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A LOCAL-GLOBAL PRINCIPLE FOR TWISTED FLAG VARIETIES

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Abstract. We prove a local-global principle for twisted flag varieties over a semiglobal field.

Keywords: Local-global principle, curves over local fields, homogeneous varieties, reductive groups.

MSC 2000: 11G99, 14G99, 14G05, 11E72, 11E12, 20G35

1. Introduction

Let T be a complete discrete valuation ring with fraction field K and residue field k. Let X be a smooth, projective, geometrically integral curve over K. Let $F = K(X)$ be the function field of X and let t be an uniformizing parameter of T. We prove the following theorem which settles a conjecture of Colliot-Thelene, Suresh and the second author for function fields of p-adic curves [C-T-P-S, conjecture 1], in the very general context of semiglobal fields.

Theorem 1.1. *(see Th. 5.4) Let G be a reductive F–group and assume that p does not divide the order of the automorphism group of the absolute root system of G_{ad}. Let Z be a twisted flag F–variety of G. Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from models of X.*

The result was known for the cases of smooth projective quadrics and Severi-Brauer varieties [C-T-P-S, th. 3.1] and also for generalized Severi-Brauer varieties (Reddy-Suresh [R-S, th. 2]).

If we use all rank one valuations, the result holds unconditionnally (Cor. 4.8). On the other hand, if G extends to a reductive group scheme over a smooth model $\mathfrak{X}$ of X, the result is unconditional (Cor. 4.6).

Let us review the contents. Given a henselian couple (A, I) and a reductive A-group scheme H, section 2 deals with generation of groups $H(A)$ by unipotent elements and also with the quotient of $H(A)/RH(A)$ by the (normal) subgroup of R-trivial elements defined in [G-S]. Section 3 is devoted to mild improvements of patching techniques of Harbater-Hartmann-Krashen involving more analytical functions and leads to various group decompositions (prop. 3.7); this permits to obtain a patching

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theorem for twisted flag varieties (th. 3.9). Section 4 deals with the setting of the beginning of the introduction and provides a weaker version of the main result. What remains to do is mostly a local study for a two dimensional complete regular local ring R with parameters t, s and torsors over its localization $R[t^{-1}, s^{-1}]$. This is achieved in section 5 by making use of the loop torsors over $R[t^{-1}, s^{-1}]$ and of the results of [Gi2].

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Conventions and notation. (a) A variety V over a field F means a separated F–scheme of finite type which is integral [St, Tag 020D].

(b) Let G be a reductive F–algebraic group and let F_s be a separable closure of F. Let (B, T) be a Killing couple of G_{k_s} and denote $\Delta(G_{F_s}) = \Delta(G_{F_s}, B, T)$ the associated Dynkin diagram (there is a canonical bijection $\Delta(G_{F_s}, B, T) \xrightarrow{\sim} \Delta(G_{F_s}, B', T')$ for another choice of Killing couple). This diagram $\Delta(G_{F_s})$ is equipped with the star action of $\mathrm{Gal}(F_s/F)$ [B-T, §6.4]. We recall that there is an order preserving bijection $I \to P_I$ between the subsets of $\Delta(G_{F_s})$ and the F_s–parabolic subgroups of G_{F_s} containing B (or equivalently with the $G(F_s)$–conjugacy classes of parabolic subgroups of G_{F_s}). Since minimal F–parabolic subgroups are $G(F)$-conjugated we denote by $\Delta_0(G) \subset \Delta(G_{F_s})$ be conjugacy class of $P_0 \times_F F_s$ where P_0 is a minimal F–parabolic subgroup of G (it is stable under the star action). The triple $(\Delta(G_{F_s}), \text{star action}, \Delta_0(G))$ is called the Tits index of G. By abuse of notation, $\Delta_0(G)$ alone is called the Tits index. By a twisted F–flag variety of G, we mean a G-variety X which becomes isomorphic over F_s to some G_{F_s}/P_I as G_{F_s}-variety (such an I is unique). According to [M-P-W, prop. 1.3], I is invariant under the star action and X is G-isomorphic to the variety of F–parabolic subgroups of type I which is denoted in this paper by $\mathrm{Par}_I(G)$.

Those varieties are called sometimes projective homogeneous G–varieties (as in [C-T-P-S]) but we warn the reader of the danger of this terminology since there exist in positive characteristic F–subgroups Q which are not smooth such that G/Q is a projective F–variety [W]. In the appendix 6, we show that a homogeneous G–variety X is a variety of parabolic subgroups if and only if the action is transitive in the sense that $G(E)$ acts transitively on $X(E)$ for each F field E (Prop. 6.2).

(c) We use mainly the terminology and notation of Grothendieck-Dieudonné [EGAI, §9.4 and 9.6] which agrees with that of Demazure-Grothendieck used in [SGA3, Exp. I.4]. Let S be a scheme and let $\mathcal{E}$ be a quasi-coherent sheaf over S. For each morphism $f : T \to S$, we denote by $\mathcal{E}_T = f^*(\mathcal{E})$ the inverse image of $\mathcal{E}$ by the morphism f. We denote by $\mathbf{V}(\mathcal{E})$ the affine S–scheme defined by $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}\big(\mathrm{Sym}^\bullet(\mathcal{E})\big)$; it is affine over S and represents the S–functor $Y \mapsto \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)$ [EGAI, 9.4.9]. We assume now that $\mathcal{E}$ is locally free of finite rank and denote by $\mathcal{E}^\vee$ its dual. In this case the affine S–scheme $\mathbf{V}(\mathcal{E})$ is of finite presentation (ibid, 9.4.11); also the S–functor

$Y \mapsto H^0(Y, \mathcal{E}_Y) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)$ is representable by the affine S–scheme $\mathbf{V}(\mathcal{E}^\vee)$ which is also denoted by $\mathbf{W}(\mathcal{E})$ [SGA3, I.4.6].

2. Around Kneser-Tits' problem

Let A be a semilocal ring, I its radical.

2.1. **Subgroups of elementary elements.** Let H be a reductive A-group scheme which is strictly A–isotropic, that is, all factors of H_{ad} are isotropic. Let H be a strictly proper parabolic A–subgroup of H and let P^- be a A–parabolic subgroup opposite to P. The subgroup of $H(A)$ generated by $R_u(P)(A)$ and $R_u(P^-)(A)$ does not depend of the choice of P and P^-, this is a normal subgroup of $H(A)$ which is denoted by $H(A)^+$ [P-S, Thm. 1 and comments].

Lemma 2.1. *We have $H(A) = H(A)^+ P(A)$.*

Proof. We put $U = R_u(P)$, $U^- = R_u(P^-)$. According to [SGA3, XXVI.5.2], we have a decomposition $H(A) = U(A)\, U^-(A) P(A)$ whence $H(A) = H(A)^+ P(A)$. □

A special case of [G-S, prop. 7.7] is the following.

Lemma 2.2. *Assume that (A, I) is an henselian couple and that H is semisimple simply connected. Then the map $H(A)/H(A)^+ \to H(A/I)/H(A/I)^+$ is an isomorphism.*

2.2. **R-equivalence.** We assume now that A is a nonzero finite k–algebra for a field k. Then A is an Artinian k–algebra. We denote by I its Jacobson radical and remind the reader that (A, I) is an henselian couple.

Let H be an A–reductive group scheme and consider the Weil restriction $G = R_{A/k}(H)$, this is a smooth affine connected algebraic k–group [C-G-P, A.5.9]. We have $G(k) = H(A)$. We use R-equivalence for group schemes over rings as defined in [G-S].

Lemma 2.3. *We decompose $A = A_1 \times \cdots \times A_d$ where each A_i is a local artinian k–algebra of residue field k_i.*
(1) The map $H(A) \to \prod_i H(k_i)$ induces isomorphisms

$$G(k)/RG(k) \xrightarrow{\sim} H(A)/RH(A) \xrightarrow{\sim} \prod_i H(k_i)/RH(k_i).$$

(2) If H is furthermore simply connected and strictly isotropic, we have a commutative diagram of isomorphisms

$$\begin{array}{ccccc} H(A)/H(A)^+ & \xrightarrow{\sim} & H(A)/RH(A) & \xleftarrow[\sim]{} & G(k)/RG(k) \\ \downarrow \wr & & \downarrow \wr & & \\ \prod_i H(k_i)/H(k_i)^+ & \xrightarrow{\sim} & \prod_i H(k_i)/RH(k_i). & & \end{array}$$

Proof. (1) The first isomorphism $G(k)/RG(k) \xrightarrow{\sim} H(A)/RH(A)$ is a formal fact [G-S, lemma 2.4]. Let n be the smallest positive integer such that $I^n = 0$. We put $A_j = A/I^{j+1}$ for $j = 0, \dots, n-1$, it comes with the ideal $J_j = I^j/I^{j+1} = I\,A_j$ which satisfies $J_j^2 = 0$. We have $A_j/J_j = A_{j-1}$ for $j = 1, \dots, n-1$.

We put $G_j = R_{A_j/k}(H_{A_j})$ for $j = 0, \dots, n-1$; we have a sequence of homomorphisms $G = G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 = \prod_i R_{k_i/k}(H_{k_i})$. It induces homomorphisms

$$G(k)/R = G_n(k)/R \to G_{n-1}(k)/R \to \cdots \to G_1(k)/R \to G_0(k)/R = \prod_i H(k_i)/R$$

and we will show by a dévissage argument that all of them are isomorphisms.

Let j be a an integer satisfying $1 \le j \le n-1$. According to a variant of [G-P-S, Lemma 8.3], we have an exact sequence of fppf (resp. étale, Zariski) sheaves on $\mathrm{Spec}(k)$

$$0 \to \mathbf{W}\Big((t_j)_*\big(\mathrm{Lie}(H)(A_{j-1}) \otimes_{A_{j-1}} J_j\big)\Big) \to R_{A_j/k}(H_{A_j})) \to R_{A_i/k}(H_{A_{j-1}}) \to 1.$$

where $t_j : \mathrm{Spec}(A_j) \to \mathrm{Spec}(k)$ is the structural morphism. We have then a sequence of k–algebraic groups

$$0 \to (\mathbb{G}_a)^{m_j} \to G_j \to G_{j-1} \to 1.$$

The map $G_j \to G_{j-1}$ is a $(\mathbb{G}_a)^{m_j}$-torsor which is trivial since G_{j-1} is affine. We have then a decomposition of k-schemes $G_j \xrightarrow{\sim} G_{j-1} \times_k (\mathbb{G}_a)^{m_j}$ which induces an isomorphism $G_j(k)/R \to G_{j-1}(k)/R$. Thus the map $G(k)/R \to G_0(k)/R = \prod_i H(k_i)/R$ is an isomorphism as desired.

(2) We have $A/I = k_1 \times \cdots \times k_d$. Lemma 2.2 provides an isomorphism $H(A)/H(A)^+ \xrightarrow{\sim} H(A/I)/H(A/I)^+ = \prod_i H(k_i)/H(k_i)^+$ and we have an isomorphism $H(k_i)/H(k_i)^+ \xrightarrow{\sim} H(k_i)/RH(k_i)$ for each i [Gi1, thm. 7.2]. This completes the proof by chasing diagram. □

3. Patching, R-equivalence and twisted flag schemes

3.1. **Using the implicit function theorem.** We consider a variation of the framework of [H-H-K, 2.4]. Let T be a complete DVR of fraction field K and t be an unformizing parameter. Let $\widehat{R}_0$ be ring containing T and which is also a complete discrete valuation ring having uniformizer t. We denote by F_0 the fraction field of $\widehat{R}_0$.

Let $\alpha > 1$ be a real number and we define the absolute value on F_0 by $|t^n u| = \alpha^{-n}$ for $n \in \mathbb{Z}$ and $u \in (\widehat{R}_0)^\times$.

Let F_1, F_2 be subfields of F_0 containing T. We further assume that we are given t-adically complete T-submodules $V \subset F_1 \cap \widehat{R}_0$ and $W \subset F_2 \cap \widehat{R}_0$ satisfying the following conditions:

$$V + W = \widehat{R}_0; \tag{3.1}$$

$$V \cap t\widehat{R}_0 = tV \quad \text{and} \quad W \cap t\widehat{R}_0 = tW. \tag{3.2}$$

Note that Condition (3.2) is equivalent to

$$V \cap t^n \widehat{R}_0 = t^n V \quad \text{and} \quad W \cap t^n \widehat{R}_0 = t^n W \quad \text{for each} \quad n \geq 1. \tag{3.3}$$

Remark 3.1. Condition (3.2) above is added there compared with [H-H-K, 2.4] but we do not require at this stage that F_1 is dense in F_0.

We equip the submodules $V[\frac{1}{t}]$ of F_0 of the induced metric (and similarly for $W[\frac{1}{t}]$).

Lemma 3.2. *(1) V is closed in $\widehat{R}_0$.*
(2) For $v \in V[\frac{1}{t}] \setminus \{0\}$, we have

$$\mid v \mid = \mathrm{Inf}\{\alpha^n \mid t^n v \in V\}.$$

(3) We have $V = \mathrm{Inf}\{v \in V[\frac{1}{t}] \mid \mid v \mid \geq 0\}$ and V is a clopen submodule of $V[\frac{1}{t}]$.
(4) $V[\frac{1}{t}]$ is closed in F_0 and is a Banach K-space.

Proof. (1) Our assumption is that the map $V \to \varprojlim V/t^{m+1}V$ is an isomorphism. Let (x_n) be a sequence of V which converges in $\widehat{R}_0$. For each $m \geq 0$, condition (3.3) shows that the map $V/t^{m+1}V \to \widehat{R}_0/t^{m+1}\widehat{R}_0$ is injective so that the sequence (x_n) modulo $t^{m+1}V$ is stationary to some $v_m \in V/t^{m+1}V$. The v_m's define a point v of V and the sequence (x_n) converges to v.
(2) We are given $v = t^m v' \in V[\frac{1}{t}]$ with $v' \in V \setminus tV$ and we have $\mid v \mid = \mathrm{Inf}\{\alpha^n \mid t^n v \in \widehat{R}_0\} = \alpha^m \mathrm{Inf}\{\alpha^n \mid t^n v' \in \widehat{R}_0\}$. Condition (II) implies that $v' \in \widehat{R}_0 \setminus t\widehat{R}_0$ so that $\mid v \mid = 0$. We conclude that $\mid v \mid = \mathrm{Inf}\{\alpha^n \mid t^n v \in V\}$.
(3) It readily follows from the assertion (2).
(4) Let (x_n) be a sequence of $V[\frac{1}{t}]$ converging to some $x \in F_0$. We want to show that x belongs to $V[\frac{1}{t}]$ so that we can assume that $x \neq 0$ and that $\mid x_n \mid = \mid x \mid = \alpha^m$ for all $n \geq 0$. Assertion (2) shows that $t^m x_n$ is a sequence of V and (1) shows that its limit $t^m x$ belongs to V. Thus $x \in V[\frac{1}{t}]$. We have shown that $V[\frac{1}{t}]$ is closed in F_0.
Finally since F_0 is a Banach K–space so is $V[\frac{1}{t}]$. □

The following statement extends partially [H-H-K, th. 2.5] and [H-H-K2, prop. 4.1].

Proposition 3.3. *Let a, b, c be positive integers. Let $\Omega \subset (F_0)^a \times (F_0)^b$ be an open neighborhood of $(0,0)$ and let $f : \Omega \to (F_0)^c$ be an analytic map. We denote by $f^a : \Omega \cap (F_0)^a \to (F_0)^c$ and $f^b : \Omega \cap (F_0)^b \to (F_0)^c$. We assume that*

(i) $f(0,0) = 0$

(ii) the differentials $Df_0^a : (F_0)^a \to (F_0)^c$ and $Df_0^b : (F_0)^a \to (F_0)^c$ satisfy

$$Df_0^a\Big(V[\frac{1}{t}]^a\Big) + Df_0^b\Big(V[\frac{1}{t}]^b\Big) = (F_0)^c.$$

Then there is a real number $\epsilon > 0$ such that for all $y \in (F_0)^c$ with $|y| \leq \epsilon$, there exist $v \in V^a$ and $w \in W^b$ such that $(v, w) \in \Omega$ and $f(v, w) = y$.

Proof. We consider the continuous embedding $i : V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \to (F_0)^a \times (F_0)^b$ and define $\widetilde{\Omega} = i^{-1}(\Omega)$ and the function $\widetilde{f} = f \circ i : \widetilde{\Omega} \to (F_0)^c$.

Claim 3.4. *The map $\widetilde{f}$ is strictly differentiable at $(0, 0)$ and $D\widetilde{f}_{(0,0)} : V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \to (F_0)^c$ is onto.*

Since f is F_0–analytic at $(0, 0)$, it is strictly differentiable [Sc, I.5.6], that is, there exists an open neighborhood Θ of $(0, 0)$ and a positive real number β such that

$$|f(x_2) - f(x_1) - Df_{(0,0)}.(x_2 - x_1)| \leq \beta |x_2 - x_1| \qquad \forall x_1, x_2 \in \Theta.$$

It is then strictly derivable as function between the Banach K–spaces $(F_0)^a \times (F_0)^b \to (F_0)^c$. On the other hand the embedding $i : V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \to (F_0)^a \times (F_0)^b$ is 1–Liftschitz so is strictly differentiable at $(0, 0)$. As composite of strictly differentiable functions, $\widetilde{f}$ is strictly differentiable at $(0, 0)$ [B, §1.3.1]. Furthermore the differential $D\widetilde{f}_{(0,0)}$ is the composite of

$$V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b \xrightarrow{\ i\ } (F_0)^a \times (F_0)^b \xrightarrow{Df_0^a + Df_0^b} (F_0)^c.$$

Condition (ii) says exactly that $D\widetilde{f}_{(0,0)}$ is surjective. The Claim is proven.

We apply the implicit function theorem to the function $\widetilde{f}$ [B, §1.5.2] (see [Sc, §4] for concocting a proof). Lemma 3.2.(4) shows that $V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b$ is a Banach K–space and so is F_0. There exists then an open neighborhood $\Upsilon \subset \widetilde{\Omega}$ of $(0, 0)$ in $V[\frac{1}{t}]^a \times W[\frac{1}{t}]^b$ such that $\widetilde{f}_{|\Upsilon}$ is open. Up to shrink Υ we can assume that $\Upsilon \subset V \times W$ according to Lemma 3.2.(3). There exists then a real number $\epsilon > 0$ such that

$$\left\{y \in (F_0)^c \mid |y| \leq \epsilon\right\} \subset \left\{y \in (F_0)^c \mid |y| < 2\epsilon\right\} \subset \widetilde{f}(\Upsilon).$$

We conclude that for all $y \in (F_0)^c$ with $|y| \leq \epsilon$, there exist $v \in V^a$ and $w \in W^b$ such that $(v, w) \in \Omega$ and $f(v, w) = y$. □

Corollary 3.5. *Let n be a positive integer. Let $\Omega \subset (F_0)^n \times (F_0)^n$ be an open neighborhood of $(0, 0)$ and let $f : \Omega \to (F_0)^n$ be an analytic map which satisfies*

(i) $f(0, 0) = 0$

(ii) $f(x, 0) = f(0, x) = x$ *over an open neighborhood* Υ *of* 0.

Then there is a real number $\epsilon > 0$ such that for all a with $|a| \leq \epsilon$, there exist $v \in V^n$ and $w \in W^n$ such that $(v, w) \in \Omega$ and $f(v, w) = a$.

Proof. In this case we have $a = b = c = n$ and $Df_0^a = Df_0^b = \mathrm{Id}_{(F_0)^n}$ so that Proposition 3.3 applies. □

3.2. **Kneser-Tits' subgroups.** Continuing in the previous setting, we assume furthermore that F_1 is t-adically dense in F_0. Let $F \subset F_1 \cap F_2$ be a subfield. For dealing later with Weil restriction issues it is convenient to deal with a finite field extension E of F.

Proposition 3.6. *Let H be a semisimple simply connected E–group scheme assumed strictly isotropic. We put $G = R_{E/F}(H)$. For each overfield L of F, we put $G(L)^+ = H(L \otimes_F E)^+$ where the second group is that defined in §2.1. Then we have the decomposition*

$$G(F_0)^+ = G(F_1)^+ \, G(F_2)^+.$$

Proof. Without lost of generality we can assume that F is infinite. The proof is based on an analytic argument requiring some preparation.

Let P be a strictly proper parabolic E–subgroup of H. Let U be its unipotent radical and U_{last} the last part of Demazure's filtration [G-P-S, §3.2]. Let $u : \mathbb{G}^d_{a,E} \xrightarrow{\sim} U_{last}$ be a E–group isomorphism. According to [G-P-S, Lemma 3.3.(3)], $E_P(E).\,\mathrm{Lie}(U_{last})(E)$ generates $\mathrm{Lie}(H)(E)$ as a vector space over E. There exists $g_1, \dots, g_n \in E_P(E)$ such that

$$\mathrm{Lie}(H)(E) = {}^{g_1}\mathrm{Lie}(U_{last})(E) \oplus {}^{g_2}\mathrm{Lie}(U_{last})(E) \oplus \cdots \oplus {}^{g_n}\mathrm{Lie}(U_{last})(E).$$

We consider the map $h : (\mathbb{G}^d_{a,E})^n \to H$, $h(x_1, \dots, x_n) = {}^{g_1}u(x_1) \dots {}^{g_n}u(x_n)$. Its differential at 0 is $dh_{0,0} : E^{dn} \cong \mathrm{Lie}(U_{last})(E)^n \to \mathrm{Lie}(H)(E)$, $(X_1, \dots, X_n) \mapsto \sum_{i=1}^n {}^{g_i}X_i$, so is an isomorphism. The map h is then étale at a neighborhood of 0. It follows that $h_\sharp = R_{E/F}(h) : R_{E/F}\big((\mathbb{G}^d_{a,E})^n\big) \to G = R_{E/F}(H)$ is étale also at a neighborhood of 0 [C-G-P, A.5.2.(4)]

Since the field F_0 is henselian, the local inversion theorem holds [G-G-MB, prop. 2.1.4]. We mean that there exists an open neighborhood $\Upsilon \subset (F_0)^n$ such that the restriction $h_{\sharp|\Omega} : \Upsilon \to G(F_0)$ is a topological open embedding.

We consider now the product morphism $q : \Upsilon \times \Upsilon \to G(F_0)$, $q(x, y) = h_\sharp(x)h_\sharp(y)$. We put $\Omega = q^{-1}(h_\sharp(\Upsilon))$, this an open subset of $(F_0)^{2n}$. Then the restriction $q_{|\Omega}$ defines an (unique) analytical map $f : \Omega \to \Upsilon$ such that $q(x, y) = h_\sharp(f(x, y))$.

By construction we have $f(0, x) = f(x, 0) = x$ for x in a neighborhood of $0 \in (F_0)^n$. We apply Corollary 3.5 to f so that there exists $\epsilon > 0$ such that for each $a \in \Upsilon$ with $\mid a \mid \leq \epsilon$, there exist $v \in V^n$ and $w \in W^n$ such that $(v, w) \in \Omega$ and $f(v, w) = a$. We denote by $\Upsilon_\epsilon = \Upsilon \cap B(0, \epsilon)$. Then $h_\sharp^{-1}(\Upsilon_\epsilon)$ is an open neighborhood of 0 in $(F_0)^n$.

Let us now prove that $G(F_0)^+ = G(F_1)^+ \times G(F_2)^+$. Since F_1 is dense in F_0, $G(F_1)^+$ is dense in $G(F_0)^+$. It is then enough to show that $\Upsilon_\epsilon \subset G(F_1)^+ \times G(F_2)^+$. Let $g = h(a) \in h_\sharp(\Upsilon_\epsilon)$. Then $a = f(v, w)$ with $(v, w) \in \Omega$. It follows that $g = h(a) = h_\sharp(f(v, w)) = q(v, w) = h_\sharp(v)h_\sharp(w) \in G(F_1)^+ \times G(F_2)^+$. □

This could be refined as follows.

Proposition 3.7. *Let H be a reductive E-group and put $G = R_{E/F}(H)$.*
(1) $RG(F_1)\, RG(F_2)$ contains an open neighborhood of 1 in $G(F_0)$.
(2) If $RG(F_1)$ is dense in $RG(F_0)$, then $RG(F_1)\, RG(F_2) = RG(F_0)$.
(3) If H is semisimple simply connected and $H_{F_1 \otimes_F E}$ is strictly isotropic, then we have

$$G(F_1)^+\, RG(F_2) = G(F_0)^+.$$

(4) If H is semisimple simply connected and $H_{F_i \otimes_F E}$ is strictly isotropic for $i = 1, 2$, then $G(F_1)^+\, G(F_2)^+ = G(F_0)^+$.

The subgroups $G(F_1)^+$, $G(F_0)^+$ are defined as in Proposition 3.6.

Proof. (1) Let $T \subset H$ be a maximal E–torus and let $1 \to S \to Q \xrightarrow{s} T \to 1$ be a resolution of T where Q is a quasitrivial torus and S is a torus. We have $Q = R_{C/E}(\mathbb{G}_m)$ where C is an étale E–algebra so that Q is an open subset of the affine E–space $\mathbf{W}(C)$.

We use now Raghunathan's technique [R, §1.2]. There exists $h_1, \dots, h_r \in H(E)$ such that $\mathrm{Lie}(H)(E) = {}^{g_1}\mathrm{Lie}(T)(E) \oplus {}^{g_2}\mathrm{Lie}(T)(E) \oplus \cdots \oplus {}^{g_r}\mathrm{Lie}(T)(E)$. We consider the map $h : (Q^n)_E \to H$, $h(x_1, \dots, x_n) = {}^{g_1}s(x_1) \dots {}^{g_n}s(x_n)$. Its differential at 0 is $dh_0 : C^r \to \mathrm{Lie}(H)(E)$, $(c_1, \dots, c_r) \mapsto \sum_i {}^{g_i}ds(c_i)$, and is onto (observe that $\mathrm{Lie}(Q)(E) \to \mathrm{Lie}(T)(E)$ is surjective).

We cut now Q^r by some suitable affine E–subspace $\mathbf{W}(C^n)$ of $\mathbf{W}(C)^r$ such that the restriction h' of h to $X = Q^r \cap \mathbf{W}(C^n)$ is such that $dh'_0 : \mathrm{Tan}_{X,1} \to \mathrm{Lie}(H)(F)$ is an isomorphism. Note that X and E have same dimension n over E.

The map h' is then étale at a neighborhood of 1. It follows that $h'_\sharp = R_{E/F}(h') : R_{E/F}\big(X\big) \to G = R_{E/F}(H)$ is étale also at a neighborhood of 0 [C-G-P, A.5.2.(4)]

Since the field F_0 is henselian, the local inversion theorem holds [G-G-MB, prop. 2.1.4]. We mean that there exists an open neighborhood $\Upsilon \subset (F_0)^n$ such that the restriction $h'_{\sharp|\Omega} : \Upsilon \to G(F_0)$ is a topological open embedding.

We consider now the product morphism $q : \Upsilon \times \Upsilon \to G(F_0)$, $q(x, y) = h'_\sharp(x)\, h'_\sharp(y)$. We put $\Omega = q^{-1}(h'_\sharp(\Upsilon))$, this an open subset of $(F_0)^{2n}$. Then the restriction $q_{|\Omega}$ defines a (unique) analytical map $f : \Omega \to \Upsilon$ such that $q(x, y) = h'_\sharp(f(x, y))$.

By construction we have $f(0, x) = f(x, 0)$ for x in a neighborhood of $0 \in F_0^n$. We apply Corollary 3.5 to f so that there exists $\epsilon > 0$ such that for each $a \in \Upsilon$ with $\mid a \mid \leq \epsilon$, there exist $v \in V^n$ and $w \in W^n$ such that $(v, w) \in \Omega$ and $f(v, w) = a$. We denote by $\Upsilon_\epsilon = \Upsilon \cap B(0, \epsilon)$. Then $h'^{-1}_\sharp(\Upsilon_\epsilon)$ is an open neighborhood of 0 in $(F_0)^n$.

We claim that $\Upsilon_\epsilon \subset RG(F_1)\, RG(F_2)$. Let $g = h'_\sharp(a) \in h(\Upsilon_\epsilon)$. Then $a = f(v, w)$ with $(v, w) \in \Omega$. It follows that $g = h'_\sharp(a) = h_\sharp(f(v, w)) = q(v, w) = h'_\sharp(v) h'_\sharp(w) \in RG(F_1) \times RG(F_2)$.

(2) If $RG(F_1)$ is furthermore dense in $RG(F_0)$, then (1) shows that $RG(F_1)\, RG(F_2)$ is a dense open subset of $RG(F_0)$. Thus $RG(F_1)\, RG(F_2) = RG(F_0)$

(3) We assume that H is semisimple simply connected and that $G_{F_1\otimes_F E_1}$ is strictly isotropic. According to Lemma 2.2.(2), we have $G(F_1)^+ = RG(F_1) = RH(F_1\otimes_F E) = H^+(F_1\otimes_F E)$ and similarly for F_0.

Since $H^+(F_1\otimes_F E)^+$ is dense in $H^+(F_0\otimes_F E)$, it follows that $RG(F_1)$ is dense in $RG(F_0)$. Assertion (2) yields then $RG(F_1)\,RG(F_2) = RG(F_0)$. Lemma 2.3.(2) states that $G(F_1)^+ = RG(F_1)$ and $G(F_0)^+ = RG(F_0)$. We conclude that $G(F_1)^+\,RG(F_2) = G(F_0)^+$.

(4) We have furthermore $G(F_2)^+ = RG(F_2)$ so that (3) yields $G(F_1)^+\,G(F_2)^+ = G(F_0)^+$. □

Remark 3.8. Note that (1) shows in particular that $RG(F_0)$ is an open subgroup of $G(F_0)$.

The main result of the section is the following patching statement on twisted flag varieties.

Theorem 3.9. *We put $F = F_1\cap F_2$. Let H be a reductive E–group and let X be a twisted flag E–variety of H. Assume that X is auto-opposite, that is, the stabilizer P of an E_s–point of X is conjugated to an opposite parabolic subgroup of P* [B-T, §4.9]. *We put $G = R_{E/F}(H)$ and $Z = R_{E/F}(X)$. If $Z(F_1)\neq\emptyset$ and $Z(F_2)\neq\emptyset$, then $Z(F)\neq\emptyset$.*

Proof. Without loss of generality, we can assume that H is semisimple simply connected and that it is absolutely E–simple. This implies that H is strictly $F_i\otimes_F E$–isotropic for $i=1,2$. Let $x_i\in Z(F_i) = X(F_i\otimes_F E)$ and denote by $P_i = \mathrm{Stab}_{G_{F_2}}(x_2)$ its stabilizer, this is parabolic $F_i\otimes_F E$–subgroup of H. We denote by U_i/E its unipotent radical.

Since the conjugacy class of P_1 is autopposite, there exists $h\in H(E\otimes_F F_0)$ such that $P_{1,E\otimes_F F_0}$ is opposite to ${}^hP_{2,E\otimes_F F_0}$ [SGA3, XXVI.5.3]. According to Lemma 2.1, we have $H(E\otimes_F F_0) = H^+(E\otimes_F F_0)\,P_2(E\otimes_F F_0)$ so we can assume that $h\in H^+(E\otimes_F F_0)$. According to Proposition 3.7.(4), we can write $h = h_1\,h_2$ with $h_i\in H^+(E\otimes_F F_i)$ for $i=1,2$. Up to replace P_1 by ${}^{h_1^{-1}}P_1$ and P_2 by ${}^{h_2^{-1}}P_2$ we can then assume that P_{1,F_0} is opposite to P_{2,F_0}. According to [SGA3, XXVI.5.1], we have a decomposition

$$H(F_0\otimes_F E) = U_2(F_0\otimes_F E)\,U_1(F_0\otimes_F E)\,P_2(F_0\otimes_F E). \tag{3.4}$$

Let $h\in H(F_0\otimes_F E)$ such that $x_1 = h.x_2$. The preceding decomposition permits to write $h = u_2\,u_1\,p_2$ with $u_2\in U_2(F_0\otimes_F E)$, $u_1\in U_1(F_0\otimes_F E)$ and $p_2\in P_2(F_0\otimes_F E)$. It follows that

$$x_1 = u_1^{-1}.x_1 = (u_1^{-1}h).x_2 = (u_1^{-1}\,u_2\,u_1)\,.\,(p_2.x_2) = (u_1^{-1}\,u_2\,u_1)\,.\,x_2$$

hence $x_1\in H(F_0\otimes_F E)^+.x_2$. According to Proposition 3.7.(4), we have $G(F_0)^+ = G(F_1)^+\,G(F_2)^+$ so that $u_1^{-1}\,u_2\,u_1 = g_1\,g_2$ with $g_i\in G(F_i)^+$ for $i=1,2$. It follows that $g_1^{-1}\,.\,x_1 = g_2^{-1}\,.\,x_2$, this defines a point of $X(E) = Z(F)$. □

4. Relation with the original HHK method

We recall the setting. Let T be an excellent complete discrete valuation ring with fraction field K, residue field k and uniformizing parameter t. Let F be a one-variable function field over K and let $\mathfrak{X}$ be a normal model of F, i.e. a normal connected projective T-curve with function field F. We denote by Y the closed fiber of $\mathfrak{X}$ and fix a separable closure F_s of F.

For each point $P \in Y$, let R_P be the local ring of $\mathfrak{X}$ at P; its completion $\widehat{R}_P$ is a domain with fraction field denoted by F_P.

For each subset U of Y that is contained in an irreducible component of Y and does not meet the other components, we define $R_U = \bigcap_{P \in U} R_P \subset F$. We denote by $\widehat{R}_U$ the t–adic completion of R_U. The rings R_U and $\widehat{R}_U$ are excellent normal domains and we denote by F_U the fraction field of $\widehat{R}_U$ [H-H-K3, Remark 3.2.(b)].

Each height one prime $\mathfrak{p}$ in $\widehat{R}_P$ that contains t defines a branch of Y at P lying on some irreducible component of Y. The t-adic completion $\widehat{R}_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$ of $\widehat{R}_P$ at $\mathfrak{p}$ is a complete DVR whose fraction field is denoted by $F_\mathfrak{p}$. The field $F_\mathfrak{p}$ contains also F_U if U is an irreducible open subset of Y such that $P \in \overline{U} \setminus U$. We have then a diagram of fields

$$\begin{tikzcd} & F_\mathfrak{p} & \\ F_P \arrow[ur] & & F_U. \arrow[ul] \end{tikzcd}$$

Example 4.1. We assume that $T = k[[t]]$ and take $X = \mathbb{P}^1_K$, $\mathfrak{X} = \mathbb{P}^1_T$, $P = \infty_k$, $U = \mathbb{A}^1_k = \mathrm{Spec}(k[x])$. The ring R_U contains $k[[t]][x]$ and is its localization with respect to the multiplicative set S of elements which are units modulo t. The t-adic completion of R_U is $\widehat{R}_U = k[x][[t]]$; we have $F_U = Frac(\widehat{R}_U) = k(x)((t))$. The local ring of $\mathfrak{X}$ at $P = \infty_k$ is $R_P = k[[t]][x^{-1}]_{(x^{-1},t)}$ so that its completion is $\widehat{R}_P = k[[t, x^{-1}]]$; in particular $F_P = k((t, x^{-1}))$.

We take $\mathfrak{p} = t\widehat{R}_P \subset \widehat{R}_P$ and the t-adic completion $\widehat{R}_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$ of $\widehat{R}_P$ at $\mathfrak{p}$ is a complete DVR which is $k((x^{-1}))[[t]]$. In particular $F_\mathfrak{p} = k((x^{-1}))((t))$.

Setting 4.2. Let $\mathcal{P}$ be a non-empty finite set of closed points of Y that contains all the closed points at which distinct irreducible components meet. Let $\mathcal{U}$ be the set of connected components of $Y \setminus \mathcal{P}$ and let $\mathcal{B}$ be the set of branches of Y at points of $\mathcal{P}$. This yields a finite inverse system of field $F_P, F_U, F_\mathfrak{p}$ (for $P \in \mathcal{P}$; $U \in \mathcal{U}$, $\mathfrak{p} \in \mathcal{B}$) where $F_P, F_U \subset F_\mathfrak{p}$ if $\mathfrak{p}$ is a branch of Y at P lying in the closure of U.

Lemma 4.3. *We assume that $X = \mathbb{P}^1_K$, $\mathfrak{X} = \mathbb{P}^1_T$, $P = \infty_k$, $U = \mathbb{A}^1_k = \mathrm{Spec}(k[x])$ and $\mathfrak{p}$ the branch of P. We put $F_1 = F_P$, $F_2 = F_U$ and $F_0 = F_\mathfrak{p}$.*

(1) F_1 is t-dense in F_0.

(2) We put $V = F_1 \cap \widehat{R}_{\mathfrak{p}}$ and $W = F_2 \cap \widehat{R}_{\mathfrak{p}}$. Then V and W satisfy conditions (3.1) *and* (3.2).

Proof. (1) We are given $u_0/v_0 \in F_0$ with $u_0, v_0 \in \widehat{R}_{\mathfrak{p}}$, $v_0 \neq 0$. There exists elements $u, v \in R_{\mathfrak{p}}$ very close respectively of u_0, v_0 with $v \neq 0$. Let $s_1, s_2 \in R_p \setminus R_p\mathfrak{p}$ such that $s_1 u \in R_P$ and $s_2 v \in R_{\mathfrak{p}}$. Then u_0/v_0 is very close of $(s_1 s_2 u)/(s_1 s_2 v) \in F_1$.

(2) Condition (3.2) is obviously fullfilled since $t \in F_1 \cap F_2$. For establishing condition (3.1), we are given an element f of $\widehat{R}_{\mathfrak{p}}$ and may write it as $f = \sum\limits_{i=0}^{\infty} x^{m_i} \Big(\sum_{j=0}^{\infty} a_{i,j} \frac{1}{x^j} \Big) t^i$ where the m_i's are non-negative integers and $a_{i,j} \in T$. We decompose

$$f = f_1 + f_2 = \sum_{i=0}^{\infty} x^{m_i} \Big(\sum_{j=m_i}^{\infty} a_{i,j} \frac{1}{x^j} \Big) t^i \quad + \quad \sum_{i=0}^{\infty} x^{m_i} \Big(\sum_{j=0}^{m_i - 1} a_{i,j} \frac{1}{x^j} \Big) t^i$$

We observe that f_2 belongs to $\widehat{R}_P$ so belongs to V. We recall that R_U is the localization of $T[x]$ with respect to the elements which are units modulo t. We conclude that f_2 belongs to W as desired. □

Theorem 4.4. *Let G be a reductive F–algebraic group.*

(1) Let Z be a twisted flag projective F–variety for G. Then $Z(F) \neq \emptyset$ if and only if $Z(F_U) \neq \emptyset$ for each $U \in \mathcal{U}$ and $Z(F_P) \neq \emptyset$ for each $P \in \mathcal{P}$.

(2) For each $U \in \mathcal{U}$ (resp. each $P \in \mathcal{P}$), we fix an F–embeddings $i_U : F_s \to F_{U,s}$ (resp. $i_P : F_s \to F_{P,s}$) providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{U,s}})$ (resp. $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{P,s}})$). The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the $\star$–action of $\mathrm{Gal}(F_s/F)$ *and such that $\Delta_0(G) \subset \Delta_0(G_{F_U})$ for each $U \in \mathcal{U}$ and $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$.*

The recollection for star action and Tits index is done in the beginning of the paper.

Proof. The outline is to prove (1) under an assumption of autoppositeness, to prove (2) and then to prove (1) in the general case.

(1) We assume that Z parameterizes an auto-opposite conjugacy class of parabolic subgroups of G. We use a Weil restriction argument as in the proof of [H-H-K2, Thm. 4.2]. This involves a finite morphism $f : \mathfrak{X} \to \mathbb{P}^1_T$ such that $\mathcal{P} = f^{-1}(\infty_k)$. Write $\underline{\mathrm{F}}$ for the function field of $\mathbb{P}^1_T$, and let $d = [F : F']$. We put $\underline{\mathrm{U}} = \mathbb{P}^1_k \setminus \{\infty\}$, $\underline{\mathrm{P}} = \infty_k$ and $\underline{\mathfrak{p}} = (U, \mathfrak{p})$ and $F_0 = \underline{\mathrm{F}}_{\underline{\mathfrak{p}}}$, $F_1 = \underline{\mathrm{F}}_{\underline{\mathrm{P}}}$ and $F_2 = F_{\underline{\mathrm{U}}}$. Also patching holds for the diamond $(\underline{\mathrm{F}}, F_1, F_2, F_0)$ according to [H-H, thm 3.9] so in particular $K(x) = \underline{\mathrm{F}} = F_1 \cap F_2 \subset F_0$. We put $V = F_1 \cap \widehat{R}_{\mathfrak{p}}$ and $W = F_2 \cap \widehat{R}_{\mathfrak{p}}$. Lemma 4.3 shows that F_1 is dense in F_0 and that V and W satisfy conditions (3.1) and (3.2).

We consider the Weil restriction $\underline{\mathrm{G}} = R_{F/\underline{\mathrm{F}}}(G)$, it is a reductive $\underline{\mathrm{F}}$–group which acts on the $\underline{\mathrm{F}}$–variety $\underline{\mathrm{Z}} = R_{F/\underline{\mathrm{F}}}(Z)$. We have

$$\underline{\mathrm{Z}}(F_1) = Z(F_1 \otimes_{\underline{\mathrm{F}}} F) = \prod_{P \in \mathcal{P}} Z(F_P)$$

according to [H-H, Lemma 6.2.(a)]. Similarly we have

$$\underline{\mathrm{Z}}(F_2) = Z(F_2 \otimes_{\underline{\mathrm{F}}} F) = \prod_{U \in \mathcal{U}} Z(F_U)$$

Our assumptions imply that $\underline{\mathrm{Z}}(F_1) \neq \emptyset$ and $\underline{\mathrm{Z}}(F_2) \neq \emptyset$. Theorem 3.9 implies that $\underline{\mathrm{Z}}(\underline{\mathrm{F}}) \neq \emptyset$. Thus $\underline{\mathrm{Z}}(\underline{\mathrm{F}}) = Z(F)$ is non-empty.

(2) Let Θ be the smallest subset of $\Delta(G_{F_s})$ which is stable under the $\star$–action of $\mathrm{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_U})$ for each $U \in \mathcal{U}$ and $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$. Since Θ is an intersection of auto-opposite subsets of $\Delta(G_{F_s})$, it is auto-opposite.

We observe that $\Delta_0(G) \subset \Theta$ since it is stable under the star action and satisfies $\Delta_0(G) \subset \Delta_0$ for each $U \in \mathcal{U}$ and $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$. For the converse inclusion we consider the F–variety Z of parabolic subgroups of type Θ (which is auto-opposite). For each $U \in \mathcal{U}$, we have $\Theta \subset \Delta_0(G_{F_U})$ so that $Z(F_U) \neq \emptyset$; similarly we have $\Theta \subset \Delta_0(G_{F_P})$ for each $P \in \mathcal{P}$ so that $Z(F_P) \neq \emptyset$. Part (1) yields that $Z(F) \neq \emptyset$. Thus $\Theta \subset \Delta_0(G)$ and $\Theta = \Delta_0(G)$

(3) We consider now the case of an arbitrary flag F–variety Z for G. It is associated to a subset Υ of $\Delta(G_{F_s})$ which is invariant under the $\star$–action of $\mathrm{Gal}(F_s/F)$. If $Z(F) \neq \emptyset$, we have that $Z(F_U) \neq \emptyset$ for each $U \in \mathcal{U}$ and that $Z(F_P) \neq \emptyset$ for each $P \in \mathcal{P}$. Conversely if $Z(F_U) \neq \emptyset$ for each $U \in \mathcal{U}$ and $Z(F_P) \neq \emptyset$ for each $P \in \mathcal{P}$. It follows that $\Upsilon \subset \Delta_0(G_{F_U})$ for each $U \in \mathcal{U}$. Part (2) of the statement yields $\Upsilon \subset \Delta_0(G)$ so that $Z(F) \neq \emptyset$. □

Corollary 4.5. *Let G be a reductive F–algebraic group.*

(1) Let Z be a twisted flag projective F–variety for G. Then $Z(F) \neq \emptyset$ if and only if $Z(F_P) \neq \emptyset$ for each $P \in Y$.

(2) For each $P \in Y$, we fix an F–embedding $i_P : F_s \to F_{P,s}$ providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{P,s}})$. The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the $\star$–action of $\mathrm{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_P})$ for each $P \in Y$.

Proof. (1) We assume $Z(F_P) \neq \emptyset$ for each $P \in Y$. Let $Y_1, \ldots, Y_d$ be the irreducible components of Y with respective generic points $\eta_1, \ldots, \eta_d$. According to [H-H-K2, prop. 5.8], there exists non-empty affine subsets $U_i \subset Y_i$ $(i = 1, \ldots, d)$ such that $Z(F_{U_i}) \neq \emptyset$ for $i = 1, \ldots, d$ and $U_i \cap U_j = \emptyset$ for $i < j$. We apply Theorem 4.4 to $\mathcal{U} = \{U_1, \ldots, U_d\}$, $\mathcal{P} = Y \setminus \cap_i U_i$ and get that $Z(F) \neq \emptyset$.

(2) This readily follows of (1).

□

Corollary 4.6. *Let G be a reductive F–algebraic group and assume that G is the generic fiber of a reductive $\mathfrak{X}$-group scheme $\mathfrak{G}$.*

(1) Let Z be a twisted flag projective F–variety for G. Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for each discrete valuation v of F.

(2) For each discrete valuation v of F, we fix an F–embedding $i_v : F_s \to F_{v,s}$ providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{v,s}})$. The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the $\star$–action of $\mathrm{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_v})$ for each discrete valuation v on F.

Proof. (1) We assume that $Z(F_v) \neq \emptyset$ for each discrete valuation v of F. Let $\mathfrak{G}$ be a reductive $\mathfrak{X}$–group of generic fiber G. Without loss of generality we can assume that $\mathfrak{G}$ is adjoint. Let $\mathfrak{G}_0$ be the Chevalley form of $\mathfrak{G}$ and let $(\mathfrak{B}_0, \mathfrak{T}_0)$ be a Killing couple for $\mathfrak{G}_0$ and let Δ_0 be the associated Dynkin diagram. Since $\mathfrak{G}_0$ is adjoint, we have an exact sequence of $\mathbb{Z}$–group schemes [SGA3, XXIV.1.3 and 3.6]

$$1 \to \mathfrak{G}_0 \to \mathrm{Aut}(\mathfrak{G}_0) \to \mathrm{Aut}(\Delta_0) \to 1$$

The $\mathrm{Aut}(\mathfrak{G}_0)$–torsor $\mathfrak{Q} = \mathrm{Isom}(\mathfrak{G}_0, \mathfrak{G})$ over $\mathfrak{X}$ defines then an $\mathrm{Aut}(\Delta_0)$-torsor over $\mathfrak{X}$. Since $\mathrm{Aut}(\Delta_0)$ is a finite constant group, Grothendieck's theory of fundamental groups tells us that this torsor is the data of a morphism $v : \Pi_1(\mathfrak{X}, \bullet) \to \mathrm{Aut}(\Delta_0)$ where the base point is $\mathrm{Spec}(F_s) \to \mathrm{Spec}(F) \to \mathfrak{X}$. The star action associated to G is the composite

$$\mathrm{Gal}(F_s/F) \to \Pi_1(\mathfrak{X}, \bullet) \to \mathrm{Aut}(\Delta_0).$$

Claim 4.7. *Z is the generic fiber of an $\mathfrak{X}$–scheme of parabolic subgroups $\mathfrak{Z}$ of $\mathfrak{G}$.*

The variety Z is a form of the variety $\mathrm{Par}_I(G_0)$ of parabolic subgroups of type I where $I \subset \Delta_0$ is stable the star action. Since $\mathfrak{X}$ is normal, the map $\mathrm{Gal}(F_s/F) \to \Pi_1(\mathfrak{X}, \bullet)$ is onto so that I is stable under the action of $\Pi_1(\mathfrak{X}, \bullet)$ on Δ_0. In particular Q admits a reduction $\mathfrak{Q}_I$ to the stabilizer $\mathrm{Aut}_I(\mathfrak{G}_0)$ for the action of $\mathrm{Aut}(\mathfrak{G}_0)$ on Δ_0. The $\mathfrak{X}$–scheme $\mathfrak{Z} = {}^{\mathfrak{Q}}\mathrm{Par}_I(\mathfrak{G}_0)$ is the scheme of parabolic subgroups of type I of $\mathfrak{G} = {}^{\mathfrak{Q}_I}\mathfrak{G}_0$, so that Z is the generic fiber of $\mathfrak{Z}$.

For applying Corollary 4.5, we have to check that $Z(F_P) \neq \emptyset$ for each $P \in Y$. If Q is a point of codimension 1 of $\mathfrak{X}$, it defines a discrete valuation v_Q on F whose completion is F_Q. Our assumption implies then that $Z(F_Q) \neq \emptyset$ in that case. We deal now with the case of a closed point P of $\mathfrak{X}$. Let D be an irreducible component of $Y = \mathfrak{X}_k$ containing P and let Q be the generic point of D. Since $\mathfrak{Z}$ is proper over T, we have $\mathfrak{Z}(\widehat{R}_Q) = \mathfrak{Z}(F_Q)$ which is not empty by the preceding case. It follows that $\mathfrak{Z}_k(k(D)) \neq \emptyset$. Again $\mathfrak{Z}_k$ is projective so that $\mathfrak{Z}_k(D) = \mathfrak{Z}_k(k(D))$ is not empty and in particular $\mathfrak{Z}_k(k(P))$ is not empty. Since $\mathfrak{Z}$ is smooth over $\mathfrak{X}$, the Hensel lemma shows that $\mathfrak{Z}_k(\widehat{R}_{\mathfrak{X},P}) \to \mathfrak{Z}_k(k(P))$ is surjective. Thus $\mathfrak{Z}(\widehat{R}_{\mathfrak{X},P})$ is not empty and so is $Z(F_P)$.
(2) It readily follows of (1).

□

Corollary 4.8. *Let G be a reductive F–algebraic group. We denote by Ω_F^1 the set of rank one valuations of F.*

(1) Let Z be a twisted flag projective F–variety for G. Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for each $v \in \Omega_F^1$.

(2) For each $v \in \Omega_F^1$, we fix an F–embedding $i_v : F_s \to F_{v,s}$ providing identifications $\Delta(G_{F_s}) \xrightarrow{\sim} \Delta(G_{F_{v,s}})$. The Tits index $\Delta_0(G)$ is the smallest subset of $\Delta(G_{F_s})$ which is stable under the $\star$–action of $\mathrm{Gal}(F_s/F)$ and such that $\Delta_0(G) \subset \Delta_0(G_{F_v})$ for each $v \in \Omega_F^1$.

Proof. (1) We assume that $Z(F_v) \neq \emptyset$ for each $v \in \Omega_F^1$. According to [H-H-K-P, Thm 2.5], there exists a regular proper model $\mathfrak{X}'$ of F such that $Z(F_P) \neq \emptyset$ for each $P \in \mathfrak{X}' \times_T k$. Then Corollary 4.6.(1) shows that $Z(F) \neq \emptyset$.

(2) This readily follows of (1). □

5. Local-global principle for discrete valuations

Let T be an excellent DVR of fraction field K and residue field k. Let X be a smooth, projective, geometrically integral curve over K. Let $F = K(X)$ be the function field of X.

Lemma 5.1. *Let $\mathfrak{X}$ be a projective, flat curve over A which is connected and regular such that $\mathfrak{X}_K = X$. Let $\mathfrak{H}$ be a flat affine $\mathfrak{X}$–group scheme of finite presentation and assume that there exists a Zariski cover $(\mathfrak{U}_i)_{i\in I}$ of $\mathfrak{X}$ such that each $\mathfrak{H}_{U_i}$ admits a closed embedding $\mathfrak{H}_{U_i} \subset \mathrm{GL}_{n_i}$ such that $\mathrm{GL}_{n_i}/\mathfrak{H}_{U_i}$ is representable by an affine U_i–scheme,*

Let $\gamma \in H^1_{\mathrm{fppf}}(F, \mathfrak{H})$ and let D be a divisor of $\mathfrak{X}$ which contains the irreducible components of $Y = \mathfrak{X}_k$ and such that γ extends to $X \setminus D_K$.

Then there exists a proper birational morphism $q : \mathfrak{X}' \xrightarrow{\sim} \mathfrak{X}$ such that $\mathfrak{X}'$ is a regular proper model of X and such that $D' = q^*D$ is a strict normal crossing divisor and

$$\gamma \in \mathrm{Im}\Big(H^1(\mathfrak{X}' \setminus D', \mathfrak{H}) \to H^1(F, \mathfrak{H})\Big).$$

Proof. Using a passage to the limit argument [Mg], there exists an open affine subscheme $\mathfrak{U}_1 \subset \mathfrak{X} \setminus D$ such that γ extends to a class $\gamma_1 \in H^1(\mathfrak{U}_1, \mathfrak{H})$. According to [G-P1, cor 1.8], γ_1 extends to a class $\gamma_2 \in H^1(\mathfrak{U}_2, \mathfrak{H})$ where $\mathfrak{U}_2$ is an open subscheme of $\mathfrak{X} \setminus D$ containing $\mathfrak{U}_1$ and $X \setminus D_K$. By purity (i.e. Theorem 7.1 of the appendix 7), we have $H^1(\mathfrak{X} \setminus D, \mathfrak{H}) = H^1(\mathfrak{U}_2, \mathfrak{H})$. Thus γ extends over $\mathfrak{X} \setminus D$. According to Lipman's theorem we can resolve the singularities of $\mathfrak{X}$ and transform D in a strict normal crossing divisor, see [H-H-K, lemma 4.7]. □

Proposition 5.2. *Let G be a reductive F–group and assume that p does not divide the order of the automorphism group of the absolute root system of G_{ad}. Let Z be a twisted flag variety of G. We assume that $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from models of X. Then there exists a regular proper model $\mathfrak{X}$ of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $Z(F_y) \neq \emptyset$.*

Proof. Without loss of generality we can assume that G is adjoint. Let G_0 be the Chevalley form of G and let (B_0, T_0) be a Killing couple for G_0 and let Δ_0 be the associated Dynkin diagram. The variety Z is a form of the variety $\mathrm{Par}_I(G_0)$ of parabolic subgroups of type I where $I \subset \Delta_0$ is stable under the star action defined by the $\mathrm{Aut}(G_0)$–torsor $Q = \mathrm{Isom}(G_0, G)$. In particular Q admits a reduction Q_I to the stabilizer $\mathrm{Aut}_I(G_0)$ for the action $\mathrm{Aut}(G_0)$ on Δ_0 through the morphism $\mathrm{Aut}(G_0) \to \mathrm{Out}(G_0) \xrightarrow{\sim} \mathrm{Aut}(\Delta_0)$. Furthermore Z is isomorphic to ${}^{Q_I}\mathrm{Par}_I(G_0)$.

We apply now Theorem 1.1.(b) of [C-G-R] to the $\mathbb{Z}$–group scheme $\mathrm{Aut}_I(G_0)$. It provides a finite $\mathbb{Z}$–subgroup S_0 of $\mathrm{Aut}_I(G_0)$ such that the map $H^1(F, S_0) \to H^1(F, \mathrm{Aut}(G_0))$ is onto. Furthermore the construction of S_0 is explicit in the proof, it is an extension of the finite constant group $\mathrm{Aut}_I(G_0)/T_0$ by a finite subgroup of T_0. In particular S_0 is finite free over $\mathbb{Z}$ and our assumption on the characteristic implies that $S_{0,T}$ is finite étale of rank prime to p. It follows that Q_I admits a reduction to an F–torsor E under S_0.

The finite flat $\mathbb{Z}$–group scheme $S_{0,A}$ admits a faithful representation $S_0 \hookrightarrow \mathrm{GL}_{N,\mathbb{Z}}$ [Br-T, §1.4.5] and the quotient $\mathrm{GL}_{N,\mathbb{Z}}/S_0$ is representable by an affine $\mathbb{Z}$-scheme [D-G, §III.2.6].

According to Lemma 5.1, there exists a regular proper model $\mathfrak{X}$ of X and a strict normal crossing divisor D containing the irreducible components of Y such that E extends to a $\mathfrak{X} \setminus D$–torsor $\mathfrak{E}$ under S_0. We put $\mathfrak{Q}_I = \mathfrak{E} \wedge^{S_0} \mathrm{Aut}_I(G_0)$ and consider the $\mathfrak{X} \setminus D$-group scheme $\mathfrak{G} = {}^{\mathfrak{Q}_I}G_0$ of generic fiber G.

We are given a closed point $P \in Y$. If y is of codimension one, by hypothesis, $Z(F_y)$ is not empty. We therefore look at a closed point $P \in Y$. We are given a point $P \in Y$ and pick a height one prime $\mathfrak{p}$ in $\widehat{R}_P$ that contains t. It defines a branch of Y at P lying on some irreducible component Y_1 of Y.

We consider the local ring $A = R_P$ of $\mathfrak{X}$ at P and denote by A_D its localization at D. Since $S_{0,T}$ is finite étale of degree prime to p, $H^1(A_D, S_0)$ consists in loop torsors as defined in [Gi2, §2.3, lemma 2.3.(2)], i.e. those arising from cocycles related to tame Galois covers of A_D. It follows that the A_D–torsor Q_I is a loop $\mathrm{Aut}_I(G)$–torsor [Gi2, lemma 2.3.(3)], so that $\mathfrak{G} \times_{\mathfrak{X}\setminus D} A_D$ is by definition a loop reductive group scheme.

Let $F_{P,v}$ be the completion of the field F_P for the valuation associated to the blow-up of $\mathrm{Spec}(A)$ at its closed point. Our assumption states in particular that $Z(F_{P,v}) \neq \emptyset$, that is, $G_{F_{P,v}}$ admits a parabolic subgroup of type I. According to [Gi2, th. 4.1 , (iii) $\Longrightarrow$ (i)], $\mathfrak{G} \times_{\mathfrak{X}\setminus D} A_D$ admits a parabolic subgroup of type I. A fortiori G_{F_P} admits a parabolic subgroup of type I so that $Z(F_P) \neq \emptyset$. $\square$

Remarks 5.3. (a) If $p = 0$, the result used [Gi2, th. 4.1] admits a simple proof, see [Gi2, Ex. 4.2], by using the analogous result over Laurent polynomials [G-P2, th. 7.1].
(b) In nice cases inspection of the proof permits to weaken the assumption on p. The precise condition is that the $\mathrm{Aut}(G_0)$–torsor $\mathrm{Isom}(G_0, G)$ admits a reduction to a finite F–subgroup whose degree is prime to p. For example in type G_2, we need to assume only that p is prime to 2.

Together with Corollary 4.5, we obtain the following consequence:

Theorem 5.4. *Let G be a reductive F–group and assume that p does not divide the order of the automorphism group of the absolute root system of G_{ad}. Let Z be a twisted flag F–variety of G. Then $Z(F) \neq \emptyset$ if and only if $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from models of X.*

6. Appendix: characterization of parabolic subgroups.

Let G be a reductive F–group over a field F. We remind the reader that an algebraic F–subgroup P of G is parabolic if P is smooth and G/P is a projective F–variety. The interest of the probably known statement below is only in positive characteristic since in this case there exist F–subgroups Q which are not smooth such that G/Q is a projective F–variety [W].

Proposition 6.1. *Let P be a k–subgroup of G such that the quotient variety G/P is projective. Then the following assertions are equivalent:*

(i) P is an F–parabolic subgroup

(ii) For each F–field E, $G(E)$ acts transitively on $(G/P)(E)$

(iii) The quotient map $G \to G/P$ admits a rational section

(iv) P is smooth connected

(v) P is smooth.

Proof. $(i) \Longrightarrow (ii)$. Since $(G/P)(E)$ parameterizes the E–parabolic subgroups of G_E of same type that P, Borel-Tits' conjugacy theorem [B-T, th. 4.13.c] shows that $G(E)$ acts transitively on $(G/P)(E)$.
$(ii) \Longrightarrow (iii)$. Our assumption rephrases by saying that the map $G(E) \to (G/P)(E)$ is onto for each F–field E. Applying that to the function field $E = F(G/P)$ of the smooth connected F- variety G/P provides a rational section of the map $G \to G/P$.
$(iii) \Longrightarrow (iv)$. To show the smoothness of P we can assume that F is algebraically closed. Then the neutral component $Q = (P_{red})^0$ of the reduced F–subgroup P_{red} of P is smooth. Furthermore the quotient F–variety P/Q is finite. It follows that the morphism $q : G/Q \to G/P$ is finite and a fortiori projective [St, Tag 0B3I]. Since the composition of projective morphisms (of qcqs schemes) is projective [St, Tag 0C4P], it follows that G/Q is projective. The F–subgroup Q of G is then parabolic. Our

assumption is that the morphism $G \to G/P$ has a rational section and so has a fortiori the finite morphism $q : G/Q \to G/P$. According to [EGAII, cor. 6.1.15], q is an isomorphism. Thus $Q = P$ and we conclude that P is smooth connected.

$(iv) \Longrightarrow (v)$. Obvious.

$(v) \Longrightarrow (i)$. This is by definition. □

A variant is the following.

Proposition 6.2. *Let X be a smooth projective G–variety. Then the following statements are equivalent:*

(i) X is a F–variety of parabolic subgroups of G

(ii) For each F–field E, $G(E)$ acts transitively on $X(E)$.

Proof. The implication $(i) \Longrightarrow (ii)$ is again Borel-Tits' conjugacy theorem. We assume (ii). According to [M-P-W, prop. 1.3], we can assume that F is separably closed. Since X is smooth, we have $X(F) \neq \emptyset$ and denote by P the stabilizer of some F–point x. According to [D-G, prop. III.3.2.1]. condition (ii) implies that the orbit map $G \to X$, $g \mapsto g.x$ induces an isomorphism $f_x : G/P \xrightarrow{\sim} X$. Proposition 6.1, $(ii) \Longrightarrow (i)$, shows that P is a F–parabolic subgroup. Thus X is a F–variety of parabolic subgroups of G. □

Remark 6.3. The condition (ii) is called *transitive action of G on X* by Harbater-Hartmann-Krashen. It occurs in [H-H-K, th. 3.7]. Projective homogeneous varieties in the result quoted above are exactly the various varieties of parabolic subgroups.

7. Appendix: extending torsors

We come back to a purity result of Colliot-Thélène and Sansuc.

Theorem 7.1. *Let X be a regular scheme of dimension 2. Let U be an open subscheme of X which contains $X^{(1)}$. Let G be an affine X–group scheme. In the following cases*

(i) G is reductive,

(ii) There exists a Zariski cover $(U_i)_{i\in I}$ of X such that each G_{U_i} admits a closed embedding $G_{U_i} \subset \mathrm{GL}_{n_i}$ such that $\mathrm{GL}_{n_i}/G_{U_i}$ is representable by an affine U_i–scheme,

then we have the equality $H^1_{\mathrm{fppf}}(X, G) \xrightarrow{\sim} H^1_{\mathrm{fppf}}(U, G)$.

Proof. The case (i) is [C-T-S, th. 6.13]. The case (ii) goes by inspection of the proof. □

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