



# A local-global principle for twisted flag varieties

Philippe Gille, Raman Parimala

## ► To cite this version:

Philippe Gille, Raman Parimala. A local-global principle for twisted flag varieties: We prove a local-global principle for twisted flag varieties over a semiglobal field.. 2023. hal-03938963v3

**HAL Id: hal-03938963**

**<https://hal.science/hal-03938963v3>**

Preprint submitted on 22 Mar 2023 (v3), last revised 6 Mar 2024 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A LOCAL-GLOBAL PRINCIPLE FOR TWISTED FLAG VARIETIES

P. GILLE AND R. PARIMALA

**ABSTRACT.** We prove a local-global principle for twisted flag varieties over a semiglobal field.

**Keywords:** Local-global principle, curves over local fields, homogeneous varieties, reductive groups.

**MSC 2000:** 11G99, 14G99, 14G05, 11E72, 11E12, 20G35

## 1. INTRODUCTION

Let  $T$  be a complete discrete valuation ring with fraction field  $K$  and residue field  $k$ . Let  $X$  be a smooth, projective, geometrically integral curve over  $K$ . Let  $F = K(X)$  be the function field of  $X$  and let  $t$  be an uniformizing parameter of  $T$ . We prove the following theorem which settles a conjecture of Colliot-Thélène, Suresh and the second author for function fields of  $p$ -adic curves [C-T-P-S, conjecture 1], in the very general context of semiglobal fields.

**Theorem 1.1.** *(see Th. 5.4) Let  $G$  be a reductive  $F$ -group and assume that  $p$  does not divide the order of the automorphism group of the absolute root system of  $G_{ad}$ . Let  $Z$  be a twisted flag  $F$ -variety of  $G$ . Then  $Z(F) \neq \emptyset$  if and only if  $Z(F_v) \neq \emptyset$  for all discrete valuations of  $F$  arising from models of  $X$ .*

The result was known for the cases of smooth projective quadrics and Severi-Brauer varieties [C-T-P-S, th. 3.1] and also for generalized Severi-Brauer varieties (Reddy-Suresh [R-S, th. 2]).

If we use all rank one valuations, the result holds unconditionnally (Cor. 4.8). On the other hand, if  $G$  extends to a reductive group scheme over a smooth model  $\mathfrak{X}$  of  $X$ , the result is unconditional (Cor. 4.6).

Let us review the contents. Given a henselian couple  $(A, I)$  and a reductive  $A$ -group scheme  $H$ , section 2 deals with generation of groups  $H(A)$  by unipotent elements and also with the quotient of  $H(A)/RH(A)$  by the (normal) subgroup of  $R$ -trivial elements defined in [G-S]. Section 3 is devoted to mild improvements of patching techniques of Harbater-Hartmann-Krashen involving more analytical functions and leads to various group decompositions (prop. 3.7); this permits to obtain a patching

theorem for twisted flag varieties (th. 3.9). Section 4 deals with the setting of the beginning of the introduction and provides a weaker version of the main result. What remains to do is mostly a local study for a two dimensional complete regular local ring  $R$  with parameters  $t, s$  and torsors over its localization  $R[t^{-1}, s^{-1}]$ . This is achieved in section 5 by making use of the loop torsors over  $R[t^{-1}, s^{-1}]$  and of the results of [Gi2].

**Acknowledgements.** We thank Venapally Suresh for a careful reading of the manuscript and fruitful suggestions. We thank David Harbater for valuable comments on a preliminary version of the paper. Finally we thank Gabriel Dospinescu for telling us about Schneider's book.

**Conventions and notation.** (a) A variety  $V$  over a field  $F$  means a separated  $F$ -scheme of finite type which is integral [St, Tag 020D].

(b) Let  $G$  be a reductive  $F$ -algebraic group and let  $F_s$  be a separable closure of  $F$ . Let  $(B, T)$  be a Killing couple of  $G_{k_s}$  and denote  $\Delta(G_{F_s}) = \Delta(G_{F_s}, B, T)$  the associated Dynkin diagram (there is a canonical bijection  $\Delta(G_{F_s}, B, T) \xrightarrow{\sim} \Delta(G_{F_s}, B', T')$  for another choice of Killing couple). This diagram  $\Delta(G_{F_s})$  is equipped with the star action of  $\text{Gal}(F_s/F)$  [B-T, §6.4]. We recall that there is an order preserving bijection  $I \rightarrow P_I$  between the subsets of  $\Delta(G_{F_s})$  and the  $F_s$ -parabolic subgroups of  $G_{F_s}$  containing  $B$  (or equivalently with the  $G(F_s)$ -conjugacy classes of parabolic subgroups of  $G_{F_s}$ ). Since minimal  $F$ -parabolic subgroups are  $G(F)$ -conjugated we denote by  $\Delta_0(G) \subset \Delta(G_{F_s})$  be conjugacy class of  $P_0 \times_F F_s$  where  $P_0$  is a minimal  $F$ -parabolic subgroup of  $G$  (it is stable under the star action). The triple  $(\Delta(G_{F_s}), \text{star action}, \Delta_0(G))$  is called the Tits index of  $G$ . By abuse of notation,  $\Delta_0(G)$  alone is called the Tits index. By a twisted  $F$ -flag variety of  $G$ , we mean a  $G$ -variety  $X$  which becomes isomorphic over  $F_s$  to some  $G_{F_s}/P_I$  as  $G_{F_s}$ -variety (such an  $I$  is unique). According to [M-P-W, prop. 1.3],  $I$  is invariant under the star action and  $X$  is  $G$ -isomorphic to the variety of  $F$ -parabolic subgroups of type  $I$  which is denoted in this paper by  $\text{Par}_I(G)$ .

Those varieties are called sometimes projective homogeneous  $G$ -varieties (as in [C-T-P-S]) but we warn the reader of the danger of this terminology since there exist in positive characteristic  $F$ -subgroups  $Q$  which are not smooth such that  $G/Q$  is a projective  $F$ -variety [W]. In the appendix 6, we show that a homogeneous  $G$ -variety  $X$  is a variety of parabolic subgroups if and only if the action is transitive in the sense that  $G(E)$  acts transitively on  $X(E)$  for each  $F$  field  $E$  (Prop. 6.2).

(c) We use mainly the terminology and notation of Grothendieck-Dieudonné [EGAI, §9.4 and 9.6] which agrees with that of Demazure-Grothendieck used in [SGA3, Exp. I.4]. Let  $S$  be a scheme and let  $\mathcal{E}$  be a quasi-coherent sheaf over  $S$ . For each morphism  $f : T \rightarrow S$ , we denote by  $\mathcal{E}_T = f^*(\mathcal{E})$  the inverse image of  $\mathcal{E}$  by the morphism  $f$ . We denote by  $\mathbf{V}(\mathcal{E})$  the affine  $S$ -scheme defined by  $\mathbf{V}(\mathcal{E}) = \text{Spec}(\text{Sym}^\bullet(\mathcal{E}))$ ; it is affine over  $S$  and represents the  $S$ -functor  $Y \mapsto \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)$  [EGAI, 9.4.9]. We

assume now that  $\mathcal{E}$  is locally free of finite rank and denote by  $\mathcal{E}^\vee$  its dual. In this case the affine  $S$ -scheme  $\mathbf{V}(\mathcal{E})$  is of finite presentation (ibid, 9.4.11); also the  $S$ -functor  $Y \mapsto H^0(Y, \mathcal{E}_Y) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)$  is representable by the affine  $S$ -scheme  $\mathbf{V}(\mathcal{E}^\vee)$  which is also denoted by  $\mathbf{W}(\mathcal{E})$  [SGA3, I.4.6].

## 2. AROUND KNESER-TITS' PROBLEM

Let  $A$  be a semilocal ring,  $I$  its radical.

**2.1. Subgroups of elementary elements.** Let  $H$  be a reductive  $A$ -group scheme which is strictly  $A$ -isotropic, that is, all factors of  $H_{ad}$  are isotropic. Let  $H$  be a strictly proper parabolic  $A$ -subgroup of  $H$  and let  $P^-$  be a  $A$ -parabolic subgroup opposite to  $P$ . The subgroup of  $H(A)$  generated by  $R_u(P)(A)$  and  $R_u(P^-)(A)$  does not depend of the choice of  $P$  and  $P^-$ , this is a normal subgroup of  $H(A)$  which is denoted by  $H(A)^+$  [P-S, Thm. 1 and comments].

**Lemma 2.1.** *We have  $H(A) = H(A)^+ P(A)$ .*

*Proof.* We put  $U = R_u(P)$ ,  $U^- = R_u(P^-)$ . According to [SGA3, XXVI.5.2], we have a decomposition  $H(A) = U(A) U^-(A) P(A)$  whence  $H(A) = H(A)^+ P(A)$ .  $\square$

A special case of [G-S, prop. 7.7] is the following.

**Lemma 2.2.** *Assume that  $(A, I)$  is an henselian couple and that  $H$  is semisimple simply connected. Then the map  $H(A)/H(A)^+ \rightarrow H(A/I)/H(A/I)^+$  is an isomorphism.*

**2.2. R-equivalence.** We assume now that  $A$  is a nonzero finite  $k$ -algebra for a field  $k$ . Then  $A$  is an Artinian  $k$ -algebra. We denote by  $I$  its Jacobson radical and remind the reader that  $(A, I)$  is an henselian couple.

Let  $H$  be an  $A$ -reductive group scheme and consider the Weil restriction  $G = R_{A/k}(H)$ , this is a smooth affine connected algebraic  $k$ -group [C-G-P, A.5.9]. We have  $G(k) = H(A)$ . We use  $R$ -equivalence for group schemes over rings as defined in [G-S].

**Lemma 2.3.** *We decompose  $A = A_1 \times \cdots \times A_d$  where each  $A_i$  is a local artinian  $k$ -algebra of residue field  $k_i$ .*

(1) *The map  $H(A) \rightarrow \prod_i H(k_i)$  induces isomorphisms*

$$G(k)/RG(k) \xrightarrow{\sim} H(A)/RH(A) \xrightarrow{\sim} \prod_i H(k_i)/RH(k_i).$$

(2) *If  $H$  is furthermore simply connected and strictly isotropic, we have a commutative diagram of isomorphisms*

$$\begin{array}{ccc}
H(A)/H(A)^+ & \xrightarrow{\sim} & H(A)/RH(A) \xleftarrow{\sim} G(k)/RG(k) \\
\downarrow \wr & & \downarrow \wr \\
\prod_i H(k_i)/H(k_i)^+ & \xrightarrow{\sim} & \prod_i H(k_i)/RH(k_i).
\end{array}$$

*Proof.* (1) The first isomorphism  $G(k)/RG(k) \xrightarrow{\sim} H(A)/RH(A)$  is a formal fact [G-S, lemma 2.4]. Let  $n$  be the smallest positive integer such that  $I^n = 0$ . We put  $A_j = A/I^{j+1}$  for  $j = 0, \dots, n-1$ , it comes with the ideal  $J_j = I^j/I^{j+1} = I A_j$  which satisfies  $J_j^2 = 0$ . We have  $A_j/J_j = A_{j-1}$  for  $j = 1, \dots, n-1$ .

We put  $G_j = R_{A_j/k}(H_{A_j})$  for  $j = 0, \dots, n-1$ ; we have a sequence of homomorphisms  $G = G_n \rightarrow G_{n-1} \rightarrow \dots \rightarrow G_1 \rightarrow G_0 = \prod_i R_{k_i/k}(H_{k_i})$ . It induces homomorphisms

$$G(k)/R = G_n(k)/R \rightarrow G_{n-1}(k)/R \rightarrow \dots \rightarrow G_1(k)/R \rightarrow G_0(k)/R = \prod_i H(k_i)/R$$

and we will show by a dévissage argument that all of them are isomorphisms.

Let  $j$  be an integer satisfying  $1 \leq j \leq n-1$ . According to a variant of [G-P-S, Lemma 8.3], we have an exact sequence of fppf (resp. étale, Zariski) sheaves on  $\text{Spec}(k)$

$$0 \rightarrow \mathbf{W}\left((t_j)_*(\text{Lie}(H)(A_{j-1}) \otimes_{A_{j-1}} J_j)\right) \rightarrow R_{A_j/k}(H_{A_j}) \rightarrow R_{A_i/k}(H_{A_{j-1}}) \rightarrow 1.$$

where  $t_j : \text{Spec}(A_j) \rightarrow \text{Spec}(k)$  is the structural morphism. We have then a sequence of  $k$ -algebraic groups

$$0 \rightarrow (\mathbb{G}_a)^{m_j} \rightarrow G_j \rightarrow G_{j-1} \rightarrow 1.$$

The map  $G_j \rightarrow G_{j-1}$  is a  $(\mathbb{G}_a)^{m_j}$ -torsor which is trivial since  $G_{j-1}$  is affine. We have then a decomposition of  $k$ -schemes  $G_j \xrightarrow{\sim} G_{j-1} \times_k (\mathbb{G}_a)^{m_j}$  which induces an isomorphism  $G_j(k)/R \rightarrow G_{j-1}(k)/R$ . Thus the map  $G(k)/R \rightarrow G_0(k)/R = \prod_i H(k_i)/R$  is an isomorphism as desired.

(2) We have  $A/I = k_1 \times \dots \times k_d$ . Lemma 2.2 provides an isomorphism  $H(A)/H(A)^+ \xrightarrow{\sim} H(A/I)/H(A/I)^+ = \prod_i H(k_i)/H(k_i)^+$  and we have an isomorphism  $H(k_i)/H(k_i)^+ \xrightarrow{\sim} H(k_i)/RH(k_i)$  for each  $i$  [Gil, thm. 7.2]. This completes the proof by chasing diagram.  $\square$

### 3. PATCHING, R-EQUIVALENCE AND TWISTED FLAG SCHEMES

**3.1. Using the implicit function theorem.** We consider a variation of the framework of [H-H-K, 2.4]. Let  $T$  be a complete DVR of fraction field  $K$  and  $t$  be an uniformizing parameter. Let  $\widehat{R}_0$  be ring containing  $T$  and which is also a complete discrete valuation ring having uniformizer  $t$ . We denote by  $F_0$  the fraction field of  $\widehat{R}_0$ .

Let  $\alpha > 1$  be a real number and we define the absolute value on  $F_0$  by  $|t^n u| = \alpha^{-n}$  for  $n \in \mathbb{Z}$  and  $u \in (\widehat{R}_0)^\times$ .

Let  $F_1, F_2$  be subfields of  $F_0$  containing  $T$ . We further assume that we are given  $t$ -adically complete  $T$ -submodules  $V \subset F_1 \cap \widehat{R}_0$  and  $W \subset F_2 \cap \widehat{R}_0$  satisfying the following conditions:

$$(3.1) \quad V + W = \widehat{R}_0;$$

$$(3.2) \quad V \cap t\widehat{R}_0 = tV \quad \text{and} \quad W \cap t\widehat{R}_0 = tW.$$

Note that Condition (3.2) is equivalent to

$$(3.3) \quad V \cap t^n \widehat{R}_0 = t^n V \quad \text{and} \quad W \cap t^n \widehat{R}_0 = t^n W \quad \text{for each } n \geq 1.$$

**Remark 3.1.** Condition (3.2) above is added there compared with [H-H-K, 2.4] but we do not require at this stage that  $F_1$  is dense in  $F_0$ .

We equip the submodules  $V[\frac{1}{t}]$  of  $F_0$  of the induced metric (and similarly for  $W[\frac{1}{t}]$ ).

**Lemma 3.2.** (1)  $V$  is closed in  $\widehat{R}_0$ .

(2) For  $v \in V[\frac{1}{t}] \setminus \{0\}$ , we have

$$|v| = \inf\{\alpha^n \mid t^n v \in V\}.$$

(3) We have  $V = \inf\{v \in V[\frac{1}{t}] \mid |v| \geq 0\}$  and  $V$  is a clopen submodule of  $V[\frac{1}{t}]$ .

(4)  $V[\frac{1}{t}]$  is closed in  $F_0$  and is a Banach  $K$ -space.

*Proof.* (1) Our assumption is that the map  $V \rightarrow \varprojlim V/t^{m+1}V$  is an isomorphism. Let  $(x_n)$  be a sequence of  $V$  which converges in  $\widehat{R}_0$ . For each  $m \geq 0$ , condition (3.3) shows that the map  $V/t^{m+1}V \rightarrow \widehat{R}_0/t^{m+1}\widehat{R}_0$  is injective so that the sequence  $(x_n)$  modulo  $t^{m+1}V$  is stationary to some  $v_m \in V/t^{m+1}V$ . The  $v_m$ 's define a point  $v$  of  $V$  and the sequence  $(x_n)$  converges to  $v$ .

(2) We are given  $v = t^m v' \in V[\frac{1}{t}]$  with  $v' \in V \setminus tV$  and we have  $|v| = \inf\{\alpha^n \mid t^n v \in \widehat{R}_0\} = \alpha^m \inf\{\alpha^n \mid t^n v' \in \widehat{R}_0\}$ . Condition (II) implies that  $v' \in \widehat{R}_0 \setminus t\widehat{R}_0$  so that  $|v| = 0$ . We conclude that  $|v| = \inf\{\alpha^n \mid t^n v \in V\}$ .

(3) It readily follows from the assertion (2).

(4) Let  $(x_n)$  be a sequence of  $V[\frac{1}{t}]$  converging to some  $x \in F_0$ . We want to show that  $x$  belongs to  $V[\frac{1}{t}]$  so that we can assume that  $x \neq 0$  and that  $|x_n| = |x| = \alpha^m$  for all  $n \geq 0$ . Assertion (2) shows that  $t^m x_n$  is a sequence of  $V$  and (1) shows that its limit  $t^m x$  belongs to  $V$ . Thus  $x \in V[\frac{1}{t}]$ . We have shown that  $V[\frac{1}{t}]$  is closed in  $F_0$ .

Finally since  $F_0$  is a Banach  $K$ -space so is  $V[\frac{1}{t}]$ .  $\square$

The following statement extends partially [H-H-K, th. 2.5] and [H-H-K2, prop. 4.1].

**Proposition 3.3.** *Let  $a, b, c$  be positive integers. Let  $\Omega \subset (F_0)^a \times (F_0)^b$  be an open neighborhood of  $(0, 0)$  and let  $f : \Omega \rightarrow (F_0)^c$  be an analytic map. We denote by  $f^a : \Omega \cap (F_0)^a \rightarrow (F_0)^c$  and  $f^b : \Omega \cap (F_0)^b \rightarrow (F_0)^c$ . We assume that*

(i)  $f(0, 0) = 0$

(ii) *the differentials  $Df_0^a : (F_0)^a \rightarrow (F_0)^c$  and  $Df_0^b : (F_0)^b \rightarrow (F_0)^c$  satisfy*

$$Df_0^a\left(V\left[\frac{1}{t}\right]^a\right) + Df_0^b\left(W\left[\frac{1}{t}\right]^b\right) = (F_0)^c.$$

*Then there is a real number  $\epsilon > 0$  such that for all  $y \in (F_0)^c$  with  $|y| \leq \epsilon$ , there exist  $v \in V^a$  and  $w \in W^b$  such that  $(v, w) \in \Omega$  and  $f(v, w) = y$ .*

*Proof.* We consider the continuous embedding  $i : V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \rightarrow (F_0)^a \times (F_0)^b$  and define  $\tilde{\Omega} = i^{-1}(\Omega)$  and the function  $\tilde{f} = f \circ i : \tilde{\Omega} \rightarrow (F_0)^c$ .

**Claim 3.4.** *The map  $\tilde{f}$  is strictly differentiable at  $(0, 0)$  and  $D\tilde{f}_{(0,0)} : V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \rightarrow (F_0)^c$  is onto.*

Since  $f$  is  $F_0$ -analytic at  $(0, 0)$ , it is strictly differentiable [Sc, I.5.6], that is, there exists an open neighborhood  $\Theta$  of  $(0, 0)$  and a positive real number  $\beta$  such that

$$|f(x_2) - f(x_1) - Df_{(0,0)} \cdot (x_2 - x_1)| \leq \beta |x_2 - x_1| \quad \forall x_1, x_2 \in \Theta.$$

It is then strictly derivable as function between the Banach  $K$ -spaces  $(F_0)^a \times (F_0)^b \rightarrow (F_0)^c$ . On the other hand the embedding  $i : V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \rightarrow (F_0)^a \times (F_0)^b$  is 1-Liftschitz so is strictly differentiable at  $(0, 0)$ . As composite of strictly differentiable functions,  $\tilde{f}$  is strictly differentiable at  $(0, 0)$  [B, §1.3.1]. Furthermore the differential  $D\tilde{f}_{(0,0)}$  is the composite of

$$V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \xrightarrow{i} (F_0)^a \times (F_0)^b \xrightarrow{Df_0^a + Df_0^b} (F_0)^c.$$

Condition (ii) says exactly that  $D\tilde{f}_{(0,0)}$  is surjective. The Claim is proven.

We apply the implicit function theorem to the function  $\tilde{f}$  [B, §1.5.2] (see [Sc, §4] for concocting a proof). Lemma 3.2.(4) shows that  $V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b$  is a Banach  $K$ -space and so is  $F_0$ . There exists then an open neighborhood  $\Upsilon \subset \tilde{\Omega}$  of  $(0, 0)$  in  $V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b$  such that  $\tilde{f}|_{\Upsilon}$  is open. Up to shrink  $\Upsilon$  we can assume that  $\Upsilon \subset V \times W$  according to Lemma 3.2.(3). There exists then a real number  $\epsilon > 0$  such that

$$\{y \in (F_0)^c \mid |y| \leq \epsilon\} \subset \{y \in (F_0)^c \mid |y| < 2\epsilon\} \subset \tilde{f}(\Upsilon).$$

We conclude that for all  $y \in (F_0)^c$  with  $|y| \leq \epsilon$ , there exist  $v \in V^a$  and  $w \in W^b$  such that  $(v, w) \in \Omega$  and  $f(v, w) = y$ .  $\square$

**Corollary 3.5.** *Let  $n$  be a positive integer. Let  $\Omega \subset (F_0)^n \times (F_0)^n$  be an open neighborhood of  $(0, 0)$  and let  $f : \Omega \rightarrow (F_0)^n$  be an analytic map which satisfies*

$$(i) \ f(0, 0) = 0$$

$$(ii) \ f(x, 0) = f(0, x) = x \text{ over an open neighborhood } \Upsilon \text{ of } 0.$$

*Then there is a real number  $\epsilon > 0$  such that for all  $a$  with  $|a| \leq \epsilon$ , there exist  $v \in V^n$  and  $w \in W^n$  such that  $(v, w) \in \Omega$  and  $f(v, w) = a$ .*

*Proof.* In this case we have  $a = b = c = n$  and  $Df_0^a = Df_0^b = \text{Id}_{(F_0)^n}$  so that Proposition 3.3 applies.  $\square$

**3.2. Kneser-Tits' subgroups.** Continuing in the previous setting, we assume furthermore that  $F_1$  is  $t$ -adically dense in  $F_0$ . Let  $F \subset F_1 \cap F_2$  be a subfield. For dealing later with Weil restriction issues it is convenient to deal with a finite field extension  $E$  of  $F$ .

**Proposition 3.6.** *Let  $H$  be a semisimple simply connected  $E$ -group scheme assumed strictly isotropic. We put  $G = R_{E/F}(H)$ . For each overfield  $L$  of  $F$ , we put  $G(L)^+ = H(L \otimes_F E)^+$  where the second group is that defined in §2.1. Then we have the decomposition*

$$G(F_0)^+ = G(F_1)^+ G(F_2)^+.$$

*Proof.* Without loss of generality we can assume that  $F$  is infinite. The proof is based on an analytic argument requiring some preparation. We put  $d = [E : F]$  and fix an isomorphism  $E \cong F^d$  of  $F$ -vector spaces.

Let  $P$  be a strictly proper parabolic  $E$ -subgroup of  $H$ . Let  $U$  be its unipotent radical and  $U_{\text{last}}$  the last part of Demazure's filtration [G-P-S, §3.2]. Let  $u : \mathbb{G}_{a,E}^d \xrightarrow{\sim} U_{\text{last}}$  be an  $E$ -group isomorphism. According to [G-P-S, Lemma 3.3.(3)],  $H(E)^+ \cdot \text{Lie}(U_{\text{last}})(E)$  generates  $\text{Lie}(H)(E)$  as a vector space over  $E$ . There exists  $h_1, \dots, h_n \in H(E)^+$  such that

$$\text{Lie}(H)(E) = {}^{h_1}\text{Lie}(U_{\text{last}})(E) + {}^{h_2}\text{Lie}(U_{\text{last}})(E) + \dots + {}^{h_n}\text{Lie}(U_{\text{last}})(E).$$

We consider the map  $b : (\mathbb{G}_{a,E}^d)^n \rightarrow H$ ,  $b(x_1, \dots, x_n) = {}^{h_1}u(x_1) \dots {}^{h_n}u(x_n)$ . Its differential at 0 is  $db_0 : E^{dn} \cong \text{Lie}(U_{\text{last}})(E)^n \rightarrow \text{Lie}(H)(E)$ ,  $(X_1, \dots, X_n) \mapsto \sum_{i=1}^n {}^{h_i}X_i$ , so is surjective. We cut now the affine space  $(\mathbb{G}_{a,E}^d)^n$  by some suitable affine  $E$ -subspace  $\mathbb{G}_{a,E}^r$  such that the restriction  $b'$  of  $b$  to  $(\mathbb{G}_{a,E}^d)^n$  is such that  $db'_0$  is an isomorphism. The map  $b'$  is then étale at a neighborhood of 0. It follows that  $b'_\sharp = R_{E/F}(b') : R_{E/F}(\mathbb{G}_{a,E}^r) \rightarrow G = R_{E/F}(H)$  is étale also at a neighborhood of 0 [C-G-P, A.5.2.(4)].

Since the field  $F_0$  is henselian, the local inversion theorem holds [G-G-MB, prop. 2.1.4]. We mean that there exists an open neighborhood  $\Upsilon \subset (E \otimes_F F_0)^r$  such that the restriction  $b'_\sharp|_\Upsilon : \Upsilon \rightarrow G(F_0)$  is a topological open embedding.

We consider now the product morphism  $q : \Upsilon \times \Upsilon \rightarrow G(F_0)$ ,  $q(x, y) = b'_\sharp(x) b'_\sharp(y)$ . We put  $\Omega = q^{-1}(b'_\sharp(\Upsilon))$ , this an open subset of  $(E \otimes_F F_0)^{2r} \cong (F_0)^{2dr}$ . Then the



restriction  $q|_{\Omega}$  defines an (unique) analytical map  $f : \Omega \rightarrow \Upsilon$  such that  $q(x, y) = b'_{\sharp}(f(x, y))$ .

By construction we have  $f(0, x) = f(x, 0) = x$  for  $x$  in a neighborhood of  $0 \in (E \otimes_F F_0)^r \cong F_0^{dr}$ . We apply Corollary 3.5 to  $f$  so that there exists  $\epsilon > 0$  such that for each  $a \in \Upsilon$  with  $|a| \leq \epsilon$ , there exist  $v \in W^{dr}$  and  $w \in W^{dr}$  such that  $(v, w) \in \Omega$  and  $f(v, w) = a$ . We denote by  $\Upsilon_{\epsilon} = \Upsilon \cap B(0, \epsilon)$ . Then  $b'_{\sharp}(\Upsilon_{\epsilon})$  is an open neighborhood of 0 in  $(F_0)^r$ .

Let us now prove that  $G(F_0)^+ = G(F_1)^+ \times G(F_2)^+$ . Since  $F_1$  is dense in  $F_0$ ,  $G(F_1)^+$  is dense in  $G(F_0)^+$ . It is then enough to show that  $\Upsilon_{\epsilon} \subset G(F_1)^+ \times G(F_2)^+$ . Let  $g = b'_{\sharp}(a) \in b'_{\sharp}(\Upsilon_{\epsilon})$ . Then  $a = f(v, w)$  with  $(v, w) \in \Omega$ . It follows that  $g = b'_{\sharp}(a) = b'_{\sharp}(f(v, w)) = q(v, w) = b'_{\sharp}(v) b'_{\sharp}(w) \in G(F_1)^+ \times G(F_2)^+$ .  $\square$

This could be refined as follows.

**Proposition 3.7.** *Let  $H$  be a reductive  $E$ -group and put  $G = R_{E/F}(H)$ .*

- (1)  *$RG(F_1)RG(F_2)$  contains an open neighborhood of 1 in  $G(F_0)$ .*
- (2) *If  $RG(F_1)$  is dense in  $RG(F_0)$ , then  $RG(F_1)RG(F_2) = RG(F_0)$ .*
- (3) *If  $H$  is semisimple simply connected and  $H_{F_1 \otimes_F E}$  is strictly isotropic, then we have*

$$G(F_1)^+ RG(F_2) = G(F_0)^+.$$

- (4) *If  $H$  is semisimple simply connected and  $H_{F_i \otimes_F E}$  is strictly isotropic for  $i = 1, 2$ , then  $G(F_1)^+ G(F_2)^+ = G(F_0)^+$ .*

The subgroups  $G(F_1)^+$ ,  $G(F_0)^+$  are defined as in Proposition 3.6.

*Proof.* Once again we put  $d = [E : F]$  and fix an isomorphism  $E \cong F^d$  of  $F$ -vector spaces.

- (1) Let  $T \subset H$  be a maximal  $E$ -torus and let  $1 \rightarrow S \rightarrow Q \xrightarrow{s} T \rightarrow 1$  be a resolution of  $T$  where  $Q$  is a quasitrivial torus and  $S$  is a torus. We have  $Q = R_{C/E}(\mathbb{G}_m)$  where  $C$  is an étale  $E$ -algebra so that  $Q$  is an open subset of the affine  $E$ -space  $\mathbf{W}(C)$ .

We use now Raghunathan's technique [R, §1.2]. There exists  $h_1, \dots, h_n \in H(E)$  such that  $\text{Lie}(H)(E) = {}^{h_1}\text{Lie}(T)(E) + {}^{h_2}\text{Lie}(T)(E) + \dots + {}^{h_n}\text{Lie}(T)(E)$ . We define the open  $E$ -subvariety  $U \subset \mathbf{W}_E(C^n)$  where  $1 + x_1, \dots, 1 + x_n$  belongs to  $Q$ . We consider the map  $b : U \rightarrow H$ ,  $b(x_1, \dots, x_n) = {}^{h_1}s(1+x_1) \dots {}^{h_n}s(1+x_n)$ . Its differential at 0 is

$$db_0 : C^n \rightarrow \text{Lie}(H)(E), \quad (c_1, \dots, c_n) \mapsto \sum_{i=1}^n {}^{h_i}ds(c_i),$$

and is onto (observe that  $\text{Lie}(Q)(E) \rightarrow \text{Lie}(T)(E)$  is surjective since  $Q \rightarrow E$  is smooth). We cut now the  $E$ -affine space  $\mathbf{W}_E(C^n)$  by some suitable affine  $E$ -subspace  $\mathbf{W}_E(E^r)$  of  $\mathbf{W}_E(C^n)$  such that the restriction  $b'$  of  $b$  to the  $E$ -variety  $Y = U \cap \mathbf{W}_E(E^r)$  is such that  $db'_0 : \text{Tan}_{Y,0} \rightarrow \text{Lie}(H)(E)$  is an isomorphism. Note that  $Y$  and  $H$  have same dimension  $r$  over  $E$ .

The map  $b'$  is then étale at a neighborhood of 0. It follows that  $b'_\# = R_{E/F}(b') : X = R_{E/F}(Y) \rightarrow G = R_{E/F}(H)$  is étale also at a neighborhood of 0 [C-G-P, A.5.2.(4)]. Note that  $X$  is an open subset of  $R_{E/F}(\mathbf{W}_E(E^r)) \cong \mathbf{W}(E^r) \cong \mathbf{A}_F^{dr}$ .

Since the field  $F_0$  is henselian, the local inversion theorem holds [G-G-MB, prop. 2.1.4]. We mean that there exists an open neighborhood  $\Upsilon \subset X(F_0) \subset (F_0 \otimes_F E)^r \cong (F_0)^{dr}$  of 0 such that the restriction  $b'_{\#|\Upsilon} : \Upsilon \rightarrow G(F_0)$  is a topological open embedding.

We consider now the product morphism  $q : \Upsilon \times \Upsilon \rightarrow G(F_0)$ ,  $q(x, y) = b'_\#(x) b'_\#(y)$ . We put  $\Omega = q^{-1}(b'_\#(\Upsilon))$ , this an open subset of  $(F_0)^{2dr}$ . Then the restriction  $q|_\Omega$  defines a (unique) analytical map  $f : \Omega \rightarrow \Upsilon$  such that  $q(x, y) = b'_\#(f(x, y))$ .

By construction we have  $f(0, x) = f(x, 0)$  for  $x$  in a neighborhood of  $0 \in (F_0 \otimes_F E)^r \cong F_0^{dr}$ . We apply Corollary 3.5 to  $f$  so that there exists  $\epsilon > 0$  such that for each  $a \in \Upsilon$  with  $|a| \leq \epsilon$ , there exist  $v \in V^{dr}$  and  $w \in W^{dr}$  such that  $(v, w) \in \Omega$  and  $f(v, w) = a$ . We denote by  $\Upsilon_\epsilon = \Upsilon \cap B(0, \epsilon)$ . Then  $b'_\#(\Upsilon_\epsilon)$  is an open neighborhood of 1 in  $(F_0)^{dr}$ .

We claim that  $\Upsilon_\epsilon \subset RG(F_1)RG(F_2)$ . Let  $g = b'_\#(a) \in b(\Upsilon_\epsilon)$ . Then  $a = f(v, w)$  with  $(v, w) \in \Omega$ . It follows that  $g = b'_\#(a) = b'_\#(f(v, w)) = q(v, w) = b'_\#(v) b'_\#(w) \in RG(F_1) \times RG(F_2)$ .

(2) If  $RG(F_1)$  is furthermore dense in  $RG(F_0)$ , then (1) shows that  $RG(F_1)RG(F_2)$  is a dense open subset of  $RG(F_0)$ . Thus  $RG(F_1)RG(F_2) = RG(F_0)$ .

(3) We assume that  $H$  is semisimple simply connected and that  $G_{F_1 \otimes_F E_1}$  is strictly isotropic. According to Lemma 2.2.(2), we have  $G(F_1)^+ = RG(F_1) = RH(F_1 \otimes_F E) = H^+(F_1 \otimes_F E)$  and similarly for  $F_0$ .

Since  $H^+(F_1 \otimes_F E)^+$  is dense in  $H^+(F_0 \otimes_F E)$ , it follows that  $RG(F_1)$  is dense in  $RG(F_0)$ . Assertion (2) yields then  $RG(F_1)RG(F_2) = RG(F_0)$ . Lemma 2.3.(2) states that  $G(F_1)^+ = RG(F_1)$  and  $G(F_0)^+ = RG(F_0)$ . We conclude that  $G(F_1)^+ RG(F_2) = G(F_0)^+$ .

(4) We have furthermore  $G(F_2)^+ = RG(F_2)$  so that (3) yields  $G(F_1)^+ G(F_2)^+ = G(F_0)^+$ .  $\square$

**Remark 3.8.** Note that (1) shows in particular that  $RG(F_0)$  is an open subgroup of  $G(F_0)$ .

The main result of the section is the following patching statement on twisted flag varieties.

**Theorem 3.9.** *We put  $F = F_1 \cap F_2$ . Let  $H$  be a reductive  $E$ -group and let  $X$  be a twisted flag  $E$ -variety of  $H$ . We put  $G = R_{E/F}(H)$  and  $Z = R_{E/F}(X)$ . If  $Z(F_1) \neq \emptyset$  and  $Z(F_2) \neq \emptyset$ , then  $Z(F) \neq \emptyset$ .*

*Proof.* Without loss of generality, we can assume that  $H$  is semisimple simply connected and that it is absolutely  $E$ -simple. This implies that  $H$  is strictly  $F_i \otimes_F E$ -isotropic for  $i = 1, 2$ . Let  $x_i \in Z(F_i) = X(F_i \otimes_F E)$  and denote by  $P_i = \text{Stab}_{G_{F_i}}(x_i)$

its stabilizer, this is parabolic  $F_i \otimes_F E$ -subgroup of  $H$ . Since  $F_i \otimes_F E$  is a semilocal algebra, we have  $x_1 = h.x_2$  for some  $h \in H(E \otimes_F F_0)$  [SGA3, XXVI.5.2]. According to Lemma 2.1, we have  $H(E \otimes_F F_0) = H^+(E \otimes_F F_0) P_2(E \otimes_F F_0)$  so we can assume that  $h \in H^+(E \otimes_F F_0)$ . According to Proposition 3.7.(4), we can write  $h = h_1 h_2$  with  $h_i \in H^+(E \otimes_F F_i)$  for  $i = 1, 2$ . Since  $x_1 = h.x_2$ , we obtain  $h_1^{-1}.x_1 = h_2^{-1}.x_2$ . This defines a point of  $X(E) = Z(F)$ .  $\square$

#### 4. RELATION WITH THE ORIGINAL HHK METHOD

We recall the setting. Let  $T$  be an excellent complete discrete valuation ring with fraction field  $K$ , residue field  $k$  and uniformizing parameter  $t$ . Let  $F$  be a one-variable function field over  $K$  and let  $\mathfrak{X}$  be a normal model of  $F$ , i.e. a normal connected projective  $T$ -curve with function field  $F$ . We denote by  $Y$  the closed fiber of  $\mathfrak{X}$  and fix a separable closure  $F_s$  of  $F$ .

For each point  $P \in Y$ , let  $R_P$  be the local ring of  $\mathfrak{X}$  at  $P$ ; its completion  $\widehat{R}_P$  is a domain with fraction field denoted by  $F_P$ .

For each subset  $U$  of  $Y$  that is contained in an irreducible component of  $Y$  and does not meet the other components, we define  $R_U = \bigcap_{P \in U} R_P \subset F$ . We denote by

$\widehat{R}_U$  the  $t$ -adic completion of  $R_U$ . The rings  $R_U$  and  $\widehat{R}_U$  are excellent normal domains and we denote by  $F_U$  the fraction field of  $\widehat{R}_U$  [H-H-K3, Remark 3.2.(b)].

Each height one prime  $\mathfrak{p}$  in  $\widehat{R}_P$  that contains  $t$  defines a branch of  $Y$  at  $P$  lying on some irreducible component of  $Y$ . The  $t$ -adic completion  $\widehat{R}_{\mathfrak{p}}$  of the local ring  $R_{\mathfrak{p}}$  of  $\widehat{R}_P$  at  $\mathfrak{p}$  is a complete DVR whose fraction field is denoted by  $F_{\mathfrak{p}}$ . The field  $F_{\mathfrak{p}}$  contains also  $F_U$  if  $U$  is an irreducible open subset of  $Y$  such that  $P \in \overline{U} \setminus U$ . We have then a diagram of fields

$$\begin{array}{ccc} & F_{\mathfrak{p}} & \\ & \swarrow \quad \nwarrow & \\ F_P & & F_U. \end{array}$$

**Example 4.1.** We assume that  $T = k[[t]]$  and take  $X = \mathbb{P}_K^1$ ,  $\mathfrak{X} = \mathbb{P}_T^1$ ,  $P = \infty_k$ ,  $U = \mathbb{A}_k^1 = \text{Spec}(k[x])$ . The ring  $R_U$  contains  $k[[t]][x]$  and is its localization with respect to the multiplicative set  $S$  of elements which are units modulo  $t$ . The  $t$ -adic completion of  $R_U$  is  $\widehat{R}_U = k[x][[t]]$ ; we have  $F_U = \text{Frac}(\widehat{R}_U) = k(x)((t))$ . The local ring of  $\mathfrak{X}$  at  $P = \infty_k$  is  $R_P = k[[t]][x^{-1}]_{(x^{-1}, t)}$  so that its completion is  $\widehat{R}_P = k[[t, x^{-1}]]$ ; in particular  $F_P = k((t, x^{-1}))$ .

We take  $\mathfrak{p} = t\widehat{R}_P \subset \widehat{R}_P$  and the  $t$ -adic completion  $\widehat{R}_{\mathfrak{p}}$  of the local ring  $R_{\mathfrak{p}}$  of  $\widehat{R}_P$  at  $\mathfrak{p}$  is a complete DVR which is  $k((x^{-1}))[[t]]$ . In particular  $F_{\mathfrak{p}} = k((x^{-1}))((t))$ .

**Setting 4.2.** Let  $\mathcal{P}$  be a non-empty finite set of closed points of  $Y$  that contains all the closed points at which distinct irreducible components meet. Let  $\mathcal{U}$  be the set of connected components of  $Y \setminus \mathcal{P}$  and let  $\mathcal{B}$  be the set of branches of  $Y$  at points of  $\mathcal{P}$ . This yields a finite inverse system of field  $F_P, F_U, F_{\mathfrak{p}}$  (for  $P \in \mathcal{P}$ ;  $U \in \mathcal{U}$ ,  $\mathfrak{p} \in \mathcal{B}$ ) where  $F_P, F_U \subset F_{\mathfrak{p}}$  if  $\mathfrak{p}$  is a branch of  $Y$  at  $P$  lying in the closure of  $U$ .

**Lemma 4.3.** *We assume that  $X = \mathbb{P}_K^1$ ,  $\mathfrak{X} = \mathbb{P}_T^1$ ,  $P = \infty_k$ ,  $U = \mathbb{A}_k^1 = \text{Spec}(k[x])$  and  $\mathfrak{p}$  the branch of  $P$ . We put  $F_1 = F_P$ ,  $F_2 = F_U$  and  $F_0 = F_{\mathfrak{p}}$ .*

(1)  $F_1$  is  $t$ -dense in  $F_0$ .

(2) We put  $V = F_1 \cap \widehat{R}_{\mathfrak{p}}$  and  $W = F_2 \cap \widehat{R}_{\mathfrak{p}}$ . Then  $V$  and  $W$  satisfy conditions (3.1) and (3.2).

*Proof.* (1) We are given  $u_0/v_0 \in F_0$  with  $u_0, v_0 \in \widehat{R}_{\mathfrak{p}}$ ,  $v_0 \neq 0$ . There exists elements  $u, v \in R_{\mathfrak{p}}$  very close respectively of  $u_0, v_0$  with  $v \neq 0$ . Let  $s_1, s_2 \in R_P \setminus R_{\mathfrak{p}}$  such that  $s_1 u \in R_P$  and  $s_2 v \in R_P$ . Then  $u_0/v_0$  is very close of  $(s_1 s_2 u)/(s_1 s_2 v) \in F_1$ .

(2) Condition (3.2) is obviously fulfilled since  $t \in F_1 \cap F_2$ . For establishing condition (3.1), we are given an element  $f$  of  $\widehat{R}_{\mathfrak{p}}$  and may write it as  $f = \sum_{i=0}^{\infty} x^{m_i} \left( \sum_{j=0}^{\infty} a_{i,j} \frac{1}{x^j} \right) t^i$  where the  $m_i$ 's are non-negative integers and  $a_{i,j} \in T$ . We decompose

$$f = f_1 + f_2 = \sum_{i=0}^{\infty} x^{m_i} \left( \sum_{j=m_i}^{\infty} a_{i,j} \frac{1}{x^j} \right) t^i + \sum_{i=0}^{\infty} x^{m_i} \left( \sum_{j=0}^{m_i-1} a_{i,j} \frac{1}{x^j} \right) t^i$$

We observe that  $f_2$  belongs to  $\widehat{R}_P$  so belongs to  $V$ . We recall that  $R_U$  is the localization of  $T[x]$  with respect to the elements which are units modulo  $t$ . We conclude that  $f_2$  belongs to  $W$  as desired.  $\square$

**Theorem 4.4.** *Let  $G$  be a reductive  $F$ -algebraic group.*

(1) *Let  $Z$  be a twisted flag projective  $F$ -variety for  $G$ . Then  $Z(F) \neq \emptyset$  if and only if  $Z(F_U) \neq \emptyset$  for each  $U \in \mathcal{U}$  and  $Z(F_P) \neq \emptyset$  for each  $P \in \mathcal{P}$ .*

(2) *For each  $U \in \mathcal{U}$  (resp. each  $P \in \mathcal{P}$ ), we fix an  $F$ -embeddings  $i_U : F_s \rightarrow F_{U,s}$  (resp.  $i_P : F_s \rightarrow F_{P,s}$ ) providing identifications  $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{U,s}})$  (resp.  $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{P,s}})$ ). The Tits index  $\Delta_0(G)$  is the smallest subset of  $\Delta(G_{F_s})$  which is stable under the  $\star$ -action of  $\text{Gal}(F_s/F)$  and such that  $\Delta_0(G) \subset \Delta_0(G_{F_U})$  for each  $U \in \mathcal{U}$  and  $\Delta_0(G) \subset \Delta_0(G_{F_P})$  for each  $P \in \mathcal{P}$ .*

The recollection for star action and Tits index is done in the beginning of the paper.

*Proof.* (1) We use a Weil restriction argument as in the proof of [H-H-K2, Thm. 4.2]. This involves a finite morphism  $f : \mathfrak{X} \rightarrow \mathbb{P}_T^1$  such that  $\mathcal{P} = f^{-1}(\infty_k)$ . Write  $\underline{F}$  for the function field of  $\mathbb{P}_T^1$ , and let  $d = [F : F']$ . We put  $\underline{U} = \mathbb{P}_k^1 \setminus \{\infty\}$ ,  $\underline{P} = \infty_k$  and  $\underline{\mathfrak{p}} = (U, \mathfrak{p})$  and  $F_0 = \underline{F}_{\underline{\mathfrak{p}}}$ ,  $F_1 = \underline{F}_{\underline{P}}$  and  $F_2 = \underline{F}_{\underline{U}}$ . Also patching holds for the diamond  $(\underline{F}, F_1, F_2, F_0)$  according to [H-H, thm 3.9] so in particular  $K(x) = \underline{F} = F_1 \cap F_2 \subset F_0$ .

We put  $V = F_1 \cap \widehat{R}_{\mathfrak{p}}$  and  $W = F_2 \cap \widehat{R}_{\mathfrak{p}}$ . Lemma 4.3 shows that  $F_1$  is dense in  $F_0$  and that  $V$  and  $W$  satisfy conditions (3.1) and (3.2).

We consider the Weil restriction  $\underline{G} = R_{F/\underline{F}}(G)$ , it is a reductive  $\underline{F}$ -group which acts on the  $\underline{F}$ -variety  $\underline{Z} = R_{F/\underline{F}}(Z)$ . We have

$$\underline{Z}(F_1) = Z(F_1 \otimes_{\underline{F}} F) = \prod_{P \in \mathcal{P}} Z(F_P)$$

according to [H-H, Lemma 6.2.(a)]. Similarly we have

$$\underline{Z}(F_2) = Z(F_2 \otimes_{\underline{F}} F) = \prod_{U \in \mathcal{U}} Z(F_U)$$

Our assumptions imply that  $\underline{Z}(F_1) \neq \emptyset$  and  $\underline{Z}(F_2) \neq \emptyset$ . Theorem 3.9 implies that  $\underline{Z}(\underline{F}) \neq \emptyset$ . Thus  $\underline{Z}(\underline{F}) = Z(F)$  is non-empty.

(2) Let  $\Theta$  be the smallest subset of  $\Delta(G_{F_s})$  which is stable under the  $\star$ -action of  $\text{Gal}(F_s/F)$  and such that  $\Delta_0(G) \subset \Delta_0(G_{F_U})$  for each  $U \in \mathcal{U}$  and  $\Delta_0(G) \subset \Delta_0(G_{F_P})$  for each  $P \in \mathcal{P}$ . We observe that  $\Delta_0(G) \subset \Theta$  since it is stable under the star action and satisfies  $\Delta_0(G) \subset \Delta_0$  for each  $U \in \mathcal{U}$  and  $\Delta_0(G) \subset \Delta_0(G_{F_P})$  for each  $P \in \mathcal{P}$ . For the converse inclusion we consider the  $F$ -variety  $Z$  of parabolic subgroups of type  $\Theta$ . For each  $U \in \mathcal{U}$ , we have  $\Theta \subset \Delta_0(G_{F_U})$  so that  $Z(F_U) \neq \emptyset$ ; similarly we have  $\Theta \subset \Delta_0(G_{F_P})$  for each  $P \in \mathcal{P}$  so that  $Z(F_P) \neq \emptyset$ . Part (1) yields that  $Z(F) \neq \emptyset$ . Thus  $\Theta \subset \Delta_0(G)$  and  $\Theta = \Delta_0(G)$   $\square$

**Corollary 4.5.** *Let  $G$  be a reductive  $F$ -algebraic group.*

(1) *Let  $Z$  be a twisted flag projective  $F$ -variety for  $G$ . Then  $Z(F) \neq \emptyset$  if and only if  $Z(F_P) \neq \emptyset$  for each  $P \in Y$ .*

(2) *For each  $P \in Y$ , we fix an  $F$ -embedding  $i_P : F_s \rightarrow F_{P,s}$  providing identifications  $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{P,s}})$ . The Tits index  $\Delta_0(G)$  is the smallest subset of  $\Delta(G_{F_s})$  which is stable under the  $\star$ -action of  $\text{Gal}(F_s/F)$  and such that  $\Delta_0(G) \subset \Delta_0(G_{F_P})$  for each  $P \in Y$ .*

*Proof.* (1) We assume  $Z(F_P) \neq \emptyset$  for each  $P \in Y$ . Let  $Y_1, \dots, Y_d$  be the irreducible components of  $Y$  with respective generic points  $\eta_1, \dots, \eta_d$ . According to [H-H-K2, prop. 5.8], there exists non-empty affine subsets  $U_i \subset Y_i$  ( $i = 1, \dots, d$ ) such that  $Z(F_{U_i}) \neq \emptyset$  for  $i = 1, \dots, d$  and  $U_i \cap U_j = \emptyset$  for  $i < j$ . We apply Theorem 4.4 to  $\mathcal{U} = \{U_1, \dots, U_d\}$ ,  $\mathcal{P} = Y \setminus \bigcup_i U_i$  and get that  $Z(F) \neq \emptyset$ .

(2) This readily follows of (1).  $\square$

**Corollary 4.6.** *Let  $G$  be a reductive  $F$ -algebraic group and assume that  $G$  is the generic fiber of a reductive  $\mathfrak{X}$ -group scheme  $\mathfrak{G}$ .*

(1) *Let  $Z$  be a twisted flag projective  $F$ -variety for  $G$ . Then  $Z(F) \neq \emptyset$  if and only if  $Z(F_v) \neq \emptyset$  for each discrete valuation  $v$  of  $F$ .*

(2) For each discrete valuation  $v$  of  $F$ , we fix an  $F$ -embedding  $i_v : F_s \rightarrow F_{v,s}$  providing identifications  $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{v,s}})$ . The Tits index  $\Delta_0(G)$  is the smallest subset of  $\Delta(G_{F_s})$  which is stable under the  $\star$ -action of  $\text{Gal}(F_s/F)$  and such that  $\Delta_0(G) \subset \Delta_0(G_{F_v})$  for each discrete valuation  $v$  on  $F$ .

*Proof.* (1) We assume that  $Z(F_v) \neq \emptyset$  for each discrete valuation  $v$  of  $F$ . Let  $\mathfrak{G}$  be a reductive  $\mathfrak{X}$ -group of generic fiber  $G$ . Without loss of generality we can assume that  $\mathfrak{G}$  is adjoint. Let  $\mathfrak{G}_0$  be the Chevalley form of  $\mathfrak{G}$  and let  $(\mathfrak{B}_0, \mathfrak{T}_0)$  be a Killing couple for  $\mathfrak{G}_0$  and let  $\Delta_0$  be the associated Dynkin diagram. Since  $\mathfrak{G}_0$  is adjoint, we have an exact sequence of  $\mathbb{Z}$ -group schemes [SGA3, XXIV.1.3 and 3.6]

$$1 \rightarrow \mathfrak{G}_0 \rightarrow \text{Aut}(\mathfrak{G}_0) \rightarrow \text{Aut}(\Delta_0) \rightarrow 1$$

The  $\text{Aut}(\mathfrak{G}_0)$ -torsor  $\mathfrak{Q} = \text{Isom}(\mathfrak{G}_0, \mathfrak{G})$  over  $\mathfrak{X}$  defines then an  $\text{Aut}(\Delta_0)$ -torsor over  $\mathfrak{X}$ . Since  $\text{Aut}(\Delta_0)$  is a finite constant group, Grothendieck's theory of fundamental groups tells us that this torsor is the data of a morphism  $v : \Pi_1(\mathfrak{X}, \bullet) \rightarrow \text{Aut}(\Delta_0)$  where the base point is  $\text{Spec}(F_s) \rightarrow \text{Spec}(F) \rightarrow \mathfrak{X}$ . The star action associated to  $G$  is the composite

$$\text{Gal}(F_s/F) \rightarrow \Pi_1(\mathfrak{X}, \bullet) \rightarrow \text{Aut}(\Delta_0).$$

**Claim 4.7.**  *$Z$  is the generic fiber of an  $\mathfrak{X}$ -scheme of parabolic subgroups  $\mathfrak{Z}$  of  $\mathfrak{G}$ .*

The variety  $Z$  is a form of the variety  $\text{Par}_I(G_0)$  of parabolic subgroups of type  $I$  where  $I \subset \Delta_0$  is stable the star action. Since  $\mathfrak{X}$  is normal, the map  $\text{Gal}(F_s/F) \rightarrow \Pi_1(\mathfrak{X}, \bullet)$  is onto so that  $I$  is stable under the action of  $\Pi_1(\mathfrak{X}, \bullet)$  on  $\Delta_0$ . In particular  $Q$  admits a reduction  $\mathfrak{Q}_I$  to the stabilizer  $\text{Aut}_I(\mathfrak{G}_0)$  for the action of  $\text{Aut}(\mathfrak{G}_0)$  on  $\Delta_0$ . The  $\mathfrak{X}$ -scheme  $\mathfrak{Z} = {}^{\mathfrak{Q}}\text{Par}_I(\mathfrak{G}_0)$  is the scheme of parabolic subgroups of type  $I$  of  $\mathfrak{G} = {}^{\mathfrak{Q}}\mathfrak{G}_0$ , so that  $Z$  is the generic fiber of  $\mathfrak{Z}$ .

For applying Corollary 4.5, we have to check that  $Z(F_P) \neq \emptyset$  for each  $P \in Y$ . If  $Q$  is a point of codimension 1 of  $\mathfrak{X}$ , it defines a discrete valuation  $v_Q$  on  $F$  whose completion is  $F_Q$ . Our assumption implies then that  $Z(F_Q) \neq \emptyset$  in that case. We deal now with the case of a closed point  $P$  of  $\mathfrak{X}$ . Let  $D$  be an irreducible component of  $Y = \mathfrak{X}_k$  containing  $P$  and let  $Q$  be the generic point of  $D$ . Since  $\mathfrak{Z}$  is proper over  $T$ , we have  $\mathfrak{Z}(\widehat{R}_Q) = \mathfrak{Z}(F_Q)$  which is not empty by the preceding case. It follows that  $\mathfrak{Z}_k(k(D)) \neq \emptyset$ . Again  $\mathfrak{Z}_k$  is projective so that  $\mathfrak{Z}_k(D) = \mathfrak{Z}_k(k(D))$  is not empty and in particular  $\mathfrak{Z}_k(k(P))$  is not empty. Since  $\mathfrak{Z}$  is smooth over  $\mathfrak{X}$ , the Hensel lemma shows that  $\mathfrak{Z}_k(\widehat{R}_{\mathfrak{X},P}) \rightarrow \mathfrak{Z}_k(k(P))$  is surjective. Thus  $\mathfrak{Z}(\widehat{R}_{\mathfrak{X},P})$  is not empty and so is  $Z(F_P)$ .

(2) It readily follows of (1). □

**Corollary 4.8.** *Let  $G$  be a reductive  $F$ -algebraic group. We denote by  $\Omega_F^1$  the set of rank one valuations of  $F$ .*

(1) *Let  $Z$  be a twisted flag projective  $F$ -variety for  $G$ . Then  $Z(F) \neq \emptyset$  if and only if  $Z(F_v) \neq \emptyset$  for each  $v \in \Omega_F^1$ .*

(2) For each  $v \in \Omega_F^1$ , we fix an  $F$ -embedding  $i_v : F_s \rightarrow F_{v,s}$  providing identifications  $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{v,s}})$ . The Tits index  $\Delta_0(G)$  is the smallest subset of  $\Delta(G_{F_s})$  which is stable under the  $\star$ -action of  $\text{Gal}(F_s/F)$  and such that  $\Delta_0(G) \subset \Delta_0(G_{F_v})$  for each  $v \in \Omega_F^1$ .

*Proof.* (1) We assume that  $Z(F_v) \neq \emptyset$  for each  $v \in \Omega_F^1$ . According to [H-H-K-P, Thm 2.5], there exists a regular proper model  $\mathfrak{X}'$  of  $F$  such that  $Z(F_P) \neq \emptyset$  for each  $P \in \mathfrak{X}' \times_T k$ . Then Corollary 4.6.(1) shows that  $Z(F) \neq \emptyset$ .

(2) This readily follows of (1).  $\square$

## 5. LOCAL-GLOBAL PRINCIPLE FOR DISCRETE VALUATIONS

Let  $T$  be an excellent DVR of fraction field  $K$  and residue field  $k$ . Let  $X$  be a smooth, projective, geometrically integral curve over  $K$ . Let  $F = K(X)$  be the function field of  $X$ .

**Lemma 5.1.** *Let  $\mathfrak{X}$  be a projective, flat curve over  $A$  which is connected and regular such that  $\mathfrak{X}_K = X$ . Let  $\mathfrak{H}$  be a flat affine  $\mathfrak{X}$ -group scheme of finite presentation and assume that there exists a Zariski cover  $(\mathfrak{U}_i)_{i \in I}$  of  $\mathfrak{X}$  such that each  $\mathfrak{H}_{\mathfrak{U}_i}$  admits a closed embedding  $\mathfrak{H}_{\mathfrak{U}_i} \subset \text{GL}_{n_i}$  such that  $\text{GL}_{n_i}/\mathfrak{H}_{\mathfrak{U}_i}$  is representable by an affine  $\mathfrak{U}_i$ -scheme,*

*Let  $\gamma \in H_{\text{fppf}}^1(F, \mathfrak{H})$  and let  $D$  be a divisor of  $\mathfrak{X}$  which contains the irreducible components of  $Y = \mathfrak{X}_k$  and such that  $\gamma$  extends to  $X \setminus D_K$ .*

*Then there exists a proper birational morphism  $q : \mathfrak{X}' \xrightarrow{\sim} \mathfrak{X}$  such that  $\mathfrak{X}'$  is a regular proper model of  $X$  and such that  $D' = q^*D$  is a strict normal crossing divisor and*

$$\gamma \in \text{Im}\left(H^1(\mathfrak{X}' \setminus D', \mathfrak{H}) \rightarrow H^1(F, \mathfrak{H})\right).$$

*Proof.* Using a passage to the limit argument [Mg], there exists an open affine subscheme  $\mathfrak{U}_1 \subset \mathfrak{X} \setminus D$  such that  $\gamma$  extends to a class  $\gamma_1 \in H^1(\mathfrak{U}_1, \mathfrak{H})$ . According to [G-P1, cor 1.8],  $\gamma_1$  extends to a class  $\gamma_2 \in H^1(\mathfrak{U}_2, \mathfrak{H})$  where  $\mathfrak{U}_2$  is an open subscheme of  $\mathfrak{X} \setminus D$  containing  $\mathfrak{U}_1$  and  $X \setminus D_K$ . By purity (i.e. Theorem 7.1 of the appendix 7), we have  $H^1(\mathfrak{X} \setminus D, \mathfrak{H}) = H^1(\mathfrak{U}_2, \mathfrak{H})$ . Thus  $\gamma$  extends over  $\mathfrak{X} \setminus D$ . According to Lipman's theorem we can resolve the singularities of  $\mathfrak{X}$  and transform  $D$  in a strict normal crossing divisor, see [H-H-K, lemma 4.7].  $\square$

**Proposition 5.2.** *Let  $G$  be a reductive  $F$ -group and assume that  $p$  does not divide the order of the automorphism group of the absolute root system of  $G_{ad}$ . Let  $Z$  be a twisted flag variety of  $G$ . We assume that  $Z(F_v) \neq \emptyset$  for all discrete valuations of  $F$  arising from models of  $X$ . Then there exists a regular proper model  $\mathfrak{X}$  of  $X$  with special fiber  $Y = \mathfrak{X}_k$  such that for every point  $y \in Y$ , then  $Z(F_y) \neq \emptyset$ .*

*Proof.* Without loss of generality we can assume that  $G$  is adjoint. Let  $G_0$  be the Chevalley form of  $G$  and let  $(B_0, T_0)$  be a Killing couple for  $G_0$  and let  $\Delta_0$  be the associated Dynkin diagram. The variety  $Z$  is a form of the variety  $\text{Par}_I(G_0)$  of

parabolic subgroups of type  $I$  where  $I \subset \Delta_0$  is stable under the star action defined by the  $\text{Aut}(G_0)$ -torsor  $Q = \text{Isom}(G_0, G)$ . In particular  $Q$  admits a reduction  $Q_I$  to the stabilizer  $\text{Aut}_I(G_0)$  for the action  $\text{Aut}(G_0)$  on  $\Delta_0$  through the morphism  $\text{Aut}(G_0) \rightarrow \text{Out}(G_0) \xrightarrow{\sim} \text{Aut}(\Delta_0)$ . Furthermore  $Z$  is isomorphic to  ${}^Q\text{Par}_I(G_0)$ .

We apply now Theorem 1.1.(b) of [C-G-R] to the  $\mathbb{Z}$ -group scheme  $\text{Aut}_I(G_0)$ . It provides a finite  $\mathbb{Z}$ -subgroup  $S_0$  of  $\text{Aut}_I(G_0)$  such that the map  $H^1(F, S_0) \rightarrow H^1(F, \text{Aut}(G_0))$  is onto. Furthermore the construction of  $S_0$  is explicit in the proof, it is an extension of the finite constant group  $\text{Aut}_I(G_0)/T_0$  by a finite subgroup of  $T_0$ . In particular  $S_0$  is finite free over  $\mathbb{Z}$  and our assumption on the characteristic implies that  $S_{0,T}$  is finite étale of rank prime to  $p$ . It follows that  $Q_I$  admits a reduction to an  $F$ -torsor  $E$  under  $S_0$ .

The finite flat  $\mathbb{Z}$ -group scheme  $S_{0,A}$  admits a faithful representation  $S_0 \hookrightarrow \text{GL}_{N,\mathbb{Z}}$  [Br-T, §1.4.5] and the quotient  $\text{GL}_{N,\mathbb{Z}}/S_0$  is representable by an affine  $\mathbb{Z}$ -scheme [D-G, §III.2.6].

According to Lemma 5.1, there exists a regular proper model  $\mathfrak{X}$  of  $X$  and a strict normal crossing divisor  $D$  containing the irreducible components of  $Y$  such that  $E$  extends to a  $\mathfrak{X} \setminus D$ -torsor  $\mathfrak{E}$  under  $S_0$ . We put  $\mathfrak{Q}_I = \mathfrak{E} \wedge^{S_0} \text{Aut}_I(G_0)$  and consider the  $\mathfrak{X} \setminus D$ -group scheme  $\mathfrak{G} = {}^{\mathfrak{Q}_I}G_0$  of generic fiber  $G$ .

We are given a closed point  $P \in Y$ . If  $y$  is of codimension one, by hypothesis,  $Z(F_y)$  is not empty. We therefore look at a closed point  $P \in Y$ . We are given a point  $P \in Y$  and pick a height one prime  $\mathfrak{p}$  in  $\widehat{R}_P$  that contains  $t$ . It defines a branch of  $Y$  at  $P$  lying on some irreducible component  $Y_1$  of  $Y$ .

We consider the local ring  $A = R_P$  of  $\mathfrak{X}$  at  $P$  and denote by  $A_D$  its localization at  $D$ . Since  $S_{0,T}$  is finite étale of degree prime to  $p$ ,  $H^1(A_D, S_0)$  consists in loop torsors as defined in [Gi2, §2.3, lemma 2.3.(2)], i.e. those arising from cocycles related to tame Galois covers of  $A_D$ . It follows that the  $A_D$ -torsor  $Q_I$  is a loop  $\text{Aut}_I(G)$ -torsor [Gi2, lemma 2.3.(3)], so that  $\mathfrak{G} \times_{\mathfrak{X} \setminus D} A_D$  is by definition a loop reductive group scheme.

Let  $F_{P,v}$  be the completion of the field  $F_P$  for the valuation associated to the blow-up of  $\text{Spec}(A)$  at its closed point. Our assumption states in particular that  $Z(F_{P,v}) \neq \emptyset$ , that is,  $G_{F_{P,v}}$  admits a parabolic subgroup of type  $I$ . According to [Gi2, th. 4.1, (iii)  $\implies$  (i)],  $\mathfrak{G} \times_{\mathfrak{X} \setminus D} A_D$  admits a parabolic subgroup of type  $I$ . A fortiori  $G_{F_P}$  admits a parabolic subgroup of type  $I$  so that  $Z(F_P) \neq \emptyset$ .  $\square$

**Remarks 5.3.** (a) If  $p = 0$ , the result used [Gi2, th. 4.1] admits a simple proof, see [Gi2, Ex. 4.2], by using the analogous result over Laurent polynomials [G-P2, th. 7.1].

(b) In nice cases inspection of the proof permits to weaken the assumption on  $p$ . The precise condition is that the  $\text{Aut}(G_0)$ -torsor  $\text{Isom}(G_0, G)$  admits a reduction to a finite  $F$ -subgroup whose degree is prime to  $p$ . For example in type  $G_2$ , we need to assume only that  $p$  is prime to 2.

Together with Corollary 4.5, we obtain the following consequence:



**Theorem 5.4.** *Let  $G$  be a reductive  $F$ -group and assume that  $p$  does not divide the order of the automorphism group of the absolute root system of  $G_{ad}$ . Let  $Z$  be a twisted flag  $F$ -variety of  $G$ . Then  $Z(F) \neq \emptyset$  if and only if  $Z(F_v) \neq \emptyset$  for all discrete valuations of  $F$  arising from models of  $X$ .*

## 6. APPENDIX: CHARACTERIZATION OF PARABOLIC SUBGROUPS.

Let  $G$  be a reductive  $F$ -group over a field  $F$ . We remind the reader that an algebraic  $F$ -subgroup  $P$  of  $G$  is parabolic if  $P$  is smooth and  $G/P$  is a projective  $F$ -variety. The interest of the probably known statement below is only in positive characteristic since in this case there exist  $F$ -subgroups  $Q$  which are not smooth such that  $G/Q$  is a projective  $F$ -variety [W].

**Proposition 6.1.** *Let  $P$  be a  $k$ -subgroup of  $G$  such that the quotient variety  $G/P$  is projective. Then the following assertions are equivalent:*

- (i)  $P$  is an  $F$ -parabolic subgroup
- (ii) For each  $F$ -field  $E$ ,  $G(E)$  acts transitively on  $(G/P)(E)$
- (iii) The quotient map  $G \rightarrow G/P$  admits a rational section
- (iv)  $P$  is smooth connected
- (v)  $P$  is smooth.

*Proof.* (i)  $\implies$  (ii). Since  $(G/P)(E)$  parameterizes the  $E$ -parabolic subgroups of  $G_E$  of same type that  $P$ , Borel-Tits' conjugacy theorem [B-T, th. 4.13.c] shows that  $G(E)$  acts transitively on  $(G/P)(E)$ .

(ii)  $\implies$  (iii). Our assumption rephrases by saying that the map  $G(E) \rightarrow (G/P)(E)$  is onto for each  $F$ -field  $E$ . Applying that to the function field  $E = F(G/P)$  of the smooth connected  $F$ -variety  $G/P$  provides a rational section of the map  $G \rightarrow G/P$ .

(iii)  $\implies$  (iv). To show the smoothness of  $P$  we can assume that  $F$  is algebraically closed. Then the neutral component  $Q = (P_{red})^0$  of the reduced  $F$ -subgroup  $P_{red}$  of  $P$  is smooth. Furthermore the quotient  $F$ -variety  $P/Q$  is finite. It follows that the morphism  $q : G/Q \rightarrow G/P$  is finite and a fortiori projective [St, Tag 0B3I]. Since the composition of projective morphisms (of qcqs schemes) is projective [St, Tag 0C4P], it follows that  $G/Q$  is projective. The  $F$ -subgroup  $Q$  of  $G$  is then parabolic. Our assumption is that the morphism  $G \rightarrow G/P$  has a rational section and so has a fortiori the finite morphism  $q : G/Q \rightarrow G/P$ . According to [EGAII, cor. 6.1.15],  $q$  is an isomorphism. Thus  $Q = P$  and we conclude that  $P$  is smooth connected.

(iv)  $\implies$  (v). Obvious.

(v)  $\implies$  (i). This is by definition. □

A variant is the following.

**Proposition 6.2.** *Let  $X$  be a smooth projective  $G$ -variety. Then the following statements are equivalent:*

- (i)  $X$  is a  $F$ -variety of parabolic subgroups of  $G$
- (ii) For each  $F$ -field  $E$ ,  $G(E)$  acts transitively on  $X(E)$ .

*Proof.* The implication (i)  $\implies$  (ii) is again Borel-Tits' conjugacy theorem. We assume (ii). According to [M-P-W, prop. 1.3], we can assume that  $F$  is separably closed. Since  $X$  is smooth, we have  $X(F) \neq \emptyset$  and denote by  $P$  the stabilizer of some  $F$ -point  $x$ . According to [D-G, prop. III.3.2.1], condition (ii) implies that the orbit map  $G \rightarrow X$ ,  $g \mapsto g.x$  induces an isomorphism  $f_x : G/P \xrightarrow{\sim} X$ . Proposition 6.1, (ii)  $\implies$  (i), shows that  $P$  is a  $F$ -parabolic subgroup. Thus  $X$  is a  $F$ -variety of parabolic subgroups of  $G$ .  $\square$

**Remark 6.3.** The condition (ii) is called *transitive action of  $G$  on  $X$*  by Harbater-Hartmann-Krashen. It occurs in [H-H-K, th. 3.7]. Projective homogeneous varieties in the result quoted above are exactly the various varieties of parabolic subgroups.

## 7. APPENDIX: EXTENDING TORSORS

We come back to a purity result of Colliot-Thélène and Sansuc.

**Theorem 7.1.** *Let  $X$  be a regular scheme of dimension 2. Let  $U$  be an open subscheme of  $X$  which contains  $X^{(1)}$ . Let  $G$  be an affine  $X$ -group scheme. In the following cases*

- (i)  $G$  is reductive,
- (ii) *There exists a Zariski cover  $(U_i)_{i \in I}$  of  $X$  such that each  $G_{U_i}$  admits a closed embedding  $G_{U_i} \subset \mathrm{GL}_{n_i}$  such that  $\mathrm{GL}_{n_i}/G_{U_i}$  is representable by an affine  $U_i$ -scheme,*  
*then we have the equality  $H_{\mathrm{fppf}}^1(X, G) \xrightarrow{\sim} H_{\mathrm{fppf}}^1(U, G)$ .*

*Proof.* The case (i) is [C-T-S, th. 6.13]. The case (ii) goes by inspection of the proof.  $\square$

## REFERENCES

- [B-T] A. Borel, J. Tits, *Groupes réductifs*, Publ. Math. IHES **27**(1965), 55–151.
- [B] N. Bourbaki, *Éléments de mathématique : fascicule de résultats, Variétés différentielles et analytiques*, Springer.
- [Br-T] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local : II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Publications Mathématiques de l'IHÉS **60** (1984), 5-184.
- [C-G-R] V. Chernousov, P. Gille, Z. Reichstein, *Reduction of structure for torsors over semi-local rings*, Manuscripta Mathematica **126** (2008), 465-480.
- [C-T-O-H-H-K-P-S] J.-L. Colliot-Thélène, D. Harbater, J. Hartmann, D. Krashen, R. Parimala, V. Suresh, *Local-global principles for tori over arithmetic curves*, preprint (2019), arXiv:1906.10672.
- [C-T-P-S] J.-L. Colliot-Thélène, R. Parimala, V. Suresh, *Patching and local-global principles for homogeneous spaces over function fields of  $p$ -adic curves*, Comment. Math. Helv. **87** (2012), 1011-1033.
- [C-T-S] J.-L. Colliot-Thélène, J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Mathematische Annalen **244** (1979), 105-134.
- [C-G-P] B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups*, Cambridge University Press, second edition (2016).
- [D-G] M. Demazure, P. Gabriel, *Groupes algébriques*, North-Holland (1970).
- [EGA I] A. Grothendieck, J.-A. Dieudonné, *Éléments de géométrie algébrique. I*, Grundlehren der Mathematischen Wissenschaften 166; Springer-Verlag, Berlin, 1971.
- [EGA II] A. Grothendieck (avec la collaboration de J. Dieudonné), *Éléments de Géométrie Algébrique II*, Publications mathématiques de l'I.H.É.S. no 8 (1961).
- [Gi1] P. Gille, *Le problème de Kneser-Tits*, exposé Bourbaki n0 983, Astérisque **326** (2009), 39-81.
- [Gi2] P. Gille, *Loop group schemes and Abhyankar lemma*, preprint (2023), hal-03933634.
- [G-G-MB] P. Gille, O. Gabber, L. Moret-Bailly, *Fibrés principaux sur les corps henséliens*, Algebraic Geometry **5** (2014), 573-612.
- [G-P-S] P. Gille, R. Parimala, V. Suresh, *Local triviality for  $G$ -torsors*, Mathematische Annalen **380** (2021), 539-567.
- [G-P1] P. Gille, A. Pianzola, *Isotriviality and étale cohomology of Laurent polynomial rings*, Journal of Pure and Applied Algebra **212** (2008), 780-800.
- [G-P2] P. Gille, A. Pianzola, *Torsors, reductive group schemes and extended affine Lie algebras*, Memoirs of AMS 1063 (2013),
- [G-S] P. Gille, A. Stavrova,  *$R$ -equivalence on reductive group schemes*, preprint (2021).
- [H-H] D. Harbater, J. Hartmann, *Patching over fields*, Israel J. Math. **176** (2010), 61–107.

- [H-H-K] D. Harbater, J. Hartmann, D. Krashen, *Applications of patching to quadratic forms and central simple algebras*, Invent. Math. **178** (2009), 231-263.
- [H-H-K2] D. Harbater, J. Hartmann, D. Krashen, *Local-global principles for torsors over arithmetic curves*, Amer. J. Math. **137** (2015), 1559-1612.
- [H-H-K3] D. Harbater, J. Hartmann, D. Krashen, *Refinements to patching and applications to field invariants*, Int. Math. Res. Not. IMRN **20** (2015), 10399-10450.
- [H-H-K-P] D. Harbater, J. Hartmann, V. Karemaker, F. Pop, *A comparison between obstructions to local-global principles over semi-global fields*, Abelian varieties and number theory, 135–146, Contemp. Math. **767** (2020), Amer. Math. Soc., RI.
- [Mg] B. Margaux, *Passage to the limit in non-abelian Čech cohomology*, J. Lie Theory **17** (2007), 591-596.
- [M-P-W] A. S. Merkurjev, I. A. Panin, A. Wadsworth, *Index reduction formulas for twisted flag varieties. I*, K-Theory **10** (1996), 517-596.
- [P-S] V. A. Petrov, A.K. Stavrova, *Elementary subgroups in isotropic reductive groups*, Algebra i Analiz **20** (2008), 160–188; translation in St. Petersburg Math. J. **20** (2009), 625–644.
- [R] M. S. Raghunathan, *Principal bundles admitting a rational section*, Invent. Math. **116** (1994), 409-423.
- [R-S] B. Surendranath Reddy, V. Suresh, *Admissibility of groups over function fields of  $p$ -adic curves*, Adv. Math. **237** (2013), 316-330.
- [Sc] P. Schneider,  *$p$ -adic Lie groups*. Grundlehren der mathematischen Wissenschaften **344**, Springer, Heidelberg, 2011.
- [SGA3] *Séminaire de Géométrie algébrique de l'I. H. E. S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).
- [St] Stacks project, <https://stacks.math.columbia.edu>
- [W] C. Wenzel, *Rationality of  $G/P$  for a nonreduced parabolic subgroup-scheme  $P$* , Proceedings of the American Mathematical Society **117** (1993), 899-904.

UMR 5208 INSTITUT CAMILLE JORDAN - UNIVERSITÉ CLAUDE BERNARD LYON 1 43 BOULEVARD DU 11 NOVEMBRE 1918 69622 VILLEURBANNE CEDEX - FRANCE  
*Email address:* gille@math.univ-lyon1.fr

DEPARTEMENT OF MATHEMATICS AND COMPUTER SCIENCE, MSC W401, 400 DOWMAN DR. EMORY UNIVERSITY ATLANTA, GA 30322 USA  
*Email address:* parimala.raman@emory.edu