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A LOCAL-GLOBAL PRINCIPLE FOR TWISTED FLAG VARIETIES

P. GILLE AND R. PARIMALA

ABSTRACT. We prove a local-global principle for twisted flag varieties over a semiglobal field.

Keywords: Local-global principle, curves over local fields, homogeneous varieties, reductive groups.

MSC 2000: 11G99, 14G99, 14G05, 11E72, 11E12, 20G35

1. INTRODUCTION

Let T be a complete discrete valuation ring with fraction field K and residue field k . Let X be a smooth, projective, geometrically integral curve over K . Let $F = K(X)$ be the function field of X and let t be an uniformizing parameter of T . We prove the following theorem which settles a conjecture of Colliot-Thélène, Suresh and the second author for function fields of p -adic curves [C-T-P-S, conjecture 1], in the very general context of semiglobal fields.

Theorem 1.1. *(see Th. 5.4) Let G be a reductive F -group and assume that p does not divide the order of the automorphism group of the absolute root system of G_{ad} . Let Z be a twisted flag F -variety of G . Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from models of X .*

The result was known for the cases of smooth projective quadrics and Severi-Brauer varieties [C-T-P-S, th. 3.1] and also for generalized Severi-Brauer varieties (Reddy-Suresh [R-S, th. 2]).

If we use all rank one valuations, the result holds unconditionnally (Cor. 4.8). On the other hand, if G extends to a reductive group scheme over a smooth model \mathfrak{X} of X , the result is unconditional (Cor. 4.6).

Let us review the contents. Given a henselian couple (A, I) and a reductive A -group scheme H , section 2 deals with generation of groups $H(A)$ by unipotent elements and also with the quotient of $H(A)/RH(A)$ by the (normal) subgroup of R -trivial elements defined in [G-S]. Section 3 is devoted to mild improvements of patching techniques of Harbater-Hartmann-Krashen involving more analytical functions and leads to various group decompositions (prop. 3.7); this permits to obtain a patching

theorem for twisted flag varieties (th. 3.9). Section 4 deals with the setting of the beginning of the introduction and provides a weaker version of the main result. What remains to do is mostly a local study for a two dimensional complete regular local ring R with parameters t, s and torsors over its localization $R[t^{-1}, s^{-1}]$. This is achieved in section 5 by making use of the loop torsors over $R[t^{-1}, s^{-1}]$ and of the results of [Gi2].

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Conventions and notation. (a) A variety V over a field F means a separated F -scheme of finite type which is integral [St, Tag 020D].

(b) Let G be a reductive F -algebraic group and let F_s be a separable closure of F . Let (B, T) be a Killing couple of G_{k_s} and denote $\Delta(G_{F_s}) = \Delta(G_{F_s}, B, T)$ the associated Dynkin diagram (there is a canonical bijection $\Delta(G_{F_s}, B, T) \xrightarrow{\sim} \Delta(G_{F_s}, B', T')$ for another choice of Killing couple). This diagram $\Delta(G_{F_s})$ is equipped with the star action of $\text{Gal}(F_s/F)$ [B-T, §6.4]. We recall that there is an order preserving bijection $I \rightarrow P_I$ between the subsets of $\Delta(G_{F_s})$ and the F_s -parabolic subgroups of G_{F_s} containing B (or equivalently with the $G(F_s)$ -conjugacy classes of parabolic subgroups of G_{F_s}). Since minimal F -parabolic subgroups are $G(F)$ -conjugated we denote by $\Delta_0(G) \subset \Delta(G_{F_s})$ be conjugacy class of $P_0 \times_F F_s$ where P_0 is a minimal F -parabolic subgroup of G (it is stable under the star action). The triple $(\Delta(G_{F_s}), \text{star action}, \Delta_0(G))$ is called the Tits index of G . By abuse of notation, $\Delta_0(G)$ alone is called the Tits index. By a twisted F -flag variety of G , we mean a G -variety X which becomes isomorphic over F_s to some G_{F_s}/P_I as G_{F_s} -variety (such an I is unique). According to [M-P-W, prop. 1.3], I is invariant under the star action and X is G -isomorphic to the variety of F -parabolic subgroups of type I which is denoted in this paper by $\text{Par}_I(G)$.

Those varieties are called sometimes projective homogeneous G -varieties (as in [C-T-P-S]) but we warn the reader of the danger of this terminology since there exist in positive characteristic F -subgroups Q which are not smooth such that G/Q is a projective F -variety [W]. In the appendix 6, we show that a homogeneous G -variety X is a variety of parabolic subgroups if and only if the action is transitive in the sense that $G(E)$ acts transitively on $X(E)$ for each F field E (Prop. 6.2).

(c) We use mainly the terminology and notation of Grothendieck-Dieudonné [EGAI, §9.4 and 9.6] which agrees with that of Demazure-Grothendieck used in [SGA3, Exp. I.4]. Let S be a scheme and let \mathcal{E} be a quasi-coherent sheaf over S . For each morphism $f : T \rightarrow S$, we denote by $\mathcal{E}_T = f^*(\mathcal{E})$ the inverse image of \mathcal{E} by the morphism f . We denote by $\mathbf{V}(\mathcal{E})$ the affine S -scheme defined by $\mathbf{V}(\mathcal{E}) = \text{Spec}(\text{Sym}^\bullet(\mathcal{E}))$; it is affine over S and represents the S -functor $Y \mapsto \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)$ [EGAI, 9.4.9]. We

assume now that \mathcal{E} is locally free of finite rank and denote by \mathcal{E}^\vee its dual. In this case the affine S -scheme $\mathbf{V}(\mathcal{E})$ is of finite presentation (ibid, 9.4.11); also the S -functor $Y \mapsto H^0(Y, \mathcal{E}_Y) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)$ is representable by the affine S -scheme $\mathbf{V}(\mathcal{E}^\vee)$ which is also denoted by $\mathbf{W}(\mathcal{E})$ [SGA3, I.4.6].

2. AROUND KNESER-TITS' PROBLEM

Let A be a semilocal ring, I its radical.

2.1. Subgroups of elementary elements. Let H be a reductive A -group scheme which is strictly A -isotropic, that is, all factors of H_{ad} are isotropic. Let H be a strictly proper parabolic A -subgroup of H and let P^- be a A -parabolic subgroup opposite to P . The subgroup of $H(A)$ generated by $R_u(P)(A)$ and $R_u(P^-)(A)$ does not depend of the choice of P and P^- , this is a normal subgroup of $H(A)$ which is denoted by $H(A)^+$ [P-S, Thm. 1 and comments].

Lemma 2.1. *We have $H(A) = H(A)^+ P(A)$.*

Proof. We put $U = R_u(P)$, $U^- = R_u(P^-)$. According to [SGA3, XXVI.5.2], we have a decomposition $H(A) = U(A)U^-(A)P(A)$ whence $H(A) = H(A)^+ P(A)$. \square

A special case of [G-S, prop. 7.7] is the following.

Lemma 2.2. *Assume that (A, I) is an henselian couple and that H is semisimple simply connected. Then the map $H(A)/H(A)^+ \rightarrow H(A/I)/H(A/I)^+$ is an isomorphism.*

2.2. R-equivalence. We assume now that A is a nonzero finite k -algebra for a field k . Then A is an Artinian k -algebra. We denote by I its Jacobson radical and remind the reader that (A, I) is an henselian couple.

Let H be an A -reductive group scheme and consider the Weil restriction $G = R_{A/k}(H)$, this is a smooth affine connected algebraic k -group [C-G-P, A.5.9]. We have $G(k) = H(A)$. We use R -equivalence for group schemes over rings as defined in [G-S].

Lemma 2.3. *We decompose $A = A_1 \times \cdots \times A_d$ where each A_i is a local artinian k -algebra of residue field k_i .*

(1) *The map $H(A) \rightarrow \prod_i H(k_i)$ induces isomorphisms*

$$G(k)/RG(k) \xrightarrow{\sim} H(A)/RH(A) \xrightarrow{\sim} \prod_i H(k_i)/RH(k_i).$$

(2) *If H is furthermore simply connected and strictly isotropic, we have a commutative diagram of isomorphisms*

$$\begin{array}{ccc}
H(A)/H(A)^+ & \xrightarrow{\sim} & H(A)/RH(A) \xleftarrow{\sim} G(k)/RG(k) \\
\downarrow \wr & & \downarrow \wr \\
\prod_i H(k_i)/H(k_i)^+ & \xrightarrow{\sim} & \prod_i H(k_i)/RH(k_i).
\end{array}$$

Proof. (1) The first isomorphism $G(k)/RG(k) \xrightarrow{\sim} H(A)/RH(A)$ is a formal fact [G-S, lemma 2.4]. Let n be the smallest positive integer such that $I^n = 0$. We put $A_j = A/I^{j+1}$ for $j = 0, \dots, n-1$, it comes with the ideal $J_j = I^j/I^{j+1} = I A_j$ which satisfies $J_j^2 = 0$. We have $A_j/J_j = A_{j-1}$ for $j = 1, \dots, n-1$.

We put $G_j = R_{A_j/k}(H_{A_j})$ for $j = 0, \dots, n-1$; we have a sequence of homomorphisms $G = G_n \rightarrow G_{n-1} \rightarrow \dots \rightarrow G_1 \rightarrow G_0 = \prod_i R_{k_i/k}(H_{k_i})$. It induces homomorphisms

$$G(k)/R = G_n(k)/R \rightarrow G_{n-1}(k)/R \rightarrow \dots \rightarrow G_1(k)/R \rightarrow G_0(k)/R = \prod_i H(k_i)/R$$

and we will show by a dévissage argument that all of them are isomorphisms.

Let j be an integer satisfying $1 \leq j \leq n-1$. According to a variant of [G-P-S, Lemma 8.3], we have an exact sequence of fppf (resp. étale, Zariski) sheaves on $\text{Spec}(k)$

$$0 \rightarrow \mathbf{W}\left((t_j)_*(\text{Lie}(H)(A_{j-1}) \otimes_{A_{j-1}} J_j)\right) \rightarrow R_{A_j/k}(H_{A_j}) \rightarrow R_{A_{j-1}/k}(H_{A_{j-1}}) \rightarrow 1.$$

where $t_j : \text{Spec}(A_j) \rightarrow \text{Spec}(k)$ is the structural morphism. We have then a sequence of k -algebraic groups

$$0 \rightarrow (\mathbb{G}_a)^{m_j} \rightarrow G_j \rightarrow G_{j-1} \rightarrow 1.$$

The map $G_j \rightarrow G_{j-1}$ is a $(\mathbb{G}_a)^{m_j}$ -torsor which is trivial since G_{j-1} is affine. We have then a decomposition of k -schemes $G_j \xrightarrow{\sim} G_{j-1} \times_k (\mathbb{G}_a)^{m_j}$ which induces an isomorphism $G_j(k)/R \rightarrow G_{j-1}(k)/R$. Thus the map $G(k)/R \rightarrow G_0(k)/R = \prod_i H(k_i)/R$ is an isomorphism as desired.

(2) We have $A/I = k_1 \times \dots \times k_d$. Lemma 2.2 provides an isomorphism $H(A)/H(A)^+ \xrightarrow{\sim} H(A/I)/H(A/I)^+ = \prod_i H(k_i)/H(k_i)^+$ and we have an isomorphism $H(k_i)/H(k_i)^+ \xrightarrow{\sim} H(k_i)/RH(k_i)$ for each i [Gil, thm. 7.2]. This completes the proof by chasing diagram. \square

3. PATCHING, R-EQUIVALENCE AND TWISTED FLAG SCHEMES

3.1. Using the implicit function theorem. We consider a variation of the framework of [H-H-K, 2.4]. Let T be a complete DVR of fraction field K and t be an uniformizing parameter. Let \widehat{R}_0 be ring containing T and which is also a complete discrete valuation ring having uniformizer t . We denote by F_0 the fraction field of \widehat{R}_0 .

Let $\alpha > 1$ be a real number and we define the absolute value on F_0 by $|t^n u| = \alpha^{-n}$ for $n \in \mathbb{Z}$ and $u \in (\widehat{R}_0)^\times$.

Let F_1, F_2 be subfields of F_0 containing T . We further assume that we are given t -adically complete T -submodules $V \subset F_1 \cap \widehat{R}_0$ and $W \subset F_2 \cap \widehat{R}_0$ satisfying the following conditions:

$$(3.1) \quad V + W = \widehat{R}_0;$$

$$(3.2) \quad V \cap t\widehat{R}_0 = tV \quad \text{and} \quad W \cap t\widehat{R}_0 = tW.$$

Note that Condition (3.2) is equivalent to

$$(3.3) \quad V \cap t^n \widehat{R}_0 = t^n V \quad \text{and} \quad W \cap t^n \widehat{R}_0 = t^n W \quad \text{for each } n \geq 1.$$

Remark 3.1. Condition (3.2) above is added there compared with [H-H-K, 2.4] but we do not require at this stage that F_1 is dense in F_0 .

We equip the submodules $V[\frac{1}{t}]$ of F_0 of the induced metric (and similarly for $W[\frac{1}{t}]$).

Lemma 3.2. (1) V is closed in \widehat{R}_0 .

(2) For $v \in V[\frac{1}{t}] \setminus \{0\}$, we have

$$|v| = \text{Inf}\{\alpha^n \mid t^n v \in V\}.$$

(3) We have $V = \text{Inf}\{v \in V[\frac{1}{t}] \mid |v| \geq 0\}$ and V is a clopen submodule of $V[\frac{1}{t}]$.

(4) $V[\frac{1}{t}]$ is closed in F_0 and is a Banach K -space.

Proof. (1) Our assumption is that the map $V \rightarrow \varprojlim V/t^{m+1}V$ is an isomorphism.

Let (x_n) be a sequence of V which converges in \widehat{R}_0 . For each $m \geq 0$, condition (3.3) shows that the map $V/t^{m+1}V \rightarrow \widehat{R}_0/t^{m+1}\widehat{R}_0$ is injective so that the sequence (x_n) modulo $t^{m+1}V$ is stationary to some $v_m \in V/t^{m+1}V$. The v_m 's define a point v of V and the sequence (x_n) converges to v .

(2) We are given $v = t^m v' \in V[\frac{1}{t}]$ with $v' \in V \setminus tV$ and we have $|v| = \text{Inf}\{\alpha^n \mid t^n v \in \widehat{R}_0\} = \alpha^m \text{Inf}\{\alpha^n \mid t^n v' \in \widehat{R}_0\}$. Condition (II) implies that $v' \in \widehat{R}_0 \setminus t\widehat{R}_0$ so that $|v| = 0$. We conclude that $|v| = \text{Inf}\{\alpha^n \mid t^n v \in V\}$.

(3) It readily follows from the assertion (2).

(4) Let (x_n) be a sequence of $V[\frac{1}{t}]$ converging to some $x \in F_0$. We want to show that x belongs to $V[\frac{1}{t}]$ so that we can assume that $x \neq 0$ and that $|x_n| = |x| = \alpha^m$ for all $n \geq 0$. Assertion (2) shows that $t^m x_n$ is a sequence of V and (1) shows that its limit $t^m x$ belongs to V . Thus $x \in V[\frac{1}{t}]$. We have shown that $V[\frac{1}{t}]$ is closed in F_0 .

Finally since F_0 is a Banach K -space so is $V[\frac{1}{t}]$. □

The following statement extends partially [H-H-K, th. 2.5] and [H-H-K2, prop. 4.1].

Proposition 3.3. *Let a, b, c be positive integers. Let $\Omega \subset (F_0)^a \times (F_0)^b$ be an open neighborhood of $(0, 0)$ and let $f : \Omega \rightarrow (F_0)^c$ be an analytic map. We denote by $f^a : \Omega \cap (F_0)^a \rightarrow (F_0)^c$ and $f^b : \Omega \cap (F_0)^b \rightarrow (F_0)^c$. We assume that*

$$(i) \ f(0, 0) = 0$$

(ii) *the differentials $Df_0^a : (F_0)^a \rightarrow (F_0)^c$ and $Df_0^b : (F_0)^b \rightarrow (F_0)^c$ satisfy*

$$Df_0^a \left(V\left[\frac{1}{t}\right]^a \right) + Df_0^b \left(V\left[\frac{1}{t}\right]^b \right) = (F_0)^c.$$

Then there is a real number $\epsilon > 0$ such that for all $y \in (F_0)^c$ with $|y| \leq \epsilon$, there exist $v \in V^a$ and $w \in W^b$ such that $(v, w) \in \Omega$ and $f(v, w) = y$.

Proof. We consider the continuous embedding $i : V\left[\frac{1}{t}\right]^a \times W\left[\frac{1}{t}\right]^b \rightarrow (F_0)^a \times (F_0)^b$ and define $\tilde{\Omega} = i^{-1}(\Omega)$ and the function $\tilde{f} = f \circ i : \tilde{\Omega} \rightarrow (F_0)^c$.

Claim 3.4. *The map \tilde{f} is strictly differentiable at $(0, 0)$ and $D\tilde{f}_{(0,0)} : V\left[\frac{1}{t}\right]^a \times W\left[\frac{1}{t}\right]^b \rightarrow (F_0)^c$ is onto.*

Since f is F_0 -analytic at $(0, 0)$, it is strictly differentiable [Sc, I.5.6], that is, there exists an open neighborhood Θ of $(0, 0)$ and a positive real number β such that

$$|f(x_2) - f(x_1) - Df_{(0,0)} \cdot (x_2 - x_1)| \leq \beta |x_2 - x_1| \quad \forall x_1, x_2 \in \Theta.$$

It is then strictly derivable as function between the Banach K -spaces $(F_0)^a \times (F_0)^b \rightarrow (F_0)^c$. On the other hand the embedding $i : V\left[\frac{1}{t}\right]^a \times W\left[\frac{1}{t}\right]^b \rightarrow (F_0)^a \times (F_0)^b$ is 1-Liftschitz so is strictly differentiable at $(0, 0)$. As composite of strictly differentiable functions, \tilde{f} is strictly differentiable at $(0, 0)$ [B, §1.3.1]. Furthermore the differential $D\tilde{f}_{(0,0)}$ is the composite of

$$V\left[\frac{1}{t}\right]^a \times W\left[\frac{1}{t}\right]^b \xrightarrow{i} (F_0)^a \times (F_0)^b \xrightarrow{Df_0^a + Df_0^b} (F_0)^c.$$

Condition (ii) says exactly that $D\tilde{f}_{(0,0)}$ is surjective. The Claim is proven.

We apply the implicit function theorem to the function \tilde{f} [B, §1.5.2] (see [Sc, §4] for concocting a proof). Lemma 3.2.(4) shows that $V\left[\frac{1}{t}\right]^a \times W\left[\frac{1}{t}\right]^b$ is a Banach K -space and so is F_0 . There exists then an open neighborhood $\Upsilon \subset \tilde{\Omega}$ of $(0, 0)$ in $V\left[\frac{1}{t}\right]^a \times W\left[\frac{1}{t}\right]^b$ such that $\tilde{f}|_{\Upsilon}$ is open. Up to shrink Υ we can assume that $\Upsilon \subset V \times W$ according to Lemma 3.2.(3). There exists then a real number $\epsilon > 0$ such that

$$\{y \in (F_0)^c \mid |y| \leq \epsilon\} \subset \{y \in (F_0)^c \mid |y| < 2\epsilon\} \subset \tilde{f}(\Upsilon).$$

We conclude that for all $y \in (F_0)^c$ with $|y| \leq \epsilon$, there exist $v \in V^a$ and $w \in W^b$ such that $(v, w) \in \Omega$ and $f(v, w) = y$. \square

Corollary 3.5. *Let n be a positive integer. Let $\Omega \subset (F_0)^n \times (F_0)^n$ be an open neighborhood of $(0, 0)$ and let $f : \Omega \rightarrow (F_0)^n$ be an analytic map which satisfies*

$$(i) f(0, 0) = 0$$

$$(ii) f(x, 0) = f(0, x) = x \text{ over an open neighborhood } \Upsilon \text{ of } 0.$$

Then there is a real number $\epsilon > 0$ such that for all a with $|a| \leq \epsilon$, there exist $v \in V^n$ and $w \in W^n$ such that $(v, w) \in \Omega$ and $f(v, w) = a$.

Proof. In this case we have $a = b = c = n$ and $Df_0^a = Df_0^b = \text{Id}_{(F_0)^n}$ so that Proposition 3.3 applies. \square

3.2. Kneser-Tits' subgroups. Continuing in the previous setting, we assume furthermore that F_1 is t -adically dense in F_0 . Let $F \subset F_1 \cap F_2$ be a subfield. For dealing later with Weil restriction issues it is convenient to deal with a finite field extension E of F .

Proposition 3.6. *Let H be a semisimple simply connected E -group scheme assumed strictly isotropic. We put $G = R_{E/F}(H)$. For each overfield L of F , we put $G(L)^+ = H(L \otimes_F E)^+$ where the second group is that defined in §2.1. Then we have the decomposition*

$$G(F_0)^+ = G(F_1)^+ G(F_2)^+.$$

Proof. Without loss of generality we can assume that F is infinite. The proof is based on an analytic argument requiring some preparation. We put $d = [E : F]$ and fix an isomorphism $E \cong F^d$ of F -vector spaces.

Let P be a strictly proper parabolic E -subgroup of H . Let U be its unipotent radical and U_{last} the last part of Demazure's filtration [G-P-S, §3.2]. Let $u : \mathbb{G}_{a,E}^d \xrightarrow{\sim} U_{last}$ be an E -group isomorphism. According to [G-P-S, Lemma 3.3.(3)], $H(E)^+ \cdot \text{Lie}(U_{last})(E)$ generates $\text{Lie}(H)(E)$ as a vector space over E . There exists $h_1, \dots, h_n \in H(E)^+$ such that

$$\text{Lie}(H)(E) = {}^{h_1}\text{Lie}(U_{last})(E) + {}^{h_2}\text{Lie}(U_{last})(E) + \dots + {}^{h_n}\text{Lie}(U_{last})(E).$$

We consider the map $b : (\mathbb{G}_{a,E}^d)^n \rightarrow H$, $b(x_1, \dots, x_n) = {}^{h_1}u(x_1) \dots {}^{h_n}u(x_n)$. Its differential at 0 is $db_0 : E^{dn} \cong \text{Lie}(U_{last})(E)^n \rightarrow \text{Lie}(H)(E)$, $(X_1, \dots, X_n) \mapsto \sum_{i=1}^n {}^{h_i}X_i$, so is surjective. We cut now the affine space $(\mathbb{G}_{a,E}^d)^n$ by some suitable affine E -subspace $\mathbb{G}_{a,E}^r$ such that the restriction b' of b to $(\mathbb{G}_{a,E}^d)^n$ is such that db'_0 is an isomorphism. The map b' is then étale at a neighborhood of 0. It follows that $b'_\sharp = R_{E/F}(b') : R_{E/F}(\mathbb{G}_{a,E}^r) \rightarrow G = R_{E/F}(H)$ is étale also at a neighborhood of 0 [C-G-P, A.5.2.(4)].

Since the field F_0 is henselian, the local inversion theorem holds [G-G-MB, prop. 2.1.4]. We mean that there exists an open neighborhood $\Upsilon \subset (E \otimes_F F_0)^r$ such that the restriction $b'_\sharp|_\Upsilon : \Upsilon \rightarrow G(F_0)$ is a topological open embedding.

We consider now the product morphism $q : \Upsilon \times \Upsilon \rightarrow G(F_0)$, $q(x, y) = b'_\sharp(x) b'_\sharp(y)$. We put $\Omega = q^{-1}(b'_\sharp(\Upsilon))$, this an open subset of $(E \otimes_F F_0)^{2r} \cong (F_0)^{2dr}$. Then the

restriction $q|_{\Omega}$ defines an (unique) analytical map $f : \Omega \rightarrow \Upsilon$ such that $q(x, y) = b'_{\sharp}(f(x, y))$.

By construction we have $f(0, x) = f(x, 0) = x$ for x in a neighborhood of $0 \in (E \otimes_F F_0)^r \cong F_0^{dr}$. We apply Corollary 3.5 to f so that there exists $\epsilon > 0$ such that for each $a \in \Upsilon$ with $|a| \leq \epsilon$, there exist $v \in W^{dr}$ and $w \in W^{dr}$ such that $(v, w) \in \Omega$ and $f(v, w) = a$. We denote by $\Upsilon_{\epsilon} = \Upsilon \cap B(0, \epsilon)$. Then $b'_{\sharp}(\Upsilon_{\epsilon})$ is an open neighborhood of 0 in $(F_0)^r$.

Let us now prove that $G(F_0)^+ = G(F_1)^+ \times G(F_2)^+$. Since F_1 is dense in F_0 , $G(F_1)^+$ is dense in $G(F_0)^+$. It is then enough to show that $\Upsilon_{\epsilon} \subset G(F_1)^+ \times G(F_2)^+$. Let $g = b'_{\sharp}(a) \in b'_{\sharp}(\Upsilon_{\epsilon})$. Then $a = f(v, w)$ with $(v, w) \in \Omega$. It follows that $g = b'_{\sharp}(a) = b'_{\sharp}(f(v, w)) = q(v, w) = b'_{\sharp}(v) b'_{\sharp}(w) \in G(F_1)^+ \times G(F_2)^+$. \square

This could be refined as follows.

Proposition 3.7. *Let H be a reductive E -group and put $G = R_{E/F}(H)$.*

- (1) $RG(F_1)RG(F_2)$ contains an open neighborhood of 1 in $G(F_0)$.
- (2) If $RG(F_1)$ is dense in $RG(F_0)$, then $RG(F_1)RG(F_2) = RG(F_0)$.
- (3) If H is semisimple simply connected and $H_{F_1 \otimes_F E}$ is strictly isotropic, then we have

$$G(F_1)^+ RG(F_2) = G(F_0)^+.$$

- (4) If H is semisimple simply connected and $H_{F_i \otimes_F E}$ is strictly isotropic for $i = 1, 2$, then $G(F_1)^+ G(F_2)^+ = G(F_0)^+$.

The subgroups $G(F_1)^+$, $G(F_0)^+$ are defined as in Proposition 3.6.

Proof. Once again we put $d = [E : F]$ and fix an isomorphism $E \cong F^d$ of F -vector spaces.

- (1) Let $T \subset H$ be a maximal E -torus and let $1 \rightarrow S \rightarrow Q \xrightarrow{s} T \rightarrow 1$ be a resolution of T where Q is a quasitrivial torus and S is a torus. We have $Q = R_{C/E}(\mathbb{G}_m)$ where C is an étale E -algebra so that Q is an open subset of the affine E -space $\mathbf{W}(C)$.

We use now Raghunathan's technique [R, §1.2]. There exists $h_1, \dots, h_n \in H(E)$ such that $\text{Lie}(H)(E) = {}^{h_1}\text{Lie}(T)(E) + {}^{h_2}\text{Lie}(T)(E) + \dots + {}^{h_n}\text{Lie}(T)(E)$. We define the open E -subvariety $U \subset \mathbf{W}_E(C^n)$ where $1 + x_1, \dots, 1 + x_n$ belongs to Q . We consider the map $b : U \rightarrow H$, $b(x_1, \dots, x_n) = {}^{h_1}s(1+x_1) \dots {}^{h_n}s(1+x_n)$. Its differential at 0 is

$$db_0 : C^n \rightarrow \text{Lie}(H)(E), \quad (c_1, \dots, c_n) \mapsto \sum_{i=1}^n {}^{h_i}ds(c_i),$$

and is onto (observe that $\text{Lie}(Q)(E) \rightarrow \text{Lie}(T)(E)$ is surjective since $Q \rightarrow E$ is smooth). We cut now the E -affine space $\mathbf{W}_E(C^n)$ by some suitable affine E -subspace $\mathbf{W}_E(E^r)$ of $\mathbf{W}_E(C^n)$ such that the restriction b' of b to the E -variety $Y = U \cap \mathbf{W}_E(E^r)$ is such that $db'_0 : \text{Tan}_{Y,0} \rightarrow \text{Lie}(H)(E)$ is an isomorphism. Note that Y and H have same dimension r over E .

The map b' is then étale at a neighborhood of 0. It follows that $b'_\sharp = R_{E/F}(b') : X = R_{E/F}(Y) \rightarrow G = R_{E/F}(H)$ is étale also at a neighborhood of 0 [C-G-P, A.5.2.(4)]. Note that X is an open subset of $R_{E/F}(\mathbf{W}_E(E^r)) \cong \mathbf{W}(E^r) \cong \mathbf{A}_F^{dr}$.

Since the field F_0 is henselian, the local inversion theorem holds [G-G-MB, prop. 2.1.4]. We mean that there exists an open neighborhood $\Upsilon \subset X(F_0) \subset (F_0 \otimes_F E)^r \cong (F_0)^{dr}$ of 0 such that the restriction $b'_{\sharp|\Upsilon} : \Upsilon \rightarrow G(F_0)$ is a topological open embedding.

We consider now the product morphism $q : \Upsilon \times \Upsilon \rightarrow G(F_0)$, $q(x, y) = b'_\sharp(x) b'_\sharp(y)$. We put $\Omega = q^{-1}(b'_\sharp(\Upsilon))$, this an open subset of $(F_0)^{2dr}$. Then the restriction $q|_\Omega$ defines a (unique) analytical map $f : \Omega \rightarrow \Upsilon$ such that $q(x, y) = b'_\sharp(f(x, y))$.

By construction we have $f(0, x) = f(x, 0)$ for x in a neighborhood of $0 \in (F_0 \otimes_F E)^r \cong F_0^{dr}$. We apply Corollary 3.5 to f so that there exists $\epsilon > 0$ such that for each $a \in \Upsilon$ with $|a| \leq \epsilon$, there exist $v \in V^{dr}$ and $w \in W^{dr}$ such that $(v, w) \in \Omega$ and $f(v, w) = a$. We denote by $\Upsilon_\epsilon = \Upsilon \cap B(0, \epsilon)$. Then $b'_\sharp(\Upsilon_\epsilon)$ is an open neighborhood of 1 in $(F_0)^{dr}$.

We claim that $\Upsilon_\epsilon \subset RG(F_1) RG(F_2)$. Let $g = b'_\sharp(a) \in b(\Upsilon_\epsilon)$. Then $a = f(v, w)$ with $(v, w) \in \Omega$. It follows that $g = b'_\sharp(a) = b'_\sharp(f(v, w)) = q(v, w) = b'_\sharp(v) b'_\sharp(w) \in RG(F_1) \times RG(F_2)$.

(2) If $RG(F_1)$ is furthermore dense in $RG(F_0)$, then (1) shows that $RG(F_1) RG(F_2)$ is a dense open subset of $RG(F_0)$. Thus $RG(F_1) RG(F_2) = RG(F_0)$

(3) We assume that H is semisimple simply connected and that $G_{F_1 \otimes_F E_1}$ is strictly isotropic. According to Lemma 2.2.(2), we have $G(F_1)^+ = RG(F_1) = RH(F_1 \otimes_F E) = H^+(F_1 \otimes_F E)$ and similarly for F_0 .

Since $H^+(F_1 \otimes_F E)^+$ is dense in $H^+(F_0 \otimes_F E)$, it follows that $RG(F_1)$ is dense in $RG(F_0)$. Assertion (2) yields then $RG(F_1) RG(F_2) = RG(F_0)$. Lemma 2.3.(2) states that $G(F_1)^+ = RG(F_1)$ and $G(F_0)^+ = RG(F_0)$. We conclude that $G(F_1)^+ RG(F_2) = G(F_0)^+$.

(4) We have furthermore $G(F_2)^+ = RG(F_2)$ so that (3) yields $G(F_1)^+ G(F_2)^+ = G(F_0)^+$. \square

Remark 3.8. Note that (1) shows in particular that $RG(F_0)$ is an open subgroup of $G(F_0)$.

The main result of the section is the following patching statement on twisted flag varieties.

Theorem 3.9. *We put $F = F_1 \cap F_2$. Let H be a reductive E -group and let X be a twisted flag E -variety of H . We put $G = R_{E/F}(H)$ and $Z = R_{E/F}(X)$. If $Z(F_1) \neq \emptyset$ and $Z(F_2) \neq \emptyset$, then $Z(F) \neq \emptyset$.*

Proof. Without loss of generality, we can assume that H is semisimple simply connected and that it is absolutely E -simple. This implies that H is strictly $F_i \otimes_F E$ -isotropic for $i = 1, 2$. Let $x_i \in Z(F_i) = X(F_i \otimes_F E)$ and denote by $P_i = \text{Stab}_{G_{F_i}}(x_i)$

its stabilizer, this is parabolic $F_i \otimes_F E$ -subgroup of H . Since $F_i \otimes_F E$ is a semilocal algebra, we have $x_1 = h.x_2$ for some $h \in H(E \otimes_F F_0)$ [SGA3, XXVI.5.2]. According to Lemma 2.1, we have $H(E \otimes_F F_0) = H^+(E \otimes_F F_0) P_2(E \otimes_F F_0)$ so we can assume that $h \in H^+(E \otimes_F F_0)$. According to Proposition 3.7.(4), we can write $h = h_1 h_2$ with $h_i \in H^+(E \otimes_F F_i)$ for $i = 1, 2$. Since $x_1 = h.x_2$, we obtain $h_1^{-1}.x_1 = h_2^{-1}.x_2$. This defines a point of $X(E) = Z(F)$. \square

4. RELATION WITH THE ORIGINAL HHK METHOD

We recall the setting. Let T be an excellent complete discrete valuation ring with fraction field K , residue field k and uniformizing parameter t . Let F be a one-variable function field over K and let \mathfrak{X} be a normal model of F , i.e. a normal connected projective T -curve with function field F . We denote by Y the closed fiber of \mathfrak{X} and fix a separable closure F_s of F .

For each point $P \in Y$, let R_P be the local ring of \mathfrak{X} at P ; its completion \widehat{R}_P is a domain with fraction field denoted by F_P .

For each subset U of Y that is contained in an irreducible component of Y and does not meet the other components, we define $R_U = \bigcap_{P \in U} R_P \subset F$. We denote by

\widehat{R}_U the t -adic completion of R_U . The rings R_U and \widehat{R}_U are excellent normal domains and we denote by F_U the fraction field of \widehat{R}_U [H-H-K3, Remark 3.2.(b)].

Each height one prime \mathfrak{p} in \widehat{R}_P that contains t defines a branch of Y at P lying on some irreducible component of Y . The t -adic completion $\widehat{R}_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$ of \widehat{R}_P at \mathfrak{p} is a complete DVR whose fraction field is denoted by $F_{\mathfrak{p}}$. The field $F_{\mathfrak{p}}$ contains also F_U if U is an irreducible open subset of Y such that $P \in \overline{U} \setminus U$. We have then a diagram of fields

$$\begin{array}{ccc} & F_{\mathfrak{p}} & \\ & \swarrow \quad \nwarrow & \\ F_P & & F_U. \end{array}$$

Example 4.1. We assume that $T = k[[t]]$ and take $X = \mathbb{P}_K^1$, $\mathfrak{X} = \mathbb{P}_T^1$, $P = \infty_k$, $U = \mathbb{A}_k^1 = \text{Spec}(k[x])$. The ring R_U contains $k[[t]][x]$ and is its localization with respect to the multiplicative set S of elements which are units modulo t . The t -adic completion of R_U is $\widehat{R}_U = k[x][[t]]$; we have $F_U = \text{Frac}(\widehat{R}_U) = k(x)((t))$. The local ring of \mathfrak{X} at $P = \infty_k$ is $R_P = k[[t]][x^{-1}]_{(x^{-1}, t)}$ so that its completion is $\widehat{R}_P = k[[t, x^{-1}]]$; in particular $F_P = k((t, x^{-1}))$.

We take $\mathfrak{p} = t\widehat{R}_P \subset \widehat{R}_P$ and the t -adic completion $\widehat{R}_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$ of \widehat{R}_P at \mathfrak{p} is a complete DVR which is $k((x^{-1}))[[t]]$. In particular $F_{\mathfrak{p}} = k((x^{-1}))((t))$.

Setting 4.2. Let \mathcal{P} be a non-empty finite set of closed points of Y that contains all the closed points at which distinct irreducible components meet. Let \mathcal{U} be the set of connected components of $Y \setminus \mathcal{P}$ and let \mathcal{B} be the set of branches of Y at points of \mathcal{P} . This yields a finite inverse system of field $F_P, F_U, F_{\mathfrak{p}}$ (for $P \in \mathcal{P}; U \in \mathcal{U}, \mathfrak{p} \in \mathcal{B}$) where $F_P, F_U \subset F_{\mathfrak{p}}$ if \mathfrak{p} is a branch of Y at P lying in the closure of U .

Lemma 4.3. *We assume that $X = \mathbb{P}_K^1, \mathfrak{X} = \mathbb{P}_T^1, P = \infty_k, U = \mathbb{A}_k^1 = \text{Spec}(k[x])$ and \mathfrak{p} the branch of P . We put $F_1 = F_P, F_2 = F_U$ and $F_0 = F_{\mathfrak{p}}$.*

(1) F_1 is t -dense in F_0 .

(2) We put $V = F_1 \cap \widehat{R}_{\mathfrak{p}}$ and $W = F_2 \cap \widehat{R}_{\mathfrak{p}}$. Then V and W satisfy conditions (3.1) and (3.2).

Proof. (1) We are given $u_0/v_0 \in F_0$ with $u_0, v_0 \in \widehat{R}_{\mathfrak{p}}, v_0 \neq 0$. There exists elements $u, v \in R_{\mathfrak{p}}$ very close respectively of u_0, v_0 with $v \neq 0$. Let $s_1, s_2 \in R_{\mathfrak{p}} \setminus R_{\mathfrak{p}\mathfrak{p}}$ such that $s_1 u \in R_P$ and $s_2 v \in R_{\mathfrak{p}}$. Then u_0/v_0 is very close of $(s_1 s_2 u)/(s_1 s_2 v) \in F_1$.

(2) Condition (3.2) is obviously fulfilled since $t \in F_1 \cap F_2$. For establishing condition (3.1), we are given an element f of $\widehat{R}_{\mathfrak{p}}$ and may write it as $f = \sum_{i=0}^{\infty} x^{m_i} \left(\sum_{j=0}^{\infty} a_{i,j} \frac{1}{x^j} \right) t^i$ where the m_i 's are non-negative integers and $a_{i,j} \in T$. We decompose

$$f = f_1 + f_2 = \sum_{i=0}^{\infty} x^{m_i} \left(\sum_{j=m_i}^{\infty} a_{i,j} \frac{1}{x^j} \right) t^i + \sum_{i=0}^{\infty} x^{m_i} \left(\sum_{j=0}^{m_i-1} a_{i,j} \frac{1}{x^j} \right) t^i$$

We observe that f_2 belongs to \widehat{R}_P so belongs to V . We recall that R_U is the localization of $T[x]$ with respect to the elements which are units modulo t . We conclude that f_2 belongs to W as desired. \square

Theorem 4.4. *Let G be a reductive F -algebraic group.*

(1) *Let Z be a twisted flag projective F -variety for G . Then $Z(F) \neq \emptyset$ if and only if $Z(F_U) \neq \emptyset$ for each $U \in \mathcal{U}$ and $Z(F_P) \neq \emptyset$ for each $P \in \mathcal{P}$.*

(2) *For each $U \in \mathcal{U}$ (resp. each $P \in \mathcal{P}$), we fix an F -embeddings $i_U : F_s \rightarrow F_{U,s}$ (resp. $i_P : F_s \rightarrow F_{P,s}$) providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{U,s}})$ (resp. $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{P,s}})$). The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the \star -action of $\text{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_U})$ for each $U \in \mathcal{U}$ and $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$.*

The recollection for star action and Tits index is done in the beginning of the paper.

Proof. (1) We use a Weil restriction argument as in the proof of [H-H-K2, Thm. 4.2]. This involves a finite morphism $f : \mathfrak{X} \rightarrow \mathbb{P}_T^1$ such that $\mathcal{P} = f^{-1}(\infty_k)$. Write \underline{F} for the function field of \mathbb{P}_T^1 , and let $d = [F : F']$. We put $\underline{U} = \mathbb{P}_k^1 \setminus \{\infty\}$, $\underline{P} = \infty_k$ and $\underline{\mathfrak{p}} = (U, \mathfrak{p})$ and $F_0 = \underline{F}_{\underline{\mathfrak{p}}}, F_1 = \underline{F}_{\underline{P}}$ and $F_2 = \underline{F}_{\underline{U}}$. Also patching holds for the diamond $(\underline{F}, F_1, F_2, F_0)$ according to [H-H, thm 3.9] so in particular $K(x) = \underline{F} = F_1 \cap F_2 \subset F_0$.

We put $V = F_1 \cap \widehat{R}_p$ and $W = F_2 \cap \widehat{R}_p$. Lemma 4.3 shows that F_1 is dense in F_0 and that V and W satisfy conditions (3.1) and (3.2).

We consider the Weil restriction $\underline{G} = R_{F/\underline{F}}(G)$, it is a reductive \underline{F} -group which acts on the \underline{F} -variety $\underline{Z} = R_{F/\underline{F}}(Z)$. We have

$$\underline{Z}(F_1) = Z(F_1 \otimes_{\underline{F}} F) = \prod_{P \in \mathcal{P}} Z(F_P)$$

according to [H-H, Lemma 6.2.(a)]. Similarly we have

$$\underline{Z}(F_2) = Z(F_2 \otimes_{\underline{F}} F) = \prod_{U \in \mathcal{U}} Z(F_U)$$

Our assumptions imply that $\underline{Z}(F_1) \neq \emptyset$ and $\underline{Z}(F_2) \neq \emptyset$. Theorem 3.9 implies that $\underline{Z}(\underline{F}) \neq \emptyset$. Thus $\underline{Z}(\underline{F}) = Z(F)$ is non-empty.

(2) Let Θ be the smallest subset of $\Delta(G_{F_s})$ which is stable under the \star -action of $\text{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_U})$ for each $U \in \mathcal{U}$ and $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$. We observe that $\Delta_0(G) \subset \Theta$ since it is stable under the star action and satisfies $\Delta_0(G) \subset \Delta_0$ for each $U \in \mathcal{U}$ and $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$. For the converse inclusion we consider the F -variety Z of parabolic subgroups of type Θ . For each $U \in \mathcal{U}$, we have $\Theta \subset \Delta_0(G_{F_U})$ so that $Z(F_U) \neq \emptyset$; similarly we have $\Theta \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$ so that $Z(F_P) \neq \emptyset$. Part (1) yields that $Z(F) \neq \emptyset$. Thus $\Theta \subset \Delta_0(G)$ and $\Theta = \Delta_0(G)$ \square

Corollary 4.5. *Let G be a reductive F -algebraic group.*

(1) *Let Z be a twisted flag projective F -variety for G . Then $Z(F) \neq \emptyset$ if and only if $Z(F_P) \neq \emptyset$ for each $P \in Y$.*

(2) *For each $P \in Y$, we fix an F -embedding $i_P : F_s \rightarrow F_{P,s}$ providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{P,s}})$. The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the \star -action of $\text{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in Y$.*

Proof. (1) We assume $Z(F_P) \neq \emptyset$ for each $P \in Y$. Let Y_1, \dots, Y_d be the irreducible components of Y with respective generic points η_1, \dots, η_d . According to [H-H-K2, prop. 5.8], there exists non-empty affine subsets $U_i \subset Y_i$ ($i = 1, \dots, d$) such that $Z(F_{U_i}) \neq \emptyset$ for $i = 1, \dots, d$ and $U_i \cap U_j = \emptyset$ for $i < j$. We apply Theorem 4.4 to $\mathcal{U} = \{U_1, \dots, U_d\}$, $\mathcal{P} = Y \setminus \bigcap_i U_i$ and get that $Z(F) \neq \emptyset$.

(2) This readily follows of (1). \square

Corollary 4.6. *Let G be a reductive F -algebraic group and assume that G is the generic fiber of a reductive \mathfrak{X} -group scheme \mathfrak{G} .*

(1) *Let Z be a twisted flag projective F -variety for G . Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for each discrete valuation v of F .*

(2) For each discrete valuation v of F , we fix an F -embedding $i_v : F_s \rightarrow F_{v,s}$ providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{v,s}})$. The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the \star -action of $\text{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_v})$ for each discrete valuation v on F .

Proof. (1) We assume that $Z(F_v) \neq \emptyset$ for each discrete valuation v of F . Let \mathfrak{G} be a reductive \mathfrak{X} -group of generic fiber G . Without loss of generality we can assume that \mathfrak{G} is adjoint. Let \mathfrak{G}_0 be the Chevalley form of \mathfrak{G} and let $(\mathfrak{B}_0, \mathfrak{T}_0)$ be a Killing couple for \mathfrak{G}_0 and let Δ_0 be the associated Dynkin diagram. Since \mathfrak{G}_0 is adjoint, we have an exact sequence of \mathbb{Z} -group schemes [SGA3, XXIV.1.3 and 3.6]

$$1 \rightarrow \mathfrak{G}_0 \rightarrow \text{Aut}(\mathfrak{G}_0) \rightarrow \text{Aut}(\Delta_0) \rightarrow 1$$

The $\text{Aut}(\mathfrak{G}_0)$ -torsor $\mathfrak{Q} = \text{Isom}(\mathfrak{G}_0, \mathfrak{G})$ over \mathfrak{X} defines then an $\text{Aut}(\Delta_0)$ -torsor over \mathfrak{X} . Since $\text{Aut}(\Delta_0)$ is a finite constant group, Grothendieck's theory of fundamental groups tells us that this torsor is the data of a morphism $v : \Pi_1(\mathfrak{X}, \bullet) \rightarrow \text{Aut}(\Delta_0)$ where the base point is $\text{Spec}(F_s) \rightarrow \text{Spec}(F) \rightarrow \mathfrak{X}$. The star action associated to G is the composite

$$\text{Gal}(F_s/F) \rightarrow \Pi_1(\mathfrak{X}, \bullet) \rightarrow \text{Aut}(\Delta_0).$$

Claim 4.7. Z is the generic fiber of an \mathfrak{X} -scheme of parabolic subgroups \mathfrak{Z} of \mathfrak{G} .

The variety Z is a form of the variety $\text{Par}_I(G_0)$ of parabolic subgroups of type I where $I \subset \Delta_0$ is stable the star action. Since \mathfrak{X} is normal, the map $\text{Gal}(F_s/F) \rightarrow \Pi_1(\mathfrak{X}, \bullet)$ is onto so that I is stable under the action of $\Pi_1(\mathfrak{X}, \bullet)$ on Δ_0 . In particular Q admits a reduction \mathfrak{Q}_I to the stabilizer $\text{Aut}_I(\mathfrak{G}_0)$ for the action of $\text{Aut}(\mathfrak{G}_0)$ on Δ_0 . The \mathfrak{X} -scheme $\mathfrak{Z} = {}^{\mathfrak{Q}}\text{Par}_I(\mathfrak{G}_0)$ is the scheme of parabolic subgroups of type I of $\mathfrak{G} = {}^{\mathfrak{Q}}\mathfrak{G}_0$, so that Z is the generic fiber of \mathfrak{Z} .

For applying Corollary 4.5, we have to check that $Z(F_P) \neq \emptyset$ for each $P \in Y$. If Q is a point of codimension 1 of \mathfrak{X} , it defines a discrete valuation v_Q on F whose completion is F_Q . Our assumption implies then that $Z(F_Q) \neq \emptyset$ in that case. We deal now with the case of a closed point P of \mathfrak{X} . Let D be an irreducible component of $Y = \mathfrak{X}_k$ containing P and let Q be the generic point of D . Since \mathfrak{Z} is proper over T , we have $\mathfrak{Z}(\widehat{R}_Q) = \mathfrak{Z}(F_Q)$ which is not empty by the preceding case. It follows that $\mathfrak{Z}_k(k(D)) \neq \emptyset$. Again \mathfrak{Z}_k is projective so that $\mathfrak{Z}_k(D) = \mathfrak{Z}_k(k(D))$ is not empty and in particular $\mathfrak{Z}_k(k(P))$ is not empty. Since \mathfrak{Z} is smooth over \mathfrak{X} , the Hensel lemma shows that $\mathfrak{Z}_k(\widehat{R}_{\mathfrak{X},P}) \rightarrow \mathfrak{Z}_k(k(P))$ is surjective. Thus $\mathfrak{Z}(\widehat{R}_{\mathfrak{X},P})$ is not empty and so is $Z(F_P)$.

(2) It readily follows of (1). □

Corollary 4.8. Let G be a reductive F -algebraic group. We denote by Ω_F^1 the set of rank one valuations of F .

(1) Let Z be a twisted flag projective F -variety for G . Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for each $v \in \Omega_F^1$.

(2) For each $v \in \Omega_F^1$, we fix an F -embedding $i_v : F_s \rightarrow F_{v,s}$ providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{v,s}})$. The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the \star -action of $\text{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_v})$ for each $v \in \Omega_F^1$.

Proof. (1) We assume that $Z(F_v) \neq \emptyset$ for each $v \in \Omega_F^1$. According to [H-H-K-P, Thm 2.5], there exists a regular proper model \mathfrak{X}' of F such that $Z(F_P) \neq \emptyset$ for each $P \in \mathfrak{X}' \times_T k$. Then Corollary 4.6.(1) shows that $Z(F) \neq \emptyset$.

(2) This readily follows of (1). □

5. LOCAL-GLOBAL PRINCIPLE FOR DISCRETE VALUATIONS

Let T be an excellent DVR of fraction field K and residue field k . Let X be a smooth, projective, geometrically integral curve over K . Let $F = K(X)$ be the function field of X .

Lemma 5.1. *Let \mathfrak{X} be a projective, flat curve over A which is connected and regular such that $\mathfrak{X}_K = X$. Let \mathfrak{H} be a flat affine \mathfrak{X} -group scheme of finite presentation and assume that there exists a Zariski cover $(\mathfrak{U}_i)_{i \in I}$ of \mathfrak{X} such that each $\mathfrak{H}_{\mathfrak{U}_i}$ admits a closed embedding $\mathfrak{H}_{\mathfrak{U}_i} \subset \text{GL}_{n_i}$ such that $\text{GL}_{n_i}/\mathfrak{H}_{\mathfrak{U}_i}$ is representable by an affine U_i -scheme,*

Let $\gamma \in H_{\text{fppf}}^1(F, \mathfrak{H})$ and let D be a divisor of \mathfrak{X} which contains the irreducible components of $Y = \mathfrak{X}_k$ and such that γ extends to $X \setminus D_K$.

*Then there exists a proper birational morphism $q : \mathfrak{X}' \xrightarrow{\sim} \mathfrak{X}$ such that \mathfrak{X}' is a regular proper model of X and such that $D' = q^*D$ is a strict normal crossing divisor and*

$$\gamma \in \text{Im}\left(H^1(\mathfrak{X}' \setminus D', \mathfrak{H}) \rightarrow H^1(F, \mathfrak{H})\right).$$

Proof. Using a passage to the limit argument [Mg], there exists an open affine subscheme $\mathfrak{U}_1 \subset \mathfrak{X} \setminus D$ such that γ extends to a class $\gamma_1 \in H^1(\mathfrak{U}_1, \mathfrak{H})$. According to [G-P1, cor 1.8], γ_1 extends to a class $\gamma_2 \in H^1(\mathfrak{U}_2, \mathfrak{H})$ where \mathfrak{U}_2 is an open subscheme of $\mathfrak{X} \setminus D$ containing \mathfrak{U}_1 and $X \setminus D_K$. By purity (i.e. Theorem 7.1 of the appendix 7), we have $H^1(\mathfrak{X} \setminus D, \mathfrak{H}) = H^1(\mathfrak{U}_2, \mathfrak{H})$. Thus γ extends over $\mathfrak{X} \setminus D$. According to Lipman's theorem we can resolve the singularities of \mathfrak{X} and transform D in a strict normal crossing divisor, see [H-H-K, lemma 4.7]. □

Proposition 5.2. *Let G be a reductive F -group and assume that p does not divide the order of the automorphism group of the absolute root system of G_{ad} . Let Z be a twisted flag variety of G . We assume that $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from models of X . Then there exists a regular proper model \mathfrak{X} of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $Z(F_y) \neq \emptyset$.*

Proof. Without loss of generality we can assume that G is adjoint. Let G_0 be the Chevalley form of G and let (B_0, T_0) be a Killing couple for G_0 and let Δ_0 be the associated Dynkin diagram. The variety Z is a form of the variety $\text{Par}_I(G_0)$ of

parabolic subgroups of type I where $I \subset \Delta_0$ is stable under the star action defined by the $\text{Aut}(G_0)$ -torsor $Q = \text{Isom}(G_0, G)$. In particular Q admits a reduction Q_I to the stabilizer $\text{Aut}_I(G_0)$ for the action $\text{Aut}(G_0)$ on Δ_0 through the morphism $\text{Aut}(G_0) \rightarrow \text{Out}(G_0) \xrightarrow{\sim} \text{Aut}(\Delta_0)$. Furthermore Z is isomorphic to ${}^Q\text{Par}_I(G_0)$.

We apply now Theorem 1.1.(b) of [C-G-R] to the \mathbb{Z} -group scheme $\text{Aut}_I(G_0)$. It provides a finite \mathbb{Z} -subgroup S_0 of $\text{Aut}_I(G_0)$ such that the map $H^1(F, S_0) \rightarrow H^1(F, \text{Aut}(G_0))$ is onto. Furthermore the construction of S_0 is explicit in the proof, it is an extension of the finite constant group $\text{Aut}_I(G_0)/T_0$ by a finite subgroup of T_0 . In particular S_0 is finite free over \mathbb{Z} and our assumption on the characteristic implies that $S_{0,T}$ is finite étale of rank prime to p . It follows that Q_I admits a reduction to an F -torsor E under S_0 .

The finite flat \mathbb{Z} -group scheme $S_{0,A}$ admits a faithful representation $S_0 \hookrightarrow \text{GL}_{N,\mathbb{Z}}$ [Br-T, §1.4.5] and the quotient $\text{GL}_{N,\mathbb{Z}}/S_0$ is representable by an affine \mathbb{Z} -scheme [D-G, §III.2.6].

According to Lemma 5.1, there exists a regular proper model \mathfrak{X} of X and a strict normal crossing divisor D containing the irreducible components of Y such that E extends to a $\mathfrak{X} \setminus D$ -torsor \mathfrak{E} under S_0 . We put $\mathfrak{Q}_I = \mathfrak{E} \wedge^{S_0} \text{Aut}_I(G_0)$ and consider the $\mathfrak{X} \setminus D$ -group scheme $\mathfrak{G} = {}^{\mathfrak{Q}_I}G_0$ of generic fiber G .

We are given a closed point $P \in Y$. If y is of codimension one, by hypothesis, $Z(F_y)$ is not empty. We therefore look at a closed point $P \in Y$. We are given a point $P \in Y$ and pick a height one prime \mathfrak{p} in \widehat{R}_P that contains t . It defines a branch of Y at P lying on some irreducible component Y_1 of Y .

We consider the local ring $A = R_P$ of \mathfrak{X} at P and denote by A_D its localization at D . Since $S_{0,T}$ is finite étale of degree prime to p , $H^1(A_D, S_0)$ consists in loop torsors as defined in [Gi2, §2.3, lemma 2.3.(2)], i.e. those arising from cocycles related to tame Galois covers of A_D . It follows that the A_D -torsor Q_I is a loop $\text{Aut}_I(G)$ -torsor [Gi2, lemma 2.3.(3)], so that $\mathfrak{G} \times_{\mathfrak{X} \setminus D} A_D$ is by definition a loop reductive group scheme.

Let $F_{P,v}$ be the completion of the field F_P for the valuation associated to the blow-up of $\text{Spec}(A)$ at its closed point. Our assumption states in particular that $Z(F_{P,v}) \neq \emptyset$, that is, $G_{F_{P,v}}$ admits a parabolic subgroup of type I . According to [Gi2, th. 4.1, (iii) \implies (i)], $\mathfrak{G} \times_{\mathfrak{X} \setminus D} A_D$ admits a parabolic subgroup of type I . A fortiori G_{F_P} admits a parabolic subgroup of type I so that $Z(F_P) \neq \emptyset$. \square

Remarks 5.3. (a) If $p = 0$, the result used [Gi2, th. 4.1] admits a simple proof, see [Gi2, Ex. 4.2], by using the analogous result over Laurent polynomials [G-P2, th. 7.1].

(b) In nice cases inspection of the proof permits to weaken the assumption on p . The precise condition is that the $\text{Aut}(G_0)$ -torsor $\text{Isom}(G_0, G)$ admits a reduction to a finite F -subgroup whose degree is prime to p . For example in type G_2 , we need to assume only that p is prime to 2.

Together with Corollary 4.5, we obtain the following consequence:

Theorem 5.4. *Let G be a reductive F -group and assume that p does not divide the order of the automorphism group of the absolute root system of G_{ad} . Let Z be a twisted flag F -variety of G . Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from models of X .*

6. APPENDIX: CHARACTERIZATION OF PARABOLIC SUBGROUPS.

Let G be a reductive F -group over a field F . We remind the reader that an algebraic F -subgroup P of G is parabolic if P is smooth and G/P is a projective F -variety. The interest of the probably known statement below is only in positive characteristic since in this case there exist F -subgroups Q which are not smooth such that G/Q is a projective F -variety [W].

Proposition 6.1. *Let P be a k -subgroup of G such that the quotient variety G/P is projective. Then the following assertions are equivalent:*

- (i) P is an F -parabolic subgroup
- (ii) For each F -field E , $G(E)$ acts transitively on $(G/P)(E)$
- (iii) The quotient map $G \rightarrow G/P$ admits a rational section
- (iv) P is smooth connected
- (v) P is smooth.

Proof. (i) \implies (ii). Since $(G/P)(E)$ parameterizes the E -parabolic subgroups of G_E of same type that P , Borel-Tits' conjugacy theorem [B-T, th. 4.13.c] shows that $G(E)$ acts transitively on $(G/P)(E)$.

(ii) \implies (iii). Our assumption rephrases by saying that the map $G(E) \rightarrow (G/P)(E)$ is onto for each F -field E . Applying that to the function field $E = F(G/P)$ of the smooth connected F -variety G/P provides a rational section of the map $G \rightarrow G/P$.

(iii) \implies (iv). To show the smoothness of P we can assume that F is algebraically closed. Then the neutral component $Q = (P_{red})^0$ of the reduced F -subgroup P_{red} of P is smooth. Furthermore the quotient F -variety P/Q is finite. It follows that the morphism $q : G/Q \rightarrow G/P$ is finite and a fortiori projective [St, Tag 0B3I]. Since the composition of projective morphisms (of qcqs schemes) is projective [St, Tag 0C4P], it follows that G/Q is projective. The F -subgroup Q of G is then parabolic. Our assumption is that the morphism $G \rightarrow G/P$ has a rational section and so has a fortiori the finite morphism $q : G/Q \rightarrow G/P$. According to [EGAII, cor. 6.1.15], q is an isomorphism. Thus $Q = P$ and we conclude that P is smooth connected.

(iv) \implies (v). Obvious.

(v) \implies (i). This is by definition. □

A variant is the following.

Proposition 6.2. *Let X be a smooth projective G -variety. Then the following statements are equivalent:*

(i) X is a F -variety of parabolic subgroups of G

(ii) For each F -field E , $G(E)$ acts transitively on $X(E)$.

Proof. The implication (i) \implies (ii) is again Borel-Tits' conjugacy theorem. We assume (ii). According to [M-P-W, prop. 1.3], we can assume that F is separably closed. Since X is smooth, we have $X(F) \neq \emptyset$ and denote by P the stabilizer of some F -point x . According to [D-G, prop. III.3.2.1]. condition (ii) implies that the orbit map $G \rightarrow X$, $g \mapsto g.x$ induces an isomorphism $f_x : G/P \xrightarrow{\sim} X$. Proposition 6.1, (ii) \implies (i), shows that P is a F -parabolic subgroup. Thus X is a F -variety of parabolic subgroups of G . \square

Remark 6.3. The condition (ii) is called *transitive action of G on X* by Harbater-Hartmann-Krashen. It occurs in [H-H-K, th. 3.7]. Projective homogeneous varieties in the result quoted above are exactly the various varieties of parabolic subgroups.

7. APPENDIX: EXTENDING TORSORS

We come back to a purity result of Colliot-Thélène and Sansuc.

Theorem 7.1. *Let X be a regular scheme of dimension 2. Let U be an open subscheme of X which contains $X^{(1)}$. Let G be an affine X -group scheme. In the following cases*

(i) G is reductive,

(ii) *There exists a Zariski cover $(U_i)_{i \in I}$ of X such that each G_{U_i} admits a closed embedding $G_{U_i} \subset \mathrm{GL}_{n_i}$ such that $\mathrm{GL}_{n_i}/G_{U_i}$ is representable by an affine U_i -scheme,*

then we have the equality $H_{\mathrm{fppf}}^1(X, G) \xrightarrow{\sim} H_{\mathrm{fppf}}^1(U, G)$.

Proof. The case (i) is [C-T-S, th. 6.13]. The case (ii) goes by inspection of the proof. \square

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