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Optimal stopping: Bermudan strategies meet non-linear evaluations

Miryana Grigorova Marie-Claire Quenez Peng Yuan

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Abstract:

We address an optimal stopping problem over the set of Bermudan-type strategies Θ (which we understand in a more general sense than the stopping strategies for Bermudan options in finance) and with non-linear operators (non-linear evaluations) assessing the rewards, under general assumptions on the non-linear operators ρ. We provide a characterization of the value family V in terms of what we call the (Θ, ρ)-Snell envelope of the pay-off family. We establish a Dynamic Programming Principle. We provide an optimality criterion in terms of a (Θ, ρ)-martingale property of V on a stochastic interval. We investigate the (Θ, ρ)-martingale structure and we show that the "first time" when the value family coincides with the pay-off family is optimal. The reasoning simplifies in the case where there is a finite number n of pre-described stopping times, where n does not depend on the scenario ω. We provide examples of non-linear operators entering our framework.

Keywords: optimal stopping, Bermudan stopping strategy, non-linear operator, non-linear evaluation, g-expectation, dynamic risk measure, dynamic programming principle, non-linear Snell envelope family, Bermudan strategy

1 Introduction

In the recent years, optimal stopping problems with non-linear evaluations have gained an increasing interest in the financial mathematics literature and in the stochastic control literature.

In the *linear* case, a classical reference are the notes by El Karoui (1981). A presentation based on families of random variables indexed by stopping times can be found in Quenez and Kobylanski (2012). In discrete time, a *non-linear* optimal stopping with dynamic monetary utilities was studied in Krätschmer and Schoenmakers (2010), and with g-evaluations (induced by Backward SDEs with Lipschitz driver g) – in Grigorova and Quenez (2016). In continuous time, a non-linear optimal stopping with dynamic convex risk measures was studied in Bayraktar et al.

(2010); with the so-called F-expectations –in Bayraktar and Yao-Part I (2011) and Bayraktar and Yao-Part II (2011); with g-evaluations–in e.g., Quenez and Sulem (2014), Grigorova et al. (2015), Grigorova et al. (2020), Klimsiak and Rzymowski (2021); with a focus on applications to American options in complete or incomplete non-linear financial markets – in Kim et al. (2021), Grigorova et al. (2021); with suprema of linear/affine operators over sets of measures- in, e.g. Ekren et al. (2014), Nutz and Zhang (2015).

In the present paper, we address an optimal stopping problem with *Bermudan-type* strategies and with general non-linear operators (non-linear evaluations) assessing the rewards.

Our purpose is two-fold:

1. We consider a modelling framework which is in-between the discrete-time and the continuous-time framework, by focusing on what we call in this paper the Bermudan-type stopping strategies [1].

   – In the discrete-time framework with finite terminal horizon $T > 0$, the agent is allowed to stop at a **finite number** only of pre-described **determinstic** times, and gain/loss processes are indexed by these pre-described determinsitic times. If we denote by $\{0 = t_0 \leqslant t_1 \leqslant ... \leqslant t_n = T\}$ the pre-defined finite deterministic grid of $n+1$ time points, the stopping strategies of the agent are of the form $\tau = \sum_{k=0}^{n} t_k \mathbf{1}_{A_k}$, where $(A_k)_{k\in\{0,1,...n\}}$ is a partition, such that A_k is $\mathcal{F}_{t_k}$-measurable, for each $k \in \{0, 1, \ldots, n\}$. Thus, for almost each scenario ω, the agent is allowed to stop only at a finite number of times (provided they do that in a non-anticipative way), where both the number of time instants (here, $n+1$) and the time instants themselves (here, the t_k's), are the same, whatever the scenario ω.

   – In the continuous-time framework (with finite horizon $T > 0$), the agent is allowed to stop continuously at any time instant $t \in [0, T]$, and the gain/loss processes are indexed by $t \in [0, T]$. The set of the agent's stopping strategies is the set of all stopping times valued in $[0, T]$. Thus, for almost each scenario ω, the agent can stop at any time instant (provided they do that in a non-anticipative way).

   – In the *intermediate* modelling framework of the current paper (with finite terminal horizon $T > 0$), in (almost) every scenario ω, the agent is allowed to stop at a finite number of times or infinite countable number of times (provided they do that in a non-anticipative way), where both the number of time instants and the time instants themselves are allowed to depend on

[1] Note that this notion is more general than the typical notion of Bermudan strategy appearing in the context of Bermudan options in mathematical finance, where the agent/buyer of the option is allowed to stop only at a pre-described **finite** number of **deterministic** times $\{0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T\}$, where n is a fixed finite number (**independent** of the scenario ω).

the scenario ω. More specifically, we are given a non-decreasing sequence of stopping times $(\theta_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\theta_k = T$. This countable stopping grid being given, the agent's stopping strategies τ are thus of the form $\tau = \sum_{k=0}^{+\infty}\theta_k \mathbf{1}_{A_k} + T\mathbf{1}_{\bar{A}}$, where $\{(A_k)_{k\in\mathbb{N}}, \bar{A}\}$ is a partition, such that A_k is $\mathcal{F}_{\theta_k}$-measurable, for each $k \in \mathbb{N}$, and $\bar{A}$ is $\mathcal{F}_T$-measurable. We call τ of this form a Bermudan stopping strategy, and we denote by Θ the set of Bermudan stopping strategies. The gain/losses are then "naturally" defined via families of random variables indexed by the stopping times τ of this form.

This modelling framework is thus closer to the real-life situations where the number of possible decision points depends on the scenario/state of nature, and so do the decision times themselves, but where the agents do not necessarily act continuously in time.

2. The second purpose of the paper is to allow for gains/losses being assessed by general non-linear evaluations $\rho = (\rho_{S,\tau}[\cdot])$, while imposing minimal assumptions on the non-linear operators, under which the results hold.

We note also that, in the above framework, working with **families of random variables** $\phi = (\phi(\tau))$ indexed by Bermudan stopping times τ, allows for an exposition in which it is not necessary to invoke any results from the theory of stochastic processes.
After formulating the non-linear optimal stopping with Bermudan-style strategies, we provide a characterization of the value family in terms of a suitably defined non-linear Snell envelope. A dynamic programming principle is established under suitable assumptions on the non-linear evaluations. An optimality criterion is proven and existence of optimal stopping times is investigated; it is shown in particular that the first "hitting time" is optimal. Examples of non-linear operators, well-known in financial mathematics and in stochastic control, entering our framework, are given, such as the non-linear evaluations induced by Backward SDEs, the non-linear F-expectations introduced by Bayraktar and Yao, as well as the dynamic concave utilities (from the risk measurement literature). In the appendix, we consider the particular case of finite number of pre-described stopping times $0 = \theta_0, \theta_1, ..., \theta_n = T$, where n does not depend on the scenario ω, and provide an explicit construction of the non-linear Snell envelope by backward induction, as well as a simpler proof of the optimality of the first hitting time.

The remainder of the paper is organized as follows: In Subsection 2.1, we set the framework and the notation. In Subsection 2.2, we formulate the optimisation problem. In Subsection 2.3, we characterize the value family of the problem in terms of the (Θ, ρ)-Snell envelope family of the pay-off family. In Subsection 2.4, we show a Dynamic Programming Principle. In Subsection 2.5, we investigate the question of optimal stopping times: we provide some technical lemmas regarding the (Θ, ρ)-martingale property, we provide an optimality criterion, as well as some useful consequences of the DPP, and we show that, under suitable assumptions,

"the first time" ν_k when the value family hits the pay-off family is optimal. Section 3 is dedicated to three examples of non-linear operators from the literature, entering our framework. The Appendix is dedicated to the particular case where in almost each scenario $\omega \in \Omega$, there are $n+1$ pre-described opportunities for stopping, where n does **not** depend on $\omega \in \Omega$. In this case, we have an explicit construction of the (Θ, ρ)-Snell envelope by backward induction and a simplified proof of the optimality of ν_k (requiring "less continuity" on ρ).

2 Optimal stopping with non-linear evaluations and Bermudan strategies

2.1 The framework

Let $T > 0$ be a **fixed finite** terminal horizon.
Let $(\Omega, \mathcal{F}, P)$ be a (complete) probability space equipped with a right-continuous complete filtration $\mathbb{F} = \{\mathcal{F}_t \colon t \in [0,T]\}$.
In the sequel, equalities and inequalities between random variables are to be understood in the P-almost sure sense. Equalities between measurable sets are to be understood in the P-almost sure sense.
Let $\mathbb{N}$ be the set of natural numbers, including 0. Let $\mathbb{N}^*$ be the set of natural numbers, excluding 0. Let $(\theta_k)_{k\in\mathbb{N}}$ be a sequence of stopping times satisfying the following properties:

(a) The sequence $(\theta_k)_{k\in\mathbb{N}}$ is non-decreasing, i.e. for all $k \in \mathbb{N}$, $\theta_k \leqslant \theta_{k+1}$, a.s.

(b) $\lim_{k\to\infty} \uparrow \theta_k = T$ a.s.

Moreover, we set $\theta_0 = 0$.
We note that the family of σ-algebras $(\mathcal{F}_{\theta_k})_{k\in\mathbb{N}}$ is non- decreasing (as the sequence (θ_k) is non-decreasing). We denote by Θ the set of stopping times τ of the form

$$\tau = \sum_{k=0}^{+\infty} \theta_k \mathbf{1}_{A_k} + T\mathbf{1}_{\bar{A}}, \tag{1}$$

where $\{(A_k)_{k=0}^{+\infty}, \bar{A}\}$ form a partition of Ω such that, for each $k \in \mathbb{N}$, $A_k \in \mathcal{F}_{\theta_k}$, and $\bar{A} \in \mathcal{F}_T$.
The set Θ can also be described as the set of stopping times τ such that for almost all $\omega \in \Omega$, either $\tau(\omega) = T$ or $\tau(\omega) = \theta_k(\omega)$, for some $k = k(\omega) \in \mathbb{N}$.
Note that the set Θ is closed under concatenation, that is, for each $\tau \in \Theta$ and each $A \in \mathcal{F}_\tau$, the stopping time $\tau\mathbf{1}_A + T\mathbf{1}_{A^c} \in \Theta$. More generally, for each $\tau \in \Theta$, $\tau' \in \Theta$ and each $A \in \mathcal{F}_{\tau\wedge\tau'}$, the stopping time $\tau\mathbf{1}_A + \tau'\mathbf{1}_{A^c}$ is in Θ. The set Θ is also closed under pairwise minimization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \wedge \tau' \in \Theta$) and under pairwise maximization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \vee \tau' \in \Theta$). Moreover, for each non-decreasing (resp. non-increasing)

sequence of stopping times $(\tau_n)_{n\in\mathbb{N}} \in \Theta^{\mathrm{N}}$, we have $\lim_{n\to+\infty} \tau_n \in \Theta$.
We note also that all stopping times in Θ are bounded from above by T.

Remark 2.1. *We have the following* canonical *writing of the sets in* (1)*:*

$$\begin{aligned} A_0 &= \{\tau = \theta_0\}; \\ A_{n+1} &= \{\tau = \theta_{n+1}, \theta_{n+1} < T\} \backslash (A_n \cup ... \cup A_0); \text{ for all } n \in \mathbb{N}^* \\ \bar{A} &= (\cup_{k=0}^{+\infty} A_k)^c \end{aligned}$$

From this writing, we have: if $\omega \in A_{k+1} \cap \{\theta_k < T\}$*, then* $\omega \notin \{\tau = \theta_k\}$.

For each $\tau \in \Theta$, we denote by Θ_τ the set of stopping times $\nu \in \Theta$ such that $\nu \geqslant \tau$ a.s. The set Θ_τ satisfies the same properties as the set Θ. We will refer to the set Θ as the set of **Bermudan stopping strategies**, and to the set Θ_τ as the set of Bermudan stopping strategies, greater than or equal to τ (or the set of Bermudan stopping strategies from time τ perspective).

Definition 2.1. *We say that a family* $\phi = (\phi(\tau),\ \tau \in \Theta)$ *is* admissible *if it satisfies the following conditions*

1. for all $\tau \in \Theta$*,* $\phi(\tau)$ *is a real valued random variable, which is* $\mathcal{F}_\tau$*-measurable.*

2. for all $\tau, \tau' \in \Theta$*,* $\phi(\tau) = \phi(\tau')$ *a.s. on* $\{\tau = \tau'\}$.

Moreover, for $p \in [1, +\infty]$ *fixed, we say that an admissible family* ϕ *is* p*-integrable, if for all* $\tau \in \Theta$*,* $\phi(\tau)$ *is in* L^p*.*

Let $\phi = (\phi(\tau),\ \tau \in \Theta)$ be an admissible family. For a stopping time τ of the form (1), we have

$$\phi(\tau) = \sum_{k=0}^{+\infty} \phi(\theta_k)\mathbf{1}_{A_k} + \phi(T)\mathbf{1}_{\bar{A}} \quad \text{a.s.} \tag{2}$$

Given two admissible families $\phi = (\phi(\tau),\ \tau \in \Theta)$ and $\phi' = (\phi'(\tau),\ \tau \in \Theta)$, we say that ϕ is *equal to* ϕ' and write $\phi = \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) = \phi'(\tau)$ a.s. We say that ϕ *dominates* ϕ' and write $\phi \geqslant \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) \geqslant \phi'(\tau)$ a.s.
The following remark is worth noting, as a consequence of the admissibility.

Remark 2.2. *Let* $\phi = (\phi(\tau),\ \tau \in \Theta)$ *be an admissible family. Let* $\tau \in \Theta$ *and let* $(\tau_n) \in \Theta^{\mathbb{N}}$ *be such that for (almost) each* $\omega \in \Omega$*, there exists* $n_0 = n_0(\omega)$ *(depending on* ω*) satisfying, for all* $n \geqslant n_0(\omega)$*,* $\tau_n(\omega) = \tau(\omega)$*. Then, for all* $n \geqslant n_0(\omega)$*,* $\phi(\tau_n)(\omega) = \phi(\tau)(\omega)$.
We show this by the following reasoning: for each fixed $n \in \mathbb{N}$*, let* $\mathcal{C}_n := \{\tau_n = \tau\}$*. For each fixed* $m \in \mathbb{N}$*, let* $\mathcal{A}_m := \cap_{n\geqslant m}\mathcal{C}_n = \cap_{n\geqslant m}\{\tau_n = \tau\}$*. Note that the set* $\mathcal{A}_m$ *might be empty. We have* $\cup_{m\in\mathbb{N}}\mathcal{A}_m = \Omega$*. Moreover, by the admissibility of* ϕ*, we have, for each fixed* $n \in \mathbb{N}$*,* $\phi(\tau_n) = \phi(\tau)$*, on* $\mathcal{C}_n = \{\tau_n = \tau\}$*. Hence, for each fixed* $m \in \mathbb{N}$*,*

$$\text{for all } n \geqslant m, \phi(\tau_n) = \phi(\tau) \text{ on } \mathcal{A}_m = \cap_{n\geqslant m}\mathcal{C}_n. \tag{3}$$

Let $\omega \in \Omega$*. By assumption, there exists* $n_0 = n_0(\omega)$ *such that* $\omega \in \mathcal{A}_{n_0}$*. By property* (3) *(applied with* $m = n_0$*), for all* $n \geqslant n_0$*,* $\phi(\tau_n)(\omega) = \phi(\tau)(\omega)$*, which is the desired conclusion.*

2.2 The optimisation problem

Let $p \in [1, +\infty]$ be fixed.
Let $\xi = (\xi(\tau),\, \tau \in \Theta)$ be **p-integrable admissible family** modelling an agent's dynamic financial position.

Remark 2.3. *For example, the family ξ can be defined via a given progressive process $(\xi_t)_{t\in[0,T]}$, corresponding to a given dynamic financial position process. For each $\tau \in \Theta$, we set $\xi(\tau) := \xi_\tau$. The family of random variables $\xi = (\xi(\tau),\, \tau \in \Theta)$ can be shown to be admissible. If for each $k \in \mathrm{N}$, $\xi_{\theta_k} \in L^p$, and $\xi_T \in L^p$, then the admissible family ξ is p-integrable. The financial interpretation of this example is as follows: the agent can choose his/her strategy only among the stopping times in Θ, that is, among the stopping times which, for almost each ω, have values in the finite grid $\{0, \theta_1(\omega), \ldots, \theta_{n(\omega)}(\omega) = T\}$, where $n(\omega)$ depends on ω, or in the infinite countable grid $\{0, \theta_1(\omega), \ldots, \theta_n(\omega), \theta_{n+1}(\omega), \ldots, T\}$. In this example, the financial position which is actually taken into account in the problem corresponds to the values of the process (ξ_t) only at times $0, \theta_1, ..., \theta_n, \theta_{n+1}, ..., T$.*

The *minimal risk at time* 0 over all Bermudan stopping strategies is defined by:

$$\tilde{V}(0) := \inf_{\tau\in\Theta} \tilde{\rho}_{0,\tau}(\xi(\tau)) = -V(0), \tag{4}$$

where

$$V(0) := \sup_{\tau\in\Theta} \rho_{0,\tau}[\xi(\tau)], \tag{5}$$

and where $\rho_{0,\tau}[\cdot] = -\tilde{\rho}_{0,\tau}[\cdot]$.

Let $p \in [1, +\infty]$. We introduce the following properties on the non-linear operators $\rho_{S,\tau}[\cdot]$, which will appear in the sequel.

For $S \in \Theta$, $S' \in \Theta$, $\tau \in \Theta$, for η, η_1 and η_2 in $L^p(\mathcal{F}_\tau)$, for $\xi = (\xi(\tau))$ an admissible p-integrable family:

(i) $\rho_{S,\tau} : L^p(\mathcal{F}_\tau) \longrightarrow L^p(\mathcal{F}_S)$
(ii) *(admissibility)* $\rho_{S,\tau}[\eta] = \rho_{S',\tau}[\eta]$ a.s. on $\{S = S'\}$.
(iii) *(knowledge preservation)* $\rho_{\tau,S}[\eta] = \eta$, for all $\eta \in L^p(\mathcal{F}_S)$, all $\tau \in \Theta_S$.
(iv) *(monotonicity)* $\rho_{S,\tau}[\eta_1] \leqslant \rho_{S,\tau}[\eta_2]$ a.s., if $\eta_1 \leqslant \eta_2$ a.s.
(v) *(consistency)* $\rho_{S,\theta}[\rho_{\theta,\tau}[\eta]] = \rho_{S,\tau}[\eta]$, for all S, θ, τ in Θ such that $S \leqslant \theta \leqslant \tau$ a.s.
(vi) *("generalized zero-one law")* $I_A\rho_{S,\tau}[\xi(\tau)] = I_A\rho_{S,\tau'}[\xi(\tau')]$, for all $A \in \mathcal{F}_S$, $\tau \in \Theta_S$, $\tau' \in \Theta_S$ such that $\tau = \tau'$ on A.
(vii) *(monotone Fatou property with respect to terminal condition)* $\rho_{S,\tau}[\eta] \leqslant \liminf_{n\to+\infty} \rho_{S,\tau}[\eta_n]$, for $(\eta_n), \eta$ such that (η_n) is non-decreasing, $\eta_n \in L^p(\mathcal{F}_\tau)$, $\sup_n \eta_n \in L^p$, and $\lim_{n\to+\infty} \uparrow \eta_n = \eta$ a.s.

Fatou property is often assumed in the literature on risk measures (particularly in the case where $p = +\infty$).

Note also that if ρ satisfies monotonocity (iv) and monotone Fatou property with respect to terminal condition (vii), then $\rho_{S,\tau}[\eta] = \lim_{n\to+\infty} \rho_{S,\tau}[\eta_n]$, for $(\eta_n), \eta$ such that (η_n) is non-decreasing, $\eta_n \in L^p(\mathcal{F}_\tau)$, $\sup_n \eta_n \in L^p$, and $\lim_{n\to+\infty} \uparrow \eta_n = \eta$ a.s. Indeed, by monotonicity of $\rho_{S,\tau}[\cdot]$, we have $\rho_{S,\tau}[\eta_n] \leqslant \rho_{S,\tau}[\eta]$; hence, $\limsup_{n\to+\infty} \rho_{S,\tau}[\eta_n] \leqslant \rho_{S,\tau}[\eta]$. On the other hand, by (vii), $\rho_{S,\tau}[\eta] \leqslant \liminf_{n\to+\infty} \rho_{S,\tau}[\eta_n]$. Hence, $\rho_{S,\tau}[\eta] = \lim_{n\to+\infty} \rho_{S,\tau}[\eta_n]$. Such type of property is also known in the literature (e.g. risk measures) as continuity from below.
Let us emphasize that no assumptions of convexity (or concavity) or translation invariance of the non-linear operators ρ are made.

2.3 (Θ, ρ)-Snell envelope family and optimal stopping

As is usual in optimal control, we embed the above optimization problem (5) in a larger class of problems by considering for each $\nu \in \Theta$, the random variable $V(\nu)$, where

$$V(\nu) := \text{ess sup}_{\tau\in\Theta_\nu} \rho_{\nu,\tau}[\xi(\tau)]. \tag{6}$$

We note that, if ρ satisfies the property of knowledge preservation (property (iii)), then $V(T) = \rho_{T,T}[\xi(T)] = \xi(T)$.

Lemma 2.1. *(Admissibility of V) Under the assumption of admissibility (ii) and "generalized zero-one law" (vi) on the non-linear operators, the family of random variables $V := (V(\nu), \nu \in \Theta)$ defined in* (6) *is* admissible *in the sense of Definition 2.1.*

The proof uses arguments similar to those of Lemma 8.1 in Grigorova et al. (2020), combined with some properties of the non-linear operators ρ.

Proof. Property 1. of the definition of admissibility follows from the definition of the essential supremum, the random variables of the family $(\rho_{\nu,\tau}[\xi(\tau)], \tau \in \Theta_\nu)$ being $\mathcal{F}_\nu$-measurable.
Let us prove Property 2. Let ν and ν' be two stopping times in Θ. We set $A := \{\nu = \nu'\}$ and we show that $V(\nu) = V(\nu')$, P-a.s. on A. We have

$$\mathbb{I}_A V(\nu) = \mathbb{I}_A \text{ess sup}_{\tau\in\Theta_\nu} \rho_{\nu,\tau}[\xi(\tau)] = \text{ess sup}_{\tau\in\Theta_\nu} \mathbb{I}_A \rho_{\nu,\tau}[\xi(\tau)] = \text{ess sup}_{\tau\in\Theta_\nu} \mathbb{I}_A \rho_{\nu',\tau}[\xi(\tau)], \tag{7}$$

where we have used the admissibility property on ρ for the last equality.
Let $\tau \in \Theta_\nu$. We set $\tau_A := \tau\mathbb{I}_A + T\mathbb{I}_{A^c}$. We note that $\tau_A \in \Theta_{\nu'}$ and $\tau_A = \tau$ p.s. on A. Using this, the admissibility of the family ξ, and the generalized zero-one law property of ρ, we get $\mathbb{I}_A \rho_{\nu',\tau}[\xi(\tau)] = \mathbb{I}_A \rho_{\nu',\tau_A}[\xi(\tau_A)] \leqslant \mathbb{I}_A V(\nu')$. As $\tau \in \Theta_\nu$ is arbitrary, we conclude that ess $\sup_{\tau\in\Theta_\nu} \mathbb{I}_A \rho_{\nu',\tau}[\xi(\tau)] \leqslant \mathbb{I}_A V(\nu')$. Combining this inequality with (7) gives $\mathbb{I}_A V(\nu) \leqslant \mathbb{I}_A V(\nu')$. We obtain the converse inequality by interchanging the roles of ν and ν'. □

Under the assumptions of the above lemma, the following remark holds true.

Remark 2.4. *As a consequence of the* ***admissibility*** *of the value family V, we have: for each $k \in \mathbb{N}$, it holds $V(\nu) = V(\theta_k)$ a.s. on $\{\nu = \theta_k\}$ and $V(\nu) = V(T)$ a.s. on $\{\nu = T\}$. Hence, under the assumptions of Lemma 2.1, for $\nu \in \Theta$ of the form $\nu = \sum_{k=0}^{+\infty} \theta_k \mathbf{1}_{A_k} + T\mathbf{1}_{\bar{A}}$, we have $V(\nu) = \sum_{k=0}^{+\infty} V(\theta_k)\mathbf{1}_{A_k} + V(T)\mathbf{1}_{\bar{A}}$.*

Remark 2.5. *1. Under the assumption of knowledge preservation (iii) on ρ, we have $V(\theta_k) \geqslant \xi(\theta_k)$, for each $k \in \mathbb{N}$.*
Indeed, $V(\theta_k) = ess\ sup_{\tau \in \Theta_{\theta_k}} \rho_{\theta_k,\tau}[\xi(\tau)] \geqslant \rho_{\theta_k,\theta_k}[\xi(\theta_k)]$, and by the property (iii) of the non-linear operators, we have $\rho_{\theta_k,\theta_k}[\xi(\theta_k)] = \xi(\theta_k)$. Hence, $V(\theta_k) \geqslant \xi(\theta_k)$.

2. If, moreover, ρ satisfies the properties of admissibility (ii) and "generalized" zero-one law (vi), then, for each $\tau \in \Theta$, $V(\tau) \geqslant \xi(\tau)$.
This follows from the first statement of the remark, and from the admissibility of ξ and that of V (cf. Lemma 2.1 and Remark 2.4).

Now, let us introduce the notion of (Θ, ρ)-*(super)martingale family.*

Definition 2.2. *Let $\phi = (\phi(\tau),\ \tau \in \Theta)$ be a p-*integrable admissible *family. We say that ϕ is a* (Θ, ρ)-supermartingale *(resp.* (Θ, ρ)–martingale*)* family *if for all σ, τ in Θ such that $\sigma \leqslant \tau$ a.s., we have*

$$\rho_{\sigma,\tau}[\phi(\tau)] \leqslant \phi(\sigma) \ \textit{(resp.} \ = \phi(\sigma)) \ \textit{a.s.}$$

We introduce the following integrability assumption on V, which is assumed in the sequel.

Assumption 2.1. *For each $\nu \in \Theta$, the random variable $V(\nu)$ is in L^p.*

Remark 2.6. *Let ρ satisfy the assumptions of admissibility (ii), knowledge preservation (iii), "generalized" zero-one law (vi), and monotonicity (iv). If the pay off family $\xi = (\xi(\tau))_{\tau \in \Theta}$ is p- integrable and dominated from above by a p-integrable (Θ, ρ)-martingale M, then the value family V satisfies the integrability Assumption 2.1.*
Indeed, let $S \in \Theta$ be given. By Remark 2.5, Statement 2, $V(S) \geqslant \xi(S)$.
On the other hand, by assumption on ξ, for each $\tau \in \Theta_S$, $\xi(\tau) \leqslant M(\tau)$. Hence, by monotonicity of ρ, we have $\rho_{S,\tau}[\xi(\tau)] \leqslant \rho_{S,\tau}[M(\tau)] = M(S)$, where we have used the (Θ, ρ)-martingale property of M for the last equality.
So, $V(S) = ess\ sup_{\tau \in \Theta_S} \rho_{S,\tau}[\xi(\tau)] \leqslant M(S)$. Hence, we get $\xi(S) \leqslant V(S) \leqslant M(S)$, which proves that $V(S) \in L^p$. Therefore, Assumption 2.1 is satisfied.

We will see in Section 3, concrete examples for which this integrability assumption on V is satisfied.

Theorem 2.1. *1.* ((Θ, ρ)-supermartingale) *Under the assumptions of admissibility (ii), consistency (v), "generalized zero-one law" (vi) and monotone Fatou property with respect to the terminal condition (vii) on the non-linear operators, the value family V is a (Θ, ρ)-supermartingale family.*

2. ((Θ,ρ)-Snell envelope) *If moreover the non-linear operators also satisfy the properties of knowledge preservation (iii) and monotonicity (iv), the value family V is equal to the (Θ,ρ)-Snell envelope of the family ξ, that is, the smallest (Θ,ρ)-supermartingale family dominating the family $\xi=(\xi(\tau),\,\tau\in\Theta)$.*

To prove this theorem, we first state a useful lemma.

Lemma 2.2. *(Maxmizing sequence lemma) Under the assumption of "generalized zero-one law" (vi) on the non-linear operators, there exists a maximizing sequence for the value $V(S)$ of problem* (6)*.*

The proof of this lemma is similar to that of Lemma 2.3 in Grigorova et al. (2020) and is given for the convenience of the reader.

Proof. It is sufficient to show that the family $(\rho_{S,\tau}[\xi(\tau)])_{\tau\in\Theta_S}$ is stable under pairwise maximization. The result then follows by a well-known property of the essential supremum. Let $\tau\in\Theta_S$ and $\tau'\in\Theta_S$. Set $A:=\{\rho_{S,\tau'}[\xi(\tau')]\leqslant\rho_{S,\tau}[\xi(\tau)]\}$ and $\nu:=\tau\mathbb{I}_A+\tau'\mathbb{I}_{A^c}$. Trivially, $A\in\mathcal{F}_S$. Moreover, $\nu\in\Theta_S$ (cf. properties of the set Θ_S). Also, $\nu=\tau$ on A, $\nu=\tau'$ on A^c. By the "generalized zero-one law" of the non-linear operators ρ, we get

$$\begin{aligned}\rho_{S,\nu}[\xi(\nu)]=\rho_{S,\nu}[\xi(\nu)]\mathbb{I}_A+\rho_{S,\nu}[\xi(\nu)]\mathbb{I}_{A^c}&=\rho_{S,\tau}[\xi(\tau)]\mathbb{I}_A+\rho_{S,\tau'}[\xi(\tau')]\mathbb{I}_{A^c}\\&=\max\big(\rho_{S,\tau}[\xi(\tau)],\rho_{S,\tau'}[\xi(\tau')]\big).\end{aligned}\tag{8}$$

This shows the stability under pairwise maximization of the value family (indexed by Θ_S). □

Let us now show the theorem. The idea of the proof is similar to that of Theorem 8.2 in Grigorova et al. (2020). The properties on ρ being weakened here, we give the proof for clarity and completeness.

Proof of Theorem 2.1. By Lemma 2.1, the value family V is admissible. By Assumption 2.1, the value family V is p-integrable.
Let now $S\in\Theta$ and $\tau\in\Theta_S$. To show the (Θ,ρ)-supermartingale property of the value family, it remains to show $\rho_{S,\tau}[V(\tau)]\leqslant V(S)$ a.s. By the maximizing sequence lemma (Lemma 2.2), there exists a sequence $(\tau_p)\in(\Theta_\tau)^{\mathrm{N}}$, such that $V(\tau)=\lim_{p\to+\infty}\uparrow\rho_{\tau,\tau_p}[\xi(\tau_p)]$. Hence, we have

$$\rho_{S,\tau}[V(\tau)]=\rho_{S,\tau}[\lim_{p\to+\infty}\uparrow\rho_{\tau,\tau_p}[\xi(\tau_p)]]\leqslant\liminf_{p\to+\infty}\rho_{S,\tau}[\rho_{\tau,\tau_p}[\xi(\tau_p)]],$$

where we have used the monotone Fatou property with respect to terminal condition (vii) to obtain the inequality. By the consistency property, we have

$$\liminf_{p\to+\infty}\rho_{S,\tau}[\rho_{\tau,\tau_p}[\xi(\tau_p)]]=\liminf_{p\to+\infty}\rho_{S,\tau_p}[\xi(\tau_p)]\leqslant V(S),$$

the last inequality being due to $\Theta_\tau \subset \Theta_S$. We conclude that $\rho_{S,\tau}[V(\tau)] \leqslant V(S)$. Hence, the value family V is a (Θ, ρ)-supermartingale family. This proves Statement 1 of the theorem.

Let us now show Statement 2. By Remark 2.5, Statement 2, we have $V \geqslant \xi$. By Statement 1, we have that V is a (Θ, ρ)-supermartingale. It remains to show that V is the smallest. Let $(V'(\tau))$ be another (Θ, ρ)-supermartingale family, such that, for each $\tau \in \Theta$, $V'(\tau) \geqslant \xi(\tau)$ (a.s.). Let $S \in \Theta$, $\tau \in \Theta_S$. By the monotonicity of the non-linear operators ρ, we have

$$\rho_{S,\tau}[V'(\tau)] \geqslant \rho_{S,\tau}[\xi(\tau)].$$

On the other hand, as $(V'(\tau))$ is a (Θ, ρ)-supermartingale family, we have $V'(S) \geqslant \rho_{S,\tau}[V'(\tau)]$. Hence, $V'(S) \geqslant \rho_{S,\tau}[V'(\tau)] \geqslant \rho_{S,\tau}[\xi(\tau)]$. By taking the essential supremum over $\tau \in \Theta_S$ in this inequality, we get

$$V'(S) \geqslant \text{ess sup}_{\tau \in \Theta_S} \rho_{S,\tau}[\xi(\tau)] = V(S) \text{ (a.s.)}.$$

The proof is complete. □

2.4 The strict value family and the Dynamic Programming Principle (DPP)

Definition 2.3 (Dynamic Programming Principle)**.** *We say that an admissible p-integrable family satisfies the Dynamic Programming Principle (abridged DPP), if the following property holds true:*
For all $k \in \mathbb{N}$,

$$\phi(\theta_k) = \max\left(\xi(\theta_k), \rho_{\theta_k,\theta_{k+1}}[\phi(\theta_{k+1})]\right), \tag{9}$$

and $\phi(T) = \xi(T)$.

The purpose of this sub-section is to investigate under which assumptions on ρ, the DPP holds. To do this, we are first interested in "what happens on the right of $V(\theta_k)$", for each $k \in \mathrm{N}$.

Let $k \in \mathrm{N}$ be fixed. We define

$$\Theta_{\theta_k^+} := \{\tau \in \Theta_{\theta_k} : \tau > \theta_k \text{ on } \{\theta_k < T\} \text{ and } \tau = T \text{ on } \{\theta_k = T\}\},$$

and we define **the strict value** $V^+(\theta_k)$ at θ_k by:

$$V^+(\theta_k) := \text{ess sup}_{\tau \in \Theta_{\theta_k}^+} \rho_{\theta_k,\tau}[\xi(\tau)].$$

Remark 2.7. *We have $\Theta_{\theta_k^+} = \Theta_{\theta_{k+1}}$.*
Indeed, let $\tau \in \Theta_{\theta_k^+}$. Then τ can be written as:

$$\tau = \sum_{i=k+1}^{+\infty} \theta_i 1_{A_i \cap \{\theta_k < T\}} + T \times 1_{\bar{A} \cap \{\theta_k < T\}} + T \times 1_{\{\theta_k = T\}},$$

where $\{(A_i)_{i\geqslant k+1}, \bar{A}\}$ *is a partition of* Ω *such that for each* $i \geqslant k+1$, $A_i \in \mathcal{F}_{\theta_i}$, *and* $\bar{A} \in \mathcal{F}_T$.
We set $B_i := A_i \cap \{\theta_k < T\}$, *for* $i \geqslant k+1$, $\bar{B} := \bar{A} \cap \{\theta_k < T\}$*and* $B_k := \{\theta_k = T\}$. *We have* $\{(B_i)_{i\geqslant k}, \bar{B}\}$ *form a partition of* Ω*; for each* $i \geqslant k$, B_i *is* $\mathcal{F}_{\theta_i}$*-measurable, and* $\bar{B}$ *is* $\mathcal{F}_T$*-measurable. Moreover,* $\tau \geqslant \theta_{k+1}$ *(indeed,* $\tau \geqslant \theta_{k+1}$ *on* $\{\theta_k < T\}$ *and* $\tau = T = \theta_k = \theta_{k+1}$ *on* $\{\theta_k = T\}$*). Hence,* $\tau \in \Theta_{\theta_{k+1}}$.
Conversely, let $\tau \in \Theta_{\theta_{k+1}}$*; then,* τ *can be written as:*

$$\begin{aligned}\tau &= \sum_{i=k+1}^{+\infty} \theta_i 1_{A_i \cap \{\theta_k < T\}} + T 1_{\bar{A} \cap \{\theta_k < T\}} + \sum_{i=k+1}^{+\infty} \theta_i 1_{A_i \cap \{\theta_k = T\}} + T 1_{\bar{A} \cap \{\theta_k = T\}} \\ &= \sum_{i=k+1}^{+\infty} \theta_i 1_{A_i \cap \{\theta_k < T\}} + T 1_{\bar{A} \cap \{\theta_k < T\}} + \sum_{i=k+1}^{+\infty} T 1_{A_i \cap \{\theta_k = T\}} + T 1_{\bar{A} \cap \{\theta_k = T\}}.\end{aligned}$$

Hence, $\tau \in \Theta_{\theta_{k^+}}$.

Due to this remark, we get

$$V^+(\theta_k) = \text{ess sup}_{\tau \in \Theta^+_{\theta_k}} \rho_{\theta_k,\tau}[\xi(\tau)] = \text{ess sup}_{\tau \in \Theta_{\theta_{k+1}}} \rho_{\theta_k,\tau}[\xi(\tau)]. \tag{10}$$

Lemma 2.3. *Under the assumption of "generalized zero-one law" (vi) on the non-linear operators, there exists a maximizing sequence for* $V^+(\theta_k)$.

Proof. The proof of this lemma is similar to the proof of the existence of a maximizing sequence for $V(\theta_k)$, and is left to the readers. (We also refer to the proof of Lemma 2.3 in Grigorova et al. (2020) for similar arguments). □

The following proposition establishes that the strict value $V^+(\theta_k)$ at θ_k is equal to the non-linear evaluation from θ_k perspective of the value $V(\theta_{k+1})$.

Proposition 2.1. *Under the assumptions of monotonicity (iv), consistency (v), "generalized zero-one law" (vi) and monotone Fatou property with respect to the terminal condition (vii) on the non-linear operators, we have*

$$V^+(\theta_k) = \rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})].$$

Proof. First we show that $V^+(\theta_k) \leqslant \rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})]$.
By Lemma 2.3, there exists a maximizing sequence $(\tau_m) \in (\Theta_{\theta_{k+1}})^{\mathrm{N}}$ such that

$$V^+(\theta_k) = \lim_{m\to+\infty} \uparrow \rho_{\theta_k,\tau_m}[\xi(\tau_m)].$$

Now, by using the consistency property of the non-linear evaluations, we get

$$V^+(\theta_k) = \lim_{m\to+\infty} \uparrow \rho_{\theta_k,\tau_m}[\xi(\tau_m)] = \lim_{m\to+\infty} \uparrow \rho_{\theta_k,\theta_{k+1}}[\rho_{\theta_{k+1},\tau_m}[\xi(\tau_m)]]. \tag{11}$$

For each $m \in \mathrm{N}$, we have

$$\rho_{\theta_{k+1},\tau_m}[\xi(\tau_m)] \leqslant \text{ess sup}_{\tau\in\Theta_{\theta_{k+1}}} \rho_{\theta_{k+1},\tau}[\xi(\tau)] = V(\theta_{k+1}).$$

Then, by the monotonicity property of $\rho_{\theta_k,\theta_{k+1}}[\cdot]$, we get

$$\rho_{\theta_k,\theta_{k+1}}[\rho_{\theta_{k+1},\tau_m}[\xi(\tau_m)]] \leqslant \rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})].$$

Hence, we have

$$\lim_{m\to+\infty} \uparrow \rho_{\theta_k,\theta_{k+1}}[\rho_{\theta_{k+1},\tau_m}[\xi(\tau_m)]] \leqslant \lim_{m\to+\infty} \uparrow \rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})] = \rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})].$$

We conclude, combining this with (11), that

$$V^+(\theta_k) \leqslant \rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})].$$

Now, let us show the converse inequality. By Lemma 2.2, there also exists a maximizing sequence $(\tau'_m) \in (\Theta_{\theta_{k+1}})^{\mathrm{N}}$ such that

$$V(\theta_{k+1}) = \lim_{m\to+\infty} \uparrow \rho_{\theta_{k+1},\tau'_m}[\xi(\tau'_m)].$$

Hence,

$$\rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})] = \rho_{\theta_k,\theta_{k+1}}[\lim_{m\to+\infty} \uparrow \rho_{\theta_{k+1},\tau'_m}[\xi(\tau'_m)]].$$

We first use the monotone Fatou property with respect to the terminal condition of the non-linear operator $\rho_{\theta_k,\theta_{k+1}}[\cdot]$; then, we apply the consistency property of the non-linear operators to get:

$$\begin{aligned}
\rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})] &\leqslant \liminf_{m\to+\infty} \rho_{\theta_k,\theta_{k+1}}[\rho_{\theta_{k+1},\tau'_m}[\xi(\tau'_m)]] \\
&= \liminf_{m\to+\infty} \rho_{\theta_k,\tau'_m}[\xi(\tau'_m)] \\
&\leqslant \text{ess sup}_{\tau\in\Theta_{\theta_{k+1}}} \rho_{\theta_k,\tau}[\xi(\tau)] = V^+(\theta_k),
\end{aligned}$$

where we have used Eq. (10) to obtain the last equality. Hence,

$$\rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})] = V^+(\theta_k).$$

The proof is complete. □

Proposition 2.2. *Under the assumptions (iii) and "generalized zero-one law" (vi) on the non-linear operators, we have*

$$V(\theta_k) = \xi(\theta_k) \vee V^+(\theta_k).$$

Proof. By Remark 2.5, first statement, which can be applied as ρ satisfies property (iii), we have $V(\theta_k) \geqslant \xi(\theta_k)$. On the other hand, since $\Theta_{\theta_{k+1}} \subset \Theta_{\theta_k}$, we have $V(\theta_k) \geqslant V^+(\theta_k)$. By combining these two inequalities, we get $V(\theta_k) \geqslant \xi(\theta_k) \vee V^+(\theta_k)$. It remains to show the converse inequality. Let $\tau \in \Theta_{\theta_k}$. We define $\bar{\tau} = \tau 1_{\{\tau > \theta_k\}} + T 1_{\{\tau \leqslant \theta_k\}}$. As $\bar{\tau} \in \Theta^+_{\theta_k}$, we have

$$V^+(\theta_k) = \text{ess sup}_{\tau \in \Theta^+_{\theta_k}} \rho_{\theta_k,\tau}[\xi(\tau)] \geqslant \rho_{\theta_k,\bar{\tau}}[\xi(\bar{\tau})].$$

Hence, we have

$$1_{\{\tau > \theta_k\}} \rho_{\theta_k,\bar{\tau}}[\xi(\bar{\tau})] \leqslant 1_{\{\tau > \theta_k\}} V^+(\theta_k). \tag{12}$$

Moreover, on the set $\{\tau > \theta_k\}$, we have $\bar{\tau} = \tau$, so the "generalized zero-one law" gives

$$1_{\{\tau > \theta_k\}} \rho_{\theta_k,\bar{\tau}}[\xi(\bar{\tau})] = 1_{\{\tau > \theta_k\}} \rho_{\theta_k,\tau}[\xi(\tau)]. \tag{13}$$

By combining (12) and (13), we get

$$1_{\{\tau > \theta_k\}} \rho_{\theta_k,\tau}[\xi(\tau)] \leqslant 1_{\{\tau > \theta_k\}} V^+(\theta_k). \tag{14}$$

On the other hand, as $\tau \in \Theta_{\theta_k}$, we have

$$\rho_{\theta_k,\tau}[\xi(\tau)] = 1_{\{\tau = \theta_k\}} \rho_{\theta_k,\tau}[\xi(\tau)] + 1_{\{\tau > \theta_k\}} \rho_{\theta_k,\tau}[\xi(\tau)].$$

By using the "generalized zero-one law" and property (iii) of the non-linear operator $\rho_{\theta_k,\tau}[\cdot]$, we get

$$1_{\{\tau = \theta_k\}} \rho_{\theta_k,\tau}[\xi(\tau)] = 1_{\{\tau = \theta_k\}} \rho_{\theta_k,\theta_k}[\xi(\theta_k)] = 1_{\{\tau = \theta_k\}} \xi(\theta_k). \tag{15}$$

From Eqs. (14) and (15), we get

$$\begin{aligned} \rho_{\theta_k,\tau}[\xi(\tau)] &= 1_{\{\tau = \theta_k\}} \xi(\theta_k) + 1_{\{\tau > \theta_k\}} \rho_{\theta_k,\tau}[\xi(\tau)] \\ &\leqslant 1_{\{\tau = \theta_k\}} \xi(\theta_k) + 1_{\{\tau > \theta_k\}} V^+(\theta_k) = \xi(\theta_k) \vee V^+(\theta_k). \end{aligned}$$

Now, by taking the essential supremum over $\tau \in \Theta_{\theta_k}$, we get $V(\theta_k) \leqslant \xi(\theta_k) \vee V^+(\theta_k)$. Hence, the proof is complete. □

We refer to Quenez and Kobylanski (2012) for a similar approach the one in the above proposition in the linear case.

By combining Proposition 2.1 and Proposition 2.2, we get:

Theorem 2.2 (DPP)**.** *Under the assumptions of knowledge preservation (iii), monotonicity (iv), consistency (v), "generalized zero-one law" (vi) and monotone Fatou property with respect to the terminal condition (vii) on the non-linear operators, the value family V satisfies the DPP:*

$$\textit{for each } k \in \mathrm{N},\ V(\theta_k) = \xi(\theta_k) \vee \rho_{\theta_k,\theta_{k+1}}[V(\theta_{k+1})],\ \textit{and } V(T) = \xi(T).$$

2.5 Optimal stopping times

For each k, let us define the random variable ν_k

$$\nu_k := \operatorname{ess\,inf} \mathcal{A}_k \quad \text{where} \quad \mathcal{A}_k := \{\, \tau \in \Theta_{\theta_k} \,:\, V(\tau) = \xi(\tau) \text{ a.s.} \,\}. \tag{16}$$

As $T < \infty$, under property (iii) on ρ, the set $\mathcal{A}_k$ is clearly non-empty (as $V(T) = \xi(T)$ in this case). Moreover, it is clearly stable by pairwise minimization. Hence, by classical properties of the essential infimum, there exists a non increasing sequence (τ_n) in $\mathcal{A}_k$ such that $\lim_{n\to+\infty} \tau_n = \nu_k$ a.s. In particular, ν_k is a stopping time and $T \geqslant \nu_k \geqslant \theta_k$ a.s., and $\nu_k \in \Theta_{\theta_k}$ (by stability of Θ_{θ_k} when passing to a monotone limit).

In the following theorem, we show that, under suitable assumptions, the stopping time ν_k defined in (16) is optimal for the optimization problem (6) at time $\nu = \theta_k$. We introduce the following assumption on the value family V.

Assumption 2.2. *We assume that the value family V is left-upper-semicontinuous (LUSC) along the sequence $(\theta_n \wedge \nu_k)_{n\in\mathbb{N}}$, that is,*

$$\limsup_{n\to+\infty} V(\theta_n \wedge \nu_k) \leqslant V(\nu_k). \tag{17}$$

Remark 2.8. *Assumption 2.2 is trivially satisfied in the following particular case on Θ: Besides the assumptions (a) and (b) on Θ, the additional assumption (c) is imposed, namely:*
(c)For almost all ω, there exits $n_0 = n_0(\omega)$ (depending on ω) such that $\theta_n(\omega) = T$, for all $n \geqslant n_0$. In other words, for almost all ω, there exists at most a finite number of time points $\theta_n(\omega)$ such that $\theta_n(\omega) < T$.
In this case, for all n after a certain rank $\bar{n} = \bar{n}(\omega)$, we have $(\theta_n \wedge \nu_k)(\omega) = \nu_k(\omega)$. Hence, as V is admissible, we have, by Remark 2.2, for all $n \geqslant \bar{n}(\omega)$, $V(\theta_n \wedge \nu_k)(\omega) = V(\nu_k)(\omega)$. Hence, Assumption 2.2 holds true.

We will see later on a further discussion on Assumption 2.2 in the case of the general Θ, and conditions (on ρ and on the pay-off family ξ) under which this assumption is satisfied.

Theorem 2.3 (Optimality of ν_k)**.** *Let $k \in \mathbb{N}$ and let ν_k be the stopping time defined by* (16)*. Let Assumption 2.2 on V be satisfied. Let ρ satisfy the properties of admissibility (ii), knowledge preservation (iii), monotonicity (iv), consistency (v), "generalized zero-one law" (vi), and monotone Fatou property with respect to the terminal condition (vii). We assume additionally that ρ satisfies the following property:*

- *(left-upper-semicontinuity (LUSC) along Bermudan stopping times with respect to the terminal condition and the terminal time at ν_k), that is,*

$$\limsup_{n\to+\infty} \rho_{S,\tau_n}[\phi(\tau_n)] \leqslant \rho_{S,\nu_k}[\limsup_{n\to+\infty} \phi(\tau_n)], \tag{18}$$

for each non-decreasing sequence $(\tau_n) \in \Theta_S^{\mathbb{N}}$ *such that* $\lim_{n\to+\infty} \uparrow \tau_n = \nu_k$ *a.s.,*
and for each p-integrable admissible family ϕ *such that* $\sup_{n\in\mathbb{N}} |\phi(\tau_n)| \in L^p$.

Then:

$$V(\theta_k) = \rho_{\theta_k,\nu_k}[\xi(\nu_k)] = ess\ sup_{\nu\in\Theta_{\theta_k}} \rho_{\theta_k,\nu}[\xi(\nu)] \quad a.s. \tag{19}$$

Note that in the case where $\rho = (\rho_S[\cdot])_{S\in\Theta}$ does not depend on the second time index, the above additional property reduces to the LUSC of $\rho_S[\cdot]$ (with respect to the terminal condition) along Bermudan stopping sequences.

2.5.1 The (Θ, ρ)-martingale property on a stochastic interval

Before proving the theorem, we give several useful technical lemmas.

The first two clarify the (Θ, ρ)-martingale structure on a stochastic interval in a more "handy" way. The third one deals with an "*if*-condition" (optimality criterion) and an "*only if*-condition" for optimality.

Lemma 2.4. *Let* ρ *satisfy the consistency property (v). Let* $\phi = (\phi(\nu))$ *be a given square-integrable admissible family. Let* $S \in \Theta$ *and* $\tau \in \Theta$ *be such that* $S \leqslant \tau$ *a.s. We assume that for any* $\sigma \in \Theta$ *such that* $S \leqslant \sigma \leqslant \tau$ *a.s., it holds*

$$\rho_{\sigma,\tau}[\phi(\tau)] = \phi(\sigma) \quad a.s. \tag{20}$$

Then, ϕ *is a* (Θ, ρ)*-martingale on the stochastic interval* $[S, \tau]$*, that is, for any* $\nu_1 \in \Theta$*,* $\nu_2 \in \Theta$*, such that* $S \leqslant \nu_1 \leqslant \nu_2 \leqslant \tau$ *a.s.,*

$$\rho_{\nu_1,\nu_2}[\phi(\nu_2)] = \phi(\nu_1) \quad a.s. \tag{21}$$

Proof. Let $\nu_1 \in \Theta$, and $\nu_2 \in \Theta$ be such that $S \leqslant \nu_1 \leqslant \nu_2 \leqslant \tau$ a.s.

Hence, by applying the equation (20) with $\sigma = \nu_1$ and by the consistency of the non-linear operators ρ , we have

$$\phi(\nu_1) = \rho_{\nu_1,\tau}[\phi(\tau)] = \rho_{\nu_1,\nu_2}[\rho_{\nu_2,\tau}[\phi(\tau)]]\ a.s.$$

Then, by applying again the equation (20) with $\sigma = \nu_2$, we have

$$\rho_{\nu_2,\tau}[\phi(\tau)] = \phi(\nu_2).$$

Hence, $\phi(\nu_1) = \rho_{\nu_1,\nu_2}[\rho_{\nu_2,\tau}[\phi(\tau)]] = \rho_{\nu_1,\nu_2}[\phi(\nu_2)]$ *a.s.* □

Definition 2.4. *(Strictly monotone operator) Let* $S \in \Theta, \tau \in \Theta_S$*. We say that* $\rho_{S,\tau}$ *is strictly monotone if the following two conditions hold:*

1. $\rho_{S,\tau}$ *is monotone.*
2. *If* $\eta_1 \leqslant \eta_2$ *and* $\rho_{S,\tau}(\eta_1) = \rho_{S,\tau}(\eta_2)$*, then* $\eta_1 = \eta_2$.

Lemma 2.5. *We assume that the non-linear operators satisfy the properties of monotonicity (iv) and consistency (v). Assume moreover that the non-linear operators ρ are strictly monotone. Let $\phi = (\phi(\nu))$ be a given p-integrable admissible family). Let $S \in \Theta$ and $\tau \in \Theta$ be such that $S \leqslant \tau$ a.s. We assume that the two conditions hold:*

1. ϕ is a (Θ, ρ)-supermartingale family on $[S, \tau]$;

2. $\phi(S) = \rho_{S,\tau}[\phi(\tau)]$ a.s.

Then, for any $\sigma \in \Theta$ such that $S \leqslant \sigma \leqslant \tau$ a.s., $\rho_{\sigma,\tau}[\phi(\tau)] = \phi(\sigma)$ a.s.

Proof. Let $\sigma \in \Theta$, such that $S \leqslant \sigma \leqslant \tau$ a.s.
By applying condition 2 of the lemma and the consistency of the non-linear operators ρ, we have

$$\phi(S) = \rho_{S,\tau}[\phi(\tau)] = \rho_{S,\sigma}[\rho_{\sigma,\tau}[\phi(\tau)]] \; a.s.$$

On the other hand, since ϕ is a (Θ, ρ)-supermartingale family on $[S, \tau]$ (by condition 1), and since the non-linear operator $\rho_{S,\sigma}$ is monotone, we have

$$\rho_{S,\sigma}[\rho_{\sigma,\tau}[\phi(\tau)]] \leqslant \rho_{S,\sigma}[\phi(\sigma)] \leqslant \phi(S) \; a.s.$$

By combining the previous two equations, we get

$$\phi(S) = \rho_{S,\sigma}[\rho_{\sigma,\tau}[\phi(\tau)]] = \rho_{S,\sigma}[\phi(\sigma)] \; a.s.$$

In particular,

$$\rho_{S,\sigma}[\rho_{\sigma,\tau}[\phi(\tau)]] = \rho_{S,\sigma}[\phi(\sigma)] \quad a.s. \tag{22}$$

Due to the additional assumption, the non-linear operators ρ are strictly monotone. From this, together with equality (22) and the inequality $\rho_{\sigma,\tau}[\phi(\tau)] \leqslant \phi(\sigma)$ a.s. (which is due to condition 1 of the lemma), we get $\phi(\sigma) = \rho_{\sigma,\tau}[\phi(\tau)]$. □

Lemma 2.6. *Let $\nu_k^* \in \Theta_{\theta_k}$. We introduce the following two conditions:*

i) $\rho_{\theta_k,\nu_k^*}[V(\nu_k^*)] = \rho_{\theta_k,\nu_k^*}[\xi(\nu_k^*)]$ a.s.

ii) The family $(V(\nu \wedge \nu_k^*))_{\nu \in \Theta_{\theta_k}}$ is a (Θ, ρ)-martingale family.

1. (Optimality criterion) If i) and ii) are satisfied, then ν_k^* is optimal for problem (6).

2. If, moreover, the non-linear operator ρ_{θ_k,ν_k^*} is assumed to be strictly monotone and satisfies the assumptions of admissibility (ii), knowledge preservation (iii), consistency (v), "generalized zero-one law" (vi) and monotone Fatou property (vii), then the converse statement is also true.

Remark 2.9. *We note that the property $V(\nu_k^*) = \xi(\nu_k^*)$ a.s. implies that $\rho_{\theta_k,\nu_k^*}[V(\nu_k^*)] = \rho_{\theta_k,\nu_k^*}[\xi(\nu_k^*)]$ a.s. The converse implication is true under the additional assumption: ρ_{θ_k,ν_k^*} is strictly monotone.*

Proof. First, let us show statement 1.

Let $\nu_k^* \in \Theta_{\theta_k}$ be such that the two conditions i) and ii) introduced above are satisfied. By condition ii) the family $(V(\nu \wedge \nu_k^*))_{\nu \in \Theta_{\theta_k}}$ is a (Θ, ρ)-martingale family.

Hence, for any $\nu \in \Theta_{\theta_k}$, we have

$$V(\theta_k \wedge \nu_k^*) = \rho_{\theta_k, \nu \wedge \nu_k^*}[V(\nu \wedge \nu_k^*)] \; a.s.,$$

which implies

$$V(\theta_k) = \rho_{\theta_k, \nu \wedge \nu_k^*}[V(\nu \wedge \nu_k^*)] \; a.s.$$

In particular, for $\nu = \nu_k^*$, we get

$$V(\theta_k) = \rho_{\theta_k, \nu_k^* \wedge \nu_k^*}[V(\nu_k^* \wedge \nu_k^*)] = \rho_{\theta_k, \nu_k^*}[V(\nu_k^*)] \; a.s.$$

From this, together with condition i), we have

$$V(\theta_k) = \rho_{\theta_k, \nu_k^*}[V(\nu_k^*)] = \rho_{\theta_k, \nu_k^*}[\xi(\nu_k^*)] \; a.s.,$$

which implies that the stopping time ν_k^* is an optimal stopping time for problem (6).

Now, let us show statement 2. Let $\nu_k^* \in \Theta_{\theta_k}$ be an optimal stopping time for problem (6). Hence, we have

$$V(\theta_k) = \rho_{\theta_k, \nu_k^*}[\xi(\nu_k^*)] \; a.s.$$

By the first part of Theorem 2.1 (which is applicable as ρ satisfies the assumptions), the value family V is a (Θ, ρ)-supermartingale family. Thus, by the (Θ, ρ)-supermartingale property of V, and as $\nu_k^* \in \Theta_{\theta_k}$, we have

$$V(\theta_k) \geqslant \rho_{\theta_k, \nu_k^*}[V(\nu_k^*)] \; a.s.$$

On the other hand, due to the fact that $\xi \leqslant V$ (cf. Remark 2.5, Statement 2) and to the monotonicity of the non-linear operator ρ_{θ_k, ν_k^*}, it holds

$$\rho_{\theta_k, \nu_k^*}[\xi(\nu_k^*)] \leqslant \rho_{\theta_k, \nu_k^*}[V(\nu_k^*)] \; a.s.,$$

Thus, we get

$$V(\theta_k) = \rho_{\theta_k, \nu_k^*}[V(\nu_k^*)] = \rho_{\theta_k, \nu_k^*}[\xi(\nu_k^*)] \; a.s.$$

Moreover, since $V(\theta_k) = \rho_{\theta_k, \nu_k^*}[V(\nu_k^*)]$, by applying Lemmas 2.4 and 2.5 (the latter is applicable as ρ is assumed to be strictly monotone) with $S = \theta_k$, $\tau = \nu_k^*$, we conclude that V is a (Θ, ρ)-martingale on $[\theta_k, \nu_k^*]$.

The proof is complete. □

2.5.2 Two useful consequences of the DPP

The following two results hold, if a given admissible p-integrable family ϕ satisfies the (DPP) from Eq.(9), and if $\tilde{\nu}_k$ is defined by

$$\tilde{\nu}_k := \operatorname{ess\,inf}\ \tilde{\mathcal{A}}_k \quad \text{where} \quad \tilde{\mathcal{A}}_k := \{\, \tau \in \Theta_{\theta_k} \,:\, \phi(\tau) = \xi(\tau) \text{ a.s.} \,\}. \tag{23}$$

The following lemma is a consequence of the definition of $\tilde{\nu}_k$ and of the DPP.

Lemma 2.7. *Assume that ϕ satisfies the DPP holds, and let $\tilde{\nu}_k$ be defined by* (23). *Then, for each $l \in \{k, k+1, ...\}$, $\phi(\theta_l) = \rho_{\theta_l,\theta_{l+1}}[\phi(\theta_{l+1})]$ on the set $\{\tilde{\nu}_k > \theta_l\}$.*

Proof. Let $l \in \{k, k+1, ...\}$. By the definition of $\tilde{\nu}_k$, on the set $\{\tilde{\nu}_k > \theta_l\}$, we have $\phi(\theta_l) > \xi(\theta_l)$. From this and from the DPP, we conclude that on the set $\{\tilde{\nu}_k > \theta_l\}$, $\phi(\theta_l) = \rho_{\theta_l,\theta_{l+1}}[\phi(\theta_{l+1})]$. □

Lemma 2.8. *Assume that the (DPP) from Eq.*(9) *holds. Under the assumptions of (iii) and "generalized zero-one law" (vi) on ρ, it holds:*

1. *For each $l \in \mathrm{N}$, $\phi(\theta_l \wedge \tilde{\nu}_k) = \rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)]$.*
2. *For each $l \in \mathrm{N}$, $\phi(\theta_l \wedge \tilde{\nu}_k) = \rho_{\theta_l\wedge\tilde{\nu}_k,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)]$.*

Proof. First, we show Statement 1 of the Lemma.

$$\begin{aligned}\rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)] &= 1_{\{\tilde{\nu}_k \leqslant \theta_l\}}\rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)] \\ &\quad + 1_{\{\tilde{\nu}_k > \theta_l\}}\rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)].\end{aligned} \tag{24}$$

For the first summand in Eq.(24), we note that on the set $\{\tilde{\nu}_k \leqslant \theta_l\}$, $\theta_{l+1} \wedge \tilde{\nu}_k = \theta_l \wedge \tilde{\nu}_k = \tilde{\nu}_k$. Hence, by the "generalized zero-one law", we have

$$1_{\{\tilde{\nu}_k \leqslant \theta_l\}}\rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)] = 1_{\{\tilde{\nu}_k \leqslant \theta_l\}}\rho_{\theta_l,\theta_l\wedge\tilde{\nu}_k}[\phi(\theta_l \wedge \tilde{\nu}_k)] = 1_{\{\tilde{\nu}_k \leqslant \theta_l\}}\phi(\theta_l \wedge \tilde{\nu}_k), \tag{25}$$

where we have used property (iii) to obtain the last equality (as $\theta_l \wedge \tilde{\nu}_k \leqslant \theta_l$).
For the second summand in Eq.(24), we use the "generalized zero-one law" to write:

$$1_{\{\tilde{\nu}_k > \theta_l\}}\rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)] = 1_{\{\tilde{\nu}_k > \theta_l\}}\rho_{\theta_l,\theta_{l+1}}[\phi(\theta_{l+1})] = 1_{\{\tilde{\nu}_k > \theta_l\}}\phi(\theta_l), \tag{26}$$

where we have applied Lemma 2.7 to obtain the last equality.
Hence, by replacing Eqs (25) and (26) in Eq. (24), we get

$$\rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)] = 1_{\{\tilde{\nu}_k \leqslant \theta_l\}}\phi(\theta_l \wedge \tilde{\nu}_k) + 1_{\{\tilde{\nu}_k > \theta_l\}}\phi(\theta_l \wedge \tilde{\nu}_k) = \phi(\theta_l \wedge \tilde{\nu}_k).$$

We now prove Statement 2 of the Lemma.

$$\begin{aligned}\rho_{\theta_l\wedge\tilde{\nu}_k,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)] &= 1_{\{\tilde{\nu}_k \leqslant \theta_l\}}\rho_{\theta_l\wedge\tilde{\nu}_k,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)] \\ &\quad + 1_{\{\tilde{\nu}_k > \theta_l\}}\rho_{\theta_l\wedge\tilde{\nu}_k,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1} \wedge \tilde{\nu}_k)].\end{aligned} \tag{27}$$

For the second summand of Eq.(27), by Statement 1 of the Lemma, we get

$$\begin{aligned}1_{\{\tilde{\nu}_k>\theta_l\}}\rho_{\theta_l\wedge\tilde{\nu}_k,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1}\wedge\tilde{\nu}_k)] &= 1_{\{\tilde{\nu}_k>\theta_l\}}\rho_{\theta_l,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1}\wedge\tilde{\nu}_k)]\\ &= 1_{\{\tilde{\nu}_k>\theta_l\}}\phi(\theta_l\wedge\tilde{\nu}_k).\end{aligned}$$

For the first summand of Equation (27), we apply the "generalized zero-one law" (as the set $\{\tilde{\nu}_k \leqslant \theta_l\}$ is $\mathcal{F}_{\theta_l\wedge\tilde{\nu}_k}$-measurable and on the set $\{\tilde{\nu}_k \leqslant \theta_l\}$, we have $\theta_{l+1}\wedge\tilde{\nu}_k = \theta_l\wedge\tilde{\nu}_k$), we have

$$\begin{aligned}1_{\{\tilde{\nu}_k\leqslant\theta_l\}}\rho_{\theta_l\wedge\tilde{\nu}_k,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1}\wedge\tilde{\nu}_k)] &= 1_{\{\tilde{\nu}_k\leqslant\theta_l\}}\rho_{\theta_l\wedge\tilde{\nu}_k,\theta_l\wedge\tilde{\nu}_k}[\phi(\theta_l\wedge\tilde{\nu}_k)]\\ &= 1_{\{\tilde{\nu}_k\leqslant\theta_l\}}\phi(\theta_l\wedge\tilde{\nu}_k),\end{aligned}$$

where we have used property (iii) on ρ to obtain the last equality.
Finally, we get

$$\begin{aligned}\rho_{\theta_l\wedge\tilde{\nu}_k,\theta_{l+1}\wedge\tilde{\nu}_k}[\phi(\theta_{l+1}\wedge\tilde{\nu}_k)] &= 1_{\{\tilde{\nu}_k\leqslant\theta_l\}}\phi(\theta_l\wedge\tilde{\nu}_k) + 1_{\{\tilde{\nu}_k\geqslant\theta_{l+1}\}}\phi(\theta_l\wedge\tilde{\nu}_k)\\ &= \phi(\theta_l\wedge\tilde{\nu}_k).\end{aligned}$$

□

2.5.3 The proof of the optimality of ν_k

We are now ready to prove Theorem 2.3 on the optimality of the Bermudan stopping time ν_k, defined by (16). We will need also the following remark:

Remark 2.10. *Any admissible family $(\phi(\tau), \tau\in\Theta)$ in our framework is **right-continuous along Bermudan stopping times**, that is, for all $\tau\in\Theta$, and for all non-increasing sequences of Bermudan stopping times $(\tau_n)\in\Theta^{\mathrm{N}}$ such that $\tau_n\downarrow\tau$, it holds $\lim_{n\to+\infty}\phi(\tau_n) = \phi(\tau)$.*

Indeed, let $\tau\in\Theta$, and let $(\tau_n)\in\Theta^{\mathrm{N}}$ be such that $\tau_n\downarrow\tau$. For each n, we have $\tau_n = \sum_{l=0}^{+\infty}\theta_l 1_{A_l^{(n)}} + T1_{\bar{A}^{(n)}}$ and $\tau = \sum_{l=0}^{+\infty}1_{A_l}\theta_l + T1_{\bar{A}}$ (cf. the canonical writing from Remark 2.1).
Let $\omega\in\Omega$. Recall that $\{(A_l)_{l\in\mathbb{N}}, \bar{A}\}$ is a partition of Ω. If $\omega\in\bar{A}$, then $\tau(\omega) = T = \tau_n(\omega)$ (as $\tau_n\downarrow\tau$). Otherwise, there exists a unique $l_0 = l_0(\omega)$, such that $\omega\in A_{l_0}$, and $\tau(\omega) = \theta_{l_0}(\omega) < T$. Then, as $\tau_n(\omega)\downarrow\tau(\omega)$ and as $\theta_k(\omega)\uparrow T$, after a certain rank $n_0 = n_0(\omega)$, $\tau_n(\omega) = \tau(\omega) = \theta_{l_0}(\omega)$.
Hence, in both cases, there exists $n_0 = n_0(\omega)$ such that for all $n\geqslant n_0$, $\tau_n(\omega) = \tau(\omega)$, and, hence, by Remark 2.2, for all $n\geqslant n_0$, $\phi(\tau_n)(\omega) = \phi(\tau)(\omega)$. We conclude that $\lim_{n\to+\infty}\phi(\tau_n)(\omega) = \phi(\tau)(\omega)$.

Proof of Theorem 2.3. By Lemma 2.6, in order to show that ν_k, defined in (16), is optimal for problem (6), it is enough to show the following two conditions:

(i) $\rho_{\theta_k,\nu_k}[V(\nu_k)] = \rho_{\theta_k,\nu_k}[\xi(\nu_k)]$ a.s.;

(ii) The family $(V(\nu))$ is a (Θ,ρ)-martingale on $[\theta_k,\nu_k]$.

We start our proof by showing the second condition first.

By Lemma 2.1, V is an admissible family, and it is also p-integrable by Assumption 2.1. By Lemma 2.4 (on the (Θ, ρ)-martingale property), in order to show the second condition, it is enough to show that: for each $\sigma \in \Theta$, such that $\theta_k \leqslant \sigma \leqslant \nu_k$,

$$\rho_{\sigma,\nu_k}[V(\nu_k)] = V(\sigma). \tag{28}$$

Let $\sigma \in \Theta_{\theta_k}$. Then σ is of the form $\sigma = \sum_{m \geqslant k} \theta_m 1_{A_m} + T 1_{\bar{A}}$, where $\{(A_m)_{m\in\mathbb{N}}, \bar{A}\}$ form a partition of Ω; A_m is $\mathcal{F}_{\theta_m}$-measurable for each m, and $\bar{A} \in \mathcal{F}_T$.
Hence, to prove Equation (28), it is enough to show that, for each $m \geqslant k$,

$$1_{A_m} \rho_{\theta_m,\nu_k}[V(\nu_k)] = 1_{A_m} V(\theta_m), \tag{29}$$

and

$$1_{\bar{A}} \rho_{T,\nu_k}[V(\nu_k)] = 1_{\bar{A}} V(T). \tag{30}$$

As $\sigma = \nu_k = T$ on $\bar{A}$, Eq.(30) holds true, by the "generalized" zero-one law (vi) and the knowledge preserving property (iii).
Let $m \geqslant k$ be fixed. The proof of Eq. (29) passes through the following steps:

1^{st} **Step:** As V satisfies the DPP (cf. Theorem 2.2), we use Lemma 2.8, Statement 2 and the consistency property on ρ, to show that, for each fixed $n \in \mathrm{N}$,

$$\rho_{\theta_m \wedge \nu_k, \theta_{m+n} \wedge \nu_k}[V(\theta_{m+n} \wedge \nu_k)] = V(\theta_m \wedge \nu_k). \tag{31}$$

Indeed, by applying successively $(n - m - 1)$ times the consistency property on ρ (if $n - m \geqslant 2$), we get

$$\begin{aligned}\rho_{\theta_m \wedge \nu_k, \theta_{m+n} \wedge \nu_k}[V(\theta_{m+n} \wedge \nu_k)] = \rho_{\theta_m \wedge \nu_k, \theta_{m+1} \wedge \nu_k}[\rho_{\theta_{m+1} \wedge \nu_k, \theta_{m+2} \wedge \nu_k}[...\\ [\rho_{\theta_{n+m-1} \wedge \nu_k, \theta_{m+n} \wedge \nu_k}[V(\theta_{m+n} \wedge \nu_k)]]]].\end{aligned}$$

By applying Lemma 2.8, Statement 2, again successively $(n - m)$ times, we get

$$\rho_{\theta_m \wedge \nu_k, \theta_{m+n} \wedge \nu_k}[V(\theta_{m+n} \wedge \nu_k)] = V(\theta_m \wedge \nu_k),$$

which proves Equation (31).

Hence, the sequence of random variables $(\rho_{\theta_m \wedge \nu_k, \theta_{m+n} \wedge \nu_k}[V(\theta_{m+n} \wedge \nu_k)])_{n\in\mathrm{N}}$ does not depend on n and is constantly equal to the random variable $V(\theta_m \wedge \nu_k)$.

2^{nd} **Step:** As V is left-upper-semicontinuous (LUSC) along the sequence $(\theta_{m+n} \wedge \nu_k)_{n\in\mathbb{N}}$ by Assumption 2.2, and as ρ is LUSC along Bermudan stopping strategies with respect to terminal condition and terminal time at (ν_k), we have

$$\begin{aligned}\limsup_{n\to+\infty} \rho_{\theta_m \wedge \nu_k, \theta_{m+n} \wedge \nu_k}[V(\theta_{m+n} \wedge \nu_k)] &\leqslant \rho_{\theta_m \wedge \nu_k, \nu_k}[\limsup_{n\to+\infty} V(\theta_{m+n} \wedge \nu_k)]\\ &\leqslant \rho_{\theta_m \wedge \nu_k, \nu_k}[V(\nu_k)],\end{aligned}$$

where we have used the monotonicity of ρ and Assumption 2.2 on V to obtain the last inequality. Hence,

$$V(\theta_m \wedge \nu_k) \leqslant \rho_{\theta_m \wedge \nu_k, \nu_k}[V(\nu_k)].$$

The opposite inequality holds true due to the (Θ, ρ)-supermartingale property of V (cf. Theorem 2.1, Statement 1). Hence, we have the equality, that is,

$$V(\theta_m \wedge \nu_k) = \rho_{\theta_m \wedge \nu_k, \nu_k}[V(\nu_k)],$$

and Eq.(29) is established.
3^{rd} **Step:** From the above Eqs (30) and (29), it follows:

$$\rho_{\sigma \wedge \nu_k, \nu_k}[V(\nu_k)] = V(\sigma \wedge \nu_k),$$

(as $\sigma = \sum_{m \geqslant k} \theta_m 1_{A_m} + T 1_{\bar{A}}$), which proves Eq.(28).

We conclude, by Lemma 2.4, that V is a (Θ, ρ)-martingale on the stochastic interval $[\theta_k, \nu_k]$. This shows condition (ii) in the optimality criterion of Lemma 2.6.

It remains for us to show condition (i) in the optimality criterion. Let us recall that $\nu_k = \operatorname{ess\,inf} \mathcal{A}_k$, where $\mathcal{A}_k = \{\tau \in \Theta_{\theta_k} : V(\tau) = \xi(\tau) \ a.s.\}$. Let (τ_n) be a non-increasing sequence in $\mathcal{A}_k$, such that $\lim_{n \to +\infty} \downarrow \tau_n = \nu_k$. As τ_n is in $\mathcal{A}_k$, we have $V(\tau_n) = \xi(\tau_n)$. By passing to the limit in this equality and by using that both families V and ξ are right-continuous along the sequence of Bermudan stopping strategies (τ_n) (cf. Remark 2.10), we obtain

$$V(\nu_k) = \xi(\nu_k), \tag{32}$$

which proves condition (i).
This concludes the proof of the optimality of ν_k. □

2.5.4 Assumption 2.2 on V: Discussion

Let us now check under which conditions Assumption 2.2 on V holds true.

Under the assumptions $(a), (b)$ on Θ, the set $\{\omega \in \Omega : \nu_k(\omega) = T, \theta_l(\omega) < T, \text{for all } l \in \mathbb{N}\}$ might be non-empty. We will show the following lemma.

Lemma 2.9. *Let ρ satisfy the properties of knowledge preservation (iii), monotonicity (iv), and consistency (v), and the following property*

$$\limsup_{n \to +\infty} \rho_{\theta_n, T}[\eta] \leqslant \rho_{T,T}[\eta], \ \textit{for all } \eta \in L^p(\mathcal{F}_T). \tag{33}$$

Then,

$$\limsup_{n \to +\infty} V(\theta_n) \leqslant \limsup_{n \to +\infty} \xi(\theta_n) \vee \xi(T). \tag{34}$$

Proof. For each $m \in \mathbb{N}$, for each $\tau \in \Theta_{\theta_m}$, $\xi(\tau) \leqslant \sup_{p \geqslant m} \xi(\theta_p) \vee \xi(T)$.

Indeed, for each $\tau \in \Theta_{\theta_m}$, we have

$$\xi(\tau) = \sum_{l \geqslant m} \xi(\theta_l) 1_{A_l} + \xi(T) 1_{\bar{A}} \leqslant \eta 1_{\bar{A}^c} + \xi(T) 1_{\bar{A}} \leqslant \eta \vee \xi(T),$$

where we have set $\eta := \sup_{p\geqslant m}\xi(\theta_p)$. Let us define $\bar{\eta} := \eta \vee \xi(T) = \sup_{p\geqslant m}\xi(\theta_p) \vee \xi(T)$.

By the monotonicity of ρ and the knowledge preserving property on ρ, we have, for all $\tau \in \Theta_{\theta_m}$,

$$\xi(\tau) = \rho_{\tau,T}[\xi(\tau)] \leqslant \rho_{\tau,T}[\eta \vee \xi(T)] = \rho_{\tau,T}[\bar{\eta}]. \tag{35}$$

Hence, for each $n \geqslant m$ (as $\Theta_n \subset \Theta_m$),

$$V(\theta_n) = \text{ess sup}_{\tau\in\Theta_{\theta_n}}\rho_{\theta_n,\tau}[\xi(\tau)] \leqslant \text{ess sup}_{\tau\in\Theta_{\theta_n}}\rho_{\theta_n,\tau}[\rho_{\tau,T}[\bar{\eta}]]$$

where we have used the monotonicity of ρ and (35) for the inequality. By the consistency property on ρ, we get $\rho_{\theta_n,\tau}[\rho_{\tau,T}[\bar{\eta}]] = \rho_{\theta_n,T}[\bar{\eta}]$.
Finally, $V(\theta_n) \leqslant \text{ess sup}_{\tau\in\Theta_{\theta_n}}\rho_{\theta_n,T}[\bar{\eta}] = \rho_{\theta_n,T}[\bar{\eta}]$.
Hence, $\limsup_{n\to+\infty} V(\theta_n) \leqslant \limsup_{n\to+\infty}\rho_{\theta_n,T}[\bar{\eta}]$.
As, by assumption on ρ, $\limsup_{n\to+\infty}\rho_{\theta_n,T}[\bar{\eta}] \leqslant \rho_{T,T}[\bar{\eta}]$, we obtain

$$\limsup_{n\to+\infty} V(\theta_n) \leqslant \rho_{T,T}[\bar{\eta}] = \bar{\eta} = \sup_{p\geqslant m}\xi(\theta_p) \vee \xi(T).$$

Hence, by passing to the limit when $m \to +\infty$, we get

$$\limsup_{n\to+\infty} V(\theta_n) \leqslant \limsup_{m\to+\infty}\xi(\theta_m) \vee \xi(T),$$

which finishes the proof of the lemma. □

Assumption 2.3. *We assume that the pay-off family ξ is LUSC along the sequence (θ_n) at T, that is, ξ is such that*

$$\limsup_{n\to+\infty}\xi(\theta_n) \leqslant \xi(T).$$

Proposition 2.3. *If the pay-off family ξ satisfies Assumption 2.3 (LUSC along the sequence (θ_n) at T), and if ρ satisfies the properties of admissibility (ii), knowledge preservation (iii), monotonicity (iv), consistency (v), "generalized" zero-one law (vi), and LUSC along the sequence (θ_n) at T (property (33)), then the value family V satisfies Assumption 2.2.*

Proof. On the set $\{\nu_k < T\}$, we have, for all n after a certain rank $\bar{n}(\omega)$ (depending on ω), $(\theta_n \wedge \nu_k)(\omega) = \nu_k(\omega)$ ((due to $\lim\uparrow \theta_n = T$). Hence, by Remark 2.2), for all $n \geqslant n_0(\omega)$, $V(\theta_n \wedge \nu_k)(\omega) = V(\nu_k)(\omega)$ Hence,

$$1_{\{\nu_k<T\}}\limsup_{n\to+\infty} V(\theta_n \wedge \nu_k) = 1_{\{\nu_k<T\}}\lim_{n\to+\infty} V(\theta_n \wedge \nu_k) = 1_{\{\nu_k<T\}}V(\nu_k).$$

On the other hand, under the Assumption 2.3 on ξ, we have, by Lemma 2.9,

$$\limsup_{n\to+\infty} V(\theta_n) \leqslant \limsup_{n\to+\infty}\xi(\theta_n) \vee \xi(T) \leqslant \xi(T) = V(T).$$

Hence, using the admissibility of V,

$$1_{\{\nu_k=T\}}\limsup_{n\to+\infty} V(\theta_n \wedge \nu_k) = 1_{\{\nu_k=T\}}\limsup_{n\to+\infty} V(\theta_n) \leqslant 1_{\{\nu_k=T\}}V(T) = 1_{\{\nu_k=T\}}V(\nu_k)$$

So, the desired LUSC property at ν_k on V (Assumption 2.2) holds true. □

3 Examples

In this section we provide some examples of non-linear operators ρ, known from the stochastic control and mathematical finance literature, which enter into our framework.

3.1 Non-linear operators induced by BSDEs

In this example, $p = 2$.

3.1.1 The g-evaluations

Peng (2004) and El Karoui and Quenez (1997) introduced a type of non-linear evaluation, now known as g-evaluation, via a non-linear backward stochastic differential equation (BSDE) with a driver g.
Let $T > 0$ be a fixed time horizon. We place ourselves in the Brownian framework (for simplicity). Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space, endowed with a d-dimensional Brownian motion $(W_t)_{t\in[0,T]}$, and let $(\mathcal{F}_t)_{t\in[0,T]}$ be the (augmented) natural filtration of the Brownian motion.

Let $g = g(\omega, t, y, z) : \Omega \times [0,T] \times \mathrm{R} \times \mathrm{R}^d \to \mathrm{R}$ be *Lipschitz driver*, that is, a function satisfying the following conditions:

① For each $y \in \mathrm{R}, z \in \mathrm{R}^d$, $g(\cdot, \cdot, y, z) \in L^2(\Omega \times [0,T])$ and g is progressively measurable;

② There exists $C > 0$ such that for each $y_1, y_2 \in \mathrm{R}$, and for each $z_1, z_2 \in \mathrm{R}^d$, $|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leqslant C(|y_1 - y_2| + \|z_1 - z_2\|)$, uniformly for a.e. $(\omega.t)$, where $\|\cdot\|$ denotes the Euclidean norm on R^d;

Let us consider the following 1-dimensional BSDE with terminal time t and terminal condition η, defined on the interval $[0,t]$, given that $0 \leqslant t \leqslant T$ and $\eta \in L^2(\mathcal{F}_t)$:

$$y_s = \eta + \int_s^t g(r, y_r, z_r)dr - \int_s^t z_r dB_r, \quad s \in [0,t].$$

Definition 3.1. *(g-evaluation) For each $0 \leqslant s \leqslant t \leqslant T$ and $\eta \in L^2(\mathcal{F}_t)$, we define*

$$\mathcal{E}^g_{s,t}[\eta] := y_s.$$

The family of operators $\mathcal{E}^g_{s,t}[\cdot] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), 0 \leqslant s \leqslant t \leqslant T$ are called g-evaluation.

We recall (cf. El Karoui et al. (1997)) that if the terminal time is given by a stopping time τ valued in $[0,T]$ and if η is $\mathcal{F}_\tau$-measurable, the solution of the BSDE with terminal time τ, terminal condition η and Lipschitz driver g is defined as the solution of the BSDE with (deterministic) terminal time T, terminal condition η

and Lipschitz driver g^τ defined by $g^\tau(t,y,z) := g(t,y,z)1_{\{t\leqslant\tau\}}$. The first component of this solution at time t is equal to $\mathcal{E}^{g^\tau}_{t,T}(\eta)$, also denoted by $\mathcal{E}^g_{t,\tau}(\eta)$. We have $\mathcal{E}^g_{t,\tau}(\eta)=\eta$ a.s. on the set $\{t\geqslant\tau\}$.
The following result summarizes some of the well-known properties of the g-evaluations.

Proposition 3.1. *Let g satisfy ① and ②. Let S,τ,θ be stopping times. Then the g-evaluation satisfies the following properties:*

(A1) (Monotonicity) $\mathcal{E}^g_{S,\tau}[\eta]\leqslant\mathcal{E}^g_{S,\tau}[\eta']$, if $\eta\leqslant\eta'$;

(A2) (Knowledge preserving) $\mathcal{E}^g_{\tau,S}[\eta]=\eta$, for all S,τ, such that $S\leqslant\tau$, for all $\eta\in L^2(\mathcal{F}_S)$.

(A3) (Time consistency) $\mathcal{E}^g_{S,\theta}[\mathcal{E}^g_{\theta,\tau}[\eta]]=\mathcal{E}^g_{S,\tau}[\eta]$, for all $S\leqslant\theta\leqslant\tau$, for all $\eta\in L^2(\mathcal{F}_\tau)$;

(A4) ("Generalized" zero-one law) $I_A\rho_{S,\tau}[\xi(\tau)]=I_A\rho_{S,\tau'}[\xi(\tau')]$, for all $A\in\mathcal{F}_S$, $\tau\in\Theta_S$, $\tau'\in\Theta_S$ such that $\tau=\tau'$ on A.

(A5) (Continuity with respect to terminal time and terminal condition)
Let $(\tau_n)_{n\in\mathrm{N}}$ be a sequence of stopping times in $\mathcal{T}_{S,\tau}$, such that $\lim_{n\to\infty}\tau_n=\tau$ a.s. Let $(\eta_n)_{n\in\mathrm{N}}$ be a sequence of random variables, such that $\eta_n\in L^2(\mathcal{F}_{\tau_n})$, $\sup_n\eta_n\in L^2$ and $\lim_{n\to\infty}\eta_n=\eta$ a.s. Then, we have $\lim_{n\to\infty}\mathcal{E}^g_{S,\tau_n}[\eta_n]=\mathcal{E}^g_{S,\tau}[\eta]$ a.s.

Remark 3.1. *For Property (A4) we refer, e.g., to Grigorova and Quenez (2016). Property (A5) was proven in Quenez and Sulem (2013) (in the case of jumps). In Peng (2004), the additional assumption $g(\cdot,0,0)=0$ is made ensuring that the g-evaluation satisfies the **usual** "zero-one law": for each $S\leqslant\tau$, for each $A\in\mathcal{F}_S$, $\mathcal{E}^g_{S,\tau}[1_A\eta]=1_A\mathcal{E}^g_{S,\tau}[\eta]$.*

Moreover, the g-evaluation $\mathcal{E}^g$ satisfies the property (18) in Theorem 2.3. Indeed, we have $\limsup_{n\to+\infty}\mathcal{E}^g_{S,\tau_n}[\phi(\tau_n)]\leqslant\mathcal{E}^g_{S,\tau^\star}[\limsup_{n\to+\infty}\phi(\tau_n)]$, for each non-decreasing sequence τ_n such that $\lim_{n\to+\infty}\tau_n=\tau^\star$, and for each square-integrable admissible family ϕ, such that $\sup_n|\phi(\tau_n)|\in L^2$. For a proof of this property, based on property (A5) of the g-evaluations, we refer to Lemma A.5 in Dumitrescu et al. (2016).

Moreover, in the Brownian framework, the first component (y_t) of the solution of the BSDE with Lipschitz driver g, terminal time T, and terminal condition $\eta\in L^2(\mathcal{F}_T)$, has continuous trajectories (in t). Hence, for any non-decreasing sequence (τ_n), such that $\lim_{n\to+\infty}\tau_n=T$,

$$\lim_{n\to+\infty}\mathcal{E}^g_{\tau_n,T}(\eta)=\lim_{n\to+\infty}y_{\tau_n}=y_T=\mathcal{E}^g_{T,T}(\eta)=\eta.$$

Thus, property (33) from Lemma 2.9 and Proposition 2.3 is satisfied.
Hence, the g-evaluation satisfies all the properties of the non-linear operators ρ used in our results.

It remains for us to argue that the integrability Assumption 2.1 (with $p = 2$) on the value family V is satisfied, under some suitable assumptions on ξ. The following remark is an application of Remark 2.6 to the framework of a complete financial market model with *imperfections encoded in the driver* g of the dynamics of self-financing portfolios.

Remark 3.2. *Let us place ourselves in a complete financial market model with possible imperfections (such as e.g. trading constraints, different interest rates for borrowing and lending, different repo rates, etc.). Let g be the driver from the dynamics of self-financing portfolios in this market. If the family $\xi = (\xi(\tau))_{\tau\in\Theta}$ is assumed to be square-integrable and super-replicable by a self-financing portfolio with wealth $X(\tau)$ at time τ, then, the family $(X(\tau))_{\tau\in\Theta}$ is a $(\Theta, \mathcal{E}^g)$-martingale. By Remark 2.6 the value $V = (V(\tau))_{\tau\in\Theta}$ is square-integrable (that is, satisfies Assumption 2.1 with $p = 2$).*

3.1.2 Peng's g-expectation

Peng's g-expectation is a particular case of the previous example, introduced in Peng (1997). In this case, the driver g is assumed to satisfy the conditions ①, ②, and condition ③' ($g(\cdot, y, 0) \equiv 0$, for all $y \in \mathrm{R}$). In this particular case, the non-linear operators do not depend on the second index, but on the first index only. Moreover, they satisfy the usual zero-one law.
More precisely, let g satisfy conditions ①, ② and ③'. The operators $\mathcal{E}_g[\cdot]$ and $\mathcal{E}_g[\cdot|\mathcal{F}_S]$ are defined by $\mathcal{E}_g[\cdot] := \mathcal{E}^g_{0,\tau}[\cdot]$, $\mathcal{E}_g[\cdot|\mathcal{F}_S] := \mathcal{E}^g_{S,\tau}[\cdot]$. The family of non-linear operators $(\mathcal{E}_g[\cdot|\mathcal{F}_S])$, indexed by the stopping times S, is called g-expectation.

The g-expectation $(\mathcal{E}_g[\cdot|\mathcal{F}_S])$ satisfies all the properties of the g-evaluation and additionally the following property:
(**usual** zero-one law) For stopping times S, τ such that $S \leqslant \tau$, for $A \in \mathcal{F}_S$, $\mathcal{E}_g[1_A\eta|\mathcal{F}_S] = 1_A\mathcal{E}_g[\eta|\mathcal{F}_S]$.

3.2 Bayraktar - Yao's non-linear expectations

Let $T > 0$ and let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, satisfying the usual conditions.
A non-linear expectation, called F-expectation, depending on one time index only, was introduced in the work by Bayraktar and Yao-Part I (2011), and optimal stopping problems in continuous time with F-expectations were studied in Bayraktar and Yao-Part II (2011). For simplicity of the exposition, we will consider the case where the domain of the F-expectation is the whole space L^∞. The non-linear operators (F-expectation) in Bayraktar and Yao-Part I (2011) are defined first for deterministic times, then extended to stopping times valued in a finite deterministic grid, then, extended to general stopping times valued in $[0, T]$. We will not repeat the construction here, but will recall the basic properties only

(cf. Bayraktar and Yao-Part I (2011) for explanations and details; in particular, cf. Propositions 2.7, 2.8, and 2.9 therein[2]).

We recall that, an F-expectation, is a family of operators $\mathcal{E} := (\mathcal{E}[\cdot|\mathcal{F}_S])$, such that $\mathcal{E}[\cdot|\mathcal{F}_S] : L^\infty(\Omega, \mathcal{F}_T, \mathrm{P}) \to L^\infty(\Omega, \mathcal{F}_S, \mathrm{P})$, satisfying the following properties, for all S,τ stopping times valued in $[0, T]$:

(C1) (Monotonicity and Positively strict monotonicity) $\mathcal{E}[\eta|\mathcal{F}_S] \leqslant \mathcal{E}[\eta'|\mathcal{F}_S]$, if $\eta \leqslant \eta'$. Moreover, if $0 \leqslant \eta \leqslant \eta'$ and $\mathcal{E}[\eta|\mathcal{F}_0] = \mathcal{E}[\eta'|\mathcal{F}_0]$, then $\eta = \eta'$;

(C2) (Consistency) $\mathcal{E}[\mathcal{E}[\eta|\mathcal{F}_\tau]|\mathcal{F}_S] = \mathcal{E}[\eta|\mathcal{F}_S]$, for any stopping times S, τ such that $0 \leqslant S \leqslant \tau \leqslant T$.

(C3) (Usual zero-one law) $\mathcal{E}[1_A \eta|\mathcal{F}_S] = 1_A \mathcal{E}[\eta|\mathcal{F}_S]$, for any $A \in \mathcal{F}_S$.

(C4) (Translation invariance) $\mathcal{E}[\eta + X|\mathcal{F}_S] = \mathcal{E}[\eta|\mathcal{F}_S] + X$, if $X \in L^\infty(\mathcal{F}_S)$.

(C5) (Knowledge preservation) $\mathcal{E}[\eta|\mathcal{F}_S] = \eta$, if $\eta \in L^\infty(\mathcal{F}_S)$.

(C6) (Local property) $\mathcal{E}[\eta 1_A + \eta' 1_{A^c}|\mathcal{F}_S] = \mathcal{E}[\eta 1_A|\mathcal{F}_S] + \mathcal{E}[\eta' 1_{A^c}|\mathcal{F}_S]$, for any $A \in \mathcal{F}_S$ and for any $\eta, \xi \in L^\infty(\mathcal{F}_T)$.

(C7) (Fatou property) Let (η_n) be a sequence in L^∞, satisfying $\inf_n \eta_n \geqslant c$, a.s. for some constant $c \in \mathbb{R}$, and such that $\lim_{n\to\infty} \eta_n = \eta$, where $\eta \in L^\infty$. Then, $\mathcal{E}[\eta|\mathcal{F}_S] \leqslant \liminf_{n\to+\infty} \mathcal{E}[\eta_n|\mathcal{F}_S]$.

(C8) (Dominated convergence) Let (η_n) be a sequence, such that $\inf_n \eta_n \geqslant c$, a.s. for some constant $c \in \mathbb{R}$, and such that $\lim_{n\to\infty} \eta_n = \eta$. If there exists $\bar{\eta} \in L^\infty$ such that $\eta_n \leqslant \bar{\eta}$ for all $n \in \mathrm{N}$, then the limit $\eta \in L^\infty$, and $\lim_{n\to\infty} \mathcal{E}[\eta_n|\mathcal{F}_S] = \mathcal{E}[\eta|\mathcal{F}_S]$.

Note that the knowledge preserving property is called constant preserving in Bayraktar and Yao-Part I (2011).

Remark 3.3. *The F-expectation satisfies property* (18) *of Theorem 2.3. Indeed, let (τ_n) be a non-decreasing sequence of stopping times, such that $\lim_{n\to+\infty} \uparrow \tau_n = \tau$. Let ϕ be an admissible L^∞-integrable family such that $\sup_{n\in\mathbb{N}} |\phi(\tau_n)| \in L^\infty$. As the F-expectation does not depend on the second index, to show property* (18)*, we need to show*

$$\limsup_{n\to+\infty} \mathcal{E}[\phi(\tau_n)|\mathcal{F}_S] \leqslant \mathcal{E}[\limsup_{n\to+\infty} \phi(\tau_n)|\mathcal{F}_S].$$

For each $n \in \mathrm{N}$, $\phi(\tau_n) \leqslant \sup_{p\geqslant n} \phi(\tau_p)$. Hence, by monotonicity of $\mathcal{E}[\cdot|\mathcal{F}_S]$, we have

$$\mathcal{E}[\phi(\tau_n)|\mathcal{F}_S] \leqslant \mathcal{E}[\sup_{p\geqslant n} \phi(\tau_p)|\mathcal{F}_S].$$

The sequence η_n defined by $\eta_n = \sup_{p\geqslant n} \phi(\tau_p)$ is non-increasing and tends to $\eta := \limsup_{n\to+\infty} \phi(\tau_n)$. Moreover, $\sup_{p\in\mathrm{N}} |\phi(\tau_p)| \in L^\infty$. Hence, by property (C8), we have $\lim_{n\to+\infty} \mathcal{E}[\eta_n|\mathcal{F}_S] = \mathcal{E}[\lim_{n\to+\infty} \eta_n|\mathcal{F}_S] = \mathcal{E}[\eta|\mathcal{F}_S]$. Hence,

$$\limsup_{n\to+\infty} \mathcal{E}[\phi(\tau_n)|\mathcal{F}_S] \leqslant \lim_{n\to+\infty} \mathcal{E}[\eta_n|\mathcal{F}_S] = \mathcal{E}[\eta|\mathcal{F}_S] = \mathcal{E}[\limsup_{n\to+\infty} \phi(\tau_n)|\mathcal{F}_S],$$

[2]Note that in the case of the present exposition, $Dom^{\#}(\mathcal{E})$ of Bayraktar and Yao-Part I (2011) is equal to L^∞.

which is the desired property.

The assumptions imposed on the pay-off in Bayraktar - Yao (Bayraktar and Yao-Part I (2011) and Bayraktar and Yao-Part II (2011)) ensure that the value in their case belongs to the domain of the operator. Hence, if we consider the Bermudan style version of their problem, Assumption 2.1 on V will also be satisfied.

3.3 Dynamic Concave Utilities

The dynamic concave utilities are among the examples of non-linear operators depending on two time indices.
In this example the space is L^∞ (that is $p = +\infty$).
We place ourselves again in the Brownian framework. A representation result, with an explicit form for the penalty term, for dynamic concave utilities was established in Delbaen et al. (2010). The optimal stopping problem with dynamic concave utilities was studied by Bayraktar, Karatzas and Yao in Bayraktar et al. (2010), where the authors rely on the representation result from Delbaen et al. (2010).[3]
We recall the following definition from Delbaen et al. (2010).

Definition 3.2. *(Dynamic concave utility) For $S, \tau \in \mathcal{T}_{0,T}$, such that $S \leqslant \tau$, let*

$$\{u_{S,\tau}(\cdot) : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_S)\}$$

be a family of operators. This family is called a dynamic concave utility, if it satisfies the following properties:

(D1) (Monotonicity) $u_{S,\tau}(\eta) \leqslant u_{S,\tau}(\eta')$, if $\eta \leqslant \eta'$;

(D2) (Translation invariance) $u_{S,\tau}(\eta + X) = u_{S,\tau}(\eta) + X$, if $\eta \in L^\infty(\mathcal{F}_\tau)$ and $X \in L^\infty(\mathcal{F}_S)$;

(D3) (Concavity) $u_{S,\tau}(\lambda\eta + (1-\lambda)\eta') \geqslant \lambda u_{S,\tau}(\eta) + (1-\lambda)u_{S,\tau}(\eta')$, for any $\lambda \in [0,1]$ and $\eta, \eta' \in L^\infty(\mathcal{F}_\tau)$;

(D4) (Normalisation) $u_{S,\tau}(0) = 0$.

Moreover, in Delbaen et al. (2010) and Bayraktar et al. (2010) the following properties on the dynamic concave utilities are assumed:

(D5) (Time consistency) for any stopping time $\sigma \in \mathcal{T}_{S,\tau}$, we have $u_{S,\sigma}(u_{\sigma,\tau}(\eta)) = u_{S,\tau}(\eta)$;

(D6) (Continuity from above) for any non-increasing sequence $(\eta_n) \subset L^\infty(\mathcal{F}_\tau)$ with $\eta = \lim_{n\to\infty} \downarrow \eta_n \in L^\infty(\mathcal{F}_\tau)$, we have $\lim_{n\to\infty} \downarrow u_{S,\tau}(\eta_n) = u_{S,\tau}(\eta)$

(D7) (Local property) $u_{S,\tau}(\eta 1_A + \xi 1_{A^c}) = u_{S,\tau}(\eta)1_A + u_{S,\tau}(\xi)1_{A^c}$, for any $A \in \mathcal{F}_S$ and for any $\eta, \xi \in L^\infty(\mathcal{F}_\tau)$;

(D8) $E_P[\eta|\mathcal{F}_t] \geqslant 0$ for any $\eta \in L^\infty(\mathcal{F}_T)$, such that $u_{t,T}(\eta) \geqslant 0$.

[3] In Bayraktar et al. (2010) the authors choose a different sign convention on ρ, and hence, study a minimization optimal stopping problem with dynamic convex risk measures.

Remark 3.4. *Note that the assumptions in (D6) imply that, for each* $n \in \mathbb{N}$, $\eta_0 \leqslant \eta_n \leqslant \eta$, *where* $\eta_0 \in L^\infty$ *and* $\eta \in L^\infty$. *Hence,* $\sup_n \eta_n \in L^\infty$ *in (D6).*

By the results of Delbaen et al. (2010), any functional ρ satisfying properties (D1) - (D8), has the following representation:

$$\begin{aligned} u_{S,\tau}(\eta) &= \operatorname*{ess\,inf}_{Q:Q\sim P, Q=P \text{ on } \mathcal{F}_S} \Big\{ E_Q[\eta|\mathcal{F}_S] + c_{S,\tau}(Q) \Big\} \\ &= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_S} E_Q[\eta + \int_S^\tau f(u,\psi_u^Q)du|\mathcal{F}_S], \end{aligned} \tag{36}$$

where the function f is such that $f(\cdot,\cdot,x)$ is predictable for any x; f is a proper, convex function in the space variable x, and valued in $[0,+\infty]$, and the process (ψ_t^Q) is the process from the Doleans-Dade exponential representation for the density process (Z_t^Q), where $Z_t^Q = \frac{dQ}{dP}|_{\mathcal{F}_t}$, and

$$\mathcal{Q}_S = \{Q : Q \sim P, \psi_t^Q(\omega) = 0, dt\otimes dP \text{ a.e. on } [[0,S[[,\ E_Q[\int_S^T f(s,\psi_s^Q)ds] < +\infty\}.$$

Remark 3.5. *A close inspection of the proof of the duality result in (Bion-Nadal (2009) and Delbaen et al. (2010)) reveals that the dynamic concave utilities depend on the second index only via their penalty term.*

It has been noted in Delbaen et al. (2010) that property (D8) is equivalent to

$$c_{t,T}(P) = 0, \text{ for all } t \in [0,T]. \tag{37}$$

The dynamic concave utilities $u_{S,\tau}$ do not enter directly into the framework of the present paper, as they are defined only for S,τ such that $S \leqslant \tau$ a.s. (cf. Delbaen et al. (2010) and Bayraktar et al. (2010)). There is, however, a "natural" extension of $u_{S,\tau}$ in view of the representation property (36). This extension is as follows:

For S and τ stopping times, and $\eta \in L^\infty(\mathcal{F}_\tau)$, we define

$$1_{\{S>\tau\}} u_{S,\tau}(\eta) = 1_{\{S>\tau\}} \times \eta. \tag{38}$$

Remark 3.6. *By properties (D4) and (D7) of the dynamic concave utilities, we get that the "generalized zero-one law" is satisfied. Indeed, let* $A \in \mathcal{F}_S$, *and let* $\tau,\tau' \in \Theta_S$ *be such that* $\tau = \tau'$ *on* A. *Let* $\eta \in L^\infty(\mathcal{F}_\tau)$. *By applying property (D7) with* $\xi = 0$, *we get* $u_{S,\tau}(1_A\eta) = 1_A u_{S,\tau}(\eta) + 1_{A^c} u_{S,\tau}(0)$. *As* $u_{S,\tau}(0) = 0$ *due to the normalisation property (D4), we obtain* $u_{S,\tau}(1_A\eta) = 1_A u_{S,\tau}(\eta)$. *Hence, the "generalized zero-one law" is satisfied.*

Remark 3.7. *Property translation invariance (D2) and property normalisation (D4) imply that* $u_{S,\tau}(\cdot)$ *satisfies the knowledge preserving property (iii) of* ρ. *Indeed, for any* $\mathcal{F}_S$*-measurable* η, *we have, by (D2) and (D4),* $u_{S,\tau}(\eta) = u_{S,\tau}(0) + \eta = 0 + \eta = \eta$, *which shows property (iii).*

Remark 3.8. *The dynamic concave utilities satisfy property* (18) *in Theorem 2.3. Indeed, let* (τ_n) *be a non-decreasing sequence of stopping times such that* $\tau_n \uparrow \tau$*, and let* ϕ *be an* L^∞*-integrable admissible family such that* $\sup_n |\phi(\tau_n)| \in L^\infty$*. Then, for each* $n \in \mathbb{N}$*,*

$$\phi(\tau_n) + \int_S^{\tau_n} f(u, \psi_u^Q)du \leqslant \sup_{p \geqslant n} \phi(\tau_p) + \int_S^{\tau} f(u, \psi_u^Q)du =: \eta_n, \tag{39}$$

where, for the inequality, we have used that f *is valued in* $[0, +\infty]$*, and* $\tau_n \leqslant \tau$*. As* $\operatorname{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\cdot|\mathcal{F}_S]$ *is non-decreasing, we get: for each* $n \in \mathbb{N}$*,*

$$\operatorname*{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\phi(\tau_n) + \int_S^{\tau_n} f(u, \psi_u^Q)du|\mathcal{F}_S] \leqslant \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\eta_n|\mathcal{F}_S]. \tag{40}$$

We have $\eta_n \downarrow \eta$*, where* $\eta := \limsup_{n \to +\infty} \phi(\tau_n) + \int_S^\tau f(u, \psi_u^Q)du$*.*
As $\operatorname{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\cdot|\mathcal{F}_S]$ *is continuous from above, we deduce*

$$\lim_{n \to +\infty} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\eta_n|\mathcal{F}_S] = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\eta|\mathcal{F}_S]. \tag{41}$$

Hence, from Eqs. (40) *and* (41)*, we get*

$$\limsup_{n \to +\infty} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\phi(\tau_n) + \int_S^{\tau_n} f(u, \psi_u^Q)du|\mathcal{F}_S] \leqslant \lim_{n \to +\infty} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\eta_n|\mathcal{F}_S] = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_S} E_Q[\eta|\mathcal{F}_S],$$

which, by the representation result (36)*, gives*

$$\limsup_{n \to +\infty} u_{S,\tau_n}(\phi(\tau_n)) \leqslant u_{S,\tau_n}(\limsup_{n \to +\infty} \phi(\tau_n))$$

. This is the desired property (18)*.*

Remark 3.9. *The dynamic concave utilities satisfy property* (33) *from Lemma 2.9 and Proposition 2.3. Indeed, for* $\tau_n \uparrow T$*,*

$$\begin{aligned} u_{\tau_n,T}(\eta) &= \operatorname*{ess\,inf}_{Q: Q \sim P, Q = P \text{ on } \mathcal{F}_{\tau_n}} \Big\{ E_Q[\eta|\mathcal{F}_{\tau_n}] + c_{\tau_n,T}(Q) \Big\} \leqslant E_P[\eta|\mathcal{F}_{\tau_n}] + c_{\tau_n,T}(P) \\ &= E_P[\eta|\mathcal{F}_{\tau_n}], \end{aligned} \tag{42}$$

where we have used that for each $n \in \mathbb{N}$*,* $c_{\tau_n,T}(P) = 0$ *by Eq.*(37)*. Hence,* $u_{\tau_n,T}(\eta) \leqslant E_P[\eta|\mathcal{F}_{\tau_n}]$*. The sequence* $(E_P[\eta|\mathcal{F}_{\tau_n}])$ *being a uniformly integrable* P*-martingale, with terminal value* $E_P[\eta|\mathcal{F}_T] = \eta$*, we get*

$$\limsup_{n \to \infty} u_{\tau_n,T}(\eta) \leqslant \lim_{n \to \infty} E_P[\eta|\mathcal{F}_{\tau_n}] = \eta,$$

which is the desired property.

To finish, the pay-off process (ξ_t) in Bayraktar et al. (2010) is assumed to be bounded. Hence, by the monotonicity and the knowledge preservation of u, if we consider the Bermudan-style version of the problem studied in Bayraktar et al. (2010), then the value V satisfies the integrability Assumption 2.1 (that is, for each $S \in \Theta, V(S) \in L^\infty$).

4 Appendix: The case of a finite number of pre-described stopping times

In this appendix, we treat the particular case where $(\theta_k)_{k\in\mathbb{N}_0}$ is constant from a certain term, ***independent of*** ω, onwards. More precisely, we place ourselves in the situation where there exists $n \in \mathbb{N}*$ (independent of ω) such that for each $m \geqslant n$, $\theta_m = T$

Theorem 4.1. *Let $\phi = (\phi(\tau),\, \tau \in \Theta)$ be a p-integrable admissible family. Under the assumptions of knowledge preservation (iii) and "generalized zero-one law" (vi) on the non-linear operators, if ϕ satisfies*

$$\rho_{\theta_k,\theta_{k+1}}[\phi(\theta_{k+1})] \leqslant \phi(\theta_k)(\text{resp. } = \phi(\theta_k)),\ \textit{for all } k, \tag{43}$$

then, for all $\tau \in \Theta$, we have

$$\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] \leqslant \phi(\theta_k \wedge \tau)(\text{resp. } = \phi(\theta_k \wedge \tau)). \tag{44}$$

Proof. Let $k \in \mathbb{N}_0$. We have

$$\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1}\wedge\tau)] = \mathbb{I}_{\{\tau\leqslant\theta_k\}}\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1}\wedge\tau)]+\mathbb{I}_{\{\tau>\theta_k\}}\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1}\wedge\tau)]. \tag{45}$$

We note that on the set $\{\tau \leqslant \theta_k\}$, $\theta_{k+1} \wedge \tau = \theta_k \wedge \tau$. Hence, by the "generalized zero-one law", we have

$$\mathbb{I}_{\{\tau\leqslant\theta_k\}}\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] = \mathbb{I}_{\{\tau\leqslant\theta_k\}}\rho_{\theta_k,\theta_k\wedge\tau}[\phi(\theta_k \wedge \tau)].$$

As $\theta_k \wedge \tau \leqslant \theta_k$, by property (iii) of the non-linear evaluation ρ, we get

$$\rho_{\theta_k,\theta_k\wedge\tau}[\phi(\theta_k \wedge \tau)] = \phi(\theta_k \wedge \tau).$$

Hence, we have

$$\mathbb{I}_{\{\tau\leqslant\theta_k\}}\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] = \mathbb{I}_{\{\tau\leqslant\theta_k\}}\phi(\theta_k \wedge \tau). \tag{46}$$

For the second term on the right-hand side of Equation (45), we note that $\tau\wedge\theta_{k+1} = \theta_{k+1}$ on $\{\tau > \theta_k\}$. Hence, by the "generalized zero-one law" of the non-linear evaluation ρ, we have

$$\mathbb{I}_{\{\tau>\theta_k\}}\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] = \mathbb{I}_{\{\tau>\theta_k\}}\rho_{\theta_k,\theta_{k+1}}[\phi(\theta_{k+1})].$$

This, together with Equation (43) on ϕ and the admissibility of ϕ, gives

$$\mathbb{I}_{\{\tau>\theta_k\}}\rho_{\theta_k,\theta_{k+1}}[\phi(\theta_{k+1})] \leqslant \mathbb{I}_{\{\tau>\theta_k\}}\phi(\theta_k) = \mathbb{I}_{\{\tau>\theta_k\}}\phi(\theta_k \wedge \tau). \tag{47}$$

By plugging in (46) and (47) in Equation (45), we get

$$\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] \leqslant \mathbb{I}_{\{\tau\leqslant\theta_k\}}\phi(\theta_k \wedge \tau) + \mathbb{I}_{\{\tau>\theta_k\}}\phi(\theta_k \wedge \tau) = \phi(\theta_k \wedge \tau).$$

This ends the proof. □

We establish a characterization of (Θ, ρ)-supermartingale (resp. (Θ, ρ)-martingale) families in the particular case where the sequence $(\theta_k)_{k\in\mathbb{N}_0}$ is constant from a certain term, *independent of* ω, onwards.

Proposition 4.1. *Assume that there exists $n \in \mathbb{N}_0$ such that $\theta_k = \theta_n = T$ a.s., for all $k \geqslant n$. Let $\phi = (\phi(\tau),\, \tau \in \Theta)$ be a p-integrable admissible family. Under the assumptions of admissibility (ii), knowledge preservation (iii), monotonicity (iv), consistency (v) and "generalized" zero-one law (vi) on the non-linear operators, if $\rho_{\theta_k,\theta_{k+1}}[\phi(\theta_{k+1})] \leqslant \phi(\theta_k)$(resp. $= \phi(\theta_k)$), for all k, then, ϕ is a (Θ, g)-supermartingale* (resp. (Θ, g)–martingale) *family.*

Proof. We prove the result for the case of a (Θ, ρ)-supermartingale family; the case of a (Θ, ρ)-martingale family can be treated similarly. Let σ, τ in Θ be such that $\sigma \leqslant \tau$ a.s. As $\sigma \in \Theta$, we have $\sigma = \sum_{k=0}^{n} \theta_k \mathbf{1}_{A_k}$, where $(A_k)_{k\in\{0,\dots,n\}}$ is a partition of Ω such that $A_k \in \mathcal{F}_{\theta_k}$. We notice that in order to prove $\rho_{\sigma,\tau}[\phi_\tau] \leqslant \phi_\sigma$, it is sufficient to prove the following property:

$$\rho_{\theta_k\wedge\tau,\tau}[\phi(\tau)] \leqslant \phi(\theta_k \wedge \tau), \text{ for all } k \in \{0, 1, \dots, n\}. \tag{48}$$

Indeed, this property proven, we will have

$$\rho_{\sigma,\tau}[\phi(\tau)] = \rho_{\sigma\wedge\tau,\tau}[\phi(\tau)] = \sum_{k=0}^{n} \mathbb{I}_{A_k}\rho_{\theta_k\wedge\tau,\tau}[\phi(\tau)] \leqslant \sum_{k=0}^{n} \mathbb{I}_{A_k}\phi(\theta_k \wedge \tau) = \phi(\sigma \wedge \tau) = \phi(\sigma),$$

where we have used the admissibility of ρ to show the second equality. This will conclude the proof. Let us now prove property (48). We proceed by backward induction. For $k = n$, we have (recall that $\theta_n = T$)

$$\rho_{\theta_n\wedge\tau,\tau}[\phi(\tau)] = \rho_{T\wedge\tau,\tau}[\phi(\tau)] = \rho_{\tau,\tau}[\phi(\tau)] = \phi(\tau) = \phi(T \wedge \tau),$$

where we have used property (iii) to obtain the last but one equality.

We suppose that the property (48) holds true for $k + 1$. Then, by using this induction hypothesis, the time-consistency and the monotonicity of the non-linear operators, we get

$$\rho_{\theta_k\wedge\tau,\tau}[\phi(\tau)] = \rho_{\theta_k\wedge\tau,\theta_{k+1}\wedge\tau}[\rho_{\theta_{k+1}\wedge\tau,\tau}[\phi(\tau)]] \leqslant \rho_{\theta_k\wedge\tau,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)].$$

In order to conclude, it remains to prove

$$\rho_{\theta_k\wedge\tau,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] \leqslant \phi(\theta_k \wedge \tau). \tag{49}$$

By Theorem 4.1, we have

$$\mathbb{I}_{\{\tau\geqslant\theta_k\}}\rho_{\theta_k\wedge\tau,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] = \mathbb{I}_{\{\tau\geqslant\theta_k\}}\rho_{\theta_k,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1} \wedge \tau)] \leqslant \mathbb{I}_{\{\tau\geqslant\theta_k\}}\phi(\theta_k \wedge \tau). \tag{50}$$

On the other hand, by the "generalized zero-one law", (applied with $A = \{\tau < \theta_k\}$ on which set, we have $\theta_{k+1} \wedge \tau = \tau$) we have

$$\mathbb{I}_{\{\tau<\theta_k\}}\rho_{\theta_k\wedge\tau,\theta_{k+1}\wedge\tau}[\phi(\theta_{k+1}\wedge\tau)] = \mathbb{I}_{\{\tau<\theta_k\}}\rho_{\tau,\tau}[\phi(\theta_{k+1}\wedge\tau)] = \phi(\tau)\mathbb{I}_{\{\tau<\theta_k\}}, \quad (51)$$

where we have used property (iii) on ρ for the last equality.

Combining Equations (50) and (51), we deduce (49). The proposition is thus proved. □

4.1 Dynamic programming principle in the case of finite number of pre-described stopping times

We first introduce an explicit construction, by backward induction, of what will turn out to be the (Θ, ρ)-Snell envelope of the pay-off family ξ, in this particular case of finite number of pre-described stopping times. Let ξ be, as before, a p-integrable admissible family.

Let us define the sequence of random variables $(U(\theta_k))_{k\in\{0,1,...,n\}}$ by backward induction as follows:

$$\begin{cases} U(\theta_n) := \xi(\theta_n), & k = n; \\ U(\theta_k) := \max(\xi(\theta_k); \rho_{\theta_k,\theta_{k+1}}[U(\theta_{k+1})]), & \text{for } k \in \{0, 1, ..., n-1\}. \end{cases} \quad (52)$$

From (52) we see, by backward induction, that for each $k \in \{0, 1, ..., n\}$, $U(\theta_k)$ is a well-defined real-valued random variable, which is $\mathcal{F}_{\theta_k}$-measurable and p-integrable. From (52), we also have $U(\theta_k) \geqslant \xi(\theta_k)$, for all $k \in \{0, 1, ..., n\}$.
Moreover, it can be shown that, for each $k \in \{0, 1, ..., n-1\}$, $U(\theta_k) = U(\theta_{k+1})$ a.s. on $\{\theta_k = \theta_{k+1}\}$. Indeed, due to the second property (admissibility) and to the third property (knowledge preservation) of the non-linear operators ρ, we have

$$\begin{aligned} U(\theta_k)1_{\{\theta_k=\theta_{k+1}\}} &= \max(\xi(\theta_k)1_{\{\theta_k=\theta_{k+1}\}}; \rho_{\theta_k,\theta_{k+1}}[U(\theta_{k+1})]1_{\{\theta_k=\theta_{k+1}\}}) \\ &= \max(\xi(\theta_{k+1})1_{\{\theta_k=\theta_{k+1}\}}; \rho_{\theta_{k+1},\theta_{k+1}}[U(\theta_{k+1})]1_{\{\theta_k=\theta_{k+1}\}}) \\ &= \max(\xi(\theta_{k+1})1_{\{\theta_k=\theta_{k+1}\}}; U(\theta_{k+1})1_{\{\theta_k=\theta_{k+1}\}}) = U(\theta_{k+1})1_{\{\theta_k=\theta_{k+1}\}}, \end{aligned}$$

where, for the last equality, we have used that $U(\theta_{k+1}) \geqslant \xi(\theta_{k+1})$.

Hence, $U(\theta_k) = U(\theta_{k+1})$ a.s. on $\{\theta_k = \theta_{k+1}\}$. We can thus "extend" U to the whole set Θ as follows. Let $\tau \in \Theta$. There exists a partition $(A_k)_{k\in\{0,1,...,n\}}$ such that, for each k, $A_k \in \mathcal{F}_{\theta_k}$, and such that $\tau = \sum_{k=0}^{n} \theta_k 1_{A_k}$. We set

$$U(\tau) := \sum_{k=0}^{n} U(\theta_k)1_{A_k}. \quad (53)$$

We show the following result:

Theorem 4.2. *Under the assumptions of admissibility (ii), knowledge preservation (iii), monotonicity (iv), consistency (v) and "generalized zero-one law" (vi) on the non-linear operators, the family* $U := (U(\tau), \tau \in \Theta)$ *defined by* (52) *and* (53) *coincides with the* (Θ, ρ)*-Snell envelope family of* ξ*.*

For this, we first show an easy lemma, based on Proposition 4.1 and on the definition of the family U.

Lemma 4.1. *Under the assumptions of admissibility (ii), knowledge preservation (iii), monotonicity (iv), consistency (v) and "generalized zero-one law" (vi) on the non-linear operators, the family* U *defined by* (52) *and* (53) *is a* (Θ, ρ)*-supermartingale family, dominating the family* ξ*.*

Proof. We already noticed, following the definition of $U(\theta_k)$, that, for each k, $U(\theta_k) \geqslant \xi(\theta_k)$. Hence, for each $\tau \in \Theta$, $U(\tau) \geqslant \xi(\tau)$ a.s. (by definition of $U(\tau)$, cf. (53)). The (Θ, ρ)-supermartingale property of U follows from the definition of $(U(\theta_k))_{k\in\{0,\dots,n\}}$ and from Proposition 4.1. □

The following proof of Theorem 4.2 is a combination of Lemma 4.1 and of a proof of the minimality property of U.

Proof of Theorem 4.2. By Lemma 4.1, U is a (Θ, ρ)-supermartingale family, dominating the family ξ. It remains to show that it is the minimal one.
Let $\hat{U}$ be (another) (Θ, ρ)-supermartingale family, such that $\hat{U}(\theta_k) \geqslant \xi(\theta_k)$, for each $k \in \{0, 1, ..., n\}$.
At the terminal time θ_n, we have $\hat{U}(\theta_n) \geqslant \xi(\theta_n) = U(\theta_n)$.
Let $k \in \{1, ..., n\}$. Suppose, by backward induction, that $\hat{U}(\theta_k) \geqslant U(\theta_k)$. We need to show that $\hat{U}(\theta_{k-1}) \geqslant U(\theta_{k-1})$.
By the backward induction hypothesis and by the monotonicity of the non-linear operators $\rho_{\theta_{k-1},\theta_k}$, we have $\rho_{\theta_{k-1},\theta_k}[\hat{U}(\theta_k)] \geqslant \rho_{\theta_{k-1},\theta_k}[U(\theta_k)]$. This, together with the definition of $U(\theta_{k-1})$, gives

$$U(\theta_{k-1}) = \max(\xi(\theta_{k-1}); \rho_{\theta_{k-1},\theta_k}[U(\theta_k)]) \leqslant \max(\xi(\theta_{k-1}); \rho_{\theta_{k-1},\theta_k}[\hat{U}(\theta_k)]).$$

Since $\hat{U}$ is a (Θ, ρ)-supermartingale family, we have $\hat{U}(\theta_{k-1}) \geqslant \rho_{\theta_{k-1},\theta_k}[\hat{U}(\theta_k)]$. Hence,

$$U(\theta_{k-1}) \leqslant \max(\xi(\theta_{k-1}); \rho_{\theta_{k-1},\theta_k}[\hat{U}(\theta_k)]) \leqslant \max(\xi(\theta_{k-1}); \hat{U}(\theta_{k-1})) = \hat{U}(\theta_{k-1}).$$

The reasoning by backward induction is thus finished and the minimality property of U shown. We conclude that the family U is equal to the smallest (Θ, ρ)-supermartingale family dominating the family ξ, that is, to the (Θ, ρ)-Snell envelope of the family ξ. □

4.2 Optimal stopping times in the case of finite number of pre-described stopping times

We define $\bar{\nu}_k$ by:

$$\bar{\nu}_k := \operatorname{ess\,inf} \bar{\mathcal{A}}_k, \quad \text{where } \bar{\mathcal{A}}_k := \{\tau \in \Theta_{\theta_k} : U(\tau) = \xi(\tau) \text{ a.s.}\}.$$

In the case of finite number of pre-described stopping times, we have:

$$\bar{\nu}_k = \inf\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l)\} = \min\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l)\}. \tag{54}$$

Indeed, as the set $\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l)\}$ is a subset of $\bar{\mathcal{A}}_k$, we have:

$$\begin{aligned}\bar{\nu}_k = \operatorname{ess\,inf}\{\tau \in \Theta_{\theta_k} : U(\tau) = \xi(\tau) \text{ a.s.}\} &\leqslant \operatorname{ess\,inf}\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l)\} \\ &= \inf\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l)\}.\end{aligned}$$

Let us now show the converse inequality: $\bar{\nu}_k \geqslant \inf\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l) \text{ a.s.}\}$. As the set $\bar{\mathcal{A}}_k$ is stable by pairwise minimization, there exists a sequence $(\tau^{(m)})_{m\in\mathrm{N}}$, such that: $(\tau^{(m)})$ is non-decreasing; for each m, $\tau^{(m)} \in \bar{\mathcal{A}}_k$; and $\lim_{m\to+\infty} \tau^{(m)} = \bar{\nu}_k$.
Let $\omega \in \Omega$ be given. For each $m \in \mathrm{N}$, $\tau^{(m)}(\omega) \in \{\theta_k(\omega), \theta_{k+1}(\omega), ..., \theta_n(\omega)\}$, and moreover, $U(\tau^{(m)})(\omega) = \xi(\tau^{(m)})(\omega)$.
As all the elements of the sequence $(\tau^{(m)}(\omega))_{m\in\mathrm{N}}$ are valued in $\{\theta_k(\omega), \theta_{k+1}(\omega), ..., \theta_n(\omega)\}$, we have, $\lim_{m\to+\infty} \tau^{(m)}(\omega) \in \{\theta_k(\omega), \theta_{k+1}(\omega), ..., \theta_n(\omega)\}$, which implies that from a certain rank onwards, the sequence is constant.
Moreover, we have $U(\lim_{m\to+\infty} \tau^{(m)})(\omega) = \xi(\lim_{m\to+\infty} \tau^{(m)})(\omega)$. Thus, we have

$$\bar{\nu}_k(\omega) = \lim_{m\to+\infty} \tau^{(m)}(\omega) \in \{\theta_k(\omega), \theta_{k+1}(\omega), ..., \theta_n(\omega)\}, \text{ and } U(\bar{\nu}_k)(\omega) = \xi(\bar{\nu}_k)(\omega).$$

Hence, $\bar{\nu}_k \geqslant \operatorname{ess\,inf}\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l) \text{ a.s.}\}$. As both inequalities hold true, we conclude

$$\bar{\nu}_k = \inf\{\theta_l \in \{\theta_k, ..., \theta_n\} : U(\theta_l) = \xi(\theta_l) \text{ a.s.}\}. \tag{55}$$

Lemma 4.2. *Under the assumptions of knowledge preservation (iii), consistency (v) and "generalized zero-one law" (vi) on ρ, the family $U = (U(\tau))$ is a (Θ, ρ)-martingale on $[\theta_k, \bar{\nu}_k]$.*

Proof. As U satisfies the DPP (which it does by definition of U), we have, by Lemma 2.8, for each $l \in \mathrm{N}$,

$$U(\theta_l \wedge \bar{\nu}_k) = \rho_{\theta_l\wedge\bar{\nu}_k, \theta_{l+1}\wedge\bar{\nu}_k}[U(\theta_{l+1} \wedge \bar{\nu}_k)]. \tag{56}$$

By Lemma 2.4, to show that U is a (Θ, ρ)-martingale on $[\theta_k, \bar{\nu}_k]$, it is sufficient to show that for any σ, such that $\theta_k \leqslant \sigma \leqslant \bar{\nu}_k$, it holds $U(\sigma) = \rho_{\sigma,\bar{\nu}_k}[U(\bar{\nu}_k)]$. Let $\sigma \in$

Θ_{θ_k}, such that $\sigma \leqslant \bar{\nu}_k$. Then, $\sigma = \sum_{i=k}^{n} \theta_i 1_{A_i}$ and $\sigma \leqslant \bar{\nu}_k$. Thus, it is sufficient to show that for $i \in \{k, ..., n\}$, such that $\theta_i \leqslant \bar{\nu}_k$, it holds $1_{A_i} \rho_{\theta_i, \bar{\nu}_k}[U(\bar{\nu}_k)] = 1_{A_i} U(\theta_i)$, which is the same as, for each $i \in \{k, ..., n\}$, such that $\theta_i \leqslant \bar{\nu}_k$,

$$1_{A_i} \rho_{\theta_i \wedge \bar{\nu}_k, \bar{\nu}_k}[U(\bar{\nu}_k)] = 1_{A_i} U(\theta_i \wedge \bar{\nu}_k).$$

We proceed by backward induction. At rank n, we have

$$\rho_{\theta_n \wedge \bar{\nu}_k, \bar{\nu}_k}[U(\bar{\nu}_k)] = \rho_{\bar{\nu}_k, \bar{\nu}_k}[U(\bar{\nu}_k)] = U(\bar{\nu}_k) = U(\theta_n \wedge \bar{\nu}_k),$$

where we have used that $\bar{\nu}_k \leqslant \theta_n = T$, and the knowledge preserving property of ρ.

We suppose, by backward induction, that the property holds true at rank $i+1$. We show it at rank i. By the consistency property and the backward induction hypothesis, we have

$$\rho_{\theta_i \wedge \bar{\nu}_k, \bar{\nu}_k}[U(\bar{\nu}_k)] = \rho_{\theta_i \wedge \bar{\nu}_k, \theta_{i+1} \wedge \bar{\nu}_k}[\rho_{\theta_{i+1} \wedge \bar{\nu}_k, \bar{\nu}_k}[U(\bar{\nu}_k)]] = \rho_{\theta_i \wedge \bar{\nu}_k, \theta_{i+1} \wedge \bar{\nu}_k}[U(\theta_{i+1} \wedge \bar{\nu}_k)].$$

By Eq. (56), $\rho_{\theta_i \wedge \bar{\nu}_k, \theta_{i+1} \wedge \bar{\nu}_k}[U(\theta_{i+1} \wedge \bar{\nu}_k)] = U(\theta_i \wedge \bar{\nu}_k)$. Hence, $\rho_{\theta_i \wedge \bar{\nu}_k, \bar{\nu}_k}[U(\bar{\nu}_k)] = U(\theta_i \wedge \bar{\nu}_k)$, which completes the reasoning by backward induction.

We conclude that the family U is a (Θ, ρ)-martingale on $[\theta_k, \bar{\nu}_k]$. □

We will now show that U coincides with V, that $\bar{\nu}_k$ is optimal for the optimal stopping problem from time θ_k-perspective, and that $\bar{\nu}_k = \nu_k$, where

$$\nu_k := \operatorname{ess\,inf} \mathcal{A}_k, \quad \text{where } \mathcal{A}_k := \{\tau \in \Theta_{\theta_k} : V(\tau) = \xi(\tau) \text{ a.s.}\}.$$

For this, we do not need any type of (Fatou)continuity assumption on ρ.

Theorem 4.3. *Under the assumptions of admissibility (ii), knowledge preservation (iii), monotonicity (iv), consistency (v), and "generalized zero-one law" (vi) on the non-linear operators, we have*

1. $U(\theta_k) = \rho_{\theta_k, \bar{\nu}_k}[\xi(\bar{\nu}_k)] = V(\theta_k)$.
2. $U = V$ *and* $\bar{\nu}_k = \nu_k$.

Proof. As $U \geqslant \xi$, and as U is a (Θ, ρ)-supermartingale (cf. Lemma 4.1), we have for any $\tau \in \Theta_{\theta_k}$, $U(\theta_k) \geqslant \rho_{\theta_k, \tau}[U(\tau)] \geqslant \rho_{\theta_k, \tau}[\xi(\tau)]$, where we have used the monotonicity of $\rho_{\theta_k, \tau}$ for the second inequality.

Hence,

$$U(\theta_k) \geqslant \operatorname{ess\,sup}_{\tau \in \Theta_{\theta_k}} \rho_{\theta_k, \tau}[\xi(\tau)] = V(\theta_k). \tag{57}$$

On the other hand, by Lemma 4.2, U is a (Θ, ρ)-martingale on $[\theta_k, \bar{\nu}_k]$. Moreover, by Eq. (55), we have $U(\bar{\nu}_k) = \xi(\bar{\nu}_k)$. Hence,

$$U(\theta_k) = \rho_{\theta_k, \bar{\nu}_k}[U(\bar{\nu}_k)] = \rho_{\theta_k, \bar{\nu}_k}[\xi(\bar{\nu}_k)] \leqslant \operatorname{ess\,sup}_{\tau \in \Theta_{\theta_k}} \rho_{\theta_k, \tau}[\xi(\tau)] = V(\theta_k).$$

We have thus showed: $U(\theta_k) = V(\theta_k) = \rho_{\theta_k, \bar{\nu}_k}[\xi(\bar{\nu}_k)]$, which proves statement 1 of the lemma.

Now, let us show statement 2. By admissibility of U and V, it follows from statement 1, that, for any $\tau \in \Theta$, $U(\tau) = V(\tau)$. Hence, $\bar{\nu}_k = \nu_k$ (from the definitions of $\bar{\nu}_k$ and ν_k), which proves statement 2 of the lemma. □

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