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EXISTENCE OF SOLUTIONS FOR A CLASS OF ONE-DIMENSIONAL MODELS OF PEDESTRIAN EVACUATIONS

BORIS ANDREIANOV* AND THEO GIRARD †

Abstract. Pedestrian evacuation in a corridor can be described mathematically by different variants of the model introduced by R. L. Hughes (Transp. Res. B, 2002). We identify a class of such models for which existence of a solution is obtained via a topological fixed point argument. In these models, the dynamics of the pedestrian density ρ (governed by a discontinuous-flux Lighthill, Whitham and Richards model $\rho_t + (\text{sign}(x - \xi(t))\rho v(\rho))_x = 0$) is coupled to the computation of a Lipschitz continuous “turning curve” ξ . We illustrate this construction by several examples, including the Hughes model with affine cost (a variant of the original problem that is encompassed in the framework of El Khatib et al. (ZAMP, 2013)). Existence holds either with open-end boundary conditions or with boundary conditions corresponding to panic behaviour with capacity drop at exits. Other examples put forward versions of the Hughes model with inertial dynamics of the turning curve and with general costs.

Key words. crowd dynamics, pedestrian evacuation, Hughes’ model, capacity drop, existence, Schauder fixed-point, admissible solution, discontinuous-flux conservation law, memory, relaxation

MSC codes. 35L65, 47H10

1. Introduction.

1.1. The Hughes model and its variants. In [21], Hughes proposed a model for pedestrian motion as a system of two equations. The first equation is a scalar conservation law on the pedestrian density ρ . We normalize this density by taking $[0, 1]$ as the range of values for ρ . The second equation is an Eikonal equation on unknown ϕ seen as a potential for the directional field governing the evacuation process. This model has been studied by many authors and we defer to [1] for an overview on the subject. In [2], the 1D Hughes model has been reformulated in terms of a “turning curve” $\xi = \xi(t)$ (denoted $x_m(t)$ in [2]) instead of the potential $\phi = \phi(t, x)$. Following the approach of [17], our prototype model in the sequel will be:

$$\begin{aligned} (1.1a) \quad & \rho_t + [\text{sign}(x - \xi(t))\rho v(\rho)]_x = 0 \\ (1.1b) \quad & \rho(0, \cdot) = \rho_0(\cdot) \\ (1.1c) \quad & \int_{-1}^{\xi(t)} c(\rho(t, x)) \, dx = \int_{\xi(t)}^1 c(\rho(t, x)) \, dx, \end{aligned}$$

with ρ defined for $t \in [0, T]$, $T > 0$, and $x \in \mathbb{R}$ and with initial datum of the form:

$$(1.2) \quad \rho_0 \in L^\infty(\mathbb{R}; [0, 1]), \quad \rho(x) = 0 \text{ for } x \notin (-1, 1);$$

the spatial subdomain $(-1, 1)$ represents the corridor that the agents want to evacuate, as if a fire alarm was ringing at $t = 0$. In (1.1), $v \geq 0$ stands for the speed of the traffic. The pedestrians move forward (with the positive flux $\rho \mapsto \rho v(\rho)$) or backward (with $\rho \mapsto \rho v(\rho)$) depending of the sign of $x - \xi(t)$. Also, c denotes a generic running cost function. The original Hughes’ model deals with the cost function $c(\rho) = \frac{1}{v(\rho)}$. However, inspired by the setting of [17], we will consider a cost verifying the following conditions:

$$(1.3) \quad \begin{cases} c \in W^{1,\infty}([0, 1]), \\ \forall \rho \in [0, 1], c(\rho) \geq 1, \\ c \text{ is non-decreasing on } [0, 1]. \end{cases}$$

Remark 1.1. It might seem strange that we consider $x \in \mathbb{R}$ for equation (1.1a) whereas equation (1.1c) takes into account the density supported in $[-1, 1]$. It is more frequent to consider instead the problem with

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boundary conditions at $x = \pm 1$,

$$(1.4) \quad \begin{cases} \rho_t + [\text{sign}(x - \xi(t))\rho v(\rho)]_x = 0 & \text{in } (-1, 1) \\ \rho(t, x = \pm 1) = 0 \\ \rho(0, \cdot) = \tilde{\rho}_0(\cdot) \in L^\infty((-1, 1)), \end{cases}$$

where the Dirichlet condition is taken in the sense of Bardos-Leroux-Nedelec (BLN in the following, see [11], [15]). However, it is shown in [10, Appendix] that one can replace the initial-boundary value problem (1.4) with $\tilde{\rho}_0 : (-1, 1) \rightarrow [0, 1]$ by the pure initial-value problem for (1.1a) with the extended datum $\rho_0 : \mathbb{R} \rightarrow [0, 1]$ (where $\rho_0 = \tilde{\rho}_0$ on $(-1, 1)$ and 0 elsewhere). We adopt this viewpoint and require, throughout the paper, the assumption (1.2). This assumption for the conservation law (1.1a) set up in the whole space can be seen as “open-end condition” at exits. We refer to Section 4 for models with more involved exit behavior.

In (1.1), ρ is considered to be an entropy solution to (1.1a)-(1.1b). Such notion of solution with a particular attention to the admissibility of the jump of ρ across the turning curve $x = \xi(t)$ was proposed in [17] (we will slightly simplify this solution notion in the sequel). On the other hand, ξ is a pointwise defined solution to (1.1c), whose existence in $L^\infty((0, T))$ and whose uniqueness follows from the intermediate values theorem under the conditions (1.3).

For notation’s sake, we consider f a generic concave positive flux such that $f(0) = f(1) = 0$. One can assume $f(\rho) = \rho v(\rho)$ to recover the Lighthill-Whitham and Richards (LWR) model (see [22], [24]). In this paper, we will consider a class of “turning curve” model’s generalisations,

$$\begin{aligned} (1.5a) & \quad \left\{ \begin{array}{l} \rho_t + [\text{sign}(x - \xi(t))f(\rho)]_x = 0 \\ \rho(0, x) = \rho_0(x) \\ \xi = \mathcal{I}(\rho). \end{array} \right. \\ (1.5b) & \\ (1.5c) & \end{aligned}$$

Here \mathcal{I} is an abstract operator mapping the density ρ to a turning curve ξ . The problem (1.1) is a particular case of (1.5) where \mathcal{I} is the solver of the integral equation (1.1c); we pay a particular attention to this case. Stating (1.5b), we mean that ρ_0 fulfills (1.2) which corresponds to open-end evacuation at exits, as discussed above.

1.2. The state of the art and the scope of the paper. Let us briefly discuss known results on the specific problem (1.1) and its variants. The survey [1] provides an extensive recent reference on the subject. First of all, uniqueness results for the coupled problem are not known, except for very particular data (see [1]). We do not address this issue in our work. In [17], uniqueness of ρ with a fixed ξ is proven (we will revisit this result in Section 2 in order to apply the fixed point argument).

In [3], global existence for Hughes’ model (with $c(\rho) = \frac{1}{v(\rho)}$) is proven under subtle assumptions that ensure that the density is zero at the turning curve for all times. In [28], [19] and [20] one can find numerical studies of the model. We defer to [1] for more references. Proof of existence and uniqueness for a regularized problem can be found in [16]. In [13], the Hughes model is also revisited with different turning curve equation. Note that in [13], the authors introduce a regularization by convolution of the density. We use the same type of idea in several situations.

The only general (with respect to the choice of the initial data) existence result is contained in [10], where solutions with BV_{loc} regularity away from the turning curve were constructed via a well-chosen many-particle approximation. The result of [10] for problem (1.1) is limited to the case of an affine cost $c(\rho) = 1 + \alpha\rho$. Our result for the original setting (1.1) will also be limited to the affine cost case. However we provide a shorter and less specific argument, compared to the many-particle approximation of [10], also we require fewer assumptions on the velocity profile v compared to [10].

The fixed-point approach we develop appears to be rather flexible since it permits to handle several models of the form (1.5). Furthermore, we adapt the arguments to exit behavior of the “capacity drop” kind (cf. [9, 8]) in the place of the “open-end” boundary behaviour implicitly contained in (1.5). This accounts for more realistic scenarios in crowd evacuation. However, we highlight the fact that our approach is restricted to situations where Lipschitz continuity of the turning curve ξ is guaranteed for the model at hand, which

appears to be a strong restriction on its applicability; this restriction also appears in [10]. We defer to Section 5 for a more detailed discussion of this important issue.

1.3. The main result. In this paper we propose an existence result for problem (1.5) elaborated through a fixed-point argument under a set of assumptions on the operator \mathcal{I} . This leads to several existence results for concrete models. In order to make these existence results precise, we present the notions of solution we use throughout this paper. First, we need to introduce some notations that will be used throughout the whole paper.

- We denote by $\{x < \xi(t)\}$ the set $\{(t, x) \in [0, T] \times \mathbb{R} \text{ s.t. } x < \xi(t)\}$. Analogously, we define $\{x = \xi(t)\}$ and $\{x > \xi(t)\}$.
- For any $r > 0$, we set

$$(1.6) \quad B_{W^{1,\infty}}(r) := \left\{ \xi \in W^{1,\infty}((0, T), \mathbb{R}) \text{ s.t. } \|\dot{\xi}\|_\infty + \|\xi\|_\infty \leq r \right\}.$$

- Analogously, we write $B_{L^1}(r)$ for the set of $\rho \in L^1((0, T) \times \mathbb{R}, [0, 1])$ such that $\|\rho\|_{L^1((0, T) \times \mathbb{R})} \leq r$.
- In the integrals, we denote by Ω the set $(0, T) \times \mathbb{R}$.

We introduce the following assumptions on f :

$$(1.7) \quad f \in W^{1,\infty}((0, 1)) \text{ is concave, non-negative and such that } f(0) = 0 = f(1),$$

$$(1.8) \quad \text{meas} \left\{ p \in [0, 1] \text{ s.t. } f'(p) = 0 \right\} = 0.$$

Given a fixed Lipschitz continuous curve ξ on $[0, T]$, we define an admissible solution ρ to the discontinuous-flux LWR equation (1.5a)-(1.5b). On the way of proving the existence result, we propose and use a notion of admissible solution for this equation that is slightly simpler than the notions used in [17], [3] and [10]. Note that all these notions of solution are equivalent, see in particular Section 2.

DEFINITION 1.2. Let $\xi \in W^{1,\infty}((0, T))$. Let $\rho_0 \in L^1(\mathbb{R}, [0, 1])$. Let f verify (1.7) and set

$$F(t, x, \rho) := \text{sign}(x - \xi(t))f(\rho).$$

We say that $\rho \in L^1((0, T) \times \mathbb{R}, [0, 1])$ is an admissible solution to

$$(1.9) \quad \begin{cases} \rho_t + F(t, x, \rho)_x = 0 \\ \rho(t = 0, \cdot) = \rho_0(\cdot) \end{cases}$$

if

- for all $\phi \in C_c^\infty((0, T) \times \mathbb{R})$,

$$(1.10) \quad \iint_{\Omega} \rho \phi_t + F(t, x, \rho) \phi_x \, dt \, dx = 0,$$

- for all positive $\phi \in C_c^\infty(\{x < \xi(t)\})$, for all $k \in [0, 1]$,

$$(1.11) \quad - \iint_{\Omega} |\rho - k| \phi_t + q(\rho, k) \phi_x \, dt \, dx - \int_{\mathbb{R}} |\rho_0 - k| \phi(0, x) \, dx \leq 0,$$

where we set

$$(1.12) \quad q(u, v) := \text{sign}(u - v) [F(t, x, u) - F(t, x, v)];$$

- for all positive $\phi \in C_c^\infty(\{x > \xi(t)\})$, for all $k \in [0, 1]$, (1.11) holds.

Remark 1.3. Note that this notion of solution makes sense for arbitrary initial datum $\rho_0 \in L^1(\mathbb{R}, [0, 1])$ but, in order to keep consistency with the standard Hughes setting, we will restrict our attention to initial data ρ_0 that fulfill (1.2).

Remark 1.4. Note that in the above definition, no admissibility condition is prescribed at $\{x = \xi(t)\}$. Only the conservativity (the Rankine-Hugoniot condition following from (1.10)) is required at the location of the turning curve. This turns out to be enough to guarantee uniqueness of an admissible solution. This fact was already observed in [2] which uses a different formalism to define solutions.

Remark 1.5. Definition 1.2 implies that $\rho \in C^0([0, T], L^1(\mathbb{R}))$. The proof is a straightforward adaptation of the argument developed in [26], on the basis of the result of [12]. This time-continuity of ρ gives sense to the notation $\rho(t, \cdot)$.

We now present the notion of solution used for the generalized Hughes' model given by system (1.5). Recalling Remark 1.5, we require that the operator equation (1.5c) is verified for all $t \in [0, T]$. However, we will require that ξ belong to $W^{1,\infty}((0, T))$ in order to obtain our main result. Note that we use the classical embedding result to identify ξ with a unique element of $C^0([0, T])$. We are in a position to introduce the definition of a solution to (1.5).

DEFINITION 1.6. *Consider $\mathcal{I} : L^1((0, T) \times \mathbb{R}) \longrightarrow C^0([0, T])$. We say that (ρ, ξ) is a solution to generalized Hughes' model (1.5) if ρ is a solution to (1.5a)-(1.5b) in the sense of Definition 1.2 and moreover, $\mathcal{I}(\rho) \in W^{1,\infty}((0, T))$ and the equality $\xi = \mathcal{I}(\rho)$ holds in $C^0([0, T])$.*

We are now in a position to state the main result of the present paper. Recall the notation (1.6).

THEOREM 1.7. *Let ρ_0 verify (1.2). Let $\mathcal{I} : L^1((0, T) \times \mathbb{R}) \longrightarrow C^0([0, T])$. Assume that f verifies (1.7)-(1.8). If there exists $r > 0$ and a subset B of $L^1((0, T) \times \mathbb{R})$ such that:*

$$(1.13a) \quad B \text{ is a convex closed bounded subset of } (L^1((0, T) \times \mathbb{R}), \|\cdot\|_{L^1((0, T) \times \mathbb{R})}),$$

$$(1.13b) \quad \forall \xi \in B_{W^{1,\infty}}(r), \text{ the unique admissible solution } \rho \text{ to (1.9) belongs to } B,$$

$$(1.13c) \quad \mathcal{I}(B) \subset B_{W^{1,\infty}}(r),$$

$$(1.13d) \quad \mathcal{I}|_B : (B, \|\cdot\|_{L^1((0, T) \times \mathbb{R})}) \longrightarrow (W^{1,\infty}((0, T)), \|\cdot\|_\infty) \text{ is a continuous operator,}$$

then there exists (ρ, ξ) a solution to the problem (1.5) in the sense of Definition 1.6.

Remark 1.8. As stated in Remark 1.3, one could chose an arbitrary initial datum $\rho_0 \in L^1(\mathbb{R}, [0, 1])$. Theorem 1.7 would still hold. In the present paper, we chose to focus on initial data satisfying (1.2), for the sake of compatibility with the setting (1.4).

We also propose a second main result (Theorem 4.6) regarding a variation of the Hughes model with different evacuation scenarios at the exits $x = \pm 1$. This modification of the model is important in view of applications. We obtain an existence result analogous to Theorem 1.7, in this modified setting. We detail this result and its proof in Section 4.

1.4. Applications. In this subsection, we will exhibit three concrete applications of Theorem 1.7. Regarding the one-dimensional Hughes' model, we obtain the following result:

PROPOSITION 1.9. *Let ρ_0 satisfy (1.2) and c verify (1.3). Let f verify (1.7)-(1.8). If the cost c is affine,*

$$(1.14) \quad c(\rho) = 1 + \alpha\rho, \quad \alpha > 0,$$

then there exists (ρ, ξ) a solution to the problem (1.1) in the sense of Definition 1.6.

This application is the most fundamental regarding the Hughes model. Under slightly stronger assumptions, this result was already proven in [10]. Our proof follows a very different approach. Note however that both [10] and our result exploit in a crucial way the claim of Lemma 3.2 below and the restriction (1.14) on the cost function.

As a second application, given $\delta > 0$, we introduce the operator $\mathcal{R} : L^1((-\infty, T)) \longrightarrow L^1((0, T))$ defined by:

$$(1.15) \quad \mathcal{R}[\rho(\cdot, x)](t) := \delta \int_{-\infty}^t \rho(s, x) e^{-\delta(t-s)} ds.$$

To make this operator well defined, we extend ρ by $\rho(t) = \rho_0$ for any $t \in (-\infty, 0]$. Then we have the following result:

PROPOSITION 1.10. *Let ρ_0 satisfy (1.2) and c verify (1.3). Let f verify (1.7)-(1.8). Then problem*

$$\begin{aligned} (1.1a) \quad & \left\{ \begin{array}{l} \rho_t + [\text{sign}(x - \xi(t))\rho v(\rho)]_x = 0 \\ \rho(0, \cdot) = \rho_0(\cdot) \end{array} \right. \\ (1.1b) \quad & \\ (1.16) \quad & \left\{ \begin{array}{l} \int_{-1}^{\xi(t)} c(\mathcal{R}[\rho(\cdot, x)](t))dx = \int_{\xi(t)}^1 c(\mathcal{R}[\rho(\cdot, x)](t))dx, \end{array} \right. \end{aligned}$$

admits at least one solution.

Let us discuss the choice of (1.16) for modeling agents' behaviour in the context of Hughes' approach. Equation (1.16) takes into account the average density over the recent past instead of the instantaneous density at a time t . The quantity $\mathcal{R}[\rho(\cdot, x)]$ models the bias towards perception of the density for the pedestrians in the corridor; it can be compared to other "subjective densities" used in the literature (cf. [13], [9, 8]).

As a third example, we consider (1.5) with a different kind of equilibrium equation (1.1c). This time, we suppose that collective behavior of pedestrians makes appear some amount of inertia in the dynamics of ξ . Fixing $\epsilon > 0$, we consider as a simplest variant of such dynamics the ODE Cauchy problem

$$\begin{aligned} (1.17a) \quad & \left\{ \begin{array}{l} -\epsilon \dot{\xi}(t) = \int_{\xi(t)}^1 c(\rho(t, x))dx - \int_{-1}^{\xi(t)} c(\rho(t, x))dx \\ \int_{\xi(0)}^1 c(\rho_0(x))dx - \int_{-1}^{\xi(0)} c(\rho_0(x))dx = 0, \end{array} \right. \\ (1.17b) \quad & \end{aligned}$$

for the ρ -driven evolution of the turning curve ξ . Formally, the case $\epsilon = 0^+$ corresponds to the standard Hughes relation between the density and the turning curve; $\epsilon > 0$ models a form of relaxation to the equilibrium given by the standard model.

PROPOSITION 1.11. *Let ρ_0 satisfy (1.2) and c verify (1.3). Let f verify (1.7)-(1.8). Then problem (1.5a)-(1.5b)-(1.17) admits at least one solution.*

Finally, another series of applications (which is an extension of all the previous results to models with different, phenomenologically relevant behavior of agents in exits) is provided in Section 4.

1.5. Outline. In Section 2, we prove the main result of this paper, Theorem 1.7. This result deals with an abstract framework where the choices of \mathcal{I} and B fulfilling (1.13) are not prescribed. Then, in Section 3, we detail the construction of the set $B = B_1$ satisfying the assumptions of Theorem 1.7 in the case of $\mathcal{I} = \mathcal{I}_0$ being the operator associated with equation (1.1c) with affine cost (1.14). This proves Proposition 1.9. We also discuss the case of a general cost satisfying (1.3) and solve it, with a different set $B = B_2$, for the modified operators $\mathcal{I} = \mathcal{I}_\delta$ and $\mathcal{I} = \tilde{\mathcal{I}}_\epsilon$, corresponding respectively to Propositions 1.10 and 1.11. In Section 4, we present a second main result, Theorem 4.6 (which is an analogue of Theorem 1.7), in a setup with constrained evacuation at exits $x = \pm 1$. Eventually, we conclude by a discussion about further prospects and generalizations of our main result.

2. Proof of the main result. In order to prove Theorem 1.7, we can reformulate the problem (1.5) as a fixed point problem. Indeed, let ρ be a fixed point of the composed operator $\mathcal{D}_0 \circ \mathcal{I}$ where \mathcal{D}_0 is the operator that maps a given turning curve ξ to the solution ρ of (1.5a)-(1.5b) in the sense of Definition 1.2 (we will prove that \mathcal{D}_0 is well defined). Then the couple $(\rho, \mathcal{I}(\rho))$ is a solution to the generalized Hughes' model (1.5) in the sense of Definition 1.6. Our strategy in the following will be to prove the existence of a fixed point thanks to a variant of the Schauder's fixed point theorem. First, we prove the Lemma 2.1 that gathers the topological assumptions on the operators that we will be using. To be specific, denoting by $\mathcal{I} : \rho \mapsto \xi$ the operator that serves to compute the interface and by $\mathcal{D} : \xi \mapsto \rho$ the one that serves to compute the density, we prove the following statement:

LEMMA 2.1. *Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \|\cdot\|_Y)$ a metric space. Assume there exists B a bounded closed convex subset of X and K a compact subset of Y such that:*

$$(2.1a) \quad \mathcal{D} : (K, \|\cdot\|_Y) \longrightarrow (X, \|\cdot\|_X) \text{ is a continuous operator,}$$

$$(2.1b) \quad \mathcal{I} : (B, \|\cdot\|_X) \longrightarrow (K, \|\cdot\|_Y) \text{ is a continuous operator,}$$

$$(2.1c) \quad \mathcal{D} \circ \mathcal{I}(B) \subset B.$$

Then $\mathcal{D} \circ \mathcal{I}$ admits a fixed point in B .

Remark 2.2. We stress that the assumption (2.1b) implies that, on the subset B , \mathcal{I} takes its values in K , making $\mathcal{D} \circ \mathcal{I}$ well-defined on B .

Proof of Lemma 2.1. We recall that, thanks to condition (2.1b), $\mathcal{D} \circ \mathcal{I}$ is well defined. Moreover, \mathcal{D} and \mathcal{I} are continuous. So $\mathcal{D} \circ \mathcal{I}$ is continuous from B into itself. Take any subset A of B . The set $\mathcal{I}(A) \subset K$ is a relatively compact set in $(Y, \|\cdot\|_Y)$. Since \mathcal{D} is continuous from $(K, \|\cdot\|_Y)$ into $(X, \|\cdot\|_X)$, $\mathcal{D} \circ \mathcal{I}(A)$ is a relatively compact subset of X . Consequently, we have that $\mathcal{D} \circ \mathcal{I}$ is a compact operator from B into itself. Furthermore, B is bounded closed convex subset of a Banach space X . We apply the Schauder fixed-point theorem (see [30, Theorem 1.C]) and deduce the existence of a fixed point of $\mathcal{D} \circ \mathcal{I}$ in B . \square

We want to apply Lemma 2.1 with \mathcal{D} being the solver that maps a turning curve ξ to a solution ρ to (1.9) in the sense of Definition 1.2. We first need to prove that this operator is well defined for ρ_0 satisfying (1.2). This is equivalent to well-posedness for the problem (1.9). To this end, we state the following facts:

- The notion of solution of Definition 1.2 is less restrictive than the one proposed in [17, 2].
 - In [2], the existence of solutions is established (for the notion of solution found in [2]).
 - From the two previous assertions, we obtain that there exists a solution in the sense of Definition 1.2.
 - We prove below (Proposition 2.3) that, thanks to the particular shape of fluxes on each side of the turning curve (emphasized in Remark 2.4), Definition 1.2 is restrictive enough to grant uniqueness¹.
 - Combining the previous assertions, we have that our notion of solution and the notion of solution used in [2] are equivalent. This is also true for the notion of solution used in [17].
- Note that one can prove the existence result for our notion of solution directly, using the convergence of finite volume approximations (we do so in Section 4, in the context of flux-limited exit behavior at the exits $x = \pm 1$).

PROPOSITION 2.3. *Fix $\xi \in W^{1,\infty}((0,T))$. Let f verify (1.7)-(1.8) and denote by L_f the Lipschitz constant of f . Let ρ (resp. $\hat{\rho}$) be an admissible solution in the sense of Definition 1.2 with initial datum ρ_0 (resp. $\hat{\rho}_0$) satisfying (1.2). We have:*

$$(2.2) \quad \text{for a.e. } t \in [0, T], \text{ for all } a, b \in \mathbb{R}, \int_a^b |\rho(t, x) - \hat{\rho}(t, x)| dx \leq \int_{a-L_f t}^{b+L_f t} |\rho_0(x) - \hat{\rho}_0(x)| dx.$$

In particular, there exists at most one solution associated to a given initial datum ρ_0 .

Remark 2.4. The uniqueness of a solution in the sense of Definition 1.2 still holds for

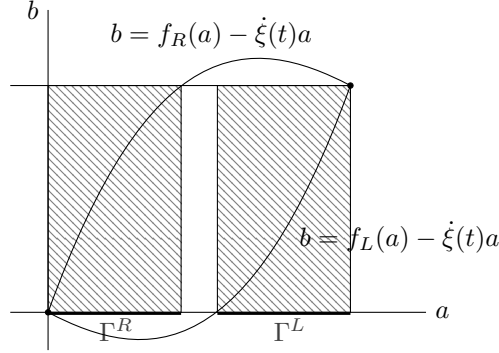
$$F(t, x, \rho) := \mathbb{1}_{\{x < \xi(t)\}}(t, x) f_L(\rho) + \mathbb{1}_{\{x > \xi(t)\}}(t, x) f_R(\rho)$$

where f_L (resp. f_R) is a convex negative (resp. concave positive) on $(0, 1)$ flux function such that $f_L(0) = f_L(1) = f_R(0) = f_R(1) = 0$. For instance, modeling a slanted corridor, we can consider $f_{L,R}(\rho) := v_{L,R} \rho(1 - \rho)$ where v_L and v_R are positive constants accounting for the difference in speed for a pedestrian when moving to the right or to the left exit.

In order to prove this Proposition, we introduce notation for the strong right and left traces of ρ along a Lipschitz curve ξ . Let $\xi \in W^{1,\infty}((0, T), \mathbb{R})$. Then, $\gamma_L \rho(t)$ (resp. $\gamma_R \rho(t)$) is a $L^\infty((0, T))$ function such that, for any $\phi \in \mathcal{C}^0([0, 1])$,

$$\text{ess} \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^T \int_{\xi(t)-\epsilon}^{\xi(t)} |\phi(\rho(t, x)) - \phi(\gamma_L \rho(t))| dx dt = 0,$$

¹Observe that in discontinuous-flux conservation law with different shape of fluxes, additional admissibility conditions at the flux interfaces may be required in order to single out a unique solution (see [5]).

FIGURE 1. Example of geometries of the sets Γ^L and Γ^R

$$\left(\text{respectively, } \operatorname{ess} \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^T \int_{\xi(t)}^{\xi(t)+\epsilon} |\phi(\rho(t, x)) - \phi(\gamma_R \rho(t))| dx dt = 0 \right).$$

Let f satisfy (1.7)-(1.8). It is proven in [29] that $\gamma_{L,R}\rho(\cdot)$ are well defined if ρ is a solution of (1.9) in the sense of Definition 1.2.

Proof of Proposition 2.3. For a more comprehensive presentation of the proof, we denote $f_R = f$ and $f_L = -f$ (see Remark 2.4 for a generalization).

The main idea of the proof consists of using Kruzhkov's doubling variable technique (see [18]) piecewise, on each side of the curve $\{x = \xi(t)\}$. Since ξ is Lipschitz continuous we can join both pieces getting left and right traces along this turning curve, following the general approach of [5, 9]. We get, for any non-negative $\phi \in \mathcal{C}_c^\infty((0, T) \times \mathbb{R})$,

$$(*) \quad - \iint_{\Omega} |\rho - \hat{\rho}| \phi_t + q(\rho, \hat{\rho}) \phi_x dt dx \leq \int_0^T \phi(t, \xi(t)) [q_R(\gamma_R \rho, \gamma_R \hat{\rho}) - q_L(\gamma_L \rho, \gamma_L \hat{\rho})] dt,$$

where $q_{L,R}(\rho, \hat{\rho}) := \operatorname{sign}(\rho - \hat{\rho}) [f_{L,R}(\rho) - f_{L,R}(\hat{\rho}) - \dot{\xi}(t)(\rho - \hat{\rho})]$.

On another side, using traces' existence, we also recover from (1.10) the Rankine-Hugoniot condition:

$$(**\rho) \quad \text{for a.e. } t \in (0, T), f_R(\gamma_R \rho(t)) - \dot{\xi}(t) \gamma_R \rho(t) = f_L(\gamma_L \rho(t)) - \dot{\xi}(t) \gamma_L \rho(t).$$

We also have the analogous relation for $\hat{\rho}$; we denote it by $(**\hat{\rho})$. We introduce the set of values for $\gamma_L \rho$ (resp. $\gamma_R \rho$) verifying $(**\rho)$:

$$\begin{aligned} \Gamma^L &:= \left\{ a \in \mathbb{R} \text{ s.t. } \exists b \in \mathbb{R}, f_L(a) - \dot{\xi}(t)a = f_R(b) - \dot{\xi}(t)b \right\}, \\ (\text{resp. } \Gamma^R &:= \left\{ a \in \mathbb{R} \text{ s.t. } \exists b \in \mathbb{R}, f_R(a) - \dot{\xi}(t)a = f_L(b) - \dot{\xi}(t)b \right\}). \end{aligned}$$

Due to the particular choice of the pair of fluxes (f_L, f_R) , those sets are non-empty. Their geometries are pictured in Figure 1, in the particular case $\dot{\xi}(t) < 0$.

Recalling the properties of f_L and f_R emphasized in Remark 2.4 and using the signs of f'_L and f'_R , we let the reader verify that, for any $\dot{\xi}(t) \in \mathbb{R}$, $\tilde{f}_R : x \mapsto f_R(x) - \dot{\xi}(t)x$ has the same monotonicity on Γ^R as the monotonicity of $\tilde{f}_L : x \mapsto f_L(x) - \dot{\xi}(t)x$ on Γ^L . We recall that:

$$q_{L,R}(a, b) = \operatorname{sign}(a - b) \left(\tilde{f}_{L,R}(a) - \tilde{f}_{L,R}(b) \right).$$

Let $(a_L, b_L) \in \Gamma^L \times \Gamma^L$ and $(a_R, b_R) \in \Gamma^R \times \Gamma^R$. Then, $q_L(a_L, b_L)$ and $q_R(a_R, b_R)$ are either both non-negative or both non-positive, whatever be the value $\xi(t) \in \mathbb{R}$. We recall that $(**\rho)$ and $(**\hat{\rho})$ hold true for a.e. t and we omit t in the following notations. Consequently, $(\gamma_L \rho, \gamma_R \rho), (\gamma_L \hat{\rho}, \gamma_R \hat{\rho}) \in \Gamma^L \times \Gamma^R$ and we have:

$q_R(\gamma_R \rho, \gamma_R \hat{\rho})$ and $q_L(\gamma_L \rho, \gamma_L \hat{\rho})$ are either both non-negative or both non-positive.

In addition, $(**\rho), (**\hat{\rho})$ imply that

$$\tilde{f}_R(\gamma_R \rho) - \tilde{f}_R(\gamma_R \hat{\rho}) = \tilde{f}_L(\gamma_L \rho) - \tilde{f}_L(\gamma_L \hat{\rho}).$$

Therefore we have:

$$\text{for a.e. } t \in (0, T), \quad q_R(\gamma_R \rho, \gamma_R \hat{\rho}) - q_L(\gamma_L \rho, \gamma_L \hat{\rho}) = 0.$$

Consequently, from $(*)$, we recover the global Kato inequality: for any non-negative $\phi \in \mathcal{C}_c^\infty((0, T) \times \mathbb{R})$,

$$- \iint_{\Omega} |\rho - \hat{\rho}| \phi_t + q(\rho, \hat{\rho}) \phi_x \, dt \, dx \leq 0.$$

The remaining arguments are identical to the classical framework of Kruzhkov. Choosing ϕ approximating the characteristic function of the trapezoid $\mathbb{1}_{[0, t]}(s) \mathbb{1}_{[a - L_f(t-s), b + L_f(t-s)]}(x)$, with L_f being the Lipschitz constant of f , we get the localized L^1 contraction property (2.2): □

$$\int_a^b |\rho(t, x) - \hat{\rho}(t, x)| \, dx \leq \int_{a - L_f t}^{b + L_f t} |\rho(0, x) - \hat{\rho}(0, x)| \, dx.$$

For a given (and fixed) $\xi \in W^{1, \infty}((0, T))$, we have shown that the notion of solution of Definition 1.2 gives a well-posed discontinuous flux conservation law. When ρ_0 satisfies (1.2), there exists a unique solution $\rho \in L^1((0, T) \times \mathbb{R})$ that continuously depends on ρ_0 (note that one can let $a \rightarrow -\infty$ and $b \rightarrow +\infty$ in (2.2)). Fix ρ_0 satisfying (1.2), we are now in a position to define the solver operator:

$$(2.3) \quad \mathcal{D}_0 : \begin{cases} W^{1, \infty}((0, T)) \longrightarrow L^1((0, T) \times \mathbb{R}) \\ \xi \longmapsto \rho. \end{cases}$$

This operator \mathcal{D}_0 maps ξ a turning curve to $\mathcal{D}_0(\xi) = \rho$ the unique admissible solution to (1.5a)-(1.5b) in the sense of Definition 1.2. In order to apply Lemma 2.1 with $\mathcal{D} = \mathcal{D}_0$ acting from $(W^{1, \infty}((0, T)), \|\cdot\|_\infty)$ on $(L^1((0, T) \times \mathbb{R}), \|\cdot\|_{L^1((0, T) \times \mathbb{R})})$, we show the continuity of this operator.

PROPOSITION 2.5. *Let ρ_0 verify (1.2). If f satisfy (1.7)-(1.8), then the solver operator*

$$\mathcal{D}_0 : (W^{1, \infty}((0, T)), \|\cdot\|_\infty) \longrightarrow (L^1((0, T) \times \mathbb{R}), \|\cdot\|_{L^1((0, T) \times \mathbb{R})})$$

defined by (2.3) is continuous.

Remark 2.6. The non-degeneracy condition (1.8) on f is a technical assumption that is often used when proving compactness of a sequence of solutions to scalar conservation law. The proof of Proposition 2.5 and the well-posedness of the strong traces are the only two places where we need this assumption.

In order to prove Proposition 2.5, we state the following useful lemma that is a refinement of (2.2). Let's denote for any $a < b \in \mathbb{R}, s < t \in [0, T]$, the trapezoid:

$$(2.4) \quad \mathcal{T}_{a,b}^{s,t} := \{(\tau, x) \in (0, T) \times \mathbb{R} \text{ s.t. } \tau \in [s, t], x \in (a + (\tau - s)L_f, b - (\tau - s)L_f)\},$$

where L_f is the Lipschitz constant of f . The following lemma states that the initial datum located on $[a, b]^c$ doesn't influence the value of the solution ρ on the trapezoid $\mathcal{T}_{a,b}^{s,t}$.

LEMMA 2.7. *Let ρ_0 satisfy (1.2), $\xi \in W^{1,\infty}((0, T))$ and ρ be the entropy solution in the sense of Definition 1.2 to (1.9) on $(0, T) \times \mathbb{R}$. Denote $\hat{\rho}$ the Kruzhkov entropy solution on $(s, t) \times \mathbb{R}$ to ²*

$$\begin{cases} \hat{\rho}_t + f(\hat{\rho})_x = 0 \\ \hat{\rho}(s, \cdot) = \rho(s, \cdot) \mathbb{1}_{(a,b)}(\cdot). \end{cases}$$

Then, for any $a < b \in \mathbb{R}, s < t \in [0, T]$, there holds

$$(2.5) \quad \mathcal{T}_{a,b}^{s,t} \subset \{x > \xi(t)\} \implies \rho = \hat{\rho} \text{ a.e. on } \mathcal{T}_{a,b}^{s,t}.$$

Proof. The proof is a simplified variant of the proof of Proposition 2.3, without the need to consider solutions' behaviour at the turning curve. The time-translated analogue of (2.2) leads to property (2.5). \square

We now prove Proposition 2.5 using this lemma.

Proof of Proposition 2.5. Consider $(\xi_n)_{n \in \mathbb{N}}$ and ξ in $W^{1,\infty}((0, T))$ such that $\|\xi_n - \xi\|_\infty \rightarrow 0$. We denote $\rho_n := \mathcal{D}_0(\xi_n)$. Let \mathcal{K} a compact subset of $\{x > \xi(t)\}$. Let $\epsilon > 0$ be such that $\mathcal{K} \subset \{x > \xi(t) + \epsilon\}$. We cover \mathcal{K} by a finite number of trapezoids of the form (2.4). Without loss of generality we can suppose that each trapezoid is contained in $\{x > \xi(t) + \epsilon\}$:

$$\mathcal{K} \subset \bigcup_{i \in I} \mathcal{T}_{a_i, b_i}^{s_i, t_i} \subset \{x > \xi(t) + \epsilon\}, \quad \mathbf{Card}(I) < +\infty.$$

Since $\|\xi_n - \xi\|_\infty \rightarrow 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall t \in [0, T], n \geq n_0 \implies |\xi_n(t) - \xi(t)| \leq \epsilon.$$

This implies $\xi_n(t) \in [\xi(t) - \epsilon; \xi(t) + \epsilon]$. Then,

$$(2.6) \quad \forall x \in \mathbb{R} \setminus [\xi(t) - \epsilon; \xi(t) + \epsilon], \quad \text{sign}(x - \xi_n(t)) = \text{sign}(x - \xi(t)).$$

Then, for such n_0 , for any $n \geq n_0$, each trapezoid $\mathcal{T}_{a_i, b_i}^{s_i, t_i}$ is included in $\{x > \xi_n(t)\}$. Using Lemma 2.7 we find that, for any $n \geq n_0$, ρ_n is equal a. e. in $\mathcal{T}_{a_i, b_i}^{s_i, t_i}$ to the Kruzhkov entropy solution of

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ \rho(s_i, \cdot) = \rho_n(s_i, \cdot) \mathbb{1}_{(a_i, b_i)}(\cdot). \end{cases}$$

Recalling (1.8), we are now in a position to apply the averaging compactness lemma (see Theorem 5.4.1 in [23]) on the trapezoid $\mathcal{T}_{a_0, b_0}^{s_0, t_0}$. We get a subsequence $(\rho_{n_k})_{k \in \mathbb{N}}$ that converges in $L^1(\mathcal{T}_{a_0, b_0}^{s_0, t_0})$. We then apply the averaging compactness lemma with $(\rho_{n_k})_k$ on $\mathcal{T}_{a_1, b_1}^{s_1, t_1}$. Repeating this process for each $i \in I$, we recover a (not relabeled) subsequence $(\rho_{n_m})_m$ that converges in $L^1(\bigcup_{i \in I} \mathcal{T}_{a_i, b_i}^{s_i, t_i})$. Then $(\rho_{n_m})_m$ converges in $L^1(\mathcal{K})$. In addition, we point out that this reasoning holds for any $\mathcal{K} \subset \{x > \xi(t)\}$.

Up to further extraction of a subsequence, we prove in the same way that $(\rho_{n_m})_m$ also converges in $L^1(\mathcal{K})$ for any compact subset \mathcal{K} of $\{x < \xi(t)\}$. Since ξ is Lipschitz, $\text{meas}(\{x = \xi(t)\}) = 0$. Consequently there exists a subsequence $(\rho_{n_k})_k$ that converges almost everywhere on $(0, T) \times \mathbb{R}$ and in $L^1_{loc}((0, T) \times \mathbb{R})$. Moreover, if $[a, b] \cap [-1, 1] = \emptyset$, $\rho_n = 0$ on $\mathcal{T}_{a, b}^{0, T}$, due to the choice of ρ_0 verifying (1.2). Then, we have that there exists $\rho \in L^1((0, T) \times \mathbb{R})$ such that $\rho_{n_k} \rightarrow \rho$ in $L^1((0, T) \times \mathbb{R})$.

Now, we prove that ρ , constructed above, is actually $\mathcal{D}_0(\xi)$. We can pass to the limit in the entropy inequalities (1.11) and pass to the limit in (1.10) by dominated convergence.

This reasoning can be reproduced for any subsequence of $(\rho_n)_n$. Thanks to a classical argument of compactness, if any converging subsequence $(\mathcal{D}_0(\xi_{n_k}))_{k \in \mathbb{N}}$ converges in $L^1((0, T) \times \mathbb{R})$ to $\mathcal{D}_0(\xi)$, the whole sequence $(\mathcal{D}_0(\xi_n))_n$ converges to $\mathcal{D}_0(\xi)$. So $\mathcal{D}_0 : (W^{1,\infty}((0, T)), \|\cdot\|_\infty) \rightarrow (L^1((0, T) \times \mathbb{R}), \|\cdot\|_{L^1((0, T) \times \mathbb{R})})$ is continuous. \square

²We recall Remark 1.5: for any $s \in [0, T]$, $\rho(s, \cdot)$ is an $L^1(\mathbb{R})$ function.

We now combine all the previous results to prove our main result.

Proof of Theorem 1.7. Suppose there exists $r > 0$ such that assumptions (1.13) are verified.

Using the notations of Theorem 2.1,

- take $Y = (\mathcal{C}^0([0, T]), \|\cdot\|_\infty)$,
- take $X = (L^1((0, T) \times \mathbb{R}), \|\cdot\|_{L^1((0, T) \times \mathbb{R})})$,
- take K the compact set of $\mathcal{C}^0([0, T])$ obtained as the image of $B_{W^{1,\infty}}(r)$ under the standard embedding.

Using Proposition 2.5 and Theorem 2.3, we know that $\mathcal{D}_0 : (K, \|\cdot\|_Y) \rightarrow (X, \|\cdot\|_X)$ is well-defined and continuous. Consequently, assumption (2.1a) holds. Further, notice that conditions (1.13c)-(1.13b) imply (2.1c). Eventually, conditions (1.13c)-(1.13d) ensure that (2.1b) is satisfied. We are now in a position to use Lemma 2.1. We deduce the existence of a solution to (1.5) in the sense of Definition 1.6. \square

3. Lipschitz continuity of the turning curve: examples. In this section, we will prove the Propositions 1.9, 1.10 and 1.11. For each problem, we will reformulate the turning curve equation (1.1) (resp. (1.16) and (1.17)) as an operator that we will denote by \mathcal{I}_0 (resp. \mathcal{I}_δ and \mathcal{I}_ϵ). The propositions then follow from a direct application of Theorem 1.7.

3.1. Proof of Proposition 1.9. We consider the model (1.1):

$$\begin{aligned} (1.1a) \quad & \rho_t + [\text{sign}(x - \xi(t))\rho v(\rho)]_x = 0 \\ (1.1b) \quad & \rho(0, \cdot) = \rho_0(\cdot) \\ (1.1c) \quad & \int_{-1}^{\xi(t)} c(\rho(t, x)) dx = \int_{\xi(t)}^1 c(\rho(t, x)) dx. \end{aligned}$$

with initial datum satisfying (1.2) where we choose, for some $\alpha > 0$, $c(\rho) = 1 + \alpha\rho$.

First, we denote \mathcal{I}_0 the operator that maps ρ to ξ the solution to (1.1c) with affine cost function (we prove that \mathcal{I}_0 is well defined in Proposition 3.1 below). In order to prove Proposition 1.9, we will exhibit the construction of a set B_1 satisfying the conditions (1.13) for this choice of $\mathcal{I} = \mathcal{I}_0$.

Notice that, using the L^1 -contraction property of the admissible solution ρ (2.2) with $\hat{\rho} = 0$ ($\hat{\rho}$ is an obvious admissible solution to (1.9)), we obtain that for all $t \in [0, T]$, $\|\rho(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|\rho_0\|_{L^1(\mathbb{R})}$. Consequently,

$$(3.2) \quad \|\rho\|_{L^1([0, T] \times \mathbb{R})} \leq T \|\rho_0\|_{L^1(\mathbb{R})}.$$

Furthermore, we will prove below that there exists a certain fixed constant $C > 0$ such that for any $\xi \in W^{1,\infty}((0, T))$, any ρ satisfying (1.10) also verifies (see Lemma 3.2 and also [10]):

$$(3.3) \quad \forall a, b \in \mathbb{R}, \forall s, t \in [0, T], \left| \int_a^b \rho(t, x) - \rho(s, x) dx \right| \leq C|t - s|.$$

Finally, considering an initial datum $0 \leq \rho_0 \leq 1$, we set:

$$(3.4) \quad B_1 = \left\{ \rho \in B_{L^1}(T \|\rho_0\|_{L^1}) \text{ s.t. } 0 \leq \rho \leq 1 \text{ and } \rho \text{ verifies (3.3)} \right\}.$$

We end up with the following proposition:

PROPOSITION 3.1. *Assume the cost is given by (1.14). Then the following properties hold:*

1. B_1 is closed convex and bounded in $L^1((0, T) \times \mathbb{R})$.
2. For any $\xi \in W^{1,\infty}((0, T))$, $\mathcal{D}_0(\xi) \in B_1$.
3. There exists $r > 0$ such that, for any $\rho \in B_1$, there exists a unique solution $\xi \in B_{W^{1,\infty}}(r)$ to (1.1c).
Consequently, the operator \mathcal{I}_0 is well-defined and single-valued.
4. $\mathcal{I}_0 : (B_1, \|\cdot\|_{L^1((0, T) \times \mathbb{R})}) \rightarrow (W^{1,\infty}([0, T]), \|\cdot\|_\infty)$ is continuous.

Admitting for a moment that Proposition 3.1 holds true, Proposition 1.9 readily follows.

Proof of Proposition 1.9. Using Proposition 3.1 we have that $\mathcal{I} = \mathcal{I}_0$ verifies (1.13) for the set $B = B_1$. We apply Theorem 1.7 and get the desired existence of a solution for the problem (1.1) with affine cost (1.14). That proves Proposition 1.9. \square

We now turn to the proof of Proposition 3.1. We rely on two lemmas that we chose to isolate in order to use them in the other examples.

LEMMA 3.2. *Let $a, b \in \mathbb{R}$, $a < b$ and $s, t \in [0, T]$, $s < t$. Fix $\xi \in W^{1,\infty}((0, T))$. We denote ρ a solution in the sense of Definition 1.2. Then, there exists $C > 0$, independent of a, b, s, t, ξ and ρ , such that:*

$$(3.5) \quad \left| \int_a^b \rho(t, x) - \rho(s, x) dx \right| \leq C|t - s|.$$

We recall that there's no ambiguity in considering $\rho(t, \cdot)$ since $\rho \in \mathcal{C}^0([0, T], L^1(\mathbb{R}))$ (see Remark 1.5).

Proof of Lemma 3.2. Let $(\kappa_n)_{n \in \mathbb{N}}$ be a sequence of mollifiers. We set

$$\Psi(\tau, x) := \mathbb{1}_{[a,b]}(x) \mathbb{1}_{[s,t]}(\tau) \quad \text{and} \quad \phi(\tau, x) := \Psi * \kappa_n(\tau, x).$$

Using ϕ as test function in (1.10), making $n \rightarrow +\infty$ we get:

$$\int_a^b \rho(s, x) - \rho(t, x) dx + \int_s^t F(\tau, a, \rho(\tau, a)) - F(\tau, b, \rho(\tau, b)) d\tau = 0.$$

Consequently,

$$\left| \int_a^b \rho(t, x) - \rho(s, x) dx \right| = \left| \int_s^t F(\tau, a, \rho(\tau, a)) - F(\tau, b, \rho(\tau, b)) d\tau \right| \leq \left(2 \sup_{p \in [0,1]} |f(p)| \right) |t - s|. \quad \square$$

LEMMA 3.3. *Let $s < t \in [0, T]$. Let ξ be a solution to (1.1c). We denote $\underline{\xi} := \min(\xi(t), \xi(s))$ and $\bar{\xi} := \max(\xi(t), \xi(s))$. Then,*

$$(3.6) \quad 2|\xi(t) - \xi(s)| \leq \left| \int_{-1}^{\bar{\xi}} c(\rho(t, x)) - c(\rho(s, x)) dx - \int_{\underline{\xi}}^1 c(\rho(t, x)) - c(\rho(s, x)) dx \right|.$$

Remark 3.4. This is a standard calculation in the context of the 1D Hughes model (see [17]). We give a proof for the sake of completeness.

Proof of Lemma 3.3. We first treat the case $\xi(s) \leq \xi(t)$.

We have:

$$\begin{aligned} \int_{-1}^{\xi(s)} c(\rho(s, x)) dx &= \int_{\xi(s)}^{\xi(t)} c(\rho(s, x)) dx + \int_{\xi(t)}^1 c(\rho(s, x)) dx, \\ \int_{-1}^{\xi(s)} c(\rho(t, x)) dx &= - \int_{\xi(s)}^{\xi(t)} c(\rho(t, x)) dx + \int_{\xi(t)}^1 c(\rho(t, x)) dx. \end{aligned}$$

Subtracting these equalities, we get

$$\int_{\xi(s)}^{\xi(t)} c(\rho(s, x)) + c(\rho(t, x)) dx = \int_{-1}^{\xi(s)} c(\rho(s, x)) - c(\rho(t, x)) dx - \int_{\xi(t)}^1 c(\rho(s, x)) - c(\rho(t, x)) dx.$$

On the contrary, if $\xi(s) \geq \xi(t)$, with an analogous argument we get:

$$\int_{\xi(t)}^{\xi(s)} c(\rho(s, x)) + c(\rho(t, x)) dx = \int_{-1}^{\xi(t)} c(\rho(t, x)) - c(\rho(s, x)) dx - \int_{\xi(s)}^1 c(\rho(t, x)) - c(\rho(s, x)) dx.$$

Using the fact that $c \geq 1$, in all cases, we get:

$$\begin{aligned} 2|\xi(t) - \xi(s)| &= 2(\bar{\xi} - \underline{\xi}) \leq \int_{\underline{\xi}}^{\bar{\xi}} c(\rho(s, x)) + c(\rho(t, x)) dx \\ &\leq \left| \int_{-1}^{\bar{\xi}} c(\rho(s, x)) - c(\rho(t, x)) dx - \int_{\underline{\xi}}^1 c(\rho(s, x)) - c(\rho(t, x)) dx \right|. \quad \square \end{aligned}$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. First, let $\rho_1, \rho_2 \in B_1$, $\lambda \in [0, 1]$; it is readily checked that $\lambda\rho_1 + (1 - \lambda)\rho_2$ still satisfies (3.5). Thus B_1 is convex. It is also readily checked that we can pass to the $L^1((0, T) \times \mathbb{R})$ limit in (3.5), proving that B_1 is closed. By construction B_1 is bounded. That ends the proof of the first assertion. Second, combining (3.2) with Lemma 3.2, we obtain the second assertion of Proposition 3.1.

Third, fix $\rho \in B_1$. We prove existence and uniqueness of $\xi \in L^\infty([0, T])$ satisfying (1.1c) for any $t \in [0, T]$. Letting $t \in [0, T]$, we set:

$$\Psi^+(a) := \int_{-1}^a c(\rho(t, x)) dx, \quad \Psi^-(a) := \int_a^1 c(\rho(t, x)) dx.$$

Because $c > 0$, Ψ^+ is a continuous strictly increasing function, while Ψ^- is continuous and strictly decreasing on $[-1, 1]$. Therefore, $a \mapsto \Psi^+(a) - \Psi^-(a)$ is continuous, strictly increasing, negative at $a = -1$ and positive at $a = 1$. Consequently, there exists one and only one $\tilde{a} \in (-1, 1)$ such that $\Psi^+(\tilde{a}) = \Psi^-(\tilde{a})$. For any $t \in [0, T]$, we set $\xi(t) = \tilde{a}$. This defines ξ , the unique pointwise solution to (1.1c) on $[0, T]$. Clearly, $\xi \in L^\infty((0, T))$.

We now prove that $\xi \in W^{1, \infty}([0, T])$. Using Lemma 3.3 we get:

$$\begin{aligned} 2|\xi(t) - \xi(s)| &\leq \left| \int_{-1}^{\xi} c(\rho(t, x)) - c(\rho(s, x)) dx - \int_{\bar{\xi}}^1 c(\rho(t, x)) - c(\rho(s, x)) dx \right| \\ &\leq \alpha \left| \int_{-1}^{\xi} \rho(t, x) - \rho(s, x) dx \right| + \alpha \left| \int_{\bar{\xi}}^1 \rho(t, x) - \rho(s, x) dx \right|. \end{aligned}$$

Using Lemma 3.2, with the choice (1.14) of the cost, we get:

$$2|\xi(t) - \xi(s)| \leq 2\alpha C |t - s|.$$

Take $r = \alpha C$; this guarantees that ξ is always in $B_{W^{1, \infty}}(r)$. This proves the third assertion of Proposition 3.1. Finally, we prove the continuity of the operator \mathcal{I}_0 . Let's consider $\rho, \rho_n \in B_1$. Then, for a given $t \in [0, T]$, using (1.1c) for both $\xi := \mathcal{I}_0(\rho)$ and $\xi_n := \mathcal{I}_0(\rho_n)$, skipping the (t, x) -dependence within the integrals, we recover:

$$\int_{\xi_n(t)}^{\xi(t)} c(\rho) + \int_{-1}^{\xi_n(t)} c(\rho) - \int_{-1}^{\xi_n(t)} c(\rho_n) = \int_{\xi(t)}^{\xi_n(t)} c(\rho) + \int_{\xi_n(t)}^1 c(\rho) - \int_{\xi_n(t)}^1 c(\rho_n).$$

Rearranging the integrals, we obtain:

$$2 \int_{\xi_n(t)}^{\xi(t)} c(\rho) = \int_{-1}^1 [c(\rho) - c(\rho_n)] \text{sign}(x - \xi_n(t)).$$

Notice that

$$\begin{aligned} \int_0^T |\xi - \xi_n| &\leq \int_0^T \left| \int_{\xi(t)}^{\xi_n(t)} c(\rho) \right| = \frac{1}{2} \int_0^T \left| \int_{-1}^1 \text{sign}(x - \xi_n(t)) [c(\rho) - c(\rho_n)] \right| \\ &\leq \frac{1}{2} \int_0^T \int_{-1}^1 |c(\rho) - c(\rho_n)| \leq \frac{\alpha}{2} \int_0^T \int_{-1}^1 |\rho - \rho_n|. \end{aligned}$$

Consequently, if $\|\rho - \rho_n\|_{L^1((0, T) \times \mathbb{R})} \rightarrow 0$, we have

$$\|\xi - \xi_n\|_{L^1((0, T))} \rightarrow 0.$$

We recall that $\xi, \xi_n \in \mathcal{I}_0(B_1)$ are r -Lipschitz. We use a finite covering of $[0, T]$ by segments of arbitrarily fixed size. On each of these segments, there exists a point where the continuous function $\xi(\cdot) - \xi_n(\cdot)$ is less or equal to its L^1 -average on the segment. In addition, the derivative of $\xi(\cdot) - \xi_n(\cdot)$ is bounded on each segment. Combining the three previous statements, we recover that $\|\xi - \xi_n\|_\infty \rightarrow 0$ when $\|\rho - \rho_n\|_{L^1((0, T) \times \mathbb{R})} \rightarrow 0$. This ends the proof of Proposition 3.1. \square

3.2. Proof of Proposition 1.10. We recall that $\mathcal{R} : L^1((-\infty, T)) \rightarrow L^1((0, T))$ is the operator defined by:

$$(1.15) \quad \mathcal{R}[\rho(\cdot, x)](t) := \delta \int_{-\infty}^t \rho(s, x) e^{-\delta(t-s)} ds.$$

We consider the problem

$$(1.1a) \quad \left\{ \begin{array}{l} \rho_t + [\text{sign}(x - \xi(t))\rho v(\rho)]_x = 0 \\ \rho(0, \cdot) = \rho_0(\cdot) \end{array} \right.$$

$$(1.1b) \quad \left\{ \begin{array}{l} \int_{-1}^{\xi(t)} c(\mathcal{R}[\rho(\cdot, x)](t)) dx = \int_{\xi(t)}^1 c(\mathcal{R}[\rho(\cdot, x)](t)) dx, \end{array} \right.$$

$$(1.16)$$

with c verifying (1.3), and with initial datum satisfying (1.2). Then, we define

$$(3.7) \quad B_2 = \{\rho \in B_{L^1}(0, T \|\rho_0\|_{L^1}) \text{ s.t. } 0 \leq \rho \leq 1\}.$$

We introduce the operator $\mathcal{I}_\delta : (B_2, \|\cdot\|_{L^1((0, T) \times \mathbb{R})}) \rightarrow (W^{1, \infty}((0, T)), \|\cdot\|_\infty)$, that maps the density ρ to ξ the solution to (1.16) with \mathcal{R} given by (1.15). We prove that this operator is well defined in a counterpart of Proposition 3.1:

PROPOSITION 3.5. *Assume the cost c satisfies (1.3). Then the following properties hold:*

1. B_2 is closed convex and bounded in $L^1((0, T) \times \mathbb{R})$.
2. For any $\xi \in W^{1, \infty}((0, T))$, $\mathcal{D}_0(\xi) \in B_2$.
3. There exists $r > 0$ such that, for any $\rho \in B_2$, there exists a unique solution $\xi \in B_{W^{1, \infty}}(r)$ to (1.16).
Consequently, the operator \mathcal{I}_δ is well-defined and single-valued.
4. $\mathcal{I}_\delta : (B_2, \|\cdot\|_{L^1((0, T) \times \mathbb{R})}) \rightarrow (W^{1, \infty}([0, T]), \|\cdot\|_\infty)$ is continuous.

Proof. The first assertion follows the same reasoning as in Proposition 3.1. The second assertion is follows the same reasoning as in Proposition 3.1, omitting the property (3.5). Also, $x \mapsto \mathcal{R}[\rho(\cdot, x)](t)$ is at least as regular as $\rho(t, \cdot)$. Following a proof analogous to the one of the third and fourth assertions of Proposition 3.1, we know that $\mathcal{I}_\delta : (B_2, \|\cdot\|_{L^1((0, T) \times \mathbb{R})}) \rightarrow (W^{1, \infty}((0, T)), \|\cdot\|_\infty)$ is well-defined, single-valued and continuous. We now prove that there exists $r > 0$ such that $\mathcal{I}_\delta(B_2) \subset B_{W^{1, \infty}}(r)$. Using Lemma 3.3 adapted to (1.16) in the place of (1.1c), we have

$$(3.8) \quad \begin{aligned} & 2|\xi(t) - \xi(s)| \\ & \leq \left| \int_{-1}^{\xi} c(\mathcal{R}[\rho(\cdot, x)](t)) - c(\mathcal{R}[\rho(\cdot, x)](s)) dx - \int_{\xi}^1 c(\mathcal{R}[\rho(\cdot, x)](t)) - c(\mathcal{R}[\rho(\cdot, x)](s)) dx \right| \\ & \leq \int_{-1}^1 \left| c(\mathcal{R}[\rho(\cdot, x)](t)) - c(\mathcal{R}[\rho(\cdot, x)](s)) \right| dx. \end{aligned}$$

We can dominate the right-hand side (3.8) by the term

$$\text{ess sup}_{p \in [0, 1]} |c'(p)| \int_{-1}^1 \left| \mathcal{R}[\rho(\cdot, x)](t) - \mathcal{R}[\rho(\cdot, x)](s) \right| dx.$$

The latter integral is controlled by $2\delta\|\rho\|_{L^\infty}|t - s|$, using $\delta\|\rho\|_{L^\infty}$ as a bound for the modulus of continuity of $\mathcal{R}[\rho(\cdot, x)]$. Consequently, assertions 3 and 4 hold true. \square

The proof of Proposition 1.10 follows from the application of Theorem 1.7 in the context of Proposition 3.5, as in the previous subsection.

3.3. Proof of Proposition 1.11. In the following, we will never use any property of ρ apart from the universal bounds $0 \leq \rho \leq 1$. Consequently, in this case we use again:

$$B_2 = \{\rho \in B_{L^1}(0, T \|\rho_0\|_{L^1}) \text{ s.t. } 0 \leq \rho \leq 1\}.$$

The proof follows the same guidelines as the two previous subsections.

PROPOSITION 3.6. *Assume the cost c satisfies (1.3). Then the following properties hold:*³

1. B_2 is closed convex and bounded in $L^1((0, T) \times \mathbb{R})$.
2. For any $\xi \in W^{1,\infty}((0, T))$, $\mathcal{D}_0(\xi) \in B_2$.
3. There exists $r > 0$ such that, for any $\rho \in B_2$, there exists a unique solution $\xi \in B_{W^{1,\infty}}(r)$ to the Cauchy problem (1.17). Then the operator

$$\tilde{\mathcal{I}}_\epsilon : B_2 \longrightarrow W^{1,\infty}((0, T))$$

that maps any ρ to the unique ξ solution to (1.17) is well-defined and single-valued.

4. Let $\rho_1, \rho_2 \in L^1((0, T) \times \mathbb{R})$. Denote $\xi_{1,2} := \tilde{\mathcal{I}}_\epsilon(\rho_{1,2})$. Then,

$$(3.9) \quad \|\xi_1 - \xi_2\|_\infty \leq \frac{\|c'\|_\infty}{\epsilon} \exp\left[\frac{2T\|c\|_\infty}{\epsilon}\right] \|\rho_1 - \rho_2\|_{L^1((0,T) \times (-1,1))}.$$

Consequently, $\mathcal{I}_\delta : (B_2, \|\cdot\|_{L^1((0,T) \times \mathbb{R})}) \longrightarrow (W^{1,\infty}([0, T]), \|\cdot\|_\infty)$ is continuous.

Proof. We only focus on solving the ODE problem (1.17). First, we prove assertion 3. Let's denote:

$$\Psi(t, a) := \frac{1}{\epsilon} \left[\int_a^1 c(\rho(t, x)) dx - \int_{-1}^a c(\rho(t, x)) dx \right].$$

Notice that for any $a, b \in [-1, 1], t \in \mathbb{R}$,

$$(3.10) \quad |\Psi(t, a) - \Psi(t, b)| \leq \frac{1}{\epsilon} \left| \int_a^b 2c(\rho(t, x)) dx \right| \leq \frac{2\|c\|_\infty}{\epsilon} |a - b|.$$

We also have, for any ξ such that $\|\xi\|_\infty \leq 1$:

$$|\Psi(t, \xi(t))| = \frac{1}{\epsilon} \left| \int_{-1}^1 \text{sign}(x - \xi(t)) c(\rho(t, x)) dx \right| \leq \frac{2\|c\|_\infty}{\epsilon}.$$

To sum up, Ψ is Lipschitz with respect to the a variable and uniformly bounded with respect to the t variable. We apply the Picard–Lindelöf theorem and recover that there exists a unique global solution to the Cauchy problem (1.17) on $[0, T]$ and that ξ is Lipschitz; moreover, the Lipschitz constant of ξ does not depend on ρ .

Second, we prove assertion 4. We denote by ξ_0 the unique solution to (1.17b). Then, for any $t \in [0, T]$,

$$\xi_{1,2} = \xi_0 - \int_0^t \Psi_{1,2}(s, \xi_{1,2}(s)) ds,$$

where $\Psi_{1,2}(t, a) := \int_{-1}^a c(\rho_{1,2}(t, x)) dx - \int_a^1 c(\rho_{1,2}(t, x)) dx$. Then, writing \wedge, \vee for min, max, respectively, we make the following calculations:

$$\begin{aligned} & \xi_2(t) - \xi_1(t) \\ &= \int_0^t \Psi_1(s, \xi_1(s)) - \Psi_2(s, \xi_2(s)) ds \\ &= \frac{1}{\epsilon} \int_0^t \left[\int_{-1}^{\xi_1(s)} c(\rho_1(s, x)) dx - \int_{\xi_1(s)}^1 c(\rho_1(s, x)) dx - \int_{-1}^{\xi_2(s)} c(\rho_2(s, x)) dx + \int_{\xi_2(s)}^1 c(\rho_2(s, x)) dx \right] ds \\ &= \frac{1}{\epsilon} \int_0^t \left[\int_{-1}^{(\xi_1 \wedge \xi_2)(s)} c(\rho_1(s, x)) - c(\rho_2(s, x)) dx \pm \int_{(\xi_1 \wedge \xi_2)(s)}^{(\xi_1 \vee \xi_2)(s)} c(\rho_1(s, x)) + c(\rho_2(s, x)) dx \right. \\ & \quad \left. + \int_{(\xi_1 \vee \xi_2)(s)}^1 c(\rho_2(s, x)) - c(\rho_1(s, x)) dx \right] ds \end{aligned}$$

³items 1 and 2 being already proven in Proposition 3.5

Consequently,

$$\begin{aligned} |\xi_1(t) - \xi_2(t)| &\leq \frac{1}{\epsilon} \int_0^t \int_{(\xi_1 \wedge \xi_2)(s)}^{(\xi_1 \vee \xi_2)(s)} c(\rho_1(s, x)) + c(\rho_2(s, x)) \, dx \, ds \\ &\quad + \frac{1}{\epsilon} \int_0^t \int_{-1}^1 |c(\rho_1(s, x)) - c(\rho_2(s, x))| \, ds \, dx =: J_1 + J_2. \end{aligned}$$

For the term J_2 we can use the Lagrange inequality denoting $\|c'\|_\infty := \sup_{p \in [0,1]} |c'(p)|$. We get:

$$J_2 \leq \frac{\|c'\|_\infty}{\epsilon} \|\rho_1 - \rho_2\|_{L^1((0,T) \times (-1,1))}.$$

For the term J_1 , notice that, thanks to the cost conditions (1.3), for any $s \in [0, t]$,

$$2|\xi_1(s) - \xi_2(s)| \leq \int_{(\xi_1 \wedge \xi_2)(s)}^{(\xi_1 \vee \xi_2)(s)} c(\rho_1(s, x)) + c(\rho_2(s, x)) \, dx \leq 2\|c\|_\infty |\xi_1(s) - \xi_2(s)|.$$

Consequently, for any $s \in [0, T]$, there exists $\beta(s) \in [2, 2\|c\|_\infty]$ such that

$$\int_{(\xi_1 \wedge \xi_2)(s)}^{(\xi_1 \vee \xi_2)(s)} c(\rho_1(s, x)) + c(\rho_2(s, x)) \, dx = \beta(s) |\xi_1(s) - \xi_2(s)|.$$

Then $\beta \in L^\infty((0, T)) \subset L^1((0, T))$. We are now in a position to use Gronwall's inequality with integrable coefficients. That inequality still holds without the continuity of β if we use the Lebesgue differentiation Theorem. Then,

$$\forall t \in [0, T], \quad |\xi_1(t) - \xi_2(t)| \leq \int_0^t \frac{\beta(s)}{\epsilon} |\xi_1(s) - \xi_2(s)| \, ds + \frac{\|c'\|_\infty}{\epsilon} \|\rho_1 - \rho_2\|_{L^1},$$

which yields the estimates

$$\begin{aligned} \forall t \in [0, T], \quad |\xi_1(t) - \xi_2(t)| &\leq \frac{\|c'\|_\infty}{\epsilon} \|\rho_1 - \rho_2\|_{L^1} \exp \left[\int_0^t \frac{\beta(s)}{\epsilon} \, ds \right], \\ \|\xi_1 - \xi_2\|_\infty &\leq \frac{\|c'\|_\infty}{\epsilon} \exp \left[\frac{2T\|c\|_\infty}{\epsilon} \right] \|\rho_1 - \rho_2\|_{L^1}. \end{aligned} \quad \square$$

The proof of Proposition 1.11 follows from the application of Theorem 1.7 in the context of Proposition 3.6, as in the two previous subsections.

4. Hughes' model with constrained evacuation at exit. In this section, we illustrate the robustness of our approach by modifying the Hughes model at the level of boundary conditions for the density, allowing for the realistic feature of capacity drop (see [9, 8] and references therein). First, consider ξ being given beforehand and the following dynamics for ρ introduced in [9] on the basis of the theory of [14, 4]:

$$\begin{aligned} (4.1a) \quad & \rho_t + [\text{sign}(x - \xi(t))f(\rho)]_x = 0 \\ (4.1b) \quad & f(\rho(t, 1)) \leq g \left(\int_\sigma^1 w_1(x)\rho(t, x) \, dx \right) \\ (4.1c) \quad & f(\rho(t, -1)) \leq g \left(\int_{-1}^{-\sigma} w_{-1}(x)\rho(t, x) \, dx \right) \\ (4.1d) \quad & \rho(0, \cdot) = \rho_0(\cdot). \end{aligned}$$

Here the functions g and $w_{1,-1}$ are parameters of the model, that we will precise below in (4.3), (4.2). The equations (4.1b)-(4.1c) prescribe the behaviour at exits situated at $x = \pm 1$; as in previous sections, we set up the conservation law for ρ in the whole space, but the initial condition (1.2) is confined to the domain

of interest $(-1, 1)$. The flux $f(\rho)$ of pedestrians going through the exits is limited by respective constraints (we take a common nonlinearity g for the sake of conciseness, but it is straightforward to extend the setting distinguishing g_1 and g_{-1}). Following the terminology of [14], the right-hand sides of (4.1b)-(4.1c) are flux constraints at the exit. These flux constraints depend non locally on $\rho(t, \cdot)$ via weights $w_{1,-1}$ supported in a vicinity of length $1 - \sigma$ around the exits. This type of constraint models the well-known phenomenon of *capacity drop* which, in extreme situations, corresponds to a panic behaviour at exits located at $x = \pm 1$, as discussed in [9] and [8]. This model is phenomenologically more relevant than the model with open-end condition considered above (and it includes the previous model, for the trivial choice $g \equiv \max_{[0,1]} f$, see Remark 4.3). As an example, such constrained evacuation model in the LWR context is able to reproduce the ‘‘Faster is Slower’’ effect at exits (see [8]).

In the following, we’ll use the results of [8] and adapt them to our framework. We use the notations proposed in [8]:

- Since f is concave, non-negative and such that $f(0) = f(1)$, there exists a $\bar{\rho} \in [0, 1]$ such that

$$f'(\rho)(\bar{\rho} - \rho) > 0 \text{ for a.e. } \rho \in [0, 1].$$

- We fix $\sigma \in (0, 1)$. This is the threshold of influence on the exit, meaning e.g. that the pedestrian located before $x = \sigma$ have no influence on the exit congestion at $x = 1$.

Let us take the strongest assumptions among those used in [9, 8]:

$$(4.2) \quad \begin{cases} w_1 \in W^{1,\infty}((\sigma, 1], \mathbb{R}^+) \text{ s.t. } \int_{\sigma}^1 w_1 = 1 \\ w_{-1} \in W^{1,\infty}([-1, -\sigma), \mathbb{R}^+) \text{ s.t. } \int_{-1}^{-\sigma} w_{-1} = 1, \end{cases}$$

$$(4.3) \quad g \in W^{1,\infty}(\mathbb{R}^+, (0, f(\bar{\rho})]) \text{ is non-increasing.}$$

We can now introduce the notion of solution we’ll use for ρ . We combine [14] and Definition 1.2:

DEFINITION 4.1. *Let $\xi \in W^{1,\infty}((0, T), (-1, 1))$ and let ρ_0 satisfy (1.2). Let f verify (1.7) and*

$$F(t, x, \rho) := \text{sign}(x - \xi(t))f(\rho).$$

Let g, ω_{-1} and ω_1 satisfy (4.2)-(4.3).

We say that $\rho \in L^1((0, T) \times \mathbb{R})$ is an admissible solution to (4.1) if:

for all $\phi \in \mathcal{C}_c^\infty((0, T) \times \mathbb{R})$,

$$(4.4) \quad \iint_{(0,T) \times \mathbb{R}} \rho \phi_t + F(t, x, \rho) \phi_x \, dt \, dx = 0,$$

moreover, setting

$$(4.5) \quad Q_{-1}(t) := g \left(\int_{-1}^{-\sigma} w_{-1}(x) \rho(t, x) \, dx \right), \quad Q_1(t) := g \left(\int_{\sigma}^1 w_1(x) \rho(t, x) \, dx \right),$$

there holds:

- for all positive $\phi \in \mathcal{C}_c^\infty(\{x > \xi(t)\})$, for all $k \in \mathbb{R}$,

$$(4.6) \quad - \iint_{(0,T) \times \mathbb{R}} |\rho - k| \phi_t + q(\rho, k) \phi_x \, dt \, dx - 2 \int_0^T \left[1 - \frac{Q_1(t)}{f(\bar{\rho})} \right] f(k) \phi(t, 1) \, dx - \int_{\mathbb{R}} |\rho_0 - k| \phi(0, x) \, dx \leq 0.$$

- for all positive $\phi \in \mathcal{C}_c^\infty(\{x < \xi(t)\})$, for all $k \in \mathbb{R}$,

$$(4.7) \quad - \iint_{(0,T) \times \mathbb{R}} |\rho - k| \phi_t + q(\rho, k) \phi_x \, dt \, dx - 2 \int_0^T \left[1 - \frac{Q_{-1}(t)}{f(\bar{\rho})} \right] (-f(k)) \phi(t, -1) \, dx - \int_{\mathbb{R}} |\rho_0 - k| \phi(0, x) \, dx \leq 0;$$

- for all positive $\phi \in C^\infty$ supported on $[a, b]$ such that $a < -1$, $1 < b$ we have:

$$(4.8a) \quad \int_0^T \int_a^{-1} \rho \phi_t + F(t, x, \rho) \phi_x \, dt \, dx \leq \int_0^T Q_{-1}(t) \phi(t, -1) \, dt,$$

$$(4.8b) \quad \int_0^T \int_1^b \rho \phi_t + F(t, x, \rho) \phi_x \, dt \, dx \leq \int_0^T Q_1(t) \phi(t, 1) \, dt.$$

Remark 4.2. As detailed in [4], using (4.8) combined with (4.4) we have that for a.e. $t \geq 0$, $f(\gamma_{L,R}^1 \rho(t)) \leq Q_1(t)$ and $-f(\gamma_{L,R}^{-1} \rho(t)) \geq -Q_{-1}(t)$. This corresponds to the formal writing in (4.1b)-(4.1c).

Remark 4.3. One can notice that if for all $t \geq 0$, $g(t) = f(\bar{\rho})$ then the flux is not limited at exits and $1 - \frac{Q_1(t)}{f(\bar{\rho})} = 1 - \frac{Q_{-1}(t)}{f(\bar{\rho})} = 0$. Then, the above definition is exactly Definition 1.2.

We state the following result:

PROPOSITION 4.4. *Let ρ_0 verify (1.2) and $\xi \in W^{1,\infty}((0, T), (-1, 1))$. Let f satisfy (1.7)-(1.8). There exists a solution to (4.1) in the sense of Definition 4.1.*

The proof of Proposition 4.4 is postponed to the Appendix. The existence is obtained via a convergent finite volume scheme.

Using the results from [14], [8], [9] and a partitioning argument we prove a corollary of Proposition 2.5:

COROLLARY 4.5. *Let ρ_0 verify (1.2). Let $\xi \in W^{1,\infty}((0, T), (-1, 1))$. There exists a unique solution ρ of (4.1) in the sense of Definition 4.1. The solver operator*

$$\mathcal{D}_g : (W^{1,\infty}((0, T), (-1, 1)), \|\cdot\|_\infty) \longrightarrow (L^1((0, T) \times (-1, 1)), \|\cdot\|_{L^1}),$$

that maps any ξ to the unique solution ρ to (4.1) is well-defined and continuous.

Proof. We use the classical compact embedding of $W^{1,\infty}([0, T], (-1, 1))$ into $C^0([0, T], (-1, 1))$: there exists I a closed interval of $(-1, 1)$ such that $\xi \in C^0([0, T], I)$. Then, the open subdomains of $(0, T) \times \mathbb{R}$ situated between $x = -\infty$ and $x = -1$, $x = -1$ and $x = \xi(t)$, $x = \xi(t)$ and $x = 1$, and finally between $x = 1$ and $x = +\infty$ are well-defined connected sets. We use the classical Kruzhkov doubling of variables (cf. [18]) in these four subdomains. Then by a limiting procedure analogous to the one employed in the proof of Proposition 2.3, we obtain the Kato inequality carrying singular terms concentrated on the three curves $\{x = \xi(t)\}$, $\{x = 1\}$ and $\{x = -1\}$: for any non-negative $\phi \in C_c^\infty((0, T) \times \mathbb{R})$,

$$(4.9a) \quad - \iint_{(0, T) \times (-1, 1)} |\rho - \hat{\rho}| \phi_t + q(\rho, \hat{\rho}) \phi_x \, dt \, dx$$

$$\leq \int_0^T \phi(t, \xi(t)) [q_R^0(\gamma_R \rho, \gamma_R \hat{\rho}) - q_L^0(\gamma_L \rho, \gamma_L \hat{\rho})] \, dt$$

$$(4.9b) \quad + \int_0^T \phi(t, 1) [q^1(\gamma_R \rho, \gamma_R \hat{\rho}) - q^1(\gamma_L \rho, \gamma_L \hat{\rho})] \, dt$$

$$(4.9c) \quad + \int_0^T \phi(t, -1) [q^{-1}(\gamma_R \rho, \gamma_R \hat{\rho}) - q^{-1}(\gamma_L \rho, \gamma_L \hat{\rho})] \, dt,$$

where the left and right traces are taken along the respective curves, and

$$q_{L,R}^0(\rho, \hat{\rho}) := \text{sign}(\rho - \hat{\rho}) [f_{L,R}(\rho) - f_{L,R}(\hat{\rho}) - \dot{\xi}(t) (\rho - \hat{\rho})],$$

$$q^1(\rho, \hat{\rho}) := \text{sign}(\rho - \hat{\rho}) [f_R(\rho) - f_R(\hat{\rho})],$$

$$q^{-1}(\rho, \hat{\rho}) := \text{sign}(\rho - \hat{\rho}) [f_L(\rho) - f_L(\hat{\rho})].$$

As in the proof of Proposition 2.3, f_L denotes $-f$ and f_R denotes f . Referring to proof of Theorem 2.3, the

integral (4.9a) is zero. Using the same argument as in the proof of Proposition 2.10 in [4], we get:

$$(4.9b) \leq 2 \int_0^T \phi(t, 1) \left| Q_1(t) - \hat{Q}_1(t) \right| dt,$$

$$(4.9c) \leq 2 \int_0^T \phi(t, -1) \left| Q_{-1}(t) - \hat{Q}_{-1}(t) \right| dt.$$

As in the proof of Proposition 2.3, we integrate (4.9) along a trapezoid $\mathcal{T}_{a,b}^{0,t}$. Then we use the definition of $Q_{\pm 1}$, $\hat{Q}_{\pm 1}$ with L_g the Lipschitz constant of g to get the following inequality:

$$\|\rho(t, \cdot) - \hat{\rho}(t, \cdot)\|_{L^1((a,b))} \leq \|\rho_0 - \hat{\rho}_0\|_{L^1((a-L_f t, b+L_f t))} + 2 \int_0^t \int_{-1}^1 L_g (\mathbb{1}_{(-1, -\sigma)} w_{-1} + \mathbb{1}_{(\sigma, 1)} w_1) |\rho - \hat{\rho}| dx ds.$$

Eventually, using Holder's inequality and Gronwall's Lemma, we get:

$$(4.10) \quad \|\rho(t, \cdot) - \hat{\rho}(t, \cdot)\|_{L^1((a,b))} \leq \|\rho_0 - \hat{\rho}_0\|_{L^1((a-L_f t, b+L_f t))} e^{Ct},$$

where $C := 2L_g \|\mathbb{1}_{(-1, -\sigma)} w_{-1} + \mathbb{1}_{(\sigma, 1)} w_1\|_\infty$. Consequently, there is at most one solution in the sense of Definition 4.1 associated to a fixed turning curve ξ and an initial datum ρ_0 .

In order to recover the continuity of the operator \mathcal{D}_g we proceed as in the proof of Proposition 2.5. We first cover any compact set contained in $\{\xi(t) < x < 1\}$ by trapezoids. Without loss of generality, we can suppose those trapezoids are at distance at least ϵ of both interfaces $\{x = \xi(t)\}$ and $\{x = 1\}$. Consequently, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$, $\text{sign}(x - \xi(t)) = \text{sign}(x - \xi_n(t))$ on any trapezoid. Then, for all $n \geq n_0$, ρ_n is a Kruzhkov entropy solution of (4.1) on each trapezoid. We recover compactness thanks to the averaging compactness lemma. This reasoning can be reproduced in the three other parts of the domain $\{x < -1\}$, $\{-1 < x < \xi(t)\}$ and $\{x > 1\}$. Then, we can pass to the limit via dominated convergence in equation (4.4) and in all the inequalities (4.6)-(4.7)-(4.8). We conclude the proof with the same classical arguments as the proof of Proposition 2.5. We use Proposition 4.4 to prove that \mathcal{D}_g is well defined. That ends the proof of Corollary 4.5. \square

We are ready to state the main result of this section which is an analog of Theorem 1.7.

THEOREM 4.6. *Let ρ_0 verify (1.2). Assume that f verifies (1.7)-(1.8). Let g (resp. $w_{1,-1}$) satisfy (4.3) (resp. (4.2)). Let $\mathcal{I} : (B, \|\cdot\|_{L^1((0,T) \times \mathbb{R})}) \rightarrow (\mathcal{C}^0([0, T], \mathbb{R}), \|\cdot\|_\infty)$. If there exists $r > 0$ and B a subset of $L^1((0, T) \times \mathbb{R})$ such that:*

- *the properties (1.13) hold,*
- *for all $\rho \in B$, for all $t \in [0, T]$, $\mathcal{I}[\rho](t) \in (-1, 1)$,*

then there exists (ρ, ξ) a solution to the problem (4.1)-(1.5c) (ρ being a solution in the sense of Definition 4.1 of (4.1)).

In particular, existence is verified for $\mathcal{I} = \mathcal{I}_0$ (for affine cost) or with $\mathcal{I} = \mathcal{I}_\delta, \tilde{\mathcal{I}}_\epsilon$ (for general cost verifying (1.3)).

Remark 4.7. More complex behavior at exits can be considered following the same guidelines. For instance, we can combine the Hughes dynamics on $(-1, 1)$ with the exit dynamics proposed in [6]. The model in [6] accounts for self-organization phenomena at exits that cannot be captured by the model of [9].

5. Conclusion and discussions. The approach of the present paper applies to many variations of Hughes' model. However, up to our knowledge, the problem (1.1) for a general cost satisfying (1.3) is still not solved. Generalization of our approach to Lipschitz cost function c would require relaxing the Lipschitz regularity of ξ . In this respect, let us point out that this regularity is used throughout this paper in two different key arguments. First, we use the strong traces along the turning curve ξ in order to get the uniqueness of the corresponding ρ (making the operator \mathcal{D}_0 (and \mathcal{D}_g) single-valued). Second, the classical compact embedding of $B_{W^{1,\infty}}(r)$ in \mathcal{C}^0 is what grants compactness to the composed operator $\mathcal{D}_0 \circ \mathcal{I}$ (and $\mathcal{D}_g \circ \mathcal{I}$).

Remark 5.1. Our fixed-point approach could be generalized with $\xi \in W^{1,1}((0, T))$, this requires some technical work on the strong traces of ρ along the ξ curve. For examples we had in mind, $W^{1,\infty}$ turned out to be sufficient.

Remark 5.2. Considering ξ less regular than $W^{1,1}((0, T))$ presents a serious difficulty for the theory of discontinuous-flux conservation laws [5]. We want to mention that, in [25], existence of a solution (ρ, ξ) is proven, where $\xi \in BV_{loc}$. However the notion of solution for ρ may not ensure uniqueness, even for a fixed $\xi \in BV_{loc}$.

Appendix A. Convergence of the finite volume scheme for (4.1). In order to prove existence of a solution to (4.1) in the sense of Definition 4.1, we construct a convergent finite volume scheme adapted to the shape of the fixed turning curve ξ , following [27]. At the exits we use an operator splitting method with a scheme that accounts for the constraints Q_1 and Q_{-1} as in [8].

We now present the scheme used in this setting. Let $T, J \in \mathbb{N}$ such that:

$$(CFL) \quad 2(\|f'\|_\infty + \|\dot{\xi}\|_\infty) \frac{J}{T} \leq 1.$$

We construct the following scheme:

$$(A.1a) \quad \Delta t = \frac{1}{T}, \quad t^n := n\Delta t,$$

$$(A.1b) \quad \Delta x = \frac{1}{J}, \quad x_j = j\Delta x,$$

$$(A.1c) \quad s^n := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \dot{\xi}(s) ds, \quad s_\Delta(t) := \sum_1^N \mathbb{1}_{[t^n, t^{n+1})}(t) s^n,$$

$$(A.1d) \quad \xi_\Delta(t) := \xi(0) + \int_0^t s_\Delta(s) ds, \quad \xi^n = \xi_\Delta(t^n).$$

The discretization (A.1c)-(A.1d) of the ξ interface is detailed in [27, Section 3.1] where it is required to construct the adapted mesh. For any n , we denote j_n the unique element of $\llbracket -J, J \rrbracket$ such that $\xi^n \in [x_{j_n}, x_{j_n+1})$. We construct the mesh $(\mathcal{P}_{j+1/2}^n)_{n,j}$ as follows:

$$(A.1e) \quad \chi_j^n := \begin{cases} (t^n, x_j) & \text{if } j \leq j_n - 1 \\ (t^n, \xi^n) & \text{if } j = j_n \\ (t^n, x_j) & \text{if } j \geq j_n + 1, \end{cases} \quad \mathcal{P}_{j+1/2}^n := \begin{cases} \text{the rectangle } \chi_j^n \chi_j^{n+1} \chi_{j+1}^{n+1} \chi_{j+1}^n & \text{if } j \leq j_n - 2 \\ \text{the trapezoid } \chi_{j_n-1}^n \chi_{j_n-1}^{n+1} \chi_{j_n+1}^{n+1} \chi_{j_n}^n & \text{if } j = j_n - 1 \\ \text{the trapezoid } \chi_{j_n}^n \chi_{j_n+1}^{n+1} \chi_{j_n+2}^{n+1} \chi_{j_n+2}^n & \text{if } j = j_n \\ \text{the rectangle } \chi_{j+1}^n \chi_{j+1}^{n+1} \chi_{j+2}^{n+1} \chi_{j+2}^n & \text{if } j \geq j_n + 1. \end{cases}$$

Notice that, thanks to the (CFL) condition, $x_{j_n-1} < \xi^{n+1} < x_{j_n+2}$ so that the trapezoids defined above are never reduced to a triangle. We denote $\overline{\mathcal{P}_{j+1/2}^n}$ (resp. $\overline{\mathcal{P}_{j+1/2}^{n+1}}$) the bottom (resp. the top) segment of the tapezoid $\mathcal{P}_{j+1/2}^n$. However, now that the mesh is modified we have two different partitions for the line $t = t^{n+1}$: $(\overline{\mathcal{P}_{j+1/2}^{n+1}})_{j \in \mathbb{Z}}$ and $(\overline{\mathcal{P}_{j+1/2}^n})_{j \in \mathbb{Z}}$. We define $(\bar{\rho}_{i+1/2}^{n+1})_{i \in \mathbb{Z}}$ corresponding to the values of ρ^{n+1} on $(\overline{\mathcal{P}_{i+1/2}^{n+1}})_{i \in \mathbb{Z}}$ and $(\rho_{j+1/2}^n)_{j \in \mathbb{Z}}$ the projection of this values on $(\overline{\mathcal{P}_{j+1/2}^n})_{j \in \mathbb{Z}}$. To be precise, we set:

$$(A.1f) \quad \bar{\rho}_{j+1/2}^{n+1} = \frac{\rho_{j+1/2}^n \left| \overline{\mathcal{P}_{j+1/2}^n} \right| - \Delta t (f_{j+1}^n - f_j^n)}{\left| \overline{\mathcal{P}_{j+1/2}^n} \right|},$$

$$(A.1g) \quad \rho_{j+1/2}^{n+1} := \frac{1}{\left| \overline{\mathcal{P}_{j+1/2}^{n+1}} \right|} \sum_{i \in \mathbb{Z}} \left| \overline{\mathcal{P}_{j+1/2}^{n+1}} \cap \overline{\mathcal{P}_{i+1/2}^n} \right| \bar{\rho}_{i+1/2}^{n+1},$$

$$(A.1h) \quad \rho_\Delta(t, x) := \sum_{n=0}^N \sum_{j \in \mathbb{Z}} \rho_{j+1/2}^n \mathbb{1}_{\overline{\mathcal{P}_{j+1/2}^n}}(t, x).$$

We now want to define the numerical fluxes $(f_j^n)_{j \in \mathbb{Z}}$ corresponding to the left and the right edges of the trapezoids. We will use Godunov numerical flux, namely,

$$\mathbf{God}_f(a, b) := \begin{cases} \min_{c \in [a, b]} f(c) & \text{if } a \leq b \\ \max_{c \in [b, a]} f(c) & \text{if } a \geq b. \end{cases}$$

We first define the non-local constraint approximation:

$$(A.1i) \quad \rho_{\Delta x}^n(\cdot) = \sum_{j \in \mathbb{Z}} \rho_{j+1/2}^n \mathbb{1}_{[x_j^n, x_{j+1}^n)}(\cdot),$$

$$(A.1j) \quad q_1^n := g \left(\int_{\sigma}^1 \rho_{\Delta x}^n(x) \omega_1(x) dx \right),$$

$$(A.1k) \quad q_{-1}^n := g \left(\int_{-1}^{-\sigma} \rho_{\Delta x}^n(x) \omega_{-1}(x) dx \right),$$

$$(A.1l) \quad F(\rho_{j-1/2}^n, \rho_{j+1/2}^n) = \begin{cases} \min \left\{ \mathbf{God}_f(\rho_{j-1/2}^n, \rho_{j+1/2}^n), q_1^n \right\} & \text{if } j-1 = J \\ \max \left\{ \mathbf{God}_{-f}(\rho_{j-1/2}^n, \rho_{j+1/2}^n), -q_{-1}^n \right\} & \text{if } j = -J \\ \mathbf{F}_{int}^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n) & \text{if } j = j_n \\ \mathbf{God}_f(\rho_{j-1/2}^n, \rho_{j+1/2}^n) & \text{if } j > j_n \text{ and } j-1 \neq J \\ \mathbf{God}_{-f}(\rho_{j-1/2}^n, \rho_{j+1/2}^n) & \text{if } j < j_n \text{ and } j \neq -J. \end{cases}$$

Eventually, we define \mathbf{F}_{int}^n as in [7] (see details in Subsections 2.5, 3.3 and 5.1):

$$(A.1m) \quad \begin{aligned} f_{L,R}^n(\rho) &:= \pm f(\rho) - s^n \rho, \\ \forall (\rho_L, \rho_R) \in [0, 1]^2, \exists k \in [0, 1] \text{ s.t. } \mathbf{God}_{f_L}^n(\rho_L, k) &= \mathbf{God}_{f_R}^n(k, \rho_R), \\ \mathbf{F}_{int}^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n) &:= \mathbf{God}_{f_L}^n(\rho_{j-1/2}^n, k) = \mathbf{God}_{f_R}^n(k, \rho_{j+1/2}^n). \end{aligned}$$

Numerical simulations with such a scheme can be found in [7, Sect. 5.1] for the case of open-end condition at exits.

We are now in a position to start the proof of convergence, which merely assembles, with the help of the partition-of-unity technique of [27, 7], the arguments from [7] (for the inner interface situated at $x = \xi(t)$) and from [8] (for the constraints set at $x = \pm 1$).

Proof of Proposition 4.4. The proof follows the general idea of [27, Sect. 4], see also [7]. Since the interfaces $\{x = -1\}$, $\{x = \xi(t)\}$ and $\{x = 1\}$ are non-intersecting, we isolate them in the supports of a partition of unity ϕ_{-1} , ϕ_0 and ϕ_1 . We fix a test function ϕ . Taking (the discretization of) the test function $\phi_0 \phi$ we can use the specific result for the Hughes' model treated in [7, Sect. 5.1] to recover the approximate entropy inequalities satisfied by the discrete solution, with the test function $\phi_0 \phi$. For test functions $\phi_{-1} \phi$ and $\phi_1 \phi$, we use in the same way the result of [8, Prop. 3.1]. Summing up the contributions of the three parts of the partition of unity, we obtain approximate entropy inequalities for the discrete solution, with arbitrary test function ϕ . In addition, the integral weak formulation for the approximate solution follows from the scheme's conservativity. We use the same compactness argument as in [27, Sect. 3.4]. We can pass to the limit in the approximate weak formulation and in the approximate entropy inequalities, for the chosen converging subsequence and arbitrary test function. This allows us to characterize the limit as an entropy solution in the sense of Definition 4.1 of the problem at hand and proves Proposition 4.4. Additionally, thanks to the uniqueness proven in Theorem 4.5, the whole sequence of discrete solutions converges to the unique solution in the sense of Definition 4.1. \square

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