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Abstract

This paper addresses the boundary null-controllability of the semi-linear heat equation $\partial_t y - \partial_{xx} y + f(y) = 0$, $(x,t) \in (0,1) \times (0,T)$. Assuming that the nonlinear function f is locally Lipschitz and satisfies $\limsup_{|r| \to +\infty} |f(r)|/(|r| \ln^{3/2}|r|) \leqslant \beta$ for some $\beta > 0$ small enough and that the initial datum belongs to $L^\infty(0,1)$, we prove the global null-controllability using the Schauder fixed point theorem and a linearization for which the term $f(y)$ is seen as a right side of the equation. Then, assuming that f is $\mathcal{C}^1$ over $\mathbb{R}$ and satisfies $\limsup_{|r| \to \infty} |f'(r)|/\ln^{3/2}|r| \leqslant \beta$ for some β small enough, we show that the fixed point application is contracting yielding a constructive method to approximate boundary controls for the semilinear equation. The crucial technical point is a regularity property of a state-control pair for a linear heat equation with L^2 right hand side obtained by using a global Carleman estimate with boundary observation. Numerical experiments illustrate the results. The arguments developed can notably be extended to the multi-dimensional case.

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1 Introduction and main results

Let $\Omega := (0,1)$ and $Q_T := \Omega \times (0,T)$ for some $T > 0$. We are concerned with the global boundary controllability of the following one-dimensional semilinear heat equation

$$\begin{cases} \partial_t y - \partial_{xx} y + f(y) = 0 & \text{in } Q_T \\ y(0,\cdot) = 0, \quad y(1,\cdot) = v, & \text{in } (0,T), \\ y(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $u_0 \in L^2(\Omega)$ is a given initial data, $f \in \mathcal{C}^1(\mathbb{R})$ a nonlinear function satisfying $f(0) = 0$ and $v \in H^1(0,T)$ a control function acting only at the boundary point $x = 1$. According to [5, Section 5], if f is locally Lipschitz-continuous and satisfies the growth condition $|f(r)| \leqslant C(1 + |r| \ln(1+|r|))$ for all $r \in \mathbb{R}$ and some $C > 0$ then the solution to (1) is globally defined in $[0,T]$ and satisfies $y \in \mathcal{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$. Without a growth condition of this kind, the solutions to (1) can blow up before $t = T$: in general, the blow up time depends on f and on the size of $\|u_0\|_{L^2(\Omega)}$. We refer to [22] and to [19, Section 2 and Section 5] for a survey on this issue.

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System (1) is said to be null-controllable at time T if for any $u_0 \in L^2(\Omega)$, there exist controls in $H^1(0,T)$ and associated state y satisfying $y \in \mathcal{C}^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and $y(\cdot,T)=0$ in Ω.

The null-controllability of the parabolic systems has been intensively studied for the past decades. We recall the pioneering work by Fattorini and Russell [14] where the so-called moments method is introduced to prove the controllability for the linear heat equation in one space dimension. The Carleman estimates initially used by Fursikov and Imanuvilov [18] is another useful tool to handle the controllability of linear heat equations. Neumann boundary controls are notably considered in [6] within this approach. We also mention the Lebeau-Robbiano spectral strategy [24] leading to distributed controllability results for the linear heat equations. Both methods are applicable in any space dimension. Another method is the flatness approach addressed in [27] for the boundary null-controllability in one space dimension. Recently, the backstepping approach has been developed by Coron and Nguyen [9] to recover the boundary null-controllability of one-dimensional linear heat equation with variable coefficients in space. More precisely, constructive controls in feedback form are given for any $u_0 \in L^2(\Omega)$ and $T>0$; it is also proved in [9] that the equation can be stabilized in finite time by means of periodic time-varying feedback laws.

Regarding the global boundary controllability problem for the semilinear heat equations, we mention the work by Fabre, Puel and Zuazua [13] where approximate controllability (boundary or interior) results has been proved for globally Lipschitz function f.

For locally Lipschitz function f, the free solution may blow up if f behaves like $|r|\ln^p(1+|r|)$ for any $p>1$ (see [19] and [22]). However, with an action of a control, the blow-up phenomenon can be compensated is $p>1$ is not too large. In this regard, we recall the work by Fernández-Cara and Zuazua where a global null-controllability result has been established for the system

$$\begin{cases} \partial_t y - \partial_{xx} y + f(y) = v\,1_\omega & \text{in } \widetilde{Q}_T := \widetilde{\Omega}\times(0,T),\\ y = 0 & \text{on } \widetilde{\Sigma}_T := \partial\widetilde{\Omega}\times(0,T),\\ y(\cdot,0) = u_0(\cdot) & \text{in } \widetilde{\Omega}, \end{cases} \tag{2}$$

where ω is any bounded subdomain of $\widetilde{\Omega}\subset\mathbb{R}^d, d\geqslant 1$. Assuming that f is $\mathcal{C}^1$, satisfies $|f'(r)|\leqslant C(1+|r|^{4+d})$ *a.e. in* $\mathbb{R}$ and the asymptotic growth condition

$$(\mathbf{H_1})\quad \exists\,\alpha>0 \text{ s.t. } |f(r)|\leqslant |r|\Big(\alpha+\beta\ln_+^{3/2}|r|\Big)\quad \forall r\in\mathbb{R},$$

for some $\beta=\beta(\widetilde{\Omega})>0$ small enough, it is proved that for any $T>0$ and any $u_0\in L^\infty(\widetilde{\Omega})$ there exists a null control function $v\in L^\infty(\omega\times(0,T))$ for (2). Here and in the sequel, we note

$$\ln_+|r| = \begin{cases} 0 \text{ if } |r|\leqslant 1\\ \ln|r| \text{ else.}\end{cases}$$

We also mention [2] which gives a similar result assuming in addition that $f(r)r\geqslant -C(1+r^2)$ for all $r\in\mathbb{R}$ and some $C>0$. On the contrary, if $|f(r)|$ grows faster than $|r|\ln_+^p(|r|)$ with $p>2$, then for some initial data, the control cannot compensate the blow-up phenomenon occurring out of $\overline{\omega}$ (see [17, Theorem 1.1]). We mention the work of Le Balc'h [23] where uniform controllability results in large time are obtained for $p\leqslant 2$ assuming additional sign conditions on f, notably that $f(r)>0$ for $r>0$ or $f(r)<0$ for $r<0$, a condition not satisfied for $f(r)=-r\ln_+^p|r|$. We also refer to [10] by Coron and Trélat where it is proved that one may reach any steady-state to any other (of the semilinear problem) by means of a boundary control for sufficiently large time T, provided that both are in the same connected component of the set of steady-states.

The main controllability result of [17] is based on a fixed point argument, initially introduced in [31] for one-dimensional semilinear wave equations. Provided refined L^1 observability inequality (see [17, Proposition 3.2]), it is shown that the operator $\Lambda: L^\infty(\widetilde{Q}_T)\to L^\infty(\widetilde{Q}_T)$, where $y:=\Lambda(z)$ is a null-controlled solution corresponding to the control of minimal L^∞ norm of the linear boundary value problem

$$\begin{cases}\partial_t y - \Delta y + y\,\widetilde{f}(z) = v1_\omega & \text{in} \quad \widetilde{Q}_T\\ y = 0 \text{ on } \widetilde{\Sigma}_T, \quad y(\cdot,0) = u_0 & \text{in} \quad \widetilde{\Omega}\end{cases}, \qquad \widetilde{f}(r) := \begin{cases} f(r)/r & r \neq 0\\ f'(0) & r = 0\end{cases} \tag{3}$$

maps a closed ball of $L^\infty(\widetilde{Q}_T)$ into itself. The Kakutani's theorem provides the existence of a fixed point for Λ which is also a controlled solution for (2).

Recently, Ervedoza along with the second and third authors presented a simpler proof of the exact controllability in [11] by Schauder fixed point approach under the assumption $(\mathbf{H_1})$, which is not based on the cost of observability of the heat equation with respect to potentials: the underlying fixed point application $\Lambda : L^\infty(\widetilde{Q}_T) \to L^\infty(\widetilde{Q}_T)$ is defined by $y = \Lambda(z)$ a controlled solution of

$$\begin{cases}\partial_t y - \Delta y = -f(z) + v1_\omega & \text{in} \quad \widetilde{Q}_T\\ y = 0 \text{ on } \widetilde{\Sigma}_T, \quad y(\cdot,0) = u_0 & \text{in} \quad \widetilde{\Omega}\end{cases}, \tag{4}$$

so that the nonlinearity is seen there as a right hand side. As in [17], the crucial technical point is to prove that the controlled solution is uniformly bounded in $\widetilde{Q}_T$. Assuming $u_0 \in L^\infty(\widetilde{\Omega})$, $\widetilde{\Omega} \subset \mathbb{R}^d$, $d \leqslant 5$, it is shown that there exists null controls $v \in L^{p_d}(\omega \times (0,T))$ where $p_d \in [2,\infty]$ depends on the dimension d such that the corresponding solution y to (4) belongs to $L^\infty(\widetilde{Q}_T)$. Then, assuming that f is locally Lipschitz continuous and satisfies the growth condition

$$(\mathbf{H_1'}) \quad \exists\, \alpha > 0 \text{ s.t. } |f'(r)| \leqslant \alpha + \beta \ln_+^{3/2}|r| \quad \forall r \in \mathbb{R},$$

[11] provides a constructive sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a state-control pair for the semilinear equation (2). Global L^2 Carleman inequalities with large enough parameters play a crucial role as they allow to prove a contraction property. We also mention [26] where in the one-dimensional case a constructive method has been developed for the internal case by introducing the following (non-convex) least-squares problem

$$\inf_{(y,v)\in\mathcal{A}} E_s(y,v); \quad E_s(y,v) := \|\partial_t y - \Delta y + f(y) - v1_\omega\|^2_{L^2(\rho_0,\widetilde{Q}_T)},$$

where $\mathcal{A}$ is a convex space which incorporates the initial and controllability requirement and ρ_0 denotes a Carleman-type weight parametrized by s and blowing up as $t \to T^-$. Assuming $(\mathbf{H_1'})$ and the following Hölder condition

$$(\overline{\mathbf{H}}_{\mathbf{p}}) \quad \exists p \in [0,1] \text{ such that } \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|f'(a) - f'(b)|}{|a-b|^p} < +\infty,$$

a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a state-control pair for (2) with $u_0 \in H_0^1(\widetilde{\Omega})$ is designed in [26]. Moreover, after a finite number of iterations, the convergence is of order at least $1+p$. A similar construction is performed in the multi-dimensional case with $d \leqslant 3$ in [25] assuming that f is globally Lipschitz.

The main purpose of the present work is to study the boundary controllability of the semilinear system (1) by following the work [11] devoted to the distributed case. To the best of our knowledge, there is no direct proof for the boundary controllability when the nonlinear function f behaves like $|r|\ln_+^{3/2}|r|$ at infinity. We mention that the distributed controllability results mentioned above imply the boundary controllability by the usual domain extension method for the scalar heat equations. We give here a direct and constructive proof of the boundary controllability and leading to accurate estimates of the state-control pair (in contrast with the indirect domain extension method).

Main results of the present work. In this paper, we prove the following result, directly in the framework of boundary controllability.

Theorem 1 (Boundary null-controllablity). *Let any $T>0$ and $u_0 \in L^\infty(\Omega)$ be given. Assume that $f \in \mathcal{C}^1(\mathbb{R})$.*

- *There exists $\beta > 0$ such that if f satisfies the assumption $(\mathbf{H_1})$, then the system (1) is null controllable at time T with a control $v \in H_0^1(0,T)$.*
- *There exists $\beta > 0$ such that if f satisfies $f(0)=0$ and the assumption $(\mathbf{H_1'})$, then one can define a sequence $(z_k, v_k)_{k\in\mathbb{N}^\star}$ that strongly converges to a pair $(z,v) \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \times H_0^1(0,T)$ such that $(\eta y^1 + z, v)$ is a control-state pair for the system (1). η denotes a C^∞ function supported in $[0,T]$ and y^1 solves the free boundary value problem (6) depending on u_0.*

  Moreover, the convergence of $(z_k, v_k)_{k\in\mathbb{N}}$ holds at least with a linear w.r.t. the norm $\|\rho\cdot\|_{L^2(Q_T)} + \|\rho_1\cdot\|_{L^2(0,T)}$, where ρ and ρ_1 are some weights defined in (9), (11) which are blowing up near the points $t=0,T$ and s is chosen sufficiently large depending on $u_0 \in L^\infty(\Omega)$.

Strategy of the proof. To prove the above theorem, we decompose the solution y of (1) as follows

$$y = \eta y^1 + z, \tag{5}$$

where y^1 solves the free boundary value problem with non zero initial condition

$$\begin{cases} \partial_t y^1 - \partial_{xx} y^1 = -f(\eta y^1) & \text{in } Q_{T^\star}, \\ y^1(0,\cdot) = y^1(1,\cdot) = 0 & \text{in } (0,T^\star), \\ y^1(\cdot,0) = u_0 & \text{in } \Omega \end{cases} \tag{6}$$

and z solves the boundary value problem

$$\begin{cases} \partial_t z - \partial_{xx} z = -f(z+\eta y^1) + \eta f(\eta y^1) - \eta' y^1 & \text{in } Q_T, \\ z(0,\cdot) = 0, \quad z(1,\cdot) = v & \text{in } (0,T), \\ z(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{7}$$

Here, η is a $\mathcal{C}^\infty(\mathbb{R})$ function supported in $[0,T^\star)$ with $T^\star \in (0,T]$ is chosen in term notably of the size of u_0 in order that y^1 exists in $Q_{T^\star}$; in particular, $0 \leqslant \eta \leqslant 1$ and $\eta(0)=1$, $\eta(T)=0$ (see (14)).

The above decomposition reduces the boundary null-controllability of the initial system (1) to the one of (7) with the function v. The main reason of this decomposition is to simplify the obtention of regularity properties of the state-control pair from global Carleman estimates. Such properties ensuring notably that the controlled state is uniformly bounded Q_T is crucial for our fixed point strategy; we refer to Remark 3 for details.

In order to analyze the controllability of system (7), we introduce the following linearized system: for given $\widehat{z}$ in some suitable class $\mathcal{C}_R(s)$ $(R>0)$ depending on the Carleman parameter $s>0$, find a control v such that the solution z to

$$\begin{cases} \partial_t z - \partial_{xx} z = -f(\widehat{z}+\eta y^1) + \eta f(\eta y^1) - \eta' y^1 & \text{in } Q_T, \\ z(0,\cdot) = 0, \quad z(1,\cdot) = v & \text{in } (0,T), \\ z(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{8}$$

satisfies $z(\cdot,T)=0$ in Ω, and (z,v) corresponds to the minimizer of a quadratic functional J_s involving the state-control pair and Carleman weight functions (defined in Remark 2). The specific forms of the solution is then $z = \rho^{-2}(-\partial_t - \partial_{xx})p$ in Q_T and the associated control is $v = s\rho_1^{-2}(1,\cdot)\partial_x p(1,\cdot)$ in $(0,T)$, where p denotes the adjoint state associated with the system (8) and the Carleman weights are defined by (9)–(11). This defines an operator $\Lambda_s : \widehat{z} \mapsto z$ from some suitable class $\mathcal{C}_R(s)$ into itself, on which we can use fixed point arguments for s sufficiently large depending on $\|u_0\|_{L^\infty(\Omega)}$, namely Schauder fixed point argument for the first item of Theorem 1, and Banach-Picard fixed point argument for the second one, allowing to exhibit a sequence of convergent approximations of the control and controlled trajectory. The analysis of the fixed point operator is based on a Carleman estimate with boundary observation introduced in [7] recalled in Proposition 1 : it allows to get precise weighted estimates on the control and controlled trajectories.

In order to get L^∞ estimate for the controlled trajectories solution of (8), the boundary control v needs to be more regular than $L^2(0,T)$. It is notable that the regularity issue is more delicate for the boundary control of the parabolic equations than hyperbolic ones. In this regards, we mention the work [12] where regularity results of the null-control v for the system $u' = Au + Bv$ have been proved assuming the initial datum regular enough and that A generates a $\mathcal{C}^0$ group. Their method applies mainly to the time-reversible infinite-dimensional systems (in particular to the wave equation) so as to extend solution out of the time interval $(0,T)$. Similar idea has been used in [4] to determine the regularity of boundary control for the one-dimensional wave equations and as application, the authors proved the boundary exact controllability of semilinear wave equations with H_0^1 initial data. Nevertheless, in the case of heat equations, the $\mathcal{C}^0$-group property of the associated diffusion operator is missing. Therefore, to get higher regularity of our control $v = s\rho_1^{-2}(1,\cdot)\partial_x p(1,\cdot)$ (see Theorem 2) for the linearized model (8), where p denotes the associated adjoint state, we need to re-introduce a state $\widetilde{p}$ in such a way that takes the value 0 outside $(0,T)$ and equals with the adjoint state p up to some weight (vanishing at $t = 0,T$) in $(0,T)$. This will allow us to define a parabolic equation in $\widetilde{p}$ in $\mathbb{R}$ and then, using a similar approach as in [4], we can obtain the H^1 regularity in time for the control v (see Theorem 3) as soon as $u_0 \in L^\infty(\Omega)$, a sufficient condition to ensure that $y^1 \in L^\infty(Q_T)$. This part is crucial in the study of the boundary controllability of our system (1).

Outline. The rest of the paper is organized as follows. In Section 2.3, we discuss the decomposition (5) of the solution y then derive in Section 2.4 a controllability result for the linear heat equation with zero initial data along with some precise estimates of the solution-control pair in term of the right hand side, provided the Carleman parameter s is large enough. Section 2.5 is devoted to prove that the optimal control for system (8), a priori in $L^2(0,T)$, belongs actually to $H_0^1(0,T)$: this result stated in Theorem 3 is proved in Appendix B. Then, in Section 3.3, we prove by using Schauder fixed point argument the uniform null controllability of (1) for any time $T > 0$ assuming that $f \in \mathcal{C}^1(\mathbb{R})$ and satisfies the condition $(\mathbf{H_1})$. Then in Section 3.4, assuming the growth condition $(\mathbf{H_1'})$ on f' and $f(0) = 0$, we show that the operator Λ_s is contracting in the set $\mathcal{C}_r(s)$ defined in (31), yielding the convergence of the Picard iterates $z_{k+1} = \Lambda_s(z_k)$, $k \geqslant 0$, for any initialization $z_0 \in \mathcal{C}_r(s)$. Section 4 illustrates the results with some numerical experiments while Section 5 concludes with some remarks.

Notations. In this article, C denote generic constants depending on Ω and T, which may vary from line to line, but are independent of the Carleman parameter s.

2 Boundary null-controllability of a linearized system with controlled solution in $L^\infty(Q_T)$

2.1 Global Carleman estimate with boundary observation

We recall in this section a global Carleman estimate with boundary observation of fundamental use in the sequel (see notably [21, 6]). We define $\theta(t) = \frac{1}{t(T-t)}$ for t in $(0,T)$. Then, for any $s > 0$, $\lambda > 0$, we set the weight functions

$$\begin{cases} \xi(x,t) = \theta(t)e^{\lambda x} & \forall (x,t) \in Q_T, \\ \varphi(x,t) = \theta(t)\left(e^{2\lambda} - e^{\lambda x}\right) =: \theta(t)\varphi_1(x) & \forall (x,t) \in Q_T, \\ \rho(x,t) = e^{s\varphi(x,t)} & \forall (x,t) \in Q_T \end{cases} \tag{9}$$

and recall the following Carleman estimate; see for instance [7, Theorem 3.4, p. 164].

Proposition 1. *There exists constants $\lambda_0 > 0$, $s_0 > 0$ and $C > 0$ such that for all $q \in P := L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, the following Carleman estimate holds*

$$\begin{aligned} s^3\lambda^4 \int_{Q_T} e^{-2s\varphi}\xi^3|q|^2 + s\lambda^2 \int_{Q_T} e^{-2s\varphi}\xi|\partial_x q|^2 + s^{-1}\int_{Q_T} e^{-2s\varphi}\xi^{-1}\left(|\partial_t q|^2 + |\partial_{xx}q|^2\right) \\ \leqslant C\int_{Q_T} e^{-2s\varphi}|\partial_t q + \partial_{xx} q|^2 + Cs\lambda \int_0^T e^{-2s\varphi}\xi|\partial_x q(1,t)|^2 \end{aligned} \tag{10}$$

for all $\lambda \geqslant \lambda_0$ and $s \geqslant s_0$.

In the sequel, we fix $\lambda = \lambda_0$ in the above Carleman estimate and consider the following weight functions

$$\rho_0 := \theta^{-3/2}\rho, \quad \rho_1 := \theta^{-1/2}\rho, \quad \rho_2 := \theta^{1/2}\rho \qquad \text{in } Q_T. \tag{11}$$

Estimate (10) then reads in term only of the parameter $s \geqslant s_0$ as follows

$$\begin{aligned} s^3 \int_{Q_T} \rho_0^{-2}|q|^2 + s\int_{Q_T} \rho_1^{-2}|\partial_x q|^2 + s^{-1}\int_{Q_T} \rho_2^{-2}\left(|\partial_t q|^2 + |\partial_{xx} q|^2\right) \\ \leqslant C\int_{Q_T} \rho^{-2}|\partial_t q + \partial_{xx} q|^2 + Cs\int_0^T \rho_1^{-2}(1,t)|\partial_x q(1,t)|^2 \end{aligned} \tag{12}$$

for all $q \in P$.

2.2 Time of existence under the growth condition $(\mathbf{H_1})$

The time of existence of the solution u of

$$\begin{cases} \partial_t u - \partial_{xx} u = -g(t,u) & \text{in } Q_T, \\ u(0,\cdot) = u(1,\cdot) = 0 & \text{in } (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{13}$$

for some continuous function g depends on the size of the data g and u_0. We refer for instance to [22] and [19, Section 5]. In the particular case of nonlinearities satisfying the growth condition $(\mathbf{H_1})$, we have the sharper result proven in Appendix A.

Proposition 2. *Let $\alpha > 0$, $\beta > 0$ and $M > 0$. There exists $T^\star > 0$ such that, for all continuous function g on $\mathbb{R}^2$ satisfying the growth condition $|g(t,r)| \leqslant |r|(\alpha + \beta\ln_+^{3/2}|r|)$ for all $t \in \mathbb{R}$, for all $u_0 \in L^2(\Omega)$ such that $\|u_0\|_{L^2(\Omega)} \leqslant M$, there exists $u \in \mathcal{C}^0([0,T^\star];L^2(\Omega)) \cap L^2(0,T^\star;H_0^1(\Omega))$ solution of (13) on $Q_{T^\star}$.*

Moreover, a lower bound of $T^\star$ is given by $\frac{1}{2\alpha(M+1)+c\beta(M+1)^{3/2}}$ for some $c = c(\Omega) > 0$.

2.3 Reformulation of the controllability problem

We now use the decomposition (5), i.e. $y = \eta y^1 + z$ for some η that we now precise and reformulate the null controllability problem in term of the variable z.

For any nonlinearity f satisfying $(\mathbf{H_1})$, we associate a time $T^\star > 0$ according to Proposition 2. Let then $\widetilde{T} := \min(T, T^\star)$ and consider a smooth function $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \leqslant \eta \leqslant 1$ and

$$\eta(t) = \begin{cases} 1 & \text{if } t \leqslant \dfrac{\widetilde{T}}{2} \\ 0 & \text{if } t \geqslant \dfrac{3\widetilde{T}}{4}. \end{cases} \tag{14}$$

We then introduce the following uncontrolled semilinear system

$$\begin{cases} \partial_t y^1 - \partial_{xx} y^1 = -f(\eta y^1) & \text{in } Q_{T^\star}, \\ y^1(0,\cdot) = y^1(1,\cdot) = 0 & \text{in } (0,T^\star), \\ y^1(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{15}$$

Since f is $\mathcal{C}^1$, $f_1 : (t,s) \mapsto f(\eta(t)s)$ is a $\mathcal{C}^1$ map and satisfies the hypothesis of Proposition 2 (since $\|\eta\|_{L^\infty(\mathbb{R})} \leqslant 1$ and $\eta \in \mathcal{C}^\infty(\mathbb{R})$): therefore, there exists $y^1 \in \mathcal{C}^0([0,T^\star]; L^2(\Omega)) \cap L^2(0,T^\star; H_0^1(\Omega))$ solution of (15) on $Q_{T^\star}$. We emphasize that $T^\star$ does not depend on η. Thus $\eta y^1 \in \mathcal{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ and moreover we have the following uniform estimate.

Lemma 1. *Let $f \in \mathcal{C}^1(\mathbb{R})$ satisfying $(\mathbf{H_1})$, η be defined by (14) and y^1 be a solution of (15). Then $y^1 \in L^\infty(Q_{T^\star})$, $\eta y^1 \in L^\infty(Q_T)$ and*

$$\|\eta y^1\|_{L^\infty(Q_T)} \leqslant \|y^1\|_{L^\infty(Q_{T^\star})} \leqslant C\left[(\alpha+\beta)(1+\|u_0\|^2_{L^2(\Omega)})\right] + \|u_0\|_{L^\infty(\Omega)}. \tag{16}$$

Proof. We define y^2 and y^3 respectively the unique weak solution of

$$\begin{cases} \partial_t y^2 - \partial_{xx} y^2 = -f(\eta y^1) & \text{in } Q_{T^\star}, \\ y^2(0,\cdot) = y^2(1,\cdot) = 0 & \text{in } (0,T^\star), \\ y^2(\cdot,0) = 0 & \text{in } \Omega \end{cases} \qquad \begin{cases} \partial_t y^3 - \partial_{xx} y^3 = 0 & \text{in } Q_{T^\star}, \\ y^3(0,\cdot) = y^3(1,\cdot) = 0 & \text{in } (0,T^\star), \\ y^3(\cdot,0) = u_0 & \text{in } \Omega. \end{cases}$$

From the maximum principle, $y^3 \in L^\infty(Q_{T^\star})$ and $\|y^3\|_{L^\infty(Q_{T^\star})} \leqslant \|u_0\|_{L^\infty(\Omega)}$. On the other hand, using the L^2 regularity result of the heat equation, $y^2 \in \mathcal{C}^0([0,T^\star]; H_0^1(\Omega)) \cap L^2(0,T^\star; H^2(\Omega)) \cap H^1(0,T^\star; L^2(\Omega))$ and satisfies $\|y^2\|_{L^\infty(Q_{T^\star})} \leqslant C\|f(\eta y^1)\|_{L^2(Q_{T^\star})}$. Then, by the uniqueness of the weak solution of the linear heat equation, we infer that y^1 coincides with the sum $y^2 + y^3$, which implies that $y^1 \in L^\infty(Q_{T^\star})$ and

$$\|y^1\|_{L^\infty(Q_{T^\star})} \leqslant C\|f(\eta y^1)\|_{L^2(Q_{T^\star})} + \|u_0\|_{L^\infty(\Omega)}.$$

We also easily check that $\|f(\eta y^1)\|_{L^2(Q_{T^\star})} \leqslant C(\alpha\|y^1\|_{L^2(Q_{T^\star})} + \beta\|\partial_x y^1\|^2_{L^2(Q_{T^\star})})$, which gives using (63) for $t = T^\star$ the estimate $\|f(\eta y^1)\|_{L^2(Q_{T^\star})} \leqslant C(\alpha+\beta)(1+\|u_0\|^2_{L^2(\Omega)})$. This implies (16) $\square$

Remark 1. *If $u_0 \in H_0^1(\Omega)$, then $\eta y^1 \in \mathcal{C}^0([0,T; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$.*

Let us now define

$$z := y - \eta y^1, \tag{17}$$

which satisfies the following set of differential equations

$$\begin{cases} \partial_t z - \partial_{xx} z = -f(z + \eta y^1) + \eta f(\eta y^1) - \eta' y^1 & \text{in } Q_T, \\ z(0,\cdot) = 0, \quad z(1,\cdot) = v & \text{in } (0,T), \\ z(\cdot,0) = 0 & \text{in } \Omega \end{cases} \tag{18}$$

and observe that a null-control $v \in L^2(0,T)$ for z is also a null boundary control for y solution for (1) starting from the initial condition $u_0 \in L^\infty(\Omega)$. The main difference is that the system in z has zero initial condition; this will allow to employ the global Carleman estimate (1) with blowing up weights at time $t = 0$ and $t = T$.

2.4 Null-controllability of a linearized system and some a priori weighted estimates

In order to obtain the null-controllability for the system (18), we first establish the null-controllability of an associated linearized model. More precisely, we consider the following linear system

$$\begin{cases} \partial_t z - \partial_{xx} z = B & \text{in } Q_T, \\ z(0,\cdot) = 0, \quad z(1,\cdot) = v & \text{in } (0,T), \\ z(\cdot,0) = 0 & \text{in } \Omega \end{cases} \tag{19}$$

and makes use of the Carleman estimate (12).

Let us denote the operator $L^\star := -\partial_t - \partial_{xx}$ and recall that $P = L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$. For any real $s \geqslant s_0$ (appearing in the Carleman estimate (10)), we define the bilinear form

$$(p,q)_s := \int_{Q_T} \rho^{-2} L^\star p\, L^\star q + s\int_0^T \rho_1^{-2}(1,t)\partial_x p(1,t)\partial_x q(1,t), \quad \forall p,q \in P. \tag{20}$$

(20) defines an inner product in P; moreover the closure P_s of P endowed with this inner product is a Hilbert space. The norm defined on P_s is

$$\|p\|_{P_s} := \left(\int_{Q_T} \rho^{-2}|L^\star p|^2 + s\int_0^T \rho_1^{-2}(1,t)|\partial_x p(1,t)|^2\right)^{1/2}. \tag{21}$$

We now prove the solvability and existence of a control function v for the linear system (19).

Theorem 2. *Let any $T>0$ be given. For any $s \geqslant s_0$ and $B \in L^2(\rho,Q_T)$, there exists a unique $p \in P_s$, depending only on B such that*

$$(p,q)_s = \int_{Q_T} Bq, \quad \forall q \in P_s. \tag{22}$$

Then $v = s\rho_1^{-2}(1,t)\partial_x p(1,t) \in L^2(\rho_1,(0,T))$ is a control function for (19) where $z = \rho^{-2}L^\star p \in L^2(\rho,Q_T)$ is the associated controlled trajectory, that is $z(x,T)=0$ for all $x\in\Omega$ and the operator defined by

$$\Lambda_s^0 : B \mapsto z \tag{23}$$

is linear, continuous from $L^2(\rho,Q_T)$ to $L^2(\rho,Q_T)$. Moreover, the following estimate holds for some constant $C>0$ independent of s:

$$\|\rho z\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1 v\|_{L^2(0,T)} \leqslant Cs^{-3/2}\|\rho B\|_{L^2(Q_T)}. \tag{24}$$

Proof. We first ensure the solvability of the variational equation (22). Since $(\cdot,\cdot)_s$ is an inner product on P_s, it suffices to check that the right hand side of (22) is a linear continuous form on P_s.

For all $q\in P_s$: since $\rho B \in L^2(Q_T)$, we have $|\int_{Q_T} Bq| \leqslant \left(\int_{Q_T}|\rho_0 B|^2\right)^{1/2}\left(\int_{Q_T}|\rho_0^{-1}q|^2\right)^{1/2}$.

Now, since q satisfies the Carleman inequality (12), one has $\|\rho_0^{-1}q\|_{L^2(Q_T)} \leqslant Cs^{-3/2}\|q\|_{P_s}$. Thus, we have

$$\left|\int_{Q_T} Bq\right| \leqslant Cs^{-3/2}\|\rho_0 B\|_{L^2(Q_T)}\|q\|_{P_s} \leqslant Cs^{-3/2}\|\rho B\|_{L^2(Q_T)}\|q\|_{P_s}$$

since $\rho_0 = \theta^{-3/2}\rho \leqslant \widetilde{C}\rho$ (for some constant $\widetilde{C}>0$ depending on T).

Thus, the right hand side of (22) corresponds to a linear functional on P_s: the Riesz representation theorem implies the existence of a unique $p \in P_s$ satisfying the formulation (22) and additionally

$$\|p\|_{P_s} \leqslant C s^{-3/2} \|\rho B\|_{L^2(Q_T)}, \tag{25}$$

where the constant $C > 0$ is independent of $s \geqslant s_0$.

Then, we set $z = \rho^{-2} L^\star p$ and $v = s\rho_1^{-2}(1,t)\partial_x p(1,t)$. From the equality (22), the pair (z, v) satisfies

$$\int_{Q_T} z L^\star q \,\mathrm{d}x\mathrm{d}t + \int_0^T v(t)\partial_x q(1,t) = \int_{Q_T} Bq, \quad \forall q \in P_s,$$

meaning that $z \in L^2(Q_T)$ is the unique solution to the linear system (19) associated with the control function $v \in L^2(0,T)$ in the sense of transposition. Eventually, using the estimate (25) for p, we get that $\rho z = \rho^{-1} L^\star p \in L^2(Q_T)$ and $s^{-1/2}\rho_1 v = s^{1/2}\rho_1^{-1}(1,t)\partial_x p(1,t) \in L^2(0,T)$ and deduce the weighted estimate (24). $\square$

Remark 2. *The pair of functions (z, v) can be characterized as the unique minimizer of the functional*

$$J_s(z,v) = \frac{s}{2}\int_{Q_T} \rho^2 z^2 + \frac{1}{2}\int_0^T \rho_1^2(1,t)|v(t)|^2, \tag{26}$$

over the set $\{(z,v) \in L^2(\rho, Q_T) \times L^2(\rho_1,(0,T))$ solution to (19) with $z(\cdot,T) = 0$ in $\Omega\}$. We refer to [18] for a detailed analysis.

Remark 3. *The decomposition $y = z + \eta y^1$ reduces the null-controllability for a system with zero initial and terminal conditions. This allows to use standard Carleman estimates with weights blowing up both at $t = 0$ and $t = T$, in contrast for instance to what have be done in [11, 26]. On the contrary, the occurence of a non-zero initial data u_0 would lead the extra term $\int_\Omega u_0 q(0)$ in the variational formulation (22), while there is no $q(0)$ type term in (1). Similarly, a zero initial condition avoids technical difficulties when one wants to get extra regularity property for the state-control pair, as done in Appendix B.*

2.5 Refined estimate of the state-control pair in $H^1(0,T;L^2(\Omega)) \times H^1(0,T)$

We refine Theorem 2 as we improve the regularity estimate for the state-control pair for (19). This technical part is crucial in our analysis.

Recall the function φ and weight functions $\rho = e^{s\varphi}$, $\rho_2 = \theta^{1/2}\rho$ from (9) and denote

$$\max_{x\in\overline{\Omega}} \varphi(x,t) = \theta(t) \max_{x\in\overline{\Omega}} \varphi_1(x) = \theta(t)\left(e^{2\lambda} - 1\right) =: \theta(t)\varphi_{1,\star} \quad t \in (0,T), \tag{27}$$

and

$$\rho_\star(t) = \max_{x\in\overline{\Omega}} \rho(x,t) = e^{s\theta(t)\varphi_{1,\star}}, \quad \rho_{2,\star}(t) = \theta^{1/2}(t)\rho_\star(t), \quad \forall t \in (0,T). \tag{28}$$

The following technical result is proved in Appendix B.

Theorem 3. *Let $(z,v) \in L^2(\rho, Q_T) \times L^2(\rho_1,(0,T))$ be the state-control for system (19) given by Theorem 2. Then, for any $B \in L^2(\rho, Q_T)$, we have the following regularity estimates*

$$\left\|\rho\rho_{2,\star}^{-1/2}\partial_t z\right\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1\rho_{2,\star}^{-1/2} v_t\|_{L^2(0,T)} \leqslant C s^{1/2}\|\rho B\|_{L^2(Q_T)}, \tag{29}$$

and

$$\|z\|_{L^2(0,T;H^2(\Omega))} + \|z\|_{H^1(0,T;L^2(\Omega))} \leqslant C s e^{-c_3 s}\|\rho B\|_{L^2(Q_T)}, \tag{30}$$

for some constants $C > 0$ and $c_3 > 0$ that do not depend on $s \geqslant s_0$.

Remark 4. *The estimate* (30) *and the compact embedding* $L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\hookrightarrow \mathcal{C}^0(\overline{Q_T})$ *in dimension one imply that the controlled solution* z *belongs to* $L^\infty(Q_T)$.

3 Null-controllability of the semilinear equation

In this section, we prove the null-controllability of the semilinear equation in z, namely of (18).

Since the initial data u_0 belongs to $L^\infty(\Omega)$, the solution y^1 to (15) in Q_T satisfies $y^1\in L^\infty(Q_{T^\star})$.

For any $R>0$ and $s\geqslant s_0$, we now introduce the class $\mathcal{C}_R(s)$, closed subset of $L^2(\rho,Q_T)$ and $L^\infty(Q_T)$, as follows

$$\mathcal{C}_R(s):=\Big\{\widehat{z}\in L^\infty(Q_T)\cap L^2(\rho,Q_T):\ \|\widehat{z}\|_{L^\infty(Q_T)}\leqslant R,\ \ \|\rho\widehat{z}\|_{L^2(Q_T)}\leqslant R^{3/4}\Big\}. \tag{31}$$

Assume that the nonlinear function f satisfies the growth condition $(\mathbf{H_1})$ and for a given $\widehat{z}\in\mathcal{C}_R(s)$, we solve the following linearized control system in z,

$$\begin{cases}\partial_t z-\partial_{xx}z=-f(\widehat{z}+\eta y^1)+\eta f(\eta y^1)-\eta' y^1 & \text{in } Q_T,\\ z(0,\cdot)=0,\quad z(1,\cdot)=v & \text{in } (0,T),\\ z(\cdot,0)=0 & \text{in } \Omega.\end{cases} \tag{32}$$

The existence of a state-control pair (z,v) such that $z(\cdot,T)=0$ in Ω is guaranteed by Theorem 2. Moreover, one has $v\in H^1_0(0,T)$ and $z\in L^\infty(Q_T)$ by means of Theorem 3 and Remark 4 respectively.

Now, our aim is to prove the existence of a fixed point for the operator

$$\Lambda_s:\mathcal{C}_R(s)\to\mathcal{C}_R(s),\qquad \Lambda_s(\widehat{z})=z.$$

Recall that $\Lambda_s(\widehat{z})=\Lambda_s^0\big(-[f(\widehat{z}+\eta y^1)-\eta f(\eta y^1)+\eta' y^1]\big)$ in terms of the notation introduced in Theorem 2.

Claim. In the spirit of [4, 11] our goal is to show the following properties ;

- there exists some $\beta>0$ in $(\mathbf{H_1})$ such that the set $\mathcal{C}_R(s)$ is stable under the map Λ_s.
-
  - $\Lambda_s(\mathcal{C}_R(s))$ is compact in $\mathcal{C}_R(s)$ w.r.t. the topology induced by the norm of $L^\infty(Q_T)$.
  - Λ_s is a continuous map in $\mathcal{C}_R(s)$ w.r.t. the topology induced by the norm of $L^\infty(Q_T)$.

Then, by the Schauder fixed point theorem, Λ_s will have a fixed point z in $\mathcal{C}_R(s)$, which will be a controlled trajectory for the semilinear equation (18).

- We also show that Λ_s is a contracting map from $\mathcal{C}_R(s)$ into itself where $\mathcal{C}_R(s)$ is endowed with the metric associated to the norm $\|\rho\cdot\|_{L^2(Q_T)}$.

One can then construct a sequence $(z_k,v_k)_{k\in\mathbb{N}^\star}$ which strongly converges in $L^2(\rho,Q_T)\times L^2(\rho_1,(0,T))$ to a controlled pair (z,v) for (18). Moreover the convergence holds at least linearly w.r.t. the norm of $L^2(\rho,Q_T)\times L^2(\rho_1,(0,T))$.

3.1 Estimate of $\|\Lambda_s(\widehat{z})\|_{L^\infty(Q_T)}$

We begin with the following lemma.

Lemma 2. *Assume* $R>0$ *such that* $\|u_0\|_{L^2(\Omega)}\leqslant R^{1/2}$, $\|u_0\|_{L^\infty(\Omega)}\leqslant R$ *and* $f\in\mathcal{C}^1(\mathbb{R})$ *satisfying the growth condition* $(\mathbf{H_1})$. *There exists a constant* $C>0$ *independent of* $s\geqslant s_0$ *and* R, *such that for any* $\widehat{z}\in\mathcal{C}_R(s)$,

$$\begin{aligned}\big\|\rho\big[f(\widehat{z}+\eta y^1)-\eta f(\eta y^1)+\eta' y^1\big]\big\|_{L^2(Q_T)}&\leqslant\|\rho\widehat{z}\|_{L^2(Q_T)}\Big(C^\star+\beta\ln_+^{3/2}R\Big)\\ &\quad+Ce^{cs}\Big(1+\alpha+\beta\ln_+^{3/2}(\alpha+\beta+1)+\beta\ln_+^{3/2}R\Big)(1+R^{1/2}),\end{aligned} \tag{33}$$

where $c:=\|\varphi\|_{L^\infty(\Omega\times(\widetilde{T}/2,3\widetilde{T}/4))}$ *and* $C^\star$ *is given in Lemma 3.*

The proof is based on the following result proved in [17, section 3.4].

Lemma 3 ([17]). *Let any $m^\star > 0$ be given and assume $f \in \mathcal{C}^1(\mathbb{R})$ satisfying $f(0) = 0$ and the growth condition $(\mathbf{H_1})$. There exists a constant $C^\star > 0$ only depending in $m^\star$ and f, such that for all $r \in \mathbb{R}$ and for all $a \in [-m^\star, m^\star]$,*

$$|f(r+a) - f(a)| \leqslant |r|(C^\star + \beta \ln_+^{3/2} |r|).$$

Proof of lemma 2. We set $m^\star = \|y^1\|_{L^\infty(Q_{T^\star})}$ and observe that

$$\begin{aligned}
&\big\|\rho\big[f(\widehat{z} + \eta y^1) - \eta f(\eta y^1) + \eta' y^1\big]\big\|_{L^2(Q_T)} \\
\leqslant &\big\|\rho\big[f(\widehat{z} + \eta y^1) - f(\eta y^1)\big]\big\|_{L^2(Q_T)} + \|(1-\eta)\rho f(\eta y^1)\|_{L^2(Q_T)} + \|\eta' \rho y^1\|_{L^2(Q_T)}.
\end{aligned}$$

Lemma 3 then implies

$$\|\rho\big(f(\widehat{z} + \eta y^1) - f(\eta y^1)\big)\|_{L^2(Q_T)} \leqslant \|\rho \widehat{z}\|_{L^2(Q_T)} \left(C^\star + \beta \ln_+^{3/2} R\right). \tag{34}$$

Moreover, using $(\mathbf{H_1})$ and the properties of the function η from (14) and (16), we get

$$\begin{aligned}
\left(\int_{Q_T} \big|(1-\eta)\rho f(\eta y^1)\big|^2\right)^{1/2} &\leqslant \left(\int_{Q_T} \rho^2 (1-\eta)^2 \eta^2 |y^1|^2 \big(\alpha + \beta \ln_+^{3/2} |\eta y^1|\big)^2\right)^{1/2} \\
&\leqslant C\left(\alpha + \beta \ln_+^{3/2}(\|y^1\|_{L^\infty(Q_T)})\right)\left(\int_{\frac{\widetilde{T}}{2}}^{\frac{3\widetilde{T}}{4}} \int_\Omega \rho^2 |y^1|^2\right)^{1/2} \\
&\leqslant C e^{cs}\left(\alpha + \beta \ln_+^{3/2}(\alpha + \beta + 1) + \beta \ln_+^{3/2} R\right)(1 + R^{1/2}),
\end{aligned} \tag{35}$$

where $c > 0$ is defined by

$$c := \|\varphi\|_{L^\infty(\Omega \times (\widetilde{T}/2, 3\widetilde{T}/4))}, \tag{36}$$

and φ is given in (9). We finally have

$$\left(\int_{Q_T} \rho^2 (\eta')^2 |y^1|^2\right)^{1/2} \leqslant \left(\int_{\frac{\widetilde{T}}{2}}^{\frac{3\widetilde{T}}{4}} \int_\Omega \rho^2 |y^1|^2\right)^{1/2} \leqslant C e^{cs}(1 + R^{1/2}). \tag{37}$$

Combining (34), (35) and (37), we get the estimate (33). □

Proposition 3. *Under the assumptions of Lemma 2, for $s \geqslant s_0$ and for all $\widehat{z} \in \mathcal{C}_R(s)$, the solution $z = \Lambda_s(\widehat{z})$ to the linearized system (32) satisfies the following estimates:*

$$\begin{aligned}
\|\rho z\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1 v\|_{L^2(0,T)} \leqslant\ & C s^{-3/2} \|\rho \widehat{z}\|_{L^2(Q_T)} \left(C^\star + \beta \ln_+^{3/2} R\right) \\
&+ C s^{-3/2} e^{cs}\left(1 + \alpha + \beta \ln_+^{3/2}(\alpha + \beta + 1) + \beta \ln_+^{3/2} R\right)(1 + R^{1/2}).
\end{aligned} \tag{38}$$

Moreover, $z \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$ and

$$\begin{aligned}
\|z\|_{L^\infty(Q_T)} \leqslant\ & C s e^{-c_3 s} \|\rho \widehat{z}\|_{L^2(Q_T)} \left(C^\star + \beta \ln_+^{3/2} R\right) \\
&+ C s e^{(c - c_3)s}\left(1 + \alpha + \beta \ln_+^{3/2}(\alpha + \beta + 1) + \beta \ln_+^{3/2} R\right)(1 + R^{1/2})
\end{aligned} \tag{39}$$

where $C > 0$ neither depends on $s \geqslant s_0$ nor on R, $C^\star$ is given in Lemma 3 and $c, c_3 > 0$ are given by (36) *and* (112) *respectively.*

Proof. Put $B = -\big(f(\widehat{z} + \eta y^1) - \eta f(\eta y^1) + \eta' y^1\big)$ in the linear model (19). Then the proof follows from Theorem 2, Theorem 3, Remark 4 and Lemma 2. □

3.2 Stability of the class $\mathcal{C}_R(s)$ for suitable choices of parameters

Lemma 4. *Let $\mathcal{C}_R(s)$ be introduced in (31). Under the assumptions of Lemma 2, if β in $(\mathbf{H_1})$ is small enough, there exist s and $R>0$ large enough such that*

$$\Lambda_s\big(\mathcal{C}_R(s)\big) \subset \mathcal{C}_R(s). \tag{40}$$

Proof. We start with any $\widehat{z} \in \mathcal{C}_R(s)$ for $s \geqslant s_0$ and we look for the bounds of the solution $z = \Lambda_s(\widehat{z})$ (to (32)) with respect to the associated norms. Since $\widehat{z} \in \mathcal{C}_R(s)$, one has $\|\rho\widehat{z}\|_{L^2(Q_T)} \leqslant R^{3/4}$ and $\|\widehat{z}\|_{L^\infty(Q_T)} \leqslant R$. Therefore, the estimate (38) yields

$$\begin{aligned}\|\rho z\|_{L^2(Q_T)} \leqslant & Cs^{-3/2}R^{3/4}\left(C^\star + \beta\ln_+^{3/2} R\right)\\ &+ Cs^{-3/2}e^{cs}\left(1+\alpha+\beta\ln_+^{3/2}(\alpha+\beta+1)+\beta\ln_+^{3/2} R\right)(1+R^{1/2}).\end{aligned} \tag{41}$$

Similarly, estimate (39) implies

$$\begin{aligned}\|z\|_{L^\infty(Q_T)} \leqslant & Cse^{-c_3 s}R^{3/4}\left(C^\star + \beta\ln_+^{3/2} R\right)\\ &+ Cse^{(c-c_3)s}\left(1+\alpha+\beta\ln_+^{3/2}(\alpha+\beta+1)+\beta\ln_+^{3/2} R\right)(1+R^{1/2}).\end{aligned} \tag{42}$$

We now fix the parameter s in term of R as follows :

$$s = \frac{1}{4c}\ln_+ R, \tag{43}$$

for some $R>0$ large enough to ensure the condition $s \geqslant s_0$.

With this choice of s, the solution $z = \Lambda_s(\widehat{z})$ satisfies, in view of (41) (since $\widehat{z}\in\mathcal{C}_R(s)$),

$$\begin{aligned}\|\rho z\|_{L^2(Q_T)} \leqslant & \frac{C(4c)^{3/2}}{\ln_+^{3/2} R}R^{3/4}\left(C^\star + \beta\ln_+^{3/2} R\right)\\ &+ \frac{C(4c)^{3/2}}{\ln_+^{3/2} R}\left(1+\alpha+\beta\ln_+^{3/2}(\alpha+\beta+1)+\beta\ln_+^{3/2} R\right)(1+R^{1/2})R^{1/4}.\end{aligned} \tag{44}$$

Now, if $\beta>0$ is small enough such that

$$C(4c)^{3/2}\beta < \frac{1}{2}, \tag{45}$$

then, it can be guaranteed for large enough $R>0$ that

$$\begin{cases}\dfrac{C(4c)^{3/2}}{\ln_+^{3/2} R}R^{3/4}\left(C^\star + \beta\ln_+^{3/2} R\right) \leqslant \dfrac{1}{2}R^{3/4},\\ \dfrac{C(4c)^{3/2}}{\ln_+^{3/2} R}\left(1+\alpha+\beta\ln_+^{3/2}(\alpha+\beta+1)+\beta\ln_+^{3/2} R\right)(1+R^{1/2})R^{1/4} \leqslant \dfrac{1}{2}R^{3/4}.\end{cases} \tag{46}$$

This yields, in view of (44) that $\|\rho z\|_{L^2(Q_T)} \leqslant R^{3/4}$.

Similarly, in view of (42) and the fact that $\widehat{z}$ belongs to $\mathcal{C}_R(s)$, we infer that

$$\begin{aligned}\|z\|_{L^\infty(Q_T)} \leqslant & \frac{C\ln_+ R}{4c}R^{-\frac{c_3}{4c}}R^{3/4}\left(C^\star + \beta\ln_+^{3/2} R\right)\\ &+ \frac{C\ln_+ R}{4c}R^{\left(\frac{1}{4}-\frac{c_3}{4c}\right)}\left(1+\alpha+\beta\ln_+^{3/2}(\alpha+\beta+1)+\beta\ln_+^{3/2} R\right)(1+R^{1/2}).\end{aligned} \tag{47}$$

Taking $\beta > 0$ as before, R large enough, we observe that

$$\begin{cases} \dfrac{C\ln_+ R}{Mc_3} R^{-\frac{c_3}{4c}} R^{3/4} \left(C^\star + \beta \ln_+^{3/2} R \right) \leqslant \dfrac{1}{2} R, \\ \dfrac{C\ln_+ R}{Mc_3} R^{\left(\frac{1}{4} - \frac{c_3}{4c}\right)} \left(1 + \alpha + \beta \ln_+^{3/2} (\alpha + \beta + 1) + \beta \ln_+^{3/2} R \right) (1 + R^{1/2}) \leqslant \dfrac{1}{2} R. \end{cases} \tag{48}$$

From (47), this implies that $\|z\|_{L^\infty(Q_T)} \leqslant R$. It follows that $z = \Lambda_s(\widehat{z}) \in \mathcal{C}_R(s)$. $\square$

Remark 5. *The smallness condition on β is explicit:*

$$\beta < \frac{1}{2C(4c)^{3/2}}, \tag{49}$$

where C is the constant appearing in Proposition 3.

Remark 6. *Within the relation (43), $\mathcal{C}_R(s)$ is stable for Λ_s for any $R \geqslant R_0$ large enough (equivalently $s \geqslant s_0$ large enough). With the above choices, in view of (46)-(48) and the conditions $\|u_0\|_{L^2(\Omega)} \leqslant R^{1/2}$, $\|u_0\|_{L^\infty(\Omega)} \leqslant R$ appearing in Lemma 2, the lower bound s_0 can be chosen as depending logarithmically on $\|u_0\|_{L^\infty(\Omega)}$.*

3.3 Proof of the first item of Theorem 1: Application of Schauder fixed point argument

We prove the first item of Theorem 1 which reads as follows.

Theorem 4. *There exists $\beta > 0$ such that if the function $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $(\mathbf{H_1})$, then for all $u_0 \in L^\infty(\Omega)$ and $T > 0$, there exists a control $v \in H_0^1(0,T)$ and a solution $y \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega))$ of (1) such that $y(\cdot,T) = 0$.*

3.3.1 Relative compactness of the set $\Lambda_s(\mathcal{C}_R(s))$

Proposition 4. *Under the assumptions of Lemma 4, $\Lambda_s(\mathcal{C}_R(s))$ is a relatively compact subset of $\mathcal{C}_R(s)$ for the $\|\cdot\|_{L^\infty(Q_T)}$ norm.*

Proof. This is a consequence of (30) and the compact embedding $\{z \in L^2(0,T;H^2(\Omega)) \mid \partial_t z \in L^2(Q_T)\} \hookrightarrow \mathcal{C}^0(\overline{Q_T})$ in space dimension one (see [29, Corollary 8 p. 90 and Lemma 12 p. 91]). $\square$

3.3.2 Continuity of the map Λ_s in $\mathcal{C}_R(s)$

Proposition 5. *Under the assumptions of Lemma 4, the map $\Lambda_s : \mathcal{C}_R(s) \to \mathcal{C}_R(s)$ is continuous with respect to the $L^\infty(Q_T)$ norm.*

Proof. Let $(\widehat{z}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_R(s)$ such that $\widehat{z}_n \to \widehat{z}$ as $n \to +\infty$ w.r.t. the $L^\infty(Q_T)$ norm for some $\widehat{z} \in \mathcal{C}_R(s)$.

Let $z_n = \Lambda_s(\widehat{z}_n)$ and show that $z_n \to z := \Lambda_s(\widehat{z})$ as $n \to +\infty$ w.r.t. the same norm.

Since $f \in \mathcal{C}^1(\mathbb{R})$, f is uniformly continuous in $[-R,R]$ implying that

$$f(\widehat{z}_n + \eta y^1) \to f(\widehat{z} + \eta y^1) \quad \text{in} \quad L^\infty(Q_T) \tag{50}$$

and since $\left(\rho\big[f(\widehat{z}_n + \eta y^1) - \eta f(\eta y^1) + \eta' y^1\big]\right)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$ (see Lemma 2), we infer that

$$\rho\big[f(\widehat{z}_n + \eta y^1) - \eta f(\eta y^1) + \eta' y^1\big] \rightharpoonup \rho\big[f(\widehat{z} + \eta y^1) - \eta f(\eta y^1) + \eta' y^1\big] \quad \text{weakly in} \quad L^2(Q_T). \tag{51}$$

On the other hand, from Theorem 2, for all $n \in \mathbb{N}$ there exists $p_n \in P_s$ and $p \in P_s$ such that $z_n = \rho^{-2}(s)L^\star p_n$ and $z = \rho^{-2}(s)L^\star p$ where p_n and p are the unique solution of (22) associated to $B_n = f(\widehat{z}_n + \eta y^1) - \eta f(\eta y^1) + \eta' y^1$ and $B = f(\widehat{z} + \eta y^1) - \eta f(\eta y^1) + \eta' y^1$ respectively. Recalling the the bound (25) for p_n, we infer that $p_n \rightharpoonup p$ weakly in P_s, which yields, since $L^2(Q_T) = L^\star(P)$ to $z_n \rightharpoonup z$ weakly in $L^2(Q_T)$. But $\Lambda_s(\mathcal{C}_R(s))$ is a relatively compact subset of $\mathcal{C}_R(s)$ for the $\|\cdot\|_{L^\infty(Q_T)}$ norm (see Proposition 4) and thus $z_n \to z$ in $L^\infty(Q_T)$. □

3.3.3 Proof of Theorem 4

Suppose that s is given by (43) and β satisfies (49). Since $\mathcal{C}_R(s)$ is a closed convex set of $L^\infty(Q_T)$, by Lemma 4, Propositions 4 and 5, we deduce from the Schauder fixed-point theorem that there exists a fixed point $z \in \mathcal{C}_R(s)$ of the map Λ_s. Therefore, by construction of Λ_s, there exists $v \in L^2(\rho_1, (0,T))$ such that $z \in L^2(\rho, Q_T)$ is the unique solution of the null controllability problem (18) associated with v. Moreover, by Theorem 3, $v \in H^1_0(0,T)$ and thus $z \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

Then, we recover the controlled solution y of our main semilinear problem (1) from the decomposition $y = z + \eta y^1$ associated to the same control function v as above, and that satisfies $y \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega))$, and $y(\cdot,T) = 0$. This ends the proof of Theorem 4, that is the first item of Theorem 1.

Remark 7. *The $\mathcal{C}^1$ regularity assumption made on the function f in Theorem 4 is technical and the consequence of the decomposition* (5) *allowing to apply to z the Carleman estimate* (12)*, which involves blowing up weights at the initial and final time. In order to avoid this decomposition and assume only $f \in \mathcal{C}^0(\mathbb{R})$ as in [11], one would need to replace the estimate* (12) *by an estimate without blowing up weight at time $t = 0$ (as the one used in [11], see Lemma 2.1 and developed in [1, Theorem 2.5] in an interior situation). The adaptation of [1, Theorem 2.5] to a boundary situation remains to be done.*

3.4 Proof of the second item of Theorem 1: Application of Banach fixed point argument

We prove the second item of Theorem 1. Precisely, we prove that Λ_s is indeed a contracting map on the class $\mathcal{C}_R(s)$ w.r.t. $L^2(\rho, Q_T)$ norm. This leads to a constructive method to find its fixed point.

We endow the convex set $\mathcal{C}_R(s)$ to the metric associated with the distance function $d(z_1, z_2) := \|\rho(z_1 - z_2)\|_{L^2(Q_T)}$.

Proposition 6. *Assume that $f \in \mathcal{C}^1(\mathbb{R})$ with $f(0) = 0$ satisfies* $(\mathbf{H_1'})$ *with $\beta > 0$ as given by* (49)*; s and R as chosen in Lemma 4. Then, the following holds*

$$d\big(\Lambda_s(\widehat{z}_2), \Lambda_s(\widehat{z}_1)\big) \leqslant \frac{1}{2} d(\widehat{z}_2, \widehat{z}_1), \quad \forall \widehat{z}_1, \widehat{z}_2 \in \mathcal{C}_R(s) \tag{52}$$

and implies that Λ_s is a contracting map from $\mathcal{C}_R(s)$ into itself w.r.t. the metric d.

Proof. Let $\widehat{z}_1, \widehat{z}_2 \in \mathcal{C}_R(s)$. We have $\Lambda_s(\widehat{z}_j) = \Lambda_s^0\big(-[f(\widehat{z}_j - \eta f(\eta y^1) + \eta' y^1)]\big)$, $j = 1, 2$. But Λ_s^0 is linear and continuous from $L^2(\rho, Q_T)$ into itself and therefore, $\Lambda_s(\widehat{z}_2) - \Lambda_s(\widehat{z}_1) = \Lambda_s^0(-[f(\widehat{z}_2 + \eta y^1) - f(\widehat{z}_1 + \eta y^1)])$.

Then, from (24) we get that

$$\|\rho(\Lambda_s(\widehat{z}_2) - \Lambda_s(\widehat{z}_1))\|_{L^2(Q_T)} \leqslant C s^{-3/2} \|\rho(f(\widehat{z}_2 + \eta y^1) - f(\widehat{z}_1 + \eta y^1))\|_{L^2(Q_T)}.$$

Then, we can use $(\mathbf{H_1})$ to deduce

$$\|\rho(\Lambda_s(\widehat{z}_2) - \Lambda_s(\widehat{z}_1))\|_{L^2(Q_T)} \leqslant C(4c)^{3/2} \left(\frac{\alpha + \beta \ln_+^{3/2}((\alpha+\beta+2))}{\ln_+^{3/2} R} + \beta \right) \|\rho(\widehat{z}_2 - \widehat{z}_1)\|_{L^2(Q_T)}, \tag{53}$$

since s is given by (43). But recall that $C(4c)^{3/2}\beta < 1/2$ from (45), which gives the required result as soon as R is large enough. □

Theorem 5. *Assume that $f \in \mathcal{C}^1(\mathbb{R})$ with $f(0)=0$ satisfying $(\mathbf{H_1'})$ with $\beta > 0$ satisfying (49); s and R as chosen in Lemma 4. Then, for any $z_0 \in \mathcal{C}_R(s)$, the sequence $(z_k)_{k\in\mathbb{N}^\star} \subset \mathcal{C}_R(s)$ given by*

$$z_{k+1} = \Lambda_s(z_k), \quad k \geqslant 0,$$

together with the associated sequence of controls $(v_k)_{k\in\mathbb{N}^\star} \subset H_0^1(0,T)$ strongly converge in $L^2(\rho, Q_T) \times L^2(\rho_1,(0,T))$ to a pair (z,v) such that $(\eta y^1 + z, v)$ is a state-control pair solution of (1). Moreover, the convergence is at least linear with respect to the distance d.

Proof. The convergence of the sequence $(z_k)_{k\in\mathbb{N}}$ toward $z = \Lambda_s(z) \in \mathcal{C}_R(s)$ with linear rate follows from the contracting property of Λ_s:

$$\begin{aligned}\|\rho(z - z_k)\|_{L^2(Q_T)} &= \|\rho(\Lambda_s(z) - \Lambda_s(z_{k-1}))\|_{L^2(Q_T)} \\ &\leqslant \frac{1}{2^k}\|\rho(z - z_0)\|_{L^2(Q_T)} \leqslant \frac{1}{2^{k-1}} R^{3/4}.\end{aligned}$$

Let now $v \in H_0^1(0,T)$ be associated with z so that $z - z_k$ satisfies, for every $k \in \mathbb{N}^\star$

$$\begin{cases} \partial_t(z - z_k) - \partial_{xx}(z - z_k) = -\big[f(z + \eta y^1) - f(z_{k-1} + \eta y^1)\big] & \text{in } Q_T, \\ (z - z_k)(0,\cdot) = 0, \quad (z - z_k)(1,\cdot) = (v - v_k) & \text{in } (0,T), \\ (z - z_k)(\cdot,0) = (z - z_k)(\cdot,T) = 0 & \text{in } \Omega. \end{cases} \tag{54}$$

Estimate (24) then implies (recall $s = \frac{1}{4c}\ln_+ R$)

$$\begin{aligned}\|\rho_1(v - v_k)\|_{L^2(0,T)} &\leqslant C s^{-1}\big\|\rho\big[f(z + \eta y^1) - f(z_{k-1} + \eta y^1)\big]\big\|_{L^2(Q_T)} \\ &\leqslant C\frac{M c_3}{\ln_+ R}\Big(\alpha + \beta \ln_+^{3/2}(\alpha + \beta + 2) + \beta \ln_+^{3/2} R\Big)\|\rho(z - z_{k-1})\|_{L^2(Q_T)}\end{aligned}$$

and therefore the convergence of the sequence $(v_k)_{k\in\mathbb{N}^\star}$ toward a null control for (1) is at least linear. □

Remark 8. *The constant appearing in front of $\|\rho(s)(\widehat{z}_2 - \widehat{z}_1)\|_{L^2(Q_T)}$ (see (53)) is getting smaller as R (consequently s) getting larger (provided β is small enough). In particular, if f satisfies $\lim_{|r|\to+\infty} \frac{|f'(r)|}{\ln_+^{3/2}|r|} = 0$, then, for any given $\epsilon > 0$ (however small), the map Λ_s is ϵ-contractive for large enough $s \geqslant s_0$. Consequently, the speed of convergence of the sequence $(z_k)_{k\geqslant 1}$ introduced by Theorem 5 increases with s.*

4 Numerical illustrations

We present some numerical illustrations of the convergence result given by Theorem 5 and emphasize the influence of the parameter s. More precisely, for s large enough, we compute the sequence $(z_k, v_k)_{k\in\mathbb{N}}$ solution to

$$\begin{cases} \partial_t z_k - \partial_{xx} z_k = -f(z_{k-1} + \eta y^1) + \eta f(\eta y^1) - \eta' y^1 & \text{in } Q_T, \\ z_k(0,\cdot) = 0, \quad z_k(1,\cdot) = v_k & \text{in } (0,T), \\ z_k(\cdot,0) = z_k(\cdot,T) = 0 & \text{in } \Omega, \end{cases} \tag{55}$$

obtained through the variational formulation (22) with the source term $B = -f(z_{k-1} + \eta y^1) + \eta f(\eta y^1) - \eta' y^1$. We first sketch the algorithm by closely following the work [11] and then discuss some numerical experiments obtained with the software FreeFem++ (see [20]).

Construction of the free solution component y^1. At first, we need to construct a component y^1 satisfying the semilinear problem (15). To do so, we employ the least squares variational approach developed in [28] (actually for studying the numerical null-controllability of heat equations) well adapted to a space-time approximation.

4.1 Construction of the sequence $(z_k, v_k)_{k\geqslant 1}$

Starting with some suitable initial guess $z_0 \in \mathcal{C}_R(s)$, we can obtain the solution z_k to (55) with a control v_k based on Theorem 2. Assume that the value of the Carleman parameter s satisfies Lemma 4. Then, for each $k \geqslant 1$, we define the unique solution $p_k \in P_s$ (see Theorem 2) of

$$(p_k, q)_s = \int_{Q_T} \left(-f(z_{k-1} + \eta y^1) + \eta f(\eta y^1) - \eta' y^1\right) q \,\mathrm{d}x\mathrm{d}t, \qquad \forall q \in P_s. \tag{56}$$

Then, we set $z_k = \rho^{-2} L^\star p_k$ (recall that $L^\star = -\partial_t - \partial_{xx}$) in Q_T and $v_k = s\rho_1^{-2}(1,\cdot)\partial_x p_k(1,\cdot)$ in $(0,T)$.

The numerical approximation of the variational formulation (56) has been addressed in [15, 16]. A conformal finite dimensional approximations, say $P_{s,h}$ of P_s, leads to a strong convergent approximation $p_{k,h}$ of p_k for the P_s norm as the discretization parameter h goes to 0. Then, from $p_{k,h}$, we can define the approximated controlled solution $z_{k,h} := \rho^{-2} L^\star p_{k,h}$ and $v_{k,h} := s\rho_1^{-2}(1,\cdot)\partial_x p_{k,h}(1,\cdot)$ in $[0,T]$.

The sequence $(z_k, v_k)_{k\geqslant 1}$ is initialized with $(z_0, v_0) = (0,0)$ so that (z_1, v_1) is the solution to the linear system (55) with the right hand side $B = -f(\eta y^1) + \eta f(\eta y^1) - \eta' y^1$. We perform the iterations until the following criterion (based on Proposition 6) is fulfilled

$$\frac{\|\rho(z_{k+1} - z_k)\|_{L^2(Q_T)}}{\|\rho z_k\|_{L^2(Q_T)}} \leqslant 10^{-6}. \tag{57}$$

We denote by $k^\star$ the smallest integer k such that (57) holds. Remark that, since the weight ρ is strictly positive, the convergence of the sequence $(\rho z_k)_{k\in\mathbb{N}}$ (stated in Proposition 6) implies the convergence of the sequence $(z_k)_{k\in\mathbb{N}}$. Finally, we express the solution-control pair $(y_{k^\star}, v_{k^\star})$ of the main semilinear problem (1) as follows:

$$\begin{cases} y_{k^\star} = \eta y^1 + z_{k^\star} = \eta y^1 + \rho^{-2} L^\star p_{k^\star} & \text{in } Q_T, \\ v_{k^\star} = s\rho_1^{-2}(1,\cdot)\partial_x p_{k^\star}(1,\cdot) & \text{in } (0,T), \end{cases} \tag{58}$$

where y^1 is obtained numerically as mentioned earlier.

Concerning the approximation of the formulation (56), we use a conformal space-time finite element method. We introduce a regular triangulation $\mathcal{T}_h$ of Q_T such that $\overline{Q}_T = \bigcup_{K\in\mathcal{T}_h} K$. We assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family, where the index h is such that $h = \max_{K\in\mathcal{T}_h} diam\,(K)$. We then approximate the variables p_k in the space $V_h := \{v_h \in \mathcal{C}^1(\overline{Q}_T) : v_h|_K \in \mathbb{P}(K), \forall K \in \mathcal{T}_h\} \subset P_s$, where $\mathbb{P}(K)$ denotes the composite Hsieh-Clough-Tocher $\mathcal{C}^1$ element defined for triangles. We refer to [8, page 356] and [3] where the implementation has been discussed.

4.2 Numerical experiments

In this section, we make several numerical experiments related to our boundary null controllability of the semilinear heat equation. In this way, we can numerically observe the convergence of the sequence $(z_k, v_k)_{k\geqslant 1}$ to a state-control pair of the semilinear system (according to Theorem 5).

4.2.1 Experiments with nonlinear growth $r\ln^{3/2}(2+|r|)$

Let us consider the nonlinear function as follows:

$$f(r) = c_f\, r\left(\alpha + \beta \ln^{3/2}\left(2 + |r|\right)\right), \tag{59}$$

with $\alpha = \beta = 1$ and $c_f \in \mathbb{R}^*$. We also consider the following data

$$\begin{cases} T = 0.5, \quad T^\star = T/2, \\ u_0(x) = c_{u_0}\sin(\pi x) \quad \forall x \in \Omega, \quad \text{with } c_{u_0} > 0. \end{cases} \tag{60}$$

Moreover, the function η introduced in the decomposition (5) and appearing notably in (56) is defined as the $C^1([0,T])$ function constant equal to one in $[0,T^\star/2]$, constant equal to zero in $[3T^\star/2,T]$ and polynomial of order 3 in $[T^\star/2,3T^\star/2]$.

Last, the value of the parameter $\lambda=\lambda_0$ is taken equal to 0.02 in (9).

Finally we employ a regular space-time mesh of the domain $Q_T=(0,1)\times(0,0.5)$ composed of 40 000 triangles and 20 301 vertices corresponding to a discretization parameter $h\approx 7.07\times 10^{-3}$.

I. Experiments for fixed (c_f,c_{u_0}) w.r.t. the parameter s. Let us make some experiments for fixed parameters c_f (associated with the nonlinear function) and c_{u_0} (associated with the initial data). We first choose $c_f=-1.5$ and $c_{u_0}=15$, and present several norms of the state-control pair in Table 1 w.r.t. $s\in\{1,2,3,4\}$.

s	$\Vert z_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert\rho z_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert v_{k^\star}\Vert_{L^2(0,T)}$	$\Vert v_{k^\star}\Vert_{L^\infty(0,T)}$	$k^\star$
1	8.658×10^{-1}	1.440	4.024	1.811	3.969	10
2	8.673×10^{-1}	2.182	4.100	2.037	4.890	9
3	8.924×10^{-1}	3.325	3.999	2.215	5.633	9
4	9.312×10^{-1}	5.086	3.992	2.382	6.335	8

Table 1: $(c_f,c_{u_0})=(-1.5,15)$; $f(r)=c_f r(1+\ln^{3/2}(2+|r|))$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. s.

But as soon as we increase the value of $|c_f|$, the convergence for a given s is lost. For instance, if we choose $c_f=-2.5$ (with the same value $c_{u_0}=15$), we check that $s=1$ is not large enough to imply the Banach contraction property (ensuring the convergence of the algorithm w.r.t. the criterion (57)). Then, by choosing $s\geqslant 2$, we recover the required convergence criterion (57). The results are provided in Table 2.

s	$\Vert z_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert\rho z_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert v_{k^\star}\Vert_{L^2(0,T)}$	$\Vert v_{k^\star}\Vert_{L^\infty(0,T)}$	$k^\star$
2	14.595	41.827	17.597	27.901	69.078	31
3	13.372	55.104	16.416	25.779	68.826	26
4	12.604	75.003	15.578	24.466	69.231	22

Table 2: $(c_f,c_{u_0})=(-2.5,15)$; $f(r)=c_f r(1+\ln^{3/2}(2+|r|))$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. s.

Figure 1 depicts the evolution of the relative error $\frac{\Vert\rho(z_{k+1}-z_k)\Vert_{L^2(Q_T)}}{\Vert\rho z_k\Vert_{L^2(Q_T)}}$ w.r.t. to the iteration number k for $s\in\{1,2,3,4\}$. In agreement with Remark 8, we observe that the decay of the error is amplified with larger values of s. We also observe in Table 2 that the weighted norm $\Vert\rho z_{k^\star}\Vert_{L^2(Q_T)}$ increases as s getting larger in agreement with the fact that the weight ρ increases with s (they behave like $e^{s\theta(t)}$). We also observe that the norms $\Vert z_{k^\star}\Vert_{L^2(Q_T)}$ (consequently $\Vert y_{k^\star}\Vert_{L^2(Q_T)}$) and $\Vert v_{k^\star}\Vert_{L^2(0,T)}$ of the control-state pair are monotonous with respect to s.

II. Evolution of the controlled solutions. In this part, we present some figures of the controlled solutions and the associated controls for the semilinear system.

We first consider $c_f=-1.5$ and $c_{u_0}=15$. Figure 2-left depicts the boundary control functions $v_{k^\star}$ for $s\in\{2,3,4\}$ for our semilinear problem (1) and in Figure 2-right, we plot the evolution of the norms $\Vert y_{k^\star}(\cdot,t)\Vert_{L^2(\Omega)}$ w.r.t. time t ($y_{k^\star}$ is given by (58)). Remark in particular that the the control functions vanish at $t=0$ and $t=T$ in accordance with our construction given by (58).

For higher values of negative c_f, the norms $\Vert v_{k^\star}\Vert_{L^\infty(0,T)}$ and $\Vert y_{k^\star}\Vert_{L^2(Q_T)}$ are comparatively larger than the ones with lower values of negative c_f as one can compare with Tables 1 and 2; see also Table 3. This can also be observed in Figure 3: Left and Right, where we consider $c_f=-2.5$ and choose the same values of c_{u_0} and s as before, and plot the control functions as well as the norms of solutions w.r.t. time.

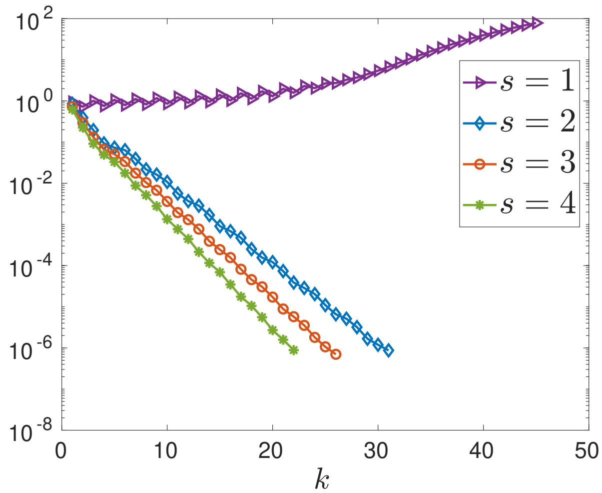

Figure 1: $\frac{\|\rho(z_{k+1}-z_k)\|_{L^2(Q_T)}}{\|\rho z_k\|_{L^2(Q_T)}}$ w.r.t. iterations k for $(c_f, c_{u_0}) = (-2.5, 15)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$.

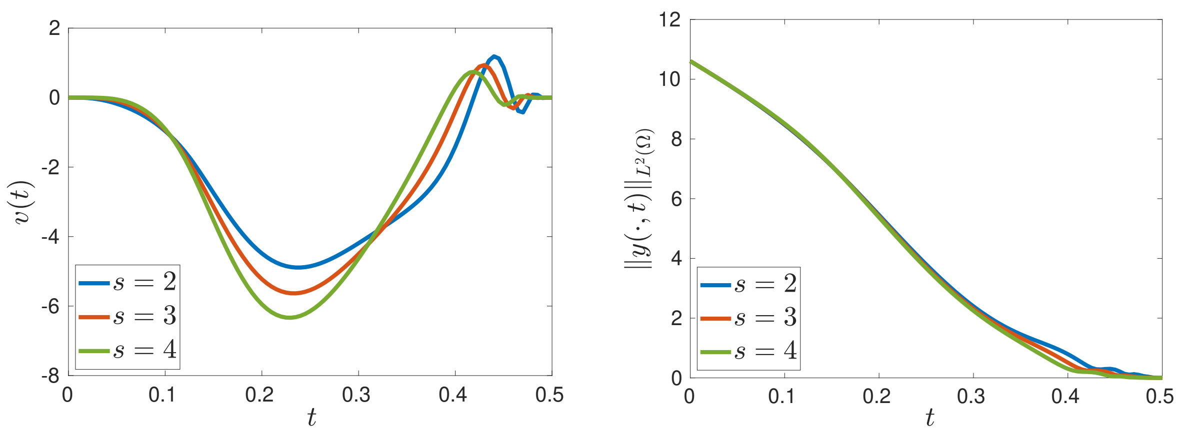

Figure 2: $c_f = -1.5$, $c_{u_0} = 15$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: Control functions v_{k^*} for different s; Right: Evolution of $\|y_{k^*}(\cdot, t)\|_{L^2(\Omega)}$ w.r.t. time t for different s.

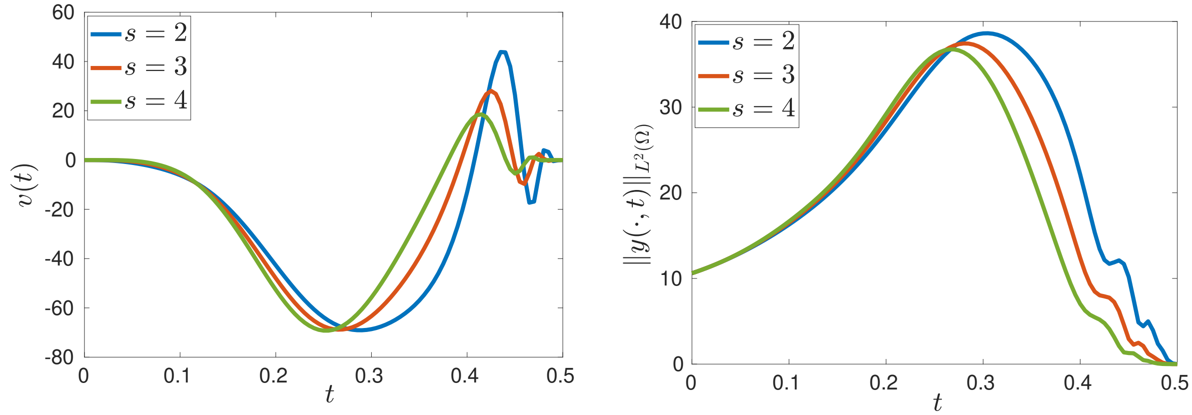

Figure 3: $c_f = -2.5$, $c_{u_0} = 15$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: Control functions v_{k^*} for different s; Right: Evolution of $\|y_{k^*}(\cdot, t)\|_{L^2(\Omega)}$ w.r.t. time t for different s.

Finally, we compare the controlled solutions for $c_f = -1.5$ and -2.5 by Figures 4-left and right respectively, when the parameters $c_{u_0} = 15$ and $s = 4$ are fixed.

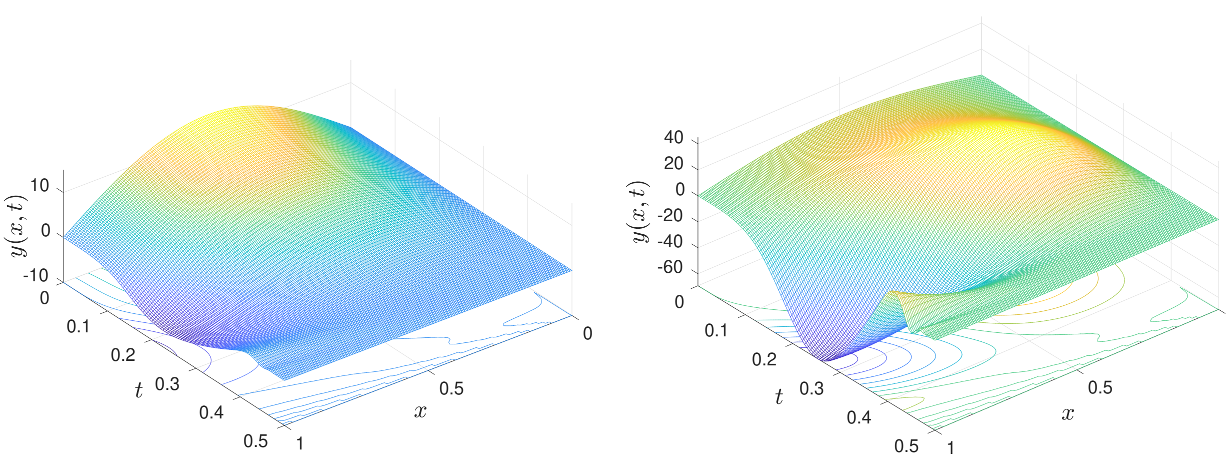

Figure 4: Controlled solutions y_{k^*} for $s = 4$, $c_{u_0} = 15$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: $c_f = -1.5$; Right: $c_f = -2.5$.

III. Experiments for fixed (s, c_{u_0}) w.r.t. c_f. In this paragraph, we fix the Carleman parameter and the initial data, and consider several values of c_f to study the number of iterations for which the solution component $z_{k^\star}$ satisfies the criterion (57). Table 3 reports some values corresponding to $s = 3$ and $c_{u_0} = 15$. As expected, larger negative values of c_f leads to larger norms of the state-control pair; we also observe that the required number of iterations $k^\star$ increases with large negative values of c_f, for example when $c_f = -2.9$, the number of iterations is $k^\star = 79$ to satisfy the convergence criterion. For larger negative c_f, for instance $c_f = -3$, the algorithm fails to converge, which is somehow in agreement with the smallness assumption on β in our Theorem 1.

On contrary, when $c_f > 0$ (implying $rf(r) \geqslant 0$), the algorithm is more favorable in the sense that the norms of the state-control pairs are much smaller. We can also observe this phenomenon in the

Figure 5: Left and Right, where we respectively plot the control functions $v_{k^\star}$ and the evolutions of $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ for $c_f \in \{-2,0,2\}$. For $c_f = -2$, we get $\|v_{k^\star}\|_{L^\infty(0,T)} \approx 15.781$ whereas for $c_f = 2$, it reduces to $\|v_{k^\star}\|_{L^\infty(0,T)} \approx 2.343 \times 10^{-1}$. The case $c_f = 0$ corresponds to the linearized model with the norm $\|v_{k^\star}\|_{L^\infty(0,T)} \approx 9.469 \times 10^{-1}$; see Table 3 for the details. In Figure we see how the L^2 and L^∞ norms of the control functions changes w.r.t. negative to positive values of c_f according to the fifth and sixth columns of Table 3.

c_f	$\|z_{k^\star}\|_{L^2(Q_T)}$	$\|\rho z_{k^\star}\|_{L^2(Q_T)}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
-2.9	92.063	586.555	91.532	128.472	463.126	79
-2.5	13.372	55.104	16.416	25.779	68.826	26
-2	2.604	9.841	6.079	6.132	15.781	13
-1	4.185×10^{-1}	1.555	3.154	1.064	2.690	7
0	1.442×10^{-1}	5.360×10^{-1}	2.371	3.750×10^{-1}	9.469×10^{-1}	1
1	6.582×10^{-2}	2.454×10^{-1}	1.979	1.729×10^{-1}	4.402×10^{-1}	6
2	3.434×10^{-2}	1.287×10^{-1}	1.735	9.067×10^{-2}	2.343×10^{-1}	7
3	1.932×10^{-2}	7.294×10^{-2}	1.564	5.110×10^{-2}	1.349×10^{-1}	8
4	1.140×10^{-2}	4.346×10^{-2}	1.436	3.014×10^{-2}	8.170×10^{-2}	9
5	6.964×10^{-3}	2.683×10^{-2}	1.337	1.835×10^{-2}	5.127×10^{-2}	12

Table 3: $(s, c_{u_0}) = (3, 15)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. c_f.

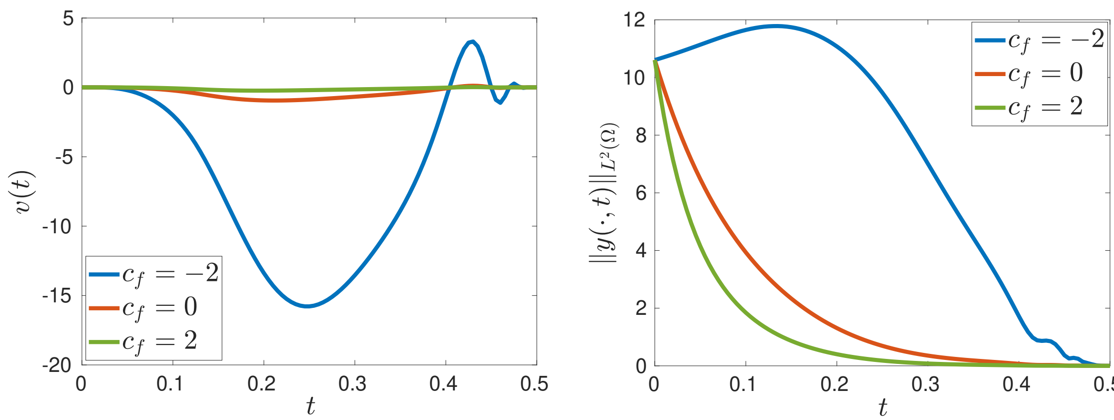

Figure 5: $(s, c_{u_0}) = (3, 15)$; $f(r) = c_f r(1+\ln^{3/2}(2+|r|))$; Left: Control functions $v_{k^\star}$ and Right: Evolution of $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. time t for $c_f \in \{-2,0,2\}$.

IV. Experiments for fixed (s, c_f) w.r.t. the parameter c_{u_0}. Let us now fix some suitable parameters s (preferably large) and c_f (preferably small) and then vary the size of the initial data u_0 in terms of the parameter c_{u_0}. We give some results in Table 4 for $(s, c_f) = (4, -1)$. One can observe that for large c_{u_0} also, the algorithm converges which gives the evidence of the global null-controllability for our semilinear system (1) with the suitable choice of Carleman parameter s and c_f (that is β, according to Remark 5).

Since we choose initial data with very large norms, the critical time $T^\star$ is going to be small for the existence of solution component y^1 to the system (15). So, in this case we set $T^\star = T/4$ in order to ensure that y^1 exists in Q_T. However, as expected, when we choose large initial data, it needs more number of iterations to fulfill the convergence criterion (57) of our algorithm; we refer Table 4 for several experiments.

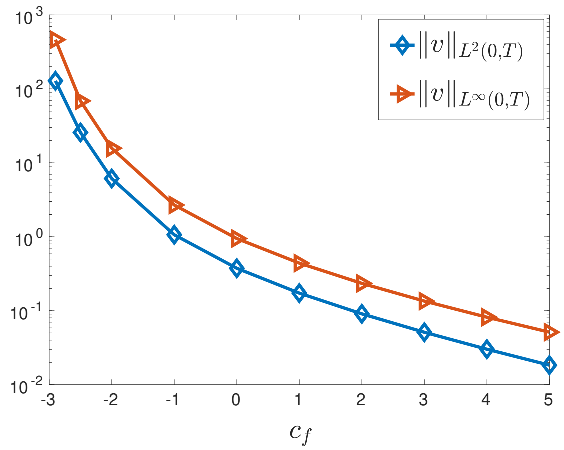

Figure 6: $(s, c_{u_0}) = (3, 15)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; $\|v_{k^\star}\|_{L^2(0,T)}$ and $\|v_{k^\star}\|_{L^\infty(0,T)}$ w.r.t. c_f as per Table 3.

c_{u_0}	$\|z_{k^\star}\|_{L^2(Q_T)}$	$\|\rho z_{k^\star}\|_{L^2(Q_T)}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
50	5.36	47.53	12.21	11.38	37.45	9
100	16.17	137.71	29.75	33.70	107.53	10
200	53.29	431.01	79.22	107.97	328.22	13
500	299.04	2237.48	354.41	559.79	1558.72	18
1000	1231.87	8861.49	1321.78	2016.46	5306.94	24
2000	5480.22	40112.3	5616.31	7053.28	18215.7	35
3000	13521.5	103650	13688.6	14821.8	36996.3	50
4000	26061.5	209478	26249.9	26927.7	68651.5	75
5000	43839.6	367751	44041.6	46173.5	132244	119

Table 4: $(s, c_f) = (4, -1)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. c_{u_0}.

V. Experiments for fixed (s, c_f, c_{u_0}) w.r.t. different diffusion coefficients. The boundary controllability of our semilinear problem still holds true for the heat operator $L_\nu = \partial_t - \nu\partial_{xx}$ with any diffusion coefficient $\nu > 0$. In this paragraph, we make some numerical experiments when the coefficient $\nu > 0$ is smaller than 1. More precisely, we make some simulations in Table 5 with the choices of data as given by (60) for some fixed s, c_{u_0} and c_f and w.r.t. ν. It is observable that for ν smaller, the norms of the solution-control pair is comparatively larger; also it needs more iterations to satisfy the convergence criterion (57) (see Table 5).

ν	$\|z_{k^\star}\|_{L^2(Q_T)}$	$\|\rho z_{k^\star}\|_{L^2(Q_T)}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
0.3	10.081	36.637	10.244	29.284	71.334	40
0.4	4.865	17.312	5.968	13.492	34.349	8
0.5	2.566	9.01	4.427	7.105	18.393	6
0.7	8.356×10^{-1}	2.974	3.342	2.301	5.691	6
1.0	2.333×10^{-1}	8.665×10^{-1}	2.680	6.013×10^{-1}	1.517	5

Table 5: $c_{u_0} = 15$; $(s, c_f) = (3, -0.5)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. ν.

In Figure 7-Left we depict the control functions for $s = 3$, $c_{u_0} = 15$ and $c_f = -0.5$ when $\nu \in \{0.3, 0.4, 0.5\}$. For the same values of s, c_{u_0} and c_f, in Figure 7-Right we compare the evolution of the

L^2-norms of the solutions w.r.t. different diffusion coefficients. Finally, we present the controlled solutions for $\nu \in \{0.3, 0.5\}$ in Figure 8.

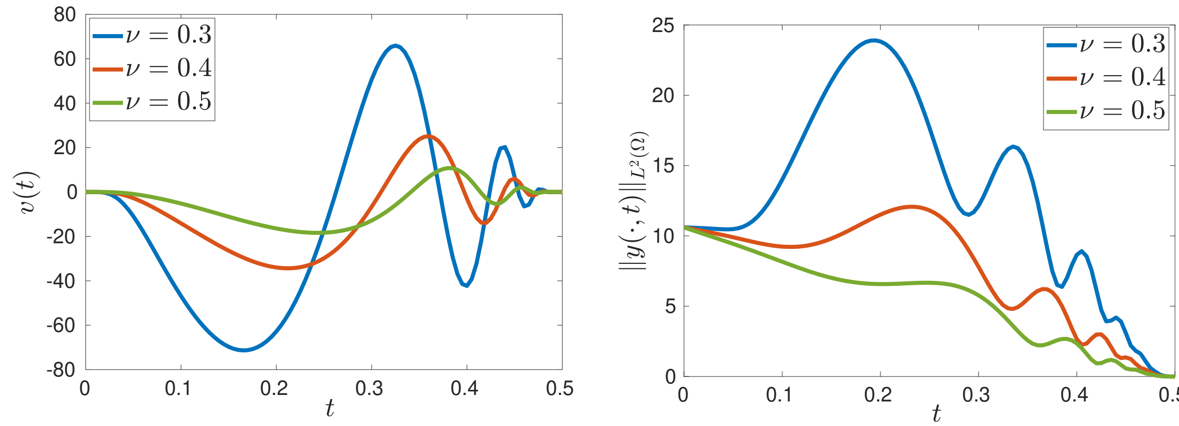

Figure 7: $c_{u_0} = 15$; $(s, c_f) = (3, -0.5)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: Control functions $v_{k^\star}$ and Right: Evolution of $\|y_{k^\star}(\cdot, t)\|_{L^2(\Omega)}$ w.r.t. time t for different diffusion coefficients ν.

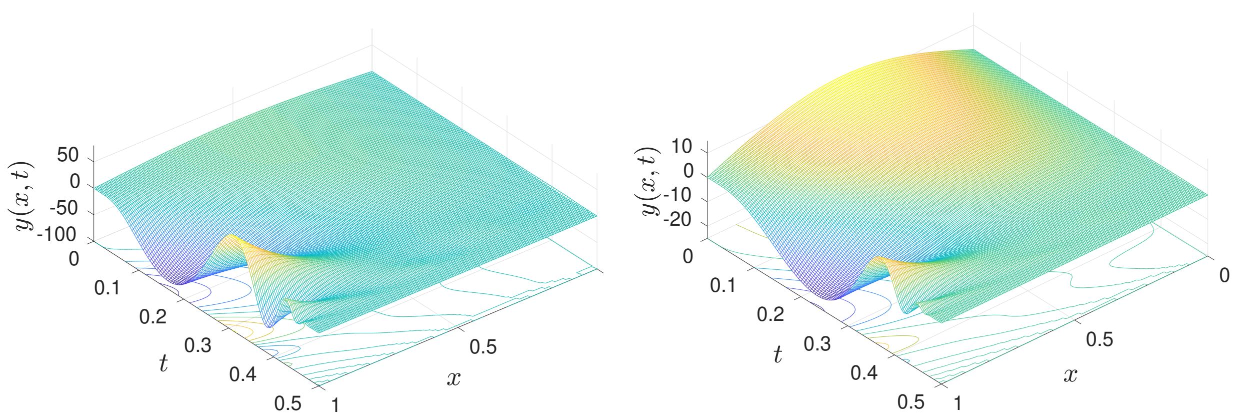

Figure 8: Controlled solutions $y_{k^\star}$ for $s = 3$, $c_{u_0} = 15$, $c_f = -0.5$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: $\nu = 0.3$; Right: $\nu = 0.5$.

VI. Experiments with some localized initial data. Let us now make some experiments with the initial $H^1(0,1)$ data

$$u_0(x) = e^{-100(x-0.7)^2}, \quad x \in (0,1). \tag{61}$$

The nonlinear function is again $f(r) = c_f(\alpha + \beta \ln^{3/2}(2 + |r|))$ with $c_f = -2.5$. In Table 6, we present the results for different values of s when the diffusion coefficient is $\nu = 0.5$. In this case, we check that $s = 1$ is not large enough to imply the Banach contraction property w.r.t. the stopping criterion (57). By choosing $s \geqslant 2$, we recover the required convergence criterion. We plot the associated control functions and the evolution of the L^2-norms of the solutions $y_{k^\star}$ w.r.t. $s \in \{2, 3, 4\}$ in Figure 9: Left and Right respectively.

s	$\|z_{k^\star}\|_{L^2(Q_T)}$	$\|\rho z_{k^\star}\|_{L^2(Q_T)}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
2	7.229×10^{-2}	1.715×10^{-1}	1.122×10^{-1}	1.923×10^{-1}	4.683×10^{-1}	27
3	7.798×10^{-2}	2.715×10^{-1}	1.156×10^{-1}	2.079×10^{-1}	5.322×10^{-1}	13
4	8.289×10^{-2}	4.219×10^{-1}	1.185×10^{-1}	2.215×10^{-1}	5.909×10^{-1}	10

Table 6: $u_0(x) = e^{-100(x-0.7)^2}$; $\nu = 0.5$; $c_f = -2.5$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. s.

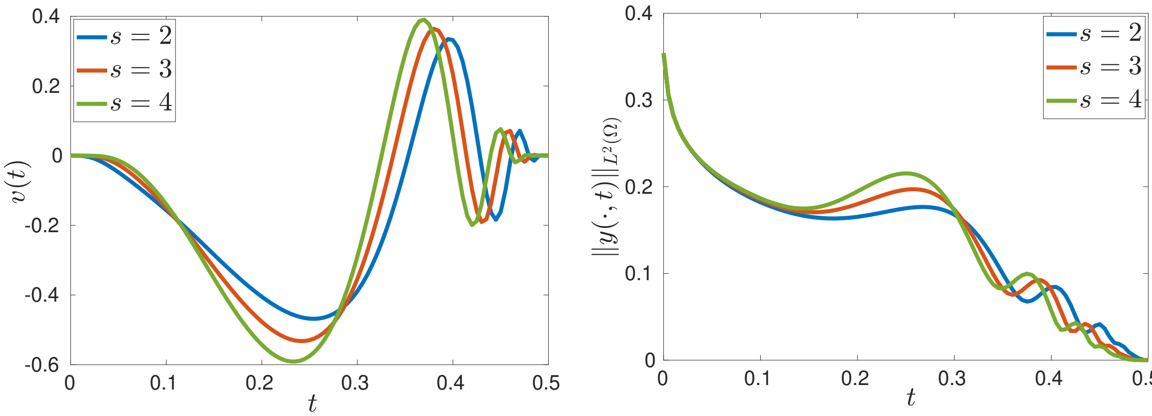

Figure 9: $u_0(x) = e^{-100(x-0.7)^2}$; $\nu = 0.5$; $c_f = -2.5$; $f(r) = c_f r(1+\ln^{3/2}(2+|r|))$; Left: Control functions $v_{k^\star}$ and Right: Evolution of $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. time t for different values of s.

We also make some experiments w.r.t. different diffusion coefficients ν when $c_f = -2.5$ and the Carleman parameter $s = 4$. We refer Table 7 for several values upon experiments and Figure 10: Left and Right respectively depicts the associated control functions $v_{k^\star}$ and the evolutions of $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. t. In Figure 11: Left and Right we respectively plot the controlled solutions $y_{k^\star}$ for $\nu \in \{0.3, 0.5\}$ when $(s, c_f) = (4, -2.5)$.

ν	$\|z_{k^\star}\|_{L^2(Q_T)}$	$\|\rho z_{k^\star}\|_{L^2(Q_T)}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
0.3	2.837×10^{-1}	1.453	2.785×10^{-1}	8.338×10^{-1}	2.080	55
0.4	1.475×10^{-1}	7.467×10^{-1}	1.653×10^{-1}	4.034×10^{-1}	1.045	16
0.5	8.289×10^{-2}	4.219×10^{-1}	1.185×10^{-1}	2.215×10^{-1}	5.909×10^{-1}	10
0.7	2.968×10^{-2}	1.545×10^{-1}	8.279×10^{-2}	7.878×10^{-2}	2.063×10^{-1}	8
1.0	8.653×10^{-3}	4.711×10^{-2}	6.343×10^{-2}	2.260×10^{-2}	5.915×10^{-2}	6

Table 7: $u_0(x) = e^{-100(x-0.7)^2}$; $(s, c_f) = (4, -2.5)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. ν.

4.2.2 Experiments with quadratic growth nonlinearity

In order to enhance the importance of the assumption $(\mathbf{H_1'})$ on the first derivative of f, let us consider the following nonlinear function

$$f(r) = -r^2, \quad \forall r \in \mathbb{R} \tag{62}$$

which does not satisfy the assumption $(\mathbf{H_1'})$. It is known that, with this kind of nonlinearity one can at most expect the local null-controllability of the system, meaning that the choice of the initial data

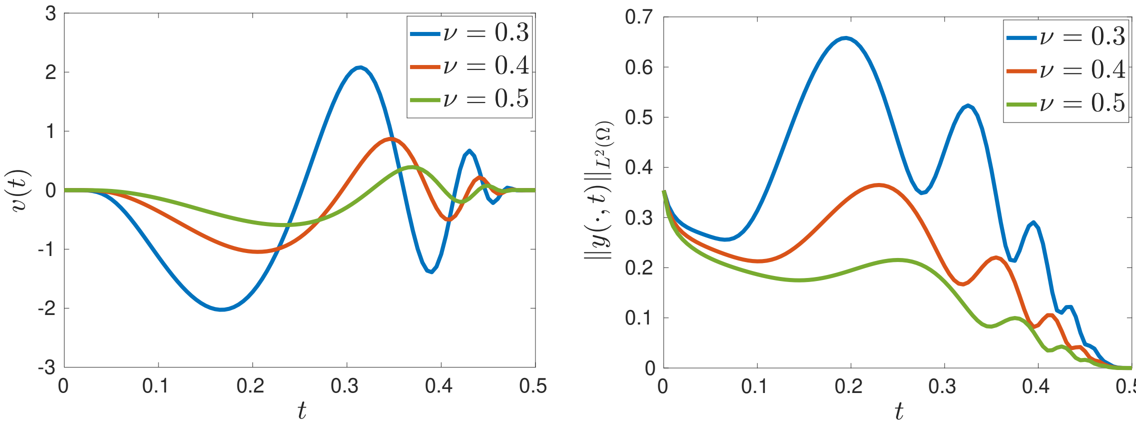

Figure 10: $u_0(x) = e^{-100(x-0.7)^2}$; $(s, c_f) = (4, -2.5)$; $f(r) = c_f r(1+\ln^{3/2}(2+|r|))$; Left: Control functions $v_{k^\star}$ and Right: Evolution of $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. time t for different values of ν.

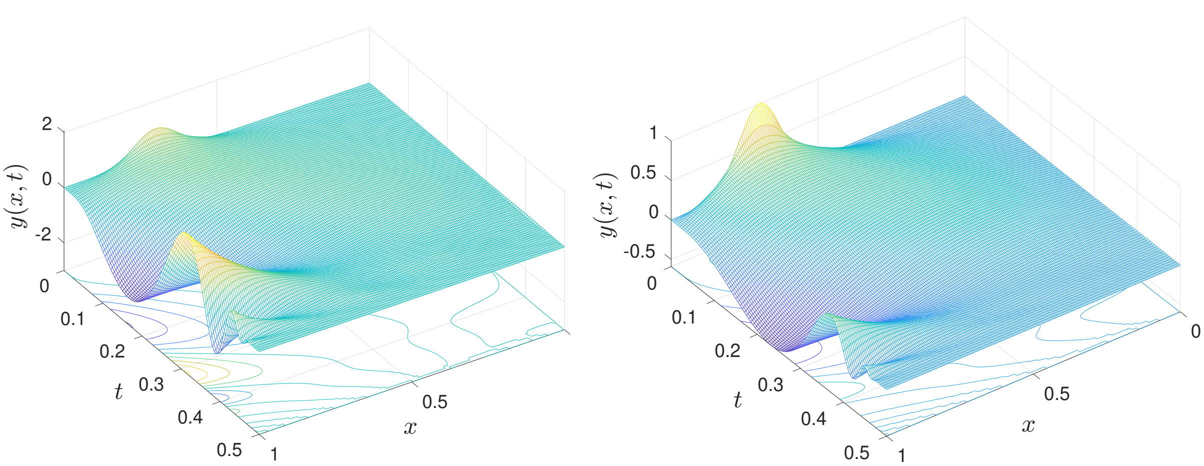

Figure 11: Controlled solutions $y_{k^\star}$ for $u_0(x) = e^{-100(x-0.7)^2}$; $(s, c_f) = (4, -2.5)$, $f(r) = c_f r(1+\ln^{3/2}(2+|r|))$; respectively for $\nu \in \{0.3, 0.5\}$.

should be small enough to achieve the controllability. We shall numerically observe this phenomenon (using our algorithm) in Table 8, where we get that the number of iterations are larger associated to larger initial data. For instance (with the initial data $u_0(x) = c_{u_0} \sin(\pi x)$), when $c_{u_0} = 12$ the number of iterations is $k^\star = 12$ to fulfill the convergence criterion (57) whilst by taking $c_{u_0} = 13$, it needs $k^\star = 31$. Moreover, the associated norms of the solution-control pairs are significantly larger for $c_{u_0} = 13$ than for $c_{u_0} = 12$. Testing the algorithm with slightly larger c_{u_0}, namely with $c_{u_0} = 13.1$, we observe that the number of iterations increse to $k^\star = 51$ and $\|v_{k^\star}\|_{L^\infty(0,T)} \approx 51.183$ whereas for $c_{u_0} = 13$, we have $\|v_{k^\star}\|_{L^\infty(0,T)} \approx 26.761$.

Figure 12-Left and Right respectively plot the control functions and the $L^2(\Omega)$ norm of the solutions w.r.t. the time variable. For larger initial data, for instance when $c_{u_0} \geqslant 13.2$, numerically we observe the blow-up phenomenon of the solution.

c_{u_0}	$\|z_{k^\star}\|_{L^2(Q_T)}$	$\|\rho z_{k^\star}\|_{L^2(Q_T)}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
10	3.545×10^{-1}	1.311	2.601	9.157×10^{-1}	2.331	8
11	5.376×10^{-1}	1.984	3.252	1.387	3.547	9
12	9.539×10^{-1}	3.508	4.373	2.465	6.374	12
12.5	1.472	5.394	5.511	3.825	10.081	16
12.7	1.875	6.861	6.328	4.903	13.156	19
12.9	2.632	9.625	7.803	6.966	19.459	26
13	3.426	12.565	9.299	9.144	26.761	31
13.1	5.927	22.307	13.432	15.512	51.183	51

Table 8: $s = 3$; $f(r) = -r^2$; $u_0(x) = c_{u_0} \sin(\pi x)$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. c_{u_0}.

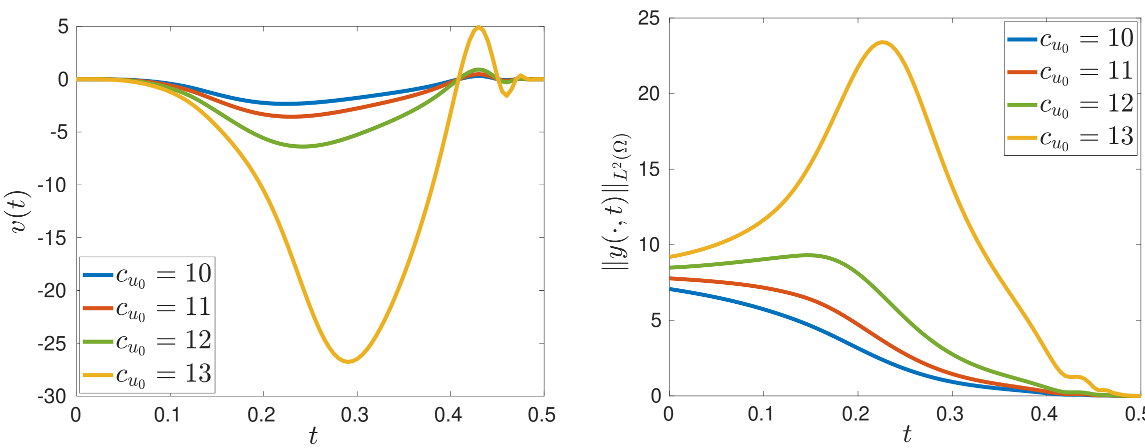

Figure 12: $s = 3$; $f(r) = -r^2$; $u_0(x) = c_{u_0} \sin(\pi x)$; Left: Control functions $v_{k^\star}$ and Right: Evolution of $\|y_{k^\star}(\cdot, t)\|_{L^2(\Omega)}$ w.r.t. time t for different values of c_{u_0}.

It is also interesting to make some experiments for fixed c_{u_0} when the Carleman parameter s varies. From Table 8, we recall that for $c_{u_0} = 13$, the number of iterations is $k^\star = 31$ to achieve the stopping criterion (57) when $s = 3$. Now, for this particular value $c_{u_0} = 13$, our goal is to see the changes of iteration numbers as well as the norms of the state-control pair for different values of s, namely for $s \in \{2, 3, 4\}$. We refer Table 9 for the results. The associated control functions and evolution of the solutions (in terms of L^2-norms) for $s \in \{2, 3, 4\}$ are given by Figure 13: Left and Right respectively.

s	$\|z_{k^\star}\|_{L^2(Q_T)}$	$\|\rho z_{k^\star}\|_{L^2(Q_T)}$	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
2	3.135	8.018	9.362	8.139	22.416	35
3	3.426	12.565	9.299	9.144	26.761	31
4	3.752	19.604	9.243	10.064	30.919	29

Table 9: $c_{u_0} = 13$; $f(r) = -r^2$; Norms of $z_{k^\star}$, $y_{k^\star}$, $v_{k^\star}$ w.r.t. s.

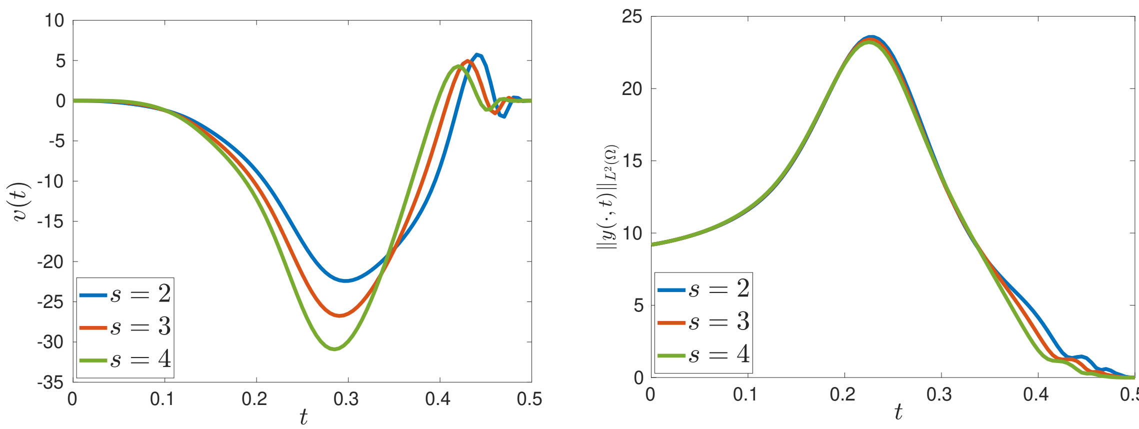

Figure 13: $f(r) = -r^2$; $u_0(x) = 13\sin(\pi x)$; Left: Control functions $v_{k^\star}$ and Right: Evolution of $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. time t for different values of s.

5 Concluding remarks

By adapting some arguments recently introduced in [4] devoted to a wave equation, we have established the null boundary controllability of a semilinear heat equation of the form $\partial_t y - \partial_{xx} y + f(y) = 0$. The proof of the controllability is direct as it is not deduced from an interior controllability result by the way of the domain extension method. This provides a direct estimate of the cost of control in term of the data. A simple fixed point operator for which the nonlinear term is seen as a right hand side is introduced and proved to satisfy the Schauder theorem as soon the C^1 function f satisfies the usual asymptotic growth condition $(\mathbf{H_1})$ considered in the interior case in the literature. Then, assuming a similar asymptotic condition on f', namely $(\mathbf{H_1'})$, we have proved that the fixed point operator is contracting yielding a strongly convergent sequence to a controlled solution for the nonlinear heat equation. As in [4] but also in [11, 26], the analysis emphasizes the role of the Carleman weights parametrized by the real s.

A key point in the application of the above fixe point theorems is a regularity property for a state-control pair of a linear heat equation. Precisely, we have proven that the unique control-state pair (y, v) for

$$\begin{cases} \partial_t z - \partial_{xx} z = B & \text{in } Q_T, \\ z(0,\cdot) = 0, \quad z(1,\cdot) = u & \text{in } (0,T), \\ z(\cdot,0) = 0 & \text{in } \Omega \end{cases}$$

with $B \in L^2(Q_T)$ which minimizes $(z,u) \to \|\rho z\|^2_{L^2(Q_T)} + \|\rho_0 u\|^2_{L^2(0,T)}$ among the admissible state-control pairs enjoys the regularity $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \times H^1_0(0,T)$; in particular, in our one dimensional setting, this optimal pair is uniformly bounded. To our knowledge, this crucial regularity property obtained from technical developments is original for the heat equation.

Following standard arguments (see for instance [11, 17]), the analysis can be extended to address the controllability to trajectories. Moreover, by extending the above regularity property, we may also consider multi-dimensional situations as well as boundary Neumann actuations.

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Appendices

A Proof of Proposition 2

Let $\alpha > 0$, $\beta > 0$, $M > 0$, g be a continuous function on $\mathbb{R}^2$ satisfying the growth condition $|g(t,r)| \leqslant |r|(\alpha + \beta \ln_+^{3/2}|r|)$ for all $t \in \mathbb{R}$, and $u_0 \in L^2(\Omega)$ such that $\|u_0\|_{L^2(\Omega)} \leqslant M$. There exists $T_{g,u_0} > 0$ and $u \in \mathcal{C}([0,T_{g,u_0}]; L^2(\Omega)) \cap L^2(0,T_{g,u_0}; H_0^1(\Omega))$ solution of (13) on $Q_{T_{g,u_0}}$. We refer for instance to [5]. Let us check that T_{g,u_0} can be chosen independently of g and u_0 but only in term of the parameter α, β and $M > 0$ where $\|u_0\|_{L^2(\Omega)} \leqslant M$.

We have, on $(0, T_{g,u_0})$

$$\frac{1}{2}\partial_t \int_\Omega |u|^2 + \int_\Omega |\partial_x u|^2 = \int_\Omega g(t,u)u$$

and thus, since $\ln_+^{3/2}|r| \leqslant \sqrt{|r|}$,

$$\begin{aligned}\frac{1}{2}\partial_t \int_\Omega |u|^2 + \int_\Omega |\partial_x u|^2 &\leqslant \|g(t,u)\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leqslant \||u|(\alpha + \beta \ln_+^{3/2}|u|)\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leqslant \alpha\|u\|^2_{L^2(\Omega)} + \beta\|u\|^{3/2}_{L^3(\Omega)}\|u\|_{L^2(\Omega)}.\end{aligned}$$

But,

$$\|u\|_{L^3(\Omega)} \leqslant \|u\|^{2/3}_{L^2(\Omega)}\|u\|^{1/3}_{L^\infty(\Omega)} \leqslant C\|u\|^{2/3}_{L^2(\Omega)}\|\partial_x u\|^{1/3}_{L^2(\Omega)}$$

and thus

$$\|u\|^{3/2}_{L^3(\Omega)} \leqslant C\|u\|_{L^2(\Omega)}\|\partial_x u\|^{1/2}_{L^2(\Omega)} \leqslant C\beta\|u\|^2_{L^2(\Omega)} + \frac{1}{\beta}\|\partial_x u\|_{L^2(\Omega)}.$$

This gives

$$\frac{1}{2}\partial_t \int_\Omega |u|^2 + \int_\Omega |\partial_x u|^2 \leqslant \alpha\|u\|^2_{L^2(\Omega)} + C\beta^2\|u\|^3_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|\partial_x u\|_{L^2(\Omega)}$$

and thus

$$\partial_t \int_\Omega |u|^2 + \int_\Omega |\partial_x u|^2 \leqslant (2\alpha+1)\|u\|^2_{L^2(\Omega)} + C\beta^2\|u\|^3_{L^2(\Omega)} = G_{\alpha,\beta}\big(\|u\|^2_{L^2(\Omega)}\big)$$

where $G_{\alpha,\beta}(r) = (2\alpha+1)r + C\beta^2|r|^{3/2}$. Since $G_{\alpha,\beta}$ is bounded in the bounded sets of $\mathbb{R}$, there exists (see [30, Lemma 6, p. 1098]) $T^\star = T_{\alpha,\beta,M}$ (so independent of the choices of g and u_0 but only on the norm $\|u_0\|_{L^2(\Omega)} \leqslant M$ and the parameters α, β) such that for all $0 \leqslant t \leqslant T^\star$:

$$\|u(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |\partial_x u|^2 \leqslant \|u_0\|^2_{L^2(\Omega)} + 1, \tag{63}$$

which shows that T_{g,u_0} can always be chosen greater than $T^\star$. Moreover, following the proof given in [30], it is easy to prove that $T^\star$ is greater than $\frac{1}{2\alpha(M+1)+c\beta(M+1)^{3/2}}$ for some $c = c(\Omega) > 0$.

B Proof of Theorem 3

Assuming the right hand side $B \in L^2(\rho, Q_T)$, we prove regularity properties for the state-control pair solution of (19) given by Theorem 2. We start with the regular case $B \in \mathcal{D}(\mathbb{R}; L^2(\Omega))$ then conclude by a density argument. The procedure is similar as the one used in [4] devoted to the wave equation (see also [12]) and consists in using appropriate test functions in the variational formulation (22) so as to make appear some derivatives of the state-control pair. The group property has been used in [4, 12] devoted to the wave equation to extend solution out of the time interval $(0, T)$, in particular for $t < 0$. As mentioned in [12, Section 6], this is no longer possible for the heat equation considered here. Hopefully, the particular structure of the Carleman weights allows to extend appropriate weighted solution.

Definition 1. *For any Banach space E, $f \in \mathcal{C}^0(\mathbb{R}; E)$ and $\tau \neq 0$, we define*

$$\begin{aligned} \delta_\tau f &:= f\left(t + \frac{\tau}{2}\right) - f\left(t - \frac{\tau}{2}\right), \\ \mathcal{T}_\tau f &:= \frac{1}{\tau}\delta_\tau\left(\frac{\delta_\tau f}{\tau}\right) = \frac{f(t+\tau) - 2f(t) + f(t-\tau)}{\tau^2}, \\ \widetilde{\delta}_\tau f(t) &:= f(t+\tau) - f(t). \end{aligned} \tag{64}$$

Remark that $\mathcal{T}_\tau(f) = \frac{1}{\tau^2}\big(\widetilde{\delta}_\tau f(t) - \widetilde{\delta}_\tau f(t-\tau)\big)$.

First, in view of the structure of the weight $\rho(x,t) = e^{s\theta(t)\varphi_1(x)}$ (see (9)), we extend by 0 on $(-\infty, 0] \cup [T, +\infty)$ any negative power of ρ : more precisely, we shall use that for all $\alpha \in \mathbb{R}$ and $\beta > 0$: $\theta^\alpha \rho^{-\beta} \in \mathcal{C}^\infty(\mathbb{R}; C^\infty(\overline{\Omega}))$ and $\theta^\alpha \rho_\star^{-\beta} \in \mathcal{C}^\infty(\mathbb{R}; C^\infty(\overline{\Omega}))$. Similarly, since $\theta_t = (2t - T)\theta^2$, we shall consider the following extension :

$$(\rho^{-1})_t = \begin{cases} s(T-2t)\theta^2\varphi_1\rho^{-1} & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-\infty, 0] \cup [T, +\infty), \end{cases} \tag{65}$$

$$(\rho_\star^{-1})_t = \begin{cases} s(T-2t)\theta^2\varphi_\star\rho_\star^{-1} & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-\infty, 0] \cup [T, +\infty), \end{cases} \tag{66}$$

and

$$(\rho^{-1})_x = \begin{cases} -s\theta(\varphi_1)_x\rho^{-1} & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-\infty, 0] \cup [T, +\infty), \end{cases} \tag{67}$$

$$(\rho^{-1})_{xx} = \begin{cases} -s\theta(\varphi_1)_{xx}\rho^{-1} + s^2\theta^2|(\varphi_1)_x|^2\rho^{-1} & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-\infty, 0] \cup [T, +\infty). \end{cases} \tag{68}$$

Recalling that $\rho_{2,\star}^{-1} = \theta^{-1/2}\rho_\star^{-1}$, we shall also use in the sequel the derivative

$$(\rho_{2,\star}^{-1})_t = \begin{cases} \dfrac{T-2t}{2}\theta^{1/2}\rho_\star^{-1} + s(T-2t)\theta^{3/2}\varphi_\star\rho_\star^{-1} := \gamma(t)\rho_{2,\star}^{-1} & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-\infty, 0] \cup [T, +\infty), \end{cases} \tag{69}$$

with

$$\gamma(t) := \frac{T-2t}{2}\theta + s(T-2t)\theta^2\varphi_\star. \tag{70}$$

Lemma 5. *Let $p \in P_s$ is given by Theorem 2. Then $\rho_{2,\star}^{-1} p \in P$ and the function $\widetilde{p}$ defined by*

$$\widetilde{p} = \begin{cases} \rho_{2,\star}^{-1} p & \text{in } (0,T), \\ 0 & \text{in } \mathbb{R} \setminus (0,T) \end{cases}$$

also satisfies $\widetilde{p} \in P$. Moreover, there exists a constant $C > 0$ such that

$$\|L^\star(\rho_{2,\star}^{-1} p)\|_{L^2(Q_T)} \leqslant C\|p\|_{P_s}. \tag{71}$$

Proof. From the Carleman estimate (12), since $\rho_{2,\star}^{-1} \leqslant C\rho_2^{-1} \leqslant C\rho_1^{-1} \leqslant C\rho_0^{-1}$ for some $C > 0$ and $(\rho_{2,\star}^{-1})_t p \in L^2(Q_T)$, we get $\rho_{2,\star}^{-1} p \in P$; thus $\rho_{2,\star}^{-1} p \in \mathcal{C}([0,T]; L^2(\Omega))$ and thus $\widetilde{p} \in L^2(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega))$. Moreover, since $\theta^2 \rho_{2,\star}^{-1} \leqslant C\rho_0^{-1}$, $\rho_{2,\star}^{-1} \leqslant C\rho_2^{-1}$ and $\rho_0^{-1} p \in L^2(Q_T)$, $\rho_2^{-1} p_t \in L^2(Q_T)$, we get that $(\rho_{2,\star}^{-1} p)(0) = (\rho_{2,\star}^{-1} p)(T) = 0$; therefore $\rho_{2,\star}^{-1} p \in H_0^1(0,T; L^2(\Omega))$. It follows that $\widetilde{p} \in H^1(\mathbb{R}; L^2(\Omega))$ and that $\widetilde{p} \in P$. Furthermore, Carleman estimate (12) yields

$$\begin{aligned}
\|L^\star(\rho_{2,\star}^{-1} p)\|_{L^2(Q_T)}^2 &\leqslant 2\|\rho_{2,\star}^{-1}(p_t + p_{xx})\|_{L^2(Q_T)}^2 + 2\|(\rho_{2,\star}^{-1})_t p\|_{L^2(Q_T)}^2 \\
&\leqslant 2\|\rho^{-1} L^\star p\|_{L^2(Q_T)}^2 + CT^2 s^2 \int_{Q_T} \theta^4 \rho_{2,\star}^{-2} |p|^2 \\
&\leqslant 2\|\rho^{-1} L^\star p\|_{L^2(Q_T)}^2 + CT^2 s^2 \int_{Q_T} \theta^3 \rho^{-2} |p|^2 \\
&\leqslant 2\|\rho^{-1} L^\star p\|_{L^2(Q_T)}^2 + CT^2 s^{-1} \|p\|_{P_s}^2,
\end{aligned}$$

that is (71). $\square$

Without confusion, we shall use in the subsequent computations the notation $\rho_{2,\star}^{-1} p$ as defined over $\Omega \times \mathbb{R}$ instead of $\widetilde{p}$.

From the previous lemma, the function

$$\mathcal{T}_\tau(\rho_{2,\star}^{-1} p) = \frac{(\rho_{2,\star}^{-1} p)(t+\tau) - 2(\rho_{2,\star}^{-1} p)(t) + (\rho_{2,\star}^{-1} p)(t-\tau)}{\tau^2}, \quad \text{for any } \tau \neq 0, \tag{72}$$

also belongs to the space P and in particular, $\mathcal{T}_\tau(\rho_{2,\star}^{-1} p) \in P_s$. Eventually, since $\rho_{2,\star}^{-1} = \theta^{-1/2} \rho_\star^{-1}$ does not depend on x (see (28)), we get that $\rho_{2,\star}^{-1} p$ satisfies the equation in $Q_T = \Omega \times (0,T)$

$$\begin{cases} L^\star(\rho_{2,\star}^{-1} p) = \rho_{2,\star}^{-1}(\rho^2 z) - (\rho_{2,\star}^{-1})_t p & \text{in } Q_T, \\ (\rho_{2,\star}^{-1} p)(0,\cdot) = (\rho_{2,\star}^{-1} p)(1,\cdot) = 0 & \text{in } (0,T), \\ (\rho_{2,\star}^{-1} p)(\cdot,0) = 0, \quad (\rho_{2,\star}^{-1} p)(\cdot,T) = 0 & \text{in } \Omega \end{cases} \tag{73}$$

where $z = \rho^{-2} L^\star p$ is the controlled solution introduced in Theorem 2.

We now proceed to the proof of Theorem 3 decomposed into four steps:

♦ **Step 1.** We prove the following estimate.

Proposition 7. *Let $s \geqslant s_0$. Let ρ_3 and ρ_4 be defined by*

$$\rho_3 = \rho^2 \rho_{2,\star}^{-1} \quad \text{and} \quad \rho_4 = \rho_1^2 \rho_{2,\star}^{-1}. \tag{74}$$

Assume $B \in \mathcal{D}(0,T; L^2(\Omega))$. The unique solution $p \in P_s$ of (22) satisfies

$$\int_{Q_T} \rho_3^{-1}(t) \left| L^\star(\rho_{2,\star}^{-1} p)_t \right|^2 \mathrm{d}x\mathrm{d}t + s\int_0^T \rho_4^{-1}(1,t) \left| (\rho_{2,\star}^{-1} p)_{xt}(1,t) \right|^2 \mathrm{d}t \leqslant Cs \int_{Q_T} |\rho B|^2 \mathrm{d}x\mathrm{d}t. \tag{75}$$

Observe first that for all $s \geqslant s_0$ and $\lambda \geqslant \lambda_0$, there exists $c>0$ such that

$$e^{cs} \leqslant \rho_3 \leqslant \rho_1 \quad \text{ and } \quad e^{cs} \leqslant \rho_4 \leqslant \theta^{-3/2}\rho = \rho_0. \tag{76}$$

Indeed, $e^{2s\theta(t)\varphi_1}e^{-s\theta(t)\varphi_\star} = e^{s\theta(t)(2e^{2\lambda}-2e^{\lambda x}-e^{2\lambda}+1)}$ and

$$0 < c_\lambda := e^{2\lambda} - 2e^{\lambda} + 1 \leqslant 2e^{2\lambda} - 2e^{\lambda x} - e^{2\lambda} + 1 \leqslant e^{2\lambda} - e^{\lambda x}, \quad \forall x \in (0,1).$$

Thus $\theta^{-1/2}e^{sc_\lambda\theta(t)} \leqslant \rho_3 = \theta^{-1/2}e^{2s\theta(t)\varphi_1}e^{-s\theta(t)\varphi_\star} \leqslant \theta^{-1/2}e^{s\theta(t)(e^{2\lambda}-e^{\lambda x})} = \rho_1$ and similarly $\theta^{-3/2}e^{sc_\lambda\theta(t)} \leqslant \rho_4 \leqslant \theta^{-3/2}e^{s\theta(t)(e^{2\lambda}-e^{\lambda x})} = \rho_0$. Since the function θ is uniformly bounded by below in $(0,T)$, (76) holds true for some $c>0$.

Now, we take $p = \rho_{2,\star}(\rho_{2,\star}^{-1}p)$ in the formulation (22) and use that $L^\star p = \rho_{2,\star}L^\star(\rho_{2,\star}^{-1}p) - \gamma p$ (γ is defined by (70)) and $p_x(1,t) = \rho_{2,\star}(1,t)(\rho_{2,\star}^{-1}p)_x(1,t)$. This implies for all test function $q \in P_s$:

$$\int_{Q_T} \rho^{-2}\rho_{2,\star}L^\star\big(\rho_{2,\star}^{-1}p\big)L^\star q + s\int_0^T \rho_1^{-2}(1,t)\rho_{2,\star}(1,t)\big(\rho_{2,\star}^{-1}p\big)_x(1,t)q_x(1,t) = \int_{Q_T}\rho^{-2}\gamma p L^\star q + \int_{Q_T} Bq. \tag{77}$$

Then, according to the Lemma 5, $T_\tau(\rho_{2,\star}^{-1}p)$ belongs to P_s for all $\tau \neq 0$: taking $q = T_\tau(\rho_{2,\star}^{-1}p)$ in (77) then leads to

$$\begin{aligned}\int_{Q_T} \rho_3^{-1}L^\star\big(\rho_{2,\star}^{-1}p\big)L^\star\big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\big) + s\int_0^T \rho_4^{-1}(1,t)\big(\rho_{2,\star}^{-1}p\big)_x(1,t)\Big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\Big)_x(1,t)\\ = \int_{Q_T}\rho^{-2}\gamma p L^\star\big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\big) + \int_{Q_T} B\mathcal{T}_\tau(\rho_2^{-1}p).\end{aligned} \tag{78}$$

We now intend to pass to the limit $\tau \to 0$ in this equality.

• **Sub-step 1.** We prove that the terms in (78) are uniformly bounded w.r.t. τ small.

• **Estimate of the first term in the l.h.s. of** (78)**.**

We have

$$\begin{aligned}
&\int_{Q_T} \rho_3^{-1}L^\star\big(\rho_{2,\star}^{-1}p\big)L^\star\mathcal{T}_\tau\big(\rho_{2,\star}^{-1}p\big)\,\mathrm{d}x\mathrm{d}t\\
&= \frac{1}{\tau}\int_{Q_T}\rho_3^{-1}(t)L^\star\big(\rho_{2,\star}^{-1}p\big)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t - \frac{1}{\tau}\int_{Q_T}\rho_3^{-1}(t)L^\star\big(\rho_{2,\star}^{-1}p\big)(t)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t-\tau)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&= \frac{1}{\tau}\int_{Q_T}\rho_3^{-1}L^\star\big(\rho_{2,\star}^{-1}p\big)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t - \frac{1}{\tau}\int_{-\tau}^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&= -\int_{Q_T}\frac{\widetilde{\delta}_\tau\big(\rho_3^{-1}L^\star\big(\rho_{2,\star}^{-1}p\big)\big)}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t - \frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega \rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&\quad - \frac{1}{\tau}\int_{T}^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{79}$$

where we have used the fact that ρ_3^{-1} and $L^\star(\rho_{2,\star}^{-1}p)$ vanish outside $(0,T)$. Observing that

$$\begin{aligned}-\int_{Q_T}\frac{\widetilde{\delta}_\tau\big(\rho_3^{-1}L^\star\big(\rho_{2,\star}^{-1}p\big)\big)}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t = -\int_{Q_T}\rho_3^{-1}\left|\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right|^2\mathrm{d}x\mathrm{d}t\\
-\int_{Q_T}L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau(\rho_3^{-1})}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{80}$$

equality (79) gives

$$
\begin{aligned}
\int_{Q_T} \rho_3^{-1} L^\star\big(\rho_{2,\star}^{-1} p\big) L^\star \mathcal{T}_\tau\big(\rho_{2,\star}^{-1} p\big)\,\mathrm{d}x\mathrm{d}t = &- \int_{Q_T} \rho_3^{-1} \left| \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1} p\big)}{\tau} \right|^2 \mathrm{d}x\mathrm{d}t \\
&- \int_{Q_T} L^\star\big(\rho_{2,\star}^{-1} p\big)(t+\tau) \frac{\widetilde{\delta}_\tau(\rho_3^{-1})}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1} p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t \\
&- \frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega \rho_3^{-1}(t+\tau) L^\star\big(\rho_{2,\star}^{-1} p\big)(t+\tau) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1} p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t \\
&- \frac{1}{\tau}\int_{T}^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau) L^\star\big(\rho_{2,\star}^{-1} p\big)(t+\tau) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1} p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{81}
$$

- **Estimate of the second term in the l.h.s. of** (78)**.** Proceeding as before we write

$$
\begin{aligned}
& s\int_0^T \rho_4^{-1}(1,t)\big(\rho_{2,\star}^{-1} p\big)_x(1,t)\big[\mathcal{T}_\tau(\rho_{2,\star}^{-1} p)\big]_x(1,t)\,\mathrm{d}x\mathrm{d}t \\
&\quad = \frac{s}{\tau}\int_0^T \rho_4^{-1}(1,t)(\rho_{2,\star}^{-1} p)_x(1,t)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t \\
&\qquad - \frac{s}{\tau}\int_{-\tau}^{T-\tau} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1} p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t \\
&\quad = - s\int_0^T \frac{\widetilde{\delta}_\tau(\rho_4^{-1}(\rho_{2,\star}^{-1} p)_x)(1,t)}{\tau}\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t \\
&\qquad - \frac{s}{\tau}\int_{-\tau}^{0} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1} p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t \\
&\qquad - \frac{s}{\tau}\int_{T}^{T-\tau} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1} p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t \\
&\quad = - s\int_0^T \rho_4^{-1}(1,t)\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\right|^2\mathrm{d}t - s\int_0^T (\rho_{2,\star}^{-1} p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_4^{-1})(1,t)}{\tau}\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t \\
&\qquad - \frac{s}{\tau}\int_{-\tau}^{0} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1} p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t \\
&\qquad - \frac{s}{\tau}\int_{T}^{T-\tau} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1} p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1} p)_x(1,t)}{\tau}\mathrm{d}t.
\end{aligned}
\tag{82}
$$

Using (81) and (82), the equality (78) then reads

$$
\begin{aligned}
&\int_{Q_T} \rho_3^{-1}(t) \left| \frac{\widetilde{\delta}_\tau L^\star(\rho_{2,\star}^{-1}p)}{\tau} \right|^2 \,\mathrm{d}x\mathrm{d}t + s\int_0^T \rho_4^{-1}(1,t) \left| \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau} \right|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad = -\int_{Q_T} B\mathcal{T}_\tau(\rho_{2,\star}^{-1}p) \,\mathrm{d}x\mathrm{d}t - \int_{Q_T} \rho^{-2}\gamma p L^\star\big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\big) \,\mathrm{d}x\mathrm{d}t \\
&\qquad - \int_{Q_T} L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau) \frac{\widetilde{\delta}_\tau(\rho_3^{-1})}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\qquad - s\int_0^T (\rho_{2,\star}^{-1}p)_x(1,t+\tau) \frac{\widetilde{\delta}_\tau(\rho_4^{-1})(1,t)}{\tau} \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau} \mathrm{d}t \\
&\qquad - \frac{1}{\tau}\int_{-\tau}^0 \int_\Omega \rho_3^{-1}(t+\tau) L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\qquad - \frac{s}{\tau}\int_{-\tau}^0 \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau) \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau} \mathrm{d}t \\
&\qquad - \frac{1}{\tau}\int_T^{T-\tau} \int_\Omega \rho_3^{-1}(t+\tau) L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\qquad - \frac{s}{\tau}\int_T^{T-\tau} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau) \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau} \mathrm{d}t \\
&\qquad := I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned}
\tag{83}
$$

- **Estimate of the term I_2.**

$$
\begin{aligned}
I_2 &= -\int_{Q_T} \rho^{-2}\gamma p L^\star\big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\big) \,\mathrm{d}x\mathrm{d}t = -\int_{Q_T} \rho_3^{-1}\gamma(\rho_{2,\star}^{-1}p) L^\star\big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\big) \,\mathrm{d}x\mathrm{d}t \\
&= -\frac{1}{\tau}\int_{Q_T} \rho_3^{-1}\gamma(\rho_{2,\star}^{-1}p) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\tau}\int_{Q_T} \rho_3^{-1}(t)\gamma(t)(\rho_{2,\star}^{-1}p)(t) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t-\tau)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&= -\frac{1}{\tau}\int_{Q_T} \rho_3^{-1}\gamma(\rho_{2,\star}^{-1}p) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\tau}\int_{-\tau}^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau) \frac{\widetilde{\delta}_\tau L\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_T} \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma\rho_{2,\star}^{-1}p)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega \rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\tau}\int_{T}^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau) \frac{\widetilde{\delta}_\tau L\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_T} \rho_3^{-1}\gamma \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} (\rho_{2,\star}^{-1}p)(t+\tau) \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega \rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau) \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\tau}\int_{T}^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau) \frac{\widetilde{\delta}_\tau L\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau} \,\mathrm{d}x\mathrm{d}t \\
&:= I_2^1 + I_2^2 + I_2^3 + I_2^4.
\end{aligned}
\tag{84}
$$

Estimate of the term I_2^1. Young inequality implies for some $\varepsilon > 0$

$$\begin{aligned}|I_2^1| =& \left|\int_{Q_T} \rho_3^{-1}\gamma \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t\right| \\ \leqslant& C(\varepsilon)\int_{Q_T} \rho_3^{-1}\gamma^2 \left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right|^2 \,\mathrm{d}x\mathrm{d}t + \varepsilon \int_{Q_T} \rho_3^{-1} \left|\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right|^2 \,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{85}$$

Estimate of the term I_2^2.

$$\begin{aligned}|I_2^2| =& \left|\int_{Q_T} (\rho_{2,\star}^{-1}p)(t+\tau) \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t\right| \\ \leqslant& \left|\int_{Q_T} \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau} \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau} \widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big) \,\mathrm{d}x\mathrm{d}t\right| + \left|\int_{Q_T} (\rho_{2,\star}^{-1}p) \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t\right|.\end{aligned}$$

We have

$$\left|\int_{Q_T} \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau} \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau} \widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big) \,\mathrm{d}x\mathrm{d}t\right| \leqslant \left\|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right\|_{L^2(Q_T)} \left\|\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\right\|_{L^\infty(Q_T)} \|\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\|_{L^2(Q_T)}.$$

Since $\rho_3^{-1}\gamma \in \mathcal{D}(\mathbb{R};\mathcal{C}^0(\overline{\Omega}))$ we get

$$|\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)(t)| \leqslant |\tau(\rho_3^{-1}\gamma)_t(t)| + \frac{\tau^2}{2}\|(\rho_3^{-1}\gamma)_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} \tag{86}$$

and thus,

$$\begin{aligned}\left|\int_{Q_T} (\rho_{2,\star}^{-1}p) \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t\right| \leqslant& \int_{Q_T} \left|(\rho_{2,\star}^{-1}p)(\rho_3^{-1}\gamma)_t \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right| \,\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\|(\rho_3^{-1}\gamma)_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} \int_{Q_T} \left|(\rho_{2,\star}^{-1}p)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right| \,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Moreover, $|(\rho_3^{-1}\gamma)_t| \leqslant C|\rho_3^{-1/2}|$ for some $C > 0$ and therefore,

$$\begin{aligned}\left|\int_{Q_T} (\rho_{2,\star}^{-1}p) \frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau} \frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau} \,\mathrm{d}x\mathrm{d}t\right| \leqslant& C(\varepsilon)\int_{Q_T} \big|(\rho_{2,\star}^{-1}p)\big|^2 \,\mathrm{d}x\mathrm{d}t + \varepsilon\int_{Q_T} \rho_3^{-1} \left|\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right| \,\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\|(\rho_3^{-1}\gamma)_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} \int_{Q_T} \left|(\rho_{2,\star}^{-1}p)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right| \,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

This gives

$$\begin{aligned}|I_2^2| \leqslant& \left\|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right\|_{L^2(Q_T)} \left\|\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\right\|_{L^\infty(\mathbb{R})} \|\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\|_{L^2(Q_T)} \\ &+ C(\varepsilon)\int_{Q_T} \big|(\rho_{2,\star}^{-1}p)\big|^2 \,\mathrm{d}x\mathrm{d}t + \varepsilon\int_{Q_T} \rho_3^{-1} \left|\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right| \,\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\|(\rho_3^{-1}\gamma)_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} \int_{Q_T} \left|(\rho_{2,\star}^{-1}p)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right| \,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{87}$$

Gathering the previous estimates of I_2^1 and I_2^2, we obtain for I_2

$$
\begin{aligned}
|I_2| \leqslant & C(\varepsilon)\int_{Q_T}\rho_3^{-1}\gamma^2\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right|^2\,\mathrm{d}x\mathrm{d}t+2\varepsilon\int_{Q_T}\rho_3^{-1}\left|\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right|^2\,\mathrm{d}x\mathrm{d}t\\
&+\left\|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right\|_{L^2(Q_T)}\left\|\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\right\|_{L^\infty(\mathbb{R})}\|\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\|_{L^2(Q_T)}+C(\varepsilon)\int_{Q_T}\left|(\rho_{2,\star}^{-1}p)\right|^2\,\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\|(\rho_3^{-1}\gamma)_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\int_{Q_T}\left|(\rho_{2,\star}^{-1}p)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right|\,\mathrm{d}x\mathrm{d}t\\
&+\left|\frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega\rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\right|\\
&+\left|\frac{1}{\tau}\int_{T}^{T-\tau}\int_\Omega\rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\right|.
\end{aligned}
\tag{88}
$$

- **Estimate of I_3.** As before, since $\rho_3^{-1}\in\mathcal{D}(\mathbb{R};\mathcal{C}^0(\overline{\Omega}))$, we get

$$
\begin{aligned}
|I_3| =& \left|\int_{Q_T}L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau(\rho_3^{-1})}{\tau}\,\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t\right|\\
\leqslant & \int_{Q_T}\left|L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)(\rho_3^{-1})_t\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right|\,\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\|(\rho_3^{-1})_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\int_{Q_T}\left|L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right|\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

and using again that $|(\rho_3^{-1})_t|\leqslant C\rho_3^{-1/2}$, we obtain

$$
\begin{aligned}
|I_3| \leqslant & C(\varepsilon)\int_{Q_T}\left|L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\right|^2dxdt+\varepsilon\int_{Q_T}\rho_3^{-1}\left|\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\right|^2\,\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\|(\rho_3^{-1})_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\int_{Q_T}\left|L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right|\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{89}
$$

- **Estimate of I_4.** Similarly, we have

$$
\begin{aligned}
|I_4| \leqslant & s\int_0^T\left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_4^{-1})(1,t)}{\tau}\,\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau}\right|\mathrm{d}t\\
\leqslant & s\int_0^T\left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)(\rho_4^{-1})_t(1,t)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau}\right|\mathrm{d}t\\
&+\frac{s}{2}\|(\rho_4^{-1})_{tt}\|_{L^\infty(\mathbb{R})}\int_0^T\left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)\right|\mathrm{d}t
\end{aligned}
$$

and since $|(\rho_4^{-1})_t|\leqslant C\rho_4^{-1/2}$, we get

$$
\begin{aligned}
|I_4| \leqslant & sC(\varepsilon)\int_0^T\left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\right|^2\mathrm{d}t+s\varepsilon\int_0^T\rho_4^{-1}(1,t)\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau}\right|^2\mathrm{d}t\\
&+\frac{s}{2}\|(\rho_4^{-1})_{tt}\|_{L^\infty(\mathbb{R})}\int_0^T\left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)\right|\mathrm{d}t.
\end{aligned}
\tag{90}
$$

• **An intermediate estimate.** Using the estimates of I_2, I_3 and I_4 from (88), (89) and (90) in (83), we get

$$
\begin{aligned}
&(1-3\varepsilon)\int_{Q_T}\rho_3^{-1}(t)\left|\frac{\widetilde{\delta}_\tau L^\star(\rho_{2,\star}^{-1}p)}{\tau}\right|^2 \mathrm{d}x\mathrm{d}t + s(1-\varepsilon)\int_0^T \rho_4^{-1}(1,t)\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau}\right|^2 \mathrm{d}t\\
&\leqslant \left|\int_{Q_T} B\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\mathrm{d}t\right| + C(\varepsilon)\int_{Q_T}\rho_3^{-1}\gamma^2\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right|^2 \mathrm{d}x\mathrm{d}t\\
&+\left\|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right\|_{L^2(Q_T)}\left\|\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\right\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\|\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\|_{L^2(Q_T)} + C(\varepsilon)\int_{Q_T}\left|(\rho_{2,\star}^{-1}p)\right|^2 \mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\|(\rho_3^{-1}\gamma)_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\int_{Q_T}\left|(\rho_{2,\star}^{-1}p)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right| \mathrm{d}x\mathrm{d}t\\
&+\left|\frac{1}{\tau}\int_{-\tau}^0\int_\Omega \rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\mathrm{d}x\mathrm{d}t\right|\\
&+\left|\frac{1}{\tau}\int_T^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\mathrm{d}x\mathrm{d}t\right|\\
&+C(\varepsilon)\int_{Q_T}\left|L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\right|^2 dxdt\\
&+\frac{1}{2}\|(\rho_3^{-1})_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\int_{Q_T}\left|L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right| \mathrm{d}x\mathrm{d}t\\
&+sC(\varepsilon)\int_0^T\left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\right|^2\mathrm{d}t + \frac{s}{2}\|(\rho_4^{-1})_{tt}\|_{L^\infty(\mathbb{R})}\int_0^T\left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)\right|\mathrm{d}t\\
&+\left|\frac{1}{\tau}\int_{-\tau}^0\int_\Omega \rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\mathrm{d}x\mathrm{d}t\right|\\
&+\left|\frac{s}{\tau}\int_{-\tau}^0 \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\mathrm{d}t\right|\\
&+\left|\frac{1}{\tau}\int_T^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\mathrm{d}x\mathrm{d}t\right|\\
&+\left|\frac{s}{\tau}\int_T^{T-\tau} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau}\mathrm{d}t\right|\\
&:= J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8+J_9+J_{10}+J_{11}+J_{12}+J_{13}+J_{14}+J_{15}.
\end{aligned}
\tag{91}
$$

Now, fix $\epsilon = \dfrac{1}{6}$ and let us obtain the suitable estimates for the terms in the right hand side of (91). (i) **Estimate of** J_1**.** Since $B \in \mathcal{D}(0,T;L^2(\Omega))$, we extend B by 0 outside $(0,T)$. This yields

$$
\begin{aligned}
&\int_{Q_T} B\mathcal{T}_\tau(\rho_{2,\star}^{-1}p) = \frac{1}{\tau^2}\int_{Q_T} B(t)(\rho_{2,\star}^{-1}p)(t+\tau) - \frac{2}{\tau^2}\int_{Q_T} B(t)(\rho_{2,\star}^{-1}p)(t) + \frac{1}{\tau^2}\int_{Q_T} B(t)(\rho_{2,\star}^{-1}p)(t-\tau)\\
&= \int_\tau^{T+\tau}\int_\Omega B(t-\tau)(\rho_{2,\star}^{-1}p)(t) - \frac{2}{\tau^2}\int_{Q_T} B(t)(\rho_{2,\star}^{-1}p)(t) + \frac{1}{\tau^2}\int_{-\tau}^{T-\tau}\int_\Omega B(t+\tau)(\rho_{2,\star}^{-1}p)(t)\\
&= \int_{Q_T}\left(\frac{B(t+\tau)-2B(t)+B(t-\tau)}{\tau^2}\right)(\rho_{2,\star}^{-1}p)(t)
\end{aligned}
$$

since $(\rho_{2,\star}^{-1}p)$ vanishes outside $(0,T)$ and thus

$$J_1=\left|\int_{Q_T}\left(\frac{B(t+\tau)-2B(t)+B(t-\tau)}{\tau^2}\right)(\rho_{2,\star}^{-1}p)(t)\right|\to\left|\int_{Q_T}B_{tt}(\rho_{2,\star}^{-1}p)\right|\quad\text{as }\tau\to 0.\tag{92}$$

It follows that the term J_1 is bounded uniformly w.r.t. $\tau\neq 0$ small.

(ii) **Estimate of J_2.** As $\tau\to 0$, we observe since $\rho_3^{-1}\gamma^2\in L^\infty(Q_T)$ that

$$J_2\leqslant C\int_{Q_T}\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right|^2\mathrm{d}x\mathrm{d}t\leqslant C\int_{Q_T}\left|(\rho_{2,\star}^{-1}p)_t\right|^2\mathrm{d}x\mathrm{d}t\tag{93}$$

and thus J_2 is bounded uniformly w.r.t $\tau\neq 0$ small.

(iii) J_3 **is bounded** uniformly w.r.t $\tau\neq 0$ small since we have

$$\left\|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right\|_{L^2(Q_T)}\left\|\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\right\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\|\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\|_{L^2(Q_T)}\to 0\quad\text{as }\tau\to 0$$

using that $\rho_3^{-1}\gamma\in\mathcal{D}(\mathbb{R};\mathcal{C}^0(\overline{\Omega}))$, $\rho_{2,\star}^{-1}p\in H^1(\mathbb{R};L^2(\Omega))$ and $L^\star\big(\rho_{2,\star}^{-1}p\big)\in L^2(\mathbb{R};L^2(\Omega))$.

(iv) J_4 **is bounded** since $\rho_{2,\star}^{-1}p\in L^2(Q_T)$.

(v) **Estimate of J_5.** We have

$$J_5\leqslant\frac{1}{2}\|(\rho_3^{-1}\gamma)_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\|\rho_{2,\star}^{-1}p\|_{L^2(Q_T)}\|\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\|_{L^2(Q_T)}\to 0\quad\text{as }\tau\to 0$$

since $\rho_{2,\star}^{-1}p\in L^2(Q_T)$ and $L^\star\big(\rho_{2,\star}^{-1}p\big)\in L^2(\mathbb{R};L^2(\Omega))$ and thus J_5 is bounded uniformly w.r.t $\tau\neq 0$ small.

(vi) **Estimate of J_6.** Observe that, if $\tau<0$, then $J_6=0$ since $\rho_3^{-1}(t+\tau)\gamma(t+\tau)=0$ on $[0,-\tau]$ and if $\tau>0$ the term J_6 satisfies, since $\rho_3^{-1}(t)\gamma(t)=0=(\rho_{2,\star}^{-1}p)(t)$ on $[-\tau,0]$

$$\begin{aligned}J_6&\leqslant\left|\frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\right|\\&\leqslant\left|\int_{-\tau}^{0}\int_\Omega\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\,\mathrm{d}x\mathrm{d}t\right|\to 0\quad\text{as }\tau\to 0,\end{aligned}\tag{94}$$

since $\gamma\rho_3^{-1}\in\mathcal{D}(\mathbb{R};\mathcal{C}^0(\overline{\Omega}))$, $\rho_{2,\star}^{-1}p\in H^1(\mathbb{R};L^2(\Omega))$ and $L^\star\big(\rho_{2,\star}^{-1}p\big)\in L^2(\mathbb{R};L^2(\Omega))$. Therefore, J_6 is uniformly bounded w.r.t. nonzero τ however small.

(vii) **Estimate of J_7.** Observe that, if $\tau>0$, then $J_7=0$ since $\rho_3^{-1}(t+\tau)\gamma(t+\tau)=0$ on $[T-\tau,T]$ and if $\tau<0$ the term J_7 satisfies, since $\rho_3^{-1}(t)\gamma(t)=0=(\rho_{2,\star}^{-1}p)(t)$ on $[T,T-\tau]$

$$\begin{aligned}J_7&\leqslant\frac{1}{\tau}\left|\int_{T}^{T-\tau}\int_\Omega\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\right|\\&\leqslant\left|\int_{T}^{T-\tau}\int_\Omega\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\,\mathrm{d}x\mathrm{d}t\right|\to 0\quad\text{as }\tau\to 0,\end{aligned}\tag{95}$$

since $\gamma\rho_3^{-1}\in\mathcal{D}(\mathbb{R};\mathcal{C}^0(\overline{\Omega}))$, $\rho_{2,\star}^{-1}p\in H^1(\mathbb{R};L^2(\Omega))$ and $L^\star\big(\rho_{2,\star}^{-1}p\big)\in L^2(\mathbb{R};L^2(\Omega))$. Therefore, J_7 is uniformly bounded w.r.t. nonzero τ however small.

(viii) J_8 **is bounded** since $L^\star\big(\rho_{2,\star}^{-1}p\big)\in L^2(\mathbb{R};L^2(\Omega))$.

(ix) J_9 **is bounded.** Indeed, we see

$$J_9=\frac{1}{2}\|(\rho_3^{-1})_{tt}\|_{L^\infty(\mathbb{R};L^\infty(\Omega))}\int_{Q_T}\left|L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\right|\mathrm{d}x\mathrm{d}t\to 0\quad\text{as }\tau\to 0$$

since $L^\star\big(\rho_{2,\star}^{-1}p\big) \in L^2(\mathbb{R};L^2(\Omega))$. So, J_9 is uniformly bounded w.r.t. nonzero τ however small.

(x) J_{10} **is bounded** since $(\rho_{2,\star}^{-1}p)_x(1,t) \in L^2(\mathbb{R})$.

(xi) J_{11} **is bounded.** We have

$$J_{11} = \frac{s}{2}\|(\rho_4^{-1})_{tt}\|_{L^\infty(\mathbb{R})}\int_0^T \left|(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)\right| \mathrm{d}t \to 0 \quad \text{as } \tau \to 0$$

since $(\rho_{2,\star}^{-1}p)_x(1,t) \in L^2(\mathbb{R})$ and $(\rho_{2,\star}^{-1}p)_x(1,t) \in L^2(\mathbb{R})$. So, J_{11} is uniformly bounded w.r.t. nonzero τ however small.

(xii) **Estimate of** J_{12}**.** Observe that, if $\tau < 0$, then $J_{12} = 0$ since $\rho_3^{-1}(t+\tau) = 0$ on $[0,-\tau]$ and if $\tau > 0$ the term J_{12} satisfies, since $\rho_3^{-1}(t) = 0$ on $[-\tau, 0]$

$$\begin{aligned} J_{12} &= \frac{1}{\tau}\left|\int_{-\tau}^0\int_\Omega \rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\right| \\ &= \left|\int_{-\tau}^0\int_\Omega \left(\frac{\widetilde{\delta}_\tau(\rho_3^{-1/2})}{\tau}\right)^2 L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)\,\mathrm{d}x\mathrm{d}t\right| \to 0 \quad \text{as } \tau \to 0, \end{aligned} \tag{96}$$

since $\rho_3^{-1/2} \in \mathcal{D}(\mathbb{R};L^\infty(\Omega))$ and $L^\star\big(\rho_{2,\star}^{-1}p\big) \in L^2(\mathbb{R};L^2(\Omega))$. Therefore, J_{12} is uniformly bounded w.r.t. nonzero τ however small.

(xiii) **Estimate of** J_{13}**.** Observe that if $\tau < 0$, then $J_{13} = 0$ since $\rho_4^{-1}(t+\tau) = 0$ on $[0,-\tau]$ and if $\tau > 0$ the term J_{13} satisfies, since $\rho_4^{-1}(t) = 0$ on $[-\tau,0]$ the term

$$\begin{aligned} J_{13} &= \left|\frac{s}{\tau}\int_{-\tau}^0 \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\mathrm{d}t\right| \\ &= \left|s\int_{-\tau}^0\left(\frac{\widetilde{\delta}_\tau\rho_4^{-1/2}}{\tau}\right)^2(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x\mathrm{d}t\right| \to 0 \quad \text{as } \tau \to 0, \end{aligned} \tag{97}$$

since $\rho_4^{-1/2} \in \mathcal{D}(\mathbb{R})$ and $(\rho_{2,\star}^{-1}p)_x(1,t) \in L^2(\mathbb{R})$. Therefore, J_{13} is uniformly bounded w.r.t. nonzero τ however small.

(xiv) **Estimate of** J_{14}**.** Observe that if $\tau > 0$, then $J_{14} = 0$ since $\rho_3^{-1}(t+\tau) = 0$ on $[T-\tau,T]$ and if $\tau < 0$ the term J_{14} satisfies, since $\rho_3^{-1}(t) = 0$ on $[T, T-\tau]$ the term

$$\begin{aligned} J_{14} &= \left|\frac{1}{\tau}\int_T^{T-\tau}\int_\Omega \rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\right| \\ &= \left|\int_T^{T-\tau}\int_\Omega\left(\frac{\widetilde{\delta}_\tau\rho_3^{-1/2}}{\tau}\right)^2 L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)\,\mathrm{d}x\mathrm{d}t\right| \to 0 \quad \text{as } \tau \to 0, \end{aligned} \tag{98}$$

since $\rho_3^{-1/2} \in \mathcal{D}(\mathbb{R};L^\infty(\Omega))$ and $L^\star\big(\rho_{2,\star}^{-1}p\big) \in L^2(\mathbb{R};L^2(\Omega))$. Therefore, J_{14} is uniformly bounded w.r.t. nonzero τ however small.

(xv) **Estimate of** J_{15}**.** Finally, we observe that if $\tau > 0$, then $J_{15} = 0$ since $\rho_4^{-1}(t+\tau) = 0$ on $[T-\tau,T]$ and if $\tau < 0$ the term J_{15} satisfies, since $\rho_4^{-1}(t) = 0$ on $[T,T-\tau]$ the term

$$\begin{aligned} J_{15} &= \left|\frac{s}{\tau}\int_T^{T-\tau} \rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau}\mathrm{d}t\right| \\ &= \left|s\int_T^{T-\tau}\left(\frac{\widetilde{\delta}_\tau\rho_4^{-1/2}}{\tau}\right)^2(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)\mathrm{d}t\right| \to 0 \quad \text{as } \tau \to 0, \end{aligned} \tag{99}$$

since $\rho_4^{-1/2} \in \mathcal{D}(\mathbb{R})$ and $(\rho_{2,\star}^{-1}p)_x(1,t) \in L^2(\mathbb{R})$. Therefore, J_{15} is uniformly bounded w.r.t. nonzero τ however small.

• We then have proved that all the terms in the right hand side of (91) are bounded w.r.t. $\tau \to 0$. This implies that the terms

$$\int_{Q_T} \rho_3^{-1} \left| \frac{\widetilde{\delta}_\tau L^\star(\rho_{2,\star}^{-1}p)}{\tau} \right|^2 \mathrm{d}x\mathrm{d}t, \quad s\int_0^T \rho_4^{-1}(1,t) \left| \frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau} \right|^2 \mathrm{d}t \tag{100}$$

are bounded uniformly w.r.t. $\tau \neq 0$ small. Therefore,

$$\rho_3^{-1/2}\big(L^\star(\rho_{2,\star}^{-1}p)\big)_t \in L^2(Q_T) \quad \text{and} \quad \rho_4^{-1/2}(\rho_{2,\star}^{-1}p)_{x,t}(1,t) \in L^2(0,T)$$

and

$$\rho_3^{-1/2}\frac{\widetilde{\delta}_\tau L^\star(\rho_{2,\star}^{-1}p)}{\tau} \to \rho_3^{-1/2}\big(L^\star(\rho_{2,\star}^{-1}p)\big)_t \text{ in } L^2(Q_T)$$

$$\rho_4^{-1/2}\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau} \to \rho_4^{-1/2}(\rho_{2,\star}^{-1}p)_{x,t}(1,t) \text{ in } L^2(0,T).$$

Indeed

$$\begin{aligned}
\rho_3^{-1/2}\frac{\widetilde{\delta}_\tau L^\star(\rho_{2,\star}^{-1}p)}{\tau} &= \frac{\widetilde{\delta}_\tau\big(\rho_3^{-1/2}\ L^\star(\rho_{2,\star}^{-1}p)\big)}{\tau} - \frac{\widetilde{\delta}_\tau(\rho_3^{-1/2})}{\tau}L^\star(\rho_{2,\star}^{-1}p)(\cdot+\tau)\\
&\to \big(\rho_3^{-1/2}\ L^\star(\rho_{2,\star}^{-1}p)\big)_t - (\rho_3^{-1/2})_t L^\star(\rho_{2,\star}^{-1}p) = \rho_3^{-1/2}\big(L^\star(\rho_{2,\star}^{-1}p)\big)_t \text{ in } L^2(Q_T)
\end{aligned}$$

since $\big(\rho_3^{-1/2}\ L^\star(\rho_{2,\star}^{-1}p)\big)_t$, $L^\star(\rho_{2,\star}^{-1}p)$ are in $L^2(\mathbb{R};L^2(\Omega))$ and $\rho_3^{-1/2}$ is in $\mathcal{D}(\mathbb{R};L^\infty(\Omega))$. We also have

$$\begin{aligned}
\rho_4^{-1/2}(1,\cdot)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}(1,\cdot) &= \frac{\widetilde{\delta}_\tau\big(\rho_4^{-1/2}(\rho_{2,\star}^{-1}p)_x\big)}{\tau}(1,\cdot) - \frac{\widetilde{\delta}_\tau(\rho_4^{-1/2})}{\tau}(1,\cdot)(\rho_{2,\star}^{-1}p)_x(1,\cdot+\tau)\\
&\to \big((\rho_{2,\star}^{-1}p)_{x,t})\big)_t(1,\cdot) - (\rho_4^{-1/2})_t(1,\cdot)(\rho_{2,\star}^{-1}p)_{x,t}(1,\cdot)\\
&= \rho_4^{-1/2}(1,\cdot)(\rho_{2,\star}^{-1}p)_{x,t}(1,\cdot) \text{ in } L^2(0,T)
\end{aligned}$$

since $\big(\rho_4^{-1/2}\ (\rho_{2,\star}^{-1}p)_{x,t}\big)_t$, $(\rho_{2,\star}^{-1}p)_{x,t}$ are in $L^2(\mathbb{R})$ and $\rho_4^{-1/2}$ is in $\mathcal{D}(\mathbb{R})$.

• **Sub-step 2.** We pass to the limit $\tau \to 0$ in the equality (83) (equivalent to (78))

Since $B \in \mathcal{D}(0,T;L^2(\Omega))$ and $\rho_{2,\star}^{-1}p \in H^2_{\mathrm{loc}}(0,T;L^2(\Omega))$ we have $\int_{Q_T} B\mathcal{T}_\tau(\rho_{2,\star}^{-1}p) \to \int_{Q_T} B(\rho_{2,\star}^{-1}p)_{tt}$.

We first pass to the limit in the equality (84), namely

$$\begin{aligned}
&\int_{Q_T}\rho^{-2}\gamma p L^\star\big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\big)\,\mathrm{d}x\mathrm{d}t\\
&\quad= -\int_{Q_T}\rho_3^{-1}\gamma\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t - \int_{Q_T}(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\frac{1}{\tau}\int_{-\tau}^0\int_\Omega\rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\frac{1}{\tau}\int_T^{T-\tau}\int_\Omega\rho_3^{-1}(t+\tau)\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{101}$$

Estimates (94) and (95) imply

$$-\frac{1}{\tau}\int_{-\tau}^0\int_\Omega\rho_3^{-1}(t+\tau))\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t \to 0,$$

and

$$-\frac{1}{\tau}\int_{T}^{T-\tau}\int_{\Omega}\rho_3^{-1}(t+\tau))\gamma(t+\tau)(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\to 0.$$

1 Moreover, since $\rho_3^{-1}\gamma\in\mathcal{D}(\mathbb{R};L^\infty(\Omega))$, for all $(t,\tau)\in\mathbb{R}^2$, there exists $\lambda(t,\tau)\in(0,1)$ such that

$$\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)(t)=\tau(\rho_3^{-1}\gamma)_t(t)+\frac{\tau^2}{2}(\rho_3^{-1}\gamma)_{tt}(t+\lambda\tau) \tag{102}$$

and thus, since $|(\rho_3^{-1}\gamma)_t|\leqslant C|\rho_3^{-1/2}|$

$$\begin{aligned}-\int_{Q_T}(\rho_{2,\star}^{-1}p)(t+\tau)\frac{\widetilde{\delta}_\tau(\rho_3^{-1}\gamma)}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t=&-\int_{Q_T}(\rho_{2,\star}^{-1}p)(t+\tau)(\rho_3^{-1}\gamma)_t\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t\\&-\frac{1}{2}\int_{Q_T}(\rho_{2,\star}^{-1}p)(t+\tau)(\rho_3^{-1}\gamma)_{tt}(t+\lambda\tau)\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)\,\mathrm{d}x\mathrm{d}t\\&\to-\int_{Q_T}(\rho_{2,\star}^{-1}p)(\rho_3^{-1}\gamma)_tL^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We also have, since $|\rho_3^{-1}\gamma|\leqslant C|\rho_3^{-1/2}|$ and $\rho_{2,\star}^{-1}p\in H^1(\mathbb{R};L^2(\Omega))$

$$-\int_{Q_T}\rho_3^{-1}\gamma\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t\to-\int_{Q_T}\rho_3^{-1}\gamma(\rho_{2,\star}^{-1}p)_tL^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t$$

2 as $\tau\to 0$. It follows that

$$\begin{aligned}&-\int_{Q_T}\rho^{-2}\gamma pL^\star\big(\mathcal{T}_\tau(\rho_{2,\star}^{-1}p)\big)\,\mathrm{d}x\mathrm{d}t\\&\to-\int_{Q_T}\rho_3^{-1}\gamma(\rho_{2,\star}^{-1}p)_tL^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t-\int_{Q_T}(\rho_{2,\star}^{-1}p)(\rho_3^{-1}\gamma)_tL^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{103}$$

Arguing as before we have

$$-\int_{Q_T}L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau(\rho_3^{-1})}{\tau}\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)}{\tau}\,\mathrm{d}x\mathrm{d}t\to-\int_{Q_T}L^\star\big(\rho_{2,\star}^{-1}p\big)(\rho_3^{-1})_tL^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t,$$

$$-s\int_0^T(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_4^{-1})(1,t)}{\tau}\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x(1,t)}{\tau}\mathrm{d}t\to-s\int_0^T(\rho_{2,\star}^{-1}p)_x(1,t)(\rho_4^{-1})_t(1,t)(\rho_{2,\star}^{-1}p)_{xt}(1,t)\mathrm{d}t$$

and from (96)-(99),

$$-\frac{1}{\tau}\int_{-\tau}^{0}\int_{\Omega}\rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\to 0,$$

$$-\frac{s}{\tau}\int_{-\tau}^{0}\rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\mathrm{d}t\to 0,$$

$$-\frac{1}{\tau}\int_{T}^{T-\tau}\int_{\Omega}\rho_3^{-1}(t+\tau)L^\star\big(\rho_{2,\star}^{-1}p\big)(t+\tau)\frac{\widetilde{\delta}_\tau L^\star\big(\rho_{2,\star}^{-1}p\big)(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\to 0,$$

and

$$-\frac{s}{\tau}\int_{T}^{T-\tau}\rho_4^{-1}(1,t+\tau)(\rho_{2,\star}^{-1}p)_x(1,t+\tau)\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\mathrm{d}t\to 0$$

as $\tau \to 0$. Therefore, the limit in the equality (83) provides

$$
\begin{aligned}
&\int_{Q_T} \rho_3^{-1}(t)\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \mathrm{d}x\mathrm{d}t + s\int_0^T \rho_4^{-1}(1,t)\left|(\rho_{2,\star}^{-1}p)_{xt}(1,t)\right|^2 \mathrm{d}t \\
&\quad = -\int_{Q_T} B(\rho_{2,\star}^{-1}p)_{tt} + \int_{Q_T} \rho_3^{-1}\gamma(\rho_{2,\star}^{-1}p)_t L^\star\left(\rho_{2,\star}^{-1}p\right)_t \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_{Q_T} (\rho_{2,\star}^{-1}p)(\rho_3^{-1}\gamma)_t L^\star\left(\rho_{2,\star}^{-1}p\right)_t \mathrm{d}x\mathrm{d}t - \int_{Q_T} L^\star\left(\rho_{2,\star}^{-1}p\right)(\rho_3^{-1})_t L^\star\left(\rho_{2,\star}^{-1}p\right)_t \mathrm{d}x\mathrm{d}t \\
&\qquad - s\int_0^T (\rho_{2,\star}^{-1}p)_x(1,t)(\rho_4^{-1})_t(1,t)(\rho_{2,\star}^{-1}p)_{xt}(1,t)\mathrm{d}t.
\end{aligned}
\tag{104}
$$

• **Sub-step 3.** Proof of the estimate (75).

First, the Carleman estimate (12) with $\dfrac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau} \in P$ (recall that, if not specified, the weights depend on the parameter s) and any parameter $\widetilde{s} \geqslant s_0$ leads to

$$
\begin{aligned}
&\widetilde{s}^3\int_{Q_T} \rho_0^{-2}(\widetilde{s})\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right|^2 + \widetilde{s}\int_{Q_T} \rho_1^{-2}(\widetilde{s})\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\right|^2 \\
&\qquad + \widetilde{s}^{-1}\int_{Q_T} \rho_2^{-2}(\widetilde{s})\left(\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_t}{\tau}\right|^2 + \left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_{xx}}{\tau}\right|^2\right) \\
&\qquad \leqslant C\int_{Q_T} \rho^{-2}(\widetilde{s})\left|L^\star\left(\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right)\right|^2 + C\widetilde{s}\int_0^T \rho_1^{-2}(\widetilde{s})(1,t)\left|\left(\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\right)(1,t)\right|^2.
\end{aligned}
$$

In order to make appear the terms $\rho_3^{-1}(s)$ and $\rho_4^{-1}(s)$ (see (75), we take for any $s \geqslant \frac{4}{3}s_0$, $\widetilde{s} = \dfrac{3s}{4}$ so that $\widetilde{s} \geqslant s_0$ and then check that there exists $C > 0$ (independent of s) such that

$$
\rho^{-2}(\widetilde{s}) \leqslant C\rho_3^{-1}(s) \text{ and } \rho_1^{-2}(\widetilde{s})(1,\cdot) \leqslant C\rho_4^{-1}(s)(1,\cdot).
$$

Thus

$$
\begin{aligned}
&\widetilde{s}^3\int_{Q_T} \rho_0^{-2}(\widetilde{s})\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right|^2 + \widetilde{s}\int_{Q_T} \rho_1^{-2}(\widetilde{s})\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\right|^2 \\
&\qquad + \widetilde{s}^{-1}\int_{Q_T} \rho_2^{-2}(\widetilde{s})\left(\left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_t}{\tau}\right|^2 + \left|\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_{xx}}{\tau}\right|^2\right) \\
&\qquad \leqslant C\int_{Q_T} \rho_3^{-1}\left|L^\star\left(\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)}{\tau}\right)\right|^2 + C\widetilde{s}\int_0^T \rho_4^{-1}(1,t)\left|\left(\frac{\widetilde{\delta}_\tau(\rho_{2,\star}^{-1}p)_x}{\tau}\right)(1,t)\right|^2.
\end{aligned}
\tag{105}
$$

Therefore, in view of (104), $\rho_0^{-1}(\widetilde{s})(\rho_{2,\star}^{-1}p)_t \in L^2(Q_T)$, $\rho_1^{-1}(\widetilde{s})(\rho_{2,\star}^{-1}p)_{xt} \in L^2(Q_T)$, $\rho_2^{-1}(\widetilde{s})(\rho_{2,\star}^{-1}p)_{tt} \in L^2(Q_T)$ and $\rho_2^{-1}(\widetilde{s})(\rho_{2,\star}^{-1}p)_{xxt} \in L^2(Q_T)$ and passing to the limit in (105) leads to

$$
\begin{aligned}
&\widetilde{s}^3\int_{Q_T} \rho_0^{-2}(\widetilde{s})\left|(\rho_{2,\star}^{-1}p)_t\right|^2 + \widetilde{s}\int_{Q_T} \rho_1^{-2}(\widetilde{s})\left|(\rho_{2,\star}^{-1}p)_{xt}\right|^2 \\
&\qquad + \widetilde{s}^{-1}\int_{Q_T} \rho_2^{-2}(\widetilde{s})\left(\left|(\rho_{2,\star}^{-1}p)_{tt}\right|^2 + \left|(\rho_{2,\star}^{-1}p)_{xxt}\right|^2\right) \\
&\qquad \leqslant C\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 + C\widetilde{s}\int_0^T \rho_4^{-1}(1,t)\left|(\rho_{2,\star}^{-1}p)_{xt}(1,t)\right|^2.
\end{aligned}
\tag{106}
$$

We now consider the r.h.s terms of (104). Using (106), we get

$$
\begin{aligned}
\left|\int_{Q_T} B(\rho_{2,\star}^{-1}p)_{tt}\right| &= \left|\int_{Q_T} \rho_2(\widetilde{s})B\rho_2^{-1}(\widetilde{s})(\rho_{2,\star}^{-1}p)_{tt}\right| \\
&\leqslant \left(s\int_{Q_T} \rho_2^2(\widetilde{s})|B|^2\right)^{1/2}\left(s^{-1}\int_{Q_T} \rho_2^{-2}(\widetilde{s})|(\rho_{2,\star}^{-1}p)_{tt}|^2\right)^{1/2} \\
&\leqslant C(\varepsilon)s\int_{Q_T} \rho^2|B|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 + \varepsilon s\int_0^T \rho_4^{-1}(1,t)\left|(\rho_{2,\star}^{-1}p)_{xt}(1,t)\right|^2
\end{aligned}
$$

since $\rho_2^2(\widetilde{s}) \leqslant \rho^2$. From (71), since $\rho_3^{-1}\gamma^2 \in L^\infty(Q_T)$, we deduce using (25), that

$$
\begin{aligned}
\left|\int_{Q_T} \rho_3^{-1}\gamma(\rho_{2,\star}^{-1}p)_t L^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t\right| &\leqslant C(\varepsilon)\int_{Q_T} \rho_3^{-1}\gamma^2|(\rho_{2,\star}^{-1}p)_t|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \\
&\leqslant C(\varepsilon)s\|p\|_{P_s}^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \\
&\leqslant C(\varepsilon)s^{-2}\int_{Q_T} \rho^2|B|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2.
\end{aligned}
$$

Since $|(\rho_3^{-1}\gamma)_t| \leqslant Cs^2\rho_3^{-1/2}$ and $\rho_{2,\star}^{-1} \leqslant C\rho_0^{-1}$, estimate (25) together with the Carleman estimate (12) imply that

$$
\begin{aligned}
\left|\int_{Q_T} (\rho_{2,\star}^{-1}p)(\rho_3^{-1}\gamma)_t L^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t\right| &\leqslant C(\varepsilon)s^4\int_{Q_T} \rho_{2,\star}^{-2}|p|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \\
&\leqslant C(\varepsilon)s^4\int_{Q_T} \rho_0^{-2}|p|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \\
&\leqslant C(\varepsilon)s\|p\|_{P_s}^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \\
&\leqslant C(\varepsilon)s^{-2}\int_{Q_T} \rho^2|B|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2.
\end{aligned}
$$

Since $|(\rho_3^{-1})_t| \leqslant Cs\rho_3^{-1/2}$, using (71) and (25), we also get

$$
\begin{aligned}
\left|\int_{Q_T} L^\star\big(\rho_{2,\star}^{-1}p\big)(\rho_3^{-1})_t L^\star\big(\rho_{2,\star}^{-1}p\big)_t\,\mathrm{d}x\mathrm{d}t\right| &\leqslant C(\varepsilon)s^2\int_{Q_T} \left|L^\star\big(\rho_{2,\star}^{-1}p\big)\right|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \\
&\leqslant C(\varepsilon)s^2\|p\|_{P_s}^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2 \\
&\leqslant C(\varepsilon)s^{-1}\int_{Q_T} \rho^2|B|^2 + \varepsilon\int_{Q_T} \rho_3^{-1}\left|L^\star(\rho_{2,\star}^{-1}p)_t\right|^2.
\end{aligned}
$$

Eventually, since $|(\rho_4^{-1})_t| \leqslant Cs\rho_4^{-1/2}$ and $\left|(\rho_{2,\star}^{-1}p)_x(1,t)\right| \leqslant C\left|\rho_1^{-1}p_x(1,t)\right|$, we deduce from the definition (21) of $\|p\|_{P_s}$ and (25) :

$$
\begin{aligned}
\left|s\int_0^T (\rho_{2,\star}^{-1}p)_x(1,t)(\rho_4^{-1})_t(1,t)(\rho_{2,\star}^{-1}p)_{xt}(1,t)\right| &\leqslant s^4C(\varepsilon)\int_0^T \left|(\rho_{2,\star}^{-1}p)_x(1,t)\right|^2 + s\varepsilon\int_0^T \rho_4^{-1}(1,t)\left|(\rho_{2,\star}^{-1}p)_{xt}\right|^2(1,t) \\
&\leqslant s^4C(\varepsilon)\int_0^T \left|(\rho_1^{-1}p_x)(1,t)\right|^2 + s\varepsilon\int_0^T \rho_4^{-1}(1,t)\left|(\rho_{2,\star}^{-1}p)_{xt}\right|^2(1,t) \\
&\leqslant C(\varepsilon)s^3\|p\|_{P_s}^2 + s\varepsilon\int_0^T \rho_4^{-1}(1,t)\left|(\rho_{2,\star}^{-1}p)_{xt}\right|^2(1,t) \\
&\leqslant C(\varepsilon)\int_{Q_T} \rho^2|B|^2 + s\varepsilon\int_0^T \rho_4^{-1}(1,t)\left|(\rho_{2,\star}^{-1}p)_{xt}\right|^2(1,t).
\end{aligned}
$$

Taking ε small enough, the estimate (75) follows.

♦ **Step 2.** We prove that $\rho_3^{1/2} z_t \ln L^2(Q_T)$, $\rho_4^{1/2} v_t \in L^2(0,T)$ and satisfy (29).

System (73) first implies that $z = \rho_3^{-1} L^\star(\rho_{2,\star}^{-1} p) + \rho_3^{-1} (\rho_{2,\star}^{-1})_t p$ and thus

$$z_t = \rho_3^{-1} L^\star(\rho_{2,\star}^{-1} p)_t + (\rho_3^{-1})_t L^\star(\rho_{2,\star}^{-1} p) + (\rho_3^{-1} (\rho_{2,\star}^{-1})_t)_t p + \rho_3^{-1} (\rho_{2,\star}^{-1})_t p_t.$$

But, since $(\rho_{2,\star}^{-1})_t = \gamma \rho_{2,\star}^{-1}$ and $(\rho_3^{-1})_t = 2s(T-2t)\theta^2 \varphi_1 \rho^{-2} \rho_{2,\star} + \rho^{-2} \gamma \rho_{2,\star} = (2s(T-2t)\theta^2\varphi_1 + \gamma)\rho_3^{-1}$, we have

$$\begin{aligned}
\left|\rho_3^{1/2} z_t\right| &\leqslant \rho_3^{-1/2} \; \left|L^\star(\rho_{2,\star}^{-1} p)_t\right| + \rho_3^{1/2} |(\rho_3^{-1})_t| \left|L^\star(\rho_{2,\star}^{-1} p)\right| + \rho_3^{1/2} |(\rho_3^{-1} (\rho_{2,\star}^{-1})_t)_t||p| + \rho_3^{-1/2} |(\rho_{2,\star}^{-1})_t| p_t| \\
&\leqslant \rho_3^{-1/2} \; \left|L^\star(\rho_{2,\star}^{-1} p)_t\right| + \rho_3^{-1/2} |2s(T-2t)\theta^2\varphi_1 + \gamma| \left|L^\star(\rho_{2,\star}^{-1} p)\right| \\
&\quad + \rho_3^{-1/2} |s(T-2t)\theta^2\varphi_1 + \gamma||\gamma| \rho_{2,\star}^{-1} |p| + \rho_3^{-1/2} |\gamma_t + \gamma^2| \rho_{2,\star}^{-1}||p| + \rho_3^{-1/2} |\gamma| \rho_{2,\star}^{-1}||p_t|,
\end{aligned}$$

yielding to

$$\begin{aligned}
\int_{Q_T} \rho_3 \left|z_t\right|^2 \leqslant C \int_{Q_T} \rho_3^{-1} \left|L^\star(\rho_{2,\star}^{-1} p)_t\right|^2 + Cs^2 \int_{Q_T} \rho_3^{-1} \theta^4 \left|L^\star(\rho_{2,\star}^{-1} p)\right|^2 \\
+ Cs^4 \int_{Q_T} \left|\rho_{2,\star}^{-1} p\right|^2 + Cs^2 \int_{Q_T} \left|\rho_{2,\star}^{-1} p_t\right|^2 \\
\leqslant Cs \|\rho B\|^2_{L^2(Q_T)}
\end{aligned} \tag{107}$$

thanks to (71), (75), the Carleman estimate (12) for p and the fact that $\rho_{2,\star}^{-1} \leqslant \rho_2^{-1} \leqslant C\rho_0^{-1}$ and $\rho_3^{-1}\theta^4 \leqslant C$.

Next, recalling that the control is given by $v = s\rho_1^{-2}(1,t) p_x(1,t) = s\rho_4^{-1}(1,t)(\rho_{2,\star}^{-1} p)_x(1,t)$, we compute

$$\begin{aligned}
v_t &= s\rho_4^{-1}(1,t)(\rho_{2,\star}^{-1} p)_{xt}(1,t) + s(\rho_4^{-1})_t(1,t)(\rho_{2,\star}^{-1} p)_x(1,t) \\
&= s\rho_4^{-1}(1,t)(\rho_{2,\star}^{-1} p)_{xt}(1,t) + (\rho_4^{-1})_t(1,t)\rho_4(1,t) v(t) \\
&= s\rho_4^{-1}(1,t)(\rho_{2,\star}^{-1} p)_{xt}(1,t) + (\rho_4^{-1})_t(1,t)\rho_4(1,t)\rho_1^{-1}(1,t)\rho_1(1,t) v(t).
\end{aligned}$$

But, $(\rho_4^{-1})_t(1,t)\rho_4(1,t)\rho_1^{-1}(1,t) \leqslant C$ so, thanks to (24) and (75) :

$$\begin{aligned}
s^{-1} \int_0^T \rho_4 \left|v_t\right|^2 \leqslant s \int_0^T \rho_4^{-1}(1,t) \left|(\rho_{2,\star}^{-1} p)_{xt}(1,t)\right|^2 + C \int_0^T \left|\rho_1(1,t) v(t)\right|^2 \\
\leqslant Cs \|\rho B\|^2_{L^2(Q_T)}.
\end{aligned} \tag{108}$$

Estimate (29) follows from (107) and (108).

♦ **Step 3.** We prove that $z \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ (thus $z \in L^\infty(Q_T)$) and satisfy (30).

As a consequence of (29), we get

$$\|v_t\|_{L^2(0,T)} \leqslant C\|\rho_4^{-1/2}\|_{L^\infty(0,T)} \|\rho_4^{1/2} v_t\|_{L^2(0,T)} \leqslant Cse^{-c_2 s} \|\rho B\|_{L^2(Q_T)}, \tag{109}$$

and then use (by means of (76)) that there exists some positive constant c_2 such that

$$\|\rho_4^{-1/2}\|_{L^\infty(0,T)} \leqslant e^{-c_2 s}, \quad \forall s \geqslant s_0. \tag{110}$$

Therefore, $v \in H^1(0,T) \subset \mathcal{C}^0([0,T])$ and since $\rho_1 v \in L^2(0,T)$ we have $v(0) = v(T) = 0$; therefore $v \in H_0^1(0,T)$. Therefore, by using the L^2 regularity result for the heat equation, we get that z solution of (19) satisfies

$$\begin{aligned}
\|z\|_{L^2(0,T;H^2(\Omega))} + \|z\|_{H^1(0,T;L^2(\Omega))} &\leqslant C\left(\|B\|_{L^2(Q_T)} + \|v\|_{H^1(0,T)}\right) \\
&\leqslant C\left(\|\rho^{-1}\|_{L^\infty(Q_T)} \|\rho B\|_{L^2(Q_T)} + se^{-c_2 s}\|\rho B\|_{L^2(Q_T)}\right) \\
&\leqslant Cse^{-c_3 s} \|\rho B\|_{L^2(Q_T)},
\end{aligned} \tag{111}$$

where

$$c_3 := \min\left\{c_2,\ \min_{\overline{Q_T}}|\varphi|\right\} > 0. \tag{112}$$

The regularity result $z \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ implies the desired property, namely that $z \in L^\infty(Q_T)$ with the same estimate as (111) (this follows from the continuous embedding $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \hookrightarrow \mathcal{C}^0(\overline{Q_T})$ in the one dimensional case).

♦ **Step 4.** We now consider the case $\rho B \in L^2(Q_T)$.

We proceed by density from the estimates (29) and (111) proved for $B \in \mathcal{D}(0,T;L^2(\Omega))$: there exists $(\widehat{B}_n)_{n\in\mathbb{N}^*} \subset \mathcal{D}(0,T;L^2(\Omega))$ such that $\widehat{B}_n \to \rho B$ in $L^2(Q_T)$ as $n \to +\infty$. This implies $\rho^{-1}\widehat{B}_n \to B$ in $L^2(Q_T)$ as $n \to +\infty$. Then $B_n := \rho^{-1}\widehat{B}_n \in \mathcal{D}(0,T;L^2(\Omega))$ satisfies $B_n \to B$ and $\rho B_n \to \rho B$ as $n \to +\infty$.

Applying Theorem 2 for all $n \in \mathbb{N}^\star$, there exists unique $(z_n, v_n) \in L^2(\rho, Q_T) \times L^2(\rho_1,(0,T))$ solution to (19) and the linearity of the map Λ_s^0 (see (23)) and (24)) imply for all $n, m \in \mathbb{N}^*$,

$$\|\rho(z_n - z_m)\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(v_n - v_m)\|_{L^2(0,T)} \leqslant Cs^{-3/2}\|\rho(B_n - B_m)\|_{L^2(Q_T)}. \tag{113}$$

Similarly, (29) implies for $\forall n, m \in \mathbb{N}^\star$

$$\|\rho_3^{1/2}(z_n - z_m)_t\|_{L^2(Q_T)} + s^{-1/2}\|\rho_4^{1/2}(v_n - v_m)_t\|_{L^2(0,T)} \leqslant Cs^{1/2}\|\rho(B_n - B_m)\|_{L^2(Q_T)}. \tag{114}$$

Finally, estimate (111) implies

$$\|z_n - z_m\|_{L^2(0,T;H^2(\Omega))} + \|z_n - z_m\|_{H^1(0,T;L^2(\Omega))} \leqslant Cse^{-c_3 s}\|\rho(B_n - B_m)\|_{L^2(Q_T)}, \quad \forall n, m \in \mathbb{N}^*$$

while estimate (109) implies

$$\|(v_n - v_m)_t\|_{L^2(0,T)} \leqslant Cse^{-c_2 s}\|\rho(B_n - B_m)\|_{L^2(Q_T)}.$$

Consequently, there exist $z \in L^2(\rho, Q_T) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $v \in L^2(\rho_1,(0,T)) \cap H^1(0,T)$ such that

$$\begin{aligned}
&z_n \to z \quad &&\text{in } L^2(\rho, Q_T) \text{ as } n \to +\infty,\\
&v_n \to v \quad &&\text{in } L^2(\rho_1,(0,T)) \text{ as } n \to +\infty,\\
&z_n \to z \quad &&\text{in } L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \text{ as } n \to +\infty,\\
&v_n \to v \quad &&\text{in } H^1(0,T) \text{ as } n \to +\infty,
\end{aligned}$$

where (z,v) is the unique state-control pair to (19) with $B \in L^2(Q_T)$ by means of Theorem 2, more precisely $z = \rho^{-2}L^\star p$, $v = s\rho_1^{-2}(1,t)p_x(1,t)$. Consequently, the pair (z,v) also satisfies the announced estimates (29) and (30) with $\rho B \in L^2(Q_T)$.

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