\beta c$  and service rate  $\alpha$ .*

**3.3.2. The unsaturated regime.** When  $\beta c/\alpha > 1$ , the number of parking spaces is of order  $N$  thus the probability for a free-floating car to find an available parking space is equal to 1. Recall that it means that the car is parked in the target zone. The free-floating car process of returns is not thinned by the mass of private cars. Thus, for the free-floating car in a zone, the results of the saturated case are valid replacing  $\beta c/\alpha$  by one. For example, in Proposition 2, just the expression of  $A$  is changed. It is written as follows.

**Proposition 5** (Stationary mean-field limit). *When  $\beta c/\alpha > 1$ , the limit as  $N$  becomes large of the joint numbers of available and reserved free-floating cars  $(V(t), R(t))$  in a given zone has the following stationary distribution*

$$(5) \quad \text{geom}(\rho) \otimes \text{Poisson} \left( \frac{\lambda \rho}{\eta} \right)$$

where  $\rho$  is given by

$$(6) \quad \rho = \frac{A + s + 1 - \sqrt{(A + s + 1)^2 - 4sA}}{2A} \quad \text{with} \quad A = \lambda \left( \frac{1}{\mu} + \frac{1}{\eta} \right).$$

**3.4. Application to dimensioning.** As in the case of station-based carsharing systems, users of free-floating carsharing systems are faced with the problem of finding available cars and parking spaces throughout the service area. To address this major issue, the first question for the operator is the dimensioning problem. The aim is to find the optimal total number of free-floating cars  $sN$  to put in circulation which corresponds to a minimum proportion of zones without cars or parking spaces.

Note that, in contrast with the case of a station-based carsharing system, the limit proportion of zones without available parking spaces given by  $1 - \beta c/\alpha$  depends only on the parameters of the environment (private cars and characteristics of the zone). The operator cannot therefore act on this proportion by varying the dimensioning parameter  $s$ . He can only act on the reduction of the number of zones where there is a lack of cars.

For free-floating carsharing systems, the fact that there is a permanent lack of cars is real. It is confirmed by our theoretical study. This limiting proportion of car-free zones, denoted by  $P_0$ , is given by  $P_0 = \mathbb{P}(V = 0) = 1 - \rho$ . This limiting proportion  $P_0$  is a decreasing function of  $s$  since equation (2) shows that  $s$  is a strictly increasing function of  $\rho$ . Therefore, the more the operator increases the number of free-floating cars per zone, the better the system performs since the probability that a user do not find an available free-floating car in a given zone becomes smaller and smaller.

**3.5. Validation by simulation.** We implemented a simulator that reproduces the dynamics of a free-floating system of  $M$  cars interacting with private cars and  $N$  stations each of capacity  $cN$ . The simulator reproduces the movement of cars following the laws of our model. In particular, it relies on generating random numbers from the exponential distribution to simulate the inter-arrival times of the Poisson processes. Two fundamental properties of the exponential distribution are the lack of memory and the fact that, the distribution of  $Y = \min\{X_1, \dots, X_n\}$  where  $X_1, \dots, X_n$  are independent random variables exponentially distributed with parameters  $\lambda_1, \dots, \lambda_n$ , is again exponential with parameter  $\lambda = \lambda_1 + \dots + \lambda_n$ . We report page 11 the algorithm used for the simulations.

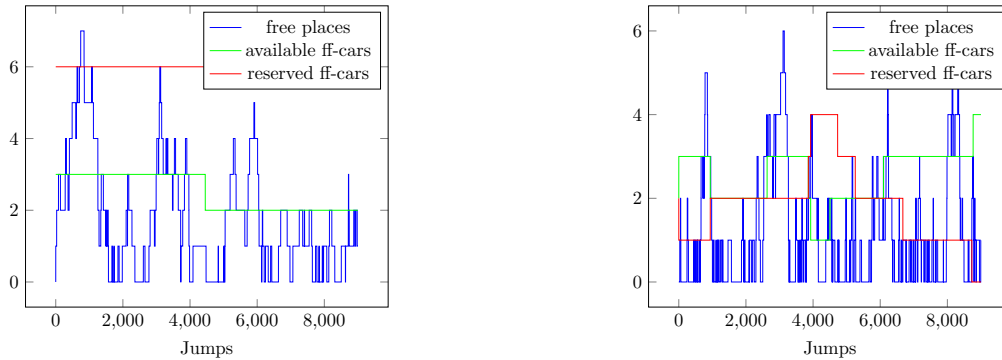


FIGURE 4. The evolution in time of the number of empty places, available and reserved FF-cars in a randomly chosen station. The figure on the left is for  $N = 50$ , on the right for  $N = 25$ .

Plotting in Figure 4 the evolution in time of the number of empty spots, available and reserved free-floating cars in a randomly chosen station, we can state what happens when the number of zones grows, i.e. when  $N$  doubles from 25 to 50. Indeed, for  $N = 25$  we can still detect jumps of the processes related to free-floating cars. But for  $N = 50$ , the number of jumps for the same processes is much smaller, approximately  $1/4$  than before. This is due to the fact that as the number of zones doubles, the jumps of the free-floating cars are spread over a larger number of zones.

In Figure 5, we plot the distribution of the number of available parking spaces and available free-floating cars and we recognize the shape of the geometric distribution. An additional verification is performed plotting the obtained distribution in logarithmic scale and observing that what we obtain is a straight line. To check that the number of reserved free-floating cars follows a Poisson distribution, we plot on the same axes the probability mass function of the Poisson with theoretical parameters and the distribution obtained by simulation. The two curves are overlapping with a small error.

#### 4. CONCLUSION AND FUTURE WORK

Until now, free-floating carsharing systems have been modeled as station-based systems with zones considered as stations with fixed capacities. But this model is not relevant since free-floating cars share the same parking spaces with private cars inducing fluctuations of random *capacities* left to the carsharing system. Moreover private cars are numerous compared to free-floating cars, ignoring intuitively their presence and behaving in our model roughly independently in each zone as an  $M/M/cN/cN$  queue. This is the simplest loss system studied. See [13][Chapter 6]. Because of the difference in order of magnitude, a stochastic averaging principle naturally governs the free-floating car behavior.

We propose a new model for carsharing specific to free-floating that we are able to analyze. The study of this model leads to the mobilisation of several probabilistic techniques, where mean-field and stochastic averaging are combined. The study of stochastic averaging principle in large systems is original, as far as we know. All technical details and theoretical arguments of the proofs are presented in [7] in preparation.

A phase transition occurs. Over a critical value for a quantity related to the *environment*, i.e. the private cars and the public space size, the private cars saturate the public space. And because they are more numerous, they prevent the free-floating cars to find a parking space with some probability. The steady-state behavior when  $N$  is large of the non-moving free-floating cars,

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**Algorithm 1** Simulate the dynamics of the model
 

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 $t \leftarrow 0$ 
 $L \leftarrow M - \sum_l (R_l + V_l)$ 
 $t_\eta \leftarrow \exp(\eta R_i, N)$ 
 $t_\alpha \leftarrow \exp(\alpha N, N)$ 
 $t_\beta \leftarrow \exp(\beta X_i, N)$ 
 $t_\lambda \leftarrow \exp(\lambda, N)$ 
 $t_\mu \leftarrow \exp(\mu, L)$ 
while number of jumps  $\leq N_j$  do
   $t \leftarrow \min(t_\eta, t_\lambda, t_\mu, t_\alpha, t_\beta)$ 
  if  $t$  is in  $t_\eta$  then
     $i \leftarrow$  station where  $t_\eta = t$ 
     $(m_i, V_i, R_i) \leftarrow (m_i + 1, R_i - 1, V_i)$ 
     $\omega_\mu \leftarrow t + \exp(\mu)$ 
    append  $\omega_\mu$  to  $t_\mu$ 
     $t_{\eta,i} \leftarrow t + \exp(\eta R_i)$  (set  $t_{\eta,i} \leftarrow \infty$  if  $R_i = 0$ )
  end if
  if  $t$  is in  $t_\mu$  then
     $i \leftarrow \mathcal{U}(\{1, \dots, N\})$ 
    if  $m_i > 0$  then
       $(m_i, V_i, R_i) \leftarrow (m_i - 1, R_i, V_i + 1)$ 
      remove  $t$  from  $t_\eta$ 
    else  $\{m_i = 0\}$ 
       $\omega_\mu \leftarrow t + \exp(\mu)$ 
      substitute  $t$  with  $\omega_\mu$  in  $t_\mu$ 
    end if
  end if
  if  $t$  is in  $t_\lambda$  then
     $i \leftarrow$  station where  $t_\lambda = t$ 
     $t_{\lambda,i} \leftarrow t + \exp(\lambda)$ 
    if  $V_i > 0$  then
       $(m_i, V_i, R_i) \leftarrow (m_i, R_i + 1, V_i - 1)$ 
       $t_{\eta,i} \leftarrow t + \exp(\eta R_i)$ 
    end if
  end if
  if  $t$  is in  $t_\alpha$  then
     $i \leftarrow$  station where  $t_\alpha = t$ 
     $t_{\alpha,i} \leftarrow t + \exp(\alpha N)$ 
    if  $m_i > 0$  then
       $(m_i, V_i, R_i) \leftarrow (m_i - 1, R_i, V_i)$ 
       $X_i \leftarrow cN - m_i - V_i - R_i$ 
       $t_{\beta,i} \leftarrow t + \exp(\beta X_i)$ 
    end if
  end if
  if  $t$  is in  $t_\beta$  then
     $i \leftarrow$  station where  $t_\beta = t$ 
     $(m_i, V_i, R_i) \leftarrow (m_i + 1, R_i, V_i)$ 
     $X_i \leftarrow cN - m_i - V_i - R_i$ 
     $t_{\beta,i} \leftarrow t + \exp(\beta X_i)$  (set  $t_{\beta,i} \leftarrow \infty$  if  $X_i = 0$ )
  end if
end while

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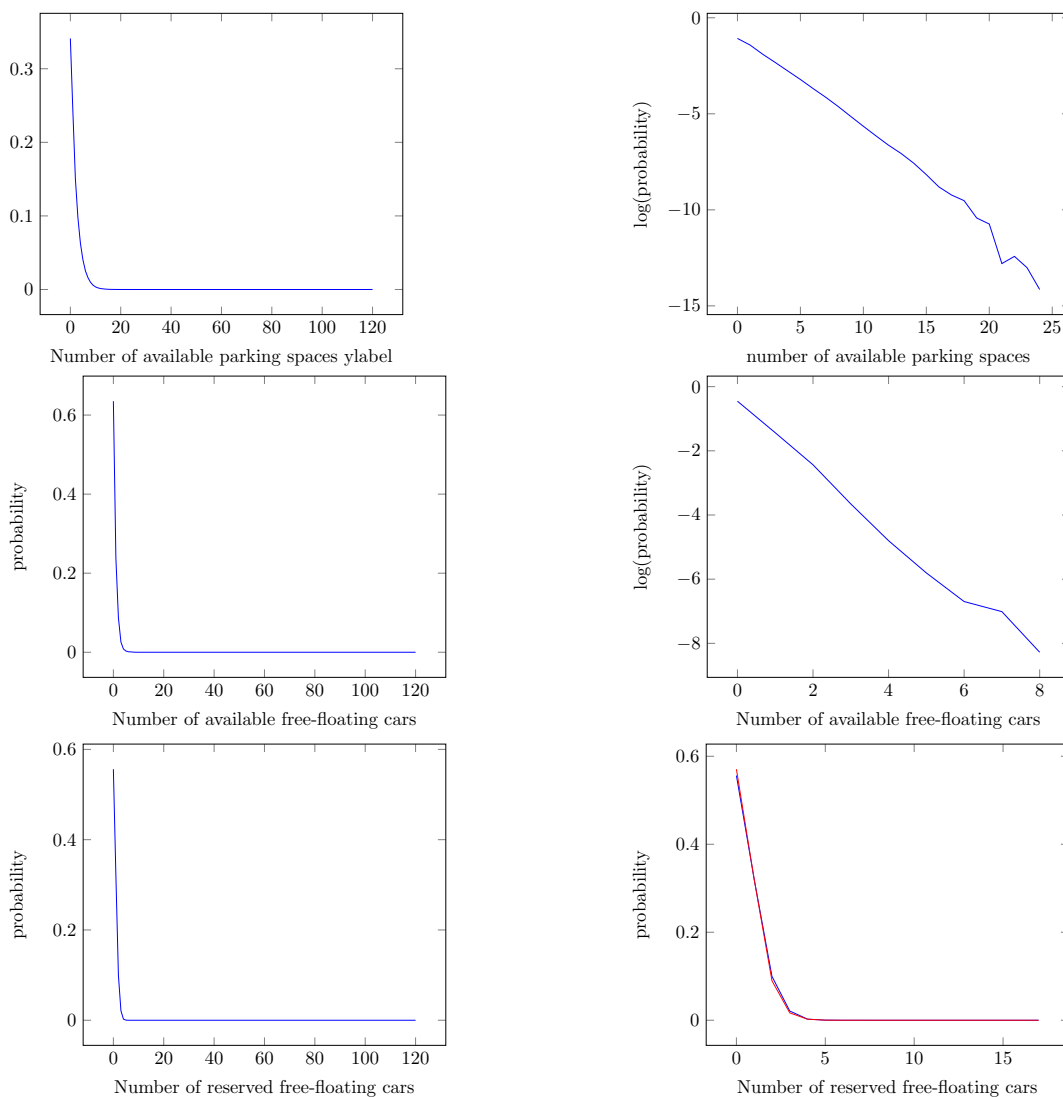


FIGURE 5. Estimation by simulation of the distribution of the number of available parking spaces, available and reserved free-floating cars in a given zone, obtained by simulation for  $N = 100$  with parameters  $\lambda = 1.5$ ,  $\mu = 1$ ,  $\alpha = 1.8$ ,  $\beta = 1$ ,  $\eta = 1$ ,  $c = 1.2$ ,  $s = 2$  (saturated regime). The respective distributions are quite close to two geometric and a Poisson distributions.

available or reserved, can be derived in a quite simple explicit form (cf Proposition 2 and 5). The technique for obtaining the long-term behavior is to accelerate the time  $t$  to  $Nt$  and let  $N$  tend to infinity. See the paper [7] for details.

As a byproduct, the paper proves the intuitive answer to the dimensioning problem for free-floating. The lack of parking spaces is not governed by free-floating carsharing but just the *environment*. The operator act only on providing cars to minimize the lack of available free-floating cars. Thus the most cars the most satisfaction for the user.

To validate our results and investigate further more realistic distributions, especially for trip times of free-floating cars and parking times of private cars, a simulator is currently written. The

algorithm is presented here and results from simulation for exponential distributions are presented in the paper. It can also manage general distributions. These distributions could be provided by data analysis of a city case-study.

Extensions could be made for more complicated dynamics of the private cars where they look for a parking space in another zone when there is no parking space in the target zone. This induces another type of interactions between zones. It is a work in progress. This work also opens many avenues for future research.

Finally, it should also be pointed out that, although free-floating carsharing networks are booming worldwide and address crucial sustainable mobility issues in the urbanized world of the 21st century, the techniques as mean-field approach and stochastic averaging principle employed here also have numerous applications in the field of interacting systems.

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(C. FRICKER) DI-ENS, CNRS, INRIA DE PARIS, PSL RESEARCH UNIVERSITY, PARIS, FRANCE  
*E-mail address:* [christine.fricker@inria.fr](mailto:christine.fricker@inria.fr)

(H. MOHAMED) MODAL’X, UMR CNRS 9023, UPL, UNIVERSITÉ PARIS NANTERRE, NANTERRE, FRANCE  
*E-mail address:* [hmrاد@parisnanterre.fr](mailto:hmrاد@parisnanterre.fr)

(A. RIGONAT) DI-ENS, CNRS, INRIA DE PARIS, PSL RESEARCH UNIVERSITY, PARIS, FRANCE

(M. TRÉPANIÉ) MATHEMATICAL AND INDUSTRIAL ENGINEERING, POLYTECHNIQUE MONTREAL, MONTREAL, CANADA,  
 H3T 1J4  
*E-mail address:* [martin.trepanier@polymtl.ca](mailto:martin.trepanier@polymtl.ca)