The Weyl law of transmission eigenvalues and the completeness of generalized transmission eigenfunctions without complementing conditions

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Abstract. The transmission eigenvalue problem is a system of two second-order elliptic equations of two unknowns equipped with the Cauchy data on the boundary. In this work, we establish the Weyl law for the eigenvalues and the completeness of the generalized eigenfunctions for a system without complementing conditions, i.e., the two equations of the system have the same coefficients for the second order terms, and thus being degenerate. These coefficients are allowed to be anisotropic and are assumed to be of class C^2. One of the keys of the analysis is to establish the well-posedness and the regularity in L^p-scale for such a system. As a result, we largely extend and rediscover known results for which the coefficients for the second order terms are required to be isotropic and of class C^∞ using a new approach.

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1. Introduction

The transmission eigenvalue problem plays a role in the inverse scattering theory for inhomogeneous media. This eigenvalue problem is connected to the injectivity of the corresponding scattering operator [12], [20]. Transmission eigenvalues are related to interrogating frequencies for which there is an incident field that is not scattered by the medium. In the acoustic setting, the transmission eigenvalue problem is a system of two second-order elliptic equations of two unknowns equipped with the Cauchy data on the boundary. After four decades of extensive study, the spectral properties are known to depend on a type of contrasts of the media near the boundary. Natural and interesting questions on the interior transmission eigenvalue problem include: the *discreteness* of the spectrum (see e.g. [6, 4, 40, 21, 33, 11]), the *location* of transmission eigenvalues (see [8, 24, 41, 42], and also [9] for the application in time domain), the *Weyl law* of transmission eigenvalues and the *completeness* of the generalized eigenfunctions (see e.g. [21, 22, 23, 39]). We refer the reader to [7] for a recent, and self-contained introduction to the transmission eigenvalue problem and its applications.

Let us describe its mathematical formulation. Let Ω be a bounded, simply connected, open subset of $\mathbb{R}^d$ of class C^3 with $d \geq 2$. Let A_1, A_2 be two real, symmetric matrix-valued functions, and let Σ_1, Σ_2 be two bounded positive functions that are all defined in Ω. Assume that A_1 and A_2 are uniformly elliptic, and Σ_1 and Σ_2 are bounded below by a positive constant in Ω, i.e., for some constant $\Lambda \geq 1$, one has, for $\ell = 1, 2$,

$$\Lambda^{-1}|\xi|^2 \leq \langle A_\ell(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \text{ for a.e. } x \in \Omega, \tag{1.1}$$

and

$$\Lambda^{-1} \leq \Sigma_\ell(x) \leq \Lambda \text{ for a.e. } x \in \Omega. \tag{1.2}$$

Here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{C}^d$ and $| \cdot |$ is the corresponding norm.

A complex number λ is called an eigenvalue of the transmission eigenvalue problem associated with the pairs (A_1, Σ_1) and (A_2, Σ_2) in Ω if there is a non-zero pair of functions $(u_1, u_2) \in [H^1(\Omega)]^2$ that satisfies the system

$$\begin{cases} \operatorname{div}(A_1\nabla u_1) - \lambda\Sigma_1 u_1 = 0 & \text{in } \Omega, \\ \operatorname{div}(A_2\nabla u_2) - \lambda\Sigma_2 u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad A_1\nabla u_1 \cdot \nu = A_2\nabla u_2 \cdot \nu & \text{on } \Gamma. \end{cases} \tag{1.3}$$

Here and in what follows, Γ denotes $\partial\Omega$, and ν denotes the outward, normal, unit vector on Γ. Such a pair (u_1, u_2) is then called an eigenfunction pair.

Assume that A_1, A_2, Σ_1, Σ_2 are continuous in $\bar{\Omega}$, and the following conditions on the boundary Γ hold, with $\nu = \nu(x)$:

$$\langle A_2(x)\nu, \nu\rangle\langle A_2(x)\xi, \xi\rangle - \langle A_2(x)\nu, \xi\rangle^2 \neq \langle A_1(x)\nu, \nu\rangle\langle A_1(x)\xi, \xi\rangle - \langle A_1(x)\nu, \xi\rangle^2, \tag{1.4}$$

for all $x \in \Gamma$ and for all $\xi \in \mathbb{R}^d \setminus \{0\}$ with $\langle \xi, \nu \rangle = 0$, and

$$\big\langle A_2(x)\nu, \nu\big\rangle\Sigma_2(x) \neq \big\langle A_1(x)\nu, \nu\big\rangle\Sigma_1(x), \ \forall x \in \Gamma. \tag{1.5}$$

(Q. H.) Nguyen and the second author [34] established the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions for transmission eigenvalue problem under conditions (1.4) and (1.5) via the Fourier analysis assuming that A_1, A_2, Σ_1, Σ_2 are continuous in $\bar{\Omega}$. Condition (1.4) is equivalent to the celebrated complementing conditions due to Agmon, Douglis, and Nirenberg [3] (see also [2]). The explicit formula given here was derived in [29] in the context of the study of negative index materials. Conditions (1.4) and (1.5) were derived by (Q. H.) Nguyen and the second author in [33] in their study of the discreteness of the eigenvalues for transmission eigenvalue problem.

In the case

$$A_1 = A_2 = A \text{ in } \Omega, \tag{1.6}$$

it was also shown by (Q. H.) Nguyen and the second author [33] (see also [40]) using the multiplier technique that the discreteness holds if

$$\Sigma_1 \neq \Sigma_2 \text{ on } \Gamma. \tag{1.7}$$

The goal of this paper is to study the Weyl law of the eigenvalues and the completeness of the generalized eigenfunctions under conditions (1.6) and (1.7). It is worth noting that results in this direction have been obtained previously with more constraints on the coefficients than (1.6) and (1.7). Robbiano [39] (see also [38]) gives the sharp order of the counting number when $A = I$ in Ω, $\Sigma_1 = 1$, $\Sigma_2 \neq \Sigma_1$ near the boundary and Σ_2 is smooth. The analysis is based on both the microanalysis (see, e.g., [16, 46]) and the regularity theory for the transmssion eigenvalue problem. In the isotropic case, the Weyl law was established by Petkov and Vodev [37] and Vodev [42, 43, 44] for C^∞ coefficients. Their analysis is heavily based on microanalysis and the smoothness condition is strongly required. In addition, their work involved a delicate analysis on the Dirichlet to Neumann maps using non-standard parametrix construction initiated by Vodev [41], which have their own interests. It is not clear how one can improve the C^∞ condition and extend their results to the anisotropic setting using their approach. Concerning the completeness of the generalized eigenfunctions, we want to mention the work of Robbiano [38] where the case $A = I$ and $\Sigma_1 \neq \Sigma_2$ in $\bar{\Omega}$ was considered.

We are ready to state the main results of this paper. From now on, we will assume in addition that

$$\|(A_1, A_2)\|_{C^2(\bar{\Omega})} + \|(\Sigma_1, \Sigma_2)\|_{C^1(\bar{\Omega})} \leq \Lambda. \tag{1.8}$$

We denote by $(\lambda_j)_j$ the set of transmission eigenvalues associated with the transmission eigenvalue problem (1.3).

Concerning the Weyl law, we have

Theorem 1.1. *Assume* (1.1)-(1.2) *and* (1.6)-(1.8). *Let $\mathcal{N}(t)$ denote the counting function, i.e.*

$$\mathcal{N}(t) = \#\{j \in \mathbb{N} : |\lambda_j| \leq t\}.$$

Then

$$\mathcal{N}(t) = \mathbf{c} t^{\frac{d}{2}} + o(t^{\frac{d}{2}}) \text{ as } t \to +\infty,$$

where

$$\mathbf{c} := \frac{1}{(2\pi)^d} \sum_{\ell=1}^{2} \int_\Omega \Big| \Big\{ \xi \in \mathbb{R}^d;\ \langle A_\ell(x)\xi, \xi \rangle < \Sigma_\ell(x) \Big\} \Big| \, dx.$$

For a measurable subset D of $\mathbb{R}^d$, we denote $|D|$ its (Lebesgue) measure.

Concerning the completeness, we obtain

Theorem 1.2. *Assume* (1.1)-(1.2) *and* (1.6)-(1.8). *The set of generalized eigenfunction pairs of* (1.3) *is complete in $L^2(\Omega) \times L^2(\Omega)$.*

Remark 1.1. As a direct consequence of either Theorem 1.1 or Theorem 1.2, the number of eigenvalues of the transmission eigenvalue problem is infinite. As far as we know, this fact is new under the assumption that A is allowed to be anisotropic and the regularity of the coefficients are only required up to the order 2.

The analysis used in the proof of Theorem 1.1 and/or Theorem 1.2 also allows us to obtain the following result on the transmission eigenvalue free region of the complex plane $\mathbb{C}$.

Proposition 1.1. *Assume* (1.1)-(1.2) *and* (1.6)-(1.8). *For $\gamma > 0$, there exists $\lambda_0 > 0$ such that if $\lambda \in \mathbb{C}$ with $|\Im(\lambda)| \ge \gamma |\lambda|$ and $|\lambda| \ge \lambda_0$, then λ is not a transmission eigenvalue.*

Here and and in what follows, for $z \in \mathbb{C}$, let $\Im(z)$ denote the imaginary part of z.

A more general result of Proposition 1.1 is given in Proposition 3.1.

Remark 1.2. Since $\gamma > 0$ can be chosen arbitrary small, combining the discreteness result in [33] mentioned above and Proposition 1.1, one derives that all the transmission eigenvalues, but finitely many, lie in a wedge of arbitrary small angle.

Some comments on Theorem 1.1 and Theorem 1.2 are in order. In the conclusion of Theorem 1.1, the multiplicity of eigenvalues is taken into account and the multiplicity is associated with some operator T_{λ^*}, which is introduced in Section 4 (see (4.5) and (4.32)). Concerning T_{λ^*}, the following facts hold (see Remark 4.3 and Remark 4.5 for more information): if μ is a characteristic value of the operator T_{λ^*} associated with an eigenfunction (u, v) and $\lambda^* + \mu \neq 0$, then $\lambda^* + \mu$ is a transmission eigenvalue of (1.3) with an eigenfunction pair (u_1, u_2) given by

$$u_1 = (\lambda^* + \mu)u + v \qquad \text{and} \qquad u_2 = v.$$

Moreover, if λ_j is a transmission eigenvalue problem, then $\lambda_j \neq \lambda^*$ and $\lambda_j - \lambda^*$ is a characteristic value of T_{λ^*}. In Theorem 1.2, the generalized eigenfunctions are also associated to such an operator T_{λ^*}. We recall that the generalized eigenfunctions are complete in $[L^2(\Omega)]^2$ if the subspace spanned by them is dense in $[L^2(\Omega)]^2$.

Theorem 1.1 and Theorem 1.2 provide the Weyl laws and the completeness under the assumptions (1.6) and (1.7) assuming the regularity conditions in (1.8). Our results hold for $A_1 = A_2 = A$ being anisotropic in contrast to the isotropic setting considered previously. Moreover, the regularity assumption (1.8) on the coefficients was out of reach previously.

Our approach is in the spirit of [34] and is hence different from the ones used to study these problems given in the previous works mentioned above. The key idea is to establish the *regularity theory* for the transmission eigenvalue problem under the stated assumptions (see Theorem 3.1). Nevertheless, several new ingredients and observations are required for the regularity theory due to the fact that (1.6), which is degenerate, is considered instead of (1.4). One of the key steps to capture the phenomena is to derive appropriate estimates in a half plane setting. It is important to note that since $A_1 = A_2 = A$, the setting is non-standard, and the classical arguments pioneered in [2, 3] cannot be applied since the role of Σ_1 and Σ_2 are ignored there. To this end, our arguments for the Cauchy problems not only require the information of the first derivatives and their structure of the data but also involve the information of the second derivatives and their structure (see, e.g., Lemma 3.2). This is quite distinct from the complementing case where the arguments for the Cauchy problems only require the information of the first derivatives and no structure of the data is required [34] (see, e.g., [34, Lemma 2 and Corollary 2]). One might note that the arguments used to derive the discreteness in [33] requires less assumption on the regularity of the coefficients but only give the information for one direction of λ ($\arg \lambda = \pi/2$) for large λ. This is not sufficient to apply the theory of Hilbert-Schmidt operators.

We have so far discuss the transmission eigenvalue problem in the acoustic setting. Known results for the transmission eigenvalue problem in the electromagnetic setting are much less. In this direction, we mention the work of Cakoni and Nguyen [10] on the state of art on the discreteness of the eigenvalues, the work of Fornerod and Nguyen [13] on the completeness of generalized of eigenfunctions and the upper bound of the eigenvalues for the setting considered in [10], and the work of Vodev [45] on the free region of eigenvalues for a setting considered in [10], and the references therein.

The Cauchy problem also naturally appears in the context of negative-index materials after using reflections as initiated in [25] (see also [31]). The well-posedness and the limiting absorption principle for the Helmholtz equation with sign-changing coefficients were developed by the second author [29] using the Fourier and multiplier approach (see also [35]). The work [29] deals with the stability question of negative index materials, and is the starting point for the analysis of the transmission eigenvalue problems in [33, 34] (see also [10]). Other aspects and applications of negative-index materials as well as the stability and instability the Cauchy problem are discussed in [27, 28, 26, 30] and the references therein. A survey is given in [32].

The paper is organized as follows. Section 2 is devoted to define some notations used throughout the paper. In Section 3, we establish the well-posedness and the regularity theory for the Cauchy systems associated with the transmission eigenvalue problems. The analysis is then developed in such a way that the theory of Hilbert-Schmidt operators can be used. This is given in Section 4 where the Weyl laws are established. The completeness is considered in Section 5.

2. Notations

Here are some useful notations used throughput this paper. We denote, for $\tau > 0$,

$$\Omega_\tau = \Big\{ x \in \Omega : \operatorname{dist}(x, \Gamma) < \tau \Big\}. \tag{2.1}$$

For $d \geq 2$, set

$$\mathbb{R}^d_+ = \Big\{ x \in \mathbb{R}^d ; x_d > 0 \Big\} \quad \text{and} \quad \mathbb{R}^d_0 = \Big\{ x \in \mathbb{R}^d ; x_d = 0 \Big\}.$$

We will identify $\mathbb{R}^d_0$ with $\mathbb{R}^{d-1}$ in several places. For $s > 0$, we denote

$$B_s = \{ x \in \mathbb{R}^d : |x| < s \}.$$

For $m \geq 1$, $p \geq 1$, and $\lambda \in \mathbb{C}^*$ and $u \in W^{m,p}(\Omega)$, we define

$$\|u\|_{W^{m,p}_\lambda(\Omega)} = \left(\sum_{j=0}^{m} \| |\lambda|^{\frac{m-j}{2}} \nabla^j u \|^p_{L^p(\Omega)} \right)^{1/p}. \tag{2.2}$$

3. Well-posedness and regularity theory for the transmission eigenvalue problems

In this section, we study the well-posedness and the regularity theory of the Cauchy problem

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = f_1 & \text{in } \Omega, \\ \operatorname{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 = f_2 & \text{in } \Omega, \\ u_1 - u_2 = 0, \quad (A_1 \nabla u_1 - A_2 \nabla u_2) \cdot \nu = 0 & \text{on } \Gamma, \end{cases} \tag{3.1}$$

under the assumptions (1.1)-(1.2), and (1.7)-(1.8), and

$$A_1 = A_2 = A \text{ in } \Omega_\tau, \tag{3.2}$$

for some $\tau > 0$, instead of (1.6) for appropriate $\lambda \in \mathbb{C}$ and (f_1, f_2) in L^p-scale.

Here is the main result of this section.

Theorem 3.1. *Assume* (1.1)-(1.2), (1.7)-(1.8), *and* (3.2). *Let* $1 < p < +\infty$ *and* $\gamma \in (0,1)$. *There exist constants* $\lambda_0 > 0$ *and* $C > 0$ *depending on* Ω, Λ, τ, p, *and* γ *such that for* $\lambda \in \mathbb{C}$ *with* $|\lambda| > \lambda_0$ *and* $|\Im(\lambda)| \geq \gamma|\lambda|$ *and for* $(f_1, f_2) \in [L^p(\Omega)]^2$, *there is a unique solution* $(u_1, u_2) \in [L^p(\Omega)]^2$ *with* $u_1 - u_2 \in W^{2,p}(\Omega)$ *of the Cauchy problem* (3.1). *Moreover,*

$$|\lambda|\|(u_1,u_2)\|_{L^p(\Omega)} + \|u_1 - u_2\|_{W^{2,p}_\lambda(\Omega)} \leq C\|(f_1,f_2)\|_{L^p(\Omega)}. \tag{3.3}$$

Assume in addition that $f_1 - f_2 \in W^{1,p}(\Omega)$. *Then* $(u_1, u_2) \in [W^{1,p}(\Omega)]^2$, $u_1 - u_2 \in W^{3,p}(\Omega_{\tau/2})$, *and*

$$|\lambda|\|(u_1,u_2)\|_{W^{1,p}_\lambda(\Omega)} + \|u_1 - u_2\|_{W^{3,p}_\lambda(\Omega_{\tau/2})} \leq C\left(|\lambda|^{1/2}\|(f_1,f_2)\|_{L^p(\Omega)} + \|f_1 - f_2\|_{W^{1,p}_\lambda(\Omega)}\right). \tag{3.4}$$

Remark 3.1. The boundary conditions must be understood as

$$u_1 - u_2 = 0 \text{ on } \Gamma \quad \text{ and } \quad A\nabla(u_1 - u_2)\cdot\nu = 0 \text{ on } \Gamma,$$

which make sense since $u_1 - u_2 \in W^{2,p}(\Omega)$.

Remark 3.2. In (3.4), we only estimate $\|u_1 - u_2\|_{W^{3,p}_\lambda(\Omega_{\tau/2})}$ not $\|u_1 - u_2\|_{W^{3,p}_\lambda(\Omega)}$ since f_1 and f_2 are not supposed to be in $W^{1,p}(\Omega)$. Nevertheless, when $A_1 = A_2$ in Ω, the estimate is also valid for $\|u_1 - u_2\|_{W^{3,p}_\lambda(\Omega)}$.

Remark 3.1. As a consequence of (3.3) and the theory of regularity of elliptic equations, one derives that $(u_1, u_2) \in [W^{2,p}_{loc}(\Omega)]^2$ and for $\Omega' \Subset \Omega$ [1], it holds

$$\|(u_1,u_2)\|_{W^{2,p}_\lambda(\Omega')} \leq C\|(f_1,f_2)\|_{L^p(\Omega)},$$

where C depends also on Ω' (see, e.g., [18, Lemma 17.1.5] and [15, Theorem 9.11]).

As a consequence of Theorem 3.1, we obtain the following result on the free-region of the eigenvalues.

Proposition 3.1. *Assume* (1.1)-(1.2), (1.7)-(1.8), *and* (3.2). *For* $\gamma > 0$, *there exists* $\lambda_0 > 0$ *such that if* $\lambda \in \mathbb{C}$ *with* $|\Im(\lambda)| \geq \gamma|\lambda|$ *and* $|\lambda| \geq \lambda_0$, *then* λ *is not a transmission eigenvalue.*

The rest of this section, containing two subsections, is devoted to the proof of Theorem 3.1. The first one is on the analysis in the half space. The proof of Theorem 3.1 is then given in the second subsection.

3.1. **Half space analysis.** Let $1 < p < +\infty$. For $j = 1, 2, \cdots$, and $\lambda \in \mathbb{C} \setminus \{0\}$, we denote

$$\|\psi\|_{W^{j-1/p,p}_\lambda(\mathbb{R}^d_0)} = |\lambda|^{1/2-1/(2p)}\|\psi\|_{W^{j-1,p}_\lambda(\mathbb{R}^d_0)} + |\nabla^{j-1}\psi|_{W^{1-1/p,p}(\mathbb{R}^d_0)},$$

where $\|\psi\|_{W^{j-1,p}_\lambda(\mathbb{R}^d_0)}$ is defined as in (2.2) with $\Omega = \mathbb{R}^d_0$, and

$$|\psi|^p_{W^{1-1/p,p}(\mathbb{R}^d_0)} = \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} \frac{|\psi(x') - \psi(y')|^p}{|x'-y'|^{d+p-2}}\,dx'\,dy'.$$

By the trace theory, there exists a positive constant C depending only on p and j such that

$$\|u\|_{W^{j-1/p,p}_\lambda(\mathbb{R}^d_0)} \leq C\|u\|_{W^{j,p}_\lambda(\mathbb{R}^d_+)} \text{ for } u \in W^{j,p}(\mathbb{R}^d_+).$$

In fact, this inequality holds for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$; the general case follows by scaling.

The starting point and the key ingredient of our analysis is Lemma 3.2. Lemma 3.1 below is a special case of Lemma 3.2 and is later used to derive Lemma 3.2.

[1] Recall that $\Omega' \Subset \Omega$ means $\overline{\Omega'} \subset \Omega$.

Lemma 3.1. *Let $A \in \mathbb{R}^{d\times d}$ be a constant symmetric matrix and let Σ_1, Σ_2 be two positive constants such that*

$$\Lambda^{-1}|\xi|^2 \le \langle A\xi, \xi\rangle \le \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d,$$
$$\Lambda^{-1} \le \Sigma_1, \Sigma_2 \le \Lambda, \quad \text{and} \quad |\Sigma_1 - \Sigma_2| \ge \Lambda^{-1},$$

for some $\Lambda \ge 1$. Let $\gamma \in (0,1)$, $1 < p < +\infty$, and let $\varphi \in W^{2-1/p,p}(\mathbb{R}^d_0)$. Given $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1$ and $|\Im(\lambda)| \ge \gamma|\lambda|$, there exists a unique solution $(u_1, u_2) \in [L^p(\mathbb{R}^d_+)]^2$ with $u_1 - u_2 \in W^{2,p}(\mathbb{R}^d_+)$ of the following Cauchy problem

$$\begin{cases} \operatorname{div}(A\nabla u_1) - \lambda\Sigma_1 u_1 = 0 & \text{in } \mathbb{R}^d_+, \\ \operatorname{div}(A\nabla u_2) - \lambda\Sigma_2 u_2 = 0 & \text{in } \mathbb{R}^d_+, \\ u_1 - u_2 = \varphi, \quad A\nabla(u_1 - u_2)\cdot e_d = 0 & \text{on } \mathbb{R}^d_0. \end{cases}$$

Moreover,

$$|\lambda|\|(u_1,u_2)\|_{L^p(\mathbb{R}^d_+)} + \|u_1 - u_2\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|\varphi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)}. \tag{3.5}$$

Assume in addition that $\varphi \in W^{3-1/p,p}(\mathbb{R}^d_0)$. Then $(u_1,u_2) \in [W^{1,p}(\mathbb{R}^d_+)]^2$ with $u_1 - u_2 \in W^{3,p}(\mathbb{R}^d_+)$, and

$$|\lambda|\|(u_1,u_2)\|_{W^{1,p}_\lambda(\mathbb{R}^d_+)} + \|u_1 - u_2\|_{W^{3,p}_\lambda(\mathbb{R}^d_+)} \le C\|\varphi\|_{W^{3-1/p,p}_\lambda(\mathbb{R}^d_0)}. \tag{3.6}$$

Here C is a positive constant depending only on Λ, γ, p, and d.

Proof. For a function $u : \mathbb{R}^d \to \mathbb{C}$ (resp. $\varphi : \mathbb{R}^{d-1} \to \mathbb{C}$) we denote by $\hat{u}$ the Fourier transform of u with respect to the first $(d-1)$ variables (resp. by $\hat{\varphi}$ the Fourier transform of φ), i.e., for $(\xi', x_d) \in \mathbb{R}^{d-1} \times (0,\infty)$,

$$\hat{u}(\xi', x_d) = \int_{\mathbb{R}^{d-1}} u(x', x_d) e^{-ix'\cdot\xi'}\, dx' \quad \text{and} \quad \hat{\varphi}(\xi') = \int_{\mathbb{R}^{d-1}} \varphi(x') e^{-ix'\cdot\xi'}\, dx'.$$

Since, for $\ell = 1, 2$,

$$\operatorname{div}(A\nabla u_\ell) - \lambda\Sigma_\ell u_\ell = 0 \text{ in } \mathbb{R}^d_+,$$

it follows that

$$a\hat{u}_\ell''(\xi', t) + 2ib(\xi')\hat{u}_\ell'(\xi', t) - (c(\xi') + \lambda\Sigma_\ell)\hat{u}_\ell(\xi', t) = 0 \text{ for } t > 0,$$

where

$$a = \langle Ae_d, e_d\rangle, \quad b(\xi') = \sum_{j=1}^{d-1} A_{jd}\xi_j', \quad c(\xi') = \sum_{i,j=1}^{d-1} A_{ij}\xi_i'\xi_j', \quad \text{and} \quad ac(\xi') - b(\xi')^2 > 0, \tag{3.7}$$

since A is symmetric and positive. One then obtains, see, e.g., [34, proof of Lemma 2] for the details,

$$\hat{u}_\ell(\xi', t) = \alpha_\ell(\xi') e^{\eta_\ell(\xi')t} \tag{3.8}$$

where

$$\eta_\ell(\xi') = \frac{1}{a}\big(-ib(\xi') - \sqrt{\Delta_\ell(\xi')}\big) \tag{3.9}$$

and

$$\alpha_\ell(\xi') = \frac{\hat{\varphi}(\xi')\sqrt{\Delta_{\ell+1}(\xi')}}{\sqrt{\Delta_2(\xi')} - \sqrt{\Delta_1(\xi')}} \quad \text{with} \quad \Delta_\ell(\xi') = -b^2(\xi') + a\big(c(\xi') + \lambda\Sigma_\ell\big). \tag{3.10}$$

Here we use the convention $\Delta_{2+\ell} = \Delta_\ell$, and $\sqrt{\Delta_\ell}$ denotes the square root of Δ_ℓ with the positive real part.

Let $v_\ell \in W^{1,p}(\mathbb{R}^d_+)$ for $\ell = 1, 2$ be the unique solution of the system

$$\begin{cases} \operatorname{div}(A\nabla v_\ell) - \lambda \Sigma_\ell v_\ell = 0 & \text{in } \mathbb{R}^d_+, \\ v_\ell = \varphi & \text{on } \mathbb{R}^d_0. \end{cases}$$

We have[2] , for $\ell = 1, 2$,

$$\|v_\ell\|_{W^{j,p}_\lambda(\mathbb{R}^d_+)} \le C\|\varphi\|_{W^{j-1/p,p}_\lambda(\mathbb{R}^d_0)} \text{ for } j = 2, 3, \tag{3.11}$$

and

$$\hat{v}_\ell(\xi', t) = \hat{\varphi}(\xi') e^{\eta_\ell(\xi')t}. \tag{3.12}$$

Extend $u_\ell(x', t)$ and $\partial^2_{tt} v_\ell(x', t)$ by 0 for $t < 0$ for $\ell = 1, 2$ and *still* denote these extensions by $u_\ell(x', t)$ and $\partial^2_{tt} v_\ell(x', t)$. Let $\mathcal{F}$ denote the Fourier transform in $\mathbb{R}^d$. We then obtain from (3.8) and (3.12) that, with $\xi = (\xi', \xi_d) \in \mathbb{R}^d$,

$$\mathcal{F}u_\ell(\xi) = -\frac{\hat{\varphi}(\xi')}{\eta_\ell(\xi') - i\xi_d} \frac{\sqrt{\Delta_{\ell+1}(\xi')}}{\sqrt{\Delta_2(\xi')} - \sqrt{\Delta_1(\xi')}} \quad \text{and} \quad \mathcal{F}\partial^2_{tt} v_\ell(\xi) = -\frac{\hat{\varphi}(\xi')\eta^2_\ell(\xi')}{\eta_\ell(\xi') - i\xi_d}.$$

It follows that

$$\mathcal{F}u_\ell(\xi) = m_{\ell,\lambda}(\xi) \mathcal{F}\partial^2_{tt} v_\ell(\xi),$$

where

$$m_{\ell,\lambda}(\xi) = \frac{\sqrt{\Delta_{\ell+1}(\xi')}}{\eta^2_\ell(\xi')(\sqrt{\Delta_2(\xi')} - \sqrt{\Delta_1(\xi')})}. \tag{3.13}$$

Note that

$$\Delta_2(\xi') - \Delta_1(\xi') = a\lambda(\Sigma_2 - \Sigma_1) \neq 0$$

and

$$\frac{1}{\eta_\ell(\xi')} \overset{(3.9)}{=} \frac{a}{-ib(\xi') - \sqrt{\Delta_\ell(\xi')}} = \frac{a\big(-ib(\xi') + \sqrt{\Delta_\ell(\xi')}\big)}{-b(\xi')^2 - \Delta_\ell(\xi')}$$

$$\overset{(3.10)}{=} \frac{a\big(-ib(\xi') + \sqrt{\Delta_\ell(\xi')}\big)}{-a\big(c(\xi') + \lambda\Sigma_\ell\big)} = \frac{ib(\xi') - \sqrt{\Delta_\ell(\xi')}}{c(\xi') + \lambda\Sigma_\ell}.$$

We derive from (3.13) that

$$m_{\ell,\lambda}(\xi) = \frac{\sqrt{\Delta_{\ell+1}(\xi')}(\sqrt{\Delta_1(\xi')} + \sqrt{\Delta_2(\xi')})(ib(\xi') - \sqrt{\Delta_\ell(\xi')})^2}{a\lambda(\Sigma_2 - \Sigma_1)(c(\xi') + \lambda\Sigma_\ell)^2}. \tag{3.14}$$

We have, by (3.7) and (3.10),[3]

$$|\Delta_\ell(\xi')| \sim (|\xi'|^2 + |\lambda|), \quad |b(\xi')| \le C|\xi'|, \quad \text{and} \quad |c(\xi') + \lambda\Sigma_\ell| \sim |\xi'|^2 + |\lambda|.$$

We then derive from (3.14) that

$$|\xi|^j |\nabla^j m_{\ell,\lambda}(\xi)| \le C_j |\lambda|^{-1} \text{ for } j \in \mathbb{N}. \tag{3.15}$$

[2]The results hold for $|\lambda| = 1$, see, e.g. [2, Theorem 14.1], the general case follows by scaling.

[3]Given two functions, $p_1(\xi', \lambda)$ and $p_2(\xi', \lambda)$ the notation $p_1(\xi, \lambda) \sim p_2(\xi', \lambda)$ means that there exists a constant $C \ge 1$ independent of ξ' and λ such that $C^{-1}|p_1(\xi', \lambda)| \le |p_2(\xi', \lambda)| \le C|p_1(\xi', \lambda)|$.

It follows from Mikhlin-Hörmander's multiplier theorem, see, e.g., [19, Theorem 7.9.5], that

$$|\lambda|\|u_\ell\|_{L^p(\mathbb{R}^d)} \le C\|\partial^2_{tt} v_\ell\|_{L^p(\mathbb{R}^d)}, \tag{3.16}$$

which implies

$$|\lambda|\|u_\ell\|_{L^p(\mathbb{R}^d)} \overset{(3.11)}{\le} C\|\varphi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d)}. \tag{3.17}$$

On the other hand, one has

$$\begin{cases} \operatorname{div}\big(A\nabla(u_1-u_2)\big) - \lambda\Sigma_1(u_1-u_2) = \lambda(\Sigma_1-\Sigma_2)u_2 \text{ in } \mathbb{R}^d_+, \\ u_1 - u_2 = 0 \text{ on } \mathbb{R}^d_0. \end{cases}$$

This yields

$$\|u_1-u_2\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|\lambda(\Sigma_1-\Sigma_2)u_2\|_{L^p(\mathbb{R}^d_+)} \overset{(3.17)}{\le} C\|\varphi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_+)}.$$

We next deal with (3.6). By taking the derivative of the system with respect to x_j for $1 \le j \le d-1$ and applying (3.5), we have, for $1 \le j \le d-1$,

$$|\lambda|\|(\partial_{x_j}u_1, \partial_{x_j}u_2)\|_{L^p(\mathbb{R}^d_+)} + \|(\partial_{x_j}u_1 - \partial_{x_j}u_2)\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|\partial_{x_j}\varphi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)}. \tag{3.18}$$

Extend $\partial_t u_\ell(x',t)$ and $\partial^3_{ttt}v_\ell(x',t)$ by 0 for $t<0$ for $\ell = 1, 2$ and *still* denote these extensions by $\partial_t u_\ell(x',t)$ and $\partial^3_{ttt}v_\ell(x',t)$. We then obtain from (3.8) and (3.12) that, with $\xi = (\xi', \xi_d) \in \mathbb{R}^d$,

$$\mathcal{F}\partial_t u_\ell(\xi) = -\frac{\hat\varphi(\xi')\eta_\ell(\xi')}{\eta_\ell(\xi') - i\xi_d}\frac{\sqrt{\Delta_{\ell+1}(\xi')}}{\sqrt{\Delta_2(\xi')} - \sqrt{\Delta_1(\xi')}} \quad \text{and} \quad \mathcal{F}\partial^3_{ttt}v_\ell(\xi) = -\frac{\hat\varphi(\xi')\eta^3_\ell(\xi')}{\eta_\ell(\xi') - i\xi_d}.$$

This yields

$$\mathcal{F}\partial_t u_\ell(\xi) = m_{\ell,\lambda}(\xi)\mathcal{F}\partial^3_{ttt}v_\ell(\xi).$$

As in the proof of (3.16), we obtain

$$|\lambda|\|\partial_t u_\ell\|_{L^p(\mathbb{R}^d)} \le C\|\partial^3_{ttt}v_\ell\|_{L^p(\mathbb{R}^d)},$$

which implies

$$|\lambda|\|\partial_t u_\ell\|_{L^p(\mathbb{R}^d)} \overset{(3.11)}{\le} C\|\varphi\|_{W^{3-1/p,p}_\lambda(\mathbb{R}^d)}. \tag{3.19}$$

Combining (3.18) and (3.19), we derive that

$$|\lambda|\|(\nabla u_1, \nabla u_2)\|_{L^p(\mathbb{R}^d_+)} + \|\nabla(u_1-u_2)\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|\varphi\|_{W^{3-1/p,p}_\lambda(\mathbb{R}^d_0)}. \tag{3.20}$$

Assertion (3.6) now follows from (3.20) and (3.5). The proof is complete. □

We now state and prove a more general version of Lemma 3.1, which is the main ingredient of the proof of Theorem 3.1.

Lemma 3.2. *Let $A \in \mathbb{R}^{d\times d}$ be a constant symmetric matrix and let Σ_1, Σ_2 be two positive constants such that*

$$\Lambda^{-1}|\xi|^2 \le \langle A\xi, \xi\rangle \le \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d,$$

and

$$\Lambda^{-1} \le \Sigma_1, \Sigma_2 \le \Lambda, \quad \text{and} \quad |\Sigma_1 - \Sigma_2| \ge \Lambda^{-1},$$

for some $\Lambda \ge 1$. Let $\gamma \in (0,1)$, $1 < p < +\infty$, and let $f_1, f_2 \in L^p(\mathbb{R}^d_+)$, $G_1, G_2 \in [L^p(\mathbb{R}^d_+)]^d$ with $G_1 - G_2 \in [W^{1,p}(\mathbb{R}^d_+)]^d$, $\varphi \in W^{2-1/p,p}(\mathbb{R}^d_0)$, $\psi \in W^{1-1/p,p}(\mathbb{R}^d_0)$, and let $r_1^{(ij)}, r_2^{(ij)} \in L^p(\mathbb{R}^d_+)$ with

$r_1^{(ij)} - r_2^{(ij)} \in W^{2,p}(\mathbb{R}^d_+)$ for $1 \le i,j \le d$. Given $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1$ and $|\Im(\lambda)| \ge \gamma|\lambda|$, there exists a unique solution $(u_1,u_2) \in [L^p(\mathbb{R}^d_+)]^2$ with $u_1 - u_2 \in W^{2,p}(\mathbb{R}^d_+)$ of the following Cauchy problem

$$
\left\{\begin{aligned}
\operatorname{div}(A\nabla u_1) - \lambda\Sigma_1 u_1 &= f_1 + \operatorname{div}(G_1) + \textstyle\sum_{i,j=1}^d \partial^2_{ij} r_1^{(ij)} && \text{in } \mathbb{R}^d_+,\\
\operatorname{div}(A\nabla u_2) - \lambda\Sigma_2 u_2 &= f_2 + \operatorname{div}(G_2) + \textstyle\sum_{i,j=1}^d \partial^2_{ij} r_2^{(ij)} && \text{in } \mathbb{R}^d_+,\\
u_1 - u_2 = \varphi, \quad A\nabla(u_1-u_2)\cdot e_d &= \psi && \text{on } \mathbb{R}^d_0.
\end{aligned}\right. \tag{3.21}
$$

Moreover,

$$
\begin{aligned}
C\Big(|\lambda|\|(u_1,u_2)\|_{L^p(\mathbb{R}^d_+)} + \|u_1-u_2\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)}\Big)
&\le \|(f_1,f_2)\|_{L^p(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|(G_1,G_2)\|_{L^p(\mathbb{R}^d_+)} + \sum_{i,j=1}^d |\lambda|\|(r_1^{(ij)}, r_2^{(ij)})\|_{L^p(\mathbb{R}^d_+)}\\
&+ \|\varphi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)} + \|\psi\|_{W^{1-1/p,p}_\lambda(\mathbb{R}^d_0)} + \|G_1-G_2\|_{W^{1,p}_\lambda(\mathbb{R}^d_+)} + \sum_{i,j=1}^d \|r_1^{(ij)} - r_2^{(ij)}\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)}.
\end{aligned} \tag{3.22}
$$

Assume in addition that $f_1 - f_2 \in W^{1,p}(\mathbb{R}^d_+)$, $G_1 - G_2 \in W^{2,p}(\mathbb{R}^d_+)$, $\varphi \in W^{3-1/p,p}(\mathbb{R}^d_0)$, $\psi \in W^{2-1/p,p}(\mathbb{R}^d_0)$, and $r_1^{(ij)} = r_2^{(ij)} = 0$ for all $1 \le i,j \le d$. Then $(u_1,u_2) \in W^{1,p}(\mathbb{R}^d_+)$ with $u_1 - u_2 \in W^{3,p}(\mathbb{R}^d_+)$, and it holds

$$
\begin{aligned}
C\Big(|\lambda|\|(u_1,u_2)\|_{W^{1,p}_\lambda(\mathbb{R}^d_+)} + \|u_1-u_2\|_{W^{3,p}_\lambda(\mathbb{R}^d_+)}\Big)
&\le |\lambda|^{1/2}\|(f_1,f_2)\|_{L^p(\mathbb{R}^d_+)} + |\lambda|\|(G_1,G_2)\|_{L^p(\mathbb{R}^d_+)} + \|\varphi\|_{W^{3-1/p,p}_\lambda(\mathbb{R}^d_0)}\\
&+ \|\psi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)} + \|f_1-f_2\|_{W^{1,p}_\lambda(\mathbb{R}^d_+)} + \|G_1-G_2\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)}.
\end{aligned} \tag{3.23}
$$

Here C denotes a positive constant depending only on Λ, γ, d, and p.

Remark 3.3. Concerning (3.23), the assumption $r_1^{(ij)} = r_2^{(ij)} = 0$ for all $1 \le i,j \le d$ is just to avoid the redundancy; the same estimate holds for the appropriate assumptions on $r_\ell^{(ij)}$ but this can be put into the conditions of f_ℓ and G_ℓ instead.

Proof. Since the problem is linear, (3.22) and (3.23) follow from the corresponding estimates in the following two cases:

- Case 1: $f_1 = f_2 = 0$, $G_1 = G_2 = 0$, and $r_1^{(ij)} = r_2^{(ij)} = 0$ for all $1 \le i,j \le d$.
- Case 2: $\varphi = 0$ and $\psi = 0$.

We now proceed the proof for these cases.

Case 1: $f_1 = f_2 = 0$, $G_1 = G_2 = 0$, and $r_1^{(ij)} = r_2^{(ij)} = 0$ for all $1 \le i,j \le d$. We have

$$
\left\{\begin{aligned}
\operatorname{div}(A\nabla u_1) - \lambda\Sigma_1 u_1 &= 0 && \text{in } \mathbb{R}^d_+,\\
\operatorname{div}(A\nabla u_2) - \lambda\Sigma_2 u_2 &= 0 && \text{in } \mathbb{R}^d_+,\\
u_1 - u_2 = \varphi, \quad A\nabla(u_1-u_2)\cdot e_d &= \psi && \text{on } \mathbb{R}^d_0.
\end{aligned}\right.
$$

Let $v \in W^{1,p}(\mathbb{R}^d_+)$ be the unique solution of

$$\begin{cases} \operatorname{div}(A\nabla v) - \lambda \Sigma_1 v = 0 & \text{in } \mathbb{R}^d_+, \\ A\nabla v \cdot e_d = \psi & \text{on } \mathbb{R}^d_0. \end{cases}$$

As a consequence of [17, Theorem 2.3.2.7] and a scaling argument, we have

$$\|v\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|\psi\|_{W^{1-1/p,p}_\lambda(\mathbb{R}^d_0)} \quad \text{and} \quad \|v\|_{W^{3,p}_\lambda(\mathbb{R}^d_+)} \le C\|\psi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)}. \tag{3.24}$$

By the trace theory, it follows that

$$\|v\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)} \le C\|\psi\|_{W^{1-1/p,p}_\lambda(\mathbb{R}^d_0)} \quad \text{and} \quad \|v\|_{W^{3-1/p,p}_\lambda(\mathbb{R}^d_0)} \le C\|\psi\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)}. \tag{3.25}$$

Considering the system of $(u_1 - v, u_2)$ and using (3.24), and (3.25), the conclusion of this case follows from Lemma 3.1.

<u>*Case 2:*</u> $\varphi = 0$, $\psi = 0$. In this case, we have

$$\begin{cases} \operatorname{div}(A\nabla u_1) - \lambda\Sigma_1 u_1 = f_1 + \operatorname{div}(G_1) + \sum_{i,j=1}^d \partial^2_{ij} r_1^{(ij)} & \text{in } \mathbb{R}^d_+, \\ \operatorname{div}(A\nabla u_2) - \lambda\Sigma_2 u_2 = f_2 + \operatorname{div}(G_2) + \sum_{i,j=1}^d \partial^2_{ij} r_2^{(ij)} & \text{in } \mathbb{R}^d_+, \\ u_1 - u_2 = 0, \quad A\nabla(u_1 - u_2)\cdot e_d = 0 & \text{on } \mathbb{R}^d_0. \end{cases}$$

For $\ell = 1, 2$, consider the following systems

$$\begin{cases} \operatorname{div}(A\nabla v_\ell^{(0)}) - \lambda\Sigma_\ell v_\ell^{(0)} &= f_\ell \quad \text{in } \mathbb{R}^d_+, \\ A\nabla v_\ell^{(0)}\cdot e_d &= 0 \quad \text{on } \mathbb{R}^d_0, \end{cases}$$

$$\begin{cases} \operatorname{div}(A\nabla v_\ell^{(j)}) - \lambda\Sigma_\ell v_\ell^{(j)} &= (G_\ell)_j \quad \text{in } \mathbb{R}^d_+, \\ A\nabla v_\ell^{(j)}\cdot e_d &= 0 \quad \text{on } \mathbb{R}^d_0 \end{cases} \qquad (1 \le j \le d),$$

where $(G_\ell)_j$ denotes the j-th component of G_ℓ, and

$$\begin{cases} \operatorname{div}(A\nabla v_\ell^{(ij)}) - \lambda\Sigma_\ell v_\ell^{(ij)} &= r_\ell^{(ij)} \quad \text{in } \mathbb{R}^d_+, \\ A\nabla v_\ell^{(ij)}\cdot e_d &= 0 \quad \text{on } \mathbb{R}^d_0 \end{cases} \qquad (1 \le i, j \le d).$$

We have, see, e.g., [2, Theorem 14.1], for $1 \le i, j \le d$,

$$\begin{cases} \|v_\ell^{(0)}\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|f_\ell\|_{L^p(\mathbb{R}^d_+)}, \\ \|v_\ell^{(j)}\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|G_\ell\|_{L^p(\mathbb{R}^d_+)}, \\ \|v_\ell^{(ij)}\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)} \le C\|r_\ell^{(ij)}\|_{L^p(\mathbb{R}^d_+)}. \end{cases} \tag{3.26}$$

Since, we have

$$\begin{cases} \operatorname{div}(A\nabla(v_1^{(0)} - v_2^{(0)})) - \lambda\Sigma_1(v_1^{(0)} - v_2^{(0)}) &= f_1 - f_2 + \lambda(\Sigma_1 - \Sigma_2)v_2^{(0)} \quad \text{in } \mathbb{R}^d_+, \\ A\nabla(v_1^{(0)} - v_2^{(0)})\cdot e_d &= 0 \quad \text{on } \mathbb{R}^d_0, \end{cases}$$

and the equations for $v_1^{(j)} - v_2^{(j)}$ and $v_1^{(ij)} - v_2^{(ij)}$ are similar, we also get, for $1 \le i, j \le d$, by using (3.26),

$$\begin{cases} C\|v_1^{(0)} - v_2^{(0)}\|_{W_\lambda^{2,p}(\mathbb{R}^d_+)} \le \|(f_1, f_2)\|_{L^p(\mathbb{R}^d_+)}, \\ C\|v_1^{(j)} - v_2^{(j)}\|_{W_\lambda^{3,p}(\mathbb{R}^d_+)} \le \|G_1 - G_2\|_{W_\lambda^{1,p}(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|G_2\|_{L^p(\mathbb{R}^d_+)}, \\ C\|v_1^{(ij)} - v_2^{(ij)}\|_{W_\lambda^{4,p}(\mathbb{R}^d_+)} \le \|r_1^{(ij)} - r_2^{(ij)}\|_{W_\lambda^{2,p}(\mathbb{R}^d_+)} + |\lambda|\|r_2^{(ij)}\|_{L^p(\mathbb{R}^d_+)}, \end{cases} \tag{3.27}$$

and

$$\begin{aligned} C\|v_1^{(0)} - v_2^{(0)}\|_{W_\lambda^{3,p}(\mathbb{R}^d_+)} &\le \|f_1 - f_2\|_{W_\lambda^{1,p}(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|f_2\|_{L^p(\mathbb{R}^d_+)}, \\ C\|v_1^{(j)} - v_2^{(j)}\|_{W_\lambda^{4,p}(\mathbb{R}^d_+)} &\le \|G_1 - G_2\|_{W_\lambda^{2,p}(\mathbb{R}^d_+)} + |\lambda|\|G_2\|_{L^p(\mathbb{R}^d_+)}. \end{aligned} \tag{3.28}$$

For $\ell = 1, 2$, set

$$w_\ell = v_\ell^{(0)} + \sum_{j=1}^d \partial_j v_\ell^{(j)} + \sum_{i,j=1}^d \partial_{ij}^2 v_\ell^{(ij)}.$$

We have

$$\operatorname{div}(A\nabla w_\ell) - \lambda\Sigma_\ell w_\ell = f_\ell + \operatorname{div}(G_\ell) + \sum_{i,j=1}^d \partial_{ij}^2 r_\ell^{(ij)} \text{ in } \mathbb{R}^d_+.$$

Moreover,

$$\begin{aligned} &C|\lambda|\|(w_1, w_2)\|_{L^p(\mathbb{R}^d_+)} \\ &\quad \overset{(3.26)}{\le} \|(f_1, f_2)\|_{L^p(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|(G_1, G_2)\|_{L^p(\mathbb{R}^d_+)} + |\lambda| \sum_{i,j=1}^d \|(r_1^{(ij)}, r_2^{(ij)})\|_{L^p(\mathbb{R}^d_+)}. \end{aligned} \tag{3.29}$$

Using (3.27) and the trace theory, we derive that

$$\begin{aligned} &\|w_1 - w_2\|_{W_\lambda^{2,p}(\mathbb{R}^d_+)} + \|w_1 - w_2\|_{W_\lambda^{2-1/p,p}(\mathbb{R}^d_0)} + \|A\nabla(w_1 - w_2)\cdot e_d\|_{W_\lambda^{1-1/p,p}(\mathbb{R}^d_0)} \\ &\le C\Big(\|f_2\|_{L^p(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|(G_1, G_2)\|_{L^p(\mathbb{R}^d_+)} + \sum_{i,j=1}^d |\lambda|\|r_2^{ij}\|_{L^p(\mathbb{R}^d_+)} \\ &\qquad + \|f_1 - f_2\|_{L^p(\mathbb{R}^d_+)} + \|G_1 - G_2\|_{W_\lambda^{1,p}(\mathbb{R}^d_+)} + \sum_{i,j=1}^d \|r_1^{(ij)} - r_2^{(ij)}\|_{W_\lambda^{2,p}(\mathbb{R}^d_+)}\Big). \end{aligned} \tag{3.30}$$

Considering the system of $(u_1 - w_1, u_2 - w_2)$, and using (3.29) and (3.30), assertion (3.22) now follows from case 1.

To deal with assertion (3.23), instead of (3.29) and (3.30), we use, since $r_1^{(ij)} = r_2^{(ij)} = 0$,

$$|\lambda|\|(w_1, w_2)\|_{W_\lambda^{1,p}(\mathbb{R}^d_+)} \overset{(3.26)}{\le} C\Big(|\lambda|^{1/2}\|(f_1, f_2)\|_{L^p(\mathbb{R}^d_+)} + |\lambda|\|(G_1, G_2)\|_{L^p(\mathbb{R}^d_+)}\Big), \tag{3.31}$$

and

$$\|w_1 - w_2\|_{W^{3,p}_\lambda(\mathbb{R}^d_+)} + \|w_1 - w_2\|_{W^{3-1/p,p}_\lambda(\mathbb{R}^d_0)} + \|A\nabla(w_1 - w_2)\cdot e_d\|_{W^{2-1/p,p}_\lambda(\mathbb{R}^d_0)} \overset{(3.28)}{\le} C\Big(\|(f_1 - f_2)\|_{W^{1,p}_\lambda(\mathbb{R}^d_+)} + \|G_1 - G_2\|_{W^{2,p}_\lambda(\mathbb{R}^d_+)}\Big). \tag{3.32}$$

By considering the system of $(u_1 - w_1, u_2 - w_2)$, assertion (3.23) now follows from case 1.

The proof is complete. □

3.2. **Proof of Theorem 3.1.** The proof is divided into two steps:

- Step 1: Assuming the solution exists, we establish (3.3) and (3.4).
- Step 2: We establish the existence of the solutions.

We now proceed these two steps.

Step 1: For $(f_1, f_2) \in [L^p(\Omega)]^2$, let $(u_1, u_2) \in [L^p(\Omega)]^2$ with $u_1 - u_2 \in W^{2,p}(\Omega)$ be a solution of (3.1). We prove that (3.3) and (3.4) hold.

Applying Lemma 3.2 and the freezing coefficient technique, we deduce that there exists $\tau_* \in (0, \tau/2)$ depending only on Ω, Λ, τ, and p, such that

$$C\left(|\lambda|\|(u_1, u_2)\|_{L^p(\Omega_{\tau_*})} + \|u_1 - u_2\|_{W^{2,p}_\lambda(\Omega_{\tau_*})}\right) \le \|(f_1, f_2)\|_{L^p(\Omega)} + |\lambda|^{1/2}\|(u_1, u_2)\|_{L^p(\Omega_\tau)} + \|u_1 - u_2\|_{W^{1,p}_\lambda(\Omega_\tau)} \tag{3.33}$$

and

$$\begin{aligned} C\left(|\lambda|\|(u_1, u_2)\|_{W^{1,p}_\lambda(\Omega_{\tau_*})} + \|u_1 - u_2\|_{W^{3,p}_\lambda(\Omega_{\tau_*})}\right) &\le |\lambda|^{1/2}\|(f_1, f_2)\|_{L^p(\Omega)} + \|f_1 - f_2\|_{W^{1,p}_\lambda(\Omega)} \\ &\quad + |\lambda|\|(u_1, u_2)\|_{L^p(\Omega_\tau)} + |\lambda|\|(u_1, u_2)\|_{L^p(\Omega_\tau)} + \|u_1 - u_2\|_{W^{2,p}_\lambda(\Omega_\tau)}, \end{aligned} \tag{3.34}$$

for every $\lambda \in \mathbb{C}$ with $|\Im(\lambda)| \ge c|\lambda|$ and $|\lambda| \ge 1$. Here and in what follows, C denotes a positive constant depending only on Ω, Λ, τ, and p.

Let us emphasize here that the terms $(r_{1,ij}, r_{2,ij})$ in Lemma 3.2, play a crucial role in the proof of (3.33) since the solutions (u_1, u_2) considered are only in $[L^p(\Omega)]^2$, but not in $[W^{1,p}(\Omega)]^2$. Indeed, let consider a small neighborhood of $x_0 \in \Gamma$. Using a change of variables, without loss of generality, one might assume that the boundary in this neighbourhood is *flat* already and $A_1 = A_2 = A$ there. In the freezing process, one has, in such a neighborhood,

$$\operatorname{div}(A(x_0)\nabla u_\ell) - \lambda\Sigma_\ell(x_0)u_\ell = \operatorname{div}\big((A(x_0) - A(x))\nabla u_\ell\big) + \operatorname{div}\big(A(x)\nabla u_\ell\big) - \lambda\Sigma_\ell(x_0)u_\ell$$

$$= \sum_{i,j=1}^d \partial^2_{ij}\Big((A_{ij}(x_0) - A_{ij}(x))u_\ell\Big) - \sum_{i,j=1}^d \partial_i\Big(u_\ell\partial_j(A_{ij}(x_0) - A_{ij}(x))\Big) + f_\ell + \lambda\big(\Sigma_\ell(x) - \Sigma_\ell(x_0)\big)u_\ell.$$

Let $\chi \in C^\infty(\mathbb{R}^d)$ with the support in a sufficiently small neighborhood of x_0, then with $v_\ell = \chi u_\ell$ for $\ell = 1, 2$, we have

$$
\begin{aligned}
(3.35)\quad & \operatorname{div}(A(x_0)\nabla v_\ell) - \lambda \Sigma_\ell(x_0) v_\ell \\
&= \chi \sum_{i,j=1}^d \partial^2_{ij}\Big((A_{ij}(x_0) - A_{ij}(x))u_\ell\Big) - \sum_{i,j=1}^d \chi\partial_i\Big(u_\ell \partial_j (A_{ij}(x_0) - A_{ij}(x))\Big) \\
&\quad + \chi f_\ell + \lambda\big(\Sigma_\ell(x) - \Sigma_\ell(x_0)\big)v_\ell - u_\ell \operatorname{div}(A(x_0)\nabla\chi) + 2\operatorname{div}(u_\ell A(x_0)\nabla\chi).
\end{aligned}
$$

The terms $r_{\ell,ij}$ are then $(A_{ij}(x_0) - A_{ij}(x))\chi u_\ell = (A_{ij}(x_0) - A_{ij}(x))v_\ell$. Since $A_1 = A_2 = A$ in Ω_τ, $u_1 - u_2 = 0$ in Γ, and $A\nabla(u_1 - u_2)\cdot \nu = 0$ on Γ, it follows that

$$
v_1 - v_2 = 0 \text{ on } \Gamma \quad \text{ and } \quad A(x_0)\nabla(v_1 - v_2)\cdot\nu = \chi(A(x_0) - A(x))\nabla(u_1 - u_2)\cdot \nu \text{ on } \Gamma.
$$

We are thus in the situation to apply Lemma 3.2 and the freezing coefficient technique to derive (3.33).

Concerning (3.34), in (3.35), one writes $\partial^2_{ij}\Big((A_{ij}(x_0) - A_{ij}(x))u_\ell(x)\Big)$ under the form

$$
\partial_i\Big((A_{ij}(x_0) - A_{ij}(x))\partial_j u_\ell\Big) + \partial_i\Big(\partial_j(A_{ij}(x_0) - A_{ij}(x))u_\ell\Big).
$$

We are thus in the situation to apply Lemma 3.2 and the freezing coefficient technique to derive (3.34). The details of the rest of the proof of (3.33) and (3.34) are omitted.

On the other hand, since

$$
\operatorname{div}(A_\ell \nabla u_\ell) - \lambda\Sigma_\ell u_\ell = f_\ell \quad \text{ in } \Omega,
$$

we have, for $|\lambda| \ge 1$,

$$
\|u_\ell\|_{W^{1,p}_\lambda(\Omega\setminus\Omega_{\tau_*/4})} \le C\Big(|\lambda|^{-1/2}\|f_\ell\|_{L^p(\Omega)} + \|u_\ell\|_{L^p(\Omega_{\tau_*})}\Big), \tag{3.36}
$$

and

$$
\|u_\ell\|_{W^{2,p}_\lambda(\Omega\setminus\Omega_{\tau_*/2})} \le C\Big(\|f_\ell\|_{L^p(\Omega)} + \|u_\ell\|_{W^{1,p}_\lambda(\Omega_{\tau_*}\setminus\Omega_{\tau_*/4})}\Big). \tag{3.37}
$$

Combining (3.36) and (3.37) yields

$$
\|u_\ell\|_{W^{2,p}_\lambda(\Omega\setminus\Omega_{\tau_*/2})} \le C\left(\|f_\ell\|_{L^p(\Omega)} + \|u_\ell\|_{L^p(\Omega_{\tau_*})}\right). \tag{3.38}
$$

From (3.33) and (3.38), we obtain

$$
|\lambda|\|(u_1, u_2)\|_{L^p(\Omega)} + \|u_1 - u_2\|_{W^{2,p}_\lambda(\Omega)} \le C\|(f_1, f_2)\|_{L^p(\Omega)}
$$

for $|\lambda| \ge \lambda_0$ and for λ_0 large enough. This completes the proof of (3.3).

From (3.34), (3.36), after using (3.3), we obtain

$$
|\lambda|\|(u_1, u_2)\|_{W^{1,p}_\lambda(\Omega)} + \|u_1 - u_2\|_{W^{3,p}_\lambda(\Omega_{\tau_*})} \le C|\lambda|^{1/2}\|(f_1, f_2)\|_{L^p(\Omega)} + \|f_1 - f_2\|_{W^{1,p}_\lambda(\Omega)}
$$

for $|\lambda| \ge \lambda_0$ and for λ_0 large enough. This completes the proof of (3.4).

Step 2: Set

$$
\begin{aligned}
X = \Big\{&(u_1, u_2) \in [L^p(\Omega)]^2 : \operatorname{div}(A_1\nabla u_1), \operatorname{div}(A_2\nabla u_2) \in L^p(\Omega), \\
& u_1 - u_2 \in W^{2,p}(\Omega), u_1 - u_2 = 0 \text{ on } \Gamma, \text{ and } (A_1\nabla u_1 - A_2\nabla u_2)\cdot\nu = 0 \text{ on } \Gamma\Big\}.
\end{aligned}
$$

The space X is a Banach space endowed with the norm

$$\|(u_1, u_2)\|_X := \|(u_1, u_2)\|_{L^p(\Omega)} + \|\operatorname{div}(A_1 \nabla u_1), \operatorname{div}(A_2 \nabla u_2)\|_{L^p(\Omega)} + \|u_1 - u_2\|_{W^{2,p}(\Omega)}. \tag{3.39}$$

Define

$$B_\lambda : X \to [L^p(\Omega)]^2$$

by

$$B_\lambda(u_1, u_2) = (\operatorname{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1, \operatorname{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2).$$

Clearly, B_λ is bilinear and continuous on X.

We claim that

$$B_\lambda \text{ has a closed and dense range.} \tag{3.40}$$

Assuming this, we derive that

$$B_\lambda(X) = [L^p(\Omega)]^2, \tag{3.41}$$

which yields the existence of the solutions.

It remains to prove (3.40).

We first prove that B_λ has a closed range. Let $((u_{1,n}, u_{2,n}))_n \subset X$ be such that $(f_{1,n}, f_{2,n}) := B_\lambda(u_{1,n}, u_{2,n}) \to (f_1, f_2)$ in $[L^p(\Omega)]^2$. It follows from (3.3) by Step 1 that $((u_{1,n}, u_{2,n}))_n$ is a Cauchy sequence in X. Let (u_1, u_2) denote its limit. One can then show that $(f_{1,n}, f_{2,n}) \to (f_1, f_2) := B_\lambda(u_1, u_2)$ since B_λ is continuous. Thus B_λ has a closed range.

We next establish that the range of B_λ is dense. To this end, it suffices to show that if $(f_1, f_2) \in [L^q(\Omega)]^2$ with $\frac{1}{p} + \frac{1}{q} = 1$ is such that

$$\int_\Omega \langle B_\lambda(u_1, u_2), (f_1, f_2) \rangle dx = 0 \qquad \text{for all } (u_1, u_2) \in X, \tag{3.42}$$

then $(f_1, f_2) = (0, 0)$.

Since (3.42) holds for all $(u_1, u_2) \in [C^\infty_c(\Omega)]^2 \subset X$, it follows that, for $\ell = 1, 2$,

$$\operatorname{div}(A_\ell \nabla f_\ell) - \bar{\lambda} \Sigma_\ell f_\ell = 0 \text{ in } \Omega. \tag{3.43}$$

Since $A_\ell \in C^1(\bar{\Omega})$ and $f_\ell \in L^q(\Omega)$, using the standard regularity theory in L^q-scale, see also [18, Lemma 17.1.5], one has

$$f_\ell \in W^{2,q}_{loc}(\Omega).$$

Set, in Ω,

$$g_1 = f_1 \quad \text{and} \quad g_2 = -f_2. \tag{3.44}$$

Then, by (3.43),

$$\operatorname{div}(A_\ell \nabla g_\ell) - \bar{\lambda} \Sigma_\ell g_\ell = 0 \text{ in } \Omega, \tag{3.45}$$

and, by (3.42), for $(u_1, u_2) \in X$,

$$\int_\Omega \operatorname{div}(A_1 \nabla u_1) \bar{g}_1 - \lambda \Sigma_1 u_1 \bar{g}_1 - \int_\Omega \operatorname{div}(A_2 \nabla u_2) \bar{g}_2 - \lambda \Sigma_2 u_2 \bar{g}_2 = 0. \tag{3.46}$$

From (3.46), we have, taking $(u_1, u_2) \in X \cap [W^{2,p}(\Omega)]^2$,

$$\int_\Omega \operatorname{div}\big(A_1 \nabla (u_1 - u_2)\big) \bar{g}_1 + \operatorname{div}(A_1 \nabla u_2)(\bar{g}_1 - \bar{g}_2) + \operatorname{div}\big((A_1 - A_2) \nabla u_2\big) \bar{g}_2 \\ - \lambda \Sigma_1 u_1 \bar{g}_1 + \lambda \Sigma_2 u_2 \bar{g}_2 = 0. \tag{3.47}$$

Using that $g_2 \in W^{2,q}_{loc}(\Omega)$ and $A_1 = A_2$ in Ω_τ, an integration by parts leads to

$$\int_\Omega \operatorname{div}\big((A_1 - A_2)\nabla u_2\big)\bar g_2 = \int_\Omega \operatorname{div}\big((A_1 - A_2)\nabla \bar g_2\big)u_2. \tag{3.48}$$

Since $u_1 - u_2 \in W^{2,p}(\Omega)$, $u_1 - u_2 = 0$ on Γ and $A\nabla(u_1 - u_2)\cdot \nu = 0$ on Γ, there exists a sequence $(v_n)_n \subset C^2_c(\Omega)$ such that $v_n \to u_1 - u_2$ in $W^{2,p}(\Omega)$. An integration by parts yields

$$\begin{aligned}\int_\Omega \operatorname{div}\big(A_1\nabla(u_1 - u_2)\big)\bar g_1 &= \lim_{n\to+\infty}\int_\Omega \operatorname{div}\big(A_1\nabla v_n\big)\bar g_1 \\ &= \lim_{n\to+\infty}\int_\Omega \operatorname{div}\big(A_1\nabla \bar g_1\big)v_n = \int_\Omega \operatorname{div}\big(A_1\nabla \bar g_1\big)(u_1 - u_2).\end{aligned} \tag{3.49}$$

Combining (3.47), (3.48), and (3.49) yields

$$\begin{aligned}\int_\Omega \operatorname{div}(A_1\nabla u_2)(\bar g_1 - \bar g_2) = &-\int_\Omega \operatorname{div}\big((A_1 - A_2)\nabla \bar g_2\big)u_2 - \int_\Omega \operatorname{div}\big(A_1\nabla \bar g_1\big)(u_1 - u_2) \\ &+ \int_\Omega \lambda\Sigma_1 u_1 \bar g_1 - \int_\Omega \lambda \Sigma_2 u_2 \bar g_2 \overset{(3.45)}{=} \int_\Omega \operatorname{div}(A_1\nabla(\bar g_1 - \bar g_2))u_2.\end{aligned} \tag{3.50}$$

Since u_2 can be chosen arbitrary[4] in $W^{2,p}(\Omega)$, and for every $\xi \in [C^1(\bar\Omega)]^d$ there exists $u_2 \in W^{2,p}(\Omega)$ with $u_2|_\Gamma = 0$ such that $\operatorname{div}(A_1\nabla u_2) = \operatorname{div}\xi$ with $\|u_2\|_{W^{2,p}(\Omega)} \le C\|\xi\|_{L^p(\Omega)}$, it follows that, see, e.g., [5, Proposition 9.18],

$$g_1 - g_2 \in W^{1,q}_0(\Omega).$$

This in turn implies, by (3.50), that $g_1 - g_2 \in W^{2,q}(\Omega)$ and $A\nabla(g_1 - g_2)\cdot\nu = 0$ on Γ. It follows that $g_1 = g_2 = 0$ in Ω after applying Step 1 to (g_1, g_2) and $\bar\lambda$ (instead of λ). Thus $f_1 = f_2 = 0$ by (3.44) and the proof of Step 2 is complete. □

4. The Weyl law for the transmission eigenvalues

Throughout this section, we assume (1.6).

4.1. **The operator T_λ and its adjoint T^*_λ.** We first reformulate the Cauchy problem (3.1) in a form for which we can apply the theory of Hilbert-Schmidt operators. Given $(f, g) \in [L^p(\Omega)]^2$ $(1 < p < +\infty)$, assume that $(u_1, u_2) \in [L^p(\Omega)]^2$ with $u_1 - u_2 \in W^{2,p}(\Omega)$ is a solution of (3.1) with $\lambda \in \mathbb{C}^*$ where, in Ω,

$$f_1 = \Sigma_1 f + \lambda^{-1}\Sigma_2 g \qquad \text{and} \qquad f_2 = \lambda^{-1}\Sigma_2 g.$$

Define, in Ω,

$$u = u_1 - u_2 \qquad \text{and} \qquad v = \lambda u_2.$$

Then the pair $(u, v) \in W^{2,p}(\Omega) \times L^p(\Omega)$ is a solution of

$$\begin{cases} \operatorname{div}(A\nabla u) - \lambda\Sigma_1 u - (\Sigma_1 - \Sigma_2)v &= \Sigma_1 f \quad \text{in } \Omega, \\ \operatorname{div}(A\nabla v) - \lambda\Sigma_2 v &= \Sigma_2 g \quad \text{in } \Omega, \\ u = 0, \ A\nabla u\cdot\nu &= 0 \quad \text{on } \Gamma. \end{cases} \tag{4.1}$$

As a direct consequence of Theorem 3.1 (see also (3.4)), we have

[4]Taking then $u_1 := u_2$ so that $(u_1, u_2) \in X$.

Proposition 4.1. *Assume* (1.1)-(1.2), *and* (1.6)-(1.8). *Let* $c \in (0,1)$ *and* $1 < p < +\infty$. *There exists* $\lambda_0 > 0$ *depending on* p, c, Λ, *and* Ω *such that the following holds: for* $(f,g) \in [L^p(\Omega)]^2$ *and for* $\lambda \in \mathbb{C}$ *with* $|\Im(\lambda)| \geq c|\lambda|$ *and* $|\lambda| > \lambda_0$, *there exists a unique solution* $(u,v) \in W^{2,p}(\Omega) \times L^p(\Omega)$ *of the Cauchy problem* (4.1)*; moreover, we have*

$$\|v\|_{L^p(\Omega)} + \|u\|_{W^{2,p}_\lambda(\Omega)} \leq C|\lambda|^{-1/2} \left(|\lambda|^{1/2} \|f\|_{L^p(\Omega)} + |\lambda|^{-1/2} \|g\|_{L^p(\Omega)} \right) \tag{4.2}$$

and

$$\|v\|_{W^{1,p}_\lambda(\Omega)} + \|u\|_{W^{3,p}_\lambda(\Omega)} \leq C \left(\|f\|_{W^{1,p}_\lambda(\Omega)} + |\lambda|^{-1/2} \|g\|_{L^p(\Omega)} \right), \tag{4.3}$$

for some positive constant C *independent of* λ, f, *and* g.

As a consequence, we have

Corollary 4.1. *Assume* (1.1)-(1.2) *and* (1.6)-(1.8). *Let* $c \in (0,1)$, *and* $1 < p < +\infty$. *There exists* $\lambda_0 > 0$ *depending on* p, c, Λ, *and* Ω *such that the following holds: for* $(f,g) \in W^{1,p}(\Omega) \times L^p(\Omega)$, *and for* $\lambda \in \mathbb{C}$ *with* $|\Im(\lambda)| \geq c|\lambda|$ *and* $|\lambda| > \lambda_0$, *there exists a unique solution* $(u,v) \in W^{3,p}(\Omega) \times W^{1,p}(\Omega)$ *of* (4.1)*; moreover, for*

(1) either $1 < p < d$ *and* $p \leq q \leq \frac{dp}{d-p}$,
(2) either $d = p \leq q < +\infty$,
(3) or $p > d$ *and* $q = +\infty$,

we have

$$\|v\|_{L^q(\Omega)} + \|u\|_{W^{2,q}_\lambda(\Omega)} \leq C|\lambda|^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}} \left(\|f\|_{W^{1,p}_\lambda(\Omega)} + |\lambda|^{-1/2} \|g\|_{L^p(\Omega)} \right), \tag{4.4}$$

for some positive constant C *independent of* λ, f, *and* g.

Remark 4.1. In case (3) of Corollary 4.1, we derive that $(u,v) \in C^2(\overline{\Omega}) \times C(\overline{\Omega})$.

Proof. Choose λ_0 such that the conclusion of Proposition 4.1 holds. By Gagliardo-Nirenberg's interpolation inequalities (see [14, 36]), we have

$$\|v\|_{L^q(\Omega)} \leq C_{p,q,\Omega} \|v\|^{1-a}_{L^p(\Omega)} \|v\|^a_{W^{1,p}(\Omega)} \leq C_{p,q,\Omega} \|v\|^{1-a}_{L^p(\Omega)} \|v\|^a_{W^{1,p}_\lambda(\Omega)},$$

where

$$a = d\left(\frac{1}{p} - \frac{1}{q}\right).$$

This implies

$$\|v\|_{L^q(\Omega)} \leq C_{p,q,\Omega} |\lambda|^{-\frac{1}{2}(1-a)} \left(\|f\|_{W^{1,p}_\lambda(\Omega)} + |\lambda|^{-1/2} \|g\|_{L^p(\Omega)} \right).$$

The other assertions can be proved similarly. □

Definition 4.1. *Assume* (1.1)-(1.2) *and* (1.6)-(1.8). *Let* $1 < p < +\infty$ *and* $\lambda \in \mathbb{C}$. *System* (4.1) *is said to be* well-posed *in* $L^p(\Omega) \times L^p(\Omega)$ *if the existence, the uniqueness, and* (4.2) *and* (4.3) *hold for* $(f,g) \in L^p(\Omega) \times L^p(\Omega)$. *For* $p = 2$ *and* $\lambda \in \mathbb{C}$ *being such that* (4.1) *is well-posed in* $L^2(\Omega) \times L^2(\Omega)$, *we define*

$$\begin{array}{rccc} T_\lambda : & L^2(\Omega) \times L^2(\Omega) & \to & L^2(\Omega) \times L^2(\Omega) \\ & (f,g) & \mapsto & (u,v) \end{array} \tag{4.5}$$

where (u,v) *is the unique solution of* (4.1).

Remark 4.2. Let $\lambda \in \mathbb{C}$ satisfy the conclusion of Proposition 4.1 with $p = 2$. Then system (4.1) is well-posed in $L^2(\Omega) \times L^2(\Omega)$ and T_λ is defined.

Remark 4.3. Let $\lambda^* \in \mathbb{C}$ be such that T^*_λ is defined. If μ is a characteristic value of the operator T_{λ^*} associated with an eigenfunction (u, v) and if $\lambda^* + \mu \neq 0$ we have

$$\lambda^* + \mu \text{ is a transmission eigenvalue of (1.3)} \tag{4.6}$$

with an eigenfunction pair (u_1, u_2) given by

$$u_1 = u + \frac{1}{\lambda^* + \mu} v \quad \text{and} \quad u_2 = \frac{1}{\lambda^* + \mu} v. \tag{4.7}$$

Moreover, the converse holds (see Remark 4.5).

Remark 4.4. Let $\lambda^* \in \mathbb{C}$ be such that T^*_λ is defined. By (4.2) and (4.3) the range of $T^2_{\lambda^*}$ is a subset of $H^1(\Omega) \times H^1(\Omega)$. It follows that the operator $T^2_{\lambda^*}$ is compact from $L^2(\Omega) \times L^2(\Omega)$ into itself. By the spectral theory of compact operators, see, e.g., [5], the spectrum of $T^2_{\lambda^*}$ consists in a discrete set of eigenvalues and the generalized eigenspace associated to each eigenvalue is of finite dimension. As a consequence, the set of eigenvalues of T_{λ^*} is discrete. This in turn implies that the set of the transmission eigenvalues of (1.3) is discrete. This fact is previously established in [33] but the arguments presented here are different.

Remark 4.5. Let $\lambda^* \in \mathbb{C}$ be such that T^*_λ is defined. If λ_j is an eigenvalue of the transmission eigenvalue problem, then $\lambda_j \neq \lambda^*$ and $\lambda_j - \lambda^*$ is a characteristic value of T_{λ^*}. One can show that the multiplicity of the characteristic values of $\lambda_j - \lambda^*$ and $\lambda_j - \hat{\lambda}$ associated with T_{λ^*} and $T_{\hat{\lambda}}$ are the same. Hence the multiplicity of the eigenvalues associated with T_{λ^*} is independent of λ^*. With this observation we define the multiplicity of λ_j as the one of the characteristic value $\lambda_j - \lambda^*$ of T_{λ^*}.

The rest of this section is devoted to characterize the adjoint T^*_λ of T_λ. This will be used in the proof of Proposition 4.2. To this end, for $(\widetilde{f}, \widetilde{g}) \in [L^p(\Omega)]^2$ with $1 < p < +\infty$, we consider the system, for $(\widetilde{u}, \widetilde{v}) \in W^{1,p}(\Omega) \times L^p(\Omega)$ [5]

$$\begin{cases} \operatorname{div}(A\nabla \widetilde{u}) - \lambda \Sigma_2 \widetilde{u} - (\Sigma_1 - \Sigma_2)\widetilde{v} = \Sigma_2 \widetilde{f} & \text{in } \Omega, \\ \operatorname{div}(A\nabla \widetilde{v}) - \lambda \Sigma_1 \widetilde{v} = \Sigma_1 \widetilde{g} & \text{in } \Omega, \\ \widetilde{u} = 0, \quad A\nabla \widetilde{u} \cdot \nu = 0 & \text{on } \Gamma. \end{cases} \tag{4.8}$$

Assume (1.1)-(1.2), and (1.6)-(1.8). Let $c \in (0, 1)$ and $1 < p < +\infty$. By Proposition 4.1, there exists $\lambda_0 > 0$ depending on p, c, Λ, and Ω such that (4.8) is well-posed in $L^p(\Omega) \times L^p(\Omega)$ for $\lambda \in \mathbb{C}$ with $|\Im(\lambda)| \geq c|\lambda|$ and $|\lambda| > \lambda_0$, i.e., for $(f, g) \in L^p(\Omega) \times L^p(\Omega)$, there exists a unique solution $(\widetilde{u}, \widetilde{v}) \in W^{1,p}(\Omega) \times L^p(\Omega)$ of (4.8); moreover,

$$\|\widetilde{u}\|_{W^{2,p}_\lambda(\Omega)} + \|\widetilde{v}\|_{L^p(\Omega)} \leq C|\lambda|^{-1/2} \left(|\lambda|^{1/2} \|\widetilde{f}\|_{L^p(\Omega)} + |\lambda|^{-1/2} \|\widetilde{g}\|_{L^p(\Omega)} \right)$$

and

$$\|\widetilde{u}\|_{W^{3,p}_\lambda(\Omega)} + \|\widetilde{v}\|_{W^{1,p}_\lambda(\Omega)} \leq C \left(\|\widetilde{f}\|_{W^{1,p}_\lambda(\Omega)} + |\lambda|^{-1/2} \|\widetilde{g}\|_{L^p(\Omega)} \right).$$

Definition 4.2. *Assume* (1.1)-(1.2) *and* (1.6)-(1.8). *For* $p = 2$ *and for* $\lambda \in \mathbb{C}$ *being such that* (4.8) *is well-posed in* $L^2(\Omega) \times L^2(\Omega)$, *we define*

$$\begin{array}{cccc} \widetilde{T}_\lambda : & L^2(\Omega) \times L^2(\Omega) & \to & L^2(\Omega) \times L^2(\Omega) \\ & (\widetilde{f}, \widetilde{g}) & \mapsto & (\widetilde{u}, \widetilde{v}) \end{array} \tag{4.9}$$

where $(\widetilde{u}, \widetilde{v})$ *is the unique solution of* (4.8).

[5] We emphasize here that in the first equation of (4.8), we have $\Sigma_2 \widetilde{u}$ not $\Sigma_1 \widetilde{u}$, compare with (4.1).

Lemma 4.1. *Assume* (1.1)-(1.2) *and* (1.6)-(1.8). *Let* $p = 2$ *and let* λ *be such that* T_λ *and* $\widetilde{T}_{\bar\lambda}$ *are defined. Set, for* $x \in \Omega$,

$$P(x) = \begin{pmatrix} 0 & \Sigma_1(x) \\ \Sigma_2(x) & 0 \end{pmatrix}. \tag{4.10}$$

We have

$$T_\lambda^* = P\widetilde{T}_{\bar\lambda}P^{-1}. \tag{4.11}$$

Proof. Fix $(f, g) \in [L^2(\Omega)]^2$ and $(f^*, g^*) \in [L^2(\Omega)]^2$. Set $(u, v) = T_\lambda(f, g)$ and $(u^*, v^*) = \widetilde{T}_{\bar\lambda}P^{-1}(f^*, g^*)$. Then

$$\int_\Omega \langle (f, g), P\widetilde{T}_{\bar\lambda}P^{-1}(f^*, g^*)\rangle = \int_\Omega \Sigma_1 f\overline{v^*} + \Sigma_2 g\overline{u^*}. \tag{4.12}$$

Since $(u, v) = T_\lambda(f, g)$, we have

$$\int_\Omega \Sigma_1 f\overline{v^*} + \Sigma_2 g\overline{u^*} = \int_\Omega (\operatorname{div}(A\nabla u) - \lambda\Sigma_1 u - (\Sigma_1 - \Sigma_2)v)\overline{v^*} + \int_\Omega (\operatorname{div}(A\nabla v) - \lambda\Sigma_2 v)\overline{u^*}. \tag{4.13}$$

As in Step 2 of the proof of Theorem 3.1, an integration by parts yields

$$\begin{aligned} &\int_\Omega (\operatorname{div}(A\nabla u) - \lambda\Sigma_1 u - (\Sigma_1 - \Sigma_2)v)\overline{v^*} + \int_\Omega (\operatorname{div}(A\nabla v) - \lambda\Sigma_2 v)\overline{u^*} \\ &\qquad = \int_\Omega u\overline{(\operatorname{div}(A\nabla v^*) - \bar\lambda\Sigma_1 v^*)} + \int_\Omega v\overline{(\operatorname{div}(A\nabla u^*) - \bar\lambda\Sigma_2 u^* - (\Sigma_1 - \Sigma_2)v^*)}. \end{aligned} \tag{4.14}$$

Since $(u^*, v^*) = \widetilde{T}_{\bar\lambda}P^{-1}(f^*, g^*)$, we have

$$\begin{aligned} &\int_\Omega u\overline{(\operatorname{div}(A\nabla v^*) - \bar\lambda\Sigma_1 v^*)} + \int_\Omega v\overline{(\operatorname{div}(A\nabla u^*) - \bar\lambda\Sigma_2 u^* - (\Sigma_1 - \Sigma_2)v^*)} \\ &\qquad = \int_\Omega \langle T_\lambda(f, g), (f^*, g^*)\rangle. \end{aligned} \tag{4.15}$$

Combining (4.12)-(4.15) yields

$$\int_\Omega \langle (f, g), P\widetilde{T}_{\bar\lambda}P^{-1}(f^*, g^*)\rangle = \int_\Omega \langle T_\lambda(f, g), (f^*, g^*)\rangle, \tag{4.16}$$

and the conclusion follows. □

4.2. **Hilbert-Schmidt operators.** In this section, we recall the definition and several properties of Hilbert-Schmidt operators. We begin with

Definition 4.3. *Let* H *be a separable Hilbert space and let* $(\phi_j)_{j=1}^\infty$ *be an orthonormal basis of* H.

(1) Let $\mathcal{T}$ *be a linear and bounded operator on* H. *We say that* $\mathcal{T}$ *is* Hilbert-Schmidt *if its* double norm *is finite, i.e.*

$$|||\mathcal{T}||| := \left(\sum_{j=1}^\infty \|\mathcal{T}\phi_j\|_H^2\right)^{1/2} < +\infty.$$

(2) *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two Hilbert-Schmidt operators on H. The* trace *of the composition $\mathcal{T}_1\mathcal{T}_2$ is defined by*

$$\operatorname{trace}(\mathcal{T}_1\mathcal{T}_2) := \sum_{j=1}^{\infty}(\mathcal{T}_1\mathcal{T}_2\phi_j, \phi_j)_H.$$

Remark 4.6. One can check that Definition 4.3 does not depend on the choice of the basis $(\phi_j)_{j=1}^{\infty}$ and the trace of $\mathcal{T}_1\mathcal{T}_2$ is well defined as an absolutely convergent series (see [1, Theorems 12.9 and 12.12]).

Let $m \in \mathbb{N}$ and $\mathbf{T} : [L^2(\Omega)]^m \to [L^2(\Omega)]^m$ be a Hilbert-Schmidt operator. There exists a unique kernel $\mathbf{K} \in [L^2(\Omega \times \Omega)]^{m\times m}$, see e.g. [1, Theorems 12.18 and 12.19], such that

$$(\mathbf{T}u)(x) = \int_\Omega \mathbf{K}(x, y)u(y)dy \quad \text{for a.e. } x \in \Omega, \text{ for all } u \in [L^2(\Omega)]^m. \tag{4.17}$$

Moreover,

$$|||\mathbf{T}|||^2 = \iint\limits_{\Omega\times\Omega} |\mathbf{K}(x, y)|^2\, dx\, dy. \tag{4.18}$$

Note that [1, Theorems 2.18 and 12.19] state for $m = 1$, nevertheless, the same arguments hold for $m \in \mathbb{N}$ as noted in [34].

We have, see [1] (see also [34, Lemma 4]):

Lemma 4.1. *Let $m \in \mathbb{N}$ and let $\mathbf{T}_1, \mathbf{T}_2$ be two Hilbert-Schmidt operators in $[L^2(\Omega)]^m$ with the corresponding kernels $\mathbf{K}_1$ and $\mathbf{K}_2$. Then $\mathbf{T} := \mathbf{T}_1\mathbf{T}_2$ is a Hilbert-Schmidt operator with the kernel $\mathbf{K}$ given by*

$$\mathbf{K}(x, y) = \int_\Omega \mathbf{K}_1(x, z)\mathbf{K}_2(z, y)\, dz. \tag{4.19}$$

Moreover,

$$\operatorname{trace}(\mathbf{T}_1\mathbf{T}_2) = \int_\Omega \operatorname{trace}(\mathbf{K}(x, x))dx. \tag{4.20}$$

We have, see, e.g., [34, Lemma 3].

Lemma 4.2. *Let $d \geq 2$, $m \in \mathbb{N}$, and $\mathbf{T} : [L^2(\Omega)]^m \to [L^2(\Omega)]^m$ be such that $\mathbf{T}(\phi) \in [C(\bar{\Omega})]^m$ for $\varphi \in [L^2(\Omega)]^m$, and*

$$\|\mathbf{T}(\phi)\|_{L^\infty(\Omega)} \leq M\|\phi\|_{L^2(\Omega)}, \tag{4.21}$$

for some $M \geq 0$. Then $\mathbf{T}$ is a Hilbert-Schmidt operator,

$$|||\mathbf{T}||| \leq C_m|\Omega|^{1/2}M, \tag{4.22}$$

and the kernel $\mathbf{K}$ of $\mathbf{T}$ satisfies

$$\sup_{x\in\Omega}\left(\int_\Omega |\mathbf{K}(x, y)|^2 dy\right)^{1/2} \leq C_m|\Omega|^{1/2}M. \tag{4.23}$$

Assume in addition that

$$\|\mathbf{T}(\phi)\|_{L^\infty(\Omega)} \leq \widetilde{M}||\phi||_{L^1(\Omega)} \text{ for } \phi \in [L^2(\Omega)]^m, \tag{4.24}$$

for some $\widetilde{M} \ge 0$, *then the kernel* $\mathbf{K}$ *of* $\mathbf{T}$ *satisfies*

$$|\mathbf{K}(x,y)| \le \widetilde{M} \quad \textit{for a.e. } x, y \in \Omega. \tag{4.25}$$

Here C_m *denotes a positive constant depending only on* m.

As a consequence of Lemma 4.2, we derive the following result.

Corollary 4.2. *Let* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *two Hilbert-Schmidt operators on* $[L^2(\Omega)]^m$ *be such that the ranges of* $\mathbf{T}_1$ *and* $\mathbf{T}_2^*$ *are in* $[C(\bar{\Omega})]^m$ *and* (4.21) *holds for* $\mathbf{T}_1$ *and* $\mathbf{T}_2^*$. *Assume that* (4.24) *holds for* $\mathbf{T} = \mathbf{T}_1\mathbf{T}_2$. *Then the kernel* $\mathbf{K}$ *of* $\mathbf{T}$ *is continuous on* $\bar{\Omega} \times \bar{\Omega}$ *and* (4.25) *holds for every* $(x,y) \in \Omega \times \Omega$.

Proof. Let $\mathbf{K}_1$ (resp. $\mathbf{K}_2$) be the kernel of $\mathbf{T}_1$ (resp. $\mathbf{T}_2$) and let $\mathbf{K}_2^*$ be the kernel of $\mathbf{T}_2^*$. We claim that for $\varepsilon > 0$, there exists $\delta > 0$ such that for every $(x, x') \in \Omega \times \Omega$ with $|x - x'| < \delta$ we have

$$\left(\int_\Omega |\mathbf{K}_1(x,z) - \mathbf{K}_1(x',z)|^2 dz \right)^{1/2} \le \varepsilon \quad \text{and} \quad \left(\int_\Omega |\mathbf{K}_2^*(x,z) - \mathbf{K}_2^*(x',z)|^2 dz \right)^{1/2} \le \varepsilon. \tag{4.26}$$

Admitting (4.26), we continue the proof. We have, see, e.g., [1, Theorem 12.20],

$$\mathbf{K}_2(z,y) = \overline{\mathbf{K}_2^*(y,z)}. \tag{4.27}$$

Since

$$\mathbf{K}(x,y) \overset{(4.19),(4.27)}{=} \int_\Omega \mathbf{K}_1(x,z)\overline{\mathbf{K}_2^*(z,y)} dz,$$

it follows from (4.26) that $\mathbf{K}$ is continuous in $\bar{\Omega} \times \bar{\Omega}$. This in turn implies (4.25) by Lemma 4.2 applied to $\mathbf{T}$.

It remains to prove (4.26). We have

$$\left(\int_\Omega |\mathbf{K}_1(x,z) - \mathbf{K}_1(x',z)|^2 dz \right)^{1/2} \le C \sup_{\varphi \in [L^2(\Omega)]^m; \|\varphi\|_{L^2(\Omega)} \le 1} \left| \int_\Omega (\mathbf{K}_1(x,z) - \mathbf{K}_1(x',z)) \varphi(z) dz \right|.$$

Given $\varepsilon > 0$, let $\varphi_\varepsilon \in [L^2(\Omega)]^m$ with $\|\varphi_\varepsilon\|_{L^2(\Omega)} \le 1$ be such that

$$\left(\int_\Omega |\mathbf{K}_1(x,z) - \mathbf{K}_1(x',z)|^2 dz \right)^{1/2} \le \left| \int_\Omega (\mathbf{K}_1(x,z) - \mathbf{K}_1(x',z)) \varphi_\varepsilon(z) dz \right| + \frac{\varepsilon}{2}.$$

This yields

$$\left(\int_\Omega |\mathbf{K}_1(x,z) - \mathbf{K}_1(x',z)|^2 dz \right)^{1/2} \le |(\mathbf{T}_1\varphi_\varepsilon)(x) - (\mathbf{T}_1\varphi_\varepsilon)(x')| + \frac{\varepsilon}{2}. \tag{4.28}$$

The first inequality of (4.26) now follows from (4.28) and the fact that $\mathbf{T}_1\varphi_\varepsilon \in [C(\bar{\Omega})]^m$.

Similarly, we obtain the second inequality of (4.26). □

4.3. **The operators $\mathbf{T}_{\theta,t}$ and their properties.** Denote

$$k = \left[\frac{d}{2}\right] + 1, \tag{4.29}$$

the smallest integer greater than $d/2$. Fix

$$2 = p_1 < p_2 < \cdots < p_k < +\infty \tag{4.30}$$

such that

$$p_{j-1} < p_j < \frac{d p_{j-1}}{d - p_{j-1}} \quad \text{and} \quad p_k > d. \tag{4.31}$$

Denote

$$\lambda^* = t^* e^{i\frac{\pi}{2}}, \tag{4.32}$$

for some large $t^* > 0$ such that, for $t \ge t^*$, (4.1) with $\lambda = te^{i\frac{\pi}{2}}$ is well-posed in $L^p(\Omega) \times L^p(\Omega)$ and (4.8) with $\lambda = te^{-i\frac{\pi}{2}}$ is well-posed in $L^p(\Omega) \times L^p(\Omega)$ with $p = p_1, \cdots, p_k$.

Let

$$\omega_j \in \mathbb{C} \text{ with } 1 \le j \le k+1 \text{ be the (distinct) } (k+1)\text{-th roots of } 1 \text{ (thus } \omega_j^{k+1} = 1) \tag{4.33}$$

and let

$$\Theta = \mathbb{R} \setminus \left\{ \frac{\pi}{k+1} \mathbb{Z} \right\}. \tag{4.34}$$

Definition 4.4. *For $\theta \in \Theta$, $1 \le j \le k+1$, and $t > 0$, we define*

$$\lambda_{j,\theta,t} = \lambda^* + \omega_j t e^{i\theta}, \tag{4.35}$$

and

$$t_\theta > t^*$$

such that the following properties hold, for $t \ge t_\theta$,

$$\text{(4.1) with } \lambda = \lambda_{j,\theta,t} \text{ is well-posed in } L^p(\Omega) \times L^p(\Omega) \\ \text{and (4.8) with } \lambda = \bar{\lambda}_{j,\theta,t} \text{ is well-posed in } L^p(\Omega) \times L^p(\Omega) \text{ with } p = p_1, \cdots, p_k, \tag{4.36}$$

and

$$\frac{t}{2} \le |\lambda_{j,\theta,t}| < 2t. \tag{4.37}$$

Such a $t_\theta > t^*$ exists by Proposition 4.1 after noting that, for $\theta \in \Theta$,

$$\Im \left(\omega_j e^{i\theta} \right) \neq 0,$$

and, for $1 \le j \le k+1$,

$$\lim_{t \to +\infty} \frac{\left| \Im \left(\lambda^* + t\omega_j e^{i\theta} \right) \right|}{|\lambda^* + t\omega_j e^{i\theta}|} = \left| \Im \left(\omega_j e^{i\theta} \right) \right| > 0.$$

Viewing (4.2) and (4.3), it is convenient to modify $T_{\lambda_{j,\theta,t}}$ to capture the scaling with respect to $t \sim \lambda_{j,\theta,t}$ there, as in [39]. Denote

$$M_t = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}. \tag{4.38}$$

Let $\theta \in \Theta$ and $t \ge t_\theta$. Define, for $1 \le j \le k+1$,

$$T_{j,\theta,t} = M_t T_{\lambda_{j,\theta,t}} M_t^{-1} \quad \text{ and } \quad \mathbf{T}_{\theta,t} = T_{k+1,\theta,t} \circ T_{k,\theta,t} \circ \cdots \circ T_{1,\theta,t}. \tag{4.39}$$

Here is the main result of this section.

Proposition 4.2. *Let $\theta \in \Theta$ and let t_θ be given in Definition 4.4. Then, for $t \ge t_\theta$,*

$$\|\mathbf{T}_{\theta,t}\|_{L^2(\Omega) \to L^2(\Omega)} \le C t^{-k-1}, \tag{4.40}$$

the range of $\mathbf{T}_{\theta,t}$ is in $[C(\bar{\Omega})]^2$,

$$\|\mathbf{T}_{\theta,t}\|_{L^2(\Omega) \to L^\infty(\Omega)} \le C t^{-k-1+\frac{d}{4}}, \tag{4.41}$$

and

$$\|\mathbf{T}_{\theta,t}\|_{L^1(\Omega) \to L^2(\Omega)} \le C t^{-k-1+\frac{d}{4}} \tag{4.42}$$

for some positive constant C independent of t. Similar facts hold for $\mathbf{T}^*_{\theta,t}$.

As a direct consequence of Lemma 4.2 and Proposition 4.2, we obtain

Corollary 4.3. *Let $\theta \in \Theta$ and let t_θ be given in Definition 4.4. Then, for $t \ge t_\theta$, the operator $\mathbf{T}_{\theta,t}$ is Hilbert-Schmidt, and*

$$\|\!|\mathbf{T}_{\theta,t}\|\!| \le C t^{-k-1+\frac{d}{4}}, \tag{4.43}$$

for some positive constant C independent of t.

We now give

Proof of Proposition 4.2. We first deal with (4.41). By using (4.37), we derive that

$$\|T_{j,\theta,t}\|_{L^2(\Omega)\to L^2(\Omega)} \overset{\text{Proposition 4.1}}{\le} Ct^{-1}$$

and hence

$$\|\mathbf{T}_{\theta,t}\|_{L^2(\Omega)\to L^2(\Omega)} \le \prod_{j=1}^{k+1} \|T_{j,\theta,t}\|_{L^2(\Omega)\to L^2(\Omega)} \le Ct^{-k-1}.$$

This establishes (4.40).

Next we deal with (4.41). For $j = 1, \cdots, k+1$ and $(f, g) \in [L^2(\Omega)]^2$, we write

$$(u^{(j)}, v^{(j)}) = T_{j,\theta,t} \circ T_{j-1,\theta,t} \circ \cdots \circ T_{1,\theta,t}(f, g). \tag{4.44}$$

By (4.37), we have

$$t^{-1/2}\|u^{(1)}\|_{W^{1,2}_t(\Omega)} + \|v^{(1)}\|_{L^2(\Omega)} \overset{\text{Proposition 4.1}}{\le} Ct^{-1}\|(f, g)\|_{L^2(\Omega)\times L^2(\Omega)}, \tag{4.45}$$

and, for $2 \le j \le k$,

$$\begin{aligned} &t^{-1/2}\|u^{(j)}\|_{W^{1,p_j}_t(\Omega)} + \|v^{(j)}\|_{L^{p_j}(\Omega)} \\ &\qquad \overset{\text{Corollary 4.1}}{\le} Ct^{-1+\frac{d}{2}\left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right)}\Big(t^{-1/2}\|u^{(j-1)}\|_{W^{1,p_{j-1}}_t(\Omega)} + \|v^{(j-1)}\|_{L^{p_{j-1}}(\Omega)}\Big), \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} &t^{-1/2}\|u^{(k+1)}\|_{W^{1,\infty}_t(\Omega)} + \|v^{(k+1)}\|_{L^\infty(\Omega)} \\ &\qquad \overset{\text{Corollary 4.1}}{\le} Ct^{-1+\frac{d}{2p_k}}\Big(t^{-1/2}\|u^{(k)}\|_{W^{1,p_k}_t(\Omega)} + \|v^{(k)}\|_{L^{p_k}(\Omega)}\Big). \end{aligned} \tag{4.47}$$

We derive from (4.45), (4.46) and (4.47),

$$\begin{aligned} &\|u^{(k+1)}\|_{L^\infty(\Omega)} + \|v^{(k+1)}\|_{L^\infty(\Omega)} \\ &\le Ct^{-1}t^{-1+\frac{d}{2p_k}} \prod_{j=2}^{k} t^{-1+\frac{d}{2}\left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right)} \|(f, g)\|_{L^2(\Omega)\times L^2(\Omega)} = Ct^{-k-1+\frac{d}{4}}\|(f, g)\|_{L^2(\Omega)\times L^2(\Omega)}. \end{aligned} \tag{4.48}$$

Thus (4.41) is proved.

We next establish (4.42). We have, by Lemma 4.1,

$$T^*_{j,\theta,t} = M_t^{-1} P \widetilde{T}_{\overline{\lambda}_{j,\theta,t}} P^{-1} M_t,$$

where P is given by (4.10). This implies

$$\mathbf{T}^*_{\theta,t} = M_t^{-1} P \widetilde{T}_{\overline{\lambda}_{1,\theta,t}} \circ \cdots \circ \widetilde{T}_{\overline{\lambda}_{k+1,\theta,t}} P^{-1} M_t.$$

Similarly to (4.48), we have

$$\|\mathbf{T}^*_{\theta,t}\|_{L^2(\Omega)\to L^\infty(\Omega)} \le C t^{-k-1+\frac{d}{4}}. \tag{4.49}$$

By a standard dual argument, we derive from (4.49) that

$$\|\mathbf{T}_{\theta,t}\|_{L^1(\Omega)\to L^2(\Omega)} \le C t^{-k-1+\frac{d}{4}}.$$

The properties for $\mathbf{T}_{\theta,t}$ are established.

The properties for $\mathbf{T}^*_{\theta,t}$ can be derived similarly. □

4.4. **The approximation of the trace of a kernel.** Denote

$$\alpha = \frac{\pi}{4(k+1)} \quad \text{and} \quad \beta = \frac{5\pi}{4(k+1)}. \tag{4.50}$$

Then

$$\alpha, \beta \in \Theta \quad \text{and} \quad e^{i\alpha(k+1)} + e^{i\beta(k+1)} = 0. \tag{4.51}$$

Recall that Θ is defined in (4.34).

Lemma 4.2. *For $t \ge \max\{t_\alpha, t_\beta\}$, where t_α and t_β are given in Definition 4.4, we have*

(1) the operator $\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}$ is Hilbert-Schmidt, and

$$|||\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}||| \le C t^{-2k-2+\frac{d}{2}}; \tag{4.52}$$

(2) the range of $\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}$ is in $[C(\bar\Omega)]^2$, and

$$\|\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}\|_{L^1(\Omega)\to L^\infty(\Omega)} \le C t^{-2k-2+\frac{d}{2}}; \tag{4.53}$$

(3) the kernel $\mathbf{K}_t$ of $\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}$ is continuous in $\Omega\times\Omega$, and

$$|\mathbf{K}_t(x,y)| \le C t^{-2k-2+\frac{d}{2}} \quad \text{for all } (x,y)\in\Omega\times\Omega; \tag{4.54}$$

for some positive constant C independent of t.

Proof. Assertion (4.52) follows from Corollary 4.3 and

$$|||\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}||| \le |||\mathbf{T}_{\alpha,t}|||\,|||\mathbf{T}_{\beta,t}|||.$$

Applying Proposition 4.2 and using the fact

$$\|\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}\|_{L^1(\Omega)\to L^\infty(\Omega)} \le \|\mathbf{T}_{\alpha,t}\|_{L^2(\Omega)\to L^\infty(\Omega)} \|\mathbf{T}_{\beta,t}\|_{L^1(\Omega)\to L^2(\Omega)},$$

we obtain (4.53).

Since both the range of $\mathbf{T}_{\alpha,t}$ and $\mathbf{T}^*_{\beta,t}$ are contained in $[C(\bar\Omega)]^2$, the continuity of $\mathbf{K}_t$ and (4.54) follow from Corollary 4.2 and (4.53). □

For $\ell = 1, 2$, $\theta \in \Theta$, and $t > 1$, consider, with $\lambda = te^{i\theta}$,

$$\begin{array}{rccc} S_{\ell,\lambda,x_0}: & L^2(\mathbb{R}^d) & \to & L^2(\mathbb{R}^d) \\ & f_\ell & \mapsto & v_\ell \end{array} \tag{4.55}$$

where $v_\ell \in H^1(\mathbb{R}^d)$ is the unique solution of

$$\operatorname{div}(A(x_0)\nabla v_\ell) - \lambda \Sigma_\ell(x_0) v_\ell = \Sigma_\ell(x_0) f_\ell \text{ in } \mathbb{R}^d. \tag{4.56}$$

One then has

$$S_{\ell,\lambda,x_0} f(x) = \int_{\mathbb{R}^d} F_{\ell,\lambda}(x_0, x-y) f(y) dy, \tag{4.57}$$

where

$$F_{\ell,\lambda}(x_0, z) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{iz\cdot\xi}}{\Sigma_\ell(x_0)^{-1} A(x_0)\xi\cdot\xi + \lambda} d\xi. \tag{4.58}$$

Set, for $\ell = 1, 2$,

$$\mathcal{S}_{\ell,t,x_0} = S_{\ell,\lambda_{k+1,\alpha,t},x_0} \circ \cdots \circ S_{\ell,\lambda_{1,\alpha,t},x_0} \circ S_{\ell,\lambda_{k+1,\beta,t},x_0} \circ \cdots \circ S_{\ell,\lambda_{1,\beta,t},x_0}.$$

Define, for $\ell = 1, 2$,

$$\mathcal{F}_{\ell,t}(x_0, z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{iz\xi}\, d\xi}{\prod_{j=1}^{k+1} \left(\Sigma_\ell(x_0)^{-1} A(x_0)\xi\cdot\xi + \lambda_{j,\alpha,t}\right) \left(\Sigma_\ell(x_0)^{-1} A(x_0)\xi\cdot\xi + \lambda_{j,\beta,t}\right)}. \tag{4.59}$$

Then

$$\mathcal{S}_{\ell,t,x_0} f_\ell(x) = \int_{\mathbb{R}^d} \mathcal{F}_{\ell,t}(x_0, x-y) f_\ell(y) dy. \tag{4.60}$$

Since $2k+2 > d$, the integrand appearing in (4.59) belongs to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and thus

$$z \mapsto \mathcal{F}_{\ell,t}(x_0, z) \text{ is continuous and belongs to } L^2(\mathbb{R}^d). \tag{4.61}$$

To introduce the freezing coefficient version of (4.1) in the whole space, we use the following result in which (4.62) is the system of $(u, v) := (v_1 - v_2, \lambda v_1)$, where v_ℓ ($\ell = 1, 2$) is defined by (4.56).

Lemma 4.3. *Let $x_0 \in \Omega$, $c \in (0, 1)$, $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ and $|\Im(\lambda)| \geq c|\lambda|$. Let $p > 1$ and let $(f, g) \in [L^p(\mathbb{R}^d)]^2$. Then there exists a unique solution $(u, v) \in [W^{1,p}(\mathbb{R}^d)]^2$ of*

$$\begin{cases} \operatorname{div}(A(x_0)\nabla u) - \lambda\Sigma_1(x_0)u - (\Sigma_1(x_0) - \Sigma_2(x_0))v &= \Sigma_1(x_0)f \quad \text{in } \mathbb{R}^d, \\ \operatorname{div}(A(x_0)\nabla v) - \lambda\Sigma_2(x_0)v &= \Sigma_2(x_0)g \quad \text{in } \mathbb{R}^d. \end{cases} \tag{4.62}$$

Moreover,

$$\|u\|_{W^{2,p}_\lambda(\mathbb{R}^d)} + |\lambda|^{-1}\|v\|_{W^{2,p}_\lambda(\mathbb{R}^d)} \leq C\left(\|f\|_{L^p(\mathbb{R}^d)} + |\lambda|^{-1}\|g\|_{L^p(\mathbb{R}^d)}\right), \tag{4.63}$$

for some $C > 0$ depending only on Λ,c and p. As a consequence,

(1) either $1 < p < d$ and $p \leq q \leq \frac{dp}{d-p}$,
(2) either $d = p \leq q < +\infty$,
(3) or $p > d$ and $q = +\infty$,

we have

$$\|u\|_{W^{1,q}_\lambda(\mathbb{R}^d)} + |\lambda|^{-1}\|v\|_{W^{1,q}_\lambda(\mathbb{R}^d)} \leq C|\lambda|^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\left(\|f\|_{L^p(\mathbb{R}^d)} + |\lambda|^{-1}\|g\|_{L^p(\mathbb{R}^d)}\right).$$

Proof. We emphasize here that (4.62) is a system with constant coefficients imposed in $\mathbb{R}^d$. The proof is quite standard. The idea is first to obtain the existence, uniqueness, and the estimate for v using the second equation of (4.62), and then using these to derive the ones for u using the first equation of (4.62). The details are omitted. $\square$

For $x_0 \in \Omega$, $j = 1, \cdots, k+1$, $\theta \in \Theta$, and $t > 1$, define, for $1 < p < +\infty$,

$$\begin{array}{rccc} R_{\lambda_{j,\theta,t},x_0}: & [L^p(\mathbb{R}^d)]^2 & \to & [L^p(\mathbb{R}^d)]^2 \\ & (f,g) & \mapsto & (u,v) \end{array}$$

where $(u, v) \in [W^{1,p}(\mathbb{R}^d)]^2$ is the unique solution (4.62) with $\lambda = \lambda_{j,\theta,t}$. Recall that $\lambda_{j,\theta,t}$ is defined in (4.35). We also introduce

$$R_{j,\theta,t,x_0} = M_t R_{\lambda_{j,\theta,t},x_0} M_t^{-1} \quad \text{and} \quad \mathbf{R}_{\theta,t,x_0} = R_{k+1,\theta,t,x_0} \circ \cdots \circ R_{1,\theta,t,x_0}. \tag{4.64}$$

As in the proof of Proposition 4.2, however, using Lemma 4.3 instead of Proposition 4.1 and Corollary 4.1, we obtain

Lemma 4.4. *Let $\theta \in \Theta$ and $t > 1$. Then, the range of $\mathbf{R}_{\theta,t,x_0}$ and $\mathbf{R}^*_{\theta,t,x_0}$ are in $[C(\mathbb{R}^d)]^2$ for all $t > 1$. Moreover,*

$$\|\mathbf{R}_{\theta,t,x_0}\|_{L^2(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} + \|\mathbf{R}_{\theta,t,x_0}\|_{L^1(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le Ct^{-k-1+\frac{d}{4}} \tag{4.65}$$

and

$$\|\mathbf{R}^*_{\theta,t,x_0}\|_{L^2(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} + \|\mathbf{R}^*_{\theta,t,x_0}\|_{L^1(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le Ct^{-k-1+\frac{d}{4}}, \tag{4.66}$$

for some positive constant C independent of t.

Define

$$\mathbf{R}_{t,x_0} = \mathbf{R}_{\alpha,t,x_0}\mathbf{R}_{\beta,t,x_0}. \tag{4.67}$$

One can then write $\mathbf{R}_{t,x_0}$ under the form

$$\mathbf{R}_{t,x_0} = \begin{pmatrix} (\mathbf{R}_{t,x_0})_{11} & (\mathbf{R}_{t,x_0})_{12} \\ (\mathbf{R}_{t,x_0})_{21} & (\mathbf{R}_{t,x_0})_{22} \end{pmatrix}.$$

Note that, by the definition of S_{ℓ,λ,x_0},

$$R_{\lambda_{j,\theta,t},x_0} = \begin{pmatrix} S_{1,\lambda_{j,\theta,t},x_0} & \Sigma_1(x_0)^{-1}(\Sigma_1(x_0) - \Sigma_2(x_0))S_{1,\lambda_{j,\theta,t},x_0}S_{2,\lambda_{j,\theta,t},x_0} \\ 0 & S_{2,\lambda_{j,\theta,t},x_0} \end{pmatrix}.$$

It follows that $R_{\lambda_{j,\theta,t},x_0}$ is an upper triangular matrix operator, and so is $\mathbf{R}_{t,x_0}$. We deduce that

$$(\mathbf{R}_{t,x_0})_{21} = 0$$

and, for $\ell = 1, 2$,

$$(\mathbf{R}_{t,x_0})_{\ell\ell} = \mathcal{S}_{\ell,t,x_0}.$$

These simple observations are useful in computing the approximation of the trace of the kernel of $\mathbf{R}_{t,x_0}$.

As an immediate consequence of (4.60), $\mathbf{R}_{t,x_0}$ is an integral operator whose kernel verifies, for $\ell = 1, 2$,

$$(\mathbf{K}_{t,x_0})_{\ell\ell}(x,y) = \mathcal{F}_\ell(x_0, x-y) \text{ for } x, y \in \mathbb{R}^d. \tag{4.68}$$

Further properties of $\mathbf{K}_{t,x_0}$ are given in the following lemma.

Lemma 4.5. *Let $t \ge 1$ and $x_0 \in \Omega$. Then $\mathbf{K}_{t,x_0}$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$, and, for $(x,y) \in \mathbb{R}^d\times\mathbb{R}^d$, it holds, for $\ell = 1, 2$,*

$$|(\mathbf{K}_{t,x_0})_{\ell\ell}(x,y)| \le Ct^{-2k-2+\frac{d}{2}}. \tag{4.69}$$

Moreover,

(4.70) $$trace(\mathbf{K}_{t,x_0}(x_0,x_0)) = \frac{t^{-2k-2+\frac{d}{2}}}{(2\pi)^d}\sum_{\ell=1}^{2}\int_{\mathbb{R}^d}\frac{d\xi}{(\Sigma_\ell(x_0)^{-1}A(x_0)\xi\cdot\xi)^{2k+2}-i} + o(t^{-2k-2+\frac{d}{2}}) \text{ as } t\to+\infty.$$

Remark 4.1. Assertion (4.70) holds uniformly with respect to $x_0\in\Omega$.

Proof. From (4.68), it follows that $(\mathbf{K}_{t,x_0})_{\ell\ell}(x,y)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$. By the choice of α, β, and ω_j in (4.50), (4.51), and (4.33), one has

$$\prod_{j=1}^{k+1}\left(\Sigma_\ell(x_0)^{-1}A(x_0)\xi\cdot\xi+\lambda^*+\omega_j te^{i\alpha}\right)\left(\Sigma_\ell(x_0)^{-1}A(x_0)\xi\cdot\xi+\lambda^*+\omega_j te^{i\beta}\right)$$
$$= (\Sigma_\ell(x_0)^{-1}A(x_0)\xi\cdot\xi+\lambda^*)^{2(k+1)} - it^{2(k+1)}.$$

It follows from (4.59) that, for every $x_0\in\Omega$ and every $z\in\mathbb{R}^d$,

$$\mathcal{F}_{\ell,t}(x_0,z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{e^{iz\cdot\xi}\,d\xi}{(\Sigma_\ell(x_0)^{-1}A(x_0)\xi\cdot\xi+\lambda^*)^{2(k+1)}-it^{2(k+1)}}.$$

A change of variables yields

(4.71) $$\mathcal{F}_{\ell,t}(x_0,z) = \frac{t^{-2k-2+\frac{d}{2}}}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{e^{it^{1/2}z\cdot\xi}\,d\xi}{(\Sigma_\ell(x_0)^{-1}A(x_0)\xi\cdot\xi+t^{-1}\lambda^*)^{2(k+1)}-i}.$$

Assertion (4.69) follows from (4.71) since $|e^{it^{1/2}z\cdot\xi}|=1$ and λ^*t^{-1} is uniformly bounded with respect to $t\geq 1$.

By taking $z=0$ in (4.71), we obtain (4.70) after using the dominated convergence theorem.

The proof is complete. □

We now prove the main result of this section concerning the trace of $\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}$ where α,β are given in (4.50) and $\mathbf{T}_{\theta,t}$ is defined in (4.39).

Proposition 4.3. *We have*

$$\mathrm{trace}(\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}) = \mathbf{c}t^{-2k-2+\frac{d}{2}} + o(t^{-2k-2+\frac{d}{2}}) \quad as \quad t\to+\infty,$$

where

(4.72) $$\mathbf{c} = \frac{1}{(2\pi)^d}\sum_{\ell=1}^{2}\int_\Omega\int_{\mathbb{R}^d}\frac{d\xi\,dx}{\left(\Sigma_\ell^{-1}(x)A(x)\xi\cdot\xi\right)^{2k+2}-i}.$$

The proof of Proposition 4.3 uses the following result.

Lemma 4.6. *Let $\delta_0\in(0,1)$ and $\theta\in\Theta$. For every $\varepsilon>0$, there exists $\delta_\varepsilon\in(0,\delta_0/2)$ depending on ε such that the following holds: There exists $t_\varepsilon>0$ depending on ε and δ_ε such that for every $t>t_\varepsilon$ and every $x_0\in\Omega\setminus\overline{\Omega_{\delta_0}}$, we have*

(4.73) $$\|\mathbf{T}_{\theta,t}-\mathbf{R}_{\theta,t,x_0}\mathbb{1}_\Omega\|_{L^2(\Omega)\to L^\infty(B(x_0,\delta_\varepsilon))}\leq\varepsilon t^{-k-1+\frac{d}{4}}$$

and

(4.74) $$\|\mathbf{T}_{\theta,t}\mathbb{1}_\Omega-\mathbf{R}_{\theta,t,x_0}\|_{L^2(\mathbb{R}^d)\to L^\infty(B(x_0,\delta_\varepsilon))}\leq\varepsilon t^{-k-1+\frac{d}{4}},$$

and similar facts for $\mathbf{T}^*_{\theta,t}$ *and* $\mathbf{R}^*_{\theta,t,x_0}$.

Recall that $\mathbf{R}_{\theta,t,x_0}$ is defined in (4.64). We admit Lemma 4.6 and give the proof of Proposition 4.3. The proof of Lemma 4.6 is presented right after the one of Proposition 4.3.

Proof of Proposition 4.3. For $\varepsilon > 0$, let $\delta_0 > 0$ be such that

$$|\Omega_{2\delta_0}| < \varepsilon, \tag{4.75}$$

where Ω_τ is given in (2.1).

We claim that there exists $\tau_* > 0$, depending on Ω, and ε but independent of x_0, and a positive constant C, independent of ε and x_0, such that, for $t > \tau_*$,

$$|\mathrm{trace}(\mathbf{K}_t(x_0, x_0)) - \mathrm{trace}(\mathbf{K}_{t,x_0}(x_0, x_0))| \le C\varepsilon t^{-2k-2+\frac{d}{2}} \quad \text{ for } x_0 \in \Omega \setminus \Omega_{\delta_0}. \tag{4.76}$$

Indeed, let $\chi \in C^\infty_c(\mathbb{R}^d)$ be such that $\chi = 1$ in B_1 and $\operatorname{supp} \chi \subset B_2$. Denote, for $\delta \in (0, \delta_0/10)$,

$$\chi_{\delta,x_0} = \chi\big((\cdot - x_0)/\delta\big),$$

and define

$$\left\{ \begin{array}{c} \mathbf{P}_{1,t,\delta} = \chi_{\delta,x_0}(\mathbf{R}_{\alpha,t,x_0}\mathbb{1}_\Omega - \mathbf{T}_{\alpha,t})\mathbf{T}_{\beta,t}\chi_{\delta,x_0}, \\ \mathbf{P}_{2,t,\delta} = \chi_{\delta,x_0}\mathbf{R}_{\alpha,t,x_0}(\mathbf{R}_{\beta,t,x_0} - \mathbb{1}_\Omega\mathbf{T}_{\beta,t})\chi_{\delta,x_0}. \end{array} \right. \tag{4.77}$$

Then

$$\chi_{\delta,x_0}(\mathbf{R}_{\alpha,t,x_0}\mathbf{R}_{\beta,t,x_0} - \mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t})\chi_{\delta,x_0} = \mathbf{P}_{1,t,\delta} + \mathbf{P}_{2,t,\delta}. \tag{4.78}$$

By applying Lemma 4.6 below with $\theta \in \{\alpha, \beta\}$, there exist $\delta_\varepsilon > 0$ and $t_\varepsilon > 0$ depending on ε such that for every $t > t_\varepsilon$,

$$\|\chi_{\delta_\varepsilon,x_0}(\mathbf{T}_{\alpha,t} - \mathbf{R}_{\alpha,t,x_0}\mathbb{1}_\Omega)\|_{L^2(\Omega)\to L^\infty(\Omega)} \le \varepsilon t^{-k-1+\frac{d}{4}} \tag{4.79}$$

and

$$\|\chi_{\delta_\varepsilon,x_0}(\mathbf{T}^*_{\beta,t}\mathbb{1}_\Omega - \mathbf{R}^*_{\beta,t,x_0})\|_{L^2(\mathbb{R}^d)\to L^\infty(\Omega)} \le \varepsilon t^{-k-1+\frac{d}{4}}. \tag{4.80}$$

Since

$$\Big((\mathbb{1}_\Omega\mathbf{T}_{\beta,t} - \mathbf{R}_{\beta,t,x_0})\chi_{\delta_\varepsilon,x_0}\Big)^* = \chi_{\delta_\varepsilon,x_0}(\mathbf{T}^*_{\beta,t}\mathbb{1}_\Omega - \mathbf{R}^*_{\beta,t,x_0}),$$

we derive from (4.80), using a dual argument, that

$$\|(\mathbb{1}_\Omega\mathbf{T}_{\beta,t} - \mathbf{R}_{\beta,t,x_0})\chi_{\delta_\varepsilon,x_0}\|_{L^1(\Omega)\to L^2(\mathbb{R}^d)} \le \varepsilon t^{-k-1+\frac{d}{4}}. \tag{4.81}$$

By Proposition 4.2 and Lemma 4.4, we have

$$\|\mathbf{T}_{\beta,t}\chi_{\delta_\varepsilon,x_0}\|_{L^1(\Omega)\to L^2(\Omega)} + \|\chi_{\delta_\varepsilon,x_0}\mathbf{R}_{\alpha,t,x_0}\|_{L^2(\mathbb{R}^d)\to L^\infty(\Omega)} \le Ct^{-k-1+\frac{d}{4}} \tag{4.82}$$

for some constant $C > 0$ independent of ε and t.

Using the fact, for appropriate linear operators L_1 and L_2,

$$\|L_1L_2\|_{L^1(\Omega)\to L^\infty(\Omega)} \le \|L_1\|_{L^2(\Omega)\to L^\infty(\Omega)}\|L_2\|_{L^1(\Omega)\to L^2(\Omega)},$$

and

$$\|L_1L_2\|_{L^1(\Omega)\to L^\infty(\Omega)} \le \|L_1\|_{L^2(\mathbb{R}^d)\to L^\infty(\Omega)}\|L_2\|_{L^1(\Omega)\to L^2(\mathbb{R}^d)},$$

we derive from (4.79), (4.81), and (4.82) that

$$\|\mathbf{P}_{1,t,\delta_\varepsilon} + \mathbf{P}_{2,t,\delta_\varepsilon}\|_{L^1(\Omega)\to L^\infty(\Omega)} \le \|\mathbf{P}_{1,t,\delta_\varepsilon}\|_{L^1(\Omega)\to L^\infty(\Omega)} + \|\mathbf{P}_{2,t,\delta_\varepsilon}\|_{L^1(\Omega)\to L^\infty(\Omega)} \le C\varepsilon t^{-2k-2+\frac{d}{2}}.$$

This yields, by (4.78),

$$\|\chi_{\delta_\varepsilon,x_0}(\mathbf{R}_{\alpha,t,x_0}\mathbf{R}_{\beta,t,x_0} - \mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t})\chi_{\delta_\varepsilon,x_0}\|_{L^1(\Omega)\to L^\infty(\Omega)} \le C\varepsilon t^{-2k-2+\frac{d}{2}}. \tag{4.83}$$

Since, for $x \in \Omega$, $\ell = 1, 2$ and $f \in L^2(\Omega)$,

$$\chi_{\delta_\varepsilon, x_0}\Big((\mathbf{R}_{\alpha,t,x_0}\mathbf{R}_{\beta,t,x_0})_{\ell\ell} - (\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t})_{\ell\ell}\Big)\chi_{\delta_\varepsilon,x_0} f(x)$$
$$= \chi_{\delta_\varepsilon,x_0}(x)\int_\Omega \chi_{\delta_\varepsilon,x_0}(y)\Big((\mathbf{K}_{t,x_0})_{\ell\ell}(x,y) - (\mathbf{K}_t)_{\ell\ell}(x,y)\Big) f(y)dy,$$

it follows that $\chi_{\delta_\varepsilon,x_0}(x)\chi_{\delta_\varepsilon,x_0}(y)((\mathbf{K}_{t,x_0})_{\ell\ell}(x,y) - (\mathbf{K}_t(x,y))_{\ell\ell})$ is the kernel of the operator

$$\chi_{\delta_\varepsilon,x_0}\Big((\mathbf{R}_{\alpha,t,x_0}\mathbf{R}_{\beta,t,x_0})_{\ell\ell} - (\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t})_{\ell\ell}\Big)\chi_{\delta_\varepsilon,x_0}.$$

By Lemma 4.2 and Lemma 4.5, this kernel is continuous on $\bar\Omega \times \bar\Omega$. Using (4.83) and applying Lemma 4.2, we derive that, since $\chi_{\delta_\varepsilon,x_0}(x_0) = 1$,

$$|\mathrm{trace}(\mathbf{K}_t(x_0,x_0)) - \mathrm{trace}(\mathbf{K}_{t,x_0}(x_0,x_0))| \le C\varepsilon t^{-2k-2+\frac{d}{2}} \quad \text{for all } t > t_\varepsilon. \tag{4.84}$$

Since the LHS of (4.84) does not depend on $\varepsilon > 0$, the claim (4.76) is proved.

By Lemma 4.2 we have, for $t > 0$ large enough,

$$\int_{\Omega_{2\delta_0}} |\mathrm{trace}(\mathbf{K}_t(x,x))|dx \le C|\Omega_{2\delta_0}|t^{-2k-2+\frac{d}{2}} \overset{(4.75)}{\le} C\varepsilon t^{-2k-2+\frac{d}{2}} \tag{4.85}$$

and, similarly by Lemma 4.5,

$$\int_{\Omega_{2\delta_0}} |\mathrm{trace}(\mathbf{K}_{t,x}(x,x))|dx \le C\varepsilon t^{-2k-2+\frac{d}{2}}. \tag{4.86}$$

Combining (4.76),(4.85), and (4.86) yields

$$\int_\Omega |\mathrm{trace}(\mathbf{K}_t(x,x)) - \mathrm{trace}(\mathbf{K}_{t,x}(x,x))|dx \le C\varepsilon t^{-2k-2+\frac{d}{2}} \quad \text{for all } t > t_\varepsilon. \tag{4.87}$$

The conclusion follows from Lemma 4.5 and (4.87). □

We now give

Proof of Lemma 4.6. Let $\varepsilon > 0$ and $\theta \in \Theta$. First, we prove (4.73). Fix $\chi \in C^\infty_c(\mathbb{R}^d)$ such that $\operatorname{supp}\chi \subset B_2$ and $\chi = 1$ in B_1. Set, for $0 < \delta < \delta_0/100$,

$$\chi_\delta = \chi\big((\cdot - x_0)/\delta\big) \text{ in } \mathbb{R}^d.$$

Define, for $(f,g) \in [L^2(\Omega)]^2$, and for $j = 1, \cdots, k+1 = [d/2]+2$,

$$(u^j, v^j) = T_{\lambda_{j,\theta,t}} \circ \cdots \circ T_{\lambda_{1,\theta,t}}(u^0, v^0) \quad \text{and} \quad (u^j_0, v^j_0) = S_{\lambda_{j,\theta,t},x_0} \circ \cdots \circ S_{\lambda_{1,\theta,t},x_0}(u^0_0, v^0_0),$$

where

$$(u^0, v^0) = (f, g) \qquad \text{and} \qquad (u^0_0, v^0_0) = (\mathbb{1}_\Omega f, \mathbb{1}_\Omega g) \quad \text{in } \Omega. \tag{4.88}$$

Set, for $0 \le j \le k+1$,

$$(u^{j,\delta}, v^{j,\delta}) = (\chi_\delta u^j, \chi_\delta v^j) \qquad \text{and} \qquad (u^{j,\delta}_0, v^{j,\delta}_0) = (\chi_\delta u^j_0, \chi_\delta v^j_0).$$

We have

$$(u^{j,\delta}, v^{j,\delta}) = S_{\lambda_{j,\theta,t},x_0}(u^{j-1,\delta}, v^{j-1,\delta}) + S_{\lambda_{j,\theta,t},x_0}(f^{j,\delta}, g^{j,\delta}), \tag{4.89}$$

where

$$\Sigma_1(x_0) f^{j,\delta} = (\Sigma_1(x) - \Sigma_1(x_0)) u^{j-1,\delta} - \lambda_{j,\theta,t} (\Sigma_1(x_0) - \Sigma_1(x)) u^{j,\delta} + A(x) \nabla \chi_\delta \cdot \nabla u^j \\ - (\Sigma_1(x_0) - \Sigma_1(x) - \Sigma_2(x_0) + \Sigma_2(x)) v^{j,\delta} + \operatorname{div} \Big((A(x_0) - A(x)) \nabla u^{j,\delta} + u^j A(x) \nabla \chi_\delta \Big) \tag{4.90}$$

and

$$\Sigma_2(x_0) g^{j,\delta} = (\Sigma_2(x) - \Sigma_2(x_0)) v^{j-1,\delta} - \lambda_{j,\theta,t} (\Sigma_2(x_0) - \Sigma_2(x)) v^{j,\delta} + A(x) \nabla \chi_\delta \cdot \nabla v^j \\ + \operatorname{div} \Big((A(x_0) - A(x)) \nabla v^{j,\delta} + v^j A(x) \nabla \chi_\delta \Big). \tag{4.91}$$

Similarly, we have

$$(u_0^{j,\delta}, v_0^{j,\delta}) = S_{\lambda_{j,\theta,t}, x_0} (u_0^{j-1,\delta}, v_0^{j-1,\delta}) + S_{\lambda_{j,\theta,t}, x_0} (f_0^{j,\delta}, g_0^{j,\delta}),$$

where

$$\Sigma_1(x_0) f_0^{j,\delta} = A(x_0) \nabla \chi_\delta \cdot \nabla u_0^j + \operatorname{div} \Big(u_0^j A(x_0) \nabla \chi_\delta \Big)$$

and

$$\Sigma_2(x_0) g_0^{j,\delta} = A(x_0) \nabla \chi_\delta \cdot \nabla v_0^j + \operatorname{div} \Big(v_0^j A(x_0) \nabla \chi_\delta \Big).$$

For $r > 0$, define

$$\Phi(r) = \min \left\{ 1, \sup_{|x-y|<r} \left(|A(x) - A(y)| + \sum_{\ell=1}^{2} |\Sigma_\ell(x) - \Sigma_\ell(y)| \right) \right\}.$$

We claim that

$$\| f^{j,\delta} \|_{L^{p_{j-1}}(\Omega \setminus \Omega_{\delta_0/2})} + t^{-1} \| g^{j,\delta} \|_{L^{p_{j-1}}(\Omega \setminus \Omega_{\delta_0/2})} \\ \le C_{\delta_0} \left(\Phi(\delta) + \frac{1}{\delta t^{1/2}} + \frac{1}{\delta^2 t} \right) \Big(\| u^{j-1} \|_{L^{p_{j-1}}(\Omega)} + t^{-1} \| v^{j-1} \|_{L^{p_{j-1}}(\Omega)} \Big), \tag{4.92}$$

and

$$\| f_0^{j,\delta} \|_{L^{p_{j-1}}(\Omega \setminus \Omega_{\delta_0/2})} + t^{-1} \| g_0^{j,\delta} \|_{L^{p_{j-1}}(\Omega \setminus \Omega_{\delta_0/2})} \\ \le C_{\delta_0} \left(\frac{1}{\delta t^{1/2}} + \frac{1}{\delta^2 t} \right) \Big(\| u_0^{j-1} \|_{L^{p_{j-1}}(\Omega)} + t^{-1} \| v_0^{j-1} \|_{L^{p_{j-1}}(\Omega)} \Big). \tag{4.93}$$

We first admit (4.92) and (4.93) and continue the proof. Since, in Ω,

$$(u^{0,\delta}, v^{0,\delta}) = (u_0^{0,\delta}, v_0^{0,\delta}),$$

using (4.89), (4.92) and (4.93) and Lemma 4.3, for $j = 1$ and then for $j = 2, \dots, k+1$, we have

$$\| u^{j,\delta} - u_0^{j,\delta} \|_{L^{p_j}(\Omega \setminus \Omega_{\delta_0/2})} + t^{-1} \| v^{j,\delta} - v_0^{j,\delta} \|_{L^{p_j}(\Omega \setminus \Omega_{\delta_0/2})} \\ \le C_{\delta_0} \left(\Phi(\delta) + \frac{1}{\delta t^{1/2}} + \frac{1}{\delta^2 t} \right) t^{-\frac{d}{2p_j} - j + \frac{d}{4}} \left(\| f \|_{L^2(\Omega)} + t^{-1} \| g \|_{L^2(\Omega)} \right). \tag{4.94}$$

Fix $\delta = \delta_\varepsilon > 0$ such that $C_{\delta_0} \Phi(\delta_\varepsilon) < \varepsilon/2$. Take $t_\varepsilon > 0$ sufficiently large such that $C_{\delta_0} (\delta_\varepsilon^{-1} t^{-1/2} + \delta_\varepsilon^{-2} t^{-1}) < \varepsilon/2$ for every $t > t_\varepsilon$. Taking $j = k+1$ in (4.94) gives (4.73).

The proof of (4.74) is similar to the one of (4.73) by considering (u^0, v^0) and (u_0^0, v_0^0) defined as follows

$$(u^0, v^0) = \mathbb{1}_\Omega(f, g) \qquad \text{and} \qquad (u_0^0, v_0^0) = (f, g) \quad \text{in } \mathbb{R}^d,$$

instead of (4.88).

Similar facts for $\mathbf{T}^*_{\theta,t}$ and $\mathbf{R}^*_{\theta,t,x_0}$ by analogous arguments.

It remains to establish (4.92) and (4.93). From the definition of (u^j, v^j) and the theory of elliptic equations (see e.g. [15, Theorem 9.11]), we have, for $\Omega_1 \Subset \Omega_2 \subset \Omega$,

$$\|u^j\|_{W_t^{2,p}(\Omega_1)} + t^{-1}\|v^j\|_{W_t^{2,p}(\Omega_1)} \le C\left(\|u^{j-1}\|_{L^p(\Omega_2)} + t^{-1}\|v^{j-1}\|_{L^p(\Omega_2)}\right) \tag{4.95}$$

and, similarly,

$$\|u_0^j\|_{W_t^{2,p}(\Omega_1)} + t^{-1}\|v_0^j\|_{W_t^{2,p}(\Omega_1)} \le C\left(\|u_0^{j-1}\|_{L^p(\Omega_2)} + t^{-1}\|v_0^{j-1}\|_{L^p(\Omega_2)}\right), \tag{4.96}$$

for some positive constant C independent of f, g, and t. It follows that

$$\|\nabla u^{j,\delta}\|_{L^p(\Omega_1)} + t^{-1}\|\nabla v^{j,\delta}\|_{L^p(\Omega_1)} \le C\left(\frac{1}{\delta t} + \frac{1}{t^{1/2}}\right)\left(\|u^{j-1}\|_{L^p(\Omega_2)} + t^{-1}\|v^{j-1}\|_{L^p(\Omega_2)}\right), \tag{4.97}$$

$$\begin{aligned}\|\nabla^2 u^{j,\delta}\|_{L^p(\Omega_1)} &+ t^{-1}\|\nabla^2 v^{j,\delta}\|_{L^p(\Omega_1)} \\ &\le C\left(1 + \frac{1}{\delta t^{1/2}} + \frac{1}{\delta^2 t}\right)\left(\|u^{j-1}\|_{L^p(\Omega_2)} + t^{-1}\|v^{j-1}\|_{L^p(\Omega_2)}\right),\end{aligned} \tag{4.98}$$

and

$$\|u_0^{j,\delta}\|_{W_t^{1,p}(\Omega_1)} + t^{-1}\|v_0^{j,\delta}\|_{W_t^{1,p}(\Omega_1)} \le C\left(\frac{1}{\delta t} + \frac{1}{t^{1/2}}\right)\left(\|u_0^{j-1}\|_{L^p(\Omega_2)} + t^{-1}\|v_0^{j-1}\|_{L^p(\Omega_2)}\right). \tag{4.99}$$

By (4.90) and (4.91), we have

$$\begin{aligned} C\big(\, \|f^{j,\delta}\|_{L^{p_{j-1}}(\Omega_1)} + t^{-1}\|g^{j,\delta}\|_{L^{p_{j-1}}(\Omega_1)}\Big) &\le \Phi(\delta)\Big(\|u^{j-1}\|_{L^{p_{j-1}}} + t^{-1}\|v^{j-1}\|_{L^{p_{j-1}}(\Omega_1)}\Big) \\ &+ \Big(\Phi(\delta) + \frac{1}{\delta^2 t} + \frac{1}{\delta t^{1/2}}\Big)\Big(\|u^j\|_{W_t^{2,p_{j-1}}(\Omega_1)} + t^{-1}\|v^j\|_{W_t^{2,p_{j-1}}(\Omega_1)}\Big) \\ &+ \|\nabla u^{j,\delta}\|_{L^{p_{j-1}}(\Omega_1)} + t^{-1}\|\nabla v^{j,\delta}\|_{L^{p_{j-1}}(\Omega_1)} + \Phi(\delta)\Big(\|\nabla^2 u^{j,\delta}\|_{L^{p_{j-1}}(\Omega_1)} + t^{-1}\|\nabla^2 v^{j,\delta}\|_{L^{p_{j-1}}(\Omega_1)}\Big). \end{aligned}$$

Combining (4.95)-(4.99) yields (4.92). Estimate (4.93) follows similarly.

The proof is complete. □

4.5. **A connection of the counting function and the trace of $\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}$ for large t.** We start this section by recalling the definition of the modified resolvent of an operator (see, e.g., [1, Definition 12.3]).

Definition 4.5. *Let H be a Hilbert space and $\mathcal{T}: H \to H$ be a linear and bounded operator. The modified resolvent set $\rho_m(\mathcal{T})$ of $\mathcal{T}$ is the set of all non-zero complex numbers s such that $I - s\mathcal{T}$ is bijective and $(I - s\mathcal{T})^{-1}$ is bounded on H. For $s \in \rho_m(\mathcal{T})$ the transformation $(\mathcal{T})_s = \mathcal{T}(I - s\mathcal{T})^{-1}$ is the modified resolvent of $\mathcal{T}$.*

Recall that, for $s \in \rho_m(\mathcal{T})$, we have

$$(\mathcal{T})_s = \mathcal{T}(I - s\mathcal{T})^{-1} = (I - s\mathcal{T})^{-1}\mathcal{T}. \tag{4.100}$$

Let $\mathcal{T} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ be a linear and bounded operator. We have for $z \in \mathbb{C}$ (see e.g. [38])

$$I - z^{k+1}\mathcal{T}^{k+1} = \prod_{j=1}^{k+1}(I - \omega_j z\mathcal{T}) \tag{4.101}$$

and

$$I - z^{k+1}\mathcal{T}^{k+1} \text{ is invertible } \iff I - \omega_j z\mathcal{T} \text{ is invertible for every } j. \tag{4.102}$$

Recall that $\omega_j^{k+1} = 1$. Using the decomposition (4.101), and the equivalence in (4.102), one can prove the following lemma.

Lemma 4.7. *Let $\widetilde{\theta} \in \mathbb{R} \setminus \{\pi\mathbb{Z}\}$. Set $\theta := \frac{\tilde{\theta}}{k+1} \in \Theta$. There exists $t_\theta > 1$ such that, for every $t > t_\theta$, it holds*

$$\gamma := t^{k+1}e^{i\widetilde{\theta}} \in \rho_m(T_{\lambda^*}^{k+1}) \tag{4.103}$$

and

$$(T_{\lambda^*}^{k+1})_\gamma = M_t^{-1}\mathbf{T}_{\theta,t}M_t. \tag{4.104}$$

Proof. We have, by the definition of γ,

$$I - \gamma T_{\lambda^*}^{k+1} \overset{(4.101)}{=} \prod_{j=1}^{k+1}(I - \omega_j t e^{i\theta}T_{\lambda^*}). \tag{4.105}$$

By Proposition 4.1, there exists $t_\theta > 0$ such that $T_{\lambda_* + \omega t e^{i\theta}}$ is defined for $t \geq t_\theta$. Hence

$$\omega_j t e^{i\theta} \in \rho_m(T_{\lambda^*}) \quad \text{ and } \quad (T_{\lambda^*})_{\omega_j t e^{i\theta}} = T_{\lambda^* + \omega_j t e^{i\theta}} = T_{\lambda_{j,\theta,t}} \text{ for } t \geq t_\theta. \tag{4.106}$$

(see, e.g. [13, Lemma 3.1] for the arguments in a similar setting). Combining (4.105) and (4.106) leads $\gamma \in \rho_m(T_{\lambda^*}^{k+1})$ for $t \geq t_\theta$.

It follows from (4.100) that, for $t \geq t_\theta$,

$$T_{\lambda_{j,\theta,t}} = T_{\lambda^*}(I - \omega_j t e^{i\theta}T_{\lambda^*})^{-1} = (I - \omega_j t e^{i\theta}T_{\lambda^*})^{-1}T_{\lambda^*} \tag{4.107}$$

and thus,

$$\begin{aligned} M_t^{-1}\mathbf{T}_{\theta,t}M_t \overset{(4.39)}{=} \prod_{j=1}^{k+1} T_{\lambda_{j,\theta,t}} \overset{(4.107)}{=} T_{\lambda^*}^{k+1}\prod_{j=1}^{k+1}(I - \omega_j e^{i\theta}tT_{\lambda^*})^{-1} \\ = T_{\lambda^*}^{k+1}(I - \gamma T_{\lambda^*}^{k+1})^{-1} \overset{\text{def.}}{=} (T_{\lambda^*}^{k+1})_\gamma. \end{aligned} \tag{4.108}$$

The proof is complete. $\square$

The following proposition establishes a connection between the trace of the operator $\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}$ and the counting function for large t. The arguments of the proof are in the spirit of [34] (see also [38]).

Proposition 4.4. *We have*

$$\mathcal{N}(t) = \frac{\Im(\mathbf{c})}{\frac{d}{8(k+1)}\int_0^\infty s^{\frac{d}{8(k+1)}-1}(1+s)^{-1}ds} t^{\frac{d}{2}} + o(t^{\frac{d}{2}}) \qquad as\ t \to +\infty,$$

where $\mathbf{c}$ *is given by* (4.72).

Proof. For t sufficiently large, by Lemma 4.7, we have

$$(T_{\lambda^*}^{k+1})_{t^{k+1}e^{i(k+1)\alpha}} = M_t^{-1}\mathbf{T}_{\alpha,t}M_t.$$

Note that

$$\Big((T_{\lambda^*}^{k+1})_{\gamma_1}\Big)_{\gamma_2} = (T_{\lambda^*}^{k+1})_{\gamma_1+\gamma_2} \tag{4.109}$$

provided that $\gamma_1,\gamma_1+\gamma_2 \in \rho_m(T_{\lambda^*}^{k+1})$. It follows from Lemma 4.7 that, for large t and for $s \ge 0$,

$$-2(t+s)^{k+1}e^{i(k+1)\alpha} \in \rho_m\Big(M_t^{-1}\mathbf{T}_{\alpha,t}M_t\Big).$$

Let $s_1, s_2, \dots$ be the characteristic values of $M_t^{-1}\mathbf{T}_{\theta,t}M_t$ repeated a number of times equal to their multiplicities. Applying [1, Theorem 12.17], we have

$$\operatorname{trace}\Big(M_t^{-1}\mathbf{T}_{\alpha,t}M_t(M_t^{-1}\mathbf{T}_{\alpha,t}M_t)_{-2(t+s)^{k+1}e^{i(k+1)\alpha}}\Big) = \sum_j \frac{1}{s_j(s_j + 2e^{i\alpha(k+1)}(t+s)^{k+1})} + c_t. \tag{4.110}$$

We claim that

$$c_t = 0. \tag{4.111}$$

Assume this, we continue the proof. As a consequence of (4.110) with $s=0$, we have

$$\operatorname{trace}\Big(M_t^{-1}\mathbf{T}_{\alpha,t}M_t(M_t^{-1}\mathbf{T}_{\alpha,t}M_t)_{-2t^{k+1}e^{i(k+1)\alpha}}\Big) = \sum_j \frac{1}{s_j(s_j + 2e^{i\alpha(k+1)}t^{k+1})}. \tag{4.112}$$

Let $(\mu_j)_j$ be the set of characteristic values of T_{λ^*} repeated according to their multiplicity. It is well-known that μ_j^{k+1} are the characteristic values of $T_{\lambda^*}^{k+1}$ and the multiplicity of μ_j^{k+1} is equal to the sum of the one of the characteristic values μ of T_{λ^*} such that $\mu^{k+1} = \mu_j^{k+1}$. By Lemma 4.7, for large t, $e^{i\alpha(k+1)}t^{k+1}$ is not a characteristic value of $T_{\lambda^*}^{k+1}$. We obtain, by [1, Theorem 12.4], that the set of the characteristic values of $(T_{\lambda^*}^{k+1})_{t^{k+1}e^{i(k+1)\alpha}}$ is given by

$$\Big\{\mu_j^{k+1} - t^{k+1}e^{i(k+1)\alpha}\ ;\ j \ge 1\Big\}.$$

We now derive from (4.112) that

$$\operatorname{trace}\Big(M_t^{-1}\mathbf{T}_{\alpha,t}M_t(M_t^{-1}\mathbf{T}_{\alpha,t}M_t)_{-2t^{k+1}e^{i(k+1)\alpha}}\Big)$$
$$= \sum_j \frac{1}{(\mu_j^{k+1} - t^{k+1}e^{i(k+1)\alpha})(\mu_j^{k+1} + t^{k+1}e^{i(k+1)\alpha})},$$

which yields, since $\alpha = \frac{\pi}{4(k+1)}$,

$$\operatorname{trace}\Big(M_t^{-1}\mathbf{T}_{\alpha,t}M_t(M_t^{-1}\mathbf{T}_{\alpha,t}M_t)_{-2t^{k+1}e^{i(k+1)\alpha}}\Big) = \sum_j \frac{1}{\mu_j^{2(k+1)} - it^{2(k+1)}}. \tag{4.113}$$

We have, by Proposition 4.1,

$$\limsup_{|\mu_j|\to+\infty} \left|\frac{\Im(\mu_j)}{\mu_j}\right| = 0.$$

As a consequence and as in [34, Proof of Corollary 3], we derive that

$$\sum_j \frac{1}{\mu_j^{2k+2} - it^{2k+2}} - \sum_j \frac{1}{|\mu_j|^{2k+2} - it^{2k+2}} = o(t^{-2k-2+\frac{d}{2}}) \text{ as } t \to +\infty. \tag{4.114}$$

Combining (4.113) and (4.114) yield

$$\operatorname{trace}\Big(M_t^{-1}\mathbf{T}_{\alpha,t}M_t(M_t^{-1}\mathbf{T}_{\alpha,t}M_t)_{-2t^{k+1}e^{i(k+1)\alpha}}\Big) = \sum_j \frac{1}{|\mu_j|^{2(k+1)} - it^{2(k+1)}} + o(t^{-2k-2+\frac{d}{2}}) \text{ as } t \to +\infty. \tag{4.115}$$

Applying (4.109) with $\gamma_1 = t^{k+1}e^{i(k+1)\alpha}$ and $\gamma_2 = -2t^{k+1}e^{i(k+1)\alpha}$ and using Lemma 4.7, we derive that

$$(M_t^{-1}\mathbf{T}_{\alpha,t}M_t)_{-2t^{k+1}e^{i(k+1)\alpha}} = M_t^{-1}\mathbf{T}_{\beta,t}M_t. \tag{4.116}$$

Since

$$\operatorname{trace}\left(M_t^{-1}\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}M_t\right) = \operatorname{trace}\left(\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}\right),$$

it follows from (4.115) and (4.116) that

$$\operatorname{trace}\left(\mathbf{T}_{\alpha,t}\mathbf{T}_{\beta,t}\right) = \sum_j \frac{1}{|\mu_j|^{2(k+1)} - it^{2(k+1)}} + o(t^{-2k-2+\frac{d}{2}}) \text{ as } t \to +\infty. \tag{4.117}$$

Applying Proposition 4.3, we derive from (4.117) that, as $t \to +\infty$

$$\sum_j \frac{1}{|\mu_j|^{2k+2} - it^{2k+2}} = \mathbf{c}t^{-2k-2+\frac{d}{2}} + o(t^{-2k-2+\frac{d}{2}}). \tag{4.118}$$

Considering the imaginary part of (4.118) we get, for $\tau = t^{4k+4}$,

$$\sum_j \frac{1}{|\mu_j|^{4k+4} + \tau} = \Im(\mathbf{c})\tau^{\frac{d}{8k+8}-1} + o(\tau^{\frac{d}{8k+8}-1}) \text{ as } \tau \to +\infty.$$

Since $\lambda_j = \mu_j + \lambda^*$, it follows that, as $\tau \to +\infty$,

$$\sum_j \frac{1}{|\lambda_j|^{4k+4} + \tau} = \Im(\mathbf{c})\tau^{\frac{d}{8k+8}-1} + o(\tau^{\frac{d}{8k+8}-1}). \tag{4.119}$$

Using the fact

$$\sum_j \frac{1}{|\lambda_j|^{4k+4} + \tau} = \int_0^\infty \frac{d\mathcal{N}(s^{\frac{1}{4(k+1)}})}{s+\tau},$$

we derive that

$$\int_0^\infty \frac{d\mathcal{N}(s^{\frac{1}{4(k+1)}})}{s+\tau} = \Im(\mathbf{c})\tau^{\frac{d}{8k+8}-1} + o(\tau^{\frac{d}{8k+8}-1}) \text{ as } \tau \to +\infty. \tag{4.120}$$

Applying a Tauberian Theorem of Hardy and Littlewood (see, e.g., [1, Theorem 14.5]), we obtain

$$\mathcal{N}(t) = \frac{\Im(\mathbf{c})}{\frac{d}{8(k+1)} \int_0^\infty s^{\frac{d}{8(k+1)}-1}(1+s)^{-1} ds} t^{\frac{d}{2}} + o(t^{\frac{d}{2}}) \qquad \text{as } t \to +\infty,$$

which is the conclusion.

It remains to prove (4.111). Applying (4.109) with $\gamma_1 = t^{k+1} e^{i(k+1)\alpha}$ and $\gamma_2 = -2(t+s)^{k+1} e^{i(k+1)\alpha}$ and using Lemma 4.7, we derive that

$$(M_t^{-1} \mathbf{T}_{\alpha,t} M_t)_{-2(t+s)^{k+1} e^{i(k+1)\alpha}} = M_r^{-1} \mathbf{T}_{\widetilde{\alpha},r} M_r.$$

where

$$\widetilde{\alpha} = \alpha + \frac{\pi}{k+1} \quad \text{and} \quad r = (2(t+s)^{k+1} - t^{k+1})^{\frac{1}{k+1}}.$$

Thus by [1, Theorem 12.14],

$$\Big\{ s_j + 2(t+s)^{k+1} e^{i\alpha(k+1)} \ ; \ j \ge 1 \Big\} \text{ is the set of characteristic values of } M_r^{-1} \mathbf{T}_{\widetilde{\alpha},r} M_r. \tag{4.121}$$

Applying Corollary 4.3 and using (4.38), we have

$$|||M_t^{-1} \mathbf{T}_{\alpha,t} M_t||| \le C t^{-k+\frac{d}{4}} \quad \text{and} \quad |||M_r^{-1} \mathbf{T}_{\widetilde{\alpha},r} M_r||| \le C r^{-k+\frac{d}{4}} \tag{4.122}$$

for some constant $C > 0$ which does not depend on s (and t). By [1, Theorem 12.12] we have

$$|\mathrm{trace}\Big(M_t^{-1} \mathbf{T}_{\alpha,t} M_t (M_t^{-1} \mathbf{T}_{\alpha,t} M_t)_{-2\alpha^{k+1}(t+s)^{k+1}} \Big)| \le |||M_t^{-1} \mathbf{T}_{\alpha,t} M_t||| \, |||M_r^{-1} \mathbf{T}_{\widetilde{\alpha},r} M_r|||. \tag{4.123}$$

Since $-k + \frac{d}{4} < 0$ it follows from (4.122) and (4.123) that

$$\lim_{s \to +\infty} \mathrm{trace}\Big(M_t^{-1} \mathbf{T}_{\alpha,t} M_t (M_t^{-1} \mathbf{T}_{\alpha,t} M_t)_{-2\alpha^{k+1}(t+s)^{k+1}} \Big) = 0. \tag{4.124}$$

On the other hand, by [1, Theorem 12.14],

$$\begin{aligned} \Big| \sum_j \frac{1}{s_j (s_j + 2e^{i\alpha(k+1)}(t+s)^{k+1})} \Big|^2 &\le \sum_j \frac{1}{|s_j|^2} \sum_j \frac{1}{|s_j + 2e^{i\alpha(k+1)}(t+s)^{k+1}|^2} \\ &\overset{(4.121)}{\le} |||M_t^{-1} \mathbf{T}_{\alpha,t} M_t|||^2 \, |||M_r^{-1} \mathbf{T}_{\widetilde{\alpha},r} M_r|||^2 \overset{(4.122)}{\to} 0 \text{ as } s \to +\infty. \end{aligned} \tag{4.125}$$

Combining (4.124) and (4.125) yields $c_t = 0$, which is (4.111) .

The proof is complete. □

4.6. **Proof of Theorem 1.1.** As in [34, p.34], we derive from Proposition 4.3 that

$$\begin{aligned} \Im(\mathbf{c}) &= \frac{1}{(2\pi)^d} \int_\Omega \sum_{\ell=1}^2 \int_{\mathbb{R}^d} \frac{1}{(\Sigma_\ell(x_0)^{-1} A(x_0) \xi \cdot \xi)^{4k+4} + 1} d\xi dx \\ &= \frac{1}{(2\pi)^d} \sum_{\ell=1}^2 \int_\Omega |\{\xi : A(x)\xi \cdot \xi < \Sigma_\ell(x)\}| \, dx \frac{d}{8(k+1)} \int_0^\infty s^{\frac{d}{8(k+1)}-1}(1+s)^{-1} ds. \end{aligned}$$

The conclusion now follows from Proposition 4.4. □

5. Completeness of the generalized eigenfunctions of the transmission eigenvalue problem - Proof of Theorem 1.2

By Lemma 4.7, for all $\widetilde{\theta} \in (0, 2\pi) \setminus \{\pi\}$, there exists $t_{\widetilde{\theta}} > 0$ such that, for $t > t_{\widetilde{\theta}}$,

$$(T_{\lambda^*}^{k+1})_{te^{i\widetilde{\theta}}} = M_{t_k}^{-1} \mathbf{T}_{\theta, t_k} M_{t_k}, \tag{5.1}$$

where

$$\theta = \frac{\widetilde{\theta}}{k+1} \quad \text{and} \quad t_k = t^{\frac{1}{k+1}}.$$

By Proposition 4.1 and Corollary 4.3,

$$|||M_{t_k}^{-1} \mathbf{T}_{\theta, t_k} M_{t_k}||| \le C t_k^{-k+\frac{d}{4}} \quad \text{and} \quad \|M_{t_k}^{-1} \mathbf{T}_{\theta, t_k} M_{t_k}\|_{L^2(\Omega) \to L^2(\Omega)} \le C t_k^{-k}. \tag{5.2}$$

In particular, $T_{\lambda^*}^{k+1}$ is a Hilbert-Schmidt operator; moreover, for $t > t_\theta$,

$$|||(T_{\lambda^*}^{k+1})_{te^{i\widetilde{\theta}}}||| \le C_{\widetilde{\theta}} t^{-1 + \frac{1}{k+1} + \frac{d}{4(k+1)}}. \tag{5.3}$$

Since $k = [d/2] + 1$, it follows that $-1 + \frac{1}{k+1} + \frac{d}{4(k+1)} \le 0$. This implies that

$$\text{for all } \widetilde{\theta} \in (0, 2\pi) \setminus \{\pi\} \text{ there exist } t_{\widetilde{\theta}} > 0 \text{ and } C_{\widetilde{\theta}} > 0 \text{ such that } \sup_{t > t_{\widetilde{\theta}}} |||(T_{\lambda^*}^{k+1})_{te^{i\widetilde{\theta}}}||| \le C_{\widetilde{\theta}}. \tag{5.4}$$

Since $T_{\lambda^*}^{k+1}$ is a Hilbert-Schmidt operator, it follows from [1, Theorem 16.4] that

i) the space spanned by the generalized eigenfunctions of $T_{\lambda^*}^{k+1}$ is equal to $\overline{\text{range}(T_{\lambda^*}^{k+1})}$, the closure of the range of $T_{\lambda^*}^{k+1}$ with respect to the L^2-topology.

In fact, in order to be able to apply [1, Theorem 16.4], one requires the assumptions on the directions of the minimal growth of the modified resolvent of $T_{\lambda^*}^{k+1}$. We have only proved (5.3) and (5.4) instead of this requirement. Nevertheless, this is sufficient to derive 1) using almost the same arguments in [1] (see also [39]).

The rest of the proof is as in [34, 13]. We have

ii) range $T_{\lambda^*}^{k+1}$ is dense in $[L^2(\Omega)]^2$ since range T_{λ^*} is dense in $[L^2(\Omega)]^2$ and T_{λ^*} is continuous,

ii) the space spanned by the general eigenfunctions of $T_{\lambda^*}^{k+1}$ associated to the non-zero eigenvalues of $T_{\lambda^*}^{k+1}$ is equal to the space spanned by the general eigenfunctions of T_{λ^*} associated to the non-zero eigenvalues of T_{λ^*} . This can be done as in the last part of the proof of [1, Theorem 16.5]. Consequently, the space spanned by all generalized eigenfunctions of $T_{\lambda^*}^{k+1}$ is equal to the space spanned by all generalized eigenfunctions of T_{λ^*}.

The conclusion now follows from i), ii), and iii). □

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