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Upper bound on saturation time of metric graphs by intervals moving on them

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Abstract

In this paper we study a dynamical system of intervals moving on a metric graph with incommensurable edge lengths. This system can be generally viewed as a collection of congruent intervals moving around the network at unit speed, propagating along all respective incident edges whenever they pass through a vertex. In particular, we analyse the phenomenon of saturation: a state when the entire graph is covered by the moving intervals. Our main contributions are as follows: (1) we prove the existence of the finite moment of permanent saturation for any metric graph with incommensurable edge lengths and any positive length of the intervals; (2) we present an upper bound on the moment of permanent saturation. To show the validity of our results, we reduce the system of moving intervals to the system of dispersing moving points, for the analysis of which we mainly use methods from discrepancy theory and number theory, in particular Kronecker sequences and the famous Three Gap Theorem.

Keywords: metric graph, wave packet, dynamical system of points, random walk, epsilon-net, saturation

1. Introduction

Recent studies of wave packets distribution in hybrid topological spaces and on metric graphs have raised the problem of identifying the time moment

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at which networks of waveguides get completely covered by waves, leaving no idle points in the system [1, 2, 3]. In this setting, we typically consider a wave packet of relatively small width, initially located in the vicinity of some node of the network, which then begins to travel freely around it. We assume that the waves travel without dispersion and propagate along all incident edges when they hit any node of the graph, creating new waves of the same width.

The specific shape of the waves is not important in this context, thereby, for the analysis of the given problem, we replace the wave packets by moving intervals of a fixed length, which gradually propagate along all the respective incident edges as they pass through any vertex of the network, thus creating new moving intervals of the same length. This idea allows us to mimic the behaviour of waves, yet get rid of all their properties that are unnecessary for tracking the coverage of a metric graph by them.

If we assume the length of these intervals to be 2ε (the width of the wave packets to be 2ε) for some positive ε , we can simplify our model even further by representing each moving interval as the ε -neighbourhood of its central point, leading to the dynamical system of points moving on a metric graph, which can be briefly described as follows.

Moving points move back and forth along the edges of the graph in continuous time at unit speed. Since we talk about metric graphs, we consider the lengths of the edges to be the actual distances that these points have to bridge between respective vertices. Whenever a moving point hits a vertex of the graph, it propagates along all the incident edges, creating new moving points on them.

At the initial time moment one moving point is artificially placed at one of the vertices: it immediately spreads over all the incident edges and the points begin to move. Before the initial time moment, the system is empty (there are no moving points on the graph).

Although the study of this process is mainly motivated by its physical applications, it is worth noting that Izmaylov and Dworzanski mentioned in [9] that it can also be used as an ‘overapproximation’ of message exchange in computer networks.

Previously, the problem of representing the number of moving points in this dynamical system (the number of wave packets in the network) as a function of time was studied in detail, e.g. in [4, 5, 6, 7, 8, 9, 14]. In order to avoid possible confusion when looking through the mentioned papers, we explicitly warn the reader that there is no standard notation among the authors: in some papers the above process is interpreted as the ‘endpoints of

a random walk on a metric graph' (e.g. [6, 14]), in others it is called either a 'DP-system' (e.g. [8, 9]) or simply 'moving points' (e.g. [4, 5]).

The problem we approach in this paper seems to be in a completely different category, but is in fact related to the problem of counting points. In [3] it was shown that if the edges of the metric graph have commensurable lengths (have rank 1 over \mathbb{Q}), then after a while a phenomenon of *stabilisation* occurs, by which we understand a situation where the number of points in the system remains constant forever. Thus, given a sufficiently small value of ε (sufficiently small width of the wave packets), under these circumstances it is possible that the system will never become saturated. On the other hand, if the edges of the graph have incommensurable lengths (have a rank greater than 1 over \mathbb{Q}), the stabilisation never occurs and the number of moving points goes to infinity over time. As Dworzanski says in [8], '... it is possible to extend the notion of stabilization time to systems with incommensurable edges using the notion of ε -net...'. In the same paper he proposes the notion of ε -saturation, which we develop in this text.

Thus, we can also consider saturation as an alternative stabilisation property for graphs with incommensurable edge lengths, since the original concept does not apply to them.

Indeed, in this paper we show that ε -saturation exists for all finite undirected connected simple metric graphs with incommensurable edge lengths and for all positive ε . Furthermore, we give a general upper bound on the permanent ε -saturation moment of dispersing moving points systems or, equivalently, a general upper bound on the permanent saturation moment of moving intervals of length 2ε (wave packets of width 2ε).

The rest of the paper is structured as follows. Section 2 contains all the necessary definitions and presents our main result. Section 3 presents the proofs of the main result and all auxiliary statements. Section 4 discusses possible directions for further research.

2. Definitions and the main result

2.1. Metric graphs and dispersing points

Let us commence with a rigorous definition of the dynamical system that is the focus of this work and all the concepts associated with it.

Definition 1 (metric graph). A *metric graph* is a tuple (V, E) where $V = \{v_1, \dots, v_n\}$ is the set of *vertices* and E is the set of *edges* with every $e_{ij} \in$

E being an interval $(0, |e_{ij}|) \subset \mathbb{R}$ connecting vertices $v_i \in V$ and $v_j \in V$ associated with endpoints 0 and $|e_{ij}|$ respectively. In this context, a positive number $|e_{ij}|$ is called the *length* of e_{ij} . We also denote $e_{ij} \cup \{v_i, v_j\} = [0, |e_{ij}|]$ as \bar{e}_{ij} .

In this work we only consider *undirected* graphs, which means that we do not distinguish between e_{ij} and e_{ji} . Furthermore, we are not strict about which vertex incident on e_{ij} is associated with the endpoint 0 and which vertex is associated with $|e_{ij}|$. In fact, later in the text we will conveniently switch these associations as needed to simplify our calculations.

Besides, all graphs in this paper are *simple*. This means that there is at most one edge between any two vertices and no vertex can be adjacent to itself.

Definition 2 (adjoined path). Let $G = (V, E)$ be a metric graph, $e_{ij} \in E$, $v \in \{v_i, v_j\}$, $k \in \mathbb{Z}_{\geq 2}$, $p_1, \dots, p_k \in \{1, \dots, |V|\}$, $\{v_{p_1}, \dots, v_{p_k}\} \subseteq V$ such that all v_{p_i} and $v_{p_{i+1}}$ for $i \in \{1, \dots, k-1\}$ are adjacent. A simple path $(v_{p_1}, \dots, v_{p_k})$ in G is (e_{ij}, v) -*adjoined* if:

1. it has v as one of its endpoints, i.e. $v \in \{v_{p_1}, v_{p_k}\}$,
2. it does not go through e_{ij} , i.e. $e_{ij} \notin \{e_{p_1 p_2}, \dots, e_{p_{k-1} p_k}\}$,
3. $(|e_{p_1 p_2}| + \dots + |e_{p_{k-1} p_k}|) / |e_{ij}|$ is irrational.

Adjoined paths may not exist in general, however, we will show later that in our case we will always be able to find them.

Definition 3 (system of dispersing moving points). Let $G = (V, E)$ be a metric graph, $v^* \in V$. Dynamical *system of dispersing moving points* $\mathcal{W}(G, v^*)$ is defined by the following rules:

1. Time and space are continuous, with $\mathbb{R}_{\geq 0}$ and $V \cup (\bigsqcup_{e \in E} e)$ being their respective domains.
2. Each moving point *lies* on some edge $e \in E$ and moves back and forth at unit speed along \bar{e} .
3. Suppose there are two distinct edges $e \in E$ and $e' \in E$ both incident on vertex $v \in V$. If at time moment $t \in \mathbb{R}_{\geq 0}$ some moving point lying on e is located at v and at the same time no moving point lying on e' is at v , then a new moving point gets spawned on edge e' at vertex v . The time moment t will be called the *birth time* of the spawned moving point.

4. At time moment 0 a moving point gets spawned at vertex v^* on any of the edges incident on v^* .

In this definition, symbol ‘ \sqcup ’ stands for disjoint union.

Note also that the choice of a particular edge in rule 4 does not matter, because thanks to rule 3, new moving points will be created on all incident edges anyway.

The position of a moving point spawned at time $\tau \in \mathbb{R}_{\geq 0}$ on edge $e \in E$ at vertex associated with point 0 of $\bar{e} = [0, |e|]$ can be expressed as the following function of time:

$$a_{e,\tau}(t) = 2|e| \left\| \frac{t - \tau}{2|e|} \right\| \quad (1)$$

for $t \in \mathbb{R}_{\geq \tau}$ where $\| \cdot \|$ is the distance to the nearest integer.

Several plots of $a_{e,\tau}$ for some abstract edge e with $|e| = 2$ can be found in Figure 1.

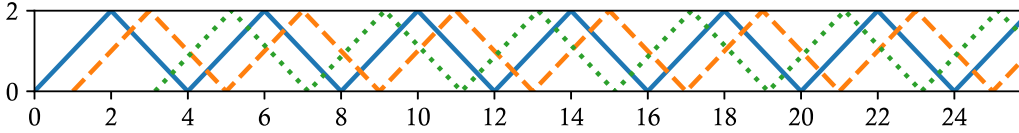


Figure 1: Plots of $a_{e,\tau}(t)$ with $|e| = 2$ and $\tau = 0$ (blue, solid), $\tau = 1$ (orange, dashed), $\tau = \pi$ (green, dotted).

Definition 4 (ε -net). Let (M, ρ) be a metric space, $\varepsilon \in \mathbb{R}_{>0}$. A subset $N \subseteq M$ is called an ε -net on M if $M \subseteq \bigcup_{x \in N} B_\varepsilon(x)$ where $B_\varepsilon(x) = \{m \in M \mid \rho(m, x) < \varepsilon\}$ is the ε -neighbourhood of x .

Every metric graph $G = (V, E)$ forms a generalised metric space together with the generalised metric ρ_G defined for the points of $V \cup (\bigsqcup_{e \in E} e)$. Specifically, $\rho_G(x, y)$ for any $x, y \in V \cup (\bigsqcup_{e \in E} e)$ is the length of the shortest curve connecting x and y , or $+\infty$ if no such curve exists. Obviously, if $x, y \in V$, this is equivalent to the length of the shortest path between x and y . Moreover, ρ_G is a proper metric if and only if G is *connected*, making (G, ρ_G) a metric space in this case. A detailed exploration of metric graphs can be found, e.g. in [13].

Note that from now on the connectedness of metric graphs will also be assumed throughout the text, along with undirectedness and simplicity.

Definition 5 (ε -saturation). Let $G = (V, E)$ be a metric graph, $v^* \in V$, $\varepsilon \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geq 0}$. The system $\mathcal{W}(G, v^*)$ is said to be ε -saturated at time t if its moving points form an ε -net on $V \cup (\bigsqcup_{e \in E} e)$ at time t .

Definition 6 (ε -saturation interval). Let $G = (V, E)$ be a metric graph, $v^* \in V$, $\varepsilon \in \mathbb{R}_{>0}$. A connected non-empty set $T \subseteq \mathbb{R}_{\geq 0}$ is called an ε -saturation interval of $\mathcal{W}(G, v^*)$ if $\mathcal{W}(G, v^*)$ is ε -saturated at any $t \in T$ and if for any $T' \subseteq \mathbb{R}_{\geq 0}$ with $T \subset T'$ there exists $t' \in T'$ such that $\mathcal{W}(G, v^*)$ is not ε -saturated at time moment t' . We call $\inf T$ and $\sup T$ an ε -saturation moment and an ε -desaturation moment of $\mathcal{W}(G, v^*)$ respectively.

Definition 7 (permanent ε -saturation). Let $G = (V, E)$ be a metric graph, $v^* \in V$, $\varepsilon \in \mathbb{R}_{>0}$, T be an ε -saturation interval of $\mathcal{W}(G, v^*)$. We call T the *interval of permanent ε -saturation*, if $\sup T = +\infty$. Likewise, we call $\inf T$ the *moment of permanent ε -saturation*.

2.2. Commensurability and continued fractions

Let us now briefly enumerate some basic notions of number theory that will be used in the following text.

Definition 8 (commensurability). Set $W \subseteq \mathbb{R}$ is called *commensurable* if there exists $x \in W$ such that for every $y \in W$ the ratio $y/x \in \mathbb{Q}$. If there are no such x , W is said to be *incommensurable*. Furthermore, if $W = \{x_1, x_2\}$ contains only two elements, we can say that x_1 is commensurable/incommensurable with x_2 .

To shorten our explanations, we say ‘commensurable/incommensurable graph’ instead of ‘graph with commensurable/incommensurable edge lengths’.

Definition 9 (simple continued fraction). Let $m \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$, $a_0 \in \mathbb{Z}_{\geq 0}$, $a_i \in \mathbb{Z}_{>0}$ for $i \in \{1, \dots, m\}$. A *simple continued fraction* is an expression of the form

$$[a_0; a_1, \dots, a_m] = a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_m}}}.$$

Every positive real number $x \in \mathbb{R}_{>0}$ has the unique simple continued fraction representation $x = [a_0; a_1, \dots, a_m]$ and, furthermore, $m = +\infty$ if and only if x is irrational.

Definition 10 (total convergent). Let $[a_0; a_1, \dots, a_m]$ (m can be $+\infty$) be a simple continued fraction, $n \in \{0, \dots, m\}$. The n^{th} total convergent of $[a_0; a_1, \dots, a_m]$ is $[a_0; a_1, \dots, a_n]$.

The n^{th} total convergent of any simple continued fraction $[a_0; a_1, \dots, a_m]$ can be expressed as p_n/q_n where p_n and q_n are members of the respective sequences $(p)_k$ and $(q)_k$ defined as follows:

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_k = a_k p_{k-1} + p_{k-2} \quad \forall k \in \{0, \dots, m\}, \quad (2)$$

$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_k = a_k q_{k-1} + q_{k-2} \quad \forall k \in \{0, \dots, m\}. \quad (3)$$

These and other facts about continued fractions and their properties, together with all the necessary proofs, can be found in [10]. In particular, the following results presented in this book will be of interest to us.

Theorem 11 ([10]). Let $x = [a_0; a_1, \dots, a_m]$ (m can be $+\infty$), sequences $(p)_k$ and $(q)_k$ be defined for x as in (2) and (3) respectively, $n \in \{-1, \dots, m\}$. Then $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$.

Theorem 12 ([10]). Let $x = [a_0; a_1, \dots, a_m]$ (m can be $+\infty$), $n \in \{0, \dots, m\}$. If n is even, then the n^{th} total convergent of x is strictly less than x and if n is odd, then the n^{th} total convergent of x is strictly greater than x .

Theorem 13 ([10]). Let $x = [a_0; a_1, \dots, a_m]$ (m can be $+\infty$), sequences $(p)_k$ and $(q)_k$ be defined for x as in (2) and (3) respectively, $n \in \{0, \dots, m\}$. Then,

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

2.3. Special sequences and discrepancy

We will require some basic elements of discrepancy theory to prove our main result. We restrict all the definitions and results to one-dimensional case as we do not need anything more general than this. A thorough exploration of discrepancy theory can be found in [11].

Definition 14 (discrepancy). Let $W \subset [a, b] \subset \mathbb{R}$ be a finite set. The discrepancy of W on $[a, b]$ is

$$D(a, b, W) = \sup_{a \leq u \leq v \leq b} \left| \frac{|W \cap (u, v)|}{|W|} - \frac{v - u}{b - a} \right|.$$

The discrepancy shows ‘how far’ the set of points is from the perfect uniform distribution. We also define Kronecker and bouncing Kronecker sequences.

Definition 15 (Kronecker sequence). Let $\alpha, \beta \in \mathbb{R}$. *Kronecker sequence* $(d^{\alpha, \beta})_k$ is defined for $k \in \mathbb{Z}_{\geq 0}$ as

$$d_k^{\alpha, \beta} = \langle \alpha k + \beta \rangle$$

where $\langle \cdot \rangle$ is the fractional part of a number.

Definition 16 (bouncing Kronecker sequence). Let $\alpha, \beta \in \mathbb{R}$. *Bouncing Kronecker sequence* $(b^{\alpha, \beta})_k$ is defined for $k \in \mathbb{Z}_{\geq 0}$ as

$$b_k^{\alpha, \beta} = \|\alpha k + \beta\| = \min\{\langle \alpha k + \beta \rangle, \langle -\alpha k - \beta \rangle\}.$$

From these definitions it follows that Kronecker sequences are completely contained in the range $[0, 1]$ and that all bouncing Kronecker sequences lie in the range $[0, 1/2]$.

The following two theorems are well-known results from discrepancy theory.

Theorem 17 ([11]). *Let $\alpha \in \mathbb{R}$ be irrational, $N \in \mathbb{Z}_{\geq 0}$. Then discrepancy $D(0, 1, \{d_k^{\alpha, 0}\}_{k=0}^N) \rightarrow 0$ as $N \rightarrow +\infty$.*

Theorem 18 (Three Gap Theorem, [12, 15]). *Let $\alpha = [a_0; a_1, a_2, \dots]$ be irrational, $N \in \mathbb{Z}_{\geq 0}$, sequence $(q)_k$ be defined for α as in (3). Let $z \in \mathbb{Z}_{\geq 0}$ be defined so that $q_{z-1} + q_z \leq N < q_z + q_{z+1}$ while $r \in \mathbb{Z}_{\geq 0}$ and $s \in \{0, \dots, q_z - 1\}$ be defined so that $N - q_{z-1} = rq_z + s$ is the Euclidean division of $N - q_{z-1}$ by q_z .*

Then $\{d_k^{\alpha, 0}\}_{k=0}^N$ contains $N + 1$ points that subdivide $[0, 1]$ into gaps with at most three distinct lengths, namely:

- $N + 1 - q_z$ gaps of length $\ell_1 = \langle q_z \alpha \rangle$ if z is even or $\ell_1 = 1 - \langle q_z \alpha \rangle$ if z is odd,
- $s + 1$ gaps of length $\ell_2 = 1 - \langle (N - s)\alpha \rangle$ if z is even or $\ell_2 = \langle (N - s)\alpha \rangle$ if z is odd,
- $q_z - s - 1$ gaps of length $\ell_3 = \ell_1 + \ell_2$.

By a *gap* induced by some finite set $W \subset [a, b] \subset \mathbb{R}$ we understand any maximal (by inclusion) interval (u, v) such that $(u, v) \cap W = \emptyset$.

2.4. Main result

With all the definitions in place, we are ready to present the statement of the main result of this work.

Theorem 19. *Let $G = (V, E)$ be an incommensurable metric graph, $v^* \in V$, $\varepsilon \in \mathbb{R}_{>0}$. Then the following two statements are true for $\mathcal{W}(G, v^*)$:*

1. *There exists the finite moment $t_{\text{perm}}(\varepsilon, G, v^*) \in \mathbb{R}_{\geq 0}$ of permanent ε -saturation.*
2. *$t_{\text{perm}}(\varepsilon, G, v^*)$ (and, hence, all other ε -saturation moments of $\mathcal{W}(G, v^*)$) admits the upper bound:*

$$t_{\text{perm}}(\varepsilon, G, v^*) \leq \max_{e_{ij} \in E} \min\{t_{\text{edge}}(\varepsilon, e_{ij}, v_i), t_{\text{edge}}(\varepsilon, e_{ij}, v_j)\}.$$

Here $t_{\text{edge}}(\varepsilon, e_{ij}, v)$ for $v \in \{v_i, v_j\}$ is defined as

$$t_{\text{edge}}(\varepsilon, e_{ij}, v) = \rho_G(v^*, v) + 2(q_z + q_{z-1})\sigma$$

where σ is the length of the shortest (e_{ij}, v) -adjoined path and q_z is chosen from $(q)_k$ defined for $\sigma/|e_{ij}|$ as in (3) with $z \geq 2$ such that

$$q_z > \frac{2|e_{ij}|}{\varepsilon}.$$

If (e_{ij}, v) -adjoined path does not exist, $t_{\text{edge}}(\varepsilon, e_{ij}, v) = +\infty$.

3. Proof of the main result

The proof of the main result will involve three successive steps. First, we establish a direct connection between the points of $\mathcal{W}(G, v^*)$ and the bouncing Kronecker sequences. Second, we prove the existence of the finite moment of permanent ε -saturation by inducing some properties about the distance between neighbouring moving points from the discrepancy of Kronecker sequences. Third, we prove some lemmata that allow us to deduce the presented upper bound from the Three Gap Theorem. Note that when we use symbols without explicitly defining them, then their definitions from the statement of Theorem 19 apply.

3.1. Connection with the bouncing Kronecker sequences

In the beginning, let us look closely at adjoined paths in G .

Lemma 20. *Let $e_{ij} \in E$. Then there is a vertex $v \in \{v_i, v_j\}$ for which an (e_{ij}, v) -adjoined path exists.*

PROOF. We present a constructive proof for this statement.

We know that G is incommensurable, hence, for e_{ij} there exists another edge $e \in E$ such that $|e|/|e_{ij}|$ is irrational. Since G is connected, there is a simple path p connecting v_i with e and going through e itself, which means that vertices incident on e are the two final vertices of p . This path can only be of length commensurable with $|e_{ij}|$ if it contains another edge of length incommensurable with $|e_{ij}|$. In this case, redefine e with the latter edge and start the process again. Since we only have a finite number of edges, sooner or later we will be able to find e such that there is a path connecting v_i with e that has a cumulative length incommensurable with $|e_{ij}|$.

This path satisfies conditions 1 and 3 from Definition 2. If it does not contain e_{ij} , it is an (e_{ij}, v_i) -adjoined path and the statement of the lemma is true. If it goes through e_{ij} , we can satisfy condition 2 by simply removing v_i from the beginning of p , thus getting an (e_{ij}, v_j) -adjoined path and confirming the statement of the lemma in this case as well. \square

Lemma 20 shows that the value $\min\{t_{\text{edge}}(\varepsilon, e_{ij}, v_i), t_{\text{edge}}(\varepsilon, e_{ij}, v_j)\}$ from the statement of Theorem 19 is always finite for all edges of the graph.

Fix now until the end of the section some edge $e_{ij} \in E$ and vertex $v \in \{v_i, v_j\}$ so that there is an (e_{ij}, v) -adjoined path. Let σ be the length of the shortest (e_{ij}, v) -adjoined path p . Obviously, the first hit of vertex v by a moving point will happen at time $\rho_G(v^*, v)$. After that, the points will begin to move along all edges incident on v , including the one belonging to p . Applying the analogous reasoning to further vertices of p , we deduce that there will be a moving point hitting the last vertex of p at time moment $\rho_G(v^*, v) + \sigma$ and, following the process back, we see that there will be another hit with v at time $\rho_G(v^*, v) + 2\sigma$. Continuing in the same way, we conclude that hits with v will occur in particular at time moments $\rho_G(v^*, v) + 2n\sigma$ for $n \in \mathbb{Z}_{\geq 0}$.

Consider now a moving point lying on the edge e_{ij} that is located at v at time moment $\rho_G(v^*, v) + 2n\sigma$ for some n . Note that from the time moment $\rho_G(v^*, v) + 2n\sigma$ on, this point will behave exactly as if it had been spawned

at vertex v at the time $\rho_G(v^*, v) + 2n\sigma$, even if it was not. Thus, if we ignore the position of this point at the time before $\rho_G(v^*, v) + 2n\sigma$, then the birth time of this moving point can be assumed to be $\tau = \rho_G(v^*, v) + 2n\sigma$ and we can think that it was spawned at v . If we associate v with the endpoint 0 of \bar{e}_{ij} , then using (1) we can derive the following information about the position of this moving point at any time $t = \rho_G(v^*, v) + 2N\sigma + \Delta$ for $N \in \mathbb{Z}_{\geq n}$ and $\Delta \in \mathbb{R}_{\geq 0} \cap \mathbb{R}_{< 2\sigma}$:

$$\begin{aligned} a_{e_{ij}, \tau}(t) &= 2|e_{ij}| \left\| \frac{t - \tau}{2|e_{ij}|} \right\| = 2|e_{ij}| \left\| \frac{\rho_G(v^*, v) + 2N\sigma + \Delta - \rho_G(v^*, v) - 2n\sigma}{2|e_{ij}|} \right\| \\ &= 2|e_{ij}| \left\| (N - n) \frac{\sigma}{|e_{ij}|} + \frac{\Delta}{2|e_{ij}|} \right\| = 2|e_{ij}| b_{N-n}^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}. \end{aligned} \quad (4)$$

Thus, if we fix the time moment $t = \rho_G(v^*, v) + 2N\sigma + \Delta$ and vary n from 0 to N , we get (up to a constant factor $2|e_{ij}|$) exactly $\{b_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N$. Here the sequence parameter $\alpha = \sigma/|e_{ij}| \notin \mathbb{Q}$.

We can think of this set as the positions of a certain subset of moving points on the given edge at the given time. Note that we do not necessarily capture the positions of all the points that could be on e_{ij} at that moment, we only capture the positions of some of them.

3.2. Proof of existence

Let us now show that there exists the finite moment of permanent ε -saturation. To simplify the formulations ahead, we will denote the length of the longest gap induced by some finite set $W \subset [a, b] \subset \mathbb{R}$ on $[a, b]$ as $\ell_{\max}(a, b, W)$. We also mention for further reference that if $[u, v] \subseteq [a, b]$, then $\ell_{\max}(u, v, W \cap [u, v]) \leq \ell_{\max}(a, b, W)$ and if there is another finite set $W' \subset [a, b]$, then $\ell_{\max}(a, b, W \cup W') \leq \ell_{\max}(a, b, W)$.

Lemma 21. *Let $W \subset [a, b] \subset \mathbb{R}$ be a finite set. Then the length of the longest gap induced by it is related to its discrepancy: $\ell_{\max}(a, b, W) \leq (b - a) \cdot D(a, b, W)$.*

PROOF. Let (u', v') be a gap induced by W on $[a, b]$. Then we can show for the length of (u', v') :

$$\begin{aligned} v' - u' &= (b - a) \left| 0 - \frac{v' - u'}{b - a} \right| = (b - a) \left| \frac{|W \cap (u', v')|}{|W|} - \frac{v' - u'}{b - a} \right| \\ &\leq (b - a) \cdot \sup_{a \leq u \leq v \leq b} \left| \frac{|W \cap (u, v)|}{|W|} - \frac{v - u}{b - a} \right| = (b - a) \cdot D(a, b, W). \end{aligned}$$

The statement is proven. \square

Lemma 22. *Let $\alpha, \beta \in \mathbb{R}$, $N \in \mathbb{Z}_{\geq 0}$. Then for the Kronecker sequence $(d^{\alpha, \beta})_k$ it holds that $\ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N) \leq \ell_{\max}(0, 1, \{d_k^{\alpha, 0}\}_{k=0}^N)$.*

PROOF. Although $\{d_k^{\alpha, \beta}\}_{k=0}^N \subset [0, 1]$, the value 1 is actually unreachable for $(d^{\alpha, \beta})_k$, hence, we can also say that $\{d_k^{\alpha, \beta}\}_{k=0}^N \subset [0, 1)$.

We begin then by mapping $\{d_k^{\alpha, \beta}\}_{k=0}^N$ from $[0, 1)$ onto a unit circle using the invertible transformation $x \mapsto 2\pi x$. In general, this transformation merges the leftmost and rightmost gaps on $[0, 1]$, however, when $\beta = 0$, the gaps induced by $\{d_k^{\alpha, 0}\}_{k=0}^N$ on $[0, 1]$ and on the unit circle remain mutually correspondent since point $d_0^{\alpha, 0} = 0$ on the circle doesn't 'cut' any gaps.

Observe that $\{d_k^{\alpha, \beta}\}_{k=0}^N$ can be derived from $\{d_k^{\alpha, 0}\}_{k=0}^N$ by rotating each point by $2\pi\beta$, preserving the gap lengths. However, this rotation may make point 0 'cut' one gap on the circle into two smaller gaps on $[0, 1]$. This will not increase the length of the longest gap on $[0, 1]$ beyond what occurs when $\beta = 0$. \square

Lemma 23. *Let $\alpha, \beta \in \mathbb{R}$, $N \in \mathbb{Z}_{\geq 0}$. Then for bouncing Kronecker sequence $(b^{\alpha, \beta})_k$ it is true that $\ell_{\max}(0, 1/2, \{b_k^{\alpha, \beta}\}_{k=0}^N) \leq \ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N)$.*

PROOF. Due to the definition of bouncing Kronecker sequences (see Definition 16) we can represent $\{b_k^{\alpha, \beta}\}_{k=0}^N$ as $\{b_k^{\alpha, \beta}\}_{k=0}^N = (\{d_k^{\alpha, \beta}\}_{k=0}^N \cap [0, 1/2]) \cup R$ where $R = \{b_k^{\alpha, \beta}\}_{k=0}^N \setminus \{d_k^{\alpha, \beta}\}_{k=0}^N$. Hence,

$$\begin{aligned} \ell_{\max}\left(0, \frac{1}{2}, \{b_k^{\alpha, \beta}\}_{k=0}^N\right) &\leq \ell_{\max}\left(0, \frac{1}{2}, \{d_k^{\alpha, \beta}\}_{k=0}^N \cap \left[0, \frac{1}{2}\right]\right) \\ &\leq \ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N). \end{aligned}$$

This finishes the proof. \square

Combining the Lemmata 21, 22 and 23, we can show for all irrational $\alpha \in \mathbb{R}$, all $\beta \in \mathbb{R}$ and $N \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} 0 < \ell_{\max}\left(0, \frac{1}{2}, \{b_k^{\alpha, \beta}\}_{k=0}^N\right) &\leq \ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N) \leq \ell_{\max}(0, 1, \{d_k^{\alpha, 0}\}_{k=0}^N) \\ &\leq D(0, 1, \{d_k^{\alpha, 0}\}_{k=0}^N) \xrightarrow{N \rightarrow +\infty} 0. \end{aligned}$$

Thus, $\ell_{\max}(0, 1/2, \{b_k^{\alpha, \beta}\}_{k=0}^N) \rightarrow 0$ as $N \rightarrow +\infty$ uniformly over β . Knowing that $\{b_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N$ represents positions of certain moving points on e_{ij} (up to a constant factor) at time moment $t = \rho_G(v^*, v) + 2N\sigma + \Delta$, we conclude that the lengths of gaps induced by all moving points on $\overline{e_{ij}}$ also tend to 0 as t goes to infinity.

This means that after a certain moment in time $\overline{e_{ij}}$ is covered indefinitely by an ε -net made of moving points lying on it. Since we have a finite number of edges, taking the maximum of these time moments proves the existence of the finite moment of permanent ε -saturation.

3.3. Proof of the upper bound

Finally, we show the validity of the claimed upper bound. The general idea is to find an upper bound on the time when each edge gets permanently covered by an ε -net made of moving points lying on it and then take a maximum of these upper bounds over all edges of G .

To do this, we restrict the set of time moments from which we choose our upper bounds. Recall that we can represent all time moments in the form $t = \rho_G(v^*, v) + 2N\sigma + \Delta$. Recall also that if $\Delta = 0$, then there is a moving point at vertex v and the parameter $\beta = \Delta/(2|e_{ij}|)$ of the bouncing Kronecker sequence $(b_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)})_k$, which represents positions of certain moving points on e_{ij} at that time moment, is equal to 0. This means that we can use the Three Gap Theorem to precisely calculate the lengths of gaps induced by $\{d_k^{\sigma/|e_{ij}|, 0}\}_{k=0}^N$ on $[0, 1]$, the maximum of which will be an upper bound on the length of the longest gap induced by $\{b_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N$ on $[0, 1/2]$ for any $\Delta \in [0, 2\sigma)$ (due to Lemma 23).

Define now $(q)_k$ for $\alpha = \sigma/|e_{ij}|$ as in (3). From the Three Gap Theorem we know that gaps of length ℓ_3 exist only if $s < q_z - 1$. Assuming this inequality holds, we can easily see that $\ell_{\max}(0, 1, \{d_k^{\sigma/|e_{ij}|, 0}\}_{k=0}^N) = \ell_3$ and $\ell_{\max}(0, 1, \{d_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N) \leq \ell_3$, again, for all $\Delta \in [0, 2\sigma)$.

Finally, we want to find the time moment $t_{\text{edge}}(\varepsilon, e_{ij}, v) = \rho_G(v^*, v) + 2N'\sigma$ such that for all moments $t > t_{\text{edge}}(\varepsilon, e_{ij}, v)$ edge $\overline{e_{ij}}$ is guaranteed to be covered by an ε -net made of moving points lying on it. To create a condition

for the edge e_{ij} , we multiply everything by $2|e_{ij}|$:

$$\begin{cases} 2|e_{ij}|\ell_3 < \varepsilon \\ s < q_z - 1 \end{cases} \Leftrightarrow \begin{cases} 1 + \langle q_z \alpha \rangle - \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} & , \text{ if } z \text{ is even} \\ 1 - \langle q_z \alpha \rangle + \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} & , \text{ if } z \text{ is odd} \\ s < q_z - 1. \end{cases} \quad (5)$$

We begin with the following lemmata.

Lemma 24. *Let $x = [a_0; a_1, \dots, a_m]$ (m can be $+\infty$), sequences $(p)_k$ and $(q)_k$ be defined for x as in (2) and (3) respectively, $n \in \{0, \dots, m\}$. Then*

$$q_{n-1} \frac{p_n}{q_n} = p_{n-1} + \frac{(-1)^{n+1}}{q_n}.$$

PROOF. This statement can be proven with the following sequence of equalities:

$$q_{n-1} \frac{p_n}{q_n} \stackrel{(*)}{=} \frac{p_{n-1}q_n + (-1)^{n+1}}{q_n} = p_{n-1} + \frac{(-1)^{n+1}}{q_n}.$$

Here equality $(*)$ follows from Theorem 11. \square

Lemma 25. *Let $x = [a_0; a_1, \dots, a_m]$ (m can be $+\infty$), sequences $(p)_k$ and $(q)_k$ be defined for x as in (2) and (3) respectively, $n \in \{2, \dots, m\}$. Then*

$$\left\langle q_{n-1} \frac{p_n}{q_n} \right\rangle \begin{cases} \leq \langle q_{n-1}x \rangle & , \text{ if } n \text{ is even} \\ \geq \langle q_{n-1}x \rangle & , \text{ if } n \text{ is odd.} \end{cases}$$

PROOF. From Theorem 12 we know that

$$q_{n-1} \frac{p_n}{q_n} \begin{cases} \leq q_{n-1}x & , \text{ if } n \text{ is even} \\ \geq q_{n-1}x & , \text{ if } n \text{ is odd.} \end{cases}$$

Note also from (3) and Definition 9 that $q_0 = 1$, $q_0 \leq q_1$ and that $q_i < q_{i+1}$ for all $i \in \mathbb{Z}_{\geq 1}$. Hence, if $n \geq 2$, then $q_n > 1$. Using Lemma 24 and knowing that $n \geq 2$, we can show:

$$\left[q_{n-1} \frac{p_n}{q_n} \right] = \begin{cases} \left[p_{n-1} - \frac{1}{q_n} \right] = p_{n-1} - 1 & , \text{ if } n \text{ is even} \\ \left[p_{n-1} + \frac{1}{q_n} \right] = p_{n-1} & , \text{ if } n \text{ is odd.} \end{cases}$$

Consider the first case of an even n . Because the floor function is monotonic,

$$\lfloor q_{n-1}x \rfloor \geq \left\lfloor q_{n-1} \frac{p_n}{q_n} \right\rfloor = p_{n-1} - 1.$$

At the same time,

$$\begin{aligned} \lfloor q_{n-1}x \rfloor - p_{n-1} &= \lfloor q_{n-1}x - p_{n-1} \rfloor = \left\lfloor q_{n-1} \left(x - \frac{p_{n-1}}{q_{n-1}} \right) \right\rfloor \\ &= - \left\lfloor q_{n-1} \left(\frac{p_{n-1}}{q_{n-1}} - x \right) \right\rfloor \stackrel{(*)}{\leq} - \left\lfloor \frac{q_{n-1}}{q_{n-1}(q_{n-1} + q_n)} \right\rfloor \\ &= - \left\lfloor \frac{1}{q_{n-1} + q_n} \right\rfloor = -1. \end{aligned}$$

The inequality $(*)$ here is true because of Theorem 12 and Theorem 13. From this chain it follows that $\lfloor q_{n-1}x \rfloor \leq p_{n-1} - 1$.

Thus, we have shown that $\lfloor q_{n-1}x \rfloor \leq p_{n-1} - 1$ and $\lfloor q_{n-1}x \rfloor \geq p_{n-1} - 1$, hence, $\lfloor q_{n-1}x \rfloor = p_{n-1} - 1$. Since

$$q_{n-1} \frac{p_n}{q_n} \leq q_{n-1}x$$

yet

$$\left\lfloor q_{n-1} \frac{p_n}{q_n} \right\rfloor = \lfloor q_{n-1}x \rfloor,$$

we conclude that

$$\left\langle q_{n-1} \frac{p_n}{q_n} \right\rangle \leq \langle q_{n-1}x \rangle.$$

For the second case with an odd n , the proof is similar. \square

Now, according to the statement of Theorem 19, set $N' = q_z + q_{z-1}$ with $q_z > 2|e_{ij}|/\varepsilon$. We will show that such N' satisfies (5).

Recall from the Three Gap Theorem that in this case r and s are defined as the Euclidean division of $N' - q_{z-1}$ by q_z : $N' - q_{z-1} = rq_z + s$. Hence, $r = 1$ and $s = 0$.

Note also from the proof of Lemma 25 that $q_0 = 1$ and that $q_0 \leq q_1$ and $q_i < q_{i+1}$ for all $i \in \mathbb{Z}_{\geq 1}$. This means that if $z \geq 2$, then the condition

$s < q_z - 1$ is guaranteed to be satisfied. According to the statement of Theorem 19, $z \geq 2$.

Let us now consider the case of even z and take a closer look at the constraint set for q_z :

$$q_z > \frac{2|e_{ij}|}{\varepsilon} \Leftrightarrow \frac{1}{q_z} < \frac{\varepsilon}{2|e_{ij}|} \Leftrightarrow 1 - \frac{1}{q_z} > 1 - \frac{\varepsilon}{2|e_{ij}|} \stackrel{(*)}{\Leftrightarrow} \left\langle q_{z-1} \frac{p_z}{q_z} \right\rangle > 1 - \frac{\varepsilon}{2|e_{ij}|}.$$

Here the equivalence $(*)$ holds due to Lemma 24. With the help of Lemma 25 we continue:

$$\langle q_{z-1}\alpha \rangle \geq \left\langle q_{z-1} \frac{p_z}{q_z} \right\rangle > 1 - \frac{\varepsilon}{2|e_{ij}|}.$$

From this:

$$\begin{aligned} 1 - \langle q_{z-1}\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} &\Leftrightarrow 1 - \langle (q_z + q_{z-1} - q_z)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\ &\Leftrightarrow 1 - \langle N'\alpha - q_z\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\ &\Leftrightarrow 1 - \langle (N' - s)\alpha \rangle - \langle q_z\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\ &\stackrel{(**)}{\Leftrightarrow} 1 + \langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \end{aligned}$$

which is exactly the condition for even z from (5). The equivalence $(**)$ is true because $1 + \langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle = \ell_3$, hence, it must be in $(0, 1)$, thereby, $\langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle \in (-1, 0)$ and $\langle (N' - s)\alpha \rangle - \langle q_z\alpha \rangle = \langle (N' - s)\alpha \rangle - \langle q_z\alpha \rangle$.

We have shown that for even $z \geq 2$ conditions from Theorem 19 imply the part of (5) relevant to even z , i.e.

$$\begin{cases} q_z > \frac{2|e_{ij}|}{\varepsilon} \\ z \text{ is even} \end{cases} \Rightarrow \begin{cases} 1 + \langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\ s < q_z - 1. \end{cases}$$

The proof for an odd z repeats the same actions.

All in all, both $t_{\text{edge}}(\varepsilon, e_{ij}, v_i)$ and $t_{\text{edge}}(\varepsilon, e_{ij}, v_j)$, given the existence of the respective adjoined paths computed in this way, do indeed provide upper bounds on the time when $\overline{e_{ij}}$ becomes permanently covered by an ε -net made of moving points lying on it. Thus, by choosing the largest upper bound for this event among all the edges of the graph, we obtain the upper bound on the moment of permanent ε -saturation of G .

The theorem is proven.

4. Discussion

The upper bound presented in this paper is, as to our knowledge, the first of its kind. It can be used to estimate the time of permanent ε -saturation for any given system of dispersing moving points (time of permanent saturation for any given system of moving intervals of length 2ε) by mechanically computing several values one after the other, however, our results do not give any insight into the general change of the ε -saturation time depending on the value of ε . In particular, it would be interesting to derive the asymptotic behaviour of the moment of permanent ε -saturation for any fixed $\mathcal{W}(G, v^*)$ as $\varepsilon \rightarrow 0$.

Moreover, we see a possibility of improvement for the presented bound. This improvement can be achieved by thoroughly studying the gap lengths of bouncing Kronecker sequences (instead of classical Kronecker sequences as we have done in this text) and deriving more precise upper bounds or even exact formulae for them. Another way of enhancement is to develop discrepancy theory for more general measurable spaces such as metric graphs. This route may potentially allow us to estimate the moment of permanent ε -saturation with one inequality instead of, de-facto, a system of separate inequalities for all edges.

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