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# Upper bound for saturation time of metric graphs by intervals moving on them

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## Abstract

In this paper we study dynamical systems of intervals moving on incommensurable metric graphs. They can be generally viewed as congruent intervals moving around some graph with unit velocity and propagating on all respective incident edges whenever sliding through any vertex. In particular, we analyse the phenomenon of saturation: a state of the system when the entire graph is covered by these moving intervals. Our main contributions are the following: (1) we prove the existence of the finite moment of permanent saturation for any incommensurable metric graph and any positive length of these intervals; (2) we present an upper bound for the moment of permanent saturation. To show the validity of our results, we reduce the system of moving intervals to the system of dispersing moving points for the analysis of which we primarily use methods of discrepancy theory and number theory, specifically Kronecker sequences and the celebrated Three Gap Theorem.

**Keywords:** metric graph, wave packet, dynamical system of points, random walk, epsilon-net, saturation.  
**MSC2010:** 37E25, 37A45. **MSC2020:** 37E25, 37A44.

## 1 INTRODUCTION

Recent studies of wave packets distribution in hybrid topological spaces and on metric graphs have brought up the problem of identifying the time moment when networks of waveguides get completely covered by waves leaving no idle points in the system [2, 3, 4]. In this setting we typically consider a wave packet of relatively small width initially located in the vicinity of some node of the network that then starts freely travelling around it. We also state that waves travel with no dispersion and propagate on all incident edges when hitting any node of the graph thus creating new waves of the same width.

Particular shapes of waves do not play significant role in this context, thereby, for the analysis of the given problem we replace wave packets with moving intervals of some fixed length that gradually propagate on all respective incident edges as they slide through any vertex of the network creating this way new moving intervals of the same length. This idea allows us to mimic the behaviour of waves yet get rid of all their properties that are unnecessary for tracking the coverage of a metric graph by them.

If we assume the length of these intervals to be  $2\varepsilon$  (the width of the wave packets to be  $2\varepsilon$ ) for some positive  $\varepsilon$ , we can simplify our model even further by presenting each moving interval as the  $\varepsilon$ -neighbourhood of its central point thus leading to the dynamical system of dispersing moving points on metric graphs that can be briefly described as follows.

Moving points move back and forth along the edges of the graph in continuous time with unit velocity. Since we talk about metric graphs, we view lengths of edges as actual distances that these points have to bridge between respective vertices. Whenever any moving point hits any vertex of the graph, it propagates on all incident edges thus spawning new moving points on them.

At the initial time moment one moving point is artificially placed at one of the vertices: it immediately spreads over all incident edges and the points begin their movement. Before the initial time moment, the system is empty (there are no moving points on the graph).

Even though the study of this process is mainly motivated by its physical applications, it is worth mentioning that Izmaylov and Dworzanski mentioned in [10] that it can also be used as an ‘overapproximation’ of message exchange in computer networks.

Previously, the problem of representing the number of moving points in this dynamical system (the number of wave packets in the network) as a function of time was thoroughly investigated [5, 6, 7, 8, 9, 10, 15]. To avoid confusion, we point out that among authors there is no standardised notation: in some works the aforementioned process is interpreted as ‘endpoints of a random walk on metric graph’ (e.g. [7, 15]), in others it is called either a ‘DP-system’ (e.g. [9, 10]) or simply ‘moving points’ (e.g. [5, 6]).

The problem we approach in this paper seemingly belongs to a completely different category yet is, in fact, associated with the problem of counting points. In [4] it was shown that if the edges of the metric graph have commensurable lengths (have rank 1 over  $\mathbb{Q}$ ), then after a while a phenomenon of *stabilisation* occurs by which we understand a situation when the number of points in the system remains constant forever. Hence, given a sufficiently small value of  $\varepsilon$  (sufficiently small width of wave packets), under these circumstances it is possible for the system to never get saturated. On the other hand, if the edges of the graph have incommensurable lengths (have rank greater than 1 over  $\mathbb{Q}$ ), the stabilisation never arrives and the number of moving points grows into infinity as time goes to infinity. As Dworzanski says in [9], ‘... it is possible to extend the notion of stabilization time to systems with incommensurable edges using the notion of  $\varepsilon$ -net...’ In the very same paper he proposes the concept of  $\varepsilon$ -saturation that we develop in this text.

Thus, we can view saturation as an alternative stabilisation characteristic for graphs with incommensurable edges as the original concept does not apply to them.

Indeed, in this paper we demonstrate that  $\varepsilon$ -saturation exists for all finite incommensurable unoriented connected simple metric graphs and all positive  $\varepsilon$ . Besides, we show a general upper bound for the permanent  $\varepsilon$ -saturation time moment of dispersing moving points systems or, equivalently, a general upper bound for the permanent saturation time moment of moving intervals of length  $2\varepsilon$  (wave packets of width  $2\varepsilon$ ).

The rest of the paper is structured as follows. Section 2 contains all necessary definitions and presents our main result. Section 3 features proofs of the main result and all auxiliary statements. Section 4 discusses possible directions of further research.

## 2 DEFINITIONS AND MAIN RESULT

### 2.1 Metric graphs and dispersing points

Let us commence with rigorously defining the dynamical system in focus of this work and all related concepts.

**Definition 1** (metric graph). *A metric graph is a tuple  $(V, E)$  where  $V = \{v_1, \dots, v_n\}$  is a set of vertices and  $E$  is a set of edges with every  $e_{ij} \in E$  being an interval  $(0, |e_{ij}|) \subset \mathbb{R}$  connecting vertices  $v_i \in V$  and  $v_j \in V$  associated with endpoints 0 and  $|e_{ij}|$ , respectively. In this context, a positive number  $|e_{ij}|$  is called the length of  $e_{ij}$ . We also denote  $e_{ij} \cup \{v_i, v_j\} = [0, |e_{ij}|]$  as  $\bar{e}_{ij}$ .*

In this work we only consider *unoriented* graphs meaning that we do not distinguish between  $e_{ij}$  and  $e_{ji}$ . Furthermore, we are not strict about which vertex incident to  $e_{ij}$  is associated with endpoint 0 and which vertex is associated with  $|e_{ij}|$ . In fact, in the further text we will conveniently switch these associations when needed to simplify our calculations.

Besides, all graphs in this paper are *simple*. This means that there is at most one edge between any two vertices and any vertex cannot be adjacent to itself.

**Definition 2** (adjoined path). *Let  $G = (V, E)$  be a metric graph,  $e_{ij} \in E$ ,  $v \in \{v_i, v_j\}$ ,  $k \in \mathbb{Z}_{\geq 2}$ ,  $p_1, \dots, p_k \in \mathbb{Z}_{\geq 1} \cap \mathbb{Z}_{\leq |V|}$ ,  $\{v_{p_1}, \dots, v_{p_k}\} \subseteq V$  such that all  $v_{p_i}$  and  $v_{p_{i+1}}$  for  $i \in \mathbb{Z}_{\geq 1} \cap \mathbb{Z}_{< k}$  are adjacent. A simple path  $(v_{p_1}, \dots, v_{p_k})$  in  $G$  is  $(e_{ij}, v)$ -adjoined if:*

1. *it has  $v$  as one of its endpoints, i.e.  $v \in \{v_{p_1}, v_{p_k}\}$ ,*
2. *it does not go through  $e_{ij}$ , i.e.  $e_{ij} \notin \{e_{p_1 p_2}, \dots, e_{p_{k-1} p_k}\}$ ,*
3.  *$(|e_{p_1 p_2}| + \dots + |e_{p_{k-1} p_k}|) / |e_{ij}|$  is irrational.*

Adjoined paths, in general, may not exist, however, later we will show that in our case we will always be able to find them.

**Definition 3** (dispersing moving points system). Let  $G = (V, E)$  be a metric graph,  $v^* \in V$ . Dynamical system  $\mathcal{W}(G, v^*)$  is defined by the following rules:

1. Time and space are continuous with  $\mathbb{R}_{\geq 0}$  and  $V \cup (\bigsqcup_{e \in E} e)$  being their respective domains.
2. Each moving point lies on some edge  $e \in E$  and moves back and forth with unit velocity along  $\bar{e}$ .
3. Say, there are two distinct edges  $e \in E$  and  $e' \in E$  incident to vertex  $v \in V$ . If at time moment  $t \in \mathbb{R}_{\geq 0}$  some moving point lying on  $e$  is located at  $v$  and, at the same time, no moving point from  $e'$  is located at  $v$ , then a new moving point gets spawned at vertex  $v$  on edge  $e'$ . Time moment  $t$  will be called the birth-time of the latter moving point.
4. At time moment 0 a moving point gets spawned at vertex  $v^*$  on any of the edges incident to  $v^*$ .

In this definition symbol ‘ $\sqcup$ ’ stands for disjoint union.

Note also, that the choice of a particular edge in rule 4 does not play any role because thanks to rule 3 new moving points will be spawned on all incident edges anyway.

Location of a moving point spawned at time moment  $\tau \in \mathbb{R}_{\geq 0}$  on edge  $e \in E$  at vertex associated with point 0 of  $\bar{e} = [0, |e|]$  can be expressed as the following function of time:

$$a_{e,\tau}(t) = 2|e| \left\| \frac{t - \tau}{2|e|} \right\| \quad (1)$$

for  $t \in \mathbb{R}_{\geq \tau}$  where  $\|\cdot\|$  is the distance to the nearest integer.

Several plots of  $a_{e,\tau}$  for some abstract edge  $e$  with  $|e| = 2$  can be found in Figure 1.

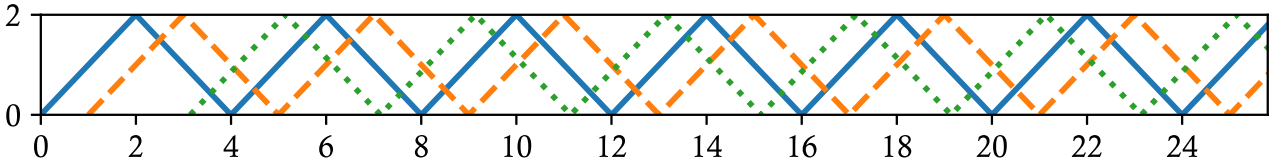


Figure 1: Plots of  $a_{e,\tau}(t)$  with  $|e| = 2$  and  $\tau = 0$  (blue, solid),  $\tau = 1$  (orange, dashed),  $\tau = \pi$  (green, dotted).

**Definition 4** ( $\varepsilon$ -net). Let  $(M, \rho)$  be a metric space,  $\varepsilon \in \mathbb{R}_{> 0}$ . Subset  $N \subseteq M$  is called an  $\varepsilon$ -net on  $M$  if  $M \subseteq \bigcup_{x \in N} B_\varepsilon(x)$  where  $B_\varepsilon(x) = \{m \in M | \rho(m, x) < \varepsilon\}$  is the  $\varepsilon$ -neighbourhood of  $x$ .

Every metric graph  $G = (V, E)$  makes up a generalised metric space together with generalised metric  $\rho_G$  defined for points from  $V \cup (\bigsqcup_{e \in E} e)$ . Precisely,  $\rho_G(x, y)$  for any  $x, y \in V \cup (\bigsqcup_{e \in E} e)$  is the length of the shortest curve connecting  $x$  and  $y$  or  $+\infty$ , if such curve does not exist. Evidently, when  $x, y \in V$  this is equivalent to the length of the shortest path between  $x$  and  $y$ . Besides,  $\rho_G$  is a proper metric if and only if  $G$  is *connected* thus making  $(G, \rho_G)$  a metric space in this case. A detailed exploration of metric graphs can be found, for example, in [14].

Note, that from now on connectivity of metric graphs will also be implied throughout the text alongside with unorientedness and simplicity.

**Definition 5** ( $\varepsilon$ -saturation). Let  $G = (V, E)$  be a metric graph,  $v^* \in V$ ,  $\varepsilon \in \mathbb{R}_{> 0}$ ,  $t \in \mathbb{R}_{\geq 0}$ . System  $\mathcal{W}(G, v^*)$  is said to be  $\varepsilon$ -saturated at time moment  $t$  if its moving points make up an  $\varepsilon$ -net on  $V \cup (\bigsqcup_{e \in E} e)$  at time moment  $t$ .

**Definition 6** ( $\varepsilon$ -saturation interval). Let  $G = (V, E)$  be a metric graph,  $v^* \in V$ ,  $\varepsilon \in \mathbb{R}_{> 0}$ . Connected non-empty set  $T \subseteq \mathbb{R}_{\geq 0}$  is called an  $\varepsilon$ -saturation interval of  $\mathcal{W}(G, v^*)$  if  $\mathcal{W}(G, v^*)$  is  $\varepsilon$ -saturated at any  $t \in T$  and if for any  $T' \subseteq \mathbb{R}_{\geq 0}$  with  $T \subset T'$  there exists  $t' \in T'$  such that  $\mathcal{W}(G, v^*)$  is not  $\varepsilon$ -saturated at time moment  $t'$ . We call  $\inf T$  and  $\sup T$  an  $\varepsilon$ -saturation moment and an  $\varepsilon$ -desaturation moment of  $\mathcal{W}(G, v^*)$ , respectively.

**Definition 7** (permanent  $\varepsilon$ -saturation). Let  $G = (V, E)$  be a metric graph,  $v^* \in V$ ,  $\varepsilon \in \mathbb{R}_{> 0}$ ,  $T$  be an  $\varepsilon$ -saturation interval of  $\mathcal{W}(G, v^*)$ . We call  $T$  the interval of permanent  $\varepsilon$ -saturation, if  $\sup T = +\infty$ . Likewise, we call  $\inf T$  the moment of permanent  $\varepsilon$ -saturation.

## 2.2 Commensurability and continued fractions

Let us now briefly enumerate several basic notions of number theory that are going to be used in the following text.

**Definition 8** (commensurability). *Set  $W \subseteq \mathbb{R}$  is called commensurable if there exists  $x \in W$  so that for every  $y \in W$  there is some  $k \in \mathbb{Q}$  with  $y = kx$ . If such  $x$  does not exist,  $W$  is called incommensurable. Furthermore, if  $W = \{x_1, x_2\}$  only contains two elements, we may also say that  $x_1$  is commensurable/incommensurable with  $x_2$ .*

To shorten our explanations, we say ‘commensurable/incommensurable graph’ instead of ‘graph with commensurable/incommensurable lengths of edges’.

**Definition 9** (simple continued fraction). *Let  $m \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ ,  $a_0 \in \mathbb{Z}_{\geq 0}$ ,  $a_i \in \mathbb{Z}_{>0}$  for  $i \in \mathbb{Z}_{\geq 1} \cap \mathbb{Z}_{\leq m}$ . A simple continued fraction is an expression of a form*

$$[a_0; a_1, \dots, a_m] = a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_m}}}.$$

Every positive real number  $x \in \mathbb{R}_{>0}$  has a unique simple continued fraction representation  $x = [a_0; a_1, \dots, a_m]$  and, moreover,  $m = +\infty$  if and only if  $x$  is irrational.

**Definition 10** (total convergent). *Let  $[a_0; a_1, \dots, a_m]$  ( $m$  can be  $+\infty$ ) be a simple continued fraction,  $n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq m}$ . The  $n^{\text{th}}$  total convergent of  $[a_0; a_1, \dots, a_m]$  is  $[a_0; a_1, \dots, a_n]$ .*

The  $n^{\text{th}}$  total convergent of any simple continued fraction  $[a_0; a_1, \dots, a_m]$  can be expressed as  $p_n/q_n$  where  $p_n$  and  $q_n$  are members of respective sequences  $(p)_k$  and  $(q)_k$  defined as follows:

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_k = a_k p_{k-1} + p_{k-2} \quad \forall k \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq m}, \quad (2)$$

$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_k = a_k q_{k-1} + q_{k-2} \quad \forall k \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq m}. \quad (3)$$

These and other facts about continued fractions and their properties together with all the necessary proofs can be found in [11]. Specifically, the following results presented in this book will be of interest to us.

**Theorem 1** ([11]). *Let  $x = [a_0; a_1, \dots, a_m]$  ( $m$  can be  $+\infty$ ),  $n \in \mathbb{Z}_{\geq -1} \cap \mathbb{Z}_{\leq m}$ , sequences  $(p)_k$  and  $(q)_k$  be defined for  $x$  as in (2) and (3), respectively. Then,  $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ .*

**Theorem 2** ([11]). *Let  $x = [a_0; a_1, \dots, a_m]$  ( $m$  can be  $+\infty$ ),  $n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq m}$ . If  $n$  is even, then the  $n^{\text{th}}$  total convergent of  $x$  is strictly less than  $x$  and if  $n$  is odd, then the  $n^{\text{th}}$  total convergent of  $x$  is strictly greater than  $x$ .*

**Theorem 3** ([11]). *Let  $x = [a_0; a_1, \dots, a_m]$  ( $m$  can be  $+\infty$ ),  $n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq m}$ , sequences  $(p)_k$  and  $(q)_k$  be defined for  $x$  as in (2) and (3), respectively. Then,*

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

## 2.3 Special sequences and discrepancy

We will require some basic elements of discrepancy theory to prove our main result. We restrict all the definitions and results to one-dimensional case as we do not need anything more general than this. A thorough exploration of discrepancy theory can be found in [12].

**Definition 11** (discrepancy). *Let  $W \subset [a, b] \subset \mathbb{R}$  be a finite set. The discrepancy of  $W$  on  $[a, b]$  is*

$$D(a, b, W) = \sup_{a \leq u \leq v \leq b} \left| \frac{|W \cap (u, v)|}{|W|} - \frac{v - u}{b - a} \right|.$$

Discrepancy shows ‘how far’ the set of points is from the perfect uniform distribution. We also define Kronecker and bouncing Kronecker sequences.

**Definition 12** (Kronecker sequences). Let  $\alpha, \beta \in \mathbb{R}$ . Kronecker sequence  $(d^{\alpha, \beta})_k$  is defined for  $k \in \mathbb{Z}_{\geq 0}$  as

$$d_k^{\alpha, \beta} = \langle \alpha k + \beta \rangle$$

where  $\langle \cdot \rangle$  is the fractional part of a number.

**Definition 13** (bouncing Kronecker sequences). Let  $\alpha, \beta \in \mathbb{R}$ . Bouncing Kronecker sequence  $(b^{\alpha, \beta})_k$  is defined for  $k \in \mathbb{Z}_{\geq 0}$  as

$$b_k^{\alpha, \beta} = \|\alpha k + \beta\| = \min\{\langle \alpha k + \beta \rangle, \langle -\alpha k - \beta \rangle\}.$$

From these definitions it is evident that Kronecker sequences are fully contained in range  $[0, 1]$  and all bouncing Kronecker sequences lie in the range  $[0, 1/2]$ .

The following two theorems are celebrated results from discrepancy theory.

**Theorem 4** ([12]). Let  $\alpha \in \mathbb{R}$  be irrational,  $N \in \mathbb{Z}_{\geq 0}$ . Then,  $D(0, 1, \{d_k^{\alpha, 0}\}_{k=0}^N) \rightarrow 0$  as  $N \rightarrow +\infty$ .

**Theorem 5** (Three Gap Theorem, [13, 16]). Let  $\alpha = [a_0; a_1, a_2, \dots]$  be irrational,  $N \in \mathbb{Z}_{\geq 0}$ , sequence  $(q)_k$  be defined for  $\alpha$  as in (3). Let  $z \in \mathbb{Z}_{\geq 0}$  be defined so that  $q_{z-1} + q_z \leq N < q_z + q_{z+1}$  while  $r \in \mathbb{Z}_{\geq 0}$  and  $s \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{< q_z}$  be defined so that  $N - q_{z-1} = r q_z + s$  is the Euclidean division of  $N - q_{z-1}$  by  $q_z$ . Then,  $\{d_k^{\alpha, 0}\}_{k=0}^N$  contains  $N + 1$  points that subdivide  $[0, 1]$  into gaps with at most three distinct lengths, namely:

- $N + 1 - q_z$  gaps of length  $\ell_1 = \langle q_z \alpha \rangle$  if  $z$  is even or  $\ell_1 = 1 - \langle q_z \alpha \rangle$  if  $z$  is odd,
- $s + 1$  gaps of length  $\ell_2 = 1 - \langle (N - s)\alpha \rangle$  if  $z$  is even or  $\ell_2 = \langle (N - s)\alpha \rangle$  if  $z$  is odd,
- $q_z - s - 1$  gaps of length  $\ell_3 = \ell_1 + \ell_2$ .

By a *gap* we understand an interval between two neighbouring points of a set contained within some  $[a, b] \subset \mathbb{R}$  or between a point of a set and a border point of  $[a, b]$  if such an interval does not contain other points of this set.

## 2.4 Main result

With all the definitions being given, we are finally ready to present the statement of the main result of this work.

**Theorem 6** (Upper bound for  $\varepsilon$ -saturation moments). Let  $G = (V, E)$  be an incommensurable metric graph,  $v^* \in V$ ,  $\varepsilon \in \mathbb{R}_{> 0}$ . Then, the following two statements are true for  $\mathcal{W}(G, v^*)$ :

1. There exists the finite moment of permanent  $\varepsilon$ -saturation  $t_{perm}(\varepsilon, G, v^*) \in \mathbb{R}_{\geq 0}$ .
2.  $t_{perm}(\varepsilon, G, v^*)$  (and, hence, all other  $\varepsilon$ -saturation moments of  $\mathcal{W}(G, v^*)$ ) admits the upper bound:

$$t_{perm}(\varepsilon, G, v^*) \leq \max_{e_{ij} \in E} \min\{t_{edge}(\varepsilon, e_{ij}, v_i), t_{edge}(\varepsilon, e_{ij}, v_j)\}.$$

Here  $t_{edge}(\varepsilon, e_{ij}, v)$  for  $v \in \{v_i, v_j\}$  is defined as

$$t_{edge}(\varepsilon, e_{ij}, v) = \rho_G(v^*, v) + 2(q_z + q_{z-1})\sigma$$

where  $\sigma$  is the length of the shortest  $(e_{ij}, v)$ -adjoined path and  $q_z$  is chosen from  $(q)_k$  defined for  $\sigma/|e_{ij}|$  as in (3) with  $z \geq 2$  such that

$$q_z > \frac{2|e_{ij}|}{\varepsilon}.$$

If  $(e_{ij}, v)$ -adjoined path does not exist,  $t_{edge}(\varepsilon, e_{ij}, v) = +\infty$ .

### 3 PROOF OF MAIN RESULT

The proof of the main result will involve three consecutive steps. First, we establish a direct connection between  $\mathcal{W}(G, v^*)$  and bouncing Kronecker sequences. Second, we prove the existence of the finite moment of permanent  $\varepsilon$ -saturation by inducing some properties about the distance between neighbouring moving points from the discrepancy of Kronecker sequences. Third, we prove several lemmata that will enable us to deduce the presented upper bound from the Three Gap Theorem. Note, that if in this section we use symbols without explicitly defining them, then definitions from the statement of Theorem 6 apply.

#### 3.1 Connection with bouncing Kronecker sequences

In the beginning, let us look closely at adjoined paths in  $G$ .

**Lemma 1.** *Let  $e_{ij} \in E$ . Then, there is an incident vertex  $v \in \{v_i, v_j\}$  for which an  $(e_{ij}, v)$ -adjoined path exists.*

*Proof.* We present a constructive proof for this statement.

We know that  $G$  is incommensurable, hence, for  $e_{ij}$  there exists another edge  $e \in E$  such that  $|e|/|e_{ij}|$  is irrational. Since  $G$  is connected, there is a simple path  $p$  connecting  $v_i$  with  $e$  and going through  $e$  itself meaning that vertices incident to  $e$  are two final vertices of  $p$ . This path can only have length commensurable with  $|e_{ij}|$  if it includes another edge of length incommensurable with  $|e_{ij}|$ . In this case, redefine  $e$  with the latter edge and start the process over. As we only have a finite number of edges, sooner or later we will be able to find  $e$  such that there is a path connecting  $v_i$  with  $e$  that has a cumulative length incommensurable with  $|e_{ij}|$ . This path satisfies requirements 1 and 3 from Definition 2. If it does not contain  $e_{ij}$ , this is an  $(e_{ij}, v_i)$ -adjoined path and the statement of the lemma is true.

If it goes through  $e_{ij}$ , though, we can proceed as follows. Since we consider a simple path, vertices in it cannot be repeated, thereby, the only way for  $p$  to go through  $e_{ij}$  is to have  $v_j$  as its second vertex. By cutting off  $v_i$  from the beginning of the path we get an  $(e_{ij}, v_j)$ -adjoined path and confirm the statement of the lemma in this case as well.  $\square$

**Lemma 2.** *Let  $e_{ij} \in E$ ,  $v \in \{v_i, v_j\}$  such that there exists an  $(e_{ij}, v)$ -adjoined path. Then, the shortest  $(e_{ij}, v)$ -adjoined path only contains one edge of length incommensurable with  $|e_{ij}|$ .*

*Proof.* This can be easily proven by contradiction.

Assume the shortest  $(e_{ij}, v)$ -adjoined path  $(v_{p_1}, \dots, v_{p_k})$  with  $p_1, \dots, p_k$  meeting all the requirements from Definition 2 to contain multiple edges of length incommensurable with  $|e_{ij}|$ . If we cut off the part of this path following the first edge  $e_{p_{k'-1}p_{k'}}$  for some  $k' \in \mathbb{Z}_{\geq 1} \cap \mathbb{Z}_{< k}$  with length incommensurable with  $|e_{ij}|$ , then the ratio of its cumulative length to  $|e_{ij}|$  will be:

$$\frac{|e_{p_1 p_2}| + \dots + |e_{p_{k'-2} p_{k'-1}}| + |e_{p_{k'-1} p_{k'}}|}{|e_{ij}|} = \underbrace{\frac{|e_{p_1 p_2}|}{|e_{ij}|} + \dots + \frac{|e_{p_{k'-2} p_{k'-1}}|}{|e_{ij}|}}_{\in \mathbb{Q}} + \underbrace{\frac{|e_{p_{k'-1} p_{k'}}|}{|e_{ij}|}}_{\notin \mathbb{Q}} \notin \mathbb{Q}.$$

Hence, this path is also an  $(e_{ij}, v)$ -adjoined path and, furthermore, it is shorter than  $(v_{p_1}, \dots, v_{p_k})$  which contradicts the initial assumption. Thus, the shortest  $(e_{ij}, v)$ -adjoined path can only contain one edge of length incommensurable with  $|e_{ij}|$ .  $\square$

Lemma 1 confirms that the value  $\min\{t_{\text{edge}}(\varepsilon, e_{ij}, v_i), t_{\text{edge}}(\varepsilon, e_{ij}, v_j)\}$  from the statement of Theorem 6 is always finite for all edges of the graph while Lemma 2 provides an insight into the structure of the shortest adjoined paths.

Fix now till the end of the entire section some edge  $e_{ij} \in E$  and vertex  $v \in \{v_i, v_j\}$  such that there is an  $(e_{ij}, v)$ -adjoined path. Let  $\sigma$  be the length of the shortest  $(e_{ij}, v)$ -adjoined path  $p$ . Evidently, the first hit of vertex  $v$  by a moving point will happen at time moment  $\rho_G(v^*, v)$ . After this, points will start moving along all edges incident to  $v$  including the one belonging to  $p$ . By applying the analogous reasoning to further vertices of  $p$ , we deduce that there will be a moving point hitting the last vertex of  $p$  at time moment  $\rho_G(v^*, v) + \sigma$  and, if we track the process back, we see that there will be another hit with  $v$  at time moment  $\rho_G(v^*, v) + 2\sigma$ . By continuing in the same fashion, we conclude that hits with  $v$  will occur, in particular, at time moments  $\rho_G(v^*, v) + 2n\sigma$  for  $n \in \mathbb{Z}_{\geq 0}$ .

Consider now a moving point lying on edge  $e_{ij}$  that is located at  $v$  at some time moment of a form  $\rho_G(v^*, v) + 2n\sigma$ . Such a point, due to the previous paragraph, will always exist. Note, that starting from time moment  $\rho_G(v^*, v) + 2n\sigma$  this point will behave exactly as if it was spawned at vertex  $v$  at time moment  $\rho_G(v^*, v) + 2n\sigma$  even if it was not. Thus, if we restrict ourselves from considering the position of this point at time before  $\rho_G(v^*, v) + 2n\sigma$ , then the birth-time of this moving point can be assumed to be  $\tau = \rho_G(v^*, v) + 2n\sigma$  and we can think that it was spawned at  $v$ . If we associate  $v$  with endpoint 0 of  $\overline{e_{ij}}$ , using (1) we will be able to deduce the following information about the position of this moving point at time moment  $t = \rho_G(v^*, v) + 2N\sigma + \Delta$  for  $N \in \mathbb{Z}_{\geq n}$  and  $\Delta \in \mathbb{R}_{\geq 0} \cap \mathbb{R}_{< 2\sigma}$ :

$$\begin{aligned} a_{e_{ij}, \tau}(t) &= 2|e_{ij}| \left\| \frac{t - \tau}{2|e_{ij}|} \right\| = 2|e_{ij}| \left\| \frac{\rho_G(v^*, v) + 2N\sigma + \Delta - \rho_G(v^*, v) - 2n\sigma}{2|e_{ij}|} \right\| \\ &= 2|e_{ij}| \left\| (N - n) \frac{\sigma}{|e_{ij}|} + \frac{\Delta}{2|e_{ij}|} \right\| = 2|e_{ij}| b_{N-n}^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}. \end{aligned} \quad (4)$$

Thus, if we fix time moment  $t = \rho_G(v^*, v) + 2N\sigma + \Delta$  and vary  $n$  from 0 to  $N$ , we get (up to a constant factor  $2|e_{ij}|$ ) exactly  $\{b_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N$ . Here, sequence parameter  $\alpha = \sigma/|e_{ij}| \notin \mathbb{Q}$ .

We can think of this set as of positions of a certain subset of moving points on the given edge at the given moment in time. Pay attention on the fact that we do not necessarily capture positions of all points that might be on  $e_{ij}$  at this moment in time, we only capture positions of some of them.

### 3.2 Proof of existence

Let us now show that there exists a finite moment of permanent  $\varepsilon$ -saturation. To simplify formulations ahead, we will denote the length of the longest gap induced by some finite set  $W \subset [a, b] \subset \mathbb{R}$  on  $[a, b]$  as  $\ell_{\max}(a, b, W)$ . We also mention for further reference that if  $[u, v] \subseteq [a, b]$ , then  $\ell_{\max}(u, v, W \cap [u, v]) \leq \ell_{\max}(a, b, W)$  and if there is another finite set  $W' \subset [a, b]$ , then  $\ell_{\max}(a, b, W \cup W') \leq \ell_{\max}(a, b, W)$ .

**Lemma 3.** *Let  $W \subset [a, b] \subset \mathbb{R}$  be a finite set. Then,  $\ell_{\max}(a, b, W) \leq (b - a)D(a, b, W)$ .*

*Proof.* Let  $(u', v')$  be a gap induced by  $W$  on  $[a, b]$ . Then, we can show for the length of  $(u', v')$ :

$$\begin{aligned} v' - u' &= (b - a) \left| 0 - \frac{v' - u'}{b - a} \right| = (b - a) \left| \frac{|W \cap (u', v')|}{|W|} - \frac{v' - u'}{b - a} \right| \\ &\leq (b - a) \cdot \sup_{a \leq u \leq v \leq b} \left| \frac{|W \cap (u, v)|}{|W|} - \frac{v - u}{b - a} \right| = (b - a)D(a, b, W). \end{aligned}$$

The statement is proven.  $\square$

**Lemma 4.** *Let  $\alpha, \beta \in \mathbb{R}$ ,  $N \in \mathbb{Z}_{\geq 0}$ . Then,  $\ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N) \leq \ell_{\max}(0, 1, \{d_k^{\alpha, 0}\}_{k=0}^N)$ .*

*Proof.* Even though  $\{d_k^{\alpha, \beta}\}_{k=0}^N \subset [0, 1]$ , value 1 is actually unreachable for  $(d_k^{\alpha, \beta})_k$ , hence, we can also say that  $\{d_k^{\alpha, \beta}\}_{k=0}^N \subset [0, 1)$ .

We start by mapping  $\{d_k^{\alpha, \beta}\}_{k=0}^N$  from  $[0, 1)$  onto a unit circle using invertible transformation  $x \mapsto 2\pi x$ . Note, that under this transformation the leftmost and the rightmost gaps on  $[0, 1)$  get merged breaking the mutual correspondence between gaps induced by  $\{d_k^{\alpha, \beta}\}_{k=0}^N$  on these two domains. However, in case  $\beta = 0$  we can see that  $d_0^{\alpha, 0} = 0$  making the leftmost gap on  $[0, 1)$  degenerate and, thus, preserving the mutual correspondence in this particular situation as none of the gaps gets ‘cut’ by point 0 on the circle.

Next, we notice that  $\{d_k^{\alpha, \beta}\}_{k=0}^N$  can be obtained from  $\{d_k^{\alpha, 0}\}_{k=0}^N$  by rotating each point from  $\{d_k^{\alpha, 0}\}_{k=0}^N$  by  $2\pi\beta$ . Rotation preserves distances, however, after this rotation point 0 can actually ‘cut’ one of the gaps on the circle into two gaps on  $[0, 1)$  of smaller lengths but lengths of all other gaps will remain the same. Thus, the length of the longest gap on  $[0, 1)$  cannot get bigger than in case  $\beta = 0$ .  $\square$

**Lemma 5.** *Let  $\alpha, \beta \in \mathbb{R}$ ,  $N \in \mathbb{Z}_{\geq 0}$ . Then,  $\ell_{\max}(0, 1/2, \{b_k^{\alpha, \beta}\}_{k=0}^N) \leq \ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N)$ .*

*Proof.* Due to the definition of bouncing Kronecker sequences (see Definition 13) we can represent  $\{b_k^{\alpha, \beta}\}_{k=0}^N$  as  $\{b_k^{\alpha, \beta}\}_{k=0}^N = (\{d_k^{\alpha, \beta}\}_{k=0}^N \cap [0, 1/2]) \cup R$  where  $R = \{b_k^{\alpha, \beta}\}_{k=0}^N \setminus \{d_k^{\alpha, \beta}\}_{k=0}^N$ . Hence,

$$\ell_{\max}\left(0, \frac{1}{2}, \{b_k^{\alpha, \beta}\}_{k=0}^N\right) \leq \ell_{\max}\left(0, \frac{1}{2}, \{d_k^{\alpha, \beta}\}_{k=0}^N \cap \left[0, \frac{1}{2}\right]\right) \leq \ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N).$$

This finishes the proof.  $\square$



Putting together Lemmata 3, 4, and 5, for all irrational  $\alpha \in \mathbb{R}$ , all  $\beta \in \mathbb{R}$ , and  $N \in \mathbb{Z}_{\geq 0}$  we can show:

$$\begin{aligned} 0 < \ell_{\max} \left( 0, \frac{1}{2}, \{b_k^{\alpha, \beta}\}_{k=0}^N \right) &\leq \ell_{\max}(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N) \leq \ell_{\max}(0, 1, \{d_k^{\alpha, 0}\}_{k=0}^N) \\ &\leq D(0, 1, \{d_k^{\alpha, \beta}\}_{k=0}^N) \xrightarrow{N \rightarrow +\infty} 0. \end{aligned}$$

Thus,  $\ell_{\max}(0, 1/2, \{b_k^{\alpha, \beta}\}_{k=0}^N) \rightarrow 0$  as  $N \rightarrow +\infty$  uniformly over  $\beta$ . Knowing that  $\{b_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N$  represents positions of certain moving points on  $e_{ij}$  (up to a constant factor) at time moment  $t = \rho_G(v^*, v) + 2N\sigma + \Delta$ , we conclude that lengths of gaps induced by all moving points on  $\overline{e_{ij}}$  also tend to 0 as  $t$  goes to infinity.

This means that after a certain moment in time  $\overline{e_{ij}}$  becomes indefinitely covered by an  $\varepsilon$ -net made of moving points lying on it. As we have a finite number of edges, by taking the maximum of these time moments the existence of the finite moment of permanent  $\varepsilon$ -saturation is proven.

### 3.3 Proof of upper bound

Finally, we show the validity of the claimed upper bound. The general idea is to find an upper bound for the time moment when each edge gets permanently covered by an  $\varepsilon$ -net made of moving points lying on it and then take a maximum of these upper bounds over all edges of  $G$ .

To accomplish this, we restrict the set of time moments we choose our upper bounds from. Recall that we can represent all time moments in the form  $t = \rho_G(v^*, v) + 2N\sigma + \Delta$ . Recall also that if  $\Delta = 0$ , then there is a moving point at vertex  $v$  and parameter  $\beta = \Delta/(2|e_{ij}|)$  of bouncing Kronecker sequence  $(b^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)})_k$  representing positions of certain moving points on  $e_{ij}$  at this time moment equals 0. This means that we can use the Three Gap Theorem to precisely calculate the lengths of gaps induced by  $\{d_k^{\sigma/|e_{ij}|, 0}\}_{k=0}^N$  on  $[0, 1]$  the maximum of which will be an upper bound for the length of the longest gap induced by  $\{b_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N$  on  $[0, 1/2]$  for any  $\Delta \in \mathbb{R}_{\geq 0} \cap \mathbb{R}_{< 2\sigma}$  (due to Lemma 5).

Define now  $(q)_k$  for  $\alpha = \sigma/|e_{ij}|$  as in (3). From the Three Gap Theorem we know that gaps of length  $\ell_3$  only exist if  $s < q_z - 1$ . Given this, we easily see that  $\ell_{\max}(0, 1, \{d_k^{\sigma/|e_{ij}|, \Delta/(2|e_{ij}|)}\}_{k=0}^N) \leq \ell_3$ , once again, for all  $\Delta \in \mathbb{R}_{\geq 0} \cap \mathbb{R}_{< 2\sigma}$ .

In the end, we would like to find time moment  $t_{\text{edge}}(\varepsilon, e_{ij}, v) = \rho_G(v^*, v) + 2N'\sigma$  such that for all time moments  $t > t'$  edge  $\overline{e_{ij}}$  is definitely covered by an  $\varepsilon$ -net made of moving points lying on it. To make up a condition for edge  $e_{ij}$  we multiply everything by  $2|e_{ij}|$ :

$$\begin{cases} 2|e_{ij}|\ell_3 < \varepsilon \\ s < q_z - 1 \end{cases} \Leftrightarrow \begin{cases} 1 + \langle q_z \alpha \rangle - \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} & , \text{ if } z \text{ is even} \\ 1 - \langle q_z \alpha \rangle + \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} & , \text{ if } z \text{ is odd} \\ s < q_z - 1. \end{cases} \quad (5)$$

We start with the following lemmata.

**Lemma 6.** Let  $x = [a_0; a_1, \dots, a_m]$  ( $m$  can be  $+\infty$ ), sequences  $(p)_k$  and  $(q)_k$  be defined for  $x$  as in (2) and (3), respectively,  $n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq m}$ . Then,

$$q_{n-1} \frac{p_n}{q_n} = p_{n-1} + \frac{(-1)^{n+1}}{q_n}.$$

*Proof.* This statement can be proven with the following sequence of equalities:

$$q_{n-1} \frac{p_n}{q_n} \stackrel{(*)}{=} \frac{p_{n-1}q_n + (-1)^{n+1}}{q_n} = p_{n-1} + \frac{(-1)^{n+1}}{q_n}.$$

Here, equality  $(*)$  follows from Theorem 1. □

**Lemma 7.** Let  $x = [a_0; a_1, \dots, a_m]$  ( $m$  can be  $+\infty$ ), sequences  $(p)_k$  and  $(q)_k$  be defined for  $x$  as in (2) and (3), respectively,  $n \in \mathbb{Z}_{\geq 2} \cap \mathbb{Z}_{\leq m}$ . Then,

$$\left\langle q_{n-1} \frac{p_n}{q_n} \right\rangle \begin{cases} \leq \langle q_{n-1} x \rangle & , \text{ if } n \text{ is even} \\ \geq \langle q_{n-1} x \rangle & , \text{ if } n \text{ is odd.} \end{cases}$$

*Proof.* From Theorem 2 we know that

$$q_{n-1} \frac{p_n}{q_n} \begin{cases} \leq q_{n-1}x & , \text{ if } n \text{ is even} \\ \geq q_{n-1}x & , \text{ if } n \text{ is odd.} \end{cases}$$

Notice also from (3) and Definition 9 that  $q_0 = 1$  and that  $q_0 \leq q_1$  and  $q_i < q_{i+1}$  for all  $i \in \mathbb{Z}_{\geq 1}$ . Hence, if  $n \geq 2$ , then  $q_n > 1$ . With the help of Lemma 6 and knowing that  $n \geq 2$  we can show:

$$\left\lfloor q_{n-1} \frac{p_n}{q_n} \right\rfloor = \begin{cases} \left\lfloor p_{n-1} - \frac{1}{q_n} \right\rfloor = p_{n-1} - 1 & , \text{ if } n \text{ is even} \\ \left\lfloor p_{n-1} + \frac{1}{q_n} \right\rfloor = p_{n-1} & , \text{ if } n \text{ is odd.} \end{cases}$$

Consider the first case when  $n$  is even. Since the floor function is monotonous,

$$\lfloor q_{n-1}x \rfloor \geq \left\lfloor q_{n-1} \frac{p_n}{q_n} \right\rfloor = p_{n-1} - 1.$$

At the same time,

$$\begin{aligned} \lfloor q_{n-1}x \rfloor - p_{n-1} &= \lfloor q_{n-1}x - p_{n-1} \rfloor = \left\lfloor q_{n-1} \left( x - \frac{p_{n-1}}{q_{n-1}} \right) \right\rfloor = - \left\lceil q_{n-1} \left( \frac{p_{n-1}}{q_{n-1}} - x \right) \right\rceil \\ &\stackrel{(*)}{\leq} - \left\lceil \frac{q_{n-1}}{q_{n-1}(q_{n-1} + q_n)} \right\rceil = - \left\lceil \frac{1}{q_{n-1} + q_n} \right\rceil = -1. \end{aligned}$$

Inequality (\*) here is true due to Theorem 2 and Theorem 3. From this chain it follows that  $\lfloor q_{n-1}x \rfloor \leq p_{n-1} - 1$ . Thus, we have shown that  $\lfloor q_{n-1}x \rfloor \leq p_{n-1} - 1$  and  $\lfloor q_{n-1}x \rfloor \geq p_{n-1} - 1$ , hence,  $\lfloor q_{n-1}x \rfloor = p_{n-1} - 1$ . Since

$$q_{n-1} \frac{p_n}{q_n} \leq q_{n-1}x$$

yet

$$\left\lfloor q_{n-1} \frac{p_n}{q_n} \right\rfloor = \lfloor q_{n-1}x \rfloor,$$

we conclude that

$$\left\langle q_{n-1} \frac{p_n}{q_n} \right\rangle \leq \langle q_{n-1}x \rangle.$$

For the second case with odd  $n$  the proof is alike. □

Set now, in accordance with the statement of Theorem 6,  $N' = q_z + q_{z-1}$  with  $q_z > 2|e_{ij}|/\varepsilon$ . We will show that such  $N'$  satisfies (5).

Recall from the Three Gap Theorem that  $r$  and  $s$  will be defined in this case as the Euclidean division of  $N' - q_{z-1}$  by  $q_z$ :  $N' - q_{z-1} = rq_z + s$ . Hence,  $r = 1$  and  $s = 0$ .

Recall also from the proof of Lemma 7 that  $q_0 = 1$  and that  $q_0 \leq q_1$  and  $q_i < q_{i+1}$  for all  $i \in \mathbb{Z}_{\geq 1}$ . This means that if  $z \geq 2$ , then condition  $s < q_z - 1$  is guaranteed to be satisfied. According to the statement of Theorem 6,  $z \geq 2$ .

Let us now consider the case of even  $z$  and take a closer look at the constraint set for  $q_z$ :

$$q_z > \frac{2|e_{ij}|}{\varepsilon} \Leftrightarrow \frac{1}{q_z} < \frac{\varepsilon}{2|e_{ij}|} \Leftrightarrow 1 - \frac{1}{q_z} > 1 - \frac{\varepsilon}{2|e_{ij}|} \stackrel{(*)}{\Leftrightarrow} \left\langle q_{z-1} \frac{p_z}{q_z} \right\rangle > 1 - \frac{\varepsilon}{2|e_{ij}|}.$$

Here, equivalence (\*) holds due to Lemma 6. With the help of Lemma 7 we continue:

$$\langle q_{z-1}\alpha \rangle \geq \left\langle q_{z-1} \frac{p_z}{q_z} \right\rangle > 1 - \frac{\varepsilon}{2|e_{ij}|}.$$

From this:

$$\begin{aligned}
1 - \langle q_{z-1}\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} &\Leftrightarrow 1 - \langle (q_z + q_{z-1} - q_z)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\
&\Leftrightarrow 1 - \langle N'\alpha - q_z\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\
&\Leftrightarrow 1 - \langle (N' - s)\alpha \rangle - \langle q_z\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\
&\stackrel{(**)}{\Leftrightarrow} 1 + \langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|}
\end{aligned}$$

which is exactly the condition for even  $z$  from (5). Equivalence (\*\*) is true because  $1 + \langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle = \ell_3$ , hence, it must not be greater than 1, thereby,  $\langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle \in (-1, 0)$  and  $\langle (N' - s)\alpha \rangle - \langle q_z\alpha \rangle = \langle (N' - s)\alpha \rangle - \langle q_z\alpha \rangle$ .

We have shown that for even  $z \geq 2$  conditions from Theorem 6 imply the part of (5) relevant to even  $z$ , i.e.

$$\left\{ \begin{array}{l} q_z > \frac{2|e_{ij}|}{\varepsilon} \\ z \text{ is even} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 1 + \langle q_z\alpha \rangle - \langle (N' - s)\alpha \rangle < \frac{\varepsilon}{2|e_{ij}|} \\ s < q_z - 1. \end{array} \right.$$

The proof for odd  $z$  repeats the same actions.

All in all, both  $t_{\text{edge}}(\varepsilon, e_{ij}, v_i)$  and  $t_{\text{edge}}(\varepsilon, e_{ij}, v_j)$ , given the existence of the respective adjoined paths, calculated in this fashion indeed provide upper bounds for the moment in time when  $\overline{e_{ij}}$  gets permanently covered by an  $\varepsilon$ -net made of moving points lying on it. Thus, by choosing the greatest upper bound for this event among all edges of the graph we get the upper bound for the moment of permanent  $\varepsilon$ -saturation of  $G$ .

The theorem is proven.

## 4 DISCUSSION

The upper bound presented in this paper, as to our knowledge, is the first of a kind. It can be used to estimate the time of permanent  $\varepsilon$ -saturation for any particular dispersing moving points system (time of permanent saturation for any particular system of moving intervals of length  $2\varepsilon$ ) by mechanically computing several values one after another, however, our results do not provide any insight on the general change of  $\varepsilon$ -saturation time depending on the value of  $\varepsilon$ . In particular, it would be interesting to derive the asymptotic behaviour of the moment of permanent  $\varepsilon$ -saturation for any fixed  $\mathcal{W}(G, v^*)$  as  $\varepsilon \rightarrow 0$ .

Besides, we see a possibility of improvement for the presented bound. This improvement can be achieved by thoroughly studying lengths of gaps of bouncing Kronecker sequences (instead of classical Kronecker sequences as we did in this text) and deriving more precise upper bounds or even exact formulae for them. Another direction of enhancement corresponds to the development of discrepancy theory for more general measurable spaces like metric graphs. This path can potentially allow us to estimate the moment of permanent  $\varepsilon$ -saturation with one inequality instead of, de-facto, a system of separate inequalities for all edges.

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## CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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