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# A Kalman condition for the controllability of a coupled system of Stokes equations

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## Abstract

We consider the controllability of a class of systems of  $n$  Stokes equations, coupled through terms of order zero and controlled by  $m$  distributed controls. Our main result states that such a system is null-controllable if and only if a Kalman type condition is satisfied. This generalizes the case of finite-dimensional systems and the case of systems of coupled linear heat equations. The proof of the main result relies on the use of the Kalman operator introduced in [1] and on a Carleman estimate for a cascade type system of Stokes equations. Using a fixed-point argument, we also obtain that if the Kalman condition is verified, then the corresponding system of Navier-Stokes equations is locally null-controllable.

**Keywords:** Null controllability, Navier-Stokes systems, Carleman estimates

**2010 Mathematics Subject Classification.** 76D05, 35Q30, 93B05, 93B07, 93C10

## 1 Introduction

In the last fifteen years the challenging issue of controlling systems of coupled equations of the same nature has attracted the interest of the control community. In optimal control problems this kind of systems appears as a characterization of the optimal control, with one equation coupled to its adjoint. Also coupled parabolic equations appear as a model for chemical reactions, population models, cancer and other physical situations (see e.g. [9, 13] and [23]).

The firsts results treating the null controllability for a single parabolic equations appear in the seminal papers of [14, 19, 26] for the heat equation and in [10, 17, 24] for the Navier-Stokes system. That means that it is possible to drive an initial datum to zero at time  $T > 0$  acting locally or on the boundary of the domain. The development in this area has been intense in the last 50 years and it is impossible to mention all the contributions on the subject.

In the case of scalar (heat) coupled equations an important number of challenging problems has been solved (see [2] for a survey of results until 2011) and sometimes the results have been surprising [3–5]. In the case of coupled Stokes or Navier-Stokes systems, to our knowledge, mainly the cases of *two* coupled systems have been treated [7, 8, 21, 30]. Recently (see [33]), the authors of the present paper considered the case of  $m$  coupled Stokes

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(Navier-Stokes) systems acting with only a control on the first equation. In [33] the coupling matrix on the zero order terms has a *cascade* structure and the coupling matrix for the first order terms is triangular superior.

Here, our aim is to generalize the classical Kalman rank condition for the controllability of a finite dimensional system to the case of Stokes or Navier-Stokes equations. More precisely, we consider a system coupling  $n$  systems of Navier-Stokes equations, that we want to control by using  $m$  distributed controls:

$$\begin{cases} \partial_t y^{(i)} - \mu_i \Delta y^{(i)} + \nabla p^{(i)} + \left( y^{(i)} \cdot \nabla \right) y^{(i)} + \sum_{j=1}^n q_{i,j} y^{(j)} = \sum_{j=1}^m r_{i,j} v^{(j)} 1_\omega & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ \nabla \cdot y^{(i)} = 0 & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ y^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, & (1 \leq i \leq n) \\ y^{(i)}(0, \cdot) = y_0^{(i)} & \text{in } \Omega, & (1 \leq i \leq n). \end{cases} \quad (1.1)$$

Here  $\Omega$  is a smooth open bounded domain of  $\mathbb{R}^N$ ,  $N = 2, 3$ . For each  $i \in \{1, \dots, n\}$ ,  $y^{(i)}$  and  $p^{(i)}$  are the velocity and the pressure of a fluid and we assume that the viscosity  $\mu_i > 0$  is constant. These Navier-Stokes equations are coupled through a matrix  $(q_{i,j})_{i,j=1,\dots,n}$  assumed to be constant (in time and space). The controls are denoted by  $v^{(j)}$ ,  $j = 1, \dots, m$  and appear through a matrix  $(r_{i,j})_{i \in \{1,\dots,n\}, j \in \{1,\dots,m\}}$ , also constant in time and space. We impose that our controls have a vanishing component, for instance

$$v^{(j)} \cdot e^{(N)} = 0 \quad (j = 1, \dots, m), \quad (1.2)$$

where  $(e^{(1)}, \dots, e^{(N)})$  is the canonical basis of  $\mathbb{R}^N$ .

Using the coupling between the equations, we aim at using less controls than the number of states ( $m < n$ ) and our objective is to show that the well-known Kalman condition in finite dimension can be extended in this context. Such an extension is already known for systems of heat type equations (see [1]) and we show here the same result for Stokes equations and for controls with a vanishing component.

First, let us consider a linearization of (1.1):

$$\begin{cases} \partial_t y^{(i)} - \mu_i \Delta y^{(i)} + \nabla p^{(i)} + \sum_{j=1}^n q_{i,j} y^{(j)} = \sum_{j=1}^m r_{i,j} v^{(j)} 1_\omega & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ \nabla \cdot y^{(i)} = 0 & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ y^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, & (1 \leq i \leq n) \\ y^{(i)}(0, \cdot) = y_0^{(i)} & \text{in } \Omega, & (1 \leq i \leq n) \end{cases} \quad (1.3)$$

that couples a system of Stokes equations. We can write the above system in an abstract way: first, we introduce the Hilbert space

$$\mathcal{H} := \left\{ y \in [L^2(\Omega)]^N : \nabla \cdot y = 0 \text{ in } \Omega, \quad y \cdot \nu = 0 \text{ on } \partial\Omega \right\}, \quad (1.4)$$

where we denote by  $\nu$  the unit outward normal vector field on  $\partial\Omega$ . We consider the classical Stokes operator

$$\mathbb{A} := -\mathbb{P}\Delta, \quad \mathcal{D}(\mathbb{A}) := \left\{ y \in [H^2(\Omega) \cap H_0^1(\Omega)]^N : \nabla \cdot y = 0 \text{ in } \Omega \right\}, \quad (1.5)$$

where  $\mathbb{P} : [L^2(\Omega)]^N \rightarrow \mathcal{H}$  is the orthogonal projection (Leray projector). The control operator  $\mathbb{B} \in \mathcal{L}(\mathcal{U}, \mathcal{H})$  of our system can be defined as follows:

$$\mathbb{B}v := \mathbb{P} \left( 1_\omega \sum_{i=1}^{N-1} v_i e^{(i)} \right), \quad \text{for } v = (v_1, \dots, v_{N-1}) \in \mathcal{U} := L^2(\omega)^{N-1}.$$

We also denote by  $D, Q \in \mathcal{M}_n(\mathbb{R})$  and  $R \in \mathcal{M}_{n,m}(\mathbb{R})$  respectively the matrices  $\text{diag}(\mu_1, \dots, \mu_n)$ ,  $(q_{i,j})_{i,j}$  and  $(r_{i,j})_{i,j}$ . Finally, we consider the operator of the coupled system (1.3):

$$\mathcal{D}(\mathbb{L}) := \mathcal{D}(\mathbb{A})^n, \quad \mathbb{L} := D\mathbb{A} + Q, \quad (1.6)$$

that is for  $y = (y^{(1)}, \dots, y^{(n)}) \in \mathcal{D}(\mathbb{A})^n$ ,

$$\mathbb{L}y := \left( \mu_i \mathbb{A}y^{(i)} + \sum_{j=1}^n q_{i,j} y^{(j)} \right)_{i \in \{1, \dots, n\}}.$$

The control operator of the coupled system is  $R\mathbb{B} \in \mathcal{L}(\mathcal{U}^m, \mathcal{H}^n)$ : for  $v = (v^{(1)}, \dots, v^{(m)}) \in \mathcal{U}^m$ ,

$$R\mathbb{B}v = \left( \sum_{j=1}^m r_{i,j} \mathbb{B}v^{(j)} \right)_{i \in \{1, \dots, n\}}.$$

Then, we can write (1.3) as

$$y' + \mathbb{L}y = R\mathbb{B}v, \quad y(0) = y_0 \tag{1.7}$$

with

$$y_0 = (y_0^{(1)}, \dots, y_0^{(n)}).$$

In order to study the null-controllability of (1.7), we introduce the Kalman operator, as in [1]:

$$\mathbb{K} : \mathcal{H}^m \times \mathcal{D}(\mathbb{A})^m \times \dots \times \mathcal{D}(\mathbb{A}^{n-1})^m \rightarrow \mathcal{H}^n, \quad (w^{(1)}, \dots, w^{(n)}) \mapsto \sum_{i=1}^n \mathbb{L}^{i-1} R w^{(i)}. \tag{1.8}$$

In the above definition, note that  $w^{(i)} = (w_1^{(i)}, \dots, w_m^{(i)}) \in \mathcal{D}(\mathbb{A}^{i-1})^m$  so that

$$R w^{(i)} = \left( \sum_{j=1}^m r_{k,j} w_j^{(i)} \right)_{k \in \{1, \dots, n\}} \in \mathcal{D}(\mathbb{A}^{i-1})^n = \mathcal{D}(\mathbb{L}^{i-1})$$

and thus the Kalman operator is well-defined. The adjoint of  $\mathbb{K}$  is given by

$$\mathcal{D}(\mathbb{K}^*) = [\mathcal{D}(\mathbb{A}^{n-1})]^n, \quad \mathbb{K}^* \varphi = [R^* \varphi, R^* \mathbb{L}^* \varphi, \dots, R^* (\mathbb{L}^*)^{n-1} \varphi]. \tag{1.9}$$

We say that (1.7) is null-controllable in time  $T > 0$  if for any  $y_0 \in \mathcal{H}^n$ , there exists a control  $v \in L^2(0, T; L^2(\omega)^{N-1})^m$  such that the corresponding solution  $y$  to (1.7) satisfies

$$y(T, \cdot) = 0.$$

Our main result states this property is equivalent to the injectivity of  $\mathbb{K}^*$ :

**Theorem 1.1.** *Assume  $\omega \subset \Omega$  is a nonempty open set. Then*

$$\text{Ker } \mathbb{K}^* = \{0\} \tag{1.10}$$

*if and only if (1.7) is null-controllable in time  $T > 0$ .*

In order to prove the above result, we show that (1.10) is equivalent to the final-state observability for the adjoint system of (1.3) or equivalently (1.7):

$$\begin{cases} \partial_t \varphi^{(i)} - \mu_i \Delta \varphi^{(i)} + \nabla \pi^{(i)} + \sum_{j=1}^n q_{j,i} \varphi^{(j)} = 0 & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ \nabla \cdot \varphi^{(i)} = 0 & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ \varphi^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, & (1 \leq i \leq n) \\ \varphi^{(i)}(0, \cdot) = \varphi_0^{(i)} & \text{in } \Omega, & (1 \leq i \leq n). \end{cases} \tag{1.11}$$

In an abstract form, the adjoint system writes as follows:

$$\varphi' + \mathbb{L}^* \varphi = 0, \quad \varphi(0) = \varphi_0. \quad (1.12)$$

The adjoint of the control operator is  $\mathbb{B}^* R^* \in \mathcal{L}(\mathcal{H}^n, \mathcal{U}^m)$ . More precisely, if  $\varphi = (\varphi^{(1)}, \dots, \varphi^{(n)}) \in \mathcal{H}^n$  and if we denote by  $\varphi_k^{(i)}$ ,  $k = 1, \dots, N$  the components of  $\varphi^{(i)} \in \mathcal{H}$ , then

$$\mathbb{B}^* R^* \varphi = \left( (\mathbb{B}^* R^* \varphi)^{(1)}, \dots, (\mathbb{B}^* R^* \varphi)^{(m)} \right) \in \mathcal{U}^m, \quad (\mathbb{B}^* R^* \varphi)^{(i)} = \left( \sum_{j=1}^n r_{j,i} \sum_{k=1}^{N-1} \varphi_k^{(j)} e^{(k)} \right) \Big|_{\omega} \quad (i = 1, \dots, m).$$

Following a standard duality argument (see, for instance, [35, Theorem 11.2.1, p.357]), Theorem 1.1 is a consequence of the two following results:

**Lemma 1.2.** *If  $\text{Ker } \mathbb{K}^* \neq \{0\}$ , then (1.11) is not final-state observable: there exists  $\varphi_0 \in \mathcal{H}^n$ ,  $\varphi_0 \neq 0$  such that the solution  $\varphi$  of (1.12) satisfies  $\varphi(T, \cdot) \neq 0$  and*

$$R^* \varphi = 0 \quad \text{in } (0, T) \times \omega.$$

We show this result in Section 2.

**Theorem 1.3.** *Assume  $\omega \subset \Omega$  is a nonempty open set and  $T > 0$ . If (1.10) holds, then there exist  $C > 0$  and  $\ell > 0$  such that for any  $\varphi_0 \in \mathcal{H}^n$ ,*

$$\int_{\Omega} |\varphi(T, \cdot)|^2 dx \leq C e^{C/T^\ell} \iint_{(0,T) \times \omega} \sum_{k=1}^{N-1} |R^* \varphi_k|^2 dx dt. \quad (1.13)$$

In the above statement, we can write the right-hand side with

$$\iint_{(0,T) \times \omega} \sum_{k=1}^{N-1} |R^* \varphi_k|^2 dx dt = \iint_{(0,T) \times \omega} \sum_{k=1}^{N-1} \sum_{i=1}^m \left| \sum_{j=1}^n r_{j,i} \varphi_k^{(j)} \right|^2 dx dt.$$

In order to prove (1.13), we will show a Carleman estimate of the form

$$\iint_{(0,T) \times \Omega} \rho_{\sharp}^2 |\varphi|^2 dx dt \leq C \iint_{(0,T) \times \omega} \rho_{\flat}^2 \sum_{k=1}^{N-1} |R^* \varphi_k|^2 dx dt. \quad (1.14)$$

where  $\rho_{\sharp}$  and  $\rho_{\flat}$  are regular functions on  $[0, T]$ , positive in  $(0, T)$  and vanishing in 0 and  $T$ . There expressions are given through weights defined in Section 4.1: for some  $\kappa \in (0, 1)$ , and for some  $\delta_n \in \mathbb{N}^*$ :

$$\rho_{\sharp} = e^{-s\alpha^*} (s\xi^*)^{3/2-1/\ell}, \quad \rho_{\flat} = e^{-\kappa s\hat{\alpha}} \left( s\hat{\xi} \right)^{\delta_n/2}.$$

The proof of (1.13) from (1.14) is quite standard so that we skip its proof and we refer the reader to [33] (see, also, [16]). The proof of (1.14) is the consequence of a Carleman estimate stated and proved in Section 4 and of a general method using the Kalman criterion. This method is presented in Section 3.

**Remark 1.4.** Note that this method follows the proof proposed in [1] in the case of the heat equation. The idea is to reduce the problem to the case where (1.11) is a cascade system with a particular right-hand side that can be absorbed by using the properties of the weights in the Carleman estimates. However, in the case of Stokes equations, the same method can not be applied with the Carleman estimates that are available in the literature.

More precisely, with the notation for the weights that are described in Section 4.1, a standard Carleman estimate for the heat equation can be written as

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha} (s\xi)^3 |\psi|^2 dxdt \leq C \left( \iint_{(0,T)\times\Omega} e^{-2s\alpha} |g|^2 dxdt + \iint_{(0,T)\times\omega} e^{-2s\alpha} (s\xi)^3 |\psi|^2 dxdt \right) \quad (1.15)$$

for  $\psi$  solution of

$$\begin{cases} \partial_t \psi - \Delta \psi = g & \text{in } (0, T) \times \Omega, \\ \psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ \psi(0, \cdot) = \psi_0 & \text{in } \Omega. \end{cases}$$

In particular, if  $g$  is of the form  $a\psi$  with  $a \in \mathbb{R}$ , the integral of  $g$  in (1.15) can be absorbed by the left-hand side. For the Stokes equations, the observability obtained in [6] in the case of a control satisfying (1.2) can be written as

$$\begin{aligned} \iint_{(0,T)\times\Omega} e^{-5s\alpha^*} (s\xi^*)^4 |\psi|^2 dxdt \\ \leq C \left( \iint_{(0,T)\times\Omega} e^{-3s\alpha^*} |g|^2 dxdt + \sum_{i < N} \iint_{(0,T)\times\omega} e^{-2s\hat{\alpha} - 3s\alpha^*} (s\hat{\xi})^7 |\psi_i|^2 dxdt \right) \end{aligned} \quad (1.16)$$

for  $\psi$  solution of

$$\begin{cases} \partial_t \psi - \Delta \psi + \nabla p = g & \text{in } (0, T) \times \Omega, \\ \nabla \cdot \psi = 0 & \text{in } (0, T) \times \Omega, \\ \psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ \psi(0, \cdot) = \psi_0 & \text{in } \Omega. \end{cases}$$

In that case, one can see that if  $g$  is of the form  $a\psi$  with  $a \in \mathbb{R}$ , the integral of  $g$  in (1.16) can not be absorbed anymore by the left-hand side. The consequence is that in our Kalman method we need to consider a generalized cascade system and we will then show a Carleman estimate for this cascade system.

Finally using a general method (see, [29]), one can also deduce from Theorem 1.3 a result of local controllability for the system (1.1):

**Theorem 1.5.** *Assume  $\omega \subset \Omega$  is a nonempty open set and  $T > 0$ . If (1.10) holds, then there exists  $c > 0$  such that for any  $y_0 \in \mathcal{V}^n$  satisfying*

$$\|y_0\|_{\mathcal{V}^n} \leq c,$$

*there exists a control  $v \in L^2(0, T; L^2(\omega)^{N-1})^m$  such that the corresponding solution  $y$  to (1.1) satisfies*

$$y(T, \cdot) = 0.$$

We skip the proof of Theorem 1.5 since the corresponding proof is completely similar to the proof of Theorem 1.3 in [33].

Our results are based on a Kalman method and on *(global) Carleman inequalities*. Such estimates were introduced in [19] for the controllability of parabolic equations and have been used by many authors. In particular, one can quote [25] or [17] among the large number articles devoted to the controllability of Stokes or Navier-Stokes systems. Several works are devoted to the case where the controls have one vanishing component: [18], [11], [6] and [8]. Note that in [12], the authors obtain the local null controllability of the Navier-Stokes system in dimension 3 with a control having two vanishing components but their method is different, using a particular linearization combined with results of Gromov. Here we follow the proof we did in [33] for a standard cascade system to obtain our Carleman estimate.

This paper is organized as follows. In Section 2, we recall some results of [1] that allow us to show Lemma 1.2 and present several important examples of systems of the form (1.3). The Kalman method is described in Section 3 and we show the Carleman estimates needed to show Theorem 1.3 in Section 4.

## 2 Preliminaries and examples

Let us recall that the Stokes operator  $\mathbb{A}$  (defined by (1.4), (1.5)) is a positive self-adjoint operator with compact resolvents. In particular, its spectrum is composed by positive eigenvalues  $(\gamma_p)_{p \geq 1}$ . Let us consider an orthonormal basis  $(\phi_p)_{p \geq 1}$  of  $\mathcal{H}$  composed by eigenvectors of  $\mathbb{A}$  associated with the eigenvalues  $(\gamma_p)_{p \geq 1}$ . Then, we can define for all  $p \geq 1$  the matrix

$$K_p := \left[ R \mid (\gamma_p D + Q) R \mid \cdots \mid (\gamma_p D + Q)^{n-1} R \right] \in \mathcal{M}_{n, nm}(\mathbb{R}) \quad (2.1)$$

and its adjoint

$$K_p^* = \begin{bmatrix} R^* \\ R^* (\gamma_p D + Q^*) \\ \vdots \\ R^* (\gamma_p D + Q^*)^{n-1} \end{bmatrix} \in \mathcal{M}_{nm, n}(\mathbb{R}).$$

Then, we have the following result (which is Proposition 2.2 in [1]):

**Proposition 2.1.** *With the above notation, we have the following equivalencies:*

$$\text{Ker } \mathbb{K}^* = \{0\} \iff \forall p \geq 1, \text{ Ker } K_p^* = \{0\} \iff \forall p \geq 1, \text{ rank } K_p = n.$$

The above result allows us to show Lemma 1.2 exactly as in [1]. We only repeat the proof for sake of completeness:

*Proof of Lemma 1.2.* Assume  $\text{Ker } \mathbb{K}^* \neq \{0\}$ . Then from Proposition 2.1, there exist  $p_0 \geq 1$  and  $z_0 \in \text{Ker } K_{p_0}^* \subset \mathbb{R}^n$ ,  $z_0 \neq 0$ . Then, from the standard Kalman condition (see, for instance, [35, Proposition 1.4.7, p.13]), we have

$$R^* e^{-(\gamma_{p_0} D + Q^*)t} z_0 = 0 \quad (t \in \mathbb{R}).$$

For  $z = (z^{(1)}, \dots, z^{(n)}) \in \mathbb{R}^n$  and  $\phi \in \mathcal{H}$ , we set

$$z\phi := (z^{(1)}\phi, \dots, z^{(n)}\phi) \in \mathcal{H}^n.$$

We can check that

$$\varphi(t) := \left( e^{-(\gamma_{p_0} D + Q^*)t} z_0 \right) \phi_{p_0} \quad (t \geq 0)$$

is the solution of (1.12) with  $\varphi_0 := z_0 \phi_{p_0}$ . We have

$$\varphi(T) = \left( e^{-(\gamma_{p_0} D + Q^*)T} z_0 \right) \phi_{p_0} \neq 0 \quad \text{and} \quad R^* \varphi(t) = \left( R^* e^{-(\gamma_{p_0} D + Q^*)t} z_0 \right) \phi_{p_0} = 0 \quad (t \geq 0).$$

□

Using Proposition 2.1, we can also consider the following particular cases of Theorem 1.1. These cases are already discussed in [1] (see Remark 1.1 in [1]).

**Case 1: all the viscosities are equal.** Assume

$$\forall i \in \{1, \dots, n\} \quad \mu_i = \mu > 0.$$

Then we deduce from Proposition 2.1 that (1.10) is equivalent to

$$\text{rank} \left[ R \mid QR \mid \cdots \mid Q^{n-1} R \right] = n, \quad (2.2)$$

that is the original Kalman condition for the matrices  $Q$  and  $R$ . In particular,

$$\begin{cases} \partial_t y^{(i)} - \mu \Delta y^{(i)} + \nabla p^{(i)} + \sum_{j=1}^n q_{i,j} y^{(j)} = \sum_{j=1}^m r_{i,j} v^{(j)} 1_\omega & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ \nabla \cdot y^{(i)} = 0 & \text{in } (0, T) \times \Omega, & (1 \leq i \leq n) \\ y^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, & (1 \leq i \leq n) \\ y^{(i)}(0, \cdot) = y_0^{(i)} & \text{in } \Omega, & (1 \leq i \leq n) \end{cases}$$

is null-controllable if and only if the finite-dimensional linear system

$$\begin{cases} \frac{d}{dt} Y^{(i)} + \sum_{j=1}^n q_{i,j} Y^{(j)} = \sum_{j=1}^m r_{i,j} V^{(j)} & \text{in } (0, T), \quad (1 \leq i \leq n) \\ Y^{(i)}(0, \cdot) = Y_0^{(i)} \in \mathbb{R} & (1 \leq i \leq n) \end{cases}$$

is controllable.

**Case 2: simultaneous controllability.** Assume

$$Q = 0, \quad m = 1, \quad R = (r_i)_{i=1, \dots, n}.$$

Then (2.1) can be written as

$$K_p := \begin{bmatrix} r_1 & (\gamma_p \mu_1) r_1 & \cdots & (\gamma_p \mu_1)^{n-1} r_1 \\ r_2 & (\gamma_p \mu_2) r_2 & \cdots & (\gamma_p \mu_2)^{n-1} r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_n & (\gamma_p \mu_n) r_n & \cdots & (\gamma_p \mu_n)^{n-1} r_n \end{bmatrix} \in \mathcal{M}_n(\mathbb{R})$$

In particular, using Vandermonde matrices, we see that (1.10) is equivalent to

$$\forall i \in \{1, \dots, n\}, \quad r_i \neq 0, \quad \mu_i \neq \mu_j \quad \text{if } i \neq j.$$

In particular, taking  $r_i = 1$  for all  $i$ , we obtain that the system

$$\begin{cases} \partial_t y^{(i)} - \mu_i \Delta y^{(i)} + \nabla p^{(i)} = v^{(1)} 1_\omega & \text{in } (0, T) \times \Omega, \quad (1 \leq i \leq n) \\ \nabla \cdot y^{(i)} = 0 & \text{in } (0, T) \times \Omega, \quad (1 \leq i \leq n) \\ y^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, \quad (1 \leq i \leq n) \\ y^{(i)}(0, \cdot) = y_0^{(i)} & \text{in } \Omega, \quad (1 \leq i \leq n) \end{cases}$$

is null-controllable with only one control  $v^{(1)}$  (with a vanishing component) provided all the viscosities  $\mu_i$  are distinct. The above system is *simultaneously* null-controllable in any time  $T > 0$ .

**Case 3: cascade systems.** Assume

$$q_{i,j} = 0 \quad \text{if } i \geq j + 2, \quad q_{i,i-1} \neq 0 \quad (2 \leq i \leq n), \quad R = (\delta_{1,i})_{i=1, \dots, n}$$

where  $\delta_{i,j}$  is the Kronecker delta. Then a proof by induction yields that  $K_p$  is an upper triangular matrix of the form

$$K_p := \begin{bmatrix} 1 & & & & \\ 0 & q_{2,1} & & & \\ \vdots & 0 & q_{3,2} q_{2,1} & & \\ \vdots & \vdots & 0 & \ddots & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \prod_{i=2}^n q_{i,i-1} \end{bmatrix} \in \mathcal{M}_n(\mathbb{R})$$

By hypothesis, the elements of diagonal are non zero. Thus in that case, (1.3) is null-controllable and we recover a part of the result of [33].



### 3 The Kalman method

In order to prove Theorem 1.1, we extend the method proposed in [1] for the coupling of parabolic equations of heat type. In this section, we can assume general hypotheses on the system. Let us consider

$$\mathbb{A} : \mathcal{D}(\mathbb{A}) \rightarrow \mathcal{H}$$

a positive self-adjoint operator in a Hilbert space  $\mathcal{H}$  and  $\mathbb{B} \in \mathcal{L}(\mathcal{U}, \mathcal{H})$  a bounded control operator. We consider the control problem

$$\partial_t y^{(i)} + \mu_i \mathbb{A} y^{(i)} + \sum_{j=1}^n q_{i,j} y^{(j)} = \sum_{j=1}^m r_{i,j} \mathbb{B} v^{(j)}, \quad y^{(i)}(0) = y_0^{(i)}, \quad (3.1)$$

where  $\mu_i > 0$  ( $i = 1, \dots, n$ ),  $Q = (q_{i,j}) \in \mathcal{M}_n(\mathbb{R})$ ,  $R = (r_{i,j}) \in \mathcal{M}_{n,m}(\mathbb{R})$ .

Defining the operator  $\mathbb{L}$  as in (1.6) with  $D = \text{diag}(\mu_1, \dots, \mu_n)$ , the above system can be written as in (1.7) and the adjoint system can be written as in (1.12). In order to study the controllability of (3.1) or equivalently the observability of (1.8), we use the Kalman operator defined by (1.8). Let us state a result for the Kalman operator obtained in [1, Theorem 2.1]. We recall that the adjoint of  $\mathbb{K}$  is given by (1.9).

**Theorem 3.1.** *Assume (1.10). Then there exists a constant  $C > 0$  such that for any  $\varphi \in [\mathcal{D}(\mathbb{A}^{2n(n-1)})]^n$ ,*

$$\|\varphi\|_{\mathcal{H}^n} \leq C \left\| \mathbb{A}^{(2n-1)(n-1)} \mathbb{K}^* \varphi \right\|_{(\mathcal{H}^m)^n}.$$

The Kalman operator allows us to reduce the problem to a family of cascade systems that we describe now. Let us denote by  $\mathcal{S}_n$  the group of permutations of the set  $\{1, \dots, n\}$ . Then, we consider the following ‘‘cascade’’ system

$$\begin{cases} \partial_t \psi^{(i,\sigma)} + \mu_{\sigma(i)} \mathbb{A} \psi^{(i,\sigma)} = \psi^{(i+1,\sigma)} & (1 \leq i \leq n-1, \sigma \in \mathcal{S}_n) \\ \partial_t \psi^{(n,\sigma)} + \mu_{\sigma(n)} \mathbb{A} \psi^{(n,\sigma)} = \sum_{\tilde{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \beta^{p,\tilde{\sigma}} \psi^{(p,\tilde{\sigma})} & (\sigma \in \mathcal{S}_n) \\ \psi^{(i,\sigma)}(0) = \psi_0^{(i,\sigma)} & (1 \leq i \leq n, \sigma \in \mathcal{S}_n) \end{cases} \quad (3.2)$$

where  $\beta^{p,\sigma} \in \mathbb{R}$  for  $p \in \{1, \dots, n\}$  and  $\sigma \in \mathcal{S}_n$ . This system couples  $n \times n!$  equations and the state is

$$\psi^{(i,\sigma)} \quad (1 \leq i \leq n, \sigma \in \mathcal{S}_n).$$

This is a generalization of the following system considered in [33] and in [20] where  $\mathbb{A}$  is the Stokes operator or the Laplace operator:

$$\begin{cases} \partial_t \psi^{(i)} + \mu_i \mathbb{A} \psi^{(i)} = \psi^{(i+1)} & (1 \leq i \leq n-1) \\ \partial_t \psi^{(n)} + \mu_n \mathbb{A} \psi^{(n)} = \sum_{p=1}^n \beta^p \psi^{(p)}. \end{cases}$$

Using a weighted estimate for the above system in the case where  $\mathbb{A}$  is the Laplace operator, the authors in [1] manage to show the Kalman rank condition for the heat equation. In the case of the Stokes equations, we need the following weighted estimate for the system (3.2):

$$\sum_{\sigma \in \mathcal{S}_n} \sum_{k=0}^{n-1} \int_0^T \rho_{\sharp}(t) \left\| \mathbb{A}^{(2n-1)(n-1)} \partial_t^k \psi^{(1,\sigma)}(t) \right\|_{\mathcal{H}}^2 dt \leq C \sum_{\sigma \in \mathcal{S}_n} \int_0^T \rho_{\flat}(t) \left\| \mathbb{B}^* \psi^{(1,\sigma)}(t) \right\|_{\mathcal{U}}^2 dt \quad (3.3)$$

where  $\rho_{\sharp}, \rho_{\flat} : [0, T] \rightarrow \mathbb{R}_+$  are continuous functions and  $C > 0$  a constant. The above estimate can be the consequence of a Carleman estimate as in the case of coupled Stokes equations or heat equations.

**Proposition 3.2.** *Assume (1.10) and that for any  $(\beta^{p,\sigma})_{p \in \{1, \dots, n\}, \sigma \in \mathcal{S}_n}$  and for any smooth solutions of (3.2), the weighted estimate (3.3) holds. Then, there exists  $C > 0$  such that for any  $\varphi_0 \in \mathcal{H}^n$ , and for any solution  $\varphi$  of (1.12),*

$$\int_0^T \rho_{\sharp}(t) \|\varphi(t)\|_{\mathcal{H}^n}^2 dt \leq C \int_0^T \rho_{\flat}(t) \|\mathbb{B}^* R^* \varphi(t)\|_{\mathcal{U}^m}^2 dt. \quad (3.4)$$

**Remark 3.3.** If  $\rho_{\sharp}$  is bounded by below on some non empty open interval of  $(0, T)$  and if  $\rho_b$  is bounded, then (3.4) implies an observability inequality

$$\|\varphi(T)\|_{\mathcal{H}^n}^2 \leq k_T^2 \int_0^T \|R^* \mathbb{B}^* \varphi\|_{\mathcal{U}^m}^2 dt. \quad (3.5)$$

*Proof of Proposition 3.2.* From Theorem 3.1 and (1.9), there exists a constant  $C > 0$  such that for any  $\varphi \in [\mathcal{D}(\mathbb{A}^{2n(n-1)})]^n$ ,

$$\|\varphi\|_{\mathcal{H}^n}^2 \leq C \sum_{k=0}^{n-1} \left\| \mathbb{A}^{(2n-1)(n-1)} R^* (\mathbb{L}^*)^k \varphi \right\|_{\mathcal{H}^m}^2.$$

Assume  $\varphi$  is solution of (1.12) with  $\varphi_0 \in [\mathcal{D}(\mathbb{A}^{2n(n-1)})]^n$ . Then for all  $t$ ,  $\varphi(t) \in [\mathcal{D}(\mathbb{A}^{2n(n-1)})]^n$  and the above estimate becomes

$$\|\varphi\|_{\mathcal{H}^n}^2 \leq C \sum_{k=0}^{n-1} \left\| \mathbb{A}^{(2n-1)(n-1)} \partial_t^k R^* \varphi \right\|_{\mathcal{H}^m}^2 = C \sum_{k=0}^{n-1} \sum_{i=1}^m \left\| \mathbb{A}^{(2n-1)(n-1)} \partial_t^k (R^* \varphi)^{(i)} \right\|_{\mathcal{H}}^2 \quad (3.6)$$

where

$$(R^* \varphi)^{(i)} = \sum_{j=1}^n r_{j,i} \varphi^{(j)} \quad (1 \leq i \leq m). \quad (3.7)$$

Now, we show that  $(R^* \varphi)^{(i)}$  satisfies an evolution equation. More generally, let us consider  $r = (r_1, \dots, r_n) \in \mathbb{R}^n$  and let us set

$$\psi^{(1)} := \sum_{j=1}^n r_j \varphi^{(j)} \quad (3.8)$$

where  $\varphi$  is the solution of (1.12). We write

$$\mathbb{E}_i := \partial_t + \mu_i \mathbb{A}, \quad \mathbb{M} := \text{diag}(\mathbb{E}_1, \dots, \mathbb{E}_n) + Q^*,$$

so that the first equation of (1.12) can be written as

$$\mathbb{M} \varphi = 0.$$

Taking the transpose of the cofactor matrix of  $\mathbb{M}$  on the above system, we deduce that for all  $j = 1, \dots, n$ ,

$$\det(\mathbb{M}) \varphi^{(j)} = 0$$

and in particular, with (3.8),

$$\det(\mathbb{M}) \psi^{(1)} = 0. \quad (3.9)$$

Moreover, there exist  $\alpha_{i_1, \dots, i_p} \in \mathbb{R}$  (depending on  $Q$ ) such that

$$\det(\mathbb{M}) = \prod_{i=1}^n \mathbb{E}_i - \sum_{p=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_p \leq n} \alpha_{i_1, \dots, i_p} \prod_{k=1}^p \mathbb{E}_{i_k}. \quad (3.10)$$

For  $\sigma \in \mathcal{S}_n$ , we set

$$\psi^{(p, \sigma)} := \prod_{k=1}^{p-1} \mathbb{E}_{\sigma(k)} \psi^{(1)}. \quad (3.11)$$

We have in particular, for any  $\sigma \in \mathcal{S}_n$ ,

$$\psi^{(1, \sigma)} = \psi^{(1)}, \quad \mathbb{E}_{\sigma(p)} \psi^{(p, \sigma)} = \psi^{(p+1, \sigma)} \quad (1 \leq p \leq n-1), \quad (3.12)$$

and, using (3.9) and (3.10),

$$\mathbb{E}_{\sigma(n)} \psi^{(n,\sigma)} = \psi^{(n+1,\sigma)} = \prod_{i=1}^n \mathbb{E}_i \psi^{(1)} = \sum_{p=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_p \leq n} \alpha_{i_1, \dots, i_p} \prod_{k=1}^p \mathbb{E}_{i_k} \psi^{(1)}. \quad (3.13)$$

Let us fix  $p \in \{0, \dots, n-1\}$ . Then for any  $1 \leq i_1 < \dots < i_p \leq n$ , there exists a permutation  $\tilde{\sigma} \in \mathcal{S}_n$  such that

$$\tilde{\sigma}(k) = i_k \quad (k = 1, \dots, p).$$

We set  $\beta^{p+1, \tilde{\sigma}} := \alpha_{i_1, \dots, i_p}$  for the corresponding permutations and  $\beta^{p+1, \tilde{\sigma}} := 0$  for any other permutations  $\tilde{\sigma} \in \mathcal{S}_n$ . Thus

$$\sum_{p=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_p \leq n} \alpha_{i_1, \dots, i_p} \prod_{k=1}^p \mathbb{E}_{i_k} \psi^{(1)} = \sum_{p=1}^n \sum_{\tilde{\sigma} \in \mathcal{S}_n} \beta^{p, \tilde{\sigma}} \psi^{(p, \tilde{\sigma})}.$$

Then combining the above relation with (3.12) and (3.13), we deduce that for any  $\sigma \in \mathcal{S}_n$ ,  $(\psi^{(p,\sigma)})_p$  satisfies the cascade system (3.2). In particular, we can use the estimate (3.3), and we deduce that for any  $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ , the function  $\psi^{(1)}$  defined by (3.8) satisfies

$$\sum_{k=0}^{n-1} \int_0^T \rho_{\sharp}(t) \left\| \mathbb{A}^{(2n-1)(n-1)} \partial_t^k \psi^{(1)}(t) \right\|_{\mathcal{H}}^2 dt \leq C \int_0^T \rho_b(t) \left\| \mathbb{B}^* \psi^{(1)}(t) \right\|_{\mathcal{U}}^2 dt. \quad (3.14)$$

Going back to (3.6) and using (3.7), we deduce (3.4). To end the proof, we use the density of  $[\mathcal{D}(\mathbb{A}^{2n(n-1)})]^n$  into  $\mathcal{H}^n$  to obtain (3.4) in the case where  $\varphi_0 \in \mathcal{H}^n$ .  $\square$

## 4 The Carleman estimate

In order to show Theorem 1.1, we will apply the abstract result Proposition 3.2 to our system. In particular, we need to show a weighted estimate of the form (3.3) for a cascade system of the form (3.2), where  $\mathbb{A}$  and  $\mathbb{B}$  are the operators introduced in Section 1. This is done by first showing a Carleman estimate on a “generalized” cascade system of Stokes equations. A similar Carleman estimate is obtained in [33] for a “standard” cascade system of Stokes equations. Here we follow the same strategy so that we skip many details and point out the main differences.

We also focus in the case  $N = 2$ , the proof for  $N = 3$  is completely similar.

### 4.1 Statement of the Carleman estimate

First, we consider the weights needed in the Carleman estimate. We start by introducing a nonempty domain  $\omega_0$  such that  $\overline{\omega_0} \subset \omega$ . Then, using [19] (see also [35, Theorem 9.4.3, p.299]), there exists  $\eta^0 \in C^2(\overline{\Omega})$  satisfying

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \partial\Omega, \quad \max_{\Omega} \eta^0 = 1, \quad \nabla \eta^0 \neq 0 \text{ in } \overline{\Omega} \setminus \overline{\omega_0}. \quad (4.1)$$

Then, we define the following functions:

$$\alpha(t, x) = \frac{\exp\{\lambda(2\ell + 2)\} - \exp\{\lambda(2\ell + \eta^0(x))\}}{t^\ell(T-t)^\ell}, \quad \xi(t, x) = \frac{\exp\{\lambda(2\ell + \eta^0(x))\}}{t^\ell(T-t)^\ell}, \quad (4.2)$$

$$\alpha^*(t) = \max_{x \in \overline{\Omega}} \alpha(t, x) = \frac{\exp\{\lambda(2\ell + 2)\} - \exp\{2\lambda\ell\}}{t^\ell(T-t)^\ell}, \quad \xi^*(t) = \min_{x \in \overline{\Omega}} \xi(t, x) = \frac{\exp\{2\lambda\ell\}}{t^\ell(T-t)^\ell}, \quad (4.3)$$

$$\widehat{\alpha}(t) = \min_{x \in \overline{\Omega}} \alpha(t, x) = \frac{\exp\{\lambda(2\ell + 2)\} - \exp\{\lambda(2\ell + 1)\}}{t^\ell(T-t)^\ell}, \quad \widehat{\xi}(t) = \max_{x \in \overline{\Omega}} \xi(t, x) = \frac{\exp\{\lambda(2\ell + 1)\}}{t^\ell(T-t)^\ell}, \quad (4.4)$$

where  $\ell \geq 11$ ,  $\lambda > 1$ . Note that we have the following useful relations: there exists  $C > 0$  depending on  $\Omega$  such that

$$|\partial_t \alpha| + |\partial_t \xi| \leq CT \xi^{1+1/\ell}, \quad (4.5)$$

$$|(\alpha^*)'| + |(\xi^*)'| \leq CT (\xi^*)^{1+1/\ell}, \quad |(\alpha^*)''| + |(\xi^*)''| \leq CT^2 (\xi^*)^{1+2/\ell}, \quad (4.6)$$

$$\xi^* \geq \frac{C}{T^{2\ell}}, \quad (4.7)$$

$$|\nabla \alpha| = |\nabla \xi| \leq C\lambda \xi, \quad |\Delta \alpha| = |\Delta \xi| \leq C\lambda^2 \xi. \quad (4.8)$$

Such weights functions were first considered in [19] and have been already used in many articles (see, for instance, [22], [21], etc.). System (3.2) in our context can be written as follows:

$$\left\{ \begin{array}{ll} \partial_t \psi^{(i,\sigma)} - \mu_{\sigma(i)} \Delta \psi^{(i,\sigma)} + \nabla \pi^{(i,\sigma)} = \psi^{(i+1,\sigma)} & \text{in } (0, T) \times \Omega, \quad (1 \leq i \leq n-1, \sigma \in \mathcal{S}_n) \\ \partial_t \psi^{(n,\sigma)} - \mu_{\sigma(n)} \Delta \psi^{(n,\sigma)} + \nabla \pi^{(n,\sigma)} = \sum_{\tilde{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \beta^{p,\tilde{\sigma}} \psi^{(p,\tilde{\sigma})} & \text{in } (0, T) \times \Omega, \quad (\sigma \in \mathcal{S}_n) \\ \nabla \cdot \psi^{(i,\sigma)} = 0 & \text{in } (0, T) \times \Omega, \quad (1 \leq i \leq n, \sigma \in \mathcal{S}_n) \\ \psi^{(i,\sigma)} = 0 & \text{on } (0, T) \times \partial\Omega, \quad (1 \leq i \leq n, \sigma \in \mathcal{S}_n) \\ \psi^{(i,\sigma)}(0, \cdot) = \psi_0^{(i,\sigma)} & \text{in } \Omega, \quad (1 \leq i \leq n, \sigma \in \mathcal{S}_n). \end{array} \right. \quad (4.9)$$

Our aim is to estimate the following quantity associated with the solutions of the system (4.9):

$$I(s, \psi) := \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left( s^{-1} \xi^{-1} |\nabla^3 \Delta \psi_1|^2 + s\xi |\nabla^2 \Delta \psi_1|^2 + s^3 \xi^3 |\nabla \Delta \psi_1|^2 + s^5 \xi^5 |\Delta \psi_1|^2 \right) dxdt + \iint_{(0,T) \times \Omega} e^{-2s\alpha^*} (s\xi^*)^5 |\psi|^2 dxdt. \quad (4.10)$$

The first Carleman estimate for the above system can be stated as follows:

**Theorem 4.1.** *There exist  $C > 0$  depending on the geometry and  $\delta_n \in \mathbb{N}^*$  such that for any  $s \geq C (T^\ell + T^{2\ell})$  and for any  $\psi_0^{(i,\sigma)} \in \mathcal{H}$ , ( $i \in \{1, \dots, n\}$ ,  $\sigma \in \mathcal{S}_n$ ), the solution  $(\psi^{(i,\sigma)})_{i,\sigma}$  of (4.9) satisfies*

$$\sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n I(s, \psi^{(i,\sigma)}) \leq C \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \omega} e^{-2s\alpha} (s\xi)^{\delta_n} \left| \psi_1^{(1,\sigma)} \right|^2 dxdt. \quad (4.11)$$

From the above result, we can deduce a weighted estimate of the form (3.3):

**Corollary 4.2.** *Assume  $\kappa \in (0, 1)$ ,  $J, K \in \mathbb{N}$ . There exist  $C > 0$  depending on the geometry and  $\delta_n \in \mathbb{N}^*$  such that for any  $s \geq C (T^\ell + T^{2\ell})$  and for any  $\psi_0^{(i,\sigma)} \in \mathcal{H}$ , ( $i \in \{1, \dots, n\}$ ,  $\sigma \in \mathcal{S}_n$ ), the solution  $(\psi^{(i,\sigma)})_{i,\sigma}$  of (4.9) satisfies*

$$\sum_{j=0}^J \sum_{k=0}^K \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \Omega} e^{-2s\alpha^*} (s\xi^*)^{3-2/\ell} \left| \partial_t^j \Delta^k \psi^{(1,\sigma)} \right|^2 dxdt \leq C \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \omega} e^{-2\kappa s \hat{\alpha}} \left( s \hat{\xi} \right)^{\delta_n} \left| \psi_1^{(1,\sigma)} \right|^2 dx dt. \quad (4.12)$$

Combining Corollary 4.2 and applying Proposition 3.2, we deduce (1.14) which implies Theorem 1.3 in a classical way.

## 4.2 Regularity results

Let us consider the ‘‘cascade’’ system (3.2). One can write it as

$$\partial_t \psi + \mathcal{A}\psi = 0, \quad \psi(0) = \psi_0, \quad (4.13)$$

where  $\psi := (\psi^{(i,\sigma)})_{1 \leq i \leq n, \sigma \in \mathcal{S}_n}$ ,  $\psi_0 := (\psi_0^{(i,\sigma)})_{1 \leq i \leq n, \sigma \in \mathcal{S}_n}$  and where the operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{X}$  is defined as follows:

$$\begin{aligned} \mathcal{X} &:= \mathcal{H}^{\{1,\dots,n\} \times \mathcal{S}_n}, \quad \mathcal{D}(\mathcal{A}) := \mathcal{D}(\mathbb{A})^{\{1,\dots,n\} \times \mathcal{S}_n}, \\ (\mathcal{A}\psi)^{(i,\sigma)} &:= \mu_{\sigma(i)} \mathbb{A}\psi^{(i,\sigma)} - \psi^{(i+1,\sigma)} \quad (1 \leq i \leq n-1, \sigma \in \mathcal{S}_n), \\ (\mathcal{A}\psi)^{(n,\sigma)} &:= \mu_{\sigma(n)} \mathbb{A}\psi^{(n,\sigma)} - \sum_{\bar{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \beta^{p,\bar{\sigma}} \psi^{(p,\bar{\sigma})} \quad (\sigma \in \mathcal{S}_n). \end{aligned}$$

We can write  $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$  where

$$\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}), \quad (\mathcal{A}_0\psi)^{(i,\sigma)} = \mu_{\sigma(i)} \mathbb{A}\psi^{(i,\sigma)} \quad (1 \leq i \leq n, \sigma \in \mathcal{S}_n).$$

The operator  $\mathcal{A}_0$  is self-adjoint and positive (see, for instance, [32, Theorem 2.1.1, pp.128–129]) and  $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$ . Thus (see, for instance, [31, Theorem 2.1, p.80])  $-\mathcal{A}$  is the infinitesimal generator of an analytic semigroup. Applying the elliptic regularity of the Stokes system (see, for instance [34, Proposition 2.2, p.33]), we have also that

$$\mathcal{D}(\mathcal{A}^2) \subset [H^4(\Omega)^N]^{\{1,\dots,n\} \times \mathcal{S}_n}, \quad \mathcal{D}(\mathcal{A}^3) \subset [H^6(\Omega)^N]^{\{1,\dots,n\} \times \mathcal{S}_n}.$$

Following the proof of [33, Lemma 2.4], we have the following result:

**Lemma 4.3.** *Assume  $T_0 > 0$  and  $T \in (0, T_0)$ . Let us consider  $\theta_0, \theta_1, \theta_2, \theta_3 \in C^3([0, T])$  and a constant  $c > 0$  such that*

$$\theta_i(0) = 0 \quad (i \in \{0, \dots, 3\}), \quad |\theta_3''| + |\theta_2''| + |\theta_1'| \leq c|\theta_0|, \quad |\theta_3''| + |\theta_2'| \leq c|\theta_1|, \quad |\theta_3'| \leq c|\theta_2|. \quad (4.14)$$

*Then, for any  $\psi_0 \in \mathcal{H}^{\{1,\dots,n\} \times \mathcal{S}_n}$ , the solution  $\psi$  of (4.13) satisfies*

$$\theta_1\psi \in H^1(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{D}(\mathcal{A})), \quad (4.15)$$

$$\theta_2\psi \in H^2(0, T; \mathcal{H}) \cap H^1(0, T; \mathcal{D}(\mathcal{A})) \cap L^2(0, T; \mathcal{D}(\mathcal{A}^2)), \quad (4.16)$$

$$\theta_3\psi \in H^3(0, T; \mathcal{H}) \cap H^2(0, T; \mathcal{D}(\mathcal{A})) \cap H^1(0, T; \mathcal{D}(\mathcal{A}^2)) \cap L^2(0, T; \mathcal{D}(\mathcal{A}^3)), \quad (4.17)$$

with

$$\begin{aligned} &\|\theta_1\psi\|_{H^1(0,T;\mathcal{H})} + \|\theta_1\psi\|_{L^2(0,T;\mathcal{D}(\mathcal{A}))} + \|\theta_2\psi\|_{H^2(0,T;\mathcal{H})} + \|\theta_2\psi\|_{H^1(0,T;\mathcal{D}(\mathcal{A}))} + \|\theta_2\psi\|_{L^2(0,T;\mathcal{D}(\mathcal{A}^2))} \\ &+ \|\theta_3\psi\|_{H^3(0,T;\mathcal{H})} + \|\theta_3\psi\|_{H^2(0,T;\mathcal{D}(\mathcal{A}))} + \|\theta_3\psi\|_{H^1(0,T;\mathcal{D}(\mathcal{A}^2))} + \|\theta_3\psi\|_{L^2(0,T;\mathcal{D}(\mathcal{A}^3))} \leq C \|\theta_0\psi\|_{L^2(0,T;\mathcal{H})}. \end{aligned} \quad (4.18)$$

## 4.3 Proof of Theorem 4.1.

The proof of the Carleman estimate follows closely the proof of a similar result done in [33]. We recall that  $I(s, \psi)$  is defined by (4.10). First we show the following result:

**Proposition 4.4.** *Let  $\widehat{\omega} \subset \Omega$  be a nonempty open set such that  $\omega_0 \in \widehat{\omega} \Subset \omega$ . Then, there exists a constant  $C$  such that for any  $s \geq C$  and for any  $(\psi^{(i,\sigma)})_{i,\sigma}$  satisfying (4.9), we have*

$$\sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n I(s, \psi^{(i,\sigma)}) \leq C \sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n \iint_{(0,T) \times \widehat{\omega}} (s\xi)^5 e^{-2s\alpha} \left| \Delta \psi_1^{(i,\sigma)} \right|^2 dx dt. \quad (4.19)$$

*Proof.* We first take the divergence of the two first equations of (4.9) to deduce that

$$\Delta\pi^{(i,\sigma)} = 0 \quad \text{in } (0, T) \times \Omega, \quad (1 \leq i \leq n).$$

Using these relations, we apply the operator  $\nabla^2\Delta$  to the two first equations of (4.9) and the corresponding first components give

$$\begin{cases} \partial_t \nabla^2 \Delta \psi_1^{(i,\sigma)} - \mu_{\sigma(i)} \Delta \nabla^2 \Delta \psi_1^{(i,\sigma)} = \nabla^2 \Delta \psi_1^{(i+1,\sigma)} & \text{in } (0, T) \times \Omega, \quad (1 \leq i \leq n-1) \\ \partial_t \nabla^2 \Delta \psi_1^{(n,\sigma)} - \mu_{\sigma(n)} \Delta \nabla^2 \Delta \psi_1^{(n,\sigma)} = \sum_{\tilde{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \beta^{p,\tilde{\sigma}} \nabla^2 \Delta \psi_1^{(p,\tilde{\sigma})} & \text{in } (0, T) \times \Omega. \end{cases} \quad (4.20)$$

Then, we use a Carleman estimate for the heat equation proved in [15]: there exists a constant  $C > 0$  such that for any  $\lambda \geq C$ ,  $s \geq C(T^\ell + T^{2\ell})$ ,  $i \in \{1, \dots, n\}$  and  $\sigma \in \mathcal{S}_n$ ,

$$\begin{aligned} & \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left( (s\xi)^{-1} \left| \nabla^3 \Delta \psi_1^{(i,\sigma)} \right|^2 + s\xi \left| \nabla^2 \Delta \psi_1^{(i,\sigma)} \right|^2 \right) dxdt \\ & \leq C \left( \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s\xi \left| \nabla^2 \Delta \psi_1^{(i,\sigma)} \right|^2 dxdt + \iint_{(0,T) \times \partial\Omega} (s\xi^*)^{-1} e^{-2s\alpha^*} \left| \frac{\partial \nabla^2 \Delta \psi_1^{(i,\sigma)}}{\partial n} \right|^2 d\gamma dt \right. \\ & \quad \left. + \sum_{\tilde{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \iint_{(0,T) \times \Omega} e^{-2s\alpha} (s\xi)^{-2} \left| \nabla^2 \Delta \psi_1^{(p,\tilde{\sigma})} \right|^2 dxdt \right). \end{aligned} \quad (4.21)$$

Next, we follow a standard method in three steps:

- Step 1 consists in completing the above left-hand side to obtain  $I(s, \psi^{(i,\sigma)})$  (see (4.10)).
- Step 2 consists in removing the boundary terms.
- Step 3 consists in removing the local terms that do not appear in (4.19).

**Step 1.** This step is completely similar to [33]. We apply two times a Carleman estimate for the gradient operator (see [11, Lemma 3]) to recover  $\nabla \Delta \psi_1^{(i,\sigma)}$  and  $\Delta \psi_1^{(i,\sigma)}$ . Then we use the divergence free condition and the boundary conditions of  $\psi^{(i,\sigma)}$  to recover  $\psi^{(i,\sigma)}$  and to deduce

$$\begin{aligned} I(s, \psi^{(i,\sigma)}) & \leq C \left( \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \left( s\xi \left| \nabla^2 \Delta \psi_1^{(i,\sigma)} \right|^2 + (s\xi)^3 \left| \nabla \Delta \psi_1^{(i,\sigma)} \right|^2 + (s\xi)^5 \left| \Delta \psi_1^{(i,\sigma)} \right|^2 \right) dxdt \right. \\ & \quad + \iint_{(0,T) \times \partial\Omega} (s\xi^*)^{-1} e^{-2s\alpha^*} \left| \frac{\partial \nabla^2 \Delta \psi_1^{(i,\sigma)}}{\partial n} \right|^2 d\gamma dt \\ & \quad \left. + \sum_{\tilde{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \iint_{(0,T) \times \Omega} e^{-2s\alpha} (s\xi)^{-2} \left| \nabla^2 \Delta \psi_1^{(p,\tilde{\sigma})} \right|^2 dxdt \right). \end{aligned} \quad (4.22)$$

**Step 2.** We apply Lemma 4.3: using (4.6), we can check that for  $s \geq T^\ell$ , then

$$\theta_0 := (s\xi^*)^{\frac{5}{2}} e^{-s\alpha^*}, \quad \theta_1(t) := (s\xi^*)^{\frac{3}{2} - \frac{1}{\ell}} e^{-s\alpha^*}, \quad \theta_2(t) := (s\xi^*)^{\frac{1}{2} - \frac{2}{\ell}} e^{-s\alpha^*}, \quad \theta_3(t) := (s\xi^*)^{-\frac{1}{2} - \frac{3}{\ell}} e^{-s\alpha^*} \quad (4.23)$$

satisfy (4.14). We deduce that

$$\begin{aligned}
& \sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n \left( \left\| \theta_1 \psi^{(i,\sigma)} \right\|_{H^1(0,T;L^2(\Omega)^2)} + \left\| \theta_1 \psi^{(i,\sigma)} \right\|_{L^2(0,T;H^2(\Omega)^2)} + \left\| \theta_2 \psi^{(i,\sigma)} \right\|_{H^2(0,T;L^2(\Omega)^2)} \right. \\
& \quad + \left\| \theta_2 \psi^{(i,\sigma)} \right\|_{H^1(0,T;H^2(\Omega)^2)} + \left\| \theta_2 \psi^{(i,\sigma)} \right\|_{L^2(0,T;H^4(\Omega)^2)} + \left\| \theta_3 \psi^{(i,\sigma)} \right\|_{H^3(0,T;L^2(\Omega)^2)} \\
& \quad \left. + \left\| \theta_3 \psi^{(i,\sigma)} \right\|_{H^2(0,T;H^2(\Omega)^2)} + \left\| \theta_3 \psi^{(i,\sigma)} \right\|_{H^1(0,T;H^4(\Omega)^2)} + \left\| \theta_3 \psi^{(i,\sigma)} \right\|_{L^2(0,T;H^6(\Omega)^2)} \right) \\
& \leq C \sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n \left\| \theta_0 \psi^{(i,\sigma)} \right\|_{L^2(0,T;\mathcal{H})}. \quad (4.24)
\end{aligned}$$

Now, we use a trace inequality (see, for instance, [28, Theorem 2.1, p.9]), an interpolation argument (see, for instance, [27, Remark 9.5, pp.46-47]) to deduce

$$\begin{aligned}
& \left| \iint_{(0,T) \times \partial\Omega} (s\xi^*)^{-1} e^{-2s\alpha^*} \left| \frac{\partial \nabla^2 \Delta \psi_1^{(i,\sigma)}}{\partial n} \right|^2 d\gamma dt \right| \\
& \leq C \int_0^T (s\xi^*)^{-1} e^{-2s\alpha^*} \left( \left\| \psi^{(i,\sigma)} \right\|_{H^4(\Omega)^2} \left\| \psi^{(i,\sigma)} \right\|_{H^6(\Omega)^2} + \left\| \psi^{(i,\sigma)} \right\|_{H^4(\Omega)^2}^{1/2} \left\| \psi^{(i,\sigma)} \right\|_{H^6(\Omega)^2}^{3/2} \right) dt \\
& \leq C \int_0^T \left( (s\xi^*)^{-1+\frac{5}{\ell}} \left\| \theta_2 \psi^{(i,\sigma)} \right\|_{H^4(\Omega)^2} \left\| \theta_3 \psi^{(i,\sigma)} \right\|_{H^6(\Omega)^2} + (s\xi^*)^{-\frac{1}{2}+\frac{11}{2\ell}} \left\| \theta_2 \psi^{(i,\sigma)} \right\|_{H^4(\Omega)^2}^{1/2} \left\| \theta_3 \psi^{(i,\sigma)} \right\|_{H^6(\Omega)^2}^{3/2} \right) dt \quad (4.25)
\end{aligned}$$

Using that  $\ell \geq 11$  and (4.7), we have for any  $C > 0$  and  $s \geq CT^{2\ell}$ ,

$$(s\xi^*)^{-1+\frac{5}{\ell}} \leq (C4^\ell)^{-1+\frac{5}{\ell}}, \quad (s\xi^*)^{-\frac{1}{2}+\frac{11}{2\ell}} \leq (C4^\ell)^{-\frac{1}{2}+\frac{11}{2\ell}}.$$

Taking  $C > 0$  large enough in the above relations, and putting together (4.22), (4.24) and (4.25), we deduce at this step the existence of  $C > 0$  such that for  $s \geq C(T^\ell + T^{2\ell})$ ,

$$\begin{aligned}
& \sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n I(s, \psi^{(i,\sigma)}) \\
& \leq C \sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \left( s\xi \left| \nabla^2 \Delta \psi_1^{(i,\sigma)} \right|^2 + (s\xi)^3 \left| \nabla \Delta \psi_1^{(i,\sigma)} \right|^2 + (s\xi)^5 \left| \Delta \psi_1^{(i,\sigma)} \right|^2 \right) dx dt. \quad (4.26)
\end{aligned}$$

**Step 3.** This step is completely similar to [33] and we refer the reader to this article for all the details. The idea is standard and consists in considering cut-off functions in compact support in  $\widehat{\omega}$  and in several integration by parts to get rid of the terms associated with  $\nabla^2 \Delta \psi_1^{(i,\sigma)}$  and  $\nabla \Delta \psi_1^{(i,\sigma)}$  in the right-hand side of (4.26). We only keep the local integral associated with  $\Delta \psi_1^{(i,\sigma)}$ , with  $\omega_0$  replaced by  $\widehat{\omega}$ . This concludes the proof of Proposition 4.4.  $\square$

Using Proposition 4.4, we can now prove Theorem 4.1.

*Proof of Theorem 4.1.* We start with (4.19) and we now remove in the right-hand side the local terms associated with  $i \geq 2$ . The proof is again very close to the proof done in [33]. First, we introduce a sequence of open sets  $\mathcal{O}_i$ , ( $0 \leq i \leq n$ ) such that

$$\widehat{\omega} =: \mathcal{O}_n \Subset \mathcal{O}_{n-1} \Subset \dots \Subset \mathcal{O}_i \Subset \dots \Subset \mathcal{O}_0 \Subset \omega$$

and functions

$$\zeta_i \in C_c^2(\mathcal{O}_{i-1}) \quad \text{such that} \quad \zeta_i \equiv 1 \text{ in } \mathcal{O}_i, \quad \zeta_i \geq 0 \quad (1 \leq i \leq n).$$

We apply  $\Delta$  on (4.9) to deduce

$$\begin{cases} \partial_t \Delta \psi_1^{(i,\sigma)} - \mu_{\sigma(i)} \Delta^2 \psi_1^{(i,\sigma)} = \Delta \psi_1^{(i+1,\sigma)} & \text{in } (0, T) \times \Omega, \quad (1 \leq i \leq n-1, \sigma \in \mathcal{S}_n) \\ \partial_t \Delta \psi_1^{(n,\sigma)} - \mu_{\sigma(n)} \Delta^2 \psi_1^{(n,\sigma)} = \sum_{\tilde{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \beta^{p,\tilde{\sigma}} \Delta \psi_1^{(p,\tilde{\sigma})} & \text{in } (0, T) \times \Omega, \quad (\sigma \in \mathcal{S}_n). \end{cases} \quad (4.27)$$

Using the above relations, we obtain, for any  $\sigma \in \mathcal{S}_n$ ,

$$\begin{aligned} \iint_{(0,T) \times \widehat{\omega}} e^{-2s\alpha} (s\xi)^5 \left| \Delta \psi_1^{(n,\sigma)} \right|^2 dx dt &\leq \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} \zeta_n (s\xi)^5 \left| \Delta \psi_1^{(n,\sigma)} \right|^2 dx dt \\ &= \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} \zeta_n (s\xi)^5 \left( \Delta \psi_1^{(n,\sigma)} \right) \left( \partial_t \Delta \psi_1^{(n-1,\sigma)} - \mu_{\sigma(n-1)} \Delta^2 \psi_1^{(n-1,\sigma)} \right) dx dt. \end{aligned} \quad (4.28)$$

Let us now estimate the last integral in the above relation. By integrating by parts and by using (4.27), we deduce

$$\begin{aligned} \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} \zeta_n (s\xi)^5 \left( \Delta \psi_1^{(n,\sigma)} \right) \left( \partial_t \Delta \psi_1^{(n-1,\sigma)} \right) dx dt \\ = - \iint_{(0,T) \times \mathcal{O}_{n-1}} \zeta_n \partial_t \left( e^{-2s\alpha} (s\xi)^5 \right) \Delta \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt \\ - \iint_{(0,T) \times \mathcal{O}_{n-1}} \zeta_n e^{-2s\alpha} (s\xi)^5 \left( \mu_{\sigma(n)} \Delta^2 \psi_1^{(n,\sigma)} + \sum_{\tilde{\sigma} \in \mathcal{S}_n} \sum_{p=1}^n \beta^{p,\tilde{\sigma}} \Delta \psi_1^{(p,\tilde{\sigma})} \right) \left( \Delta \psi_1^{(n-1,\sigma)} \right) dx dt. \end{aligned} \quad (4.29)$$

By integrating by parts, we also find

$$\begin{aligned} \iint_{(0,T) \times \mathcal{O}_{n-1}} \zeta_n e^{-2s\alpha} (s\xi)^5 \Delta \psi_1^{(n,\sigma)} \Delta^2 \psi_1^{(n-1,\sigma)} dx dt &= \iint_{(0,T) \times \mathcal{O}_{n-1}} \zeta_n e^{-2s\alpha} (s\xi)^5 \Delta^2 \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt \\ &+ \iint_{(0,T) \times \mathcal{O}_{n-1}} \Delta \left( \zeta_n e^{-2s\alpha} (s\xi)^5 \right) \Delta \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt \\ &+ 2 \iint_{(0,T) \times \mathcal{O}_{n-1}} \nabla \left( \zeta_n e^{-2s\alpha} (s\xi)^5 \right) \cdot \nabla \Delta \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt. \end{aligned} \quad (4.30)$$

Now we estimate the right-hand side of (4.29) and (4.30). Using (4.5), (4.8), we have that for any  $s \geq C(T^\ell + T^{2\ell})$ ,

$$\left| \partial_t \left( e^{-2s\alpha} s^5 \xi^5 \right) \right| \leq C e^{-2s\alpha} (s\xi)^{6+1/\ell}, \quad \left| \Delta \left( \zeta_n e^{-2s\alpha} s^5 \xi^5 \right) \right| \leq C e^{-2s\alpha} s^7 \xi^7, \quad \left| \nabla \left( \zeta_n e^{-2s\alpha} s^5 \xi^5 \right) \right| \leq C e^{-2s\alpha} s^6 \xi^6.$$

Thus,

$$\begin{aligned} \left| \iint_{(0,T) \times \mathcal{O}_{n-1}} \zeta_n \partial_t \left( e^{-2s\alpha} (s\xi)^5 \right) \Delta \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt \right| \\ \leq \varepsilon \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^5 \xi^5 \left| \Delta \psi_1^{(n,\sigma)} \right|^2 dx dt + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} (s\xi)^{7+2/\ell} \left| \Delta \psi_1^{(n-1,\sigma)} \right|^2 dx dt, \end{aligned} \quad (4.31)$$

$$\begin{aligned} \left| \iint_{(0,T) \times \mathcal{O}_{n-1}} \Delta \left( \zeta_n e^{-2s\alpha} (s\xi)^5 \right) \Delta \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt \right| \\ \leq \varepsilon \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^5 \xi^5 \left| \Delta \psi_1^{(n,\sigma)} \right|^2 dx dt + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} s^9 \xi^9 \left| \Delta \psi_1^{(n-1,\sigma)} \right|^2 dx dt, \end{aligned} \quad (4.32)$$



$$\begin{aligned}
& \left| \iint_{(0,T) \times \mathcal{O}_{n-1}} \nabla \left( \zeta_n e^{-2s\alpha} (s\xi)^5 \right) \cdot \nabla \Delta \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt \right| \\
& \leq \varepsilon \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^3 \xi^3 \left| \nabla \Delta \psi_1^{(n,\sigma)} \right|^2 dx dt + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} s^9 \xi^9 \left| \Delta \psi_1^{(n-1,\sigma)} \right|^2 dx dt, \quad (4.33)
\end{aligned}$$

$$\begin{aligned}
& \left| \iint_{(0,T) \times \mathcal{O}_{n-1}} \zeta_n e^{-2s\alpha} (s\xi)^5 \Delta^2 \psi_1^{(n,\sigma)} \Delta \psi_1^{(n-1,\sigma)} dx dt \right| \\
& \leq \varepsilon \iint_{(0,T) \times \Omega} e^{-2s\alpha} s \xi \left| \Delta^2 \psi_1^{(n,\sigma)} \right|^2 dx dt + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} s^9 \xi^9 \left| \Delta \psi_1^{(n-1,\sigma)} \right|^2 dx dt, \quad (4.34)
\end{aligned}$$

and

$$\begin{aligned}
& \left| \iint_{(0,T) \times \mathcal{O}_{n-1}} \zeta_n e^{-2s\alpha} (s\xi)^5 \Delta \psi_1^{(p,\bar{\sigma})} \left( \Delta \psi_1^{(n-1,\sigma)} \right) dx dt \right| \\
& \leq \varepsilon \iint_{(0,T) \times \Omega} e^{-2s\alpha} (s\xi)^5 \left| \Delta \psi_1^{(p,\bar{\sigma})} \right|^2 dx dt + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} (s\xi)^5 \left| \Delta \psi_1^{(n-1,\sigma)} \right|^2 dx dt. \quad (4.35)
\end{aligned}$$

Combining (4.28)–(4.30) with (4.31)–(4.35) yields the existence of a constant  $C$  such that for any  $s \geq C$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned}
\iint_{(0,T) \times \widehat{\omega}} e^{-2s\alpha} (s\xi)^5 \left| \Delta \psi_1^{(n,\sigma)} \right|^2 dx dt & \leq \varepsilon \sum_{\bar{\sigma} \in \mathcal{S}_n} \sum_{i=1}^n I(s, \psi^{(i,\bar{\sigma})}) \\
& \quad + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_{n-1}} e^{-2s\alpha} (s\xi)^9 \left| \Delta \psi_1^{(n-1,\sigma)} \right|^2 dx dt. \quad (4.36)
\end{aligned}$$

With a similar proof, we can show by induction that for all  $k \in \{2, \dots, n\}$ ,

$$\begin{aligned}
\iint_{(0,T) \times \mathcal{O}_k} e^{-2s\alpha} (s\xi)^{2^{n-k+2}+1} \left| \Delta \psi_1^{(k,\sigma)} \right|^2 dx dt & \leq \varepsilon \sum_{\bar{\sigma} \in \mathcal{S}_n} \sum_{i=1}^n I(s, \psi^{(i,\bar{\sigma})}) \\
& \quad + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_{k-1}} e^{-2s\alpha} (s\xi)^{2^{n-k+3}+1} \left| \Delta \psi_1^{(k-1,\sigma)} \right|^2 dx dt. \quad (4.37)
\end{aligned}$$

Combining this with (4.19), we deduce

$$\sum_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n I(s, \psi^{(i,\sigma)}) \leq C \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \mathcal{O}_1} (s\xi)^{2^{n+1}+1} e^{-2s\alpha} \left| \Delta \psi_1^{(1,\sigma)} \right|^2 dx dt. \quad (4.38)$$

To end the proof, we transform the above local terms into local terms of  $\psi_1^{(1,\sigma)}$ . This is standard: by integrating by parts, we obtain

$$\begin{aligned}
\iint_{(0,T) \times \mathcal{O}_1} (s\xi)^{2^{n+1}+1} e^{-2s\alpha} \left| \Delta \psi_1^{(1,\sigma)} \right|^2 dx dt & \leq \iint_{(0,T) \times \mathcal{O}_0} \zeta_1 (s\xi)^{2^{n+1}+1} e^{-2s\alpha} \left| \Delta \psi_1^{(1,\sigma)} \right|^2 dx dt \\
& = \iint_{(0,T) \times \mathcal{O}_0} \Delta \left( \zeta_1 (s\xi)^{2^{n+1}+1} e^{-2s\alpha} \right) \left( \Delta \psi_1^{(1,\sigma)} \right) \psi_1^{(1,\sigma)} dx dt \\
& \quad + \iint_{(0,T) \times \mathcal{O}_0} 2 \nabla \left( \zeta_1 (s\xi)^{2^{n+1}+1} e^{-2s\alpha} \right) \cdot \left( \nabla \Delta \psi_1^{(1,\sigma)} \right) \psi_1^{(1,\sigma)} dx dt \\
& \quad \quad \quad + \iint_{(0,T) \times \mathcal{O}_0} \zeta_1 (s\xi)^{2^{n+1}+1} e^{-2s\alpha} \left( \Delta^2 \psi_1^{(1,\sigma)} \right) \psi_1^{(1,\sigma)} dx dt
\end{aligned}$$

Considering (4.8) and using Young's inequality, we deduce the existence of  $C$  such that for any  $s \geq C(T^\ell + T^{2\ell})$ , and for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \iint_{(0,T) \times \mathcal{O}_1} (s\xi)^{2^{n+1}+1} e^{-2s\alpha} \left| \Delta \psi_1^{(1,\sigma)} \right|^2 dx dt \\ & \leq \varepsilon \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left( (s\xi)^5 \left| \Delta \psi_1^{(1,\sigma)} \right|^2 + (s\xi)^3 \left| \nabla \Delta \psi_1^{(1,\sigma)} \right|^2 + s\xi \left| \Delta^2 \psi_1^{(1,\sigma)} \right|^2 \right) dx dt \\ & \quad + \frac{C}{\varepsilon} \iint_{(0,T) \times \mathcal{O}_0} (s\xi)^{2^{n+2}+1} e^{-2s\alpha} \left| \psi_1^{(1,\sigma)} \right|^2 dx dt. \end{aligned}$$

Combining the above estimate and (4.38) implies (4.11) with  $\delta_n = 2^{n+2} + 1$ . This concludes the proof of Theorem 4.1.  $\square$

#### 4.4 Proof of Corollary 4.2

Using Theorem 4.1, we can now show Corollary 4.2:

*Proof of Corollary 4.2.* We combine (4.24) (with  $\theta_1$  defined by (4.23)) and (4.11):

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \Omega} e^{-2s\alpha^*} (s\xi^*)^{3-2/\ell} \left( \left| \partial_t \psi^{(1,\sigma)} \right|^2 + \left| \mathbb{A} \psi^{(1,\sigma)} \right|^2 \right) dx dt \\ & \leq C \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \omega} e^{-2s\hat{\alpha}} \left( s\hat{\xi} \right)^{\delta_n} \left| \psi_1^{(1,\sigma)} \right|^2 dx dt. \end{aligned} \quad (4.39)$$

Then writing (4.9) under the form (3.2) (using (1.5)), we notice that  $(\partial_t \psi^{(i,\sigma)})_{i,\sigma}$  and  $(\mathbb{A} \psi^{(i,\sigma)})_{i,\sigma}$  satisfy (4.9) (with different initial conditions). The corresponding relations (4.39) yield the following relation:

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \Omega} e^{-2s\alpha^*} (s\xi^*)^{3-2/\ell} \left( \left| \partial_t^2 \psi^{(1,\sigma)} \right|^2 + \left| \partial_t \mathbb{A} \psi^{(1,\sigma)} \right|^2 + \left| \mathbb{A}^2 \psi^{(1,\sigma)} \right|^2 \right) dx dt \\ & \leq C \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \omega} e^{-2s\hat{\alpha}} \left( s\hat{\xi} \right)^{\delta_n} \left( \left| \mathbb{A} \psi_1^{(1,\sigma)} \right|^2 + \left| \partial_t \psi_1^{(1,\sigma)} \right|^2 \right) dx dt. \end{aligned} \quad (4.40)$$

Now, for any  $\kappa \in (0, 1)$ , there exist  $\lambda_0 > 0$  and  $s_0 > 0$  such that for any  $\lambda \geq \lambda_0$  and for any  $s \geq s_0$ ,

$$e^{-2s\hat{\alpha}} \left( s\hat{\xi} \right)^{\delta_n} \leq C e^{-2\kappa s\alpha^*} (s\xi^*)^{3-2/\ell} \quad (4.41)$$

and we deduce from the above estimate, (4.39) and (4.40) that

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \Omega} e^{-2s\alpha^*} (s\xi^*)^{3-2/\ell} \left( \left| \partial_t^2 \psi^{(1,\sigma)} \right|^2 + \left| \partial_t \mathbb{A} \psi^{(1,\sigma)} \right|^2 + \left| \mathbb{A}^2 \psi^{(1,\sigma)} \right|^2 \right) dx dt \\ & \leq C \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \omega} e^{-2\kappa s\hat{\alpha}} \left( \kappa s\hat{\xi} \right)^{\delta_n} \left| \psi_1^{(1,\sigma)} \right|^2 dx dt. \end{aligned} \quad (4.42)$$

By induction, we obtain that for all  $J \in \mathbb{N}$ , there exists  $C > 0$  such that

$$\begin{aligned} & \sum_{j=0}^J \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \Omega} e^{-2s\alpha^*} (s\xi^*)^{3-2/\ell} \left| \partial_t^j \mathbb{A}^{J-j} \psi^{(1,\sigma)} \right|^2 dx dt \\ & \leq C \sum_{\sigma \in \mathcal{S}_n} \iint_{(0,T) \times \omega} e^{-2\kappa^{J-1} s\hat{\alpha}} \left( s\hat{\xi} \right)^{\delta_n} \left| \psi_1^{(1,\sigma)} \right|^2 dx dt. \end{aligned} \quad (4.43)$$

$\square$

## Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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