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WEAKLY ASYMMETRIC FACILITATED EXCLUSION PROCESS

GUILLAUME BARRAQUAND, ORIANE BLONDEL, AND MARIELLE SIMON

ABSTRACT. We consider the facilitated exclusion process, an interacting particle system on the integer line where particles hop to one of their left or right neighbouring site only when the other neighbouring site is occupied by a particle. A peculiarity of this system is that, starting from the step initial condition, the density profile develops a downward jump discontinuity around the position of the first particle, unlike other exclusion processes such as the asymmetric simple exclusion process (ASEP). In the weakly asymmetric regime, we show that the field of particle positions around the jump discontinuity converges to the solution of the multiplicative noise stochastic heat equation (*i.e.* the exponential of a solution to the KPZ equation) on a half-line subject to Dirichlet boundary condition, with initial condition given by the derivative of a Dirac delta function. We prove this result by reformulating the problem in terms of ASEP on a half-line with a boundary reservoir, for which we extend known proofs of convergence to deal with Dirichlet boundary condition and the very singular type of initial condition that arises in our case.

1. INTRODUCTION

1.1. Facilitated exclusion process. The facilitated exclusion process (FEP) was introduced in the physics literature [RPSV00] as a representative of a universality class for absorbing phase transitions. It is an interacting particle system on a lattice in which particles can jump to empty neighbors provided there is a particle in their neighborhood. So far our understanding of this model is restricted to dimension 1, where it has been studied under different lights. Its main feature is the absorbing transition mentioned above, at the critical particle density $1/2$: for particle densities below $1/2$ (subcritical regime), the system fixates on a configuration with isolated particles which cannot move, while for densities above $1/2$ (supercritical regime) it remains active forever and holes become eventually isolated.

The totally asymmetric version of this process (where particles only jump to the right) has been studied in [BM09] (approach to the phase transition) and [CZ18, GLS19, GLS21] (identification of the stationary states). Starting from a step initial condition, contrary to the well-studied totally asymmetric simple exclusion process (TASEP), a downstep leads to a rarefaction fan with a discontinuity [GKR10]. For the same initial condition, at large time t , particle positions fluctuate on the $t^{1/3}$ scale with Tracy-Widom GUE statistics, while the fluctuations of the rightmost particle, *i.e.* at the discontinuity, have Tracy-Widom GSE statistics [BBCS18a]. The symmetric FEP has been studied as well, on the periodic lattice. It was found [BESS20, BES21] that in the diffusive space-time scaling, and under the hydrodynamic limit, the macroscopic density ρ evolves according to a Stefan problem written as $\partial_t \rho = \partial_{uu}(\frac{2\rho-1}{\rho} \mathbf{1}_{\rho>1/2})$, with the space variable u belonging to the one-dimensional torus of size 1. In other words, starting the microscopic dynamics from a density profile with both supercritical and subcritical regions, the diffusive supercritical phase progressively invades the subcritical phase via moving interfaces, until one or the other phase disappears. For the partially asymmetric version, where particles jump to the right at rate $p < 1$ and to the left at rate $q < p$, invariant measures have been characterized on the torus [GKR10] and on the line [AGLS22]. Recently, [ESZ22] showed that the hydrodynamic limit (in the hyperbolic scaling) is given by the unique entropy solution of $\partial_t \rho + (2p-1)\partial_x \left(\frac{(1-\rho)(2\rho-1)}{\rho} \mathbf{1}_{\rho>1/2} \right)$, with $x \in \mathbb{R}$.

The next natural question concerns the fluctuations of the FEP in the asymmetric case (FASEP). As it has been noted for the totally asymmetric case in [BBCS18a], the problem can be reformulated in terms of the asymmetric simple exclusion process (ASEP) on the half-line with jump rates $p > q$, and with a specific boundary condition, where particles enter the

system at a rate p and can never exit (see Figure 1). For fixed p and q , we expect that, up to scaling constants, the fluctuations should be similar as for the totally asymmetric FEP¹.

In the present paper, we study the fluctuations of particle positions in the FASEP in a weakly asymmetric asymptotic regime. In the bulk (that is, far from the jump discontinuity of the density profile), we do not expect that the facilitation rule will have any effect on the scaling limit of fluctuations. In particular, we expect that if the asymmetry is properly scaled with time, the field of particle positions, appropriately rescaled, should converge to the solution of the Kardar-Parisi-Zhang (KPZ) equation on the real line with narrow wedge initial condition. This is consistent with the Tracy-Widom GUE asymptotics observed in the totally asymmetric setting. However, for the step initial condition, if we consider the field of particle positions in the FASEP around the first particle, the situation is more interesting and the facilitation rule plays a role. The limit should be described by a stochastic PDE on a semi-infinite interval with a specific boundary condition. The main goal of the present paper is to describe this stochastic PDE.

1.2. KPZ equation and Hopf-Cole transform. In order to state our main result, let us first recall how to solve the KPZ equation on the full one-dimensional line, which reads as

$$\partial_t h = \frac{1}{2} \partial_{uu} h + \frac{1}{2} (\partial_u h)^2 + \xi, \quad (1)$$

with ξ the standard space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. One usually considers the Hopf-Cole transform of a putative solution h , namely $\mathcal{Z}(t, u) = e^{h(t, u)}$. If we apply the chain rule in (1), ignoring all issues of regularity, the function \mathcal{Z} solves the Stochastic Heat Equation (SHE) with multiplicative noise

$$\partial_t \mathcal{Z} = \frac{1}{2} \partial_{uu} \mathcal{Z} + \mathcal{Z} \xi. \quad (2)$$

The latter equation can be solved through standard SPDE techniques, and whenever it can be shown that $\mathcal{Z} > 0$ [Mue91], this procedure yields a Hopf-Cole solution to (1). In their seminal paper [BG97], Bertini and Giacomin noticed that the discrete Hopf-Cole transform (introduced by Gärtner [Ga87])

$$Z_t(x) := e^{-\lambda h_t(x) + \nu t} \quad (3)$$

of ASEP height function $h_t(x)$ ($x \in \mathbb{Z}$), satisfies, for well-chosen parameters λ, ν , a martingale problem that is a discrete analogue of the martingale problem satisfied by \mathcal{Z} . In the weakly asymmetric regime, say $p = \frac{1}{2}e^\varepsilon, q = \frac{1}{2}e^{-\varepsilon}$ with $0 < \varepsilon \ll 1$, as assumed in this paper, it can then be showed that solutions of the discrete martingale problem converge to solutions of the continuous one. Given uniqueness of the solution to the continuous martingale problem, this is enough to conclude that ASEP height function, suitably rescaled, converges to a solution of the KPZ equation. Further, [DT16] identifies a whole class of models to which this method may apply, in the sense that a generalization of the discrete Hopf-Cole transform can be found and convergence to the SHE can be proved.

Other approaches to solving (1) that do not require a detour through the SHE were also looked for in the last decades, and can be useful in cases where there is no applicable discrete Hopf-Cole transform (unlike the present paper). Let us mention regularity structures [Hai13, Hai14]; energy solutions [GJ14, GP18], which have been applied *e.g.* in [BGS16, GJS17, GPS20, Yan18]; and paracontrolled distributions [GIP15, GP17].

1.3. KPZ equation on the positive half-line. A half-space analogue of the result from [BG97] was proved in [CS18]. More precisely, under some condition on injection and ejection rates at the origin of ASEP on the half-line, and assuming that the effective density at the origin imposed by those rates scales as $\rho \approx \frac{1}{2}(1 + (A + \frac{1}{2})\varepsilon)$, [CS18] shows that the height function of ASEP on the half-line converges to the KPZ equation on \mathbb{R}_+ with Neumann type boundary

¹Proving this would however require an analogous statement for ASEP on a half-line, and this has so far remained out of reach. Indeed, some exact formulas characterizing the distribution of particles have been proved in [TW13b, TW13a], and more recently in [BC22a], but these formulas are not amenable for asymptotic analysis. There exists one exception where the fluctuations of half-line ASEP have been analyzed asymptotically, in [BBCW18], but the boundary condition considered in that paper is different from the one which is relevant for the study of the FASEP, and the methods used there crucially depend on this specific boundary condition.

condition $\partial_x h(t, 0) = A$. As in [BG97], this result is proved via the discrete Hopf-Cole transform (3) of ASEP height function $h_t(x)$ ($x \in \mathbb{N}$), which is shown to converge to the SHE on \mathbb{R}_+ with Robin type boundary condition

$$\begin{cases} \partial_t \mathcal{Z} = \frac{1}{2} \partial_{uu} \mathcal{Z} + \mathcal{Z} \xi, \\ \partial_u \mathcal{Z}(t, 0) = A \mathcal{Z}(t, 0). \end{cases} \quad (4)$$

Since $\mathcal{Z}(t, \cdot)$ is not differentiable, the boundary condition should rather be imposed on the half-space heat kernel which is used to define the solution, we refer to [CS18] for details. The result of [CS18] was restricted to $A \geq 0$ and near equilibrium initial conditions (see Definition 3.8 below). It was then extended to all $A \in \mathbb{R}$ and the empty initial condition in [Par19]. Let us note that on the full-line, the extension of the convergence result of [BG97] to step initial condition was first discussed in [ACQ11], with considerably less details than in [Par19]. Some of these results were further extended in [Yan22] to generalizations of ASEP, in the spirit of [DT16]. An alternative way to make sense of the KPZ equation on a half-line via regularity structures was also considered in [GH19].

In the physics literature, the solution \mathcal{Z} to the SHE is understood as the partition function for a continuous Brownian directed polymer in a white noise potential ξ . The boundary parameter A can then be understood as controlling an extra energy collected by polymer paths, given by reflected Brownian motions, along the boundary [BBC16]. Alternatively, we may assume that there is no extra potential on the boundary, but the Brownian paths in the polymer partition function have elastic reflection on the boundary controlled by the parameter A . Discrete directed polymers are another family of models, with exclusion processes, which converge to the KPZ equation in full-space [AKQ14] or half-space [Wu20, Par22, BC22b].

Remark 1.1. The KPZ equation (and the associated SHE via Hopf-Cole transform) has also been considered on an interval with Neumann type boundary conditions. The discrete Hopf-Cole transform of open ASEP height function satisfies Robin type boundary conditions [GLM17] and converges to the KPZ equation on an interval [CS18, Par19, GPS20, Yan22].

1.4. Main results. As previously mentioned, the problem of fluctuations of the FASEP reduces to the fluctuations of ASEP on the half-line with, at the origin, injections of particles at the right-jump rate p and no ejection of particles. This translates into a microscopic boundary condition for the discrete Hopf-Cole transform (3) given by

$$Z_t(-1) = \mu Z_t(0), \quad \text{with } \mu \approx 1 - \varepsilon,$$

as stated more precisely in (11) below. In contrast, in the setting of [CS18, Par19], the boundary parameter μ is scaled as $\mu \approx 1 - \varepsilon^2 A$. We will prove below that in our case, the appropriate scaling of the solution is different from [CS18, Par19]: we will set $\mathcal{Z}_t^\varepsilon(u) = \varepsilon^{-2} Z_{\varepsilon^{-4}t}(\varepsilon^{-2}u)$ for any macroscopic point u , and prove that starting from the step initial condition for the FASEP (*i.e.* empty initial condition for the half-line ASEP), $\mathcal{Z}_t^\varepsilon(u)$ weakly converges as a continuous process (see Theorem 3.7 for a precise statement) to the solution of the SHE

$$\begin{cases} \partial_t \mathcal{Z} = \frac{1}{2} \partial_{uu} \mathcal{Z} + \mathcal{Z} \xi, \\ \mathcal{Z}(t, 0) = 0, \\ \mathcal{Z}(0, u) = -2\delta'_0(u). \end{cases} \quad (5)$$

We also prove a similar statement for near-equilibrium initial conditions (Theorem 3.9), under the same scaling as in [CS18]. The SHE with Dirichlet boundary condition ($\mathcal{Z}(t, 0) = 0$) was already considered in [Par22] in the context of directed polymer models, but only for another type of initial condition. Here the initial condition that we consider is very singular, this is the derivative of a delta function at 0. We provide a more precise definition of this stochastic PDE in Definition 3.3 and prove existence and uniqueness of the solution in Proposition 3.5 below. Dealing with this very singular initial condition is one of the main novelties of this paper. Eventually, our main result is therefore the following: the macroscopic fluctuations of the field of the first particles' positions in the weakly asymmetric FASEP are given (*via* the Hopf-Cole transform) by the solution to the KPZ equation on the half-line \mathbb{R}_+ , with initial condition being the derivative of a Dirac distribution, and with Dirichlet boundary condition at the origin.

1.5. Comparison with previous literature on half-space KPZ equation. In a sense, the microscopic boundary condition $Z_t(-1) = \mu Z_t(0)$ with $\mu \approx 1 - \varepsilon$ that we are considering is the $A \rightarrow \infty$ limit of the setting in [CS18, Par19] which considered $\mu \approx 1 - A\varepsilon^2$, and it poses no additional difficulty in terms of handling the boundary condition. However, as already said, our initial condition is much more singular than the ones in [CS18, Par19, Par22]. Proving tightness requires us to develop precise estimates about the Dirichlet heat kernel and its discrete analogue. Further, the rigorous identification of the initial condition requires some control on the second moment of $Z_t^\varepsilon(u)$, and the estimate that we need turned out to be surprisingly difficult to prove. Eventually, we use an explicit second moment integral formula for half-line ASEP from the recent preprint [BC22a], coming from a Markov duality (such use of Markov duality also arises in the proof of convergence of the stochastic six-vertex model height function to the KPZ equation in [CGST20]).

The continuous directed polymer model corresponding to the solution of (5) was considered in the physics paper [GLD12]. The polymer paths are conditioned not to hit the boundary, and [GLD12] studied the distribution of the partition function of polymers starting and ending at a location η , after letting $\eta \rightarrow 0$ and appropriately rescaling the partition function by η^2 . Restricting on test functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $f(0) = 0$, the distribution $\frac{1}{\eta} \delta_0(\cdot - \eta)$, where δ_0 is the delta Dirac distribution, converges to $-\delta'_0$. This explains why the initial condition that we consider in the present paper is the physically natural one to consider for the SHE with Dirichlet boundary condition, though the solution was never mathematically constructed before.

Finally, let us mention that the KPZ equation on a half-line with Dirichlet boundary condition is also considered in [GH19], but there the Dirichlet boundary condition $h(t, 0) = 0$ is imposed on the KPZ equation itself and not on the SHE, so that this corresponds to a completely different stochastic PDE than the one we consider in the present paper.

1.6. Outline of the paper. In Section 2, we define the FASEP and ASEP and construct a mapping that connects the two. Section 3 is devoted to the presentation of the Hopf-Cole transform, in terms of which we state our main results on the fluctuations of the particle positions in a weakly asymmetric regime. We then provide several preliminary results: first, the existence and uniqueness of the macroscopic SHE with δ'_0 initial condition and Dirichlet boundary condition (Section 4), and second, some explicit estimates on the discrete heat kernel with diverging Robin boundary condition (Section 5). Finally, the main convergence results cover two types of initial conditions which require different scalings: the step (resp. derivative of delta) initial condition (Theorem 3.7) and near-equilibrium initial conditions (Theorem 3.9), which both require new arguments. We actually need the understanding of the latter as an intermediate step towards Theorem 3.7, so we start by proving Theorem 3.9 in Section 6. Our main result, Theorem 3.7 is finally proved in Section 7.

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2. MICROSCOPIC MODELS AND MAPPING

In the following, \mathbb{N} denotes the set of non-negative integers, \mathbb{N}^* the set of positive integers.

The *facilitated asymmetric exclusion process* in dimension one (FASEP) is a Markov process on the state space $\Omega := \{0, 1\}^{\mathbb{Z}}$ which is denoted by $\{\eta_t(x) ; x \in \mathbb{Z}\}_{t \geq 0}$. Each component $\eta_t(x) \in \{0, 1\}$ is the occupation variable of the configuration of particles at site $x \in \mathbb{Z}$.

Let $p, q \in (0, 1)$ be two asymmetry parameters. The time evolution of the particle configurations is ruled by the Markov generator \mathcal{L}_F which acts on functions $f : \Omega \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} \mathcal{L}_F f(\eta) &= \sum_{x \in \mathbb{Z}} p \eta(x-1) \eta(x) (1 - \eta(x+1)) [f(\eta^{x,x+1}) - f(\eta)] \\ &\quad + \sum_{x \in \mathbb{Z}} q \eta(x+1) \eta(x) (1 - \eta(x-1)) [f(\eta^{x,x-1}) - f(\eta)] \end{aligned} \quad (6)$$

where $\eta^{x,y}$ is the configuration obtained from η after the exchange of the occupation variables $\eta(x) \leftrightarrow \eta(y)$, namely: $\eta^{x,y}(z) := \eta(x)\mathbf{1}_{z=y} + \eta(y)\mathbf{1}_{z=x} + \eta(z)\mathbf{1}_{z \notin \{x,y\}}$. In other words, as it can be read on the generator, particles are displayed on the lattice \mathbb{Z} and jump to their neighbouring sites at rates which encode the following rules:

- a jump to the right from site x to site $x+1$ occurs with rate p if and only if site x is occupied by a particle, site $x+1$ is empty *and* site $x-1$ is occupied ;
- a jump to the left from site x to site $x-1$ occurs with rate q if and only if site x is occupied by a particle, site $x-1$ is empty *and* site $x+1$ is occupied.

We say that site $x \in \mathbb{Z}$ is occupied by an *active particle* if it can jump either to $x-1$ or $x+1$ with positive rate, in other words, it is such that

$$\eta(x-1)\eta(x)(1-\eta(x+1)) + (1-\eta(x-1))\eta(x)\eta(x+1) = 1.$$

Let us now map this process onto another exclusion process. We need to introduce some notation. Let $L < R$ be two integers, and define:

$$\bar{\Omega}_{L,R} := \left\{ \eta \in \Omega ; \begin{cases} \eta(x) = 0 & \text{if } x \geq R, \\ \eta(x) = 1 & \text{if } x \leq L, \\ \eta(x) + \eta(x+1) \geq 1 & \text{if } x < R. \end{cases} \right\}$$

and

$$\bar{\Omega} := \bigcup_{L < R} \bar{\Omega}_{L,R}.$$

In other words, if $\eta \in \bar{\Omega}_{L,R}$ then the active particles are all contained in the box $\{L, L+1, \dots, R-1\}$. The set $\bar{\Omega}$ is remarkable because it is preserved along time evolution of the dynamics generated by \mathcal{L}_F , as stated in the following (straightforward) lemma:

Lemma 2.1. *If $\eta_0 \in \bar{\Omega}$, then, for any $t > 0$, $\eta_t \in \bar{\Omega}$.*

For any $\eta \in \bar{\Omega}$, we can label particles from right to left, by the following recursive procedure: assume that $\eta \in \bar{\Omega}_{L,R}$ for some $L < R$, and define²

$$\begin{aligned} X_1(\eta) &:= R-1, \\ X_{i+1}(\eta) &:= \max \{j < X_i(\eta) ; \eta(j) = 1\}. \end{aligned}$$

We are now ready to construct the mapping: let $\mathfrak{S} : \bar{\Omega} \rightarrow \{0, 1\}^{\mathbb{N}^*}$ be the application such that,

$$\text{for any } \eta \in \bar{\Omega}, \quad \sigma := \mathfrak{S}(\eta) \quad \text{satisfies} \quad \sigma(i) = 1 - \eta(X_i - 1).$$

In other words, site i is occupied by a particle in σ (*i.e.* $\sigma(i) = 1$) if and only if the i -th particle in η has an empty site to its left.

In particular, the *step initial condition* η_0 given by

$$\eta_0(x) = \mathbf{1}_{x \leq x_0} \quad \text{for some } x_0 \in \mathbb{Z},$$

belongs to $\bar{\Omega}$, and it corresponds to an *empty configuration* $\sigma_0 = \mathfrak{S}(\eta_0)$, namely

$$\sigma_0(x) = 0 \quad \text{for any } x \in \mathbb{N}^*. \quad (7)$$

These initial conditions will be important in what follows. The dynamics of the mapped process is described in the following lemma. Note that with the step initial condition, for all $t \geq 0$, $X_i(\eta_t) - X_{i+1}(\eta_t) \in \{1, 2\}$, and $\mathfrak{S}(\eta_t)$ encodes all the information about these spacings.

²Note that R is a function of η since it is the position of the right-most particle, therefore X_1 is well defined.

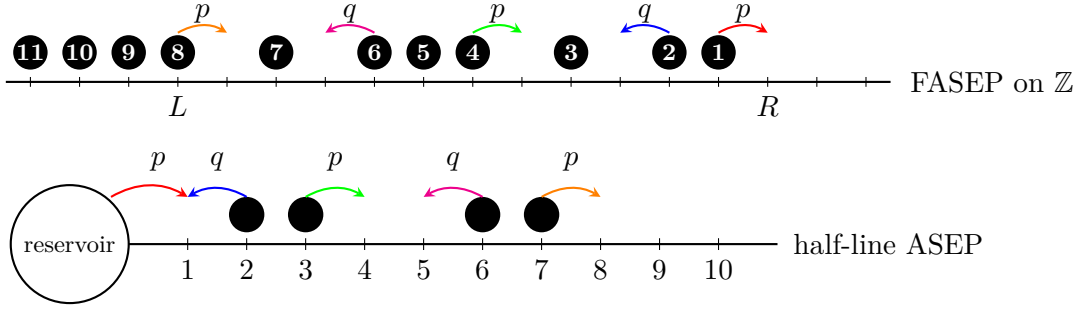


FIGURE 1. The top figure represents a configuration η in $\bar{\Omega}$ and the possible transitions with their respective rates. The bottom figure represents the mapped configuration $\mathfrak{S}(\eta)$ on $\{0, 1\}^{\mathbb{N}^*}$ and the possible transitions, in the same color as the corresponding transitions in η .

Lemma 2.2. *Assume that the generator of the Markov process $\{\eta_t\}_{t \geq 0}$ is \mathcal{L}_F given in (6), then $\{\mathfrak{S}(\eta_t)\}_{t \geq 0}$ is an asymmetric simple exclusion process on the infinite half-line \mathbb{N}^* with a boundary reservoir which injects particles at rate p . More precisely its generator is given as follows: for any $\sigma \in \Sigma := \{0, 1\}^{\mathbb{N}^*}$, for any $f : \Sigma \rightarrow \mathbb{R}$,*

$$\begin{aligned} \mathcal{L}f(\sigma) = & \sum_{x=1}^{\infty} \left(p\sigma(x)(1 - \sigma(x+1)) + q\sigma(x+1)(1 - \sigma(x)) \right) \left[f(\sigma^{x,x+1}) - f(\sigma) \right] \\ & + p(1 - \sigma(1)) \left[f(\sigma^1) - f(\sigma) \right], \end{aligned}$$

where σ^1 is obtained after the creation of one particle at site 1, namely $\sigma^1(x) = \mathbf{1}_{x=1} + \sigma(x)\mathbf{1}_{x \neq 1}$.

Proof. The proof is straightforward by looking at every possible transition, see Figure 1. \square

3. MAIN RESULTS AND STRATEGY OF THE PROOF

From now on we consider the half-line ASEP denoted by $\{\sigma_t\}_{t \geq 0}$, which is a Markov process on $\Sigma = \{0, 1\}^{\mathbb{N}^*}$ generated by \mathcal{L} given in (8).

3.1. Hopf-Cole transform. Let us define the *height function* associated with the particle system, in the following (standard) way: for any $x \in \mathbb{N}^*$,

$$h_t(x) := h_t(0) + \sum_{k=1}^x (2\sigma_t(k) - 1), \quad \text{with } h_t(0) = 2 \sum_{k=1}^{\infty} \sigma_0(k) - 2 \sum_{k=1}^{\infty} \sigma_t(k).$$

This is well-defined when $\sum_{k=1}^{\infty} \sigma_0(k) < \infty$. In that case, note that $-h_t(0)$ is equal to twice the number of particles that have entered into the system between times 0 and t (no particle can exit the system by hypothesis). For any initial condition $\sigma_0 \in \Sigma$ and any $t > 0$, this number is bounded by a Poisson random variable with parameter pt , and in particular, it is almost surely finite. In the case where the initial number of particles in the system is infinite, we can still define $h_t(0)$ as minus twice the number of particles that have entered the system. Thus, starting from any initial condition $\sigma_0 \in \Sigma$, the height function satisfies: for any $t > 0$, any $x \in \mathbb{N}$, $h_t(x) < \infty$ a.s. For $\nu, \lambda \in \mathbb{R}$ which will be chosen later we then define the *Hopf-Cole transform*

$$Z_t(x) := e^{-\lambda h_t(x) + \nu t}, \quad x \in \mathbb{N}. \quad (8)$$

Alternatively, Z_t can be defined as a function of the positions of the particles in the FASEP. It is straightforward to check that, under the coupling described in Section 2, for the FASEP started from the step initial condition, the position of the j -th particle in the FASEP is related to the height function with empty initial condition through

$$X_j(\eta_t) = \sum_{k=j}^{\infty} \sigma_t(k) - j.$$

Consequently, the Hopf-Cole transform can be recast as

$$Z_t(x) = e^{2\lambda X_{x+1}(\eta_t) + (3x+2)\lambda + \nu t}. \quad (9)$$

We will now mainly work with the Hopf-Cole transform Z and we state our results in terms of this quantity. In the following, for any function $f : \mathbb{Z} \rightarrow \mathbb{R}$ we define its left and right gradients by

$$\nabla^+ f(x) := f(x+1) - f(x), \quad \nabla^- f(x) := f(x-1) - f(x)$$

and its discrete Laplacian by $\Delta f(x) := f(x+1) + f(x-1) - 2f(x)$. One knows that Z satisfies

$$dZ_t(x) = (\nu Z_t(x) + \mathcal{L}Z_t(x))dt + dM_t(x), \quad x \in \mathbb{N},$$

where $\{M_t(x)\}_{t \geq 0}$ are martingales whose quadratic variations will be computed below. As in [CS18, Par19] (see also [GLM17]), we look for conditions on ν, λ so that $\nu Z_t(x) + \mathcal{L}Z_t(x)$ can be rewritten as $D\Delta Z_t(x)$ for some diffusion coefficient D . After straightforward computations (given in Appendix A) we choose

$$\lambda = \frac{1}{2} \log \frac{q}{p}, \quad \nu = q + p - 2\sqrt{pq}, \quad (10)$$

which imply: for any $x \in \mathbb{N}^*$,

$$dZ_t(x) = D\Delta Z_t(x)dt + dM_t(x), \quad \text{with } D = \sqrt{pq}. \quad (11)$$

It remains to define $Z_t(-1)$ in order that (11) remains valid at the boundary point $x = 0$, which can be done if we let

$$Z_t(-1) := \mu Z_t(0), \quad \text{with } \mu = \sqrt{\frac{q}{p}} = e^\lambda. \quad (12)$$

We therefore obtain the following result.

Lemma 3.1. *We assume the choice of parameters (10) and define $\mu = \sqrt{q/p}$. Let Δ^μ be the discrete Laplacian on \mathbb{N} with the following boundary condition:*

$$\Delta^\mu f(x) := \begin{cases} f(x+1) + f(x-1) - 2f(x) & \text{if } x > 0 \\ f(1) + \mu f(0) - 2f(0) & \text{if } x = 0. \end{cases} \quad (13)$$

Then, for any $x \neq y \in \mathbb{N}$, the following three quantities are martingales:

$$M_t(x) := Z_t(x) - Z_0(x) - D \int_0^t \Delta^\mu Z_s(x) dx, \quad M_t(x)^2 - \int_0^t \frac{d}{ds} [M(x)]_s ds, \quad M_t(x)M_t(y),$$

and moreover,

$$\frac{d}{dt} [M(x)]_t = \begin{cases} Z_t(x)^2 \left[\eta_t(x)(1 - \eta_t(x+1)) \frac{(p-q)^2}{p} + \eta_t(x+1)(1 - \eta_t(x)) \frac{(p-q)^2}{q} \right] & \text{if } x > 0, \\ Z_t(0)^2 (1 - \eta_t(1)) \frac{(p-q)^2}{p} & \text{if } x = 0. \end{cases}$$

Proof. This is straightforward, using the following identity:

$$\frac{Z_t(x+1)}{Z_t(x)} = \sqrt{\frac{q}{p}} (1 - \eta_t(x+1)) + \sqrt{\frac{p}{q}} \eta_t(x+1).$$

□

3.2. Weak asymmetry. From now on, we consider the half-line ASEP in the *weak asymmetry regime* where

$$p = \frac{1}{2} e^\varepsilon \quad \text{and} \quad q = \frac{1}{2} e^{-\varepsilon}, \quad \text{for } \varepsilon > 0. \quad (14)$$

Rewriting everything in terms of ε , the Hopf-Cole transform reads as

$$Z_t(x) = e^{\varepsilon h_t(x) + \nu t}, \quad \text{where } \nu = \frac{1}{2} e^\varepsilon (e^{-\varepsilon} - 1)^2 = \frac{1}{2} \varepsilon^2 + o(\varepsilon^2).$$

The boundary parameter μ appearing in (12) and the diffusion coefficient D are equal to

$$\mu = e^{-\varepsilon}, \quad D = \frac{1}{2}.$$

In this weak asymmetry regime, the quadratic variation of the above martingale satisfies:

Lemma 3.2. *As $\varepsilon \rightarrow 0$ we have*

$$\frac{d}{dt}[M(x)]_t = \varepsilon^2 Z_t(x)^2 + \nabla^+ Z_t(x) \nabla^- Z_t(x) + o(\varepsilon^2) Z_t(x)^2, \quad \text{for any } x > 0 \quad (15)$$

$$\frac{d}{dt}[M(0)]_t = \varepsilon^2 Z_t(0)^2 - \varepsilon Z_t(0) \nabla^+ Z_t(0) + o(\varepsilon^2) Z_t(0)^2. \quad (16)$$

Note that the term $\varepsilon Z_t(0) \nabla^+ Z_t(0)$ at the boundary is new in our case, it did not appear in the case of [CS18].

Proof. It is straightforward using

$$\nabla^+ Z_t(x) = \varepsilon Z_t(x) (2\eta_t(x+1) - 1 + o(1))$$

and Lemma 3.1 which implies that

$$\frac{d}{dt}[M(x)]_t = \varepsilon^2 \left(\frac{1}{2} Z_t(x)^2 + \frac{1}{2} Z_t(x+1) Z_t(x-1) \right) + \nabla^+ Z_t(x) \nabla^- Z_t(x) + o(\varepsilon^2) Z_t(x)^2.$$

□

3.3. Main theorems. Before stating our main results, let us start by defining the notion of solution for the stochastic heat equation which is at the core of the convergence results of this paper.

Definition 3.3. *Let ξ be the standard space-time white noise on $\mathbb{R}_+ \times \mathbb{R}_+$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $\mathcal{Z}_t(u)$ solves the stochastic heat equation (SHE)*

$$\partial_t \mathcal{Z} = \frac{1}{2} \partial_{uu} \mathcal{Z} + \mathcal{Z} \xi \quad (17)$$

on the time interval $[0, T]$, with Dirichlet boundary condition, and initial condition \mathcal{Z}_{ini} , if for any $t \in (0, T]$,

$$\mathcal{Z}_t(u) = \int_{\mathbb{R}_+} dv P_t^{\text{Dir}}(u, v) \mathcal{Z}_{\text{ini}}(v) + \int_0^t ds \int_{\mathbb{R}_+} dv P_t^{\text{Dir}}(u, v) \mathcal{Z}_s(v) \xi(s, v), \quad (18)$$

where the Dirichlet half-space heat kernel P_t^{Dir} is defined by

$$P_t^{\text{Dir}}(u, v) = \frac{1}{\sqrt{2\pi t}} \left(e^{-(u-v)^2/(2t)} - e^{-(u+v)^2/(2t)} \right). \quad (19)$$

3.3.1. Empty initial condition. We define, for any $u \in \varepsilon^2 \mathbb{N}$, the scaled process

$$\mathcal{Z}_t^\varepsilon(u) := \varepsilon^{-2} Z_{\varepsilon^{-4}t}(\varepsilon^{-2}u) \quad (20)$$

and we extend $\mathcal{Z}_t^\varepsilon(\cdot)$ to the continuous half-line \mathbb{R}_+ by linear interpolation.

Remark 3.4. Let us already emphasize here that the scaling in (20) is not the one which will appear later when the initial condition is supposed to be *near-equilibrium* (see Section 3.3.2, (24) and Definition 3.8), nor the one in [Par19], where the macroscopic initial condition is the delta Dirac function δ_0 and the prefactor there is ε^{-1} instead of ε^{-2} .

Let us consider the space of test functions

$$\mathcal{H} = \{ \phi \in C_c^\infty(\mathbb{R}) : \phi(0) = 0 \}. \quad (21)$$

The initial condition (7) implies that $Z_0(x) = \mu^x$, for any $x \in \mathbb{N}$, and therefore, for any $\phi \in \mathcal{H}$,

$$(\mathcal{Z}_0^\varepsilon, \phi) \xrightarrow{\varepsilon \rightarrow 0} 2\phi'(0), \quad (22)$$

where $(\mathcal{Z}, \phi) := \int_{\mathbb{R}} \mathcal{Z} \phi$ denotes the usual scalar product in $L^2(\mathbb{R})$. In other words, the initial condition of the continuous limit of the rescaled process $\mathcal{Z}_t^\varepsilon$ is $-2\delta'_0$ where δ'_0 is the derivative of the delta Dirac distribution.

Our first result, which will be proved in Section 4, is the existence and uniqueness of solutions to (17) for the δ'_0 initial condition and Dirichlet boundary condition:

Proposition 3.5. *Let ξ be the standard space-time white noise on $\mathbb{R}_+ \times \mathbb{R}_+$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There exists a $C(\mathbb{R}_+)$ -valued process $(\mathcal{Z}_t)_{t>0}$, which is adapted to the filtration $\mathcal{F}_t = \sigma(\{\xi(s, \cdot)\}_{s \leq t})$, and solves the SHE in the sense of Definition 3.3 with initial condition $\mathcal{Z}_{\text{ini}} = -2\delta'_0$. This solution is unique in the class of adapted continuous processes satisfying*

$$\sup_{\substack{X \in \mathbb{R}_+ \\ s \in (0, t)}} \left\{ s^2 \mathbb{E} \left[\mathcal{Z}_s(X)^2 \right] \right\} < \infty. \quad (23)$$

Remark 3.6. The existence and uniqueness of solutions to (17) are already proved in [Par22, Theorem 4.1, Prop. 4.2] when \mathcal{Z}_{ini} is *near equilibrium* (see Definition 3.8 below). This is not the case of the δ'_0 initial condition that we consider here, and therefore we need to provide a new proof. Existence and uniqueness is also proved in [Par19, Proposition 4.3] for Robin type boundary condition and δ_0 initial condition. As we will see below in Section 4, the proof of Proposition 3.5 involves different estimates than [Par19, Par22].

The main result of this paper is the following convergence:

Theorem 3.7. *Fix $T > 0$. Assume the initial particle configuration is empty as in (7). Then the rescaled process $\{\mathcal{Z}_s^\varepsilon\}_{s \in (0, T]}$ converges as $\varepsilon \rightarrow 0$ to the solution of the stochastic heat equation (17) on the time interval $[0, T]$ with Dirichlet boundary condition, and initial condition $\mathcal{Z}_{\text{ini}} = -2\delta'_0$ (as defined in Definition 3.3), in the sense of weak convergence of probability measures on the path space $D((0, T], C(\mathbb{R}_+))$ endowed with the Skorokhod topology.*

3.3.2. Near-equilibrium initial condition. We also study the simpler case of near-equilibrium initial condition. In this section we consider the scaled process

$$\mathcal{Z}_t^\varepsilon(u) := \varepsilon^2 \mathcal{Z}_t^\varepsilon(u) = Z_{\varepsilon^{-4}t}(\varepsilon^{-2}u), \quad (24)$$

which differs from $\mathcal{Z}_t^\varepsilon$ by a factor ε^2 , and we allow $Z_0(x)$ to be different from μ^x .

Definition 3.8. *We say that a sequence of random functions $\mathcal{F}^\varepsilon \in C(\mathbb{R}_+)$ is near-equilibrium if it satisfies the following: there exists a $a > 0$ such that, for any $n \in \mathbb{N}$, any $\alpha \in (0, \frac{1}{2})$, there exists some constant $C = C(\alpha, n) > 0$ such that, for any $u, u' \in \mathbb{R}_+$, any $\varepsilon > 0$,*

$$\|\mathcal{F}^\varepsilon(u)\|_n \leq C e^{au} \quad (25)$$

$$\|\mathcal{F}^\varepsilon(u) - \mathcal{F}^\varepsilon(u')\|_n \leq C |u - u'|^\alpha e^{a(u+v)}, \quad (26)$$

where $\|G\|_n := \mathbb{E}[|G|^n]^{1/n}$ denotes the L^n -norm with respect to the probability measure.

An intermediate – although important – result is the following:

Theorem 3.9. *Fix $T > 0$. Assume that the initial condition $\mathcal{Z}_0^\varepsilon \in C(\mathbb{R}_+)$ is near-equilibrium (in the sense of Definition 3.8), and that $\mathcal{Z}_0^\varepsilon$ weakly converges in $C(\mathbb{R}_+)$ to some initial condition \mathcal{Z}_{ini} as $\varepsilon \rightarrow 0$.*

Then the rescaled process $(\mathcal{Z}_s^\varepsilon)_{s \in [0, T]}$ converges as $\varepsilon \rightarrow 0$ to the solution of the stochastic heat equation (17) with initial condition \mathcal{Z}_{ini} in the time interval $[0, T]$ (as defined in Definition 3.3), in the sense of weak convergence of probability measures on the path space $D([0, T], C(\mathbb{R}_+))$ endowed with the Skorokhod topology.

3.3.3. An example of near-equilibrium initial condition. Theorem 3.9 may be useful independently from its application to the proof of Theorem 3.7. Let us consider the half-line ASEP generated by \mathcal{L} in the weakly asymmetric regime (14), but with initial condition given by product Bernoulli, that is, we assume that the variables $\sigma_0(k), k \in \mathbb{N}^*$, are independent Bernoulli variables with parameter ϱ . Let us scale $\varrho = (1 - \varepsilon(B + \frac{1}{2}))/2$, so that $\varepsilon h_0(\varepsilon^{-2}u)$ converges to a Brownian motion with drift $-(B + 1/2)$. Then, it can be shown that $\mathcal{Z}_0^\varepsilon$ is near-equilibrium³, and thus Theorem 3.9 can be applied, to find that $(\mathcal{Z}_s^\varepsilon)$ converges to the SHE with Dirichlet boundary condition, and initial condition given by the exponential of a Brownian motion with drift $-(B + 1/2)$.

³The term near-equilibrium actually comes from the fact that such initial condition is stationary for ASEP on \mathbb{Z} .

Denoting by $\mathcal{Z}(t, u)$ this solution, [Par22] showed that we have the identity in distribution

$$\lim_{u \rightarrow 0} \frac{\mathcal{Z}(t, u)}{u} = \widetilde{\mathcal{F}}(t, 0) \quad (27)$$

where $\widetilde{\mathcal{F}}$ is the solution of the SHE on \mathbb{R}_+ with Robin boundary parameter B and delta Dirac function as initial condition. In the special case $B = -1/2$, that is $\varrho = 1/2$, the law of $\widetilde{\mathcal{F}}(t, 0)$ is explicitly known and related to eigenvalue GOE (Gaussian Orthogonal Ensemble) statistics [BBCW18, Par19]. For other values of B , the law of $\widetilde{\mathcal{F}}(t, 0)$ was computed very recently [IMS22] (see also [KLD20]).

Remark 3.10. Discrete analogues of the identity (27), allowing to exchange the roles of the boundary and initial condition parameters, also exist for directed polymers [BBC20, Prop. 8.1], last passage percolation [BBCS18b, Lemma 6.1] or more general models defined through Pfaffian Schur measures [BR01, Corollary 7.6]. This suggests that the height function at the origin for ASEP on \mathbb{N}^* may have the same distribution in the following two situations:

- (1) The reservoir has injection rate p , ejection rate 0, and the initial condition is i.i.d. Bernoulli with parameter ϱ ;
- (2) The reservoir has injection parameter $\alpha = p\varrho$, ejection parameter $\gamma = q(1 - \varrho)$, and the initial configuration is empty.

There exists such an identity in distribution when $p = 1, q = 0$, that is in the case of TASEP⁴. In the weakly asymmetric regime $\varepsilon \rightarrow 0$, such an identity would yield (27) (which is proved in [Par22] through another route). We leave this as an open question in the case of ASEP.

3.4. Strategy of the proof. As stated in Definition 3.3, a mild solution $\mathcal{Z}_t(x)$ to the stochastic heat equation (17) with Dirichlet boundary condition satisfies (18). Solutions can equivalently be characterized by the following martingale problem. The equivalence of the two notions of solutions is proved in [Par19, Prop. 4.4] in the case of Neumann type boundary condition. The argument applies mutatis mutandis in the Dirichlet case.

Definition 3.11. A solution $\mathcal{Z}_t(u)$ to the martingale problem for the stochastic heat equation (17) with Dirichlet boundary condition, initial condition \mathcal{Z}_{ini} and time interval $[0, T]$, is a random variable \mathcal{Z} with values in $C([0, T], C(\mathbb{R}_+))$, such that for all $\phi \in \mathcal{H}$ and $0 < t \leq T$, the following quantities are martingales:

$$N_t(\phi) := (\mathcal{Z}_t, \phi) - (\mathcal{Z}_{\text{ini}}, \phi) - \frac{1}{2} \int_0^t (\mathcal{Z}_s, \phi'') ds, \quad (28)$$

$$Q_t(\phi) := N_t(\phi)^2 - \int_0^t (\mathcal{Z}_s^2, \phi^2) ds. \quad (29)$$

We divide the proof of Theorem 3.7 (Section 7) and Theorem 3.9 (Section 6) into several steps, following the strategy of [Par19]:

- (1) For a process $\mathcal{Z}_t^\varepsilon$ starting from a near-equilibrium initial condition $\mathcal{Z}_0^\varepsilon$, which satisfies that $\mathcal{Z}_0^\varepsilon \Rightarrow \mathcal{Z}_{\text{ini}}$ as $\varepsilon \rightarrow 0$, we prove its convergence towards the solution to the heat equation with Dirichlet boundary condition starting from \mathcal{Z}_{ini} (a.k.a. Theorem 3.9). This is split in two steps: proof of tightness (Section 6.1), and identification of the limit point (Section 6.2). The latter step uses the martingale problem above: we show that the discrete martingale problem gives the continuous one in the limit, and the control of the error terms is a consequence of the tightness estimates.
- (2) We come back to the initial condition $Z_0(x) = \mu^x$ of Theorem 3.7. We prove that at time $\delta > 0$, $\mathcal{Z}_\delta^\varepsilon$ defined in (20) is *near-equilibrium* in the sense of Definition 3.8. This is the purpose of Section 7.1. Similarly to the first point, this property gives us tightness in $D([\delta, T], C(\mathbb{R}_+))$ for any $\delta \in (0, T)$, and that any limit point is solution to (17) with Dirichlet boundary condition.

⁴More precisely, one needs to consider the TASEP on \mathbb{N}^* with an initial condition such that the site 1 is occupied, and all other sites are occupied according to Bernoulli i.i.d. random variables. Then, the height function for this model has the same distribution [PS01] as the height function associated to the last passage percolation model with boundary considered in [BBCS18b]. The symmetry between the boundary and the initial condition parameters follows from [BBCS18b, Lemma 6.1] (choosing the parameter α there as $\alpha = \varrho$).

- (3) The missing point is to push δ to 0. We use consistency, and identify the initial condition. The structure of the argument is similar to [Par19] but we use a different method in order to obtain second moment bounds (Lemma 7.5). We rely on the analysis of exact integral formulas for the moments of $Z_t(x)$ obtained using a Markov duality in [BC22a].

4. EXISTENCE AND UNIQUENESS OF THE SOLUTION

This section is devoted to the proof of existence and uniqueness of the solution to the SHE (Definition 3.3) with Dirichlet boundary condition and initial condition $\mathcal{Z}_{\text{ini}} = -2\delta'_0$. Although we follow a standard argument from [Wal86] of Picard iteration (see also [Par19, Section 4]), the initial condition that we consider is much more singular than initial condition considered in previous works, and this requires us to use refined estimates.

It will be convenient to introduce the notation

$$dP_t^{\text{Dir}}(u, 0) := -P_t^{\text{Dir}}(u, \cdot) * 2\delta'_0(\cdot) = 2\partial_v P_t^{\text{Dir}}(u, v)|_{v=0} = \frac{2\sqrt{\frac{2}{\pi}}ue^{-\frac{u^2}{2t}}}{t^{3/2}}. \quad (30)$$

We start with an estimate involving the quantity $dP_t^{\text{Dir}}(u, 0)$, which we will reuse in Sections 7.2 and 7.3.

Lemma 4.1. *Define, for any $s \in (0, t)$ and $u \in \mathbb{R}_+$,*

$$G_t(s, u) = \int_{\mathbb{R}_+} dv \left(P_{t-s}^{\text{Dir}}(u, v) \right)^2 \left(dP_s^{\text{Dir}}(v, 0) \right)^2. \quad (31)$$

There exists a constant C such that for any $s \in (0, t)$ and for any $u \in \mathbb{R}_+$,

$$G_t(s, u) \leq C \frac{\sqrt{t}}{\sqrt{s(t-s)}} dP_t^{\text{Dir}}(u, 0)^2. \quad (32)$$

Proof. Using the explicit expressions for $P_{t-s}^{\text{Dir}}(u, v)$ from (19) and $dP_s^{\text{Dir}}(v, 0)$ in (30), we can perform the integral (31) and obtain

$$G_t(s, u) = \frac{2e^{-\frac{u^2}{t}} \left(\frac{t(t-s)}{s} \left(1 - e^{-\frac{-su^2}{t(t-s)}} \right) + 2u^2 \right)}{\pi^{3/2} t^{5/2} \sqrt{s(t-s)} u^2}.$$

Dividing by $dP_t^{\text{Dir}}(u, 0)^2$, we get

$$\frac{G_t(s, u)}{dP_t^{\text{Dir}}(u, 0)^2} = \frac{\sqrt{t} \left(\frac{t(t-s)}{s} \left(1 - e^{-\frac{-su^2}{t(t-s)}} \right) + 2u^2 \right)}{4\sqrt{\pi} \sqrt{s(t-s)} u^2}.$$

Using the bound $1 - e^{-x} \leq x$, and simplifying the resulting expression, we obtain that

$$\frac{G_t(s, u)}{dP_t^{\text{Dir}}(u, 0)^2} \leq \frac{3}{4\sqrt{\pi}} \frac{\sqrt{t}}{\sqrt{s(t-s)}}.$$

□

Let us now turn to the proof of Proposition 3.5.

Proof of Proposition 3.5. Fix a terminal time $T > 0$ and consider the Banach space \mathcal{B} of adapted processes (\mathcal{Z}_t) satisfying

$$\|\mathcal{Z}\|_{\mathcal{B}}^2 := \sup_{\substack{X \in \mathbb{R}_+ \\ s \in (0, T)}} \left\{ s^2 \mathbb{E} \left[\mathcal{Z}_s(X)^2 \right] \right\} < \infty.$$

We define a sequence of processes defined for $t \leq T$, $u \in \mathbb{R}_+$, by

$$\begin{aligned} \mathcal{U}_0(t, u) &:= dP_t^{\text{Dir}}(u, 0). \\ \mathcal{U}_{n+1}(t, u) &:= \int_0^t \int_{\mathbb{R}_+} P_{t-s}^{\text{Dir}}(u, v) \mathcal{U}_n(s, v) \xi(s, v) dv ds. \end{aligned}$$

This implies that, if $S_n = \sum_{k=0}^n \mathcal{U}_k$, then

$$S_{n+1}(t, u) = dP_t^{\text{Dir}}(u, 0) + \int_0^t \int_{\mathbb{R}_+} P_{t-s}^{\text{Dir}}(u, v) S_n(s, v) \xi(s, v) dv ds.$$

In order to show the existence of the solution to (108), it suffices to show that the series S_n converges in the space \mathcal{B} , and for that we will show that $\sum \|\mathcal{U}_n\|_{\mathcal{B}}$ converges. Regarding uniqueness, it follows from the same argument as in [Par19, Proposition 4.2].

In order to estimate $\|\mathcal{U}_n\|_{\mathcal{B}}$, we introduce

$$f_n(s) = \sup_{v \in \mathbb{R}_+} \left\{ \frac{\mathbb{E} [\mathcal{U}_n(s, v)^2]}{dP_s^{\text{Dir}}(v, 0)^2} \right\}.$$

By Itô isometry, we have,

$$\mathbb{E} [\mathcal{U}_{n+1}(t, u)^2] = \int_0^t ds \int_{\mathbb{R}_+} dv \left(P_{t-s}^{\text{Dir}}(u, v) \right)^2 \mathbb{E} [\mathcal{U}_n(s, v)^2].$$

Thus, we may write, recalling the definition of $G_t(s, u)$ in (31),

$$\begin{aligned} \mathbb{E} [\mathcal{U}_{n+1}(t, u)^2] &\leq \int_0^t ds \int_{\mathbb{R}_+} dv \left(P_{t-s}^{\text{Dir}}(u, v) \right)^2 dP_s^{\text{Dir}}(v, 0)^2 f_n(s) \\ &\leq \int_0^t ds G_t(s, u) f_n(s). \end{aligned}$$

Using Lemma 4.1, we obtain that

$$\mathbb{E} [\mathcal{U}_{n+1}(t, u)^2] \leq C \left(dP_t^{\text{Dir}}(u, 0) \right)^2 2\sqrt{t} \int_0^t ds \frac{f_n(s)}{\sqrt{s(t-s)}}. \quad (33)$$

Dividing both sides of (33) by $(dP_t^{\text{Dir}}(u, 0))^2$, we obtain that for $t \in (0, T)$,

$$f_{n+1}(t) \leq C\sqrt{t} \int_0^t ds \frac{f_n(s)}{\sqrt{s(t-s)}}.$$

Iterating this inequality, we get

$$\begin{aligned} f_{n+2}(t) &\leq C\sqrt{t} \int_0^t dr \sqrt{r} \int_0^r ds \frac{f_n(s)}{\sqrt{r(t-r)}\sqrt{s(r-s)}} \\ &= C\sqrt{t} \int_0^t ds \frac{f_n(s)}{\sqrt{s}} \int_s^t \frac{dr}{\sqrt{t-r}\sqrt{r-s}} \\ &= C\sqrt{t} \int_0^t ds \frac{f_n(s)}{\sqrt{s}}, \end{aligned}$$

by exchanging the integration order, from which we deduce by induction that $f_n(t) \leq C^n t^{n/2} / ([n/2])!$. Hence, we have obtained that

$$\|\mathcal{U}_n\|_{\mathcal{B}}^2 \leq C \sup_{\substack{u \in \mathbb{R}_+ \\ s \in (0, T)}} \left\{ \frac{\mathbb{E} [\mathcal{U}_n(s, u)^2]}{dP_s^{\text{Dir}}(u, 0)^2} \right\} = f_n(T) \leq \frac{C^n T^{n/2}}{([n/2])!},$$

where in the first inequality we have simply used that $dP_s^{\text{Dir}}(u, 0) \leq Cs^{-1}$ (which is easy to check using the explicit expression (30)). This shows that $\sum_{n=0}^{+\infty} \|\mathcal{U}_n\|_{\mathcal{B}} < \infty$ so that the series $\sum_{n=0}^{+\infty} \mathcal{U}_n$ exists in the space \mathcal{B} , and it concludes the proof. \square

5. HEAT KERNEL WITH DIVERGING ROBIN BOUNDARY CONDITIONS

In the proofs of Theorems 3.7 and 3.9, we will need sharp estimates on the heat kernel \mathbf{p}_t^R with Robin boundary conditions. We therefore collect in this section all preliminary results on \mathbf{p}_t^R , which will be invoked later on. We start with a definition.

Definition 5.1. *Let $\mu < 1$. The discrete heat kernel with Robin boundary condition (and parameter μ), is defined as the solution to: for any $x, y \in \mathbb{N}$*

$$\partial_t \mathbf{p}_t^R(x, y) = \frac{1}{2} \Delta_y \mathbf{p}_t^R(x, y), \quad \mathbf{p}_0^R(x, y) = \mathbf{1}_{x=y}, \quad \mathbf{p}_t^R(-1, y) = \mu \mathbf{p}_t^R(0, y),$$

where Δ_y denotes the discrete Laplacian acting on functions of the variable y .

Since we assume $\mu < 1$, $\mathbf{p}_t^R(x, y)$ corresponds to the transition probability for a continuous time random walk on \mathbb{N} which behaves as the symmetric simple random walk on positive integers, while at 0, after an exponentially distributed waiting time with mean 1, it jumps to 1 with probability $\frac{1}{2}$, it stays at 0 with probability $\frac{\mu}{2}$, and it is killed with probability $\frac{1-\mu}{2}$. This gives us a representation of \mathbf{p}^R in terms of the transition probabilities (\mathbf{p}_n) of the underlying discrete random walk (which moves similarly but at integer times):

$$\mathbf{p}_t^R(x, y) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} \mathbf{p}_n(x, y). \quad (34)$$

Moreover, we have the following representation for \mathbf{p}_t^R , in terms of the kernel p_t for the continuous time symmetric simple random walk on \mathbb{Z} , see [CS18, Section 4.1]:

$$\mathbf{p}_t^R(x, y) = p_t(x - y) + \mu p_t(x + y + 1) + (1 - \mu^{-2}) \sum_{z=2}^{+\infty} \mu^z p_t(x + y + z). \quad (35)$$

In particular, $\mathbf{p}_t^R(x, y)$ is increasing in μ .

Definition 5.2. Let $\{\mathbf{p}_t^\varepsilon(x, y)\}_{x, y \in \mathbb{N}}$ be the discrete heat kernel with Robin boundary condition and parameter $\mu = e^{-\varepsilon}$.

5.1. First properties.

Lemma 5.3 (Heat kernel bounds). Fix $T > 0$. We have the following estimates.

(i) For any $a > 0$, there exists $C = C(a, T) \in \mathbb{R}_+$ such that, for all $t \leq \varepsilon^{-4}T$, $x \in \mathbb{N}$,

$$\sum_{y=0}^{\infty} \mathbf{p}_t^\varepsilon(x, y) e^{a\varepsilon^2 y} \leq C e^{a\varepsilon^2 x}. \quad (36)$$

(ii) There exists $C = C(T) \in \mathbb{R}_+$ such that, for all $t \leq \varepsilon^{-4}T$, $x, y, z \in \mathbb{N}$, for all $v \in [0, 1]$, for $|x - y| \leq \lceil \sqrt{t} \rceil$

$$|\mathbf{p}_t^\varepsilon(x, z) - \mathbf{p}_t^\varepsilon(y, z)| \leq C \left(1 \wedge \frac{1}{\sqrt{t^{1+v}}}\right) |x - y|^v. \quad (37)$$

Moreover, for all $b > 0$, there exists $C = C(T, b) \in \mathbb{R}_+$ such that for all $t \leq \varepsilon^{-4}T$, $x, z \in \mathbb{N}$, for all $v \in [0, 1]$,

$$|\nabla^\pm \mathbf{p}_t^\varepsilon(x, z)| \leq C \left(1 \wedge \frac{1}{\sqrt{t^{1+v}}}\right) e^{-b|z-x|(1 \wedge t^{-1/2})}, \quad (38)$$

and consequently, for all $a \geq 0$, there exists $C = C(T, a) \in \mathbb{R}_+$ such that

$$\sum_{y=0}^{\infty} |\nabla^\pm \mathbf{p}_t^\varepsilon(x, y)| e^{a\varepsilon^2 y} e^{a|x-y|(1 \wedge t^{-1/2})} \leq C e^{a\varepsilon^2 x} t^{-1/2}. \quad (39)$$

(iii) For all $t \geq s \geq 0$, $x, y \in \mathbb{N}$,

$$\mathbf{p}_s^\varepsilon(x, y) \leq e^{t-s} \mathbf{p}_t^\varepsilon(x, y). \quad (40)$$

Proof. A number of these bounds follow from those established for instance in [Par19]. Therein, the author considers the discrete heat kernel with Robin boundary condition with parameter $\mu_A = 1 - A\varepsilon^2$ (note that ε for us corresponds to $\sqrt{\varepsilon}$ therein). In particular, the monotonicity in μ of \mathbf{p}^R allows us to use directly the bounds established in [Par19]. More precisely:

- (36) follows from Corollary 3.3 in [Par19], where we take $a_1 = 0$ and $a_2 = a$;
- (37) and (38) follow from similar bounds that hold for the standard heat kernel p_t on the whole line \mathbb{Z} , and from (35), as in the proof of Proposition 3.2 in [Par19]. In fact, the monotonicity in μ of (35) implies a monotonicity in the upper bounds used in the proof of [Par19], so that we can use the estimates therein as an upper bound;
- (40) is an immediate consequence of (34).

□

Lemma 5.4 (Key cancellation and consequences). *We have the following estimates:*

(i) For any $x, x' \in \mathbb{N}$,

$$\sum_{y=0}^{\infty} \int_0^{\infty} \nabla^+ \mathbf{p}_t^\varepsilon(x, y) \nabla^+ \mathbf{p}_t^\varepsilon(x', y) dt = \mathbf{1}_{x=x'}. \quad (41)$$

(ii) For any $a > 0$, there exist $\varepsilon_0 = \varepsilon_0(a, T) > 0, c = c(a, T) \in (0, 1)$ such that for all $\varepsilon < \varepsilon_0, x \in \mathbb{N}^*$,

$$\sum_{y=1}^{\infty} \int_0^{\varepsilon^{-4}T} \left| \nabla^+ \mathbf{p}_r^\varepsilon(x, y) \nabla^- \mathbf{p}_r^\varepsilon(x, y) \right| e^{a\varepsilon^2|x-y|} dr \leq c. \quad (42)$$

(iii) For any $a > 0$, there exist $\varepsilon_0 = \varepsilon_0(a, T) > 0, C = C(a, T) \in \mathbb{R}_+$ such that for all $\varepsilon < \varepsilon_0, x \in \mathbb{N}^*, t \leq T$,

$$\sum_{y=1}^{\infty} \int_0^{\varepsilon^{-4}t} \left| \nabla^+ \mathbf{p}_r^\varepsilon(x, y) \nabla^- \mathbf{p}_r^\varepsilon(x, y) \right| e^{a\varepsilon^2|x-y|} \frac{1}{\sqrt{\varepsilon^{-4}t - r}} dr \leq C\varepsilon^2. \quad (43)$$

Proof. The last two bounds (42) and (43) follow from the first one (41), together with (38) and (39), by the same arguments as in [CS18, Proof of Corollary 5.4].

Let us show (41), which can be proved in a more elementary way than is done in [CS18] (where it is Proposition 5.1). Indeed, note that $\sum_{y=0}^{\infty} \mathbf{p}_t^\varepsilon(x, y) \mathbf{p}_t^\varepsilon(x', y) = \mathbf{p}_{2t}^\varepsilon(x, x')$ by symmetry of \mathbf{p}_t^ε and the Markov property of the random walk described after Definition 5.1. Moreover, $\int_0^{\infty} \mathbf{p}_{2t}^\varepsilon(x, x') dt = \frac{1}{2} G(x, x')$ is the Green's function associated with that random walk, *i.e.* the expected number of times the random walk started from x goes through x' before being killed. Elementary computations therefore yield

$$\sum_{y=0}^{\infty} \int_0^{\infty} \nabla^+ \mathbf{p}_t^\varepsilon(x, y) \nabla^+ \mathbf{p}_t^\varepsilon(x', y) dt = G(x+1, x'+1) + G(x, x') - G(x+1, x') - G(x, x'+1). \quad (44)$$

By the interpretation of G in terms of number of visits, it is clear that $G(x+1, x'+1) = G(x, x'+1)$ if $x > x'$ and $G(x+1, x') = G(x, x')$ if $x \geq x'$. Moreover, by first-step analysis, $G(x+1, x+1) = 1 + \frac{1}{2} [G(x, x+1) + G(x+2, x+1)] = 1 + \frac{1}{2} [G(x, x+1) + G(x+1, x+1)]$, and therefore $G(x+1, x+1) = 2 + G(x, x+1)$. Then (41) follows. \square

5.2. Moment estimates. In this section, we establish moment estimates on the heat kernel \mathbf{p}_t^R and its convolution with the empty initial condition $Z_0(x) = \mu^x$.

Proposition 5.5. *There exists a constant $C > 0$ such that for any $\varepsilon \in (0, 1), t > 0, \alpha \in [0, \frac{1}{2}]$ and $x, y \in \mathbb{N}$, we have that for any $\mu \in (0, 1)$,*

$$\mathbf{p}_t^R(x, y) \leq \frac{C}{\sqrt{t}}, \quad (45)$$

and for $\mu = e^{-\varepsilon}$, we have that

$$\varepsilon^{-2} \mathbb{E}[Z_t(x)] = \varepsilon^{-2} \sum_{y \geq 0} \mathbf{p}_t^\varepsilon(x, y) \mu^y \leq C \min\{(\varepsilon^4 t)^{-1}, \varepsilon^{-2}\} \quad (46)$$

and

$$\begin{aligned} |(\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x) - (\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(y)| &= \left| \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(x, z) \mu^z - \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(y, z) \mu^z \right| \\ &\leq C \left(\varepsilon^2 |x - y| \right)^\alpha (\varepsilon^4 t)^{-1-\alpha/2}. \end{aligned} \quad (47)$$

Remark 5.6. Note that the constant C in (45) is universal, and does not depend on t nor on the terminal time T . This estimate is called *long-time estimate* in [Par19], see Proposition 3.6 therein. Here, we provide a different proof and obtain an even better estimate.

Proof. Let us first establish explicit formulas for the quantities to bound in (45), (46) and (47). For $t \geq 0$ and $x \in \mathbb{Z}$, we have the integral representation of the heat kernel

$$p_t(x) = \frac{1}{2i\pi} \oint e^{\frac{1}{2}(\xi+\xi^{-1}-2)t} \xi^x \frac{d\xi}{\xi}, \quad (48)$$

where the contour is a positively oriented circle around 0. From (48) and (35), we deduce

$$\mathfrak{p}_t^R(x, y) = \frac{1}{2i\pi} \oint e^{\frac{1}{2}(\xi+\xi^{-1}-2)t} \xi^x \left(\xi^{-y} + \xi^{y+1} \frac{\mu - \xi}{1 - \mu\xi} \right) \frac{d\xi}{\xi}, \quad (49)$$

where the contour is a positively oriented circle around 0 with radius smaller than μ^{-1} . Hence, we have that

$$\sum_{y \geq 0} \mathfrak{p}_t^R(x, y) \mu^y = \mathbb{E}[Z_t(x)] = \frac{1}{2i\pi} \oint e^{\frac{1}{2}(\xi+\xi^{-1}-2)t} \xi^x \frac{(1 - \mu^2)(1 - \xi^2)}{(1 - \mu/\xi)(1 - \mu\xi)^2} \frac{d\xi}{\xi} \quad (50)$$

where the contour is a positively oriented circle around 0 with radius comprised between μ and μ^{-1} , and

$$\sum_{z \geq 0} \mathfrak{p}_t^R(x, z) \mu^z - \sum_{z \geq 0} \mathfrak{p}_t^R(y, z) \mu^z = \frac{1}{2i\pi} \oint e^{\frac{1}{2}(\xi+\xi^{-1}-2)t} (\xi^x - \xi^y) \frac{(1 - \mu^2)(1 - \xi^2)}{(1 - \mu/\xi)(1 - \mu\xi)^2} \frac{d\xi}{\xi}. \quad (51)$$

Now we estimate the integrals above. Since $\mu \in (0, 1)$, we may assume that the contour (in each formula above) is a circle of radius 1. Now, since ξ has modulus 1, we have that $|\xi^{x-y}| = |\xi^{x+y+1}| = 1$ and for any $\mu \in (0, 1)$, $|\mu - \xi| = |\mu - \bar{\xi}| = |\mu\xi - 1|$ so that $\left| \frac{\mu - \xi}{1 - \mu\xi} \right| = 1$. Using the change of variables $\xi = e^{i\theta}$ in (49), we get

$$\mathfrak{p}_t^R(x, y) \leq \frac{2}{2\pi} \int_{-\pi}^{\pi} e^{\frac{t}{2} \Re[e^{i\theta} + e^{-i\theta} - 2]} d\theta.$$

Then, we use the estimate $\cos(x) - 1 \leq \frac{-x^2}{5}$, valid for $x \in (-\pi, \pi)$, so that

$$\Re[e^{i\theta} + e^{-i\theta} - 2] = 2 \cos(\theta) - 2 \leq \frac{-2\theta^2}{5}. \quad (52)$$

Hence, setting $t = \varepsilon^{-4}T$, we obtain

$$\mathfrak{p}_t^R(x, y) \leq \frac{\varepsilon^2}{\pi} \int_{-\pi}^{\pi} e^{-\frac{\theta^2 t}{5}} d\theta.$$

Finally, using the change of variables $\theta = \tilde{\theta}t^{-1/2}$,

$$\mathfrak{p}_t^R(x, y) \leq \frac{1}{\pi\sqrt{t}} \int_{\mathbb{R}} e^{-\frac{\tilde{\theta}^2}{5}} d\tilde{\theta}$$

which proves (45) with $C = \frac{1}{\pi} \int_{\mathbb{R}} e^{-\frac{z^2}{5}} dz$. Now we turn to the proof of (46) and assume henceforth that $\mu = e^{-\varepsilon}$. Since $\mathfrak{p}_t^R(x, \cdot)$ defines a measure on \mathbb{N} with mass at most 1, and $0 < \mu < 1$, we have the trivial bound $\mathbb{E}[Z_t(x)] \leq 1$, so that $\varepsilon^{-2} \mathbb{E}[Z_t(x)] \leq \varepsilon^{-2}$. It remains to show that

$$\varepsilon^{-2} \mathbb{E}[Z_t(x)] = \varepsilon^{-2} \sum_{y \geq 0} \mathfrak{p}_t^\varepsilon(x, y) \mu^y \leq C(\varepsilon^4 t)^{-1}.$$

Using the change of variables $\xi = e^{i\theta\varepsilon^2}$, and setting $x = \varepsilon^{-2}X$, (50) becomes

$$\varepsilon^{-2} \sum_{y \geq 0} \mathfrak{p}_t^\varepsilon(x, y) \mu^y = \frac{1}{2\pi} \int_{-\varepsilon^{-2}\pi}^{\varepsilon^{-2}\pi} e^{\frac{1}{2}(e^{i\varepsilon^2\theta} + e^{-i\varepsilon^2\theta} - 2)t} e^{i\theta X} \frac{(1 - e^{-2\varepsilon})(1 - e^{2i\varepsilon^2\theta})}{(1 - e^{-\varepsilon - i\varepsilon^2\theta})(1 - e^{-\varepsilon + i\varepsilon^2\theta})^2} d\theta. \quad (53)$$

Using $x - \frac{x^2}{2} \leq 1 - e^{-x} \leq x$ for $x > 0$, and $0 \leq 1 - \cos(x) \leq \frac{x^2}{2}$ for $x \in (-\pi, \pi)$, we have

$$\left| \frac{(1 - e^{-2\varepsilon})(1 - e^{2i\varepsilon^2\theta})}{(1 - e^{-\varepsilon - i\varepsilon^2\theta})(1 - e^{-\varepsilon + i\varepsilon^2\theta})^2} \right| = \frac{(1 - e^{-2\varepsilon})\sqrt{2(1 - \cos(2\varepsilon^2\theta))}}{(1 + e^{-2\varepsilon} - 2e^{-\varepsilon} \cos(\varepsilon^2\theta))^{3/2}} \leq \frac{4\varepsilon^3|\theta|}{(\varepsilon - \varepsilon^2/2)^3} = \frac{4|\theta|}{(1 - \varepsilon/2)^3}.$$

Thus, using additionally the estimate (52) and setting $t = \varepsilon^{-4}T$, we obtain the bound

$$\varepsilon^{-2} \sum_{y \geq 0} \mathfrak{p}_t^\varepsilon(x, y) \mu^y \leq \frac{1}{2\pi} \int_{-\varepsilon^{-2}\pi}^{\varepsilon^{-2}\pi} e^{-\frac{\theta^2 T}{5}} \frac{4|\theta|}{(1 - \varepsilon/2)^3} d\theta.$$

The change of variables $\theta = \tilde{\theta}T^{-1/2}$ now yields (for $\varepsilon \in (0, 1)$)

$$\varepsilon^{-2} \sum_{y \geq 0} \mathbf{p}_t^\varepsilon(x, y) \mu^y \leq \frac{1}{T} \frac{16}{\pi} \int_{\mathbb{R}} |\tilde{\theta}| e^{-\frac{\tilde{\theta}^2}{5}} d\tilde{\theta},$$

which proves (46) with $C = \frac{16}{\pi} \int_{\mathbb{R}} |z| e^{-z^2/5} dz$. Now we turn to the proof of (47). Using (51) and the same steps as above (with $y = \varepsilon^{-2}Y$),

$$\left| \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(x, z) \mu^z - \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(y, z) \mu^z \right| \leq \frac{1}{T} \frac{16}{\pi} \int_{\mathbb{R}} |\tilde{\theta}| e^{-\frac{\tilde{\theta}^2}{5}} \left| e^{iX\tilde{\theta}T^{-1/2}} - e^{iY\tilde{\theta}T^{-1/2}} \right| d\tilde{\theta} \quad (54)$$

Hence, using the change of variables $u = 2\theta$ and choosing a constant C such that $\frac{16}{\pi} |\tilde{\theta}| e^{-\frac{\tilde{\theta}^2}{5}} \leq C e^{-\frac{u^2}{2}}$,

$$\begin{aligned} \left| \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(x, z) \mu^z - \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(y, z) \mu^z \right| &\leq \frac{C}{T} \int_{\mathbb{R}} e^{-\frac{u^2}{2}} \left| e^{2iXuT^{-1/2}} - e^{2iYuT^{-1/2}} \right| du, \\ &\leq \frac{C}{T} \int_{\mathbb{R}} e^{-\frac{u^2}{2}} \left| 1 - e^{2i(X-Y)uT^{-1/2}} \right| du, \\ &\leq \frac{2C}{T} \max\{|X - Y|T^{-1/2}, 1\} \int_{\mathbb{R}} u e^{-\frac{u^2}{2}} du. \end{aligned}$$

Thus, (47) holds for any $\alpha \in (0, 1]$. \square

6. NEAR-EQUILIBRIUM INITIAL CONDITION: PROOF OF THEOREM 3.9

In this section we assume that the initial condition is near-equilibrium, and we prove Theorem 3.9. We are thus interested in the rescaled process $\mathcal{Z}_t^\varepsilon$ defined in (24) for any $u \in \varepsilon^2\mathbb{N}$ and then extended to \mathbb{R}_+ by linear interpolation. Together with the uniqueness of the solution to the martingale problem (28)–(29), the next two propositions prove the desired result stated in Theorem 3.9.

Proposition 6.1 (Tightness). *Under the assumptions of Theorem 3.9, the sequence of processes $(\mathcal{Z}_s^\varepsilon)_{s \in [0, T]}$ is tight in the space $D([0, T], C(\mathbb{R}_+))$. Moreover, any limit point belongs to $C([0, T], C(\mathbb{R}_+))$.*

Proposition 6.2 (Identification of limit points). *Under the assumptions of Theorem 3.9, any limit point $(\mathcal{Z}_s)_{s \in [0, T]}$ of $(\mathcal{Z}_s^\varepsilon)_{s \in [0, T]}$ in the space $D([0, T], C(\mathbb{R}_+))$ satisfies the continuous martingale problem (28)–(29).*

The rest of the section is devoted to the proof of Proposition 6.1 and Proposition 6.2.

6.1. Tightness: proof of Proposition 6.1. The main ingredient is the following lemma, which is a consequence of estimates established in [CS18, Par19], adapted to the case $\mu = e^{-\varepsilon}$ instead of $1 - \varepsilon^2 A$.

Lemma 6.3. *Fix $T > 0$ and assume that the initial condition $\mathcal{Z}_0^\varepsilon \in C(\mathbb{R}_+)$ is near-equilibrium (Definition 3.8). Then for all $n \in \mathbb{N}$ and $\alpha \in [0, \frac{1}{2})$, there exists $C = C(\alpha, n, T) \in \mathbb{R}_+$ such that for all $\varepsilon > 0$, $u, u' \in \mathbb{R}_+$, $s, s' \in [0, T]$,*

$$\|\mathcal{Z}_s^\varepsilon(u)\|_n \leq C e^{au}, \quad (55)$$

$$\|\mathcal{Z}_s^\varepsilon(u) - \mathcal{Z}_s^\varepsilon(u')\|_{2n} \leq C |u - u'|^\alpha e^{a(u+v)}, \quad (56)$$

$$\|\mathcal{Z}_s^\varepsilon(u) - \mathcal{Z}_{s'}^\varepsilon(u')\|_{2n} \leq C \left(\varepsilon^{2\alpha} \vee |t - t'|^{\alpha/2} \right) e^{2au}. \quad (57)$$

These estimates, together with Arzela-Ascoli's Theorem, imply Proposition 6.1 (see [Bil68, Chapter 3]). In order to prove Lemma 6.3, we use the bounds on the heat kernel \mathbf{p}^R which have been proved in Lemma 5.3 and Lemma 5.4. Since the proof closely follows that in [Par19], we do not detail it, but just point out which ingredients are needed where. We repeatedly use the following lemma (proved in [DT16, Lemma 3.1], [CS18, Lemma 4.18], as stated in [Par19, Lemma 5.3]).

Lemma 6.4. *For any $n \in \mathbb{N}^*$, there exists $C = C(n) < \infty$ such that, for any F bounded on $\mathbb{R}_+ \times \mathbb{N}$ and any $t > 1$,*

$$\left\| \int_0^t \sum_{y=0}^{\infty} F(s, y) dM_s(y) \right\|_n^2 \leq C\varepsilon^2 \int_0^t \sum_{y=0}^{\infty} \bar{F}(s, y)^2 \|Z_s(y)\|_n^2 ds, \quad (58)$$

where $\bar{F}(s, y) = \sup_{|s'-s|<1} |F(s', y)|$. Moreover, when $F(s, y) = \mathbf{p}_s^\varepsilon(x, y)$ for some $x \in \mathbb{N}$,

$$\bar{\mathbf{p}}_s^\varepsilon(x, y)^2 \leq Cs^{-1/2} \mathbf{p}_{s+1}^\varepsilon(x, y). \quad (59)$$

The first bound (55) in Lemma 6.3 is obtained by an iteration argument, and is a consequence of (25), (36) and Lemma 6.4 (same proof as in [Par19, Proof of Proposition 5.4]).

For the second bound (56), write the consequence of Lemma 3.1

$$Z_t(x) - Z_t(y) = \sum_{z=0}^{\infty} [\mathbf{p}_t^\varepsilon(x, z) - \mathbf{p}_t^\varepsilon(y, z)] Z_0(z) + \int_0^t \sum_{z=0}^{\infty} [\mathbf{p}_{t-s}^\varepsilon(x, z) - \mathbf{p}_{t-s}^\varepsilon(y, z)] dM_s(z). \quad (60)$$

To control the first sum, we can proceed as in [Par19] and extend Z_0 into a function over \mathbb{Z} by imposing that $z \mapsto Z_0(z-1) - \mu Z_0(z)$ is odd. Note that Z_0 now depends implicitly on ε . Then it is easy to check that Z_0 still satisfies (26) (possibly changing C). The rest of the proof of (56) is exactly similar to [Par19] and uses Lemma 6.4, (36), (37), (40), and (55).

The last bound (57) is obtained as in [Par19]. It is even simpler for us because the term labeled J_2 therein vanishes. The estimates rely on Lemma 6.4, on (55) and on (36).

6.2. Identification of limit points: proof of Proposition 6.2.

6.2.1. Discrete martingale problem. Let us denote by $\Delta^\varepsilon := \Delta^\mu$ the discrete Laplacian with boundary condition defined in (13) with $\mu = e^{-\varepsilon}$. We also introduce, for any $\phi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ which are square summable, the following notation:

$$(\psi, \phi)_\varepsilon := \varepsilon^2 \sum_{x=0}^{\infty} \phi(\varepsilon^2 x) \psi(x). \quad (61)$$

From Lemma 3.1, for any $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ with compact support,

$$N_t^\varepsilon(\phi) := (Z_{\varepsilon^{-4}t}, \phi)_\varepsilon - (Z_0, \phi)_\varepsilon - \frac{1}{2} \int_0^{\varepsilon^{-4}t} (\Delta^\varepsilon Z_s, \phi)_\varepsilon ds \quad (62)$$

is a martingale. Let us compute

$$\begin{aligned} \varepsilon^{-2} (\Delta^\varepsilon Z_s, \phi)_\varepsilon &= \sum_{x=0}^{\infty} Z_s(x) \left[\phi(\varepsilon^2(x+1)) + \phi(\varepsilon^2(x-1)) - 2\phi(\varepsilon^2 x) \right] + Z_s(-1)\phi(0) - Z_s(0)\phi(-\varepsilon^2) \\ &= \sum_{x=0}^{\infty} Z_s(x) \varepsilon^4 \phi''(\varepsilon^2 x) \\ &\quad + \sum_{x=0}^{\infty} Z_s(x) \left[\Delta\phi(\varepsilon^2 \cdot)(x) - \varepsilon^4 \phi''(\varepsilon^2 x) \right] + \sqrt{\frac{q}{p}} Z_s(0)\phi(0) - Z_s(0)\phi(-\varepsilon^2), \\ &= \varepsilon^4 \sum_{x=0}^{\infty} \mathcal{Z}_{\varepsilon^4 s}^\varepsilon(\varepsilon^2 x) \phi''(\varepsilon^2 x) \\ &\quad + \sum_{x=0}^{\infty} \mathcal{Z}_{\varepsilon^4 s}^\varepsilon(\varepsilon^2 x) \left[\Delta\phi(\varepsilon^2 \cdot)(x) - \varepsilon^4 \phi''(\varepsilon^2 x) \right] + \mathcal{Z}_{\varepsilon^4 s}^\varepsilon(0) \left[\sqrt{\frac{q}{p}} \phi(0) - \phi(-\varepsilon^2) \right]. \end{aligned}$$

Therefore $N_t^\varepsilon(\phi)$ can be rewritten as

$$\begin{aligned} N_t^\varepsilon(\phi) &= \varepsilon^2 \sum_{x=0}^{\infty} \phi(\varepsilon^2 x) \mathcal{Z}_t^\varepsilon(\varepsilon^2 x) - \varepsilon^2 \sum_{x=0}^{\infty} \phi(\varepsilon^2 x) \mathcal{Z}_0^\varepsilon(\varepsilon^2 x) - \frac{1}{2} \int_0^t \varepsilon^2 \sum_{x=0}^{\infty} \mathcal{Z}_s^\varepsilon(\varepsilon^2 x) \phi''(\varepsilon^2 x) ds \\ &\quad - \frac{1}{2} \varepsilon^{-2} \left[\sqrt{\frac{q}{p}} \phi(0) - \phi(-\varepsilon^2) \right] \int_0^t \mathcal{Z}_s^\varepsilon(0) ds + \mathfrak{E}_1(\varepsilon), \end{aligned} \quad (63)$$

where the error term is

$$\mathfrak{E}_1(\varepsilon) := -\frac{1}{2}\varepsilon^{-2} \sum_{x=0}^{\infty} \left[\Delta^\varepsilon \phi(\varepsilon^2 \cdot)(x) - \varepsilon^4 \phi''(\varepsilon^2 x) \right] \int_0^t \mathcal{Z}_s^\varepsilon(\varepsilon^2 x) ds. \quad (64)$$

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be smooth and compactly supported, such that $\phi(0) = 0$. Fix $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$ smooth, compactly supported, such that $\psi(0) = 1$ and $\psi'(0) = 0$, and define

$$\phi_\varepsilon = \phi + \varepsilon \phi'(0) \psi, \quad (65)$$

so that $\phi_\varepsilon(0) = \varepsilon \phi'_\varepsilon(0)$. From the previous computation (63), the following is a martingale:

$$\begin{aligned} N_t^\varepsilon(\phi_\varepsilon) &= \varepsilon^2 \sum_{x=0}^{\infty} \phi(\varepsilon^2 x) \mathcal{Z}_t^\varepsilon(\varepsilon^2 x) - \varepsilon^2 \sum_{x=0}^{\infty} \phi(\varepsilon^2 x) \mathcal{Z}_0^\varepsilon(\varepsilon^2 x) \\ &\quad - \frac{\varepsilon^2}{2} \int_0^t \sum_{x=0}^{\infty} \mathcal{Z}_s^\varepsilon(\varepsilon^2 x) \phi''(\varepsilon^2 x) ds + R_1 + R_2 + R_3, \end{aligned}$$

where

$$R_1 := -\frac{1}{2}\varepsilon^{-2} \sum_{x=0}^{\infty} \left[\Delta^\varepsilon \phi_\varepsilon(\varepsilon^2 \cdot)(x) - \varepsilon^4 \phi_\varepsilon''(\varepsilon^2 x) \right] \int_0^t \mathcal{Z}_s^\varepsilon(\varepsilon^2 x) ds \quad (66)$$

$$R_2 := -\frac{1}{2}\varepsilon^{-2} \left[e^{-\varepsilon} \phi_\varepsilon(0) - \phi_\varepsilon(-\varepsilon^2) \right] \int_0^t \mathcal{Z}_s^\varepsilon(0) ds, \quad (67)$$

$$R_3 := \varepsilon^3 \phi'(0) \sum_{x=0}^{\infty} \psi(\varepsilon^2 x) \left[\mathcal{Z}_t^\varepsilon(\varepsilon^2 x) - \mathcal{Z}_0^\varepsilon(\varepsilon^2 x) \right]. \quad (68)$$

Moreover, the quadratic variation of $N_t^\varepsilon(\phi_\varepsilon)$ is given (see Lemma 3.1 and Lemma 3.2) by:

$$[N^\varepsilon(\phi_\varepsilon)]_t = \varepsilon^4 \sum_{x=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \int_0^{\varepsilon^{-4}t} \left[\varepsilon^2 Z_s(x)^2 + \nabla^+ Z_s(x) \nabla^- Z_s(x) + o(\varepsilon^2) Z_s(x)^2 \right] ds \quad (69)$$

$$+ \varepsilon^4 \phi_\varepsilon^2(0) \int_0^{\varepsilon^{-4}t} \left[\varepsilon^2 Z_s(0)^2 + \varepsilon Z_s(0) \nabla^+ Z_s(0) + o(\varepsilon^2) Z_s(0)^2 \right] ds. \quad (70)$$

In order to conclude, we prove in the next section that R_1, R_2, R_3 vanish in probability as $\varepsilon \rightarrow 0$, which then establishes that $N_t(\phi)$ (defined in (28)) is a martingale for any \mathcal{Z} limit point of \mathcal{Z}^ε . Then, we verify (29).

6.2.2. Proof of Proposition 6.2. In order to prove that R_1, R_2, R_3 vanish in probability, we estimate their $\|\cdot\|_n$ -norms. Let us start with R_2 , where we will see why we chose ϕ_ε as in (65).

(R_2) For any $n \in \mathbb{N}$, by Lemma 6.3,

$$\|R_2\|_n \leq C \varepsilon^{-2} t \left[e^{-\varepsilon} \varepsilon \phi'(0) - \phi(-\varepsilon^2) - \varepsilon \phi'(0) \psi(-\varepsilon^2) \right] \quad (71)$$

$$\leq C \varepsilon^{-2} t \left[(1 - \varepsilon) \varepsilon \phi'(0) + \varepsilon^2 \phi'(0) - \varepsilon \phi'(0) + o(\varepsilon^2) \right] = o(1). \quad (72)$$

Therefore R_2 vanishes in any L^n , $n \in \mathbb{N}$.

(R_3) Moreover, the same holds for R_3 , again thanks to Lemma 6.3, and because the series $\varepsilon^2 \sum_{x=0}^{\infty} \psi(\varepsilon^2 x) e^{2a\varepsilon^2 x}$ converges to $\int_{\mathbb{R}_+} \psi(u) e^{2au} du$ (the finiteness of this quantity is the reason we do not take $\psi \equiv 1$).

(R_1) Let us now consider R_1 . By Lemma 6.3, it is enough to show that, for all $x \in \mathbb{N}^*$,

$$\left| \Delta^\varepsilon \phi_\varepsilon(\varepsilon^2 \cdot)(x) - \varepsilon^4 \phi_\varepsilon''(\varepsilon^2 x) \right| = o(\varepsilon^4),$$

and

$$\left| \Delta^\varepsilon \phi_\varepsilon(\varepsilon^2 \cdot)(0) - \varepsilon^4 \phi_\varepsilon''(0) \right| = o(\varepsilon^2),$$

which can be checked by elementary computations and thanks to the compactness of the support of ϕ .

It remains to show that any limit point \mathcal{Z} of \mathcal{Z}^ε satisfies (29). Let us rewrite

$$[N^\varepsilon(\phi_\varepsilon)]_t = \int_0^t \varepsilon^2 \sum_{x=0}^{\infty} \phi^2(\varepsilon^2 x) \mathcal{Z}_s^\varepsilon(\varepsilon^2 x)^2 ds + R'_1 + R'_2 + R'_3 + R'_4, \quad (73)$$

where

$$R'_1 = \int_0^t \varepsilon^2 \sum_{x=0}^{\infty} [\phi_\varepsilon^2(\varepsilon^2 x) - \phi^2(\varepsilon^2 x)] \mathcal{Z}_s^\varepsilon(\varepsilon^2 x)^2 ds, \quad (74)$$

$$R'_2 = \varepsilon^4 \int_0^{\varepsilon^{-4}t} \sum_{x=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \nabla^+ Z_s(x) \nabla^- Z_s(x) ds, \quad (75)$$

$$R'_3 = -\varepsilon^5 \int_0^{\varepsilon^{-4}t} \phi_\varepsilon^2(0) Z_s(0) \nabla^+ Z_s(0) ds, \quad (76)$$

$$R'_4 = o(1) \int_0^t \varepsilon^2 \sum_{x=0}^{\infty} \phi^2(\varepsilon^2 x) \mathcal{Z}_s^\varepsilon(\varepsilon^2 x)^2 ds. \quad (77)$$

We have to show that R'_1, R'_2, R'_3 vanish (*e.g.* in L^2) as $\varepsilon \rightarrow 0$.

(R'_1) For the first term, the result follows from the bound $\|\phi_\varepsilon^2 - \phi^2\|_\infty = \mathcal{O}(\varepsilon)$, the fact that ϕ, ϕ_ε have compact support, and from (55).

(R'_3 and R'_4) Both terms R'_3 and R'_4 are also controlled with (55) and (56).

(R'_2) Similarly to [CS18] and [Par19, Proof of Theorem 5.7], we split R'_2 in two parts: write $R'_2 = r_1 + r_2$, where

$$r_1 = \varepsilon^4 \int_0^{\varepsilon^{-3}} \sum_{x=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \nabla^+ Z_s(x) \nabla^- Z_s(x) ds \quad (78)$$

$$r_2 = \varepsilon^4 \int_{\varepsilon^{-3}}^{\varepsilon^{-4}t} \sum_{x=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \nabla^+ Z_s(x) \nabla^- Z_s(x) ds. \quad (79)$$

By (56), $\|\nabla^\pm Z_s(x)\|_2 \leq C\varepsilon^{2\alpha} e^{2a\varepsilon^2 x}$ for any $\alpha < \frac{1}{2}$ and some $C, a > 0$. Therefore, we can bound

$$\|r_1\|_2 \leq C\varepsilon^4 \varepsilon^{-3} \varepsilon^{4\alpha} \sum_{x=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) e^{4a\varepsilon^2 x} \leq C'\varepsilon^{4\alpha-1}, \quad (80)$$

which goes to 0 if we choose for instance $\alpha = 1/3$. On the other hand,

$$\mathbb{E}[r_2^2] = 2\varepsilon^8 \int_{\varepsilon^{-3}}^{\varepsilon^{-4}t} \int_{\varepsilon^{-3}}^s \sum_{x, x'=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \phi_\varepsilon^2(\varepsilon^2 x') \mathbb{E}[\nabla^+ Z_s(x) \nabla^- Z_s(x) \nabla^+ Z_r(x') \nabla^- Z_r(x')] dr ds \quad (81)$$

$$= 2\varepsilon^8 \int_{\varepsilon^{-3}}^{\varepsilon^{-4}t} \int_{\varepsilon^{-3}}^s \sum_{x, x'=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \phi_\varepsilon^2(\varepsilon^2 x') \mathbb{E}[\nabla^+ Z_r(x') \nabla^- Z_r(x') U(x, r, s)] dr ds, \quad (82)$$

where

$$U(x, r, s) := \mathbb{E}[\nabla^+ Z_s(x) \nabla^- Z_s(x) | \mathcal{F}_r], \quad \mathcal{F}_r := \sigma(Z_s(x); x \in \mathbb{N}, s \leq r). \quad (83)$$

Therefore,

$$\mathbb{E}[r_2^2] \leq C\varepsilon^8 \varepsilon^{4\alpha} \int_{\varepsilon^{-3}}^{\varepsilon^{-4}t} \int_{\varepsilon^{-3}}^s \sum_{x, x'=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \phi_\varepsilon^2(\varepsilon^2 x') e^{4a\varepsilon^2 x'} \mathbb{E}[|U(x, r, s)|] dr ds \quad (84)$$

$$\leq C\varepsilon^{6+4\alpha} \int_{\varepsilon^{-3}}^{\varepsilon^{-4}t} \int_{\varepsilon^{-3}}^s \sum_{x=1}^{\infty} \phi_\varepsilon^2(\varepsilon^2 x) \mathbb{E}[|U(x, r, s)|] dr ds. \quad (85)$$

We will prove in Lemma 6.5 below an estimate on $\mathbb{E}[|U(x, r, s)|]$ which shows that

$$\mathbb{E}[r_2^2] \leq C\varepsilon^{4+6\alpha} \int_{\varepsilon^{-3}}^{\varepsilon^{-4}t} \int_{\varepsilon^{-3}}^s \frac{1}{\sqrt{s-r}} dr ds \leq C\varepsilon^{1+6\alpha}. \quad (86)$$

Therefore, R'_i goes to 0 in L^2 for $i = 1, \dots, 4$, and both properties (28) and (29) are satisfied, which conclude the proof of Proposition 6.2. We now prove the last needed estimate:

Lemma 6.5. *For any $\alpha < \frac{1}{2}$, there exist $a, C = C(\alpha, t) > 0$ such that, if $\varepsilon^{-3} \leq r \leq s \leq \varepsilon^{-4}t$, $x \in \mathbb{N}^*$,*

$$\mathbb{E}[|U(x, r, s)|] \leq C\varepsilon^{2\alpha} \frac{e^{a\varepsilon^2 x}}{\sqrt{s-r}}. \quad (87)$$

Proof of Lemma 6.5. As in [BG97, CS18, Par19], we write

$$\begin{aligned} U(x, r, s) &= \nabla^- I_s(x) \nabla^+ I_s(x) + \nabla^- I_s(x) \nabla^+ N_r^s(x) + \nabla^- N_r^s(x) \nabla^+ I_s(x) + \nabla^- N_r^s(x) \nabla^+ N_r^s(x) \\ &+ \mathbb{E} \left[\int_r^s \sum_{y=0}^{\infty} K_{s-\tau}(x, y) d[M(y)]_\tau \middle| \mathcal{F}_r \right], \end{aligned} \quad (88)$$

where

$$\begin{aligned} I_s(x) &= \sum_{y=0}^{\infty} \mathbf{p}_s^\varepsilon(x, y) Z_0(y) \\ N_r^s(x) &= \int_0^r \sum_{y=0}^{\infty} \mathbf{p}_{s-\tau}^\varepsilon(x, y) dM_\tau(y), \\ K_t(x, y) &= \nabla^+ \mathbf{p}_t^\varepsilon(x, y) \nabla^- \mathbf{p}_t^\varepsilon(x, y). \end{aligned}$$

By (39), (25) and Lemma 3.1, we have (see *e.g.* [CS18, Proof of Lemma 5.7])

$$\max \left(\mathbb{E} \left[(\nabla^\pm I_s(x))^2 \right], \mathbb{E} \left[(\nabla^\pm N_r^s(x))^2 \right] \right) \leq C\varepsilon^{2\alpha} \frac{e^{a\varepsilon^2 x}}{\sqrt{s-r}}. \quad (89)$$

Moreover, the expectation term in (88) can be split into

$$\begin{aligned} &(\varepsilon^2 + o(\varepsilon^2)) \sum_{y=0}^{\infty} \int_r^s K_{s-\tau}(x, y) \mathbb{E}[(Z_\tau(y))^2 | \mathcal{F}_r] d\tau - \varepsilon \int_r^s K_{s-\tau}(x, 0) \mathbb{E}[Z_\tau(0) \nabla^+ Z_\tau(0) | \mathcal{F}_r] d\tau \\ &- \sum_{y=1}^{\infty} \int_r^s K_{s-\tau}(x, y) \mathbb{E}[\nabla^- Z_\tau(y) \nabla^+ Z_\tau(y) | \mathcal{F}_r] d\tau. \end{aligned}$$

If we can bound the first two terms by $C\varepsilon^{2\alpha} e^{a\varepsilon^2 x} / \sqrt{s-r}$, using (42), (43), the same iterative procedure as described in [CS18] yields the desired result.

Let us consider the second term (which is new with respect to the case treated in [CS18]). By (55), (56) and (38), we have for any $\alpha < \frac{1}{2}, b \geq 0, \nu \leq 1$

$$\begin{aligned} \left| \varepsilon \int_r^s K_{s-\tau}(x, 0) \mathbb{E}[Z_\tau(0) \nabla^+ Z_\tau(0) | \mathcal{F}_r] d\tau \right| &\leq C\varepsilon^{1+4\alpha} e^{-2b|x|} \int_r^s \left(1 \wedge (s-\tau)^{1+\nu} \right) d\tau \\ &= C\varepsilon^{1+4\alpha} e^{-2b|x|} (s-r)^{-\nu}. \end{aligned}$$

It thus remains to check that

$$\varepsilon^2 \sum_{y=0}^{\infty} \int_u^s K_{s-\tau}(x, y) \mathbb{E}[(Z_\tau(y))^2 | \mathcal{F}_u] d\tau \leq C\varepsilon^{2\alpha} \frac{e^{a\varepsilon^2 x}}{\sqrt{s-u}}, \quad (90)$$

which is exactly Lemma 5.8 in [CS18]. In turn, this inequality is a consequence of bounds (38), (39) and Lemma 6.3 (see [CS18]). \square

7. DELTA PRIME INITIAL CONDITION: PROOF OF THEOREM 3.7

In this section we come back to our initial problem and assume that the initial condition is empty so that $Z_0(x) = \mu^x$, where we recall that $\mu = \sqrt{q/p}$. Therefore, the initial condition is not near-equilibrium, and we now use the different scaling

$$\mathcal{Z}_t^\varepsilon(u) := \varepsilon^{-2} Z_{\varepsilon^{-4}t}(\varepsilon^{-2}u). \quad (91)$$

Let us now prove Theorem 3.7.

Recall that we already know from Section 4 existence and uniqueness of the solution to the SHE (Definition 3.11) with Dirichlet boundary condition and initial condition $\mathcal{Z}_{\text{ini}} = -2\delta'_0$. Also, we have obtained in Section 5 explicit estimates involving the discrete Dirichlet heat kernel, relying on exact computations.

The estimates from Section 5 enable us to show in Section 7.1 that at time $\delta > 0$, $\mathcal{Z}_\delta^\varepsilon(\cdot)$ is near-equilibrium in the sense of Definition 3.8. This allows to follow the same strategy as in Section 6 to have tightness in the space $D([\delta, T], C(\mathbb{R}_+))$ for any $0 < \delta < T$, and show that the limit points satisfy the SHE (Proposition 7.2). Following [Par19], we will then show in Section 7.2 that there exists a limit point in the space $D((0, T], C(\mathbb{R}_+))$, which is solution to the SHE with Dirichlet boundary condition. We also prove in Section 7.2 second moment bounds satisfied by this limit point, using exact formulas from [BC22a]. We finally determine the initial condition in Section 7.3, using the moment bounds from Section 7.2.

7.1. Near-equilibrium property. Recall definition (91). We now prove the following estimates which tell us that at any positive time t , $\mathcal{Z}_t^\varepsilon$ is near-equilibrium:

Proposition 7.1. *Fix $T \geq 0$. Let $\alpha \in [0, \frac{1}{2})$ and $n \in \mathbb{N}$. There exists $C = C(\alpha, T, n) \in \mathbb{R}_+$ such that, for all $\varepsilon \in (0, 1)$, $x, x' \in \mathbb{N}$, and $t, t' \in [0, \varepsilon^{-4}T]$ with $t' < t$:*

$$\|\varepsilon^{-2}Z_t(x)\|_n \leq C(\varepsilon^4t)^{-1} \quad (92)$$

$$\|\varepsilon^{-2}(Z_t(x) - Z_t(x'))\|_n \leq C(\varepsilon^2|x - x'|)^\alpha(\varepsilon^4t)^{-1-\alpha/2} \quad (93)$$

$$\|\varepsilon^{-2}(Z_t(x) - Z_{t'}(x))\|_n \leq C(\varepsilon^4t')^{-1-\alpha/2} \varepsilon^{2\alpha}(1 \vee |t - t'|^{\alpha/2}). \quad (94)$$

Proof. The proof uses the same argument as in [Par19, Proposition 6.2].

First of all, let us note that, as in [Par19, Lemma 6.1]: for any $\tau \geq 0$, there exists C_τ such that, for any $\varepsilon \in (0, 1)$,

$$\sup_{t \leq \tau} \|Z_t(x)\|_n \leq C_\tau Z_0(x). \quad (95)$$

This is due to the fact that the initial configuration of particles is empty and the jump rate satisfies $p \leq \frac{1}{2} + \varepsilon$, therefore the position of the largest occupied site at time t is stochastically dominated by a Poisson random variable $N(3t/2)$ with mean $3t/2$.

Then, we write the martingale decomposition

$$Z_t(x) = \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(x, z) Z_0(z) + \int_0^t \sum_{z \geq 0} \mathbf{p}_{t-s}^\varepsilon(x, z) dM_s(z). \quad (96)$$

We bound the second term in the right-hand-side for $t \geq 1$, using both estimates (58) and (59) from Lemma 6.4, and get

$$\left\| \int_0^t \sum_{z \geq 0} \mathbf{p}_{t-s}^\varepsilon(x, z) dM_s(z) \right\|_n^2 \leq C\varepsilon^2 \int_0^t (t-s)^{-1/2} \sum_{z \geq 0} \mathbf{p}_{t-s+1}^\varepsilon(x, z) \|Z_s(z)\|_n^2 ds. \quad (97)$$

For the first term, from Proposition 5.5, (46) recall that

$$\varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(x, z) Z_0(z) \leq C \min\{(\varepsilon^4t)^{-1}, \varepsilon^{-2}\}, \quad (98)$$

where $C > 0$ is a universal constant, and therefore

$$\left| \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_t^\varepsilon(x, z) Z_0(z) \right|^2 \leq C \min\{(\varepsilon^4t)^{-1}, \varepsilon^{-2}\} (\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x). \quad (99)$$

Summarizing, using the convexity inequality $(a + b)^2 \leq 2a^2 + 2b^2$ we have proved, for any $t \in [1, \varepsilon^{-4}T]$,

$$\begin{aligned} \|\varepsilon^{-2}Z_t(x)\|_n^2 &\leq C \min\{(\varepsilon^4t)^{-1}, \varepsilon^{-2}\} (\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x) \\ &\quad + C\varepsilon^2 \int_0^t (t-s)^{-1/2} \sum_{z \geq 0} \mathbf{p}_{t-s+1}^\varepsilon(x, z) \|\varepsilon^{-2}Z_s(z)\|_n^2 ds. \end{aligned} \quad (100)$$

Now, for $t \leq 1$, we use (95) together with (40), and we obtain

$$\|Z_t(x)\|_n \leq CZ_0(x) \leq Ce^t \mathbf{p}_t^\varepsilon(x, x) Z_0(x) \leq C(\mathbf{p}_t^\varepsilon * Z_0)(x)$$

which implies from (99): for any $t \leq 1$,

$$\|\varepsilon^{-2} Z_t(x)\|_n^2 \leq C \min\{(\varepsilon^4 t)^{-1}, \varepsilon^{-2}\} (\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x). \quad (101)$$

Since (101) is stronger than (99), we have established that (100) holds for any $t \in [0, \varepsilon^{-4} T]$. We are now able to iterate this inequality using the semigroup property of \mathbf{p}_t^ε . We get

$$\|\varepsilon^{-2} Z_t(x)\|_n^2 \leq C \min\{(\varepsilon^4 t)^{-1}, \varepsilon^{-2}\} (\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x) + \sum_{k=0}^{+\infty} C^{k+2} \varepsilon^{2(k+1)} \mathfrak{J}_k(t) (\mathbf{p}_{t+k+1}^\varepsilon * \varepsilon^{-2} Z_0)(x)$$

where

$$\mathfrak{J}_k(t) = \int_{\Delta_k(t)} \min\{(\varepsilon^4 t_0)^{-1}, \varepsilon^{-2}\} \prod_{j=1}^k (t_j - t_{j-1})^{-1/2} (t - t_k)^{-1/2} dt_0 \cdots dt_k$$

$$\Delta_k(t) = \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : 0 < t_0 < t_1 < \cdots < t_k < t\}$$

and with the convention $\prod_{\emptyset} = 1$. We estimate the $\mathfrak{J}_k(t)$ with a change of variables, and we obtain

$$\mathfrak{J}_k(t) \leq \frac{2^{k+1} t^{(k+1)/2}}{(k/2)!}.$$

Now, recalling the bound (98), we can conclude that

$$\begin{aligned} \|\varepsilon^{-2} Z_t(x)\|_n^2 &\leq C \min\{\varepsilon^{-2}, (\varepsilon^4 t)^{-1}\} (\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x) + C' \sum_{k \geq 0} C^k \frac{(\varepsilon^4 t)^{k/2}}{(k/2)!} (\mathbf{p}_{t+k+1}^\varepsilon * \varepsilon^{-2} Z_0)(x) \\ &\leq C(\varepsilon^4 t)^{-1} ((\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x) + 1). \end{aligned}$$

where we used that $\varepsilon^4 t \leq T$ and therefore C depends on the terminal time T . We then obtain (92) from (98).

We now turn to (93). In a similar fashion we have

$$\begin{aligned} \|\varepsilon^{-2} (Z_t(x) - Z_t(x'))\|_n &\leq |(\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x) - (\mathbf{p}_t^\varepsilon * \varepsilon^{-2} Z_0)(x')| \\ &\quad + \left\| \varepsilon^{-2} \int_0^t \sum_{z \geq 0} (\mathbf{p}_{t-s}^\varepsilon(x, z) - \mathbf{p}_{t-s}^\varepsilon(x', z)) dM_s(z) \right\|_n. \quad (102) \end{aligned}$$

We call the terms on the right-hand-side J_1 and J_2 , respectively. We get from (47) that

$$J_1 \leq C(\varepsilon^2 |x - y|)^\alpha (\varepsilon^4 t)^{-1-\alpha/2}$$

Now, for J_2 , we have

$$\begin{aligned} J_2^2 &\leq C \varepsilon^2 |x - y|^\alpha \int_0^t (t-s)^{-(1+\alpha)/2} \sum_{z \geq 0} (\mathbf{p}_{t-s+1}^\varepsilon(x, z) + \mathbf{p}_{t-s+1}^\varepsilon(y, z)) \|\varepsilon^{-2} Z_s(z)\|_n^2 ds \\ &\leq C \varepsilon^{-2} |x - y|^\alpha \int_0^t (t-s)^{-(1+\alpha)/2} s^{-1} \sum_{z \geq 0} (\mathbf{p}_{t-s+1}^\varepsilon(x, z) + \mathbf{p}_{t-s+1}^\varepsilon(y, z)) ((\mathbf{p}_s^\varepsilon * \varepsilon^{-2} Z_0)(x) + 1) ds \\ &= C \varepsilon^{-2} t^{-1/2} t^{-\alpha/2} |x - y|^\alpha ((\mathbf{p}_{t+1}^\varepsilon * \varepsilon^{-2} Z_0)(x) + (\mathbf{p}_{t+1}^\varepsilon * \varepsilon^{-2} Z_0)(y) + 1 + 1) \\ &\leq C(\varepsilon^4 t)^{-1/2} (\varepsilon^4 t)^{-1-\alpha/2} (\varepsilon^2 |x - y|)^\alpha. \end{aligned}$$

So

$$J_2 \leq C(\varepsilon^2 |x - y|)^{\alpha/2} (\varepsilon^4 t)^{-3/4-\alpha/4}$$

and we conclude that (93) holds since $(\varepsilon^4 t)^{-3/4-\alpha/4} \leq T^{1/4+\alpha/4} (\varepsilon^4 t)^{-1-\alpha/2}$.

Finally we prove (94), noting similarly that

$$\|\varepsilon^{-2} (Z_t(x) - Z_{t'}(x))\|_n \leq \|(\mathbf{p}_{t-t'}^\varepsilon * \varepsilon^{-2} Z_{t'})(x) - \varepsilon^{-2} Z_{t'}(x)\|_n + \left\| \varepsilon^{-2} \int_{t'}^t \sum_{z \geq 0} \mathbf{p}_{t-u}^\varepsilon(x, z) dM_{t'}(z) du \right\|_n. \quad (103)$$

We denote the terms on the right-hand side above by I_1 and I_2 respectively. We have, using (92) and (93),

$$\begin{aligned}
I_1 &= \left\| \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_{t-t'}^\varepsilon(x, z) Z_{t'}(z) - Z_{t'}(x) \right\|_n \\
&\leq \left\| \varepsilon^{-2} \sum_{z \geq 0} \mathbf{p}_{t-t'}^\varepsilon(x, z) (Z_{t'}(z) - Z_{t'}(x)) \right\|_n + \left| \sum_{z \geq 0} \mathbf{p}_{t-t'}^\varepsilon(x, z) - 1 \right| \left\| \varepsilon^{-2} Z_{t'}(x) \right\|_n \\
&\leq \sum_{z \geq 0} \mathbf{p}_{t-t'}^\varepsilon(x, z) (\varepsilon^4 t')^{-1-\alpha/2} (\varepsilon^2 |x-z|)^\alpha + C\varepsilon |t-t'|^{1/2} (\varepsilon^4 t')^{-1} \\
&\leq C(\varepsilon^4 t')^{-1-\alpha/2} \varepsilon^{2\alpha} (\mathbf{1} \vee |t-t'|^{\alpha/2}) + C\varepsilon |t-t'|^{1/2} (\varepsilon^4 t')^{-1} \\
&\leq C(\varepsilon^4 t')^{-1-\alpha/2} \varepsilon^{2\alpha} (\mathbf{1} \vee |t-t'|^{\alpha/2}) + C(\varepsilon^4 t')^{-1-\alpha/2} \varepsilon^{2\alpha} |t-t'|^{\alpha/2}.
\end{aligned}$$

Let us explain the steps used in the above series of inequalities.

- In the second line we have simply used the triangle inequality.
- In the third line, we have used Proposition 7.1 and the bound

$$\left| \sum_{z \geq 0} 1 - \mathbf{p}_{t-t'}^\varepsilon(x, z) \right| \leq C\varepsilon^2 |t-t'|^{1/2}. \quad (104)$$

Recall that $\mathbf{p}_{t-t'}^\varepsilon(x, z)$ correspond to transition probabilities of continuous-time discrete-space random walks (see the discussion below Definition 5.1). Hence, the LHS of (104) is the probability, for a random walk started at x at time t' (let $t' = 0$ for simplicity), to have been killed by time t . This probability is bounded by the killing rate $\frac{1-\mu}{2} < C\varepsilon$ times the expectation of the time spent at 0 by an auxiliary random walk having transition kernel \mathbf{p}^R with $\mu = 1$ (that is with no killing). For the latter random walk, the expectation of the time spent at 0 is the same as for the discrete space-discrete time random walk, with kernel denoted $\mathbf{p}^n(x, z)$ in (34). It is well-known that the local time at 0 for such random walks is of order \sqrt{t} (one may for instance consider that when at 0, the random walk stays there for an independent time, geometrically distributed, while outside from 0 it behaves as the absolute value of the simple random walk). Hence, the bound (104) holds.

- In the fourth line, we have used the bound

$$\sum_{z \geq 0} \mathbf{p}_{t-t'}^\varepsilon(x, z) |x-z|^\alpha \leq C(\mathbf{1} \vee |t-t'|^{\alpha/2}). \quad (105)$$

This bound is proved in [Par19, Eq. (30) and Corollary 3.3] for the Robin heat kernel \mathbf{p}^R under a different scaling of μ , and in particular, it holds for $\mu = 1$. Now recall that for any $t > 0, x, z \in \mathbb{N}$, the Robin heat kernel \mathbf{p}^R is increasing in μ (see (35)), so that (105) holds when $\mu = e^{-\varepsilon}$.

- In the last line we used $\varepsilon^4 t, \varepsilon^4 t' \leq T$, and the fact that

$$\varepsilon^2 |t-t'|^{1/2} \times (\varepsilon^4 t')^{-1} = ((\varepsilon^4 (t-t'))^{1/2-\alpha/2} \varepsilon^{2\alpha} (t-t')^{\alpha/2}) \times ((\varepsilon^4 t)^{\alpha/2} (\varepsilon^4 t')^{-1-\alpha/2}). \quad (106)$$

This proves the desired bound for I_1 . Now we consider I_2 . We have, using (92)

$$\begin{aligned}
I_2^2 &\leq C\varepsilon^2 \int_{t'}^t (t-s)^{-1/2} \sum_{z \geq 0} \mathbf{p}_{t-s+1}^\varepsilon(x, z) \left\| \varepsilon^{-2} Z_s(z) \right\|_n^2 ds \\
&\leq C\varepsilon^2 (\varepsilon^4 t)^{-2} \int_{t'}^t (t-s)^{-1/2} \sum_{z \geq 0} \mathbf{p}_{t-s+1}^\varepsilon(x, z) ds \\
&\leq C\varepsilon^2 (\varepsilon^4 t)^{-2} \int_{t'}^t (t-s)^{-1/2} ds = C(\varepsilon^4 t)^{-2} \varepsilon^2 (t-t')^{1/2},
\end{aligned}$$

where we have used that

$$\sum_{z \geq 0} \mathbf{p}_s^\varepsilon(x, z) \leq 1, \quad (107)$$

since the $\mathbf{p}_s^\varepsilon(x, z)$ correspond to transition probabilities of a random walk with killing. Therefore

$$I_2 \leq C(\varepsilon^4 t)^{-1} \times \varepsilon(t - t')^{1/4} \leq C(\varepsilon^4 t')^{-1 - \alpha/2} \varepsilon^{2\alpha} |t - t'|^{\alpha/2},$$

by the same argument as in (106). This proves (94). \square

7.2. Construction and properties of the limit point. We are now able to obtain tightness of $\{\mathcal{Z}_t^\varepsilon\}$, and identify the law of any of its limit point, as follows:

Proposition 7.2 (Tightness). *For any $0 < \delta \leq \tau$, the laws of $\{\mathcal{Z}^\varepsilon\}$ are tight on the Skorokhod space $D([\delta, \tau], C(\mathbb{R}_+))$. Moreover, any limit point \mathbb{P} is an element of $C([\delta, \tau], C(\mathbb{R}_+))$.*

For any $\theta \in [\delta, \tau]$, define $\mathcal{L}_\theta : C([\delta, \tau], C(\mathbb{R}_+)) \rightarrow C(\mathbb{R}_+)$ as the evaluation map at time θ . Then, the process $\{\mathcal{L}_{\theta+\delta} ; \theta \in [0, \tau - \delta]\}$ has the same distribution under \mathbb{P} as the solution of the stochastic heat equation (17) as defined in Definition 3.11, with initial condition \mathcal{Z}_{ini} whose distribution is the same as the one of \mathcal{L}_θ under \mathbb{P} .

Proof. The argument is exactly the same as in [Par19, Proof of Corollary 6.3]. The tightness property is based on Proposition 7.1, together with Arzela-Ascoli's Theorem. Continuity of limit points follow from Proposition 7.1 and Kolmogorov's continuity criterion. Finally, the identification of limit points follows from the same arguments as in Section 6.2, replacing \mathcal{Z}^ε by \mathcal{Z}^ε in (63) (Proposition 7.1 replacing Lemma 6.3 in the control of the error terms). \square

The next step consists in defining a limit point in $C((0, +\infty), C(\mathbb{R}_+))$ (Lemma 7.3 below) and identifying the initial condition. This means showing that the Duhamel form of the SHE (18) is satisfied for all $t > 0$, with $\mathcal{Z}_{\text{ini}} = -2\delta'_0$, that is,

$$\mathcal{Z}_t(u) = dP_t^{\text{Dir}}(u, 0) + \int_0^t ds \int_{\mathbb{R}_+} dv P_t^{\text{Dir}}(u, v) \mathcal{Z}_s(v) \xi(s, v). \quad (108)$$

Lemma 7.3. *Let \mathbb{Q}^ε denote the law of \mathcal{Z}^ε on $C((0, +\infty), C(\mathbb{R}_+))$. Then, there exists a measure \mathbb{Q} on $C((0, +\infty), C(\mathbb{R}_+))$ which is a limit point of the sequence $\{\mathbb{Q}^\varepsilon\}_\varepsilon$ on $D((0, +\infty), C(\mathbb{R}_+))$.*

Proof. This can be proved exactly as in [Par19, Lemma 6.5] thanks to Kolmogorov's extension Theorem. \square

The identification of the initial condition follows a general argument, given in [Par19], based on estimates on the first two moments given in the next two lemmas, which however are specific to the δ'_0 initial condition and use arguments different from [Par19]. We start with some exact computation for the first moment.

Lemma 7.4. *Let $\{\mathcal{Z}_t\}$ be distributed according to the measure \mathbb{Q} defined in Lemma 7.3. For any $t > 0$, and $u \in \mathbb{R}_+$, we have*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mathcal{Z}_t^\varepsilon(u)] = \mathbb{E}[\mathcal{Z}_t(u)] = dP_t^{\text{Dir}}(u, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{t}{2}\theta^2 + i\theta u} (-4i\theta) d\theta, \quad (109)$$

where we recall that $dP_t^{\text{Dir}}(u, 0)$ is defined in (30).

Proof. Using the same steps as in the proof of Proposition 5.5 – see (53) in particular –

$$\begin{aligned} \mathbb{E}[\mathcal{Z}_t^\varepsilon(u)] &= \varepsilon^{-2} \sum_{v \geq 0} \mathbf{p}_{t\varepsilon^{-4}}^\varepsilon(\varepsilon^{-2}u, \varepsilon^{-2}v) \mu^y \\ &= \frac{1}{2\pi} \int_{-\varepsilon^{-2}\pi}^{\varepsilon^{-2}\pi} e^{\frac{1}{2}(e^{i\varepsilon^2\theta} + e^{-i\varepsilon^2\theta} - 2)} \varepsilon^{-4t} e^{i\theta u} \frac{(1 - e^{-2\varepsilon})(1 - e^{2i\varepsilon^2\theta})}{(1 - e^{-\varepsilon - i\varepsilon^2\theta})(1 - e^{-\varepsilon + i\varepsilon^2\theta})^2} d\theta. \end{aligned} \quad (110)$$

Using the estimate (52), we see that the exponential term in the integrand of (110) is dominated by $e^{-2\theta^2 t/5}$, so that we may apply dominated convergence to take the $\varepsilon \rightarrow 0$ limit. It is easy to see that the integrand converges pointwise to

$$e^{-\frac{\theta^2 t}{2} + i\theta u} (-4i\theta),$$

so that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mathcal{Z}_t^\varepsilon(u)] = \int_{-\infty}^{+\infty} e^{-\frac{\theta^2 t}{2} + i\theta u} (-4i\theta) d\theta.$$

This Gaussian type of integral can be computed explicitly as $2\sqrt{\frac{2}{\pi}}\frac{u}{t^{3/2}}e^{-\frac{u^2}{2t}}$, which, by the definition of the Dirichlet heat kernel from (19), equals $2\partial_v P_t^{\text{Dir}}(u, 0)$. Furthermore, we have $\mathbb{E}[(\mathcal{Z}_t^\varepsilon(u))^2] \leq (C/T)^2$ by (92), so that the sequence is uniformly integrable. Hence we may exchange the limit with the expectation, and obtain (109). \square

We can conclude from Lemma 7.4 that if $\mathcal{Z}_t^\varepsilon(u)$ does converge as $\varepsilon \rightarrow 0$ to some $\mathcal{Z}_t(u)$ solving the stochastic heat equation (17) with some deterministic initial condition \mathcal{Z}_{ini} , then for all $t > 0$,

$$\int_{\mathbb{R}_+} dv P_t^{\text{Dir}}(u, v) \mathcal{Z}_{\text{ini}}(v) = dP_t^{\text{Dir}}(u, 0). \quad (111)$$

This suggests that $\mathcal{Z}_{\text{ini}} = -2\delta'$. However, in order to rigorously identify the initial condition, we need to establish that (108) holds, and for that we will also need a second moment estimate.

Lemma 7.5. *Fix $T > 0$ and consider $\{\mathcal{Z}_t\}_{t \in (0, T]}$ distributed according to the measure \mathbb{Q} defined in Lemma 7.3 (restricted to the time interval $(0, T]$). There exists a constant $C = C(T)$ such that, for any $u \in \mathbb{R}_+$ and $0 < t \leq T$,*

$$\|\mathcal{Z}_t(u)\|_2^2 \leq C \left(dP_t^{\text{Dir}}(u, 0) \right)^2 \quad (112)$$

$$\|\mathcal{Z}_t(u) - dP_t^{\text{Dir}}(u, 0)\|_2^2 \leq C\sqrt{t} \left(dP_t^{\text{Dir}}(u, 0) \right)^2, \quad (113)$$

where $\|F\|_2 := \sqrt{\int |F|^2 d\mathbb{Q}}$.

Remark 7.6. The bounds (112) and (113) are different from those in [Par19, Lemma 6.6] which is the analogue of Lemma 7.5 in the case of the Neumann boundary condition with narrow wedge initial condition. The proof of [Par19, Lemma 6.6] uses the martingale decomposition (96) and equation (100) above. These ingredients allow to mimic at the microscopic level the construction of the solution to the SHE (similar to our Section 4). In our case, we did not manage to obtain the discrete analog of (32) which would have been needed to complete the proof, because this required much sharper heat kernel estimates. We eventually opted to use the exact computation of $\lim \mathbb{E}[\mathcal{Z}_t^\varepsilon(u)^2]$ as $\varepsilon \rightarrow 0$ which was done in the separate article [BC22a].

Proof. From [BC22a, Prop. 4.8], we have that for $u_1 \leq u_2$,

$$\mathbb{E}[\mathcal{Z}_t(u_1)\mathcal{Z}_t(u_2)] = 4^2 \int_{\mathbf{i}\mathbb{R}+1+\eta} \frac{dz_1}{2\mathbf{i}\pi} \int_{\mathbf{i}\mathbb{R}} \frac{dz_2}{2\mathbf{i}\pi} \frac{z_1 - z_2}{z_1 - z_2 - 1} \frac{z_1 + z_2}{z_1 + z_2 - 1} e^{\frac{tz_1^2}{2} - u_1 z_1 + \frac{tz_2^2}{2} - u_2 z_2} z_1 z_2, \quad (114)$$

where $\eta > 0$ is any positive real number, and the expectation is taken with respect to the measure \mathbb{Q} . More precisely, [BC22a, Prop. 4.8] in the case $n = 2$ states that under the measure \mathbb{Q}^ε from Lemma 7.3, the limit as $\varepsilon \rightarrow 0$ of $\mathbb{E}[\mathcal{Z}_t^\varepsilon(u_1)\mathcal{Z}_t^\varepsilon(u_2)]$ is given by the right-hand side of (114). However, the case $n > 0$ of the same equation [BC22a, Eq. (4.20)] shows that higher moments are uniformly bounded as $\varepsilon \rightarrow 0$, which implies that the sequence of random variables $\mathcal{Z}_t^\varepsilon(u_1)\mathcal{Z}_t^\varepsilon(u_2)$ is uniformly integrable as $\varepsilon \rightarrow 0$. Hence, the weak convergence to $\mathcal{Z}_t(u_1)\mathcal{Z}_t(u_2)$ implied by Proposition 7.2 holds as well in L^1 , so that (114) holds.

The evaluation of this double Gaussian integral is not trivial. To simplify its estimation, it is convenient to use the shorthand notation

$$D(t, u) = 4 \int_{\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi} e^{\frac{tz^2}{2} - uz} z = 4 \int_{\mathbf{i}\mathbb{R}+1+\eta} \frac{dz}{2\mathbf{i}\pi} e^{\frac{tz^2}{2} - uz} z = dP_t^{\text{Dir}}(u, 0). \quad (115)$$

The second equality follows from Cauchy's theorem⁵ and the lack of residues between the vertical lines $\mathbf{i}\mathbb{R}$ and $\mathbf{i}\mathbb{R} + 1 + \eta$.

Using that for $\Re\text{al}[z_1 \pm z_2 - 1] > 0$,

$$\frac{1}{z_1 \pm z_2 - 1} = \int_0^\infty dy e^{-y(z_1 \pm z_2 - 1)},$$

⁵More precisely, we may apply the Cauchy theorem on the rectangle formed by the points $-\mathbf{i}R$, $\mathbf{i}R$, $\mathbf{i}R + 1 + \eta$, $-\mathbf{i}R + 1 + \eta$. Since the function $z \mapsto e^{\frac{tz^2}{2} - uz} z$ has no residues inside the rectangle, the integral over the rectangle equal zero. Moreover, due to the exponential decay of the integrand as the imaginary part increases, we see that the difference between the integration along the segment $[-\mathbf{i}R, \mathbf{i}R]$ or the segment $[-\mathbf{i}R + 1 + \eta, \mathbf{i}R + 1 + \eta]$ goes to zero as R goes to infinity. This proves that the two integral formulas in (115) are equal.

we have that

$$\mathbb{E}[\mathcal{Z}_t(u_1)\mathcal{Z}_t(u_2)] = \int_0^\infty d\lambda \int_0^\infty d\mu e^{\lambda+\mu} \partial_\lambda \partial_\mu D(t, u_1 + \lambda + \mu) D(t, u_2 - \lambda + \mu),$$

so that using the exact explicit expression for $dP_t^{\text{Dir}}(u, 0)$ in (30),

$$\frac{\|\mathcal{Z}_t(u)\|_2^2}{dP_t^{\text{Dir}}(u, 0)^2} = \frac{1}{4} \int_0^\infty d\lambda \int_0^\infty d\mu e^{\lambda+\mu} \partial_\lambda \partial_\mu \left(\frac{(u+\mu)^2 - \lambda^2}{u^2} e^{-\frac{\lambda^2 + \mu^2 + 2\mu u}{t}} \right).$$

Computing the derivatives in λ and μ , the integral can be evaluated using Mathematica, which yields the explicit formula

$$\frac{\|\mathcal{Z}_t(u)\|_2^2}{dP_t^{\text{Dir}}(u, 0)^2} = \frac{e^{-u} \left(\sqrt{\pi} e^{t/4} \sqrt{t} \left(e^u \left(\operatorname{erf} \left(\frac{\sqrt{t}}{2} \right) + 1 \right) (t(u-1) + 2u^2) + t e^{\frac{u^2}{t}} \left(\operatorname{erf} \left(\frac{t-2u}{2\sqrt{t}} \right) + 1 \right) \right) + 2e^u u(t+2u) \right)}{4u^2}. \quad (116)$$

We see that the maximum of this function is attained as $u \rightarrow 0$, so that

$$0 \leq \frac{\|\mathcal{Z}_t(u)\|_2^2}{dP_t^{\text{Dir}}(u, 0)^2} \leq \frac{1}{8} \left(\sqrt{\pi} e^{t/4} \sqrt{t} (t+6) \left(\operatorname{erf} \left(\frac{\sqrt{t}}{2} \right) + 1 \right) + 2(t+4) \right). \quad (117)$$

It can be checked that this expression behaves as $1 + \frac{3\sqrt{\pi}}{4} \sqrt{t} + o(\sqrt{t})$ as $t \rightarrow 0$, so that on an interval $[0, T]$, (117) is bounded by $1 + C\sqrt{t}$ where the constant C depend on T . This immediately implies (112), and, using Lemma 7.4, it also implies (113). \square

7.3. Conclusion. Provided with Proposition 7.2 and Lemma 7.3, one can follow the argument of [Par19, Lemma 6.7 and Theorem 6.8], and one obtains that $\{\mathcal{Z}_s^\varepsilon\}_{s \in (0, T]}$ converges as $\varepsilon \rightarrow 0$ to a solution of the stochastic heat equation (17) on the time interval $[0, T]$ with Dirichlet boundary condition, in the sense of weak convergence of probability measures on the path space $D((0, T], C(\mathbb{R}_+))$ endowed with the Skorokhod topology.

It remains to formally check that the initial condition is $\mathcal{Z}_{\text{ini}} = -2\delta'_0$ as we have claimed. We need to check that the limit of $\{\mathcal{Z}_s^\varepsilon\}_{s \in (0, T]}$, denoted $\{\mathcal{Z}_s\}_{s \in (0, T]}$, satisfies (108). Since \mathcal{Z}_s is a solution of (2) on the space $D((0, T], C(\mathbb{R}_+))$, we already know that for any $0 < s < t < T$,

$$\mathcal{Z}_t(u) = P_{t-s}^{\text{Dir}}(u, \cdot) * \mathcal{Z}_s(\cdot) + \int_s^t d\tau \int_{\mathbb{R}_+} dv P_{t-\tau}^{\text{Dir}}(u, v) \mathcal{Z}_\tau(v) \xi(\tau, v).$$

Hence, following [Par19, Section 6], we may write

$$\left\| \mathcal{Z}_t(u) - dP_t^{\text{Dir}}(u, 0) - \int_0^t d\tau \int_{\mathbb{R}_+} dv P_{t-\tau}^{\text{Dir}}(u, v) \mathcal{Z}_\tau(v) \xi(\tau, v) \right\|_2 \leq \left\| P_{t-s}^{\text{Dir}}(u, \cdot) * \mathcal{Z}_s(\cdot) - dP_t^{\text{Dir}}(u, 0) \right\|_2 + \left\| \int_0^s d\tau \int_{\mathbb{R}_+} dv P_{t-\tau}^{\text{Dir}}(u, v) \mathcal{Z}_\tau(v) \xi(\tau, v) \right\|_2. \quad (118)$$

This inequality holds for any $0 < s < t$, so that in order to show that the LHS of (118) equal zero, it suffices to show that the RHS vanishes as $s \rightarrow 0$. Using the semi-group property,

$$\begin{aligned} \left\| P_{t-s}^{\text{Dir}}(u, \cdot) * \mathcal{Z}_s(\cdot) - dP_t^{\text{Dir}}(u, 0) \right\|_2 &\leq \left\| \int_{\mathbb{R}_+} dv P_{t-s}^{\text{Dir}}(u, v) \left(\mathcal{Z}_s(v) - dP_s^{\text{Dir}}(v, 0) \right) \right\|_2 \\ &\leq \int_{\mathbb{R}_+} dv P_{t-s}^{\text{Dir}}(u, v) \left\| \mathcal{Z}_s(v) - dP_s^{\text{Dir}}(v, 0) \right\|_2 \\ &\leq \int_{\mathbb{R}_+} dv P_{t-s}^{\text{Dir}}(u, v) s^{1/4} dP_s^{\text{Dir}}(v, 0) \\ &\leq s^{1/4} dP_t^{\text{Dir}}(v, 0) \end{aligned}$$

where in the third inequality we have used Lemma 7.5. Hence we have obtained that

$$\lim_{s \rightarrow 0} \left\| P_{t-s}^{\text{Dir}}(u, \cdot) * \mathcal{Z}_s(\cdot) - dP_t^{\text{Dir}}(u, 0) \right\|_2 = 0.$$

Now we turn to the second term to bound. By Ito isometry,

$$\begin{aligned} \left\| \int_0^s d\tau \int_{\mathbb{R}_+} dv P_{t-\tau}^{\text{Dir}}(u, v) \mathcal{Z}_\tau(v) \xi(\tau, v) \right\|_2^2 &= \int_0^s d\tau \int_{\mathbb{R}_+} dv P_{t-\tau}^{\text{Dir}}(u, v)^2 \mathbb{E} \left[\mathcal{Z}_\tau(v)^2 \right] \\ &\leq C \int_0^s d\tau \int_{\mathbb{R}_+} dv P_{t-\tau}^{\text{Dir}}(u, v)^2 dP_\tau(v, 0)^2 \\ &= C \int_0^s d\tau G_t(\tau) \\ &\leq C \sqrt{\frac{t}{t-s}} \int_0^s d\tau \frac{1}{\sqrt{\tau}} dP_t(u, 0)^2 \\ &\leq C \sqrt{\frac{st}{t-s}} dP_t(u, 0)^2, \end{aligned}$$

where in the first inequality we have used Lemma 7.5, and in the second inequality we have used Lemma 4.1 (recall that the function $G_t(\tau)$ is defined in (31)). We conclude that

$$\lim_{s \rightarrow 0} \left\| \int_0^s d\tau \int_{\mathbb{R}_+} dv P_{t-\tau}^{\text{Dir}}(u, v) \mathcal{Z}_\tau(v) \xi(\tau, v) \right\|_2 = 0,$$

so that (108) is satisfied for all $0 < t \leq T$. This concludes the proof of Theorem 3.7.

APPENDIX A. MICROSCOPIC COLE-HOPF TRANSFORM

Here we explain our choice of parameters (10) which permits to obtain the discrete stochastic heat equation as in (11). We observe that $Z_t(x)$ is affected only by exchanging values $\eta(x)$, $\eta(x+1)$. Therefore, one can check that

$$\mathcal{L}Z_t(x) = \left[p\sigma_t(x)(1 - \sigma_t(x+1))(e^{2\lambda} - 1) + q\sigma_t(x+1)(1 - \sigma_t(x))(e^{-2\lambda} - 1) \right] Z_t(x)$$

for any $x > 0$ and moreover

$$LZ_t(0) = p(1 - \sigma_t(1))(e^{2\lambda} - 1)Z_t(0).$$

Besides, the discrete Laplacian acts as:

$$\Delta Z_t(x) = [e^{-\lambda(2\sigma_t(x+1)-1)} + e^{\lambda(2\sigma_t(x)-1)} - 2] Z_t(x)$$

for any $x > 0$. By identification we obtain the following conditions

$$\nu = D(e^\lambda + e^{-\lambda} - 2) \tag{119}$$

$$\nu = D(2e^\lambda - 2) - p(e^{2\lambda} - 1) \tag{120}$$

$$\nu = D(2e^{-\lambda} - 2) - q(e^{-2\lambda} - 1). \tag{121}$$

which, after resolution, give

$$\lambda = \frac{1}{2} \log \frac{q}{p}, \quad D = \sqrt{pq}, \quad \nu = q + p - 2\sqrt{pq}. \tag{122}$$

Finally, we need to define $Z(-1)$ such that, at the boundary point $x = 0$, we get $(\nu Z(0) + LZ(0)) = D\Delta Z(0)$. This gives

$$\nu = D \left(\sqrt{\frac{p}{q}} - 2 + \frac{Z(-1)}{Z(0)} \right) \quad \text{if } \eta(1) = 1 \tag{123}$$

$$\nu + q - p = D \left(\sqrt{\frac{q}{p}} - 2 + \frac{Z(-1)}{Z(0)} \right) \quad \text{if } \eta(1) = 0. \tag{124}$$

With the choice of D made in (122), the last two conditions are in fact the same, and read

$$Z(-1) = \mu Z(0) \quad \text{with } \mu = \sqrt{\frac{q}{p}}. \tag{125}$$

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G. BARRAQUAND, LABORATOIRE DE PHYSIQUE DE L’ÉCOLE NORMALE SUPÉRIEURE, ENS, UNIVERSITÉ PSL, CNRS, SORBONNE UNIVERSITÉ, UNIVERSITÉ PARIS-CITÉ, PARIS, FRANCE

Email address: `guillaume.barraquand@ens.fr`

O. BLONDEL, UNIV LYON, CNRS, UNIVERSITÉ CLAUDE BERNARD LYON 1, UMR5208, INSTITUT CAMILLE JORDAN, F-69622 VILLEURBANNE, FRANCE

Email address: `blondel@math.univ-lyon1.fr`

M. SIMON, UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1, CNRS UMR 5208, INSTITUT CAMILLE JORDAN, F-69622 VILLEURBANNE, FRANCE

Email address: `msimon@math.univ-lyon1.fr`