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CLT for real β-ensembles at high temperature*

Charlie Dworaczek Guera†, Ronan Memin‡

Abstract

We establish a central limit theorem for the fluctuations of the empirical measure in the β-ensemble of dimension N at a temperature proportional to N and with confining smooth potential. The space of test functions for which the CLT holds includes $\mathcal{C}^1$, vanishing functions at infinity. It is obtained by the inversion of an operator which is a pertubation of a Sturm-Liouville operator. The method that we use is based on a change of variables introduced in [BFG15] and in [Shc14].

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1 Introduction and main result

The β-ensemble of dimension $N \geqslant 1$ with parameter $\beta > 0$ and potential V is the probability measure on $\mathbb{R}^N$ given by

$$d\mathbf{P}_N^{\beta,V}(x_1,\dots,x_N) = \frac{1}{Z_N(V,\beta)} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,. \tag{1}$$

The potential V has to be chosen so that the partition function

$$Z_N(V,\beta) = \int_{\mathbb{R}^N} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N$$

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is finite. This is the case for example if for some $\beta' > \max(1, \beta)$,

$$\liminf_{|x|\to\infty} \frac{V(x)}{N\beta' \ln|x|} > 1\,, \tag{2}$$

see [AGZ10, equation (2.6.2)]. The parameter β, which is allowed to depend on N, is the so-called inverse temperature.

Under the special choice of $V_G(x) = \dfrac{x^2}{2}$, the measure (1) can be seen as the joint law of the (unordered) eigenvalues of certain matrix models:

- For $\beta = 1$ (resp. $\beta = 2$), it is the law of the eigenvalues of the Gaussian Orthogonal Ensemble (resp. Gaussian Unitary Ensemble), [AGZ10][Theorem 2.5.2].
- For general $\beta > 0$, potentially depending on N, it is the law of the spectrum of certain tri-diagonal random matrices as shown by Dumitriu and Edelman in [DE02].

We consider here the *high temperature regime* where β scales as $1/N$, and write $\beta = \frac{2P}{N}$ for some $P > 0$. The corresponding measure is therefore

$$d\mathbb{P}_N^{V,P}(x_1, \dots, x_N) = \frac{1}{Z_N^{V,P}} \prod_{i<j} |x_i - x_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N\,, \tag{3}$$

with partition function

$$Z_N^{V,P} = \int_{\mathbb{R}^N} \prod_{i<j} |x_i - x_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N\,. \tag{4}$$

It was shown in [GZ19] that under $\mathbb{P}_N^{V,P}$, the sequence of empirical measures

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$$

satisfies a large deviation principle at speed N with strictly convex, good rate function. As a consequence, $\hat{\mu}_N$ converges almost surely in distribution towards a deterministic measure μ_P^V as N goes to infinity, meaning that almost surely, for every bounded continuous $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f d\hat{\mu}_N \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} f d\mu_P^V\,.$$

The limiting measure μ_P^V can be seen to have a density ρ_P^V which satisfies for almost every $x \in \mathbb{R}$

$$V(x) - 2P \int_{\mathbb{R}} \ln|x - y| \rho_P^V(y) dy + \ln \rho_P^V(x) = \lambda_P^V\,, \tag{5}$$

where λ_P^V is constant (see [GM22, Lemma 3.2] for example).

The β-ensemble in the regime $\beta N \underset{N\to\infty}{\longrightarrow} 2P > 0$ has drawn a lot of attention from the random matrix and statistical physics communities lately. This regime was first considered by [CL97] with the study of Dyson Brownian motion with vanishing repulsive coefficient scaled like $\dfrac{1}{N}$. Gases of vortices were also studied with temperature proportional to N in [BG99]. The limiting density was then described in the case of the quadratic potential in [ABG12], as a crossover between the Wigner semicircle law (fixed $\beta > 0$ case) and the Gaussian density (case $\beta = 0$). The fluctuations of the eigenvalues in the bulk and at the edge of a configuration were studied for example in [BGP15],[NT18],[NT20],[Pak18], [Lam21]. These fluctuations were shown to be described by Poisson statistics in this regime. Recently, Spohn

uncovered in [Spo20] a link between the study of the Classical Toda chain and the β-ensemble in the high temperature regime, showing that the limiting density of states of the classical Toda chain, distributed according to the generalized Gibbs ensemble with polynomial potential, can be computed by means of the limiting empirical measure of the β-ensemble at high temperature. In [Maz22], the author established this relation using the matrix representation of the β-ensemble and a moment method, and in [GM22] the authors proved a large deviation principle for the empirical measure of the Toda chain, establishing the previous result for potentials with polynomial growth. See also [Spo22],[GM23],[MM22] for a similar link between the Ablowitz-Ladik lattice and the circular β-ensemble at high temperature. This relation can be further pushed to compute the limiting *currents* of the Toda chain through the central limit theorem for the empirical measure in the β ensemble. The computation of these currents is a crucial step to the derivation of a hydrodynamic equation for the Toda chain, and to the analysis of the correlations of the locally conserved quantities at equilibrium through *linearized hydodynamics*, see [Spo21].

The Central Limit Theorem for the fluctuations of the linear statistics of β-ensembles was first established by [Joh98] for $\beta = 2$ polynomial potential, then generalized and further developed in the regime where β is fixed in [Shc13], [BG13a], [BG13b],[BLS18],[LLW19]. Also an optimal local law was found in this regime in [BMP22]. The CLT was obtained in the high-temperature regime $\beta N \to 2P > 0$ by Nakano and Trinh in [NT18, Theorem 4.9] for quadratic V, relying on the tridiagonal representation for the β-ensemble with quadratic potential in [DE02]. In [HL21], the authors prove the CLT in the case of the *circular* β-ensemble at high temperature with general potential, using a normal approximation method involving the spectral analysis of an operator associated to the limiting covariance structure. Their method allowed them to derive a Berry-Esseen bound, *i.e.* a speed of convergence of the fluctuations towards a Gaussian variable.

In this paper, we adapt part of the arguments of [HL21] to our setup. More precisely, we show that for a class of regular, convex potentials V satisfying a growth condition of the type

$$\lim_{|x|\to\infty} \frac{V''(x)}{V'(x)^2} = 0\,,$$

denoting $\nu_N = \hat{\mu}_N - \mu_P^V$ and considering test functions f belonging to the range of a certain integro-differential operator, the scaled fluctuations of $\hat{\mu}_N$, defined by

$$\sqrt{N}\nu_N(f) := \sqrt{N}\left(\int_{\mathbb{R}} f d\mu_N - \int_{\mathbb{R}} f d\mu_P^V\right)\,,$$

converge in law towards centered Gaussian law with variance depending on f.

When considering the fixed temperature regime, *i.e.* β fixed, one has to renormalize the x_i's by $\sqrt{N}$. It is shown in [AGZ10][Theorem 2.6.1] that the measure

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i/\sqrt{N}}$$

satisfies a large deviation principle, and the limiting measure is characterized in [AGZ10][Lemma 2.6.2] by an equation similar to (5). In fact, the term $\ln \rho_P^V$ in the left-hand side of (5) is the only difference in the equation characterizing the limiting measure in the fixed β case. We point out the very similar characterization of the equilibrium measure corresponding to the minimization problem arising in [BGK16]. There again, the limiting measure is compactly supported. The term $\ln \rho_P^V$ is of prime importance because its presence implies that the support of ρ_P^V is the whole real line. It leads to technicalities to deal with the behavior at infinity of most of the associated objects, namely dealing with weighted Lebesgue spaces $L^2(\mu_P^V)$ and the corresponding Sobolev spaces $H^k(\mu_P^V)$.

Our strategy is based on a change of variables in the partition function $Z_N^{V,P}$ (4), used for the β-ensemble at fixed temperature introduced in [BFG15] and [Shc14], and used in [Gui19] and in [BGK16] to derive the loop equations and in [BLS18] to derive a CLT in the β-ensemble with β fixed. The outline

of the argument goes as follows: Take $\phi : \mathbb{R} \to \mathbb{R}$ smooth, vanishing fast enough at infinity, and do the change of variables in $Z_N^{V,P}$, $x_i = y_i + \frac{t}{\sqrt{N}}\phi(y_i)$, $1 \leqslant i \leqslant N$, to get

$$Z_N^{V,P} = \int_{\mathbb{R}^N} \prod_{i<j} \left| y_i - y_j + \frac{t}{\sqrt{N}}(\phi(y_i) - \phi(y_j)) \right|^{2P/N} e^{-\sum_{i=1}^N V\left(y_i + \frac{t}{\sqrt{N}}\phi(y_i)\right)} \prod_{i=1}^N \left(1 + \frac{t}{\sqrt{N}}\phi'(y_i)\right) d^N\mathbf{y}\,.$$

Expanding the different terms in this integral, one gets

$$Z_N^{V,P} = \int_{\mathbb{R}^N} \prod_{i<j} |y_i - y_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(y_i)} e^{\frac{t}{\sqrt{N}}\left[\frac{2P}{N}\sum_{i<j}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j} + \sum_{i=1}^N \left(\phi'(y_i) - V'(y_i)\phi(y_i)\right)\right]} e^{-\frac{t^2}{2}\sigma_N^2(\phi)} d^N\mathbf{y}\,,$$

where the term $\sigma_N^2(\phi)$ converges towards a limiting variance $\sigma^2(\phi)$ depending on ϕ, P and V. After dividing both parts of the equation by $Z_N^{V,P}$, and because of equation (5) characterizing μ_P^V, one can deduce from the last equation the convergence of the Laplace transform

$$\mathbb{E}\left[e^{t\sqrt{N}(\nu_N(\Xi\phi) + \text{error term})}\right] \underset{N\to\infty}{\longrightarrow} \exp\left(\frac{t^2}{2}\sigma^2(\phi)\right), \tag{6}$$

where Ξ is a linear operator acting on test functions and defined by

$$(\Xi\phi)(x) = 2P\int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{x - y} d\mu_P^V(y) + \phi'(x) - V'(x)\phi(x)\,. \tag{7}$$

Once the error term is taken care of, (6) shows the central limit theorem for test functions of the form $\Xi\phi$. Following [HL21], the operator $\mathcal{L}$ given by

$$\mathcal{L}\phi = \Xi\phi' \tag{8}$$

can be analyzed using Hilbert space techniques. In particular, the operator $\mathcal{L}$, seen as an unbounded operator of the Hilbert space

$$\mathsf{H} = \left\{ u \in L^2(\mu_P^V) \;\middle|\; u' \in L^2(\mu_P^V), \int_{\mathbb{R}} u\rho_P^V dx = 0 \right\}, \qquad \langle u, v\rangle_{\mathsf{H}} = \langle u', v'\rangle_{L^2(\rho_P^V)}\,,$$

can be decomposed as

$$-\mathcal{L} = \mathcal{A} + 2P\mathcal{W}\,,$$

where $\mathcal{A}$ is a positive Sturm-Liouville operator and $\mathcal{W}$ is positive and self-adjoint. Such a writing allows us to show that $-\mathcal{L}$ is invertible, see Theorem 6.7.

We now state the assumptions we make on the potential V. Recall that a probability measure μ supported on $\mathbb{R}$ satisfies the Poincaré inequality if there exists $C > 0$ such that for all $f \in \mathcal{C}^1(\mathbb{R})$ with compact support:

$$\mathrm{Var}_\mu(f) := \int_{\mathbb{R}} \left(f - \int_{\mathbb{R}} f d\mu\right)^2 d\mu \leqslant C\int f'^2 d\mu\,. \tag{9}$$

Assumptions 1.1. *The potential V satisfies:*

i) $V \in \mathcal{C}^3(\mathbb{R})$, $V(x) \underset{|x|\to+\infty}{\longrightarrow} +\infty$, $|V'(x)| \underset{|x|\to+\infty}{\longrightarrow} +\infty$ and is such that μ_P^V satisfies the Poincaré inequality (9).

ii) For all polynomial $Q \in \mathbb{R}[X]$ and $\alpha > 0$, $Q\Big(V'(x)\Big)e^{-V(x)} = \underset{|x|\to\infty}{o}(x^{-\alpha})$.

iii) Furthermore, for any sequence x_N such that $|x_N|$ goes to infinity, and for all real $a < b$, we have, as N goes to infinity,

$$\frac{1}{V'(x_N)^2} \sup_{a\leqslant x\leqslant b} |V''(x_N + x)| \underset{N\to\infty}{\longrightarrow} 0\,.$$

iv) The function $\frac{1}{V'^2}$ *is integrable at infinity.* $\frac{V''(x)}{V'(x)} = \underset{|x|\to\infty}{O}(1)$ *and* $\frac{V^{(3)}(x)}{V'(x)} = \underset{|x|\to\infty}{O}(1)$.

Taking $V = V_{\text{conv}} + \phi$ with V_{conv}, $\phi \in \mathcal{C}^3(\mathbb{R})$ such that $\phi^{(k)}$ is bounded for $k = 0, \dots, 3$, V_{conv} is convex with $|V'_{\text{conv}}| \to +\infty$ at infinity, satisfying hypotheses *ii)*, *iii)* and *iv)* such that there exists $\varepsilon > 0$ such that $V_{\text{conv}} - 2Pf_\varepsilon$ is convex (see Lemma 2.4), then V satisfies Assumptions 1.1.

Because *i)* implies that V goes to infinity faster than linearly, we will see that it ensures exponential decay at infinity of ρ_P^V. Recalling the sufficient condition for $\mathbb{P}_N^{V,P}$ of equation (2) to be defined, this first assumption implies that there exists $\alpha > 0$ such that $\liminf_{|x|\to\infty} \frac{V(x)}{|x|} > \alpha$. This guarantees in particular that the β-ensemble (3) is well-defined for all $N \geqslant 1$ and $P \geqslant 0$. We will use the fact that μ_P^V satisfies the Poincaré inequality to ensure that H endowed with $\langle \cdot, \cdot \rangle_{\mathsf{H}}$ is a Hilbert space.

The second assumption ensures that any power of V' and V'' is in $L^2(\mu_P^V)$ and ρ_P^V, which behaves like e^{-V} up to a sub-exponential factor, belongs to the Sobolev space $H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$. Indeed, for $k \leqslant 2$, using *iv)*, $\rho_P^{V\,(k)}$ behaves at infinity like $(V')^k \rho_P^V$ as shown in Lemma 2.2 which is in $L^2(\mathbb{R})$ by assumption *ii)*.

Assumption *iii)* will be used to localize the minimum/maximum point of a typical configuration $(x_1, \dots, x_N)$ following the law $\mathbb{P}_N^{V,P}$: this will be done in Corollary 4.2, which comes as a consequence of [Lam21][Theorem 3.4]. More precisely, Corollary 4.2 establishes that for sequences $(\alpha_N^+)_N, (\alpha_N^-)_N$ going to infinity, the random variables

$$\alpha_N^+ \left(\max_{1 \leqslant j \leqslant N} x_j - E_N^+ \right) \qquad \text{and} \qquad \alpha_N^- \left(\max_{1 \leqslant j \leqslant N} x_j - E_N^- \right)$$

converge in distribution. For large N, the scalars E_N^+ and E_N^- can thus be seen as the edges of a typical configuration. Furthermore,

$$V(E_N^\pm) \sim \ln N \,. \tag{10}$$

We refer to Section 4 for detailed statements. The final step in the proof of Theorem 1.3 consists in lifting the result of Proposition 5.1 from compactly supported functions to more general functions.

We use Assumption *iv)* to control integral remainders in the proof of Theorem 7.1, ensuring that $\mathcal{L}^{-1}$ is regular enough *i.e.* that for sufficiently smooth functions f, $\left(\mathcal{L}^{-1} f\right)' \in H^2(\mathbb{R})$.

We will need another technical assumption to ensure that Taylor remainders arising in the proof of Theorem 5.2 are negligible.

Assumption 1.2. *With the notations of Theorem 4.1, we have*

$$\sup_{d(x, I_N) \leqslant 1} \left| V^{(3)}(x) \right| = o(N^{1/2}) \,,$$

where $I_N = \left[E_N^- - 2; E_N^+ + 2 \right]$.

Again taking $V = V_{\text{conv}} + \phi$ with V_{conv}, $\phi \in \mathcal{C}^3(\mathbb{R})$ such that $\phi^{(k)}$ is bounded for $k = 0, \dots, 3$, V_{conv} is convex with $|V'_{\text{conv}}| \to +\infty$ at infinity, satisfying hypotheses *ii)*, *iii)*, *iv)* and Assumption 1.2, such that there exists $\varepsilon > 0$ such that $V_{\text{conv}} - 2Pf_\varepsilon$ is convex (see Lemma 2.4), then V satisfies Assumptions 1.1.

This last hypothesis is satisfied whenever V_{conv} is sufficiently convex in a compact centered at 0 and is made so that V_{conv} compensate the *small* lack of concavity in the bulk of a function behaving like $-\ln|x|$ at infinity (note that assuming that $V'' \geqslant \alpha$ for some $\alpha > 0$ is sufficient). This is the reason why we introduce the function f_ε, which are functions with required growth at infinity and with second derivative as small as desired. The main point that needs to be checked is that the measure μ_P^V satisfies the Poincaré inequality, this will be done in Proposition 2.6.

The type of potential V_{conv} that one can consider is typically the convex polynomials or $\cosh(\alpha x)$. On the other hand a *scaled* potentials like e^{x^2} which have a faster growing derivative at infinity and so doesn't satisfy assumptions *iii)* and *iv)*.

We are now able to state the main result, *ie* the central limit theorem for functions belonging to the image of the operator $\mathcal{L}$ introduced in (8).

Theorem 1.3. *Assume that V satisfies Assumptions 1.1 and Assumption 1.2. Then for ϕ verifying the following conditions:*

- $\phi \in \mathcal{C}^1(\mathbb{R})$
- *there exists $\varepsilon > 0$, $\phi(x) = \underset{|x|\to\infty}{O}(x^{-\frac{1}{2}-\varepsilon})$ and $\phi'(x) = \underset{|x|\to\infty}{O}(x^{\frac{1}{2}-\varepsilon})$ at infinity*
- $\displaystyle\int_{\mathbb{R}} \phi(x) d\mu_P^V(x) = 0$

we have the convergence in law

$$\sqrt{N}\nu_N(\phi) \to \mathcal{N}\left(0, (\sigma_P^V)^2(\phi)\right) \tag{11}$$

where the limiting variance $(\sigma_P^V)^2(\phi)$ is given by

$$(\sigma_P^V)^2(\phi) = \langle \phi, \mathcal{L}^{-1}\phi\rangle_{\mathsf{H}} = \int_{\mathbb{R}} \left(\left(\mathcal{L}^{-1}\phi\right)''(x)^2 + V''(x)\left(\mathcal{L}^{-1}\phi\right)'(x)^2 \right) d\mu_P^V(x) + P \iint_{\mathbb{R}^2} \left(\frac{\left(\mathcal{L}^{-1}\phi\right)'(x) - \left(\mathcal{L}^{-1}\phi\right)'(y)}{x-y} \right)^2 d\mu_P^V(x) d\mu_P^V(y)\,. \tag{12}$$

Remark 1.4. *Since $\nu_N(\phi + c) = \nu_N(\phi)$ for all constant $c \in \mathbb{R}$, the assumption $\displaystyle\int_{\mathbb{R}} \phi(x) d\mu_P^V = 0$ can be dropped by replacing ϕ by $\phi - \displaystyle\int_{\mathbb{R}} \phi(x) d\mu_P^V$ in the expression of the limiting variance.*

As a tool to deal with the error term of equation (6), we establish a concentration inequality for the empirical measure. This inequality is stated in terms of the following distance over the set of probability distributions $\mathcal{P}(\mathbb{R})$.

For $\mu, \mu' \in \mathcal{P}(\mathbb{R})$ we define the distance

$$d(\mu, \mu') = \sup_{\substack{\|f\|_{\mathrm{Lip}} \leqslant 1 \\ \|f\|_{1/2} \leqslant 1}} \left\{ \left| \int f d\mu - \int f d\mu' \right| \right\}, \tag{13}$$

where $\|f\|_{\mathrm{Lip}}$ denotes the Lipschitz constant of f, and $\|f\|_{1/2}^2 = \displaystyle\int_{\mathbb{R}} |t|\, |\mathcal{F}[f](t)|^2\, dt$, where $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R})$ which takes the following expression $\mathcal{F}[f](t) = \displaystyle\int_{\mathbb{R}} f(x) e^{-\mathrm{i}tx} dx$ for f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

We then have

Theorem 1.5. *There exists $K \in \mathbb{R}$ (depending on P and on V), such that for any $N \geqslant 1$ and $r > 0$,*

$$\mathbb{P}_N^{V,P}\left(d(\hat{\mu}_N, \mu_P^V) > r\right) \leqslant e^{-Nr^2\frac{P\pi^2}{2} + 5P\ln N + K}\,. \tag{14}$$

This result is the analog of [HL21, Theorem 1.4].

The paper is organized as follows. In Section 2 we discuss the regularity of the equilibrium density ρ_P^V under Assumption 1.1. In Section 3 we prove Theorem 1.5. Section 4 is dedicated to the localization of the edge of a typical configuration, mentioned in the discussion preceding the statement of Assumption 1.2. We next prove in Section 5 the convergence of the Laplace transform of $\sqrt{N}\nu_N(\mathcal{L}\phi)$ for general functions ϕ which establishes Theorem 1.3 for functions of the form $\mathcal{L}\phi$. Section 6 is dedicated to the diagonalization and inversion of $\mathcal{L}$ given by (8). In Section 7, we show regularity properties of $\mathcal{L}^{-1}$ to establish Theorem 1.3. We detail in Appendix A elements of proof for the spectral theory of Schrödinger operators, used in Section 6.

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2 Regularity of the equilibrium measure and Hilbert transform

In this section, we discuss the regularity properties of the equilibrium density ρ_P^V, namely its decay at infinity and its smoothness, and give formulas for its two first derivatives.

The Hilbert transform, whose definition we recall, plays a central role in the analysis of the equilibrium measure. It is first defined on the Schwartz class through $\forall \phi \in \mathcal{S}(\mathbb{R})$, $\forall x \in \mathbb{R}$,

$$\mathcal{H}[\phi](x) := \fint_{\mathbb{R}} \frac{\phi(t)}{t-x} dt = \lim_{\varepsilon \downarrow 0} \int_{|t-x|>\varepsilon} \frac{\phi(t)}{t-x} dt = \int_0^{+\infty} \frac{\phi(x+t)-\phi(x-t)}{t} dt, \tag{15}$$

where $\fint$ denotes the Cauchy principal value integral, and then extended to $L^2(\mathbb{R})$ thanks to property *ii)* of Lemma 2.1: $\|f\|_{L^2(dx)} = \frac{1}{\pi}\|\mathcal{H}[f]\|_{L^2(dx)}$. The last expression in (15) is a definition where the integral converges in the classical sense. We also recall the definition of the logarithmic potential U^f of a density of probability $f : \mathbb{R} \to \mathbb{R}$, given for $x \in \mathbb{R}$ by

$$U^f(x) = -\int_{\mathbb{R}} \ln|x-y| f(y) dy\,. \tag{16}$$

Because we assume $f \in L^1(\mathbb{R})$ to be nonnegative, U^f takes values in $[-\infty, +\infty)$. If f integrates the function ln, *i.e* $\int_{\mathbb{R}} \ln|x| f(x) dx < +\infty$, then U^f takes real values. Additionally, one can check that the logarithmic potential and the Hilbert transform of f are linked through the distributional identity $\left(U^f\right)' = \mathcal{H}[f]$.

We recall in the next lemma some properties of the Hilbert transform that we will use in the rest of the paper.

Lemma 2.1 (Properties of the Hilbert transform)**.**

i) Fourier transform: For all $\phi \in L^2(\mathbb{R})$, $\mathcal{F}\Big[\mathcal{H}[\phi]\Big](\omega) = \mathrm{i}\pi sgn(\omega)\mathcal{F}[\phi](\omega)$ for all $\omega \in \mathbb{R}$.

ii) As a consequence, $\frac{1}{\pi}\mathcal{H}$ is an isometry of $L^2(\mathbb{R})$, and $\mathcal{H}$ satisfies on $L^2(\mathbb{R})$ the identity $\mathcal{H}^2 = -\pi^2 I$.

iii) Derivative: For any $f \in H^1(\mathbb{R})$, $\mathcal{H}[f]$ is also $H^1(\mathbb{R})$ and $\mathcal{H}[f]' = \mathcal{H}[f']$.

iv) For all $p > 1$, the Hilbert transform can be extended as a bounded operator $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$.

v) Skew-self adjointness: For any $f, g \in L^2(\mathbb{R})$, $\langle \mathcal{H}[f], g\rangle_{L^2(\mathbb{R})} = -\langle f, \mathcal{H}[g]\rangle_{L^2(\mathbb{R})}$.

Proof. We refer to [Kin09] for the proofs of these properties. □

As a consequence of [GZ19], $\hat{\mu}_N$ converges almost surely under $\mathbb{P}_N^{V,P}$ towards the unique minimizer of the energy-functional $\mathcal{E}_P^V$, defined for $\mu \in \mathcal{P}(\mathbb{R})$ by

$$\mathcal{E}_P^V(\mu) = \begin{cases} \int_{\mathbb{R}} \Big[V + \ln\Big(\frac{d\mu}{dx}\Big)\Big] d\mu - P \iint_{\mathbb{R}^2} \ln\big|x-y\big| d\mu(x) d\mu(y) \text{ if } \mu \ll dx \\ +\infty \text{ otherwise} \end{cases}. \tag{17}$$

(Here we wrote $\mu \ll dx$ for "μ is absolutely continuous with respect to Lebesgue measure")

Consequently, following [GM22, Lemma 3.2], the density ρ_P^V of μ_P^V satisfies equation (5), which we rewrite here for convenience.

$$V(x) - 2P \int_{\mathbb{R}} \ln|x-y| \rho_P^V(y) dy + \ln \rho_P^V(x) = \lambda_P^V \,, \tag{18}$$

where λ_P^V is a constant (depending on V and P). Using this equation, we will show in the next lemma that ρ_P^V decays exponentially and is twice continuously differentiable via the representation:

$$\forall x \in \mathbb{R},\ \rho_P^V(x) = \exp\Big(- V(x) - 2PU^{\rho_P^V}(x) - \lambda_P^V \Big)$$

In the Gaussian potential case *ie* $V_G(x) = \dfrac{x^2}{2}$, an explicit formula has been found [ABG12]:

$$\rho_P^{V_G}(x) = \frac{\Gamma(P)}{P\sqrt{2\pi}} \frac{\exp\Big(-\dfrac{x^2}{2}\Big)}{\displaystyle\int_0^{+\infty} t^{P-1} e^{-\frac{t^2}{2}+ixt} dt}$$

It has been established in [BGP15] that $\sqrt{P+1}\rho_P^{V_G}(\sqrt{P+1}x)$ converges to the Gaussian distribution when P goes to zero and the semi-circle law when P goes to infinity. So in the Gaussian case, μ_P can be seen as an interpolation between the Gaussian distribution and the semi-circular one. **We now drop the superscript of ρ_P^V and μ_P^V and denote it ρ_P and μ_P for convenience.** In the next lemma, we prove that ρ_P has the same regularity as V.

Lemma 2.2. *Under Assumption 1.1,*

- *The support of μ_P is $\mathbb{R}$ and there exists a constant C_P^V such that for all $x \in \mathbb{R}$,*

$$\rho_P(x) \leqslant C_P^V (1+|x|)^{2P} e^{-V(x)} \,.$$

- *The density ρ_P is in $\mathcal{C}^3(\mathbb{R})$ and we have*

$$\rho_P' = -\Big(V' + 2P\mathcal{H}[\rho_P]\Big)\rho_P \tag{19}$$

and

$$\rho_P'' = \Big(- 2P\mathcal{H}[\rho_P]' - V'' + V'^2 + 4P^2\mathcal{H}[\rho_P]^2 + 4PV'\mathcal{H}[\rho_P]\Big)\rho_P \,. \tag{20}$$

Proof. For the first point, [GM22, Lemma 3.2] establishes that the support of μ_P is the whole real axis, and that under the first condition of Assumptions 1.1, we have the bound, valid for all $x \in \mathbb{R}$

$$\rho_P(x) \leqslant \frac{K_P^V}{(1+|x|)^2} \,, \tag{21}$$

with K_P^V a positive constant. Using (18) and the fact that

$$\ln|x-y| \leqslant \ln\big(1+|x|\big) + \ln\big(1+|y|\big) \,,$$

we see that for all $x \in \mathbb{R}$,

$$\rho_P(x) \leqslant C_P^V \exp\Big(- V(x) + 2P\ln(1+|x|)\Big) \,, \tag{22}$$

with

$$C_P^V = \exp\Big(2P\int_{\mathbb{R}} \ln(1+|y|)\rho_P(y)dy + \lambda_P^V\Big)$$

which is indeed finite by (21).

For the second point, we use that $\big(U^{\rho_P}\big)' = \mathcal{H}[\rho_P]$ weakly and equation (18) to conclude on the distributional identity

$$\rho_P' = \Big(- V' - 2P\mathcal{H}[\rho_P]\Big)\rho_P\,.$$

By the second point of Assumption 1.1, $V'(x)e^{-V(x)+2P\ln(1+|x|)} = o(x^{-1})$ as $|x| \to \infty$, thus by (22), $V'\rho_P \in L^2(\mathbb{R})$. Also since ρ_P is $L^2(\mathbb{R})$ and bounded, we deduce, by using that $\mathcal{H}\big[L^2(\mathbb{R})\big] = L^2(\mathbb{R})$, that $\mathcal{H}[\rho_P]\rho_P \in L^2(\mathbb{R})$. Adding up these terms we get $\rho_P \in H^1(\mathbb{R})$. Because $\mathcal{H}[\rho_P]' = \mathcal{H}[\rho_P']$ in a weak sense by Lemma 2.1, $\mathcal{H}[\rho_P] \in H^1(\mathbb{R})$. By the classical fact that $H^1(\mathbb{R})$ is contained in the set of 1/2-Hölder functions $\mathcal{C}^{1/2}(\mathbb{R})$, we have $\mathcal{H}[\rho_P] \in \mathcal{C}^{1/2}(\mathbb{R})$ and so $U^{\rho_P} \in \mathcal{C}^{1,1/2}(\mathbb{R})$, the set of functions in $\mathcal{C}^1(\mathbb{R})$ with derivative of class 1/2-Hölder.
Using the fact that V is continuously differentiable, the previous equation for the weak derivative of ρ_P then ensures that $\rho_P \in \mathcal{C}^1(\mathbb{R})$ and equation (19) holds in the strong sense.

Differentiating (in a weak sense) equation (19) we obtain

$$\rho_P'' = \Big(- 2P\mathcal{H}[\rho_P]' - V'' + V'^2 + 4P^2\mathcal{H}[\rho_P]^2 + 4PV'\mathcal{H}[\rho_P]\Big)\rho_P\,.$$

The three first terms belong to $L^2(\mathbb{R})$ for the same reasons as before. Since $\rho_P \in H^1(\mathbb{R})$n by Lemma 2.1*iii)* so is $\mathcal{H}[\rho_P] \in H^1(\mathbb{R})$, it is then bounded over $\mathbb{R}$ hence the two last term are in $L^2(\mathbb{R})$ when multiplied by ρ_P. Finally, we can conclude that $\rho_P \in H^2(\mathbb{R})$ and so that $\mathcal{H}[\rho_P] \in H^2(\mathbb{R})$ with $\mathcal{H}[\rho_P]'' = \mathcal{H}[\rho_P'']$ (in a weak sense). As before, we conclude that $\rho_P \in \mathcal{C}^2(\mathbb{R})$ and that equation (20) holds in a strong sense. By the exact same method, we can show that $\rho_P \in \mathcal{C}^3(\mathbb{R})$. □

We next show that the Hilbert transform of ρ_P is continuous and decays at infinity.

Lemma 2.3. *Let $u \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} u(t)dt$ exists and $f : t \mapsto tu(t) \in H^1(\mathbb{R})$ then*

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} u(t)dt}{x}.$$

Moreover if $\displaystyle\int_{\mathbb{R}} u(t)dt = 0$, $\int_{\mathbb{R}} f(t)dt$ exists and $g : t \mapsto t^2u(t) \in H^1(\mathbb{R})$, then

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} tu(t)dt}{x^2}.$$

As a consequence, we obtain that $\mathcal{H}[\rho_P](x) \underset{|x|\to\infty}{\sim} x^{-1}$ and the logarithmic potential U^{ρ_P} is Lipschitz bounded, with bounded derivative $\mathcal{H}[\rho_P]$.

Proof. Let $u \in L^2(\mathbb{R})$, such that $\int_{\mathbb{R}} u(t)dt$ exists and $f : t \mapsto tu(t) \in H^1(\mathbb{R})$. Then

$$x\mathcal{H}[u](x) + \int_{\mathbb{R}} u(t)dt = \int_{\mathbb{R}} \Big[\frac{xu(x+t) - xu(x-t)}{2t} + \frac{u(x+t)}{2} + \frac{u(x-t)}{2}\Big]dt = \mathcal{H}[f](x).$$

Since $f \in H^1(\mathbb{R})$, so is $\mathcal{H}[f]$, proving that it goes to zero at infinity. Hence

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} u(t)dt}{x}$$

Moreover if $\displaystyle\int_{\mathbb{R}} u(t)dt = 0$, $\int_{\mathbb{R}} f(t)dt$ exists and $g : t \mapsto t^2u(t) \in H^1(\mathbb{R})$, then by the same argument:

$$x^2\mathcal{H}[u](x) = x\mathcal{H}[f](x) = \mathcal{H}[g](x) - \int_{\mathbb{R}} f(t)dt$$

where $g(t) = t^2u(t)$. We deduce that $\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \dfrac{-\int_{\mathbb{R}} tu(t)dt}{x^2}$ since $\mathcal{H}[g]$ goes to zero at infinity. □

Lemma 2.4 (Asymptotic of the logarithmic potential)**.** *We have the following asymptotic expansion at infinity* $U^{\rho_P} = \ln|x| + \underset{|x|\to\infty}{O}(1)$.

Proof. Since $\mathcal{H}[\rho_P](x) = x^{-1} + \underset{|x|\to\infty}{O}(x^{-2})$, and recalling that $U^{\rho_P^V}$ (defined by (16)) satisfies $(U^{\rho_P^V})' = \mathcal{H}[\rho_P^V]$, we deduce the result by integrating $t \mapsto \mathcal{H}[\rho_P^V](t) - 1/t$ in a neighborhood of infinity. □

We conclude this section by stating the Poincaré inequality for the measure μ_P under the assumption that V is a bounded perturbation of a strictly convex potential V_{conv}.

Lemma 2.5. *Let* $\varepsilon > 0$*, there exists a function* $f \in \mathcal{C}^2(\mathbb{R})$ *such that* $f_\varepsilon(x) - \ln|x| = \underset{|x|\to\infty}{O}(1)$*, and* $\|f_\varepsilon''\|_\infty \leqslant \varepsilon$.

Proposition 2.6. *Assume that* $V = V_{\text{conv}} + \phi$*, where* $V_{\text{conv}} \in \mathcal{C}^3(\mathbb{R})$ *with* V_{conv} *convex, such that there exists* $\varepsilon > 0$ *such that* ϕ *is bounded and* $V_{\text{conv}} - 2Pf_\varepsilon$ *is convex (*f_ε *being given by Lemma 2.5). Then, the measure* μ_P *satisfies the Poincaré inequality: there exists a constant* $C > 0$ *such that for all* $f \in \mathcal{C}_c^1(\mathbb{R})$*,*

$$\mathrm{Var}_{\mu_P}(f) \leqslant C \int_{\mathbb{R}} |f'|^2 d\mu_P \,. \tag{23}$$

Proof. We use the fact that if μ_1, μ_2 are two absolutely continuous probability measures supported on $\mathbb{R}$ such that $\dfrac{1}{C} \leqslant \dfrac{d\mu_1}{d\mu_2} \leqslant C$ for some $C > 0$ and μ_1 satisfies Poincaré inequality with constant C_1 then so does μ_2 for some other constant. Indeed, in that case let $f \in \mathcal{C}_c^1(\mathbb{R})$, we have

$$\mathrm{Var}_{\mu_2}(f) = \inf_a \int_{\mathbb{R}} (f-a)^2 \, d\mu_2 \leqslant C \,\mathrm{Var}_{\mu_1}(f) \leqslant C^2 C_1 \int_{\mathbb{R}} f' d\mu_2.$$

Here we take $d\mu_2(x) := \rho_P^V(x)dx$ and we want to compare it to a measure μ_1 supported on $\mathbb{R}$ defined by $d\mu_1(x) = \dfrac{1}{Z}\exp\big(-W(x)\big)dx$ for some convex function W. The measure μ_1 then clearly verifies the Poincaré inequality. This fact comes as a direct consequence of [BBCG08][Corollary 1.9], which states that if a probability measure μ has a log-concave density on $\mathbb{R}$, then it satisfies (23). With the definition $W := V_{\text{conv}} - 2Pf_\varepsilon$ with $\varepsilon > 0$ such that $V_{\text{conv}} - 2Pf_\varepsilon$ is convex, $W - V - 2PU^{\rho_P}$ is bounded on $\mathbb{R}$. It is then not hard to see that $\dfrac{1}{C} \leqslant \dfrac{d\mu_1}{d\mu_P} \leqslant C$ for some $C > 0$ which allows to conclude that μ_P satisfies the Poincaré inequality.

□

Remark 2.7. *We will apply later inequality* (23) *to more general functions than* $\mathcal{C}_c^1(\mathbb{R})$*, namely functions of the weighted Sobolev space* $H^1(\rho_P)$*, defined in Section 6; which can be seen as the completion of* $\mathcal{C}_c^\infty(\mathbb{R})$ *with respect to the norm* $\|u\|_{L^2(\rho_P)} + \|u'\|_{L^2(\rho_P)}$.

3 Concentration inequality, proof of Theorem 1.5

We prove in this section the concentration Theorem 1.5. Its proof is a direct adaptation of Theorem 1.4 of [HL21], which shows the analogous estimate in the circular setup. It is inspired by [MMS14] and based on a comparison between a configuration $\mathbf{x}_N = (x_1, \dots, x_N)$ sampled with $\mathbb{P}_N^{V,P}$ and a regularized version $\mathbf{y}_N = (y_1, \dots, y_N)$, which we describe here.

Definition 3.1. $y_1 := x_1$*, and for* $0 \leqslant k \leqslant N-1$*,* $y_{k+1} := y_k + \max\{x_{k+1} - x_k, N^{-3}\}$.

Note that the configuration $\mathbf{y}_N$ given by the previous definition satisfies $y_{k+1} - y_k \geqslant N^{-3}$, and $\mathbf{y}_N$ is close to $\mathbf{x}_N$ in the sense that

$$\sum_{k=1}^N |x_k - y_k| \leqslant \frac{1}{2N} \,. \tag{24}$$

Indeed, by construction we have $|x_k - y_k| = y_k - x_k \leqslant (k-1)N^{-3}$, and we get the bound by summing these inequalities.
The key point of the proof of Theorem 1.5 is comparing the empirical measure $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, where $\mathbf{x}_N$ follows $\mathbb{P}_N^{V,P}$, to the regularized measure

$$\widetilde{\mu}_N := \lambda_{N^{-5}} * \frac{1}{N}\sum_{i=1}^N \delta_{y_i}, \tag{25}$$

ie the convolution of $\lambda_{N^{-5}}$ and the empirical measure, where $\lambda_{N^{-5}}$ is the uniform measure on $[0, N^{-5}]$. The interest of introducing the measure $\widetilde{\mu}_N$ is that it is close to $\hat{\mu}_N$, while having a finite energy $\mathcal{E}_P^V(\widetilde{\mu}_N)$, given by (17). Finally, notice that the empirical measure doesn't change when reordering $x_1, \ldots, x_N$, and thus we do not lose in generality for our purposes in assuming that $x_1 \leqslant \ldots \leqslant x_N$ in definition 3.1.

We now introduce a distance on $\mathcal{P}(\mathbb{R})$ which is well-suited to our context.

Definition 3.2. *For $\mu, \mu' \in \mathcal{P}(\mathbb{R})$ we define the distance (possibly infinite) $D(\mu, \mu')$ by*

$$D(\mu, \mu') := \left(-\int \ln|x-y| d(\mu-\mu')(x) d(\mu-\mu')(y)\right)^{1/2} \tag{26}$$

$$= \left(\int_0^{+\infty} \frac{1}{t}\left|\mathcal{F}[\mu-\mu'](t)\right|^2 dt\right)^{1/2}. \tag{27}$$

where the Fourier transform of a signed measure ν is defined by $\mathcal{F}[\nu](x) := \int_{\mathbb{R}} e^{-\mathrm{i}tx} d(\mu-\mu')(x)$

Let $f : \mathbb{R} \to \mathbb{R}$ with finite 1/2 norm $\|f\|_{1/2} := \left(\int_{\mathbb{R}} |t| \left|\mathcal{F}[f](t)\right|^2 dt\right)^{1/2}$. By Plancherel theorem and Hölder inequality, for any $\mu, \mu' \in \mathcal{P}(\mathbb{R})$, setting $\nu = \mu - \mu'$,

$$\left|\int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f d\mu'\right|^2 = \left|\frac{1}{2\pi}\int_{\mathbb{R}} |t|^{1/2}\mathcal{F}[f](t)\frac{\overline{\mathcal{F}[\nu](t)}}{|t|^{1/2}} dt\right|^2 \leqslant \frac{1}{2\pi^2}\|f\|_{1/2}^2 D^2(\mu, \mu').$$

Therefore the metric d defined in (13) is dominated by D:

$$d(\mu, \mu') \leqslant \frac{1}{\sqrt{2}\pi} D(\mu, \mu'). \tag{28}$$

The following lemma shows how the distance D is related to the energy-functional $\mathcal{E}_P^V$ defined in (17), we will write $\mathcal{E}_P$ for simplicity.

Lemma 3.3. *We have for any absolutely continuous $\mu \in \mathcal{P}(\mathbb{R})$ with finite energy $\mathcal{E}_P^V(\mu)$,*

$$\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P) = PD^2(\mu, \mu_P) + \int \ln\left(\frac{d\mu}{d\mu_P}\right) d\mu\,. \tag{29}$$

Proof of Lemma 3.3. Subtracting $\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P)$ we find

$$\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P) = \int V d(\mu-\mu_P) + \int \ln\frac{d\mu}{dx} d\mu - \int \ln\rho_P d\mu_P - P\iint \ln|x-y| d\mu(x) d\mu(y) + P\iint \ln|x-y| d\mu_P(x) d\mu_P(y)\,. \tag{30}$$

Now, if ν is a signed measure of mass zero, integrating (18) we get

$$\int V(x) d\nu(x) - 2P\iint \ln|x-y| d\nu(x) d\mu_P(y) + \int \ln(\rho_P)(x) d\nu(x) = 0\,.$$

We take $\nu = \mu - \mu_P$, and get

$$\int V(x)d(\mu-\mu_P)(x) = 2P \iint \ln|x-y|d\mu(x)d\mu_P(y) - 2P \iint \ln|x-y|d\mu_P(x)d\mu_P(y) \\ - \int \ln(\rho_P)(x)d\mu(x) + \int \ln(\rho_P)(x)d\mu_P(x)\,.$$

Plugging this last identity in (30), we find

$$\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P) = -P \iint \ln|x-y|d\nu(x)d\nu(y) + \int \ln\left(\frac{d\mu}{d\mu_P}\right)(x)d\mu(x)$$

which establishes the result. □

Proof of Theorem 1.5. We first give a lower bound for the partition function $Z_N^{V,P}$ (4) of $\mathbb{P}_N^{V,P}$. We rewrite it as

$$Z_N^{V,P} = \int_{\mathbb{R}^N} \exp\left(\frac{2P}{N}\sum_{i<j}\ln|x_i-x_j| - \sum_{i=1}^N \Big[V(x_i)+\ln\rho_P(x_i)\Big]\right) d\rho_P(x_1)\dots d\rho_P(x_N)\,,$$

and apply Jensen inequality to obtain:

$$\begin{aligned}\ln Z_N^{V,P} &\geqslant \int_{\mathbb{R}^N}\left(\frac{2P}{N}\sum_{i<j}\ln|x_i-x_j| - \sum_{i=1}^N \Big[V(x_i)+\ln\rho_P(x_i)\Big]\right) d\rho_P(x_1)\dots d\rho_P(x_N)\\ &\geqslant P(N-1)\iint \ln|x-y|d\rho_P(x)d\rho_P(y) - N\int_{\mathbb{R}}\Big[V+\ln\rho_P\Big]d\rho_P\\ &\geqslant -N\mathcal{E}_P^V\big[\mu_P\big] - P\iint \ln|x-y|d\rho_P(x)d\rho_P(y).\end{aligned}$$

Using this estimate and the fact that for $1 \leqslant i,j \leqslant N$ we have $|x_i - x_j| \leqslant |y_i - y_j|$, with $\mathbf{y}_N = (y_1,\dots,y_N)$ of definition 3.1, we deduce the bound on the density of probability

$$\frac{d\mathbb{P}_N^{V,P}}{d\mathbf{x}}(x_1,\dots,x_N) \leqslant e^{N\mathcal{E}_P(\mu_P)+P\iint \ln|x-y|d\mu_P(x)d\mu_P(y)+\frac{P}{N}\sum_{i\neq j}\ln|y_i-y_j|-\sum_{i=1}^N V(x_i)}\,. \tag{31}$$

Recalling (25), we now show the following estimate

$$\sum_{i\neq j}\ln|y_i-y_j| \leqslant 2 + N^2\iint \ln|x-y|d\widetilde{\mu}_N(x)d\widetilde{\mu}_N(y) + 5N\ln N + \frac{3}{2}N\,. \tag{32}$$

Let $i \neq j$ and $u,v \in [0,N^{-5}]$. Since for $x \neq 0$ and $|h| \leqslant \frac{|x|}{2}$, we have $\big|\ln|x+h| - \ln|x|\big| \leqslant \frac{2|h|}{|x|}$, we deduce

$$\big|\ln|y_i-y_j+u-v| - \ln|y_i-y_j|\big| \leqslant \frac{2|u-v|}{|y_i-y_j|} \leqslant \frac{2N^{-5}}{N^{-3}} = \frac{2}{N^2}.$$

Thus, summing over $i \neq j$ and integrating with respect to u and v, we get

$$\begin{aligned}\sum_{i\neq j}\ln|y_i-y_j| &\leqslant 2 + \sum_{i\neq j}\iint \ln|y_i-y_j+u-v|d\lambda_{N^{-5}}(u)d\lambda_{N^{-5}}(v)\\ &= 2 + N^2\iint \ln|x-y|d\widetilde{\mu}_N(x)d\widetilde{\mu}_N(y) - N\iint \ln|u-v|d\lambda_{N^{-5}}(u)d\lambda_{N^{-5}}(v)\,.\end{aligned}$$

The last integral is equal to $-\frac{3}{2} - 5\ln N$, so we deduce (32). We now combine (31) and (32). Recall (17) and set

$$c_N = P\left(\iint \ln|x-y| d\mu_P(x) d\mu_P(y) + 3/2 + 2/N\right).$$

Then we get

$$\begin{aligned}\frac{d\mathbb{P}_N^{V,P}}{d\mathbf{x}}(x_1,\dots,x_N) &\leqslant e^{c_N+5P\ln N}\exp\left[N\left\{\mathcal{E}_P(\mu_P) - \mathcal{E}_P(\widetilde{\mu}_N) + \int\left(V + \ln\frac{d\widetilde{\mu}_N}{dx}\right)d\widetilde{\mu}_N\right\} - \sum_{i=1}^N V(x_i)\right]\\ &= e^{c_N+5P\ln N}\exp\left[-NPD^2(\widetilde{\mu}_N,\mu_P) + N\int(V+\ln\rho_P)\,d\widetilde{\mu}_N - \sum_{i=1}^N V(x_i)\right]\end{aligned}$$

where we used equation (29) in the last equality. Using again equation (18) we then see that the density $\frac{d\mathbb{P}_N^{V,P}}{d\mathbf{x}}(x_1,\dots,x_N)$ is bounded by

$$e^{c_N+5P\ln N}\exp\left[-NPD^2(\widetilde{\mu}_N,\mu_P) + 2PN\iint\ln|x-y|d(\widetilde{\mu}_N-\hat{\mu}_N)(x)d\mu_P(y)\right]\prod_{i=1}^N\rho_P(x_i)\,.$$

Recalling (16), we used that $\iint\ln|x-y|d(\widetilde{\mu}_N-\hat{\mu}_N)(x)d\mu_P(y) = -\int U^{\rho_P}d(\widetilde{\mu}_N-\hat{\mu}_N)$. As a consequence of the bound on the density $\frac{d\mathbb{P}_N^{V,P}}{d\mathbf{x}}(x_1,\dots,x_N)$ we established, we have for all $r>0$

$$\mathbb{P}_N^{V,P}\left(D^2(\widetilde{\mu}_N,\mu_P)>r\right) \leqslant e^{-NPr+c_N+5P\ln N}\int_{\mathbb{R}^N}\exp\left\{-2PN\int U^{\rho_P}d(\widetilde{\mu}_N-\hat{\mu}_N)\right\}\prod_{i=1}^N\rho_P(x_i)dx_i\,. \quad (33)$$

Next, we show that $-N\int U^{\rho_P}d(\widetilde{\mu}_N-\hat{\mu}_N)$ is bounded. By Lemma 2.3, U^{ρ_P} is differentiable with bounded derivative $\mathcal{H}[\rho_P]$ on $\mathbb{R}$. As a consequence,

$$\begin{aligned}\left|N\int U^{\rho_P}d(\widetilde{\mu}_N-\hat{\mu}_N)\right| &\leqslant \sum_{i=1}^N\int\left|U^{\rho_P}(y_i+u)-U^{\rho_P}(x_i)\right|d\lambda_{N^{-5}}(u)\\ &\leqslant \|\mathcal{H}[\rho_P]\|_\infty\left(\sum_{i=1}^N|y_i-x_i| + N\int u d\lambda_{N^{-5}}(u)\right)\\ &\leqslant \|\mathcal{H}[\rho_P]\|_\infty\Big(\frac{1}{2N}+N^{-4}/2\Big),\end{aligned}$$

where we used (24) in the last inequality. Therefore, we deduce from (33)

$$\mathbb{P}_N^{V,P}\left(D^2(\widetilde{\mu}_N,\mu_P)>r\right) \leqslant e^{-NPr+c_N+5P\ln N+\frac{2P}{N}\|\mathcal{H}[\rho_P]\|_\infty} = e^{-NPr+5P\ln N+K_N} \quad (34)$$

with $K_N := c_N + \dfrac{2P}{N}\|\mathcal{H}[\rho_P]\|_\infty$. Since $(c_N)_N$ is bounded, so is $(K_N)_N$.
Finally, let f be a Lipschitz bounded function with $\|f\|_{\mathrm{Lip}}\leqslant 1$, then, we have (as we did for U^{ρ_P})

$$\left|\int f d\hat{\mu}_N - \int f d\widetilde{\mu}_N\right| \leqslant N^{-2}\,.$$

Thus by (28)

$$d(\hat{\mu}_N,\mu_P) \leqslant d(\hat{\mu}_N,\widetilde{\mu}_N) + d(\widetilde{\mu}_N,\mu_P) \leqslant N^{-2} + \frac{1}{\sqrt{2}\pi}D(\widetilde{\mu}_N,\mu_P)\,,$$

and for any N such that $r - N^{-2} \geqslant r/2$ (in particular $r - N^{-2} > 0$) we get

$$\mathbb{P}_N^{V,P}\left(d(\hat{\mu}_N, \mu_P) > r\right) \leqslant \mathbb{P}_N^{V,P}\left(\frac{1}{2\pi^2} D^2(\widetilde{\mu}_N, \mu_P) > (r - N^{-2})^2\right) \leqslant \mathbb{P}_N^{V,P}\left(\frac{1}{2\pi^2} D^2(\widetilde{\mu}_N, \mu_P) > r^2/4\right),$$

and the last term is bounded by $e^{-Nr^2 \frac{P\pi^2}{2} + 5P \ln N + K}$ for some K large enough, which concludes the proof. □

As a consequence of Theorem 1.5, we are able to control the quantities

$$\zeta_N(\phi) := \iint_{\mathbb{R}^2} \frac{\phi(x) - \phi(y)}{x - y} d(\hat{\mu}_N - \mu_P)(x) d(\hat{\mu}_N - \mu_P)(y) \tag{35}$$

for a certain class of test functions ϕ.

Corollary 3.4. *There exists $C, K > 0$ such that for all $\phi \in \mathcal{C}^2(\mathbb{R}) \cap H^2(\mathbb{R})$ with bounded second derivative, we have for $\varepsilon > 0$ and N large enough,*

$$\mathbb{P}_N^{V,P}\left(\sqrt{N}|\zeta_N(\phi)| \leqslant N^{-1/2+\varepsilon}\right) \geqslant 1 - \exp\left\{-\frac{PN^{\varepsilon}}{2C\|\psi\|_{H^2(\mathbb{R})}} + 5P \ln N + K\right\}$$

with $N_2(\phi) = \|\phi'\|_{L^2(dx)} + \|\phi''\|_{L^2(dx)}$.

Proof. We follow the proof given in [Gui19][Cor. 4.16] and adapt it to our setting. Let us denote by $\widetilde{\zeta_N}(\phi)$ the quantity

$$\iint_{\mathbb{R}^2} \frac{\phi(x) - \phi(y)}{x - y} d(\widetilde{\mu}_N - \mu_P)(x) d(\widetilde{\mu}_N - \mu_P)(y).$$

We have the almost sure inequality, by a Taylor estimate

$$|\zeta_N(\phi) - \widetilde{\zeta_N}(\phi)| \leqslant 2N^{-2}\|\phi''\|_{\infty}. \tag{36}$$

Thus, for any $\delta > 0$,

$$\begin{aligned}\mathbb{P}_N^{V,P}\left(|\zeta_N(\phi)| > \delta\right) &\leqslant \mathbb{P}_N^{V,P}\left(|\zeta_N(\phi) - \widetilde{\zeta_N}(\phi)| > \delta/2\right) + \mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \delta/2\right) \\ &\leqslant \mathbb{P}_N^{V,P}\left(2N^{-2}\|\phi''\|_{\infty} > \delta/2\right) + \mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \delta/2\right),\end{aligned}$$

where the first term of the right-hand side is either 0 or 1. With $\delta = N^{-1+\varepsilon}$, $\varepsilon > 0$, it is zero for N large enough. For such a choice of δ, and for N large enough,

$$\mathbb{P}_N^{V,P}\left(|\zeta_N(\phi)| > N^{-1+\varepsilon}\right) \leqslant \mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \frac{1}{2}N^{-1+\varepsilon}\right).$$

We next show that, for some $C > 0$ independent of ϕ, we have

$$|\widetilde{\zeta_N}(\phi)| \leqslant CD^2(\widetilde{\mu}_N, \mu_P)\|\phi\|_{H^2(\mathbb{R})}. \tag{37}$$

We begin by showing this inequality for $\psi \in \mathcal{S}(\mathbb{R})$. By using the inverse Fourier transform we have

$$\begin{aligned}\widetilde{\zeta}_N(\psi) &= \frac{1}{2\pi} \iint \frac{\int dt \mathcal{F}[\psi](t) e^{\mathrm{i}tx} - \int dt \mathcal{F}[\psi](t) e^{\mathrm{i}ty}}{x - y} d\big(\widetilde{\mu}_N - \mu_P\big)(x) d\big(\widetilde{\mu}_N - \mu_P\big)(y) \\ &= \frac{1}{2\pi} \int dt \mathrm{i}t \mathcal{F}[\psi](t) \iint e^{\mathrm{i}ty} \frac{e^{\mathrm{i}t(x-y)} - 1}{\mathrm{i}t(x - y)} d\big(\widetilde{\mu}_N - \mu_P\big)(x) d\big(\widetilde{\mu}_N - \mu_P\big)(y) \\ &= \frac{1}{2\pi} \int dt \mathrm{i}t \mathcal{F}[\psi](t) \iint e^{\mathrm{i}ty} \int_0^1 d\alpha e^{\mathrm{i}\alpha t(x-y)} d\big(\widetilde{\mu}_N - \mu_P\big)(x) d\big(\widetilde{\mu}_N - \mu_P\big)(y) \\ &= \frac{1}{2\pi} \int dt \mathrm{i}t \mathcal{F}[\psi](t) \int_0^1 d\alpha \int e^{\mathrm{i}\alpha tx} d\big(\widetilde{\mu}_N - \mu_P\big)(x) \int e^{\mathrm{i}(1-\alpha)ty} d\big(\widetilde{\mu}_N - \mu_P\big)(y)\end{aligned}$$

We then apply in order the triangular inequality, Cauchy-Schwarz inequality, a change of variable and the fact that $|\mathcal{F}[\widetilde{\mu}_N - \mu_P]|^2$ is an even function.

$$
\begin{aligned}
|\widetilde{\zeta}_N(\psi)| &\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)| \int_0^1 d\alpha\, |\mathcal{F}[\widetilde{\mu}_N - \mu_P](\alpha t)| \,.\, \left|\mathcal{F}[\widetilde{\mu}_N - \mu_P]\left((1-\alpha)t\right)\right| \\
&\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)| \Big(\int_0^1 d\alpha\, |\mathcal{F}[\widetilde{\mu}_N - \mu_P](\alpha t)|^2\Big)^{\frac{1}{2}} \Big(\int_0^1 d\alpha\, \left|\mathcal{F}[\widetilde{\mu}_N - \mu_P]\left((1-\alpha)t\right)\right|^2\Big)^{\frac{1}{2}} \\
&\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)| \int_0^1 d\alpha\, |\mathcal{F}[\widetilde{\mu}_N - \mu_P](\alpha t)|^2 \\
&\leqslant \frac{1}{2\pi}\int_0^{+\infty} dt\, |t\mathcal{F}[\psi](t)| \int_0^1 \frac{t d\alpha}{t\alpha}\, |\mathcal{F}[\widetilde{\mu}_N - \mu_P](\alpha t)|^2 + \frac{1}{2\pi}\int_{-\infty}^0 dt\, |t\mathcal{F}[\phi](t)| \int_0^1 \frac{-t d\alpha}{-t\alpha}\, |\mathcal{F}[\widetilde{\mu}_N - \mu_P](\alpha t)|^2 \\
&\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)|\, D^2(\widetilde{\mu}_N, \mu_P) \\
&\leqslant \frac{1}{2\pi}\Big(\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)|^2 (1+t^2)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \frac{dt}{1+t^2}\Big)^{\frac{1}{2}} D^2(\widetilde{\mu}_N, \mu_P) \\
&\leqslant \frac{1}{2\sqrt{\pi}} D^2(\widetilde{\mu}_N, \mu_P) N_2(\psi) \\
&\leqslant \frac{1}{2\sqrt{\pi}} D^2(\widetilde{\mu}_N, \mu_P) \|\psi\|_{H^2(\mathbb{R})}
\end{aligned}
$$

By density of $\mathcal{S}(\mathbb{R})$ in $H^2(\mathbb{R})$, and since $\widetilde{\zeta}_N : \Big(H^2(\mathbb{R}), \|\cdot\|_{H^2(\mathbb{R})}\Big) \to \mathbb{R}$ is continuous, the inequality still holds for ϕ. Thus, using equation (34),

$$
\mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \frac{1}{2}N^{-1+\varepsilon}\right) \leqslant \mathbb{P}_N^{V,P}\left(D^2(\widetilde{\mu}_N, \mu_P) > \frac{N^{-1+\varepsilon}}{2C\|\phi\|_{H^2(\mathbb{R})}}\right) \leqslant \exp\left\{-P\frac{N^{\varepsilon}}{2C\|\phi\|_{H^2(\mathbb{R})}} + 5P\ln N + K\right\},
$$

which concludes the proof. □

4 Localization of the edge of a configuration

In [Lam21][Theorem 1.8, Theorem 3.4], Lambert was able to control the edge (*i.e* the minimum and the maximum) of a typical configuration $(x_1, \dots, x_N)$ distributed according to $\mathbb{P}_N^{V,P}$, by showing that the random measure

$$
\Xi_N := \sum_{j=1}^N \delta_{\varphi_N^{-1}(x_j)}
$$

converges in distribution towards a Poisson point process for a function φ_N which takes the form

$$
\varphi_N(x) := E_N + \alpha_N^{-1} x\,.
$$

Before being more precise on the construction of $(E_N)_N$ and $(\alpha_N)_N$, we explain, following [Lam21], how one can use this convergence to localize the edge of a typical configuration $(x_1, \dots, x_N)$. Let us assume for a moment that Ξ_N converges towards a Poisson point process with intensity $\theta(x) = e^{-x}$, with $E_N \to +\infty$. In particular, the random variable

$$
\Xi_N(t, +\infty)
$$

converges in distribution towards a Poisson random variable with mean $\int_t^{+\infty} e^{-x} dx$. Combined with the equalities

$$
\begin{aligned}
\mathbb{P}_N^{V,P}\Big(\Xi_N(t,+\infty) = 0\Big) &= \mathbb{P}_N^{V,P}\Big(\forall\, 1 \leqslant j \leqslant N,\ \varphi_N^{-1}(x_j) = \alpha_N(x_j - E_N) \leqslant t\Big) \\
&= \mathbb{P}_N^{V,P}\Big(\alpha_N\Big(\max_{1\leqslant j\leqslant N} x_j - E_N\Big) \leqslant t\Big),
\end{aligned}
$$

we deduce that for all $t \in \mathbb{R}$

$$\mathbb{P}_N^{V,P}\left(\alpha_N\left(\max_{1\leqslant j\leqslant N} x_j - E_N\right) \leqslant t\right) \underset{N\to\infty}{\longrightarrow} \exp(-e^{-t})\,.$$

Therefore, the random variable

$$\alpha_N\left(\max_{1\leqslant j\leqslant N} x_j - E_N\right)$$

converges in distribution to the Gumbel law, showing that the maximum of a configuration is of order E_N. Furthermore, as will be clear from the construction of α_N and E_N, α_N is positive, and goes to infinity as N goes to infinity.

Replacing in the previous analysis $\theta(x) = e^x$ and $E_N \to -\infty$, we would have deduced in the same fashion that

$$\alpha_N\left(\min_{1\leqslant j\leqslant N} x_j - E_N\right)$$

converges in law.

With the above notations, we can apply [Lam21][Theorem 3.4] to our context.

Theorem 4.1. *Let $v = \pm$. There exists $(E_N^v)_N$, $(\alpha_N^v)_N$ sequences of real numbers with $|E_N^v| \to +\infty$, $\alpha_N^v > 0$ for large enough N, satisfying $V'(E_N^v) = \alpha_N^v v$, such that:*

a) $\dfrac{Ne^{-V(E_N^v)+2P\ln|E_N^v|+\lambda_V^P}}{\alpha_N^v} \underset{N\to\infty}{\longrightarrow} 1$ *(recall λ_P^V is defined through equation (5)),*

b) $\frac{\ln(\alpha_N^v)}{N} \underset{N\to\infty}{\longrightarrow} 0$ *and* $\alpha_N^v|E_N^v| \underset{N\to\infty}{\longrightarrow} +\infty$ *,*

c) For all compact $K \subset \mathbb{R}$,

$$(\alpha_N^v)^{-2}\sup_{x\in K}|V''(\varphi_N(x))| \underset{N\to\infty}{\longrightarrow} 0\,.$$

As a consequence, the random measure Ξ_N converges in distribution as $N \to \infty$ to a Poisson point process with intensity $\theta(x) = e^{-vx}$.

Proof. We prove it in the case $v = +$, the case where $v = -$ being similar. We show that there exists a sequence $(E_N^+)_N$ going to $+\infty$ satisfying $f(E_N^+) = -\ln N$, where we defined the function f by

$$f(x) = -V(x) + 2P\ln|x| + \lambda_P^V - \ln|V'(x)|\,.$$

Recalling Assumption 1.1 *i)*, $|V'|$ goes to infinity at infinity, thus $\alpha_N^+ = V'(E_N^+) \to +\infty$ (in the case $v = -1$ we would have looked for a sequence $(E_N^-)_N$ going to $-\infty$ and $\alpha_N^- = -V'(E_N^-)$).
As a consequence of Assumptions 1.1,*ii)*, one shows that $\ln|V'|$ is negligible with respect to V at infinity. Therefore, because $\dfrac{\ln|x|}{V(x)} \underset{|x|\to\infty}{\longrightarrow} 0$,

$$f(x) = -V(x) + \underset{x\to+\infty}{o}(V(x))\,. \tag{38}$$

Because $f(x) \underset{x\to+\infty}{\longrightarrow} -\infty$ there exists $(E_N^+)_N$ going to infinity such that for all $N \geqslant 1$, $f(E_N^+) = -\ln N$. Setting $x = E_N^+$ in (38), we obtain that $-V(E_N^+) \sim f(E_N^+) = -\ln N$. Property c) follows from Assumptions 1.1, point *ii)*, along with the fact that α_N^{-1} stays bounded.

It remains to show that $\dfrac{\ln(\alpha_N^+)}{N} = \dfrac{\ln|V'(E_N^+)|}{N} \underset{N\to\infty}{\longrightarrow} 0$. By construction, we have

$$\frac{\ln|V'(E_N^+)|}{N} = \frac{\ln\left(Ne^{-V(E_N^+)+2P\ln N+\lambda_P}\right)}{N} = -\frac{V(E_N^+)}{N} + o(1)\,.$$

Using that $V(E_N^+) \sim \ln N$, we can conclude that $\ln|V'(E_N^+)| = o(N)$ which concludes the proof. □

By the discussion preceding Theorem 4.1, we deduce

Corollary 4.2 (Edge of a configuration)**.** *Let $E_N^{\pm}$, $\alpha_N^{\pm} := |V'(E_N^{\pm})|$ be the sequences of Theorem 4.1 associated with $v = \pm 1$. Then, both random variables*

$$\alpha_N^{+}\left(\max_{1\leqslant j\leqslant N} x_j - E_N^{+}\right)$$

and

$$\alpha_N^{-}\left(\min_{1\leqslant j\leqslant N} x_j - E_N^{-}\right)$$

converge to a Gumbel law, whose distribution function is given for $t \geqslant 0$ by $\mathcal{G}([0,t]) = \exp(e^{-t})$. Furthermore, $V(E_N^{\pm}) \sim \ln N$ and $\alpha_N^{\pm} \underset{N\to\infty}{\longrightarrow} \pm\infty$.

Remark 4.3. *Note that[Lam21][Theorem 3.4] applies for V of class $\mathcal{C}^2$ outside of a compact set, allowing to take $V(x) = |x|^a$ for $a > 1$. In this case, we find $E_N^{\pm} \sim \pm(\ln N)^{1/a}$. If $V(x) = \cosh(x)$, we find $E_N^{+} \sim -E_N^{-} \sim \arg\cosh(\ln N) \sim \ln\ln N$.*

The next lemma will be convenient in the proof of Theorem 5.2 when dealing with error terms.

Lemma 4.4. *With the notations of Corollary 4.2, we have*

$$\mu_P([E_N^{-}, E_N^{+}]^c) = o(N^{-1/2})\,.$$

Proof. Let $0 < \delta < 1$, to be specified later. We have

$$\int_{E_N^{+}}^{+\infty} \rho_P dx = \int_{E_N^{+}}^{+\infty} (\rho_P)^{\delta}(\rho_P)^{1-\delta} dx \leqslant \int_{\mathbb{R}} (\rho_P)^{\delta} dx \sup_{[E_N^{+},+\infty[} (\rho_P)^{1-\delta}\,.$$

By the first inequality of Lemma 2.2, the integral is finite. Also from the same inequality, we have for some constant C' and x big enough $\rho_P(x) \leqslant C' e^{-\frac{3}{4}V(x)}$. Because V is increasing in a neighborhood of $+\infty$, we get for N large enough

$$\sup_{[E_N^{+},+\infty[} (\rho_P)^{1-\delta} \leqslant C'^{1-\delta} e^{-(1-\delta)\frac{3}{4}V(E_N^{+})}\,.$$

Taking $\delta > 0$ such that $\frac{1}{2} - (1-\delta)\frac{3}{4} =: -\gamma < 0$ and using that $V(E_N^{+}) = \ln N + o(\ln N)$ (established in the proof of Theorem 4.1),

$$\sqrt{N}\int_{E_N^{+}}^{+\infty} \rho_P dx \leqslant K e^{-\gamma \ln N + (1-\delta)\frac{3}{4}o(\ln N)}\,,$$

and the right-hand side goes to zero as N goes to infinity. We deal with the integral $\int_{-\infty}^{E_N^{-}} \rho_P dx$ in the same way.

□

Remark 4.5. *We could improve the proof to show that $\mu_P([E_N^{-}, E_N^{+}]^c) \sim \dfrac{1}{N}$ but showing that it is $o(N^{\frac{1}{2}})$ is sufficient for what we need and requires less carefulness.*

5 Laplace transform for smooth test functions, proof of Theorem 1.3

Section 3 allows us to justify in Proposition 5.1 the heuristics we gave in equation (6) for ϕ having compact support. We will then extend in Theorem 5.2 this result to a more general set of functions, by an approximation by compactly supported functions, using Corollary 4.2.

Proposition 5.1. *For $\phi \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ with compact support, we have for any real t, as N goes to infinity,*

$$\mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\nu_N(\Xi\phi)}\right] \to \exp\left\{\frac{t^2}{2}q_P(\phi)\right\}, \tag{39}$$

where $\Xi\phi$ is given by equation (7), and $q_P(\phi)$ is given by

$$q_P(\phi) := \int_{\mathbb{R}}\left(\phi'(x)^2 + V''(x)\phi(x)^2\right)d\mu_P(x) + P\iint_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\mu_P(x)d\mu_P(y)\,. \tag{40}$$

Proof. Let $\phi \in \mathcal{C}_c^1(\mathbb{R},\mathbb{R})$, and let $t \in \mathbb{R}$. We perform in equation (4) the change of variables $x_i = y_i + \frac{t}{\sqrt{N}}\phi(y_i)$, $1 \leqslant i \leqslant N$, which is a diffeomorphism for N big enough. We thus have

$$Z_N^{V,P} = \int \prod_{1\leqslant i<j\leqslant N}\left|y_i - y_j + \frac{t}{\sqrt{N}}\big(\phi(y_i)-\phi(y_j)\big)\right|^{2P/N}.e^{-\sum_{i=1}^N V\left(y_i+\frac{t}{\sqrt{N}}\phi(y_i)\right)}.\prod_{i=1}^N\left(1+\frac{t}{\sqrt{N}}\phi'(y_i)\right)d^N\mathbf{y}, \tag{41}$$

and we develop separately the different terms of this integral. The first term can be written as:

$$\prod_{i<j}|y_i-y_j|^{2P/N}\prod_{i<j}\left|1+\frac{t}{\sqrt{N}}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}\right|^{2P/N},$$

The second product above, setting $\Delta\phi_{i,j} := \frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}$ and using Taylor-Lagrange theorem, equals

$$\exp\left(\frac{2P}{N}\sum_{i<j}\ln\left|1+\frac{t}{\sqrt{N}}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}\right|\right) = \exp\left(\frac{2P}{N}\sum_{i<j}\left(\frac{t}{\sqrt{N}}\Delta\phi_{i,j} - \frac{t^2}{2N}(\Delta\phi_{i,j})^2 + R_{N,1}(i,j)\right)\right),$$

where we noticed that $1+\frac{t}{\sqrt{N}}\Delta\phi_{i,j} \geqslant 1-\frac{t}{\sqrt{N}}\|\phi'\|_\infty > 0$ if N is big enough, and where

$$|R_{N,1}(i,j)| \leqslant \frac{|t|^3}{3N^{3/2}}\|\phi'\|_\infty^3.$$

Again by Taylor-Lagrange theorem, the second term in (41) equals

$$\exp\left(-\sum_{i=1}^N\left(V(y_i) + \frac{t}{\sqrt{N}}V'(y_i)\phi(y_i) + \frac{t^2}{2N}V''(y_i)\phi(y_i)^2 + R_{N,2}(i)\right)\right)$$

where $R_{N,2}(i) = \frac{t^3}{6N^{3/2}}V^{(3)}\left(y_i + \frac{t\theta_i}{\sqrt{N}}\phi(y_i)\right)\phi(y_i)^3$ for some $\theta_i \in [0,1]$, thus for N large enough

$$|R_{N,2}(i)| \leqslant \frac{|t|^3}{6N^{3/2}}\|\phi\|_\infty^3 \sup_{d(x,\text{supp }\phi)\leqslant 1}|V^{(3)}(x)|.$$

The last term reads

$$\prod_{i=1}^N\left(1+\frac{t}{\sqrt{N}}\phi'(y_i)\right) = \exp\left(\sum_{i=1}^N\left(\frac{t}{\sqrt{N}}\phi'(y_i) - \frac{t^2}{2N}\phi'(y_i)^2 + R_{N,3}(i)\right)\right),$$

with $|R_{N,3}(i)| \leqslant \frac{t^3}{3N^{3/2}}\|\phi'\|_\infty^3$. Dividing both sides of equation (41) by $Z_N^{V,P}$ we get

$$\mathbb{E}_N^{V,P}\Bigg[\exp\left\{t\sqrt{N}\left(P\iint_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\hat{\mu}_N(x)d\hat{\mu}_N(y) + \int_{\mathbb{R}}(\phi' - V'\phi)d\hat{\mu}_N\right)\right\}\times\exp\left\{K_N(t,\phi)\right\}$$
$$\times\exp\left\{\frac{t^2}{2}\left(-P\iint_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\hat{\mu}_N(x)d\hat{\mu}_N(y) - \int_{\mathbb{R}}(V''\phi^2 + \phi'^2)d\hat{\mu}_N\right)\right\}\Bigg] = 1,$$

with $|K_N(t,\phi)| \leqslant \frac{c(t,\phi)}{\sqrt{N}}$ where $c(t,\phi) \geqslant 0$ is independent of N. This bound shows that taking the limit $N \to \infty$ we can get rid of K_N:

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\Bigg[\exp\left\{ t\sqrt{N}\Big(P \iint_{\mathbb{R}^2} \frac{\phi(x)-\phi(y)}{x-y} d\hat{\mu}_N(x) d\hat{\mu}_N(y) + \int_{\mathbb{R}} (\phi' - V'\phi) d\hat{\mu}_N \Big) \right\} \\ \times \exp\left\{ \frac{t^2}{2} \left(-P \iint_{\mathbb{R}^2} \left(\frac{\phi(x)-\phi(y)}{x-y} \right)^2 d\hat{\mu}_N(x) d\hat{\mu}_N(y) - \int_{\mathbb{R}} (V''\phi^2 + \phi'^2) d\hat{\mu}_N \right) \right\} \Bigg] = 1.$$

Using Fubini's theorem (the function $(x,y) \mapsto \frac{\phi(x)-\phi(y)}{x-y}$ being bounded continuous on $\mathbb{R}^2$), the first line in the expectation value can be rewritten as $e^{t\sqrt{N}\Lambda_N}$ with

$$\Lambda_N := 2P \iint_{\mathbb{R}^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x) d(\hat{\mu}_N - \mu_P)(y) + \int_{\mathbb{R}} (\phi' - V'\phi) d(\hat{\mu}_N - \mu_P) + P\zeta_N(\phi) \tag{42}$$

where we used equation (5) and $\zeta_N(\phi)$ is given by (35). Let $F : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$F(\mu) = -P \iint_{\mathbb{R}^2} \left(\frac{\phi(x)-\phi(y)}{x-y} \right)^2 d\mu(x) d\mu(y) - \int_{\mathbb{R}} (V''\phi^2 + \phi'^2) d\mu\,. \tag{43}$$

It is continuous for the topology of weak convergence since all the functions in the integrals are bounded continuous. So far we have established that

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)} \right] = 1,$$

with Λ_N given by (42). We now replace in the latter equation the term $F(\hat{\mu}_N)$ by its limiting expression, $F(\mu_P)$. Fix a metric that is compatible with the weak convergence of probability measures on $\mathbb{R}$. For example,

$$d_{\mathrm{Lip}}(\mu,\nu) = \sup \left| \int f d\mu - \int f d\nu \right|, \tag{44}$$

where the supremum runs over $f : \mathbb{R} \to \mathbb{R}$ bounded and Lipschitz with $\|f\|_\infty \leqslant 1$ and Lipschitz constant $|f|_{\mathrm{Lip}} \leqslant 1$. By the large deviations principle for $(\hat{\mu}_N)_N$ under the probability (3) established by [GZ19, Theorem 1.1], for all $\delta > 0$ the event $\{d_{\mathrm{Lip}}(\hat{\mu}_N, \mu_P) > \delta\}$ has (for N big enough) probability smaller than e^{-Nc_δ} where $c_\delta > 0$. Hence,

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)} \right] = \lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_P)\leqslant\delta\}} e^{t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)} \right].$$

By continuity of F there is some $\varepsilon(\delta)$ which goes to 0 as $\delta \to 0$ such that, for $d_{\mathrm{Lip}}(\nu,\mu_P) \leqslant \delta$, we have $|F(\nu) - F(\mu_P)| \leqslant \varepsilon(\delta)$. Taking the (decreasing) limit as δ goes to zero we deduce

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)} \right] = \lim_{\delta\to 0} \lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_P)\leqslant\delta\}} e^{t\sqrt{N}\Lambda_N} \right] e^{\frac{t^2}{2}F(\mu_P)}.$$

But the same large deviations argument shows that

$$\lim_{\delta\to 0} \lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_P)\leqslant\delta\}} e^{t\sqrt{N}\Lambda_N} \right] = \lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N} \right].$$

Thus, we have shown that

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\left(2P\iint_{\mathbb{R}^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x) d(\hat{\mu}_N-\mu_P)(y) + \int_{\mathbb{R}} (\phi'-V'\phi) d(\hat{\mu}_N-\mu_P) + P\zeta_N(\phi)\right)} \right] = e^{-\frac{t^2}{2}F(\mu_P)}, \tag{45}$$

Which establishes that $\sqrt{N}\Lambda_N = \sqrt{N}\Big(\nu_N(\Xi\phi) + P\zeta_N(\phi)\Big)$ converges in law towards a centered Gaussian random variable with announced variance. We finally get rid of the remaining term $\zeta_N(\phi)$, using Corollary 3.4: taking $\varepsilon = 1/4$ for example, we see in particular that $\sqrt{N}\zeta_N(\phi)$ converges in probability towards zero. The conclusion follows from Slutsky's lemma. □

We now extend the result of Proposition 5.1 to a more general set of functions. With the notations of Proposition 5.1, we have

Theorem 5.2. *Let $\phi \in H^2(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ such that ϕ'' is bounded. Additionally, suppose that $V^{(3)}\phi^2$, $V''\phi\phi'$, $V''\phi^2$ and $V'\phi$ are bounded. Then, recalling* (40) *we have the convergence in distribution as N goes to infinity*

$$\sqrt{N}\nu_N(\Xi\phi) \to \mathcal{N}(0, q_P(\phi))\,.$$

Proof. For $N \geqslant 1$, let E_N^-, E_N^+ be given by Corollary 4.2. Let $\chi_N : \mathbb{R} \to [0,1]$ be $\mathcal{C}^2$ with compact support such that

$$\chi_N(x) = 1 \text{ for } x \in [E_N^- - 1, E_N^+ + 1] \text{ and } \chi_N(x) = 0 \text{ for } x \in [E_N^- - 2, E_N^+ + 2]^c$$

and such that, denoting $\phi_N = \phi\chi_N$, $\sup_N \|\phi_N^{(k)}\|_\infty + \|\phi_N^{(k)}\|_{L^2(\mathbb{R})} < +\infty$ for $k = 0,1,2$ (we assumed $\phi \in H^2(\mathbb{R})$, in particular ϕ and ϕ' is bounded and such a χ_N exists). The point of cutting ϕ outside the set $[E_N^- - 1, E_N^+ + 1]$ is that with high probability, the empirical measure $\hat{\mu}_N$ doesn't see the difference between ϕ and ϕ_N.

The support of ϕ_N is then contained in $[E_N^- - 2, E_N^+ + 2]$, and we now argue that the proof of Proposition 5.1 can be adapted so that

$$\sqrt{N}\nu_N(\Xi\phi_N) \to \mathcal{N}(0, q_P(\phi))\,. \tag{46}$$

Similarly as in Proposition 5.1, we perform in $Z_N^{V,P}$ the change of variables $x_i = y_i + \frac{t}{\sqrt{N}}\phi_N(y_i)$, $1 \leqslant i \leqslant N$, which is the same as before, but with ϕ replaced by ϕ_N. First, with $I_N := [E_N^- - 2, E_N^+ + 2]$, the error term

$$K_N(t, \phi_N) \leqslant 2\frac{t^3}{3N^{1/2}}\|\phi_N'\|_\infty^3 + \frac{t^3}{6N^{1/2}}\|\phi_N\|_\infty \sup_{d(x, I_N) \leqslant 1} |V^{(3)}(x)|$$

of the proof of Proposition 5.1 is still going to zero, because of our choice of χ_N and Assumption 1.2. As previously, we then have

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N(\phi_N) + \frac{t^2}{2}F_N(\hat{\mu}_N)}\right] = 1 \tag{47}$$

with

$$\Lambda_N(\phi_N) := 2P \iint_{\mathbb{R}^2} \frac{\phi_N(x) - \phi_N(y)}{x - y} d\mu_P(x) d(\hat{\mu}_N - \mu_P)(y) + \int_{\mathbb{R}} (\phi_N' - V'\phi_N) d(\hat{\mu}_N - \mu_P) + P\zeta_N(\phi_N)\,,$$

where ζ_N is given by (35), and

$$F_N(\hat{\mu}_N) = -P \iint_{\mathbb{R}^2} \left(\frac{\phi_N(x) - \phi_N(y)}{x - y}\right)^2 d\hat{\mu}_N(x) d\hat{\mu}_N(y) - \int_{\mathbb{R}} (V''\phi_N^2 + \phi_N'^2) d\hat{\mu}_N\,.$$

Taking again the distance d_{Lip} defined in (44), one can check that for μ, ν probability measures over $\mathbb{R}$,

$$|F_N(\mu) - F_N(\nu)| \leqslant C_N d_{\mathrm{Lip}}(\mu, \nu)\,,$$

where C_N is a term depending on the norms $\|\phi_N'\|_\infty, \|\phi_N''\|_\infty$, $\|V''\phi_N^2\|_\infty$ and $\|(V''\phi_N^2)'\|_\infty$. The choice of χ_N and the fact that ϕ is chosen so that $V^{(3)}\phi^2$ and $V''\phi\phi'$ are bounded guarantee that $\|(V''\phi_N^2)'\|_\infty$ is bounded in N. The other norms are easily bounded by hypothesis. Therefore C_N can be seen to be uniformly bounded in N, and we find some $C \geqslant 0$ independent of N such that

$$|F_N(\mu) - F_N(\nu)| \leqslant C d_{\mathrm{Lip}}(\mu, \nu)\,.$$

As in proposition 5.1, we use the large deviation principle for $(\hat{\mu}_N)$ to deduce

$$\lim_{N\to+\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N(\phi_N) + \frac{t^2}{2}F_N(\hat{\mu}_N)}\right] = \lim_{N\to+\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N(\phi_N)}\right] e^{\frac{t^2}{2}F_N(\mu_P)}\,.$$

By dominated convergence, $F_N(\mu_P)$ converges to $F(\mu_P)$, the function F being given by (43). This shows the convergence as N goes to infinity

$$\lim_{N\to+\infty} \mathbb{E}_N^{V,P}\left[e^{t\sqrt{N}\Lambda_N(\phi_N)}\right] = e^{-\frac{t^2}{2}F(\mu_P)}\,,$$

and $\sqrt{N}\Big(\nu_N(\Xi\phi_N)+P\zeta_N(\phi_N)\Big)$ converges towards a centered Gaussian variable with variance $-F(\mu_P) = q_P(\phi)$. Because $\sup_N \|\phi_N\|_{H^2(\mathbb{R})}$ is finite, we can apply again Corollary 3.4 to deduce the convergence in law (46). We now have the ingredients to conclude, by showing that the characteristic function

$$\mathbb{E}_N^{V,P}\left[e^{\mathrm{i}t\sqrt{N}\nu_N(\Xi\phi)}\right] = \mathbb{E}_N^{V,P}\left[e^{\mathrm{i}t\sqrt{N}\int \Xi\phi d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int \Xi\phi d\mu_P}$$

converges to the characteristic of a Gaussian variable with appropriate variance. By Corollary 4.2, the probability under $\mathbb{P}_N^{V,P}$ of the event $\mathcal{E}_N = \Big\{x_1,\dots,x_N \in [E_N^- - 1, E_N^+ + 1]\Big\}$ converges to 1. Along with the convergence (46), we deduce

$$e^{-\frac{t^2}{2}q_P(\phi)} = \lim_N \mathbb{E}_N^{V,P}\left[e^{\mathrm{i}t\sqrt{N}\int \Xi\phi_N d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int \Xi\phi_N d\mu_P} = \lim_N \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\mathcal{E}_N} e^{\mathrm{i}t\sqrt{N}\int \Xi\phi_N d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int \Xi\phi_N d\mu_P}\,,$$

Where we used

$$\left|\mathbb{E}_N^{V,P}\left[\mathbf{1}_{\mathcal{E}_N^c} e^{\mathrm{i}t\sqrt{N}\int \Xi\phi_N d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int \Xi\phi_N d\mu_P}\right| \leqslant \mathbb{P}_N^{V,P}(\mathcal{E}_N^c) \xrightarrow[N\to+\infty]{} 0\,.$$

Using that $\phi_N = \phi$ on $J_N = [E_N^- - 1, E_N^+ + 1]$,

$$\begin{aligned}\int \Xi\phi_N d\mu_P &= 2P \iint \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x)d\mu_P(y) + \int (\phi_N' - V'\phi_N) d\mu_P \\ &= 2P \iint_{J_N^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x)d\mu_P(y) + 2P \iint_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x)d\mu_P(y) \\ &+ \int_{J_N} (\phi' - V'\phi) d\mu_P + \int_{J_N^c} (\phi\chi_N' + \phi'\chi_N - V'\phi\chi_N) d\mu_P\,.\end{aligned}$$

By boundedness of $(\|\phi_N'\|_\infty)_N$, the second term is bounded by

$$C_P \iint_{(J_N^2)^c} d\mu_P d\mu_P \leqslant 2C_P \mu_P(J_N^c) = o(N^{-1/2})\,,$$

where we used the union bound and Lemma 4.4. By the same estimate and the fact that χ_N can be chosen so that $(\|\chi_N'\|_\infty)_N$ is bounded, and because ϕ', $V'\phi$ are bounded, the last term is also $o(N^{-1/2})$. By the previous arguments, we also conclude that

$$2P \iint_{(J_N^2)^c} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x)d\mu_P(y) + \int_{J_N^c} (\phi' - V'\phi) d\mu_P = o(N^{-1/2})\,,$$

thus

$$\int \Xi\phi_N d\mu_P = \int \Xi\phi d\mu_P + o(N^{-1/2})\,,$$

and so far we have

$$e^{-\frac{t^2}{2}q_P(\phi)} = \lim_N \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\mathcal{E}_N} e^{\mathrm{i}t\sqrt{N}\int \Xi\phi_N d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int \Xi\phi d\mu_P}\,.$$

Finally, on $\mathcal{E}_N$, using $\phi_N = \phi$ and that $\hat{\mu}_N$ is supported in J_N,

$$\begin{aligned}\int \Xi\phi_N d\hat{\mu}_N &= 2P \iint_{J_N^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) + 2P \iint_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) + \int_{J_N} (\phi' - V'\phi) d\hat{\mu}_N \\ &= 2P \iint \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) + \int (\phi' - V'\phi) d\hat{\mu}_N + o(N^{-1/2}),\end{aligned}$$

Where in the second line we used, using Lemma 4.4 again, that

$$\iint_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) = \iint_{J_N \times J_N^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) = o(N^{-1/2}),$$

and the same estimate holds for ϕ_N replaced by ϕ. Therefore,

$$e^{-\frac{t^2}{2} q_P(\phi)} = \lim_N \mathbb{E}_N^{V,P} \left[\mathbf{1}_{\mathcal{E}_N} e^{\mathrm{i}t\sqrt{N} \int \Xi\phi d\hat{\mu}_N} \right] e^{-\mathrm{i}t\sqrt{N} \int \Xi\phi d\mu_P}.$$

This establishes that

$$\lim_N \mathbb{E}_N^{V,P} \left[e^{\mathrm{i}t\sqrt{N} \int \Xi\phi d\hat{\nu}_N} \right] = e^{-\frac{t^2}{2} q_P(\phi)},$$

which concludes the proof. □

Remark 5.3. *Taking ϕ such that ϕ' satisfies the conditions of Theorem 5.2, we then have*

$$\mathbb{E}_N^{V,P} \left[e^{t\sqrt{N}\nu_N(\mathcal{L}\phi)} \right] \underset{N\to\infty}{\longrightarrow} \exp\left\{ \frac{t^2}{2} q_P(\phi') \right\}, \tag{48}$$

where the operator $\mathcal{L}$ is defined as $\mathcal{L}\phi := \Xi\phi'$, ie

$$\mathcal{L}\phi = 2P \int_{\mathbb{R}} \frac{\phi'(x)-\phi'(y)}{x-y} d\mu_P(y) + \phi''(x) - V'(x)\phi'(x). \tag{49}$$

Note that $q_P^V(\phi') = \left(\sigma_P^V\right)^2(\mathcal{L}\phi)$ where σ_P^V is defined in (12). By Theorem 7.1, the class of functions in $\mathcal{L}^{-1}(\mathcal{T})$ where

$$\mathcal{T} := \left\{ f \in \mathcal{C}^1(\mathbb{R}), \exists \varepsilon > 0,\ f(x) = \underset{|x|\to\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right),\ f'(x) = \underset{|x|\to\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right), \int_{\mathbb{R}} f\rho_P = 0 \right\}$$

satisfies (48). This proves Theorem 1.3.

We now prove a more compact formula for the variance such as the one appearing in [HL21].

Lemma 5.4. *The following equality holds for all $\phi \in \mathcal{T}$*

$$\begin{aligned}\langle \mathcal{L}^{-1}\phi, \phi \rangle_{\mathsf{H}} = \left(\sigma_P^V\right)^2(\phi) := \int_{\mathbb{R}} \left(\left(\mathcal{L}^{-1}\phi\right)''(x)^2 + V''(x)\left(\mathcal{L}^{-1}\phi\right)'(x)^2 \right) d\mu_P(x) \\ + P \iint_{\mathbb{R}^2} \left(\frac{\left(\mathcal{L}^{-1}\phi\right)'(x) - \left(\mathcal{L}^{-1}\phi\right)'(y)}{x-y} \right)^2 d\mu_P(x) d\mu_P(y)\end{aligned} \tag{50}$$

Proof. It suffices to show that $\left(\sigma_P^V\right)^2(\mathcal{L}\phi) = \langle \mathcal{L}\phi, \phi \rangle_{\mathsf{H}}$ for all ϕ, such that $\phi' \in H^2(\mathbb{R})$.

$$\langle \mathcal{L}\phi, \phi \rangle_{\mathsf{H}} = -\int_{\mathbb{R}} \left(\frac{(\phi'\rho_P)'}{\rho_P} \right)' \phi'\rho_P - 2P \int_{\mathbb{R}} \mathcal{H}[\phi'\rho_P]'\phi'\rho_P$$

Proceeding to integration by parts in the first integral leads to

$$-\int_{\mathbb{R}}\Big(\frac{(\phi'\rho_P)'}{\rho_P}\Big)'\phi'\rho_P=\int_{\mathbb{R}}\Big(\frac{(\phi'\rho_P)'}{\rho_P}\Big)^2\rho_P=\int_{\mathbb{R}}\phi''^2\rho_P+2\phi'\phi''\rho_P'+\phi'^2\big(\frac{\rho_P'}{\rho_P}\big)^2\rho_P$$
$$=\int_{\mathbb{R}}\phi''^2\rho_P-\phi'^2\frac{\rho_P''}{\rho_P}\rho_P+\phi'^2\big(\frac{\rho_P'}{\rho_P}\big)^2\rho_P$$

Since

$$\frac{\rho_P''}{\rho_P}=\Big(-V''-2P\mathcal{H}[\rho_P]'+V'^2+4P^2\mathcal{H}[\rho_P]^2+4PV'\mathcal{H}[\rho_P]\Big)=-V''-2P\mathcal{H}[\rho_P]'+\Big(\frac{\rho_P'}{\rho_P}\Big)^2$$

we obtain

$$\langle\mathcal{L}\phi,\phi\rangle_{\mathsf{H}}=\int_{\mathbb{R}}\phi''^2\rho_P+V''\phi^2\rho_P-2P\int_{\mathbb{R}}\mathcal{H}[\phi'\rho_P]'\phi'\rho_P+2P\int_{\mathbb{R}}\mathcal{H}[\rho_P]'\phi'^2\rho_P$$

To conclude, we just have to show that

$$\iint_{\mathbb{R}^2}\Big(\frac{\phi'(x)-\phi'(y)}{x-y}\Big)^2d\mu_P(x)d\mu_P(y)=2\int_{\mathbb{R}}\mathcal{H}[\rho_P]'\phi'^2\rho_P-\mathcal{H}[\phi'\rho_P]'\phi'\rho_P$$

First

$$\int_{\mathbb{R}}\mathcal{H}[\phi'\rho_P]'\phi'\rho_P=\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\phi'(x)\phi'(y)}{(y-x)^2}d\mu_P(x)d\mu_P(y)$$

Secondly

$$\int_{\mathbb{R}}\mathcal{H}[\rho_P]'\phi'^2\rho_P=\frac{1}{2}\iint_{\mathbb{R}^2}\frac{\phi'^2(x)+\phi'^2(y)}{(y-x)^2}d\mu_P(x)d\mu_P(y)$$

which allows to conclude that $\big(\sigma_P^V\big)^2(\mathcal{L}\phi)=\langle\mathcal{L}\phi,\phi\rangle_{\mathsf{H}}$. □

6 Inversion of $\mathcal{L}$

This section is dedicated to the definition of $\mathcal{L}$ given by (8) and its domain and then we focus on its inversion. We rely heavily on results of Appendix A: the diagonalization of the operator $\mathcal{A}$ by the use of the theory of Schrödinger operators.

Let $P>0$ be fixed. We introduce the operators $\mathcal{A}$ and $\mathcal{W}$, acting on sufficiently smooth functions of $L^2(\rho_P)$, by

$$\mathcal{A}\phi=-\frac{\Big(\phi'\rho_P\Big)'}{\rho_P}=-\Big(\phi''+\frac{\rho_P'}{\rho_P}\phi'\Big)\quad\text{and}\quad\mathcal{W}\phi=-\mathcal{H}\big[\phi'\rho_P\big]\,. \tag{51}$$

One can show that the operator $\mathcal{A}$ corresponds to the operator verifying:

$$\langle\phi,\psi\rangle_{\mathsf{H}}=\int_{\mathbb{R}}\phi'\psi'd\mu_P=\int_{\mathbb{R}}\phi\mathcal{A}\psi d\mu_P=\langle\phi,\mathcal{A}\psi\rangle_{L^2(\mu_P)}$$

We first show the following decomposition of $\mathcal{L}$.

Lemma 6.1. *For ϕ twice differentiable we have the following pointwise identity*

$$-\mathcal{L}\phi=\mathcal{A}\phi+2P\mathcal{W}\phi\,. \tag{52}$$

Proof. We write for $x\in\mathbb{R}$

$$2P\int_{\mathbb{R}}\frac{\phi'(x)-\phi'(y)}{x-y}\rho_P(y)dy=-2P\phi'(x)\mathcal{H}[\rho_P](x)+2P\mathcal{H}[\phi'\rho_P](x)\,. \tag{53}$$

Then,

$$\mathcal{L}\phi = \phi'' - V'\phi' - 2P\phi'\mathcal{H}[\rho_P] + 2P\mathcal{H}\left[\phi'\rho_P\right].$$

By (19) we have $-V' - 2P\mathcal{H}[\rho_P] = \frac{\rho_P'}{\rho_P}$, which concludes the proof. □

In order to state the next theorem, whose proof we detail in the Appendix, we introduce the following Sobolev-type spaces. Let

$$H^1_{V'}(\mathbb{R}) := \left\{u \in H^1(\mathbb{R}),\, uV' \in L^2(\mathbb{R})\right\}.$$

We now define

$$\mathcal{D}(\mathcal{S}) = \left\{u \in H^1_{V'}(\mathbb{R}), -u'' + (w_P + \alpha)u \in L^2(\mathbb{R})\right\}$$

and

$$\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) := \rho_P^{-1/2}\mathcal{D}(\mathcal{S})$$

and its homogeneous counterpart

$$\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) := \left\{u \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}),\, \int_{\mathbb{R}} u\rho_P dx = 0\right\}.$$

Finally, we let $L^2_0(\rho_P)$ be the subset of $L^2(\rho_P)$ of zero mean functions with respect to ρ_P.

We detail the proof of the following theorem in Appendix A which is based on Schrödinger operators theory.

Theorem 6.2 (Diagonalization of $\mathcal{A}$ in $L^2_0(\rho_P)$)**.** *There exists a sequence $0 < \lambda_1 < \lambda_2 < \dots$ going to infinity, and a complete orthonormal set $(\phi_n)_{n\geqslant 1}$ of $L^2_0(\rho_P)$ of associated eigenfunctions for $\mathcal{A}$, meaning that*

- $\mathrm{span}\{\phi_n,\, n \geqslant 1\}$ *is dense in* $L^2_0(\rho_P)$,
- *For all* i, j, $\langle \phi_i, \phi_j\rangle_{L^2(\rho_P)} = \delta_{i,j}$,
- *For all* $n \geqslant 1$, $\mathcal{A}\phi_n = \lambda_n\phi_n$.

Furthermore, each ϕ_n is in $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \to L^2_0(\rho_P)$ is bijective, and we have the writing, for $u \in L^2_0(\rho_P)$

$$\mathcal{A}^{-1}u = \sum_{n\geqslant 1} \lambda_n^{-1} \langle u, \phi_n\rangle_{L^2(\rho_P)} \phi_n.$$

We see the operators $\mathcal{A}$ and $\mathcal{W}$ as unbounded operators on the space

$$\mathsf{H} = \left\{u \in H^1(\rho_P) \mid \int_{\mathbb{R}} u\rho_P dx = 0\right\}$$

endowed with the inner product $\langle u, v\rangle_{\mathsf{H}} = \langle u', v'\rangle_{L^2(\rho_P)}$. This defines an inner product on H and makes it a complete space: it can be seen that $H^1(\rho_P)$ is the completion of $\mathcal{C}^\infty_c(\mathbb{R})$ with respect to the inner product $\langle u, v\rangle_{L^2(\rho_P)} + \langle u', v'\rangle_{L^2(\rho_P)}$. The space H is then the kernel of the bounded (with respect to $\|\cdot\|_{\mathsf{H}}$) linear form, $\langle \widetilde{1}, \cdot\rangle_{L^2(\rho_P)}$ on $H^1(\rho_P)$, and both inner products are equivalent on H because of the Poincaré inequality, Proposition 2.6. The use of H is motivated by the fact that both $\mathcal{A}$ and $\mathcal{W}$ are self-adjoint positive on this space as we show in Lemma 6.4.

In the next proposition, we deduce from Theorem 6.2 the diagonalization of $\mathcal{A}$ in H.

Proposition 6.3 (Diagonalization of $\mathcal{A}$ in H)**.** *With the same eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ as in Theorem 6.2, there exists a complete orthonormal set $(\psi_n)_{n\geqslant 1}$ of H formed by eigenfunctions of $\mathcal{A}$.*

Proof. With $(\phi_n)_{n\geqslant 1}$ of Theorem 6.2,

$$\begin{aligned}\delta_{i,j} = \langle \phi_i, \phi_j\rangle_{L^2(\rho_P)} &= \frac{1}{\lambda_j}\langle \phi_i, \mathcal{A}\phi_j\rangle_{L^2(\rho_P)} \\ &= \frac{1}{\lambda_j}\langle \phi_i', \phi_j'\rangle_{L^2(\rho_P)} \\ &= \frac{1}{\lambda_j}\langle \phi_i, \phi_j\rangle_{\mathsf{H}}.\end{aligned}$$

With $\psi_n = \frac{1}{\sqrt{\lambda_n}}\phi_n$, $(\psi_n)_{n\geqslant 1}$ is then orthonormal with respect to the inner product of H. To show that $\mathrm{span}\{\psi_n, n \geqslant 1\}$ is dense in H, let $u \in \mathsf{H}$ be such that for all $j \geqslant 1$, $\langle u, \phi_j\rangle_{\mathsf{H}} = 0$. In the last series of equalities, replace ϕ_i by u: we see that u is orthogonal to each ϕ_j in $L^2(\rho_P)$, thus u is a constant as shown in the proof of Lemma A.10, and because $u \in \mathsf{H}$ it has zero mean against ρ_P. This shows that $u = 0$. □

We set for what follows $\mathcal{D}(\mathcal{A}) := \left\{u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \,|\, \mathcal{A}u \in \mathsf{H}\right\}$ and $\mathcal{D}(\mathcal{W}) := \{u \in \mathsf{H} \,|\, \mathcal{W}u \in \mathsf{H}\}$.

Lemma 6.4. *The following properties hold:*

- *The operator $\mathcal{W} : \mathcal{D}(\mathcal{W}) \to \mathsf{H}$ is positive: for all $\phi \in \mathcal{D}(\mathcal{W})$,*

$$\langle \mathcal{W}\phi, \phi\rangle_{\mathsf{H}} = \frac{1}{2}\|\phi'\rho_P\|_{1/2}^2 \geqslant 0\,,$$

  with equality only for $\phi = 0$, where the $1/2$-norm of u is given by

$$\|u\|_{1/2}^2 = \int_{\mathbb{R}} |x|.\,|\mathcal{F}[u](x)|^2\,dx\,.$$

- *Both $\mathcal{A}$ and $\mathcal{W}$ are self-adjoint for the inner product of H.*

Proof. To prove the first point, let $\phi \in \mathcal{D}(\mathcal{W})$. Then,

$$\begin{aligned}2\pi\,\langle \mathcal{W}\phi, \phi\rangle_{\mathsf{H}} = -2\pi\,\langle \mathcal{H}[\phi'\rho_P]', \phi'\rho_P\rangle_{L^2(dx)} &= -\left\langle ix\mathcal{F}\left[\mathcal{H}[\phi'\rho_P]\right], \mathcal{F}[\phi'\rho_P]\right\rangle_{L^2(dx)} \\ &= \pi\,\langle\,|\,x\,|\,\mathcal{F}[\phi'\rho_P], \mathcal{F}[\phi'\rho_P]\rangle_{L^2(dx)} = \pi\|\phi'\rho_P\|_{1/2}^2 \geqslant 0\,,\end{aligned}$$

and because ϕ is in H, this last quantity is zero if and only if ϕ vanishes.

For the second point, let $u, v \in \mathcal{D}(\mathcal{W})$. Using Plancherel's isometry and *i)* of Lemma 2.1,

$$\langle \mathcal{W}u, v\rangle_{\mathsf{H}} = \langle (\mathcal{W}u)', v'\rho_P\rangle_{L^2(dx)} = \frac{1}{2}\,\langle\,|\,x\,|\,\mathcal{F}[u'\rho_P], \mathcal{F}[v'\rho_P]\rangle_{L^2(dx)}\,,$$

and this last expression is symmetric in (u, v). The proof of the self-adjointness of $\mathcal{A}$ follows from integration by parts. □

Definition 6.5 (Quadratic form associated to $-\mathcal{L}$)**.** *We define for all $u, v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$ the quadratic form associated to $-\mathcal{L}$ by*

$$q_{-\mathcal{L}}(u, v) = \langle \mathcal{A}u, \mathcal{A}v\rangle_{L^2(\rho_P)} + 2P\,\langle \mathcal{F}[u'\rho_P], \mathcal{F}[v'\rho_P]\rangle_{L^2(|x|dx)}$$

Note that for all $u, v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$, $q_{-\mathcal{L}}(u, v) = \langle -\mathcal{L}u, v\rangle_{\mathsf{H}}$ and that whenever $u \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{W})$,

$$q_{-\mathcal{L}}(u, u) = \langle \mathcal{A}u, u\rangle_{\mathsf{H}} + 2P\,\langle \mathcal{W}u, u\rangle_{\mathsf{H}} \geqslant \lambda_1(\mathcal{A})\|u\|_{\mathsf{H}}^2 \tag{54}$$

by Proposition 6.3 and Lemma 6.4. After extending the $q_{-\mathcal{L}}$ to its form domain $Q(\mathcal{L})$ which is equal to $\left\{u \in \mathsf{H}, \mathcal{A}u \in L^2(\rho_P),\, \mathcal{F}[u'\rho_P] \in L^2(|x|dx)\right\} = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$. The equality comes from the fact

that $\mathcal{A}^{-1}\Big(L^2_0(\rho_P)\Big) = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, that $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \subset \mathsf{H}$ and that $\mathcal{F}[u'\rho_P] \in L^2(x^2dx)$ whenever $u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, indeed $u'\rho_P \in H^1(\mathbb{R})$ because $(u'\rho_P)' = -\rho_P\mathcal{A}u \in L^2(\mathbb{R})$. We now define $\mathcal{D}(\mathcal{L})$ the domain of definition of $-\mathcal{L}$ by:

$$\mathcal{D}(\mathcal{L}) := \Big\{u \in Q(\mathcal{L}), v \mapsto q_{-\mathcal{L}}(u,v) \text{ can be extended to a continuous linear form on } \mathsf{H}\Big\}$$

Proposition 6.6. *$\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{W})$.*

Proof. Let $u \in \mathcal{D}(\mathcal{L})$, by Riesz's theorem there exists $f_u \in \mathsf{H}$, such that $q_{-\mathcal{L}}(u,v) = \langle f_u, v\rangle_{\mathsf{H}}$ for all $v \in \mathsf{H}$, we set $-\mathcal{L}u := f_u$, it is called the Friedrichs extension of $-\mathcal{L}$. Then for all $v \in \mathsf{H} \cap \mathcal{C}^\infty_c(\mathbb{R})$, by integration by part we get:

$$\langle -\mathcal{L}u, v\rangle_{\mathsf{H}} = q_{-\mathcal{L}}(u,v) = \langle u, \mathcal{A}v\rangle_{\mathsf{H}} + 2P\,\langle u, \mathcal{W}v\rangle_{\mathsf{H}}\,,$$

hence we deduce the distributional identity $-\mathcal{L}u = \mathcal{A}u + 2P\mathcal{W}u$. Since $u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, $\mathcal{W}u \in H^1(\rho_P)$ implying that $\mathcal{A}u \in \mathsf{H}$ and then that $\mathcal{W}u \in \mathsf{H}$. The converse is trivially true. □

We are now ready to state the main theorem of this section, that is the inversion of $\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$.

Theorem 6.7 (Inversion of $\mathcal{L}$)**.** *$-\mathcal{L} : \mathcal{D}(\mathcal{L}) \longrightarrow \mathsf{H}$ is bijective. Furthermore, $(-\mathcal{L})^{-1}$ is continuous from $(\mathsf{H}, \|.\|_{\mathsf{H}})$ to $(\mathcal{D}(\mathcal{L}), q_{-\mathcal{L}})$.*

Proof. Let $f \in \mathsf{H}$, since $\langle f, .\rangle_{\mathsf{H}}$ is a linear form on $Q(\mathcal{L}) = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ which is, by (54), continuous with respect to $q_{-\mathcal{L}}$, one can applies Riesz's theorem so there exists a unique $u_f \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, such that for all $v \in \mathsf{H}$, $\langle f, v\rangle_{\mathsf{H}} = q_{-\mathcal{L}}(u_f, v)$. Since, u_f is clearly in $\mathcal{D}(\mathcal{L})$ by definition of the Friedrichs extension of $-\mathcal{L}$, we have $-\mathcal{L}u = f$. □

Remark 6.8. *We can diagonalize $\mathcal{L}$ by the same argument we used in Appendix A to diagonalize $\mathcal{A}$ in $L^2_0(\rho_P)$.*

We now state a result that could allow one to extract more regularity for $\mathcal{L}^{-1}$ by the help of an explicit form that uses Fredholm determinant theory for Hilbert-Schmidt operators, the reader can refer to [GGK12].

Definition 6.9 (Fredholm determinant)**.** *Let $\mathcal{U}$ be a self-adjoint Hilbert-Schmidt operator, we denote the Fredholm determinant by* $\det(I + \mathcal{U})$.

Theorem 6.10 (Determinant formula for $\mathcal{L}^{-1}$)**.** *For all $u \in \mathsf{H}$, such that $x \mapsto \dfrac{1}{\rho_P(x)}\displaystyle\int_x^{+\infty} u(t)\rho_P(t)dt$ is integrable at $+\infty$, we have:*

$$\mathcal{L}^{-1}u = \mathcal{A}^{-1}u - \rho_P^{-1/2}\mathcal{R}\big[\rho_P^{1/2}\mathcal{A}^{-1}u\big] \tag{55}$$

where $\mathcal{R}$ is the kernel operator defined for all $v \in L^2(\mathbb{R})$ by:

$$\mathcal{R}[v](x) = \int_{\mathbb{R}} R(x,y)v(y)dy$$

where

$$R(x,y) = \frac{1}{\det(I+\mathcal{K})}\sum_{n\geqslant 0}\frac{1}{n!}\int_{\mathbb{R}^n}\det_{n+1}\begin{bmatrix} K(x,y) & K(x,\lambda_b) \\ K(\lambda_a,y) & K(x,\lambda_b)\end{bmatrix}_{a,b=1\dots n} d\lambda_1\dots d\lambda_n$$

where $\mathcal{K}$ is the kernel operator defined for all $w \in L^2(\rho_P)$ by:

$$\mathcal{K}[w](x) = \int_{\mathbb{R}} K(x,y)w(y)dy \tag{56}$$

with

$$K(x,y) = -2P\rho_P(x)\rho_P(y)\ln\Big|1 - \frac{y}{x}\Big|. \tag{57}$$

Proof. Let $f \in \mathsf{H}$, there exists a unique $u \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}u = f$. Since $(u'\rho_P)' = \rho_P \mathcal{A}u \in L^2(\mathbb{R})$, hence $u'\rho_P \in H^1(\mathbb{R})$ so $u'(x)\rho_P(x) \underset{|x|\to+\infty}{\longrightarrow} 0$. By definition, $-\dfrac{(u'\rho_P)'}{\rho_P} = f$ hence

$$(\mathcal{A}^{-1}f)'(x)\rho_P(x) = u'(x)\rho_P(x) = \int_x^{+\infty} f(t)\rho_P(t)dt. \tag{58}$$

Using the fact that $\int_{\mathbb{R}} u(x)\rho_P(x)dx = 0$, integrating again we get:

$$u(x) = -\int_x^{+\infty} \frac{ds}{\rho_P(s)} \int_s^{+\infty} f(t)\rho_P(t)dt + C$$

where $C = \displaystyle\int_{\mathbb{R}} \rho_P(x)dx \int_x^{+\infty} \frac{ds}{\rho_P(s)} \int_s^{+\infty} f(t)\rho_P(t)dt$. Now let $g \in \mathsf{H}$, there exists a unique $v \in \mathcal{D}(\mathcal{L})$, such that $-\mathcal{L}v = \mathcal{A}v + 2P\mathcal{W}v = g$ and then $v + 2P\mathcal{W}\mathcal{A}^{-1}v = \mathcal{A}^{-1}g$. using (58), we get:

$$\mathcal{W}\mathcal{A}^{-1}v(x) = \fint_{\mathbb{R}} \frac{ds}{s-x} \int_s^{+\infty} dtv(t)\rho_P(t)$$

By Sokhotski-Plejmel formula, we have:

$$\fint_{\mathbb{R}} \frac{ds}{s-x} \int_s^{+\infty} dtv(t)\rho_P(t) = \lim_{M\to+\infty} \lim_{\varepsilon\to 0} \int_{-M}^{M} \frac{ds}{2} \Big\{ \frac{1}{x-s+\mathrm{i}\varepsilon} + \frac{1}{x-s-\mathrm{i}\varepsilon} \Big\} \int_s^{+\infty} dtv(t)\rho_P(t)$$

We then proceed to an integration by part:

$$\fint_{\mathbb{R}} \frac{ds}{s-x} \int_s^{+\infty} dtv(t)\rho_P(t) = \lim_{M\to+\infty} \lim_{\varepsilon\to 0} \Big[-\frac{\ln\big((x-s)^2+\varepsilon^2\big)}{2} \int_s^{+\infty} dtv(t)\rho_P(t) \Big]_{-M}^{M} - \int_{\mathbb{R}} ds \ln|x-s| v(s)\rho_P(s)$$

To conclude that $\mathcal{W}\mathcal{A}^{-1}v(x) = -\int_{\mathbb{R}} ds \ln|x-s|v(s)\rho_P(s)$, we just need to show that

$$\ln(x) \int_x^{+\infty} dtv(t)\rho_P(t) \underset{|x|\to\infty}{\longrightarrow} 0$$

which can be seen by Cauchy-Schwarz inequality:

$$\Big| \ln(x) \int_x^{+\infty} dtv(t)\rho_P(t) \Big| \leqslant |\ln(x)| \|v\|_{L^2(\rho_P)} \cdot \rho_P(x)^{1/4} \Big(\int_{\mathbb{R}} \rho_P(t)^{1/2} dt \Big)^{1/2}.$$

In this inequality, we used that ρ_P is decreasing in a neighborhood of $+\infty$, hence

$$\ln|x| \int_s^{+\infty} dtv(t)\rho_P(t) \underset{x\to+\infty}{\longrightarrow} 0$$

the exact same argument allows us to conclude when x goes to $-\infty$. Using the fact that $\int_{\mathbb{R}} v(t)\rho_P(t)dt = 0$, we obtain the following equality:

$$v - 2P \int_{\mathbb{R}} ds \ln|x-s| v(s)\rho_P(s) = \mathcal{A}^{-1}g := h.$$

Now setting $\tilde{v}(t) = \rho_P^{1/2}(t)v(t)$ and $\tilde{h} = \rho_P^{1/2}(t)h(t)$, we obtain $\tilde{v} + \mathcal{K}[\tilde{v}] = \tilde{h}$ where $\mathcal{K}$ is defined in (56). Since its kernel (defined in (57)) belongs to $L^2(\mathbb{R}^2)$, $\mathcal{K}$ is Hilbert-Schmidt. Hence by Fredholm determinant theory:

$$\tilde{v} = \tilde{h} - \mathcal{R}[\tilde{h}]$$

or $\mathcal{L}^{-1}g = \mathcal{A}^{-1}g - \rho_P^{-1/2}\mathcal{R}\big[\rho_P^{1/2}\mathcal{A}^{-1}g\big]$ as expected. □

7 Regularity of the inverse of $\mathcal{L}$ and completion of the proof of Theorem 1.3

Since we have proven the central limit theorem for functions of the type $\mathcal{L}\phi$ with ϕ regular enough and satisfying vanishing asymptotic conditions at infinity, we exhibit a class of functions f for which $\mathcal{L}^{-1}f$ is regular enough to satisfy conditions of Theorem 5.2. We define $\mathcal{T}$ the subset of H defined by

$$\mathcal{T} := \left\{ f \in \mathcal{C}^1(\mathbb{R}), \exists \varepsilon > 0,\ f(x) = \underset{|x|\to\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right),\ f'(x) = \underset{|x|\to\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right), \int_{\mathbb{R}} f\rho_P = 0 \right\}$$

Theorem 7.1. *For all $f \in \mathcal{T}$, there exists a unique $u \in \mathcal{C}^3(\mathbb{R})$ such that $u' \in H^2(\mathbb{R})$ with $u^{(3)}$ bounded wich verifies:*

- $u'(x) = \underset{|x|\to\infty}{O}\left(\dfrac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\right)$
- $u''(x) = \underset{|x|\to\infty}{O}\left(\dfrac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\right)$
- $u^{(3)}(x) = \underset{|x|\to\infty}{O}\left(\dfrac{1}{x^{\frac{1}{2}+\varepsilon}}\right)$

such that $f = \mathcal{L}u$.

Proof. Let $f \in \mathcal{T} \subset \mathsf{H}$, then since $-\mathcal{L}$ is bijective from $\mathcal{D}(\mathcal{L}) \to \mathsf{H}$, there exists a unique $u \in \mathcal{D}(\mathcal{L})$ such that $-\mathcal{L}u = f$ *ie*:

$$-u'' - \frac{\rho_P'}{\rho_P}u' - 2P\mathcal{H}[u'\rho_P] = f \tag{59}$$

Hence we have

$$-(u'\rho_P)' = \rho_P\Big(f + 2P\mathcal{H}[u'\rho_P]\Big). \tag{60}$$

Since $u \in \mathcal{D}(\mathcal{L}) \subset \{u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}), \mathcal{A}u \in \mathsf{H}\}$, the functions $u'\rho_P$ and its distributional derivatives $(u'\rho_P)' = -\rho_P\mathcal{A}u$ and $(u'\rho_P)'' = -\dfrac{\rho_P'}{\rho_P}\rho_P^{1/2}.\Big(\rho_P^{1/2}\mathcal{A}u\Big) - \rho_P\Big(\mathcal{A}u\Big)'$ are in $L^2(\mathbb{R})$. In particular $u'\rho_P$ goes to zero at infinity, and $\mathcal{H}[u'\rho_P] \in H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$. So we can integrate (60) on $[x, +\infty[$, since by Lemma 2.3, the right-hand side behaves like a $\underset{|x|\to\infty}{O}\left(\dfrac{\rho_P(x)}{|x|^{\frac{1}{2}+\varepsilon}}\right)$, to get the following expression

$$u'(x)\rho_P(x) = \int_x^{+\infty} \frac{\rho_P(t)}{\rho_P'(t)}(f + 2P\mathcal{H}[u'\rho_P]).\rho_P'(t)dt \tag{61}$$

From this expression, we can see that $u' \in \mathcal{C}^2(\mathbb{R})$ so we just have to check the integrability condition at infinity and the fact that $u^{(3)}$ is bounded. After proceeding to an integration by parts, which is permitted by the previous argument, we obtain:

$$u'(x) = -\frac{\rho_P(x)}{\rho_P'(x)}\Big(f(x) + 2P\mathcal{H}[u'\rho_P](x)\Big) - \frac{1}{\rho_P(x)}\int_x^{+\infty}\left(\frac{\rho_P(t)}{\rho_P'(t)}(f + 2P\mathcal{H}[u'\rho_P])\right)'\rho_P(t)dt \tag{62}$$

and we define $R_1(x) := \dfrac{1}{\rho_P(x)}\displaystyle\int_x^{+\infty}\left(\frac{\rho_P(t)}{\rho_P'(t)}(f + 2P\mathcal{H}[u'\rho_P])\right)'\rho_P(t)dt$, we will need to show that R_1 is a remainder of order $\underset{x\to+\infty}{O}\left(\dfrac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\right)$ at infinity. In this case, we will have $u'(x) = \underset{x\to+\infty}{O}\left(\dfrac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\right)$ which will be useful for the following. If we reinject (62) in (59), we find:

$$u'' = -(f + 2P\mathcal{H}[u'\rho_P]) - \frac{\rho_P'}{\rho_P}\Big(-\frac{\rho_P}{\rho_P'}\Big(f + 2P\mathcal{H}[u'\rho_P]\Big) - R_1\Big) = \frac{\rho_P'}{\rho_P}R_1 \tag{63}$$

Hence

$$u''(x) = \frac{\rho_P'}{\rho_P^2}(x)\int_x^{+\infty}\rho_P(t)dt\Bigg\{\underbrace{\Big(\frac{\rho_P}{\rho_P'}\Big)'(t)}_{=\underset{t\to+\infty}{O}\left(\frac{V''(t)}{V'(t)^2}\right)}\underbrace{\big[f+2P\mathcal{H}[u'\rho_P]\big](t)}_{=\underset{t\to+\infty}{O}\left(t^{-\frac{1}{2}-\varepsilon}\right)}+\underbrace{\frac{\rho_P}{\rho_P'}(t)}_{=\underset{t\to+\infty}{O}\left(\frac{1}{V'(t)}\right)}\underbrace{\big[f'-2P\mathcal{H}[\rho_P\mathcal{A}u]\big](t)}_{=\underset{t\to+\infty}{O}\left(t^{-\frac{1}{2}-\varepsilon}\right)}\Bigg\}.$$

The fact that $\mathcal{H}[\rho_P\mathcal{A}u](t) = \underset{t\to+\infty}{O}(t^{-2})$ comes again from lemma 2.3. Finally we have that,

$$\begin{aligned}
u^{(3)}(x) &= \Big(\frac{\rho_P'}{\rho_P^2}\Big)'(x)\rho_P(x)R_1(x) - \Big(\frac{\rho_P'}{\rho_P^2}\Big)(x)\left(\frac{\rho_P}{\rho_P'}(f+2P\mathcal{H}[u'\rho_P])\right)'(x)\rho_P(x)\\
&= \underbrace{\Big(\frac{\rho_P''}{\rho_P}-2\frac{\rho_P'^2}{\rho_P^2}\Big)(x)}_{=\underset{x\to+\infty}{O}\left(V'(x)^2\right)}R_1(x) - \underbrace{\Big(\frac{\rho_P'}{\rho_P}\Big)(x)\left(\frac{\rho_P}{\rho_P'}(f+2P\mathcal{H}[u'\rho_P])\right)'(x)}_{=\underset{x\to+\infty}{O}\left(\frac{V''(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)}+x^{-\frac{1}{2}-\varepsilon}\right)}
\end{aligned}$$

The second term is $\underset{x\to+\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right)$ by the assumption that $\dfrac{V''}{V'}(x) = \underset{|x|\to\infty}{O}(1)$. Hence, we just have to check that $R_1(x) = \underset{x\to+\infty}{O}\left(\dfrac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\right)$ to establish that u', u'', $u^{(3)}$ are in $L^2(\mathbb{R})$. By a comparison argument, we control R_1 by controlling

$$I_1(x) := \int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)}dt$$

By integration by parts:

$$\begin{aligned}
I_1(x) &:= -\underbrace{\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)}\frac{\rho_P}{\rho_P'}(x)}_{=\underset{x\to+\infty}{O}\left(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\right)} - \int_x^{+\infty}\rho_P(t)\Big(\frac{1}{t^{\frac{1}{2}+\varepsilon}V'}\frac{\rho_P}{\rho_P'}\Big)'(t)dt\\
&= \underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big) - \int_x^{+\infty}\rho_P(t)dt\left\{\frac{-\frac{1}{2}-\varepsilon}{t^{\frac{3}{2}+\varepsilon}V'(t)}\frac{\rho_P}{\rho_P'}(t) - \frac{V''(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^2}\frac{\rho_P}{\rho_P'} + \frac{1}{t^{\frac{1}{2}+\varepsilon}V'(t)}\Big(\frac{\rho_P}{\rho_P'}\Big)'(t)\right\}
\end{aligned} \tag{64}$$

By the same argument as before, the last integral is of the form $\displaystyle\int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^2}\Big)dt$ so if $I_2(x) := \displaystyle\int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^2}dt = \underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big)$ then so is I_1. By integration by parts, we obtain:

$$\begin{aligned}
I_2(x) &= \rho_P(x)\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\frac{\rho_P}{\rho_P'}(x) - \int_x^{+\infty}\rho_P(t)dt\left\{\Big(\frac{\rho_P}{\rho_P'}\Big)'(t)\frac{1}{t^{\frac{1}{2}+\varepsilon}V'(t)^2} - \frac{\rho_P}{\rho_P'}(t)\Big(\frac{1}{t^{\frac{3}{2}+\varepsilon}V'(t)^2}+\frac{2V''(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^3}\Big)\right\}\\
&= \underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big) + \int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^3}\Big)dt
\end{aligned}$$

The last integral is a $\underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac12+\varepsilon}V'(x)^2}\Big)$ because, again, by integration by part:

$$\int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac12+\varepsilon}V'(t)^3}dt=\rho_P(x)\frac{1}{x^{\frac12+\varepsilon}V'(x)^3}\frac{\rho_P}{\rho_P'}(x)-\int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_P(t)}{t^{\frac12+\varepsilon}V'(t)^4}\Big)$$

and finally

$$\int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac12+\varepsilon}V'(t)^4}dt\leqslant\frac{\rho_P(x)}{x^{\frac12+\varepsilon}V'(x)^2}\int_x^{+\infty}\frac{dt}{V'(t)^2}=\underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac12+\varepsilon}V'(x)^2}\Big)$$

In the final step, we used the fact that $x\mapsto\frac{\rho_P(x)}{x^{\frac12+\varepsilon}V'(x)}$ is decreasing in a neighborhood of $+\infty$ (which can be checked by differentiating) and that $x\mapsto\frac{1}{V'(x)^2}$ is integrable at ∞ by assumption *iv)*. Hence $R_1(x)=\underset{x\to+\infty}{O}\Big(\frac{1}{x^{\frac12+\varepsilon}V'(x)^2}\Big)$ (the exact same result can be shown at $-\infty$), which leads to the fact

$$u'(x)=\underset{|x|\to+\infty}{O}\Big(\frac{1}{x^{\frac12+\varepsilon}V'(x)}\Big),\qquad u''(x)=\underset{|x|\to+\infty}{O}\Big(\frac{1}{x^{\frac12+\varepsilon}V'(x)}\Big)\quad\text{and}\quad u^{(3)}(x)=\underset{|x|\to+\infty}{O}\Big(\frac{1}{x^{\frac12+\varepsilon}}\Big)\tag{65}$$

which establishes that these functions are in L^2 in a neighborhood of ∞. Since we already showed that $u\in\mathcal{C}^3(\mathbb{R})\subset H^3_{\text{loc}}(\mathbb{R})$, it establishes that $u\in H^3(\mathbb{R})\cap\mathcal{C}^3(\mathbb{R})$ with $u^{(3)}$ bounded. To complete the proof we just have to show that $(u')^2V^{(3)}$, $u'u''V''$, $(u')^2V''$ and $u'V'$ are bounded which is easily checked by (65) and Assumption 1.1 *iv)*. □

Remark 7.2. *We choose here the functions that vanishes at infinity at worst like $|x|^{-1/2-\varepsilon}$, but functions like $x\mapsto|x|^{-1/2}\ln^{-1/2-\varepsilon}|x|$ or $x\mapsto|x|^{-1/2}\ln^{-1/2}|x|\ln^{-1/2-\varepsilon}\ln|x|$ also work, the proof being the same. The only hypotheses that we use is that $f\in H^1(\mathbb{R})\cap\mathcal{C}^1(\mathbb{R})$, that $f'=\underset{|x|\to+\infty}{O}(f(x))$ and that f is decreasing (resp. increasing) in a neighborhood of + (resp.-) ∞.*

A Appendix: proof of Theorem 6.2

In order to analyze $\mathcal{A}$, we let, for $u\in L^2(\mathbb{R})$,

$$\mathcal{S}u:=\rho_P^{1/2}\mathcal{A}\rho_P^{-1/2}u\,.$$

Note that $u\in\big(L^2(\mathbb{R}),\|.\|_{L^2(dx)}\big)\mapsto\rho_P^{-1/2}u\in\big(L^2(\rho_P),\|.\|_{L^2(\rho_P)}\big)$ is an isometry. It turns out that it will be easier to study first the operator $\mathcal{S}$ in order to get the spectral properties of $\mathcal{A}$.

Proposition A.1. *The operator $\mathcal{S}$ is a Schrödinger operator: it admits the following expression for all $u\in C_c^2(\mathbb{R})$: $\mathcal{S}u=-u''+w_Pu$ with*

$$w_P=\frac12\left(\frac12V'^2-V''+2PV'\mathcal{H}[\rho_P]-2P\mathcal{H}[\rho_P']+2P^2\mathcal{H}[\rho_P]^2\right)=\frac12\Big[(\ln\rho_P)''+\frac12(\ln\rho_P)'^2\Big]\,.$$

Furthermore, w_P is continuous and we have $w_P(x)\underset{\infty}{\sim}\frac{V'(x)^2}{4}\underset{|x|\to\infty}{\longrightarrow}+\infty$.

Proof. We compute directly

$$\begin{aligned}
\frac{\Big(\rho_P\big(\rho_P^{-1/2}u\big)'\Big)'}{\rho_P} &= \big(\rho_P^{-1/2}u\big)'' + \frac{\rho_P'}{\rho_P}\big(\rho_P^{-1/2}u\big)' \\
&= \big(\rho_P^{-1/2}u' - \frac{1}{2}\rho_P^{-3/2}\rho_P'u\big)' + \rho_P'\rho_P^{-3/2}u' - \frac{1}{2}\rho_P^{-5/2}\big(\rho_P'\big)^2u \\
&= \rho_P^{-1/2}u'' + \frac{1}{4}\rho_P^{-5/2}\big(\rho_P'\big)^2u - \frac{1}{2}\rho_P^{-3/2}\rho_P''u \\
&= \rho_P^{-1/2}\Big[u'' + \frac{1}{4}\rho_P^{-2}\big(\rho_P'\big)^2u - \frac{1}{2}\rho_P^{-1}\rho_P''u\Big] \\
&= \rho_P^{-1/2}\left(u'' - \frac{1}{2}\Big[\big(\frac{\rho_P''}{\rho_P}\big) - \frac{1}{2}\big(\frac{\rho_P'}{\rho_P}\big)^2\Big]u\right) \\
&= \rho_P^{-1/2}\left(u'' - \frac{1}{2}\Big[(\ln\rho_P)'' + \frac{1}{2}(\ln\rho_P)'^2\Big]u\right) = \rho_P^{-1/2}\Big(u'' - w_Pu\Big).
\end{aligned}$$

Now, using Lemma 2.2, we have

$$w_P = \frac{1}{2}\left(\frac{1}{2}V'^2 - V'' + 2PV'\mathcal{H}[\rho_P] - 2P\mathcal{H}[\rho_P'] + 2P^2\mathcal{H}[\rho_P]^2\right).$$

Notice that $\mathcal{H}[\rho_P']$ and $\mathcal{H}[\rho_P]$ are bounded since they belong to $H^1(\mathbb{R})$, as we showed in Lemma 2.2 that ρ_P is $H^2(\mathbb{R})$. Along with Assumption 1.1 *iii)* and Lemma 2.3, we deduce $w_P(x) \underset{\infty}{\sim} \frac{1}{4}V'^2(x)$. □

Remark A.2. *Note that the function w_P need not be positive on $\mathbb{R}$. In fact, neglecting the terms involving the Hilbert transforms of ρ_P and ρ_P', w_P would only be positive outside of a compact set. However, using the positivity of $\mathcal{A}$, which will be shown further in the article, we can show that the operator $-u'' + w_Pu$ is itself positive on $L^2(\mathbb{R})$. It can also be checked that, by integration by parts, $\mathcal{S}$ is self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R})$ with the inner product of $L^2(\mathbb{R})$.*

We now introduce an extension of $\mathcal{S}$ by defining its associated bilinear form.

Definition A.3 (Quadratic form associated to $\mathcal{S}$)**.**
Let $\alpha \geqslant 0$ such that $w_P + \alpha \geqslant 1$. We define the quadratic form associated to $\mathcal{S} + \alpha I$, defined for all $u \in \mathcal{C}_c^\infty(\mathbb{R})$ by

$$q_\alpha(u,u) := \int_{\mathbb{R}} u'^2dx + \int_{\mathbb{R}} u^2(w_P + \alpha)dx$$

This quadratic form can be extended to a larger domain denoted by $Q(\mathcal{S}+\alpha I)$, called the *form domain* of the operator $\mathcal{S} + \alpha I$. By the theory of Schrödinger operators, it is well-known (see [Dav96][Theorem 8.2.1]) that such a domain is given by

$$Q(\mathcal{S} + \alpha I) = \Big\{u \in H^1(\mathbb{R}), u(w_P + \alpha)^{1/2} \in L^2(\mathbb{R})\Big\} = \big\{u \in H^1(\mathbb{R}), uV' \in L^2(\mathbb{R})\big\} =: H^1_{V'}(\mathbb{R}).$$

The space $H^1_{V'}(\mathbb{R})$ can be seen to be the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ under the norm q_α. Now that the quadratic form associated to $\mathcal{S}+\alpha I$ has been extended to its form domain, it is possible to go back to the operator and extend it by its Friedrichs extension.

Theorem A.4 (Friedrichs extension of $\mathcal{S} + \alpha I$)**.**
There exists an extension $(S + \alpha I)_F$ of the operator $\mathcal{S} + \alpha I$, called the Friedrichs extension of $\mathcal{S} + \alpha I$ defined on $\mathcal{D}\Big((S + \alpha I)_F\Big) = \Big\{u \in H^1_{V'}(\mathbb{R}), -u'' + (w_P + \alpha)u \in L^2(\mathbb{R})\Big\}$.

Proof. We denote

$$\mathcal{D}\Big((S+\alpha I)_F\Big)$$
$$:=\Big\{v\in H^1_{V'}(\mathbb{R}), u\in H^1_{V'}(\mathbb{R})\longmapsto q_\alpha(u,v) \text{ can be extended to a continuous linear form on } L^2(\mathbb{R})\Big\}$$

If $v\in\mathcal{D}\Big((S+\alpha I)_F\Big)$, by Riesz's theorem there exists a unique $f_v\in L^2(\mathbb{R})$ such that $q_\alpha(u,v)=\langle u,f_v\rangle_{L^2(dx)}$ holds for all $u\in L^2(\mathbb{R})$ and we can set $(S+\alpha I)_F v:=f_v$. Note that it is indeed a way of extending $\mathcal{S}+\alpha I$ since for all $u,v\in\mathcal{C}_c^\infty(\mathbb{R})$, $q_\alpha(u,v)=\langle u,(\mathcal{S}+\alpha I)v\rangle_{L^2(dx)}$.

We want to show that $\mathcal{D}\Big((S+\alpha I)_F\Big)=\Big\{u\in H^1_{V'}(\mathbb{R}), -u''+(w_P+\alpha)u\in L^2(\mathbb{R})\Big\}$. Let $f\in\mathcal{D}\Big((S+\alpha I)_F\Big)$ and $g:=(S+\alpha I)_F f\in L^2(\mathbb{R})$. By definition of q_α, for all $u\in\mathcal{C}_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} gudx=\int_{\mathbb{R}} f'u'dx+\int_{\mathbb{R}}(w_P+\alpha)fudx=-\int_{\mathbb{R}} fu''dx+\int_{\mathbb{R}}(w_P+\alpha)fudx$$

Therefore in the sense of distributions, we get $-f''+(w_P+\alpha)=g$ which is a function in $L^2(\mathbb{R})$, hence $f\in\Big\{u\in H^1_{V'}(\mathbb{R}), -u''+(w_P+\alpha)u\in L^2(\mathbb{R})\Big\}$. Conversely, if $f\in H^1_{V'}(\mathbb{R})$ such that $-f''+(w_P+\alpha)f\in L^2(\mathbb{R})$, it is possible to extend $u\mapsto q_\alpha(f,u)$ to a continuous linear form on $L^2(\mathbb{R})$ by

$$u\mapsto\int_{\mathbb{R}} u\Big(-f''+f(w_P+\alpha)\Big)dx$$

which concludes the fact that $\mathcal{D}\Big((S+\alpha I)_F\Big)=\Big\{u\in H^1_{V'}(\mathbb{R}), -u''+(w_P+\alpha)u\in L^2(\mathbb{R})\Big\}$. □

In the following, we will deal only with $(\mathcal{S}+\alpha I)_F:\mathcal{D}\Big((S+\alpha I)_F\Big)\longrightarrow L^2(\mathbb{R})$ and denote it $\mathcal{S}+\alpha I:\mathcal{D}(\mathcal{S}+\alpha I)$.

Remark A.5. *Note that in the previous proof, the application of Riesz's theorem doesn't allow to say that* $(\mathcal{S}+\alpha I):v\in\Big(\mathcal{D}(S+\alpha I),\|.\|_{q_\alpha}\Big)\mapsto f_v\in\Big(L^2(\mathbb{R}),\|.\|_{L^2(dx)}\Big)$*, where* $\|.\|_{q_\alpha}$ *stands for the norm associated to the bilinear positive definite form* q_α*, is continuous. It can be seen by the fact that* $v\in\Big(\mathcal{D}(S+\alpha I),\|.\|_{q_\alpha}\Big)\mapsto q(.,v)\in\Big(L^2(\mathbb{R})',\|.\|_{L^2(dx)'}\Big)$*, where* $L^2(\mathbb{R})'$ *stands for the topological dual of* $L^2(\mathbb{R})$ *equipped with its usual norm, is not continuous. Indeed the* $\|.\|_{q_\alpha}$ *norm doesn't control the second derivative of* v *and hence doesn't provide any module of continuity for the* $L^2(\mathbb{R})$*-extended linear form* $q(.,v)$*.*

Also note that, even though it would be convenient that $\mathcal{D}\Big((S+\alpha I)_F\Big)=L^2(\mathbb{R},(w_P+\alpha)^2dx)\cap H^2(\mathbb{R})$ *it is not true without more properties on* w_P*. Such a result holds, for example when* w_P *belongs to* B_2*, the class of reverse Hölder weights, see [ABA07][Theorem 1.1].*

Theorem A.6 (Inversion of $\mathcal{S}+\alpha I$)**.**
For every $f\in L^2(\mathbb{R})$*, there exists a unique* $u\in\mathcal{D}\Big((S+\alpha I)_F\Big)$ *such that* $(\mathcal{S}+\alpha I)u=f$*. Furthermore, the map* $(S+\alpha I)^{-1}$ *is continuous from* $\big(L^2(\mathbb{R}),\|.\|_{L^2(dx)}\big)$ *to* $\big(\mathcal{D}(S+\alpha I),\|.\|_{q_\alpha}\big)$*.*

Proof. Let $f\in L^2(\mathbb{R})$, the map $u\longmapsto\langle u,f\rangle_{L^2(dx)}$ is continuous on $\big(H^1_{V'}(\mathbb{R}),\|.\|_{q_\alpha}\big)$ which is a Hilbert space. Therefore by Riesz's theorem, there exists a unique $v_f\in H^1_{V'}(\mathbb{R})$ such that for all $u\in H^1_{V'}(\mathbb{R})$, $\langle f,u\rangle_{L^2(dx)}=q_\alpha(v_f,u)$ from which we deduce that, in the sense of distributions, $f=-v_f''+(w_P+\alpha)v_f$ which implies that $v_f\in\mathcal{D}(S+\alpha I)$. Since $v_f\in\mathcal{D}(S+\alpha I)$, we have then for all $u\in L^2(\mathbb{R})$, $\langle f,u\rangle_{L^2(dx)}=q_\alpha(v_f,u)=\langle(\mathcal{S}+\alpha)v_f,u\rangle_{L^2(dx)}$, hence $(\mathcal{S}+\alpha I)v_f=f$. Finally, by Riesz's theorem, $f\in L^2(\mathbb{R})\mapsto v_f\in H^1_{V'}(\mathbb{R})$ is continuous hence so is $(\mathcal{S}+\alpha I)^{-1}$. □

Remark A.7. *It would be tempting to use Banach's isomorphism theorem to say that since $(\mathcal{S}+\alpha I)^{-1}$ is bijective and continuous, so must be $\mathcal{S}+\alpha I$. But since $\big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\big)$ is not a Banach space (it's not closed in $H^1_{V'}(\mathbb{R})$) we can't apply it.*

We are now able to diagonalize the resolvent of $\mathcal{S}$.

Theorem A.8 (Diagonalization of $(\mathcal{S}+\alpha I)^{-1}$)**.**
There exists a complete orthonormal set $(\psi_n)_{n\geqslant 0}$ of $L^2(\mathbb{R})$ (meaning that

$$\overline{\text{span}\{\psi_n,\ n\geqslant 0\}}^{\|.\|_{L^2(dx)}} = L^2(\mathbb{R})$$

and $\langle \psi_i, \psi_j\rangle_{L^2(dx)} = \delta_{i,j}$), where each $\psi_n \in \mathcal{D}(S+\alpha I)$ and $\big(\mu_n(\alpha)\big)_{n\geqslant 0} \in [0,1]^{\mathbb{N}}$ with $\mu_n(\alpha) \underset{N\to\infty}{\longrightarrow} 0$ such that $(\mathcal{S}+\alpha I)^{-1}\psi_n = \mu_n(\alpha)\psi_n$ for all $n\geqslant 0$. We also have $\Big|\Big|\Big|\big(\mathcal{S}+\alpha I\big)^{-1}\Big|\Big|\Big|_{\mathcal{L}\big(L^2(dx)\big)} \leqslant 1$.

Proof. By Proposition A.1, $w_P+\alpha$ is continuous and goes to infinity at infinity. By Rellich criterion [RS78][see Theorem XIII.65], the unit ball of $\mathcal{D}(S+\alpha I)$, *ie* the set

$$\Big\{u\in\mathcal{D}(S+\alpha I),\ \int_{\mathbb{R}} u'^2 + \int_{\mathbb{R}} (w_P+\alpha)u^2 \leqslant 1\Big\}$$

considered as a subset of $L^2(\mathbb{R})$ is relatively compact in $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$. Hence, we can conclude that the injection $\iota : \big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\big) \longrightarrow \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ is a compact operator. Since $(S+\alpha I)^{-1} : \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big) \longrightarrow \big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\big)$ is continuous then $(\mathcal{S}+\alpha I)^{-1}$ is compact from $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ to itself. The fact that $(\mathcal{S}+\alpha I)^{-1}$ is self-adjoint and positive allows us to apply the spectral theorem to obtain $\big(\mu_n(\alpha)\big)_{n\geqslant 0}$ positive eigenvalues verifying $\mu_n(\alpha) \underset{N\to\infty}{\longrightarrow} 0$ by compactness and a Hilbertian basis $(\psi_n)_{n\geqslant 0} \in L^2(\mathbb{R})^{\mathbb{N}}$, such that for all $n\geqslant 0$, $(\mathcal{S}+\alpha I)^{-1}\psi_n = \mu_n(\alpha)\psi_n$. It is then easy to see that for all n, $\psi_n \in \mathcal{D}(S+\alpha I)$ since they belong to the range of $(\mathcal{S}+\alpha I)^{-1}$. Finally, since for all $\phi\in L^2(\mathbb{R})$, $\langle(\mathcal{S}+\alpha I)\phi,\phi\rangle_{L^2(dx)} \geqslant \|\phi\|^2_{L^2(dx)}$, the spectrum of $(\mathcal{S}+\alpha I)^{-1}$ is contained in $[0,1]$. It allows us to conclude that $\Big|\Big|\Big|(\mathcal{S}+\alpha I)^{-1}\Big|\Big|\Big|_{L^2(dx)} \leqslant 1$. □

Since for all $u\in H^1_{V'}(\mathbb{R})$, $(S+\alpha)u \in L^2(\mathbb{R})$ *iff* $\mathcal{S}u\in L^2(\mathbb{R})$, if we define $\mathcal{D}(\mathcal{S})$ in the same manner that we did before, $\mathcal{D}(\mathcal{S}) = \mathcal{D}(S+\alpha I)$. It is now straightforward to see how to extend $\mathcal{A} = \rho_P^{-1/2}\mathcal{S}\rho_P^{1/2}$ on $\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) := \rho_P^{-1/2}\mathcal{D}(\mathcal{S})$ equipped with the norm $\|.\|_{q_\alpha,\rho_P}$ to $\big(L^2(\rho_P), \|.\|_{L^2(\rho_P)}\big)$. The norm $\|.\|_{q_\alpha,\rho_P}$ is defined for all $u\in\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ by

$$\|u\|_{q_\alpha,\rho_P} = \int_{\mathbb{R}} u'^2\rho_P dx + \int_{\mathbb{R}} u^2(w_P+\alpha)\rho_P dx\,.$$

It is easy to see that $(\mathcal{A}+\alpha I)^{-1}$ is continuous.

Remark A.9. *The kernel of $\mathcal{A}$ is generated by the function $\widetilde{1}$. Indeed if $\phi\in\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ is in the kernel of $\mathcal{A}$ then*

$$0 = -\frac{\big(\phi'\rho_P\big)'}{\rho_P} \Rightarrow \exists c\in\mathbb{R},\ \phi' = \frac{c}{\rho_P}$$

But since ϕ' is in $L^2(\rho_P)$ then $c=0$ which implies that ϕ is constant. We must restrict $\mathcal{A}$ to the orthogonal of $Ker\mathcal{A}$ with respect to the inner product of $L^2(\rho_P)$, ie

$$\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) := \Big\{u\in\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})\,|\,\int_{\mathbb{R}} u\rho_P = 0\Big\}\,.$$

Doing so makes $\mathcal{A}$ injective.

Before inverting $\mathcal{A}$, we need the following lemma:

Lemma A.10. *The following equality holds*

$$(\mathcal{A}+\alpha I)\Big(\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})\Big) = L^2_0(\rho_P) := \Big\{u \in L^2(\rho_P),\ \int_{\mathbb{R}} u\rho_P dx = 0\Big\}$$

Proof. Let $\phi = \widetilde{c}$ for $c \in \mathbb{R}$, $(\mathcal{A}+\alpha I)\phi = \widetilde{\alpha c}$ then $(\mathcal{A}+\alpha I)(\mathbb{R}.\widetilde{1}) = \mathbb{R}\widetilde{1}$. Hence since $\mathcal{A}+\alpha I$ is self-adjoint with respect to the inner product of $L^2(\rho_P)$ and that $\mathbb{R}\widetilde{1}$ is stable by $\mathcal{A}+\alpha I$, then $(\mathcal{A}+\alpha I)\Big((\mathbb{R}.\widetilde{1})^{\perp} \cap \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})\Big) \subset (\mathbb{R}.\widetilde{1})^{\perp}$. For the converse, let $u \in (\mathbb{R}.\widetilde{1})^{\perp}$, since $\mathcal{A}+\alpha I$ is bijective, there exists $v \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ such that $u = (\mathcal{A}+\alpha I)v$. For all $w \in \mathbb{R}.\widetilde{1}$,

$$0 = \langle u, w\rangle_{L^2(\rho_P)} = \langle (\mathcal{A}+\alpha I)v, w\rangle_{L^2(\rho_P)} = \langle v, (\mathcal{A}+\alpha I)w\rangle_{L^2(\rho_P)}$$

Hence $v \in \big[(\mathcal{A}+\alpha I)(\mathbb{R}\widetilde{1})\big]^{\perp} = \mathbb{R}\widetilde{1}^{\perp}$ and so $(\mathbb{R}.\widetilde{1})^{\perp} \subset (\mathcal{A}+\alpha I)\Big((\mathbb{R}.\widetilde{1})^{\perp}\Big)$. □

It is easy to see that $L^2_0(\rho_P)$ is a closed subset of $L^2(\rho_P)$ as it is the kernel of the linear form $\phi \in L^2(\rho_P) \mapsto \Big\langle \phi, \widetilde{1}\Big\rangle_{L^2(\rho_P)}$, making it a Hilbert space.

Proposition A.11 (Diagonalization and invertibility of $\mathcal{A}$)**.** *There exists a complete orthonormal set of* $\Big(L^2_0(\rho_P), \langle .,.\rangle_{L^2(\rho_P)}\Big)$, $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})^{\mathbb{N}}$ *such that* $\mathcal{A}\phi_n = \lambda_n\phi_n$ *(meaning that*

$$\overline{\mathrm{span}\{\phi_n,\ n \geqslant 0\}}^{\|.\|_{L^2(\rho_P)}} = L^2_0(\rho_P)$$

and $\langle \phi_i, \phi_j\rangle_{L^2(\rho_P)} = \delta_{i,j}$*). Furthermore,* $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \longrightarrow L^2_0(\rho_P) := \Big\{u \in L^2(\rho_P),\ \int_{\mathbb{R}} u\rho_P dx = 0\Big\}$ *is bijective,* $\mathcal{A}^{-1}$ *is continuous when considered as an operator of* $L^2_0(\rho_P)$*.*

Proof. Since $(\mathcal{S}+\alpha I)^{-1}$ considered as an operator of $L^2(\mathbb{R})$, is compact so is $(\mathcal{A}+\alpha I)^{-1}$ on $L^2(\rho_P)$ and since $\mathcal{A}$ is self-adjoint, by the spectral theorem, $(\mathcal{A}+\alpha I)^{-1}$ is diagonalizable. With the notations of Theorem A.8, $(\mathcal{A}+\alpha I)^{-1}$ has eigenvalues $\big(\mu_n(\alpha)\big)_{n\geqslant 0}$ and corresponding eigenfunctions $\phi_n = \rho_P^{-1/2}\psi_n \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$. Hence for all $n \in \mathbb{N}$, $\mathcal{A}\phi_n = \lambda_n\phi_n$ with $\lambda_n := \big(\dfrac{1}{\mu_n(\alpha)} - \alpha\big)$. Now,

$$\lambda_n\|\phi_n\|_{L^2(\rho_P)} = \int_{\mathbb{R}} (\mathcal{A}\phi_n)\phi_n\rho_P dx = -\int_{\mathbb{R}} (\rho_P\phi_n')'\phi_n = \int_{\mathbb{R}} \phi_n'^2\rho_P \geqslant 0\,.$$

Furthermore, the kernel of $\mathcal{A}$ is $\mathbb{R}.\widetilde{1}$, thus the spectrum of $\mathcal{A}$ restricted to $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ is positive. But since $(\mathcal{A}+\alpha I)^{-1}$ is a compact operator of $L^2(\rho_P)$ and that $(\mathcal{A}+\alpha I)$ maps $\mathbb{R}.\widetilde{1}^{\perp}$ to $\mathbb{R}.\widetilde{1}^{\perp}$ with respect to the inner product of $L^2(\rho_P)$ (see lemma A.10), then $\big(\mathcal{A}+\alpha I\big)^{-1}$ is compact as an operator from $L^2_0(\rho_P)$ to itself. By Fredholm alternative, for every $\lambda \in \mathbb{R}$ $\lambda \neq 0$, either $(\mathcal{A}+\alpha I)^{-1} - \lambda I$ is bijective either $\lambda \in Sp\big((\mathcal{A}+\alpha I)^{-1}\big)$. These conditions are equivalent to: either $\mathcal{A} + (\alpha - \dfrac{1}{\lambda})I$ is bijective as an operator from $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ to $L^2_0(\rho_P)$, either $-\alpha + \dfrac{1}{\lambda} \in Sp\big(\mathcal{A}\big)$. If we set $\lambda = \dfrac{1}{\alpha}$ then either $\mathcal{A}$ is bijective either $0 \in Sp(\mathcal{A})$, since the latter is wrong then $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \to L^2_0(\rho_P)$ is bijective. The spectrum of $\mathcal{A}$ is $\left(\dfrac{1}{\mu_n(\alpha)} - \alpha\right)_{n\geqslant 0} \subset (\lambda_1, +\infty) \subset (0, +\infty)$, where λ_1 is the smallest eigenvalue, hence we deduce that $|||\mathcal{A}^{-1}|||_{\mathcal{L}(L^2(\rho_P))} \leqslant \lambda_1^{-1}$. □

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