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Optimal guaranteed estimation methods for the Cox - Ingersoll - Ross models.

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Abstract

In this paper we study parameter estimation problems for the Cox - Ingersoll - Ross (CIR) processes. For the first time for such models sequential estimation procedures are proposed. In the non-asymptotic setting, the proposed sequential procedures provide the estimation with non-asymptotic fixed mean square accuracy. For the scalar parameter estimation problems non-asymptotic normality properties for the proposed estimators are established even in the cases when the classical non sequential maximum likelihood estimators can not be calculated. Moreover, the Laplace transformations for the mean observation durations are obtained. In the asymptotic setting, the limit forms for the mean observation durations are founded and it is shown, that the constructed sequential estimators uniformly converge in distribution to normal random variables. Then using the Local Asymptotic Normality (LAN) property it is obtained asymptotic sharp lower bound for the minimax risks in the class of all sequential procedures with the same mean observation duration and as consequence, it is established, that the proposed sequential procedures are optimal in the minimax sens in this class.

Keywords: Cox-Ingersoll-Ross processes; Sequential estimation; Parameter estimation; Minimax estimation.
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1. Introduction

1.1. Motivations

In this paper we consider parameter estimation problems for the Cox-Ingersoll-Ross (CIR) processes defined through the following stochastic differential equation

$$dX_t = (a - bX_t)dt + \sqrt{\sigma X_t}dW_t, \quad X_0 = x > 0, \quad (1.1)$$

where $a > 0$, $b \in \mathbb{R}$ and $\sigma > 0$ are fixed parameters and $(W_t)_{t \geq 0}$ is a standard Brownian motion. In all these cases we assume that the diffusion parameter σ is known. Indeed, it can be founded explicitly through the quadratic characteristic $\langle X \rangle_t = \sigma \int_0^t X_s ds$. In this paper, we consider the estimation problems for the parameters a and b on the basis of the observations $(X_t)_{t \geq 0}$. This model was introduced in [6] as a modification of the well-known the Vasicek model to describe the dynamics of the interest rates for the bond markets (see also in [24] and the references therein). Moreover, in [10] such processes were used to introduce some class of the stochastic volatility financial stock markets, for which later optimal consumption - investment and hedging methods were developed (see, for example, in [4, 28]). In addition, recently, as it turned out in [31] discrete versions of such models play a key role in the statistical epidemic analysis. Therefore, for their implementation in practice, one needs to estimate the unknown parameters with a good admissible accuracy in the non-asymptotic setting. As to the asymptotic setting, recently in [1, 2] the maximum likelihood estimators are studied for this model in the ergodic and non-ergodic cases. Unfortunately, for the model of type (1.1) it is not possible to resolve the estimation problem in the framework of the classical non-asymptotic estimation approach since the usual maximum likelihood estimator is a nonlinear function of observations of complex nonlinear form. As is shown in [5] to keep the optimality properties for the financial strategies in the presence of unknown parameters in the market models there is only one way to use sequential analysis methods. It should be noted, that one of the main properties of sequential estimators is that they provide non-asymptotic fixed accuracy, in contrast to classical maximum likelihood estimators. It is this property that plays a key role in the analysis of the optimality of strategies in stochastic volatility financial markets since it allows one to analyse the error in the deviation of objective functions from their optimal values when unknown parameters are replaced with their sequential estimators in strategies. Firstly, estimators having the non-asymptotic fixed accuracy were proposed for the scalar parameter estimation problems in [26, 29] for the model in continuous time and in [3] for the processes in discrete time in the non-asymptotic setting and in [23] in the asymptotic one. Later, such sequential procedures were called *guaranteed* (see, for example, in [18] and the references therein). For the multidimensional parameter, the sequential fixed accuracy estimation methods were developed in [13, 14, 19] for the observations in discrete and continuous time respectively. Moreover, as is shown in [21] only sequential procedure can provide guaranteed estimation property in the non-asymptotic setting for statistical model with dependent observations. So, the main goal of this paper is to develop guaranteed estimation methods for the coefficients a and b on the basis of the observations $(X_t)_{t \geq 0}$ of the process (1.1).

1.2. Basic tools

To estimate the parameters of the process (1.1) we develop fixed accuracy estimation methods through the sequential analysis approach firstly proposed in [26, p. 244] and [29] for the scalar Ornstein - Uhlenbeck process. The main idea is to replace the observation duration in the maximum likelihood estimation (MLE) with a special stopping time to transform the random denominator in MLE into a non-random constant. It turned out that the estimate thus obtained possesses very nice non-asymptotic properties, namely it is unbiased and is Gaussian with a known variance which presents mean square estimation accuracy. More precisely, for the parameter estimation problem for the Ornstein - Uhlenbeck process

$$dX_t = \theta X_t dt + dW_t, \quad (1.2)$$

the Maximum Likelihood Estimator (MLE) for the parameter θ defined as

$$\widehat{\theta}_T = \frac{\int_0^T X_s dX_s}{\int_0^T X_s^2 ds},$$

is non-linear function of the observations $(X_t)_{0 \leq t \leq T}$ and it is not possible to study its properties in the non-asymptotic setting, i.e. for any fixed $T > 0$. In the sequential analysis setting we can transform the random denominator into non-random fixed threshold replacing the observation duration T with the stopping time defined as

$$\tau_H^* = \inf \left\{ t \geq 0 : \int_0^t X_s^2 ds \geq H \right\}, \quad (1.3)$$

where $H > 0$ is some fixed non random arbitrary constant. Using the properties of the stochastic integrals one can check directly, that $\widehat{\theta}_{\tau_H^*}$ is $\mathcal{N}(\theta, H^{-1})$ Gaussian random variable for any $H > 0$. This means, that this sequential estimator guarantees any mean square non-asymptotic accuracy by choosing the level $H > 0$. It should be noted, that this simple way can be used only for the scalar parameter estimation problem. In the case when one has to estimate a multidimensional parameter, as in the process (1.1) with the unknown parameters a and b , one should use the two-step estimation method developed in [8, 13, 14, 16] for general stochastic processes in discrete and continuous time. Later, for the proposed sequential estimation methods asymptotic properties, as $H \rightarrow \infty$, were studied (see, for example, in [15–17, 30]). Moreover, truncated versions of developed sequential estimators were proposed in [19, 20]. Unfortunately, there are still no results on a guaranteed estimation for stochastic differential equations with degenerate diffusion coefficients.

1.3. Main contributions

In this paper for the first time, the sequential guaranteed methods were developed for the models (1.1) with the degenerated diffusion coefficient. It turned out, that the scalar sequential estimators for the parameters a and b are unbiased and have a non-asymptotic Gaussian distribution even in the case when the classical non-sequential maximal likelihood estimators do not exist. For the vector estimation problem the constructed sequential estimators have a guaranteed non-asymptotic mean square estimation accuracy as well. Furthermore, in asymptotic setting through the LAN property and lower bound methods for quadratic risks it is shown, that the proposed sequential estimators are optimal in the minimax sense. It should be emphasized that also for the first time sharp lower bounds for stochastic differential equations are obtained in the class of all sequential procedures with an arbitrary stopping time not exceeding in mean a given fixed value tending to infinity. As a consequence, the optimality properties for guaranteed estimates are also established in this class. This is an essential new property since usually optimality properties for guaranteed sequential procedures for continuous time statistical models are obtained only in the class of sequential procedures having the same rule determining the observation duration. For example, for the model (1.2) in [26] it is shown that the sequential maximum likelihood estimator $\widehat{\theta}_{\tau_H^*}$ is optimal only in the class of sequential estimators in which the observation duration defined by the stopping time (1.3).

1.4. Organisation of the paper

In Section 2 we develop the scalar sequential estimation methods for the model (1.1). In Section 3 we develop the two step sequential estimation method for the vector $\theta = (a, b)^\top$ in the model (1.1). In Section 5 we present the results on LAN properties for the process (1.1). In Section 6 we find conditions on the parameters of the process (1.1) which provide the optimality properties in minimax sense for the proposed sequential procedures. Appendix contains all the auxiliary technical results used in the proofs.

2. Scalar sequential procedures

First we consider the the estimation problem for the parameter b in the process (1.1) in the case, when a is known, i.e. $\theta = b$. Note, that in this case the Maximum Likelihood Estimator (MLE) (see, for example, in [1]) is defined as

$$\widehat{\theta}_T = \frac{aT - X_T + x}{\int_0^T X_s ds}. \quad (2.1)$$

In this case this estimator is non a liner function of observations and, therefore, it is not possible to study its non asymptotic properties, i.e. for fixed duration of the observation $0 < T < \infty$. To overcome this difficulty we use the

approach based on the sequential analysis methods and proposed in [29] for the scalar diffusion process. To this end, we define the sequential procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ as

$$\tau_H^* = \inf \left\{ t : \int_0^t X_s ds \geq H \right\} \quad \text{and} \quad \theta_H^* = \frac{a\tau_H^* - X_{\tau_H^*} + x}{H}, \quad (2.2)$$

where $H > 0$ is a fixed threshold, which will be clarified later.

Theorem 1. For any $a > 0$, $\theta \in \mathbb{R}$ and any $H > 0$ the sequential procedure (2.2) possesses the following properties:

- 1) $\mathbf{P}_\theta(\tau_H^* < \infty) = 1$;
- 2) the sequential estimator θ_H^* is normally distributed with parameters

$$\mathbf{E}_\theta \theta_H^* = \theta \quad \text{and} \quad \mathbf{E}_\theta (\theta_H^* - \theta)^2 = \frac{\sigma}{H}, \quad (2.3)$$

where \mathbf{E}_θ is the expectation with respect to the distribution \mathbf{P}_θ of the process (1.1) with the parameters $\theta = (a, b)^\top$.

Proof. First note, that Proposition 4 implies the first property, i.e. for any $H > 0$ the stopping time $\tau_H^* < \infty$ \mathbf{P}_θ - a.s. Moreover, from (1.1) we obtain, that

$$\begin{aligned} \theta_H^* &= \frac{1}{H} \left(a\tau_H^* - \int_0^{\tau_H^*} (a - \theta X_s) ds - \int_0^{\tau_H^*} \sqrt{\sigma X_s} dW_s \right) \\ &= \theta - \frac{\sqrt{\sigma}}{H} \int_0^{\tau_H^*} \sqrt{X_s} dW_s. \end{aligned}$$

Then, in view of Lemma 17.4 from [26], the normalized stochastic integral $H^{-1/2} \int_0^{\tau_H^*} \sqrt{X_s} dW_s$ is a $\mathcal{N}(0, 1)$ Gaussian random variable. \square

Now using Lemma 3, we can calculate the Laplace transformation for the mean observation time of the sequential procedure δ_H^* .

Proposition 1. For any $\mu > 0$ the Laplace transformation of the expected stopping time $\mathbf{m}(H) = \mathbf{E}_\theta \tau_H^*$ defined in (2.2) is represented as

$$\widehat{\mathbf{m}}(\mu) = \int_0^{+\infty} e^{-\mu z} \mathbf{m}(z) dz = \frac{1}{\mu} \int_0^{+\infty} \Lambda(t, \mu) dt, \quad (2.4)$$

where the functions $\Lambda(t, \mu)$ are given in (A.2).

Proof. First of all, note, that for any $z > 0$

$$\mathbf{m}(z) = \mathbf{E}_\theta \tau_z^* = \int_0^{+\infty} \mathbf{P}_\theta(\tau_z^* > t) dt = \int_0^{+\infty} \mathbf{P}_\theta \left(\int_0^t X_u du < z \right) dt. \quad (2.5)$$

Moreover, note that

$$\Lambda(t, \mu) = \mathbf{E}_\theta e^{-\mu \int_0^t X_s ds} = \int_0^1 \mathbf{P}_\theta \left(e^{-\mu \int_0^t X_s ds} > y \right) dy = \mu \int_0^{+\infty} e^{-\mu z} \mathbf{P}_\theta \left(\int_0^t X_s ds < z \right) dz$$

and from (2.5) it follows that

$$\int_0^{+\infty} \mathbf{E}_\theta e^{-\mu \int_0^t X_s ds} dt = \mu \int_0^{+\infty} e^{-\mu z} \mathbf{m}(z) dz.$$

This completes the proof. \square

Now we study the asymptotic properties, as $H \rightarrow \infty$, of the observation time τ_H^* .

Theorem 2. For any $a > \sigma/2$, any compact set $\Theta \subset]0, +\infty[$ and any $r > 0$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left| \frac{\tau_H^*}{H} - \frac{1}{I_0(\theta)} \right|^r = 0, \quad (2.6)$$

where $I_0(\theta) = a/\theta$.

Proof. First, setting $\mathbf{D}_T = \int_0^T (X_s - I_0(\theta)) ds$, we obtain from (1.1), that for any $T > 0$,

$$\mathbf{D}_T = \frac{x - X_T}{\theta} + \frac{\sqrt{\sigma}}{\theta} \int_0^T \sqrt{X_s} dW_s. \quad (2.7)$$

Now, by [25, Lemma 4.12] and the upper bound (A.6), we have for any $m > 1$ and $\theta \in \Theta$,

$$\begin{aligned} \mathbf{E}_\theta \mathbf{D}_T^{2m} &\leq c_m \left(\frac{x^{2m} + \mathbf{E}_\theta X_T^{2m}}{\theta^{2m}} + \left(\frac{\sqrt{\sigma}}{\theta} \right)^{2m} \mathbf{E}_\theta \left(\int_0^T \sqrt{X_s} dW_s \right)^{2m} \right) \\ &\leq c_m (1 + T^m), \end{aligned} \quad (2.8)$$

where c_m is some positive constant not depending on θ and T that can change from line to line. Now, for any $T > I_0^{-1}(\theta)H$, we have

$$\begin{aligned} \mathbf{P}_\theta(\tau_H^* > T) &= \mathbf{P}_\theta \left(\int_0^T X_s ds < H \right) = \mathbf{P}_\theta(I_0(\theta)T + \mathbf{D}_T < H) \\ &\leq \mathbf{P}_\theta(|\mathbf{D}_T| > I_0(\theta)T - H) \leq \frac{\mathbf{E}|\mathbf{D}_T|^{2m}}{(I_0(\theta)T - H)^{2m}} \leq \frac{c_m(1 + T^m)}{(T - I_0^{-1}(\theta)H)^{2m}} \end{aligned} \quad (2.9)$$

and for any $T < I_0^{-1}(\theta)H$,

$$\begin{aligned} \mathbf{P}_\theta(\tau_H^* < T) &= \mathbf{P}_\theta \left(\int_0^T X_s ds > H \right) = \mathbf{P}_\theta(I_0(\theta)T + \mathbf{D}_T > H) \\ &\leq \mathbf{P}_\theta(|\mathbf{D}_T| > H - I_0(\theta)T) \leq \frac{\mathbf{E}_\theta |\mathbf{D}_T|^{2m}}{(H - I_0(\theta)T)^{2m}} \leq \frac{c_m(1 + T^m)}{(I_0^{-1}(\theta)H - T)^{2m}}. \end{aligned}$$

Then, for any $0 < \varepsilon < \inf_{\theta \in \Theta} I_0^{-1}(\theta)$ setting $T_1 = (I_0^{-1}(\theta) + \varepsilon)H$ and $T_2 = (I_0^{-1}(\theta) - \varepsilon)H$ we obtain, that for $\theta \in \Theta$

$$\mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - I_0^{-1}(\theta) \right| > \varepsilon \right) = \mathbf{P}_\theta(\tau_H^* > T_1) + \mathbf{P}_\theta(\tau_H^* < T_2) \leq c_m \frac{(1 + H^m)}{(\varepsilon H)^{2m}},$$

i.e. for any $\varepsilon > 0$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - I_0^{-1}(\theta) \right| > \varepsilon \right) = 0.$$

Now, for any $H > 1$, $r > 0$, using the upper bound (2.9) with $m > 2r$, we can obtain, that

$$\begin{aligned} \mathbf{E}_\theta(\tau_H^*)^r &= r \int_0^\infty t^{r-1} \mathbf{P}_\theta(\tau_H^* > t) dt \leq r \int_0^{I_0^{-1}(\theta)H + H^{3/4}} t^{r-1} dt \\ &\quad + c_m \int_{I_0^{-1}(\theta)H + H^{3/4}}^\infty \frac{t^{m+r-1}}{(t - I_0^{-1}(\theta)H)^{2m}} dt = (I_0^{-1}(\theta)H + H^{3/4})^r + c_m J_H, \end{aligned}$$

where for some positive constant c_m

$$\begin{aligned} J_H &= \int_{H^{3/4}}^{\infty} \frac{(s + I_0^{-1}(\theta)H)^{m+r-1}}{s^{2m}} ds \leq c_m \left(\int_{H^{3/4}}^{\infty} \frac{1}{s^{m-r+1}} ds + H^{m+r-1} \int_{H^{3/4}}^{\infty} \frac{1}{s^{2m}} ds \right) \\ &\leq c_m \left(\frac{1}{H^{3(m-r)/4}} + \frac{1}{H^{m/2-r+1/4}} \right) \rightarrow 0 \quad \text{as } H \rightarrow \infty. \end{aligned}$$

From this, we can conclude, that for any $r > 0$

$$\sup_{H>1} \sup_{\theta \in \Theta} \mathbf{E}_{\theta} \left(\frac{\tau_H^*}{H} \right)^r < \infty,$$

i.e. for any $r > 0$ the family $((\tau_H^*/H)^r)_{H>1}$ is uniformly integrable and we obtain (2.6), hence Theorem 2. \square
It should be noted, that Theorem 1 and Theorem 2 immediately imply the following asymptotic result.

Corollary 1. For any $a > \sigma/2$ and any compact set $\Theta \subset]0, +\infty[$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_{\theta} \left(\Upsilon_H(\theta)(\theta_H^* - \theta) \right)^2 = 1, \quad (2.10)$$

where $\Upsilon_H(\theta) = \sqrt{I(\theta)\mathbf{E}_{\theta}\tau_H^*}$ and $I(\theta) = \sigma^{-1}I_0(\theta)$.

Now we consider the estimation problem for the parameter a in (1.1) when the coefficient b is known, i.e. $\theta = a$. In this case the Maximum Likelihood estimator is given as

$$\widehat{\theta}_T = \frac{bT + \int_0^T X_t^{-1} dX_t}{\int_0^T X_t^{-1} dt}. \quad (2.11)$$

using the same way as in (2.2) we defined the sequential estimation procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ with $H > 0$ for the parameter θ as

$$\tau_H^* = \inf \left(t : \int_0^t X_s^{-1} ds \geq H \right) \quad \text{and} \quad \theta_H^* = \frac{b\tau_H^* + \int_0^{\tau_H^*} X_s^{-1} dX_s}{H}. \quad (2.12)$$

Firs we study non asymptotic properties properties of this procedures, i.e. for any fixed threshold $H > 0$.

Theorem 3. For any $b \geq 0$, $\theta > 0$ and for any fixed $H > 0$ the sequential procedure (2.12) possesses the following properties:

- 1) $\mathbf{P}_{\theta}(\tau_H^* < \infty) = 1$;
- 2) the sequential estimator θ_H^* is normally distributed with parameters

$$\mathbf{E}_{\theta}\theta_H^* = \theta \quad \text{and} \quad \mathbf{E}_{\theta}(\theta_H^* - \theta)^2 = \frac{\sigma}{H}. \quad (2.13)$$

Proof. Proposition 5 implies directly the firs point. Moreover, from (1.1) it follows that

$$\theta_H^* = \theta + \frac{\sqrt{\sigma}}{H} \int_0^{\tau_H^*} \frac{1}{\sqrt{X_s}} dW_s.$$

Therefore, in view of Lemma 17.4 from [26], the stochastic integral $H^{-1/2} \int_0^{\tau_H^*} X_s^{-1/2} dW_s$ is the $\mathcal{N}(0, 1)$ Gaussian distributed and we obtain our results. \square

Now, similarly to Proposition 1 and using Lemma 4 we get the following result.

Proposition 2. For any $\mu > 0$ the Laplace transformation of the expected stopping time $\mathbf{m}(H) = \mathbf{E}_\theta \tau_H^*$ defined in (2.12) is represented as

$$\widehat{\mathbf{m}}(\mu) = \int_0^{+\infty} e^{-\mu z} \mathbf{m}(z) dz = \frac{1}{\mu} \int_0^{+\infty} \Upsilon(t, \mu) dt, \quad (2.14)$$

where the function $\Upsilon(t, \mu)$ is given in (A.4).

Now we study the asymptotic, as $H \rightarrow \infty$ of the sequential procedure (2.12).

Theorem 4. For any $b > 0$, any compact set $\Theta \subset]\sigma/2, +\infty[$ and for any $r > 0$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left| \frac{\tau_H^*}{H} - I_0^{-1}(\theta) \right|^r = 0, \quad (2.15)$$

where $I_0(\theta) = 2b/(2\theta - \sigma)$.

Proof. From (1.1), by Itô's formula, we have for any $T > 0$,

$$\ln X_T = \ln x + \frac{2\theta - \sigma}{2} \int_0^T (X_t^{-1} - I_0(\theta)) dt + \sqrt{\sigma} \int_0^T X_t^{-1/2} dW_t. \quad (2.16)$$

From here, setting $\mathbf{D}_T = \int_0^T (X_t^{-1} - I_0(\theta)) dt$, we get

$$\mathbf{D}_T = \frac{2(\ln X_T - \ln x)}{2\theta - \sigma} - \frac{2\sqrt{\sigma}}{2\theta - \sigma} \int_0^T X_t^{-1/2} dW_t. \quad (2.17)$$

Now we can deduce, that there exists some constant $\mathbf{c} > 0$, not depending on θ and T that can change from line to line, such, that for any $\theta \in \Theta$

$$\mathbf{E}_\theta \mathbf{D}_T^2 \leq \mathbf{c} \left((\ln x)^2 + \mathbf{E}_\theta (\ln X_T)^2 + \int_0^T \mathbf{E}_\theta X_t^{-1} dt \right).$$

Now, taking into account here, that for any $\epsilon > 0$

$$\sup_{x > 0} \frac{|\ln x|}{x^\epsilon + x^{-\epsilon}} < \infty, \quad (2.18)$$

we can conclude through (A.6) that there exists some constant $\mathbf{c} > 0$ such that for any $T > 0$

$$\sup_{\theta \in \Theta} \mathbf{E}_\theta \mathbf{D}_T^2 \leq \mathbf{c}(1 + T). \quad (2.19)$$

Now, from here, we get, that for any $\theta \in \Theta$ and for $T > I_0^{-1}(\theta)H$, the probability

$$\begin{aligned} \mathbf{P}_\theta(\tau_H^* > T) &= \mathbf{P}_\theta \left(\int_0^T X_s^{-1} ds < H \right) = \mathbf{P}_\theta (I_0(\theta)T + \mathbf{D}_T < H) \\ &\leq \mathbf{P}_\theta (|\mathbf{D}_T| > I_0(\theta)T - H) \leq \frac{\mathbf{E}_\theta \mathbf{D}_T^2}{(I_0(\theta)T - H)^2} \leq \frac{\mathbf{c}(1 + T)}{(T - I_0^{-1}(\theta)H)^2} \end{aligned} \quad (2.20)$$

and for $T < I_0^{-1}(\theta)H$,

$$\begin{aligned} \mathbf{P}_\theta(\tau_H^* < T) &= \mathbf{P}_\theta \left(\int_0^T X_s^{-1} ds > H \right) = \mathbf{P}_\theta (I_0(\theta)T + \mathbf{D}_T > H) \\ &\leq \mathbf{P}_\theta (|\mathbf{D}_T| > H - I_0(\theta)T) \leq \frac{\mathbf{E}_\theta \mathbf{D}_T^2}{(H - I_0(\theta)T)^2} \leq \frac{\mathbf{c}(1 + T)}{(I_0^{-1}(\theta)H - T)^2}. \end{aligned}$$

Therefore, setting here $T_1 = (I_0^{-1}(\theta) + \varepsilon)H$ and $T_2 = (I_0^{-1}(\theta) - \varepsilon)H$, we get, that for any $0 < \varepsilon < \inf_{\theta \in \Theta} I_0^{-1}(\theta) \varepsilon > 0$ and any $\theta \in \Theta$

$$\mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - I_0^{-1}(\theta) \right| > \varepsilon \right) = \mathbf{P}_\theta(\tau_H^* > T_1) + \mathbf{P}_\theta(\tau_H^* < T_2) \leq \frac{\mathbf{c}(1+H)}{(\varepsilon H)^2},$$

i.e. for any $\varepsilon > 0$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - I_0^{-1}(\theta) \right| > \varepsilon \right) = 0. \quad (2.21)$$

Now we have to show, that for any $r > 0$

$$\sup_{H > 1} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left(\frac{\tau_H^*}{H} \right)^r < \infty. \quad (2.22)$$

To this end, note, that

$$\mathbf{P}_\theta(\tau_H^* > t) = \mathbf{P}_\theta \left(\int_0^t X_s^{-1} ds < H \right) \leq \mathbf{P}_\theta \left(\int_0^t \phi(X_s) ds < H \right), \quad (2.23)$$

where $\phi(x) = (1+x)^{-1} \leq 1$. Now one needs to use the deviations in the ergodic theorem for the process (1.1). To this end, we set

$$\Delta_t(\phi) = \int_0^t (\phi(X_s) - \mathbf{m}_\theta(\phi)) ds \quad \text{and} \quad \mathbf{m}_\theta(\phi) = \int_{\mathbb{R}_+} \phi(x) \mathbf{q}_\theta(x) dx, \quad (2.24)$$

where

$$\mathbf{q}_\theta(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z} \mathbf{1}_{\{z \geq 0\}}, \quad \alpha = \frac{2\theta}{\sigma} \quad \text{and} \quad \beta = \frac{2b}{\sigma}. \quad (2.25)$$

Here $\mathbf{1}_A$ is the indicator of the set A . Taking into account here, that $\alpha > 1$ for $\theta > \sigma/2$, we obtain that

$$\mathbf{m}_\theta(\phi) > \int_1^2 \frac{1}{1+x^2} \mathbf{q}_\theta(x) dx > \frac{\beta^\alpha}{5\Gamma(\alpha)} e^{-2\beta},$$

i.e. for any compact set $\Theta \subset]\sigma/2, +\infty[$ we get $\mathbf{m}_* = \inf_{\theta \in \Theta} \mathbf{m}_\theta(\phi) > 0$. Therefore, for $t > H/\mathbf{m}_*$ through the Chebyshev inequality the last probability in (2.23) can be estimated as

$$\mathbf{P}_\theta \left(\int_0^t \phi(X_s) ds < H \right) = \mathbf{P}_\theta(t\mathbf{m}_\theta(\phi) - H < -\Delta_t(\phi)) \leq \mathbf{P}_\theta(|\Delta_t(\phi)| > t\mathbf{m}_* - H) \leq \frac{\mathbf{E}_\theta \Delta_t^{2q}(\phi)}{(t\mathbf{m}_* - H)^{2q}}.$$

Using here the concentration inequality (4.2), we obtain, that for any $q > 1$ and $t > 1$

$$\mathbf{P}_\theta(\tau_H^* > t) \leq \frac{\mathbf{c}_q t^q}{(t\mathbf{m}_* - H)^{2q}}.$$

Therefore, for any $r > 0$, $\theta \in \Theta$ and $H > 1$ using this upper bound for $q > 2r$, we have

$$\begin{aligned} \mathbf{E}_\theta(\tau_H^*)^r &= r \int_0^\infty t^{r-1} \mathbf{P}_\theta(\tau_H^* > t) dt \leq r \int_0^{H/\mathbf{m}_* + H^{3/4}} t^{r-1} dt \\ &\quad + \mathbf{c}_q \int_{H/\mathbf{m}_* + H^{3/4}}^\infty \frac{t^{q+r-1}}{(t - H/\mathbf{m}_*)^{2q}} dt = (H/\mathbf{m}_* + H^{3/4})^r \\ &\quad + \mathbf{c}_q \int_{H^{3/4}}^\infty \frac{(s + H/\mathbf{m}_*)^{q+r-1}}{s^{2m}} ds \leq (H/\mathbf{m}_* + H^{3/4})^r + \frac{\mathbf{c}_q}{H^{3(q-r)/4}} + \frac{\mathbf{c}_q}{H^{q/2-r+1/4}}. \end{aligned}$$

This bound yields immediately, the (A.7). Therefore, the family $(\tau_H^*/H)_{H \geq 1}^r$ is uniformly integrable and the convergence (2.21) implies the property (2.15). \square

Now the last property in (2.13) implies the following result.

Corollary 2. For any $b > 0$, any compact set $\Theta \subset]\sigma/2, +\infty[$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left(\Upsilon_H(\theta)(\theta_H^* - \theta) \right)^2 = 1, \quad (2.26)$$

where $\Upsilon_H(\theta) = \sqrt{I(\theta)\mathbf{E}_\theta \tau_H^*}$ and $I(\theta) = \sigma^{-1}I_0(\theta)$.

Remark 1. It should be noted, that in view of Proposition 4 from [2] in the case when $a < \sigma/2$ in the model (1.1) the stochastic integral $\int_0^t X_s^{-1} ds$ is not defined for any fixed non random $t > 0$. Therefore, the non sequential MLE (2.11) can not be calculated for this case, but in view of Theorem 3 the sequential procedure (2.12) is well defined for any $a > 0$.

3. Multidimensional sequential estimation method

In this section we consider the estimation problem for the two dimension parameter $\theta = (a, b)^\top$. To this end we rewrite the process (1.1) as

$$dX_t = \mathbf{g}_t^\top \theta dt + \sqrt{\sigma X_t} dW_t, \quad (3.1)$$

where $\mathbf{g}_t = (1, -X_t)^\top$. In this section we assume, that $b > 0$ and $a > \sigma/2$. Then, in view of the results from [1] the random matrix

$$G_t = \int_0^t X_s^{-1} \mathbf{g}_s \mathbf{g}_s^\top ds = \begin{pmatrix} \int_0^t X_s^{-1} ds & -t \\ -t & \int_0^t X_s ds \end{pmatrix} \quad (3.2)$$

possesses the following asymptotic property

$$\lim_{t \rightarrow \infty} \frac{1}{t} G_t = F = \begin{pmatrix} f_1 & -1 \\ -1 & f_2 \end{pmatrix} \quad \mathbf{P}_\theta - \text{ a.s.}, \quad (3.3)$$

where $f_1 = 2b/(2a - \sigma)$ and $f_2 = a/b$. It should be noted, that the matrix F is positive defined. To construct the fixed accuracy estimator for the vector θ we use the two-step sequential fixed accuracy estimation method developed in [14, 16] for multidimensional parameter estimation problems under scalar observations in continuous time. To this end, we fix a non random sequence of non-decreasing positive numbers $(\kappa_n)_{n \geq 1}$ for which

$$\rho = \sum_{n \geq 1} \frac{1}{\kappa_n} < \infty. \quad (3.4)$$

Now for any $z > 0$ we set

$$\mathbf{t}_z = \inf \left\{ t \geq 0 : \int_0^t X_s^{-1} |\mathbf{g}_s|^2 ds \geq z \right\}, \quad (3.5)$$

where $\inf\{\emptyset\} = +\infty$ and $|x|$ is the usual euclidean norm of the vector $x = (x_1, x_2)$, i.e. $|x|^2 = x_1^2 + x_2^2$. In the sequel we denote by $\mathbf{t}_n = \mathbf{t}_{\kappa_n}$. Now on the set $\{\mathbf{t}_n < +\infty\}$ we defined the sequential MLE as

$$\widehat{\theta}_{\mathbf{t}_n} = G_{\mathbf{t}_n}^+ \int_0^{\mathbf{t}_n} X_s^{-1} \mathbf{g}_s dX_s \quad (3.6)$$

and the matrix $G^+ = G^{-1}$ if the inverse matrix G^{-1} exists and $G^+ = 0$ otherwise. So, on the first step we constructed the sequence of the sequential procedures $(\delta_n = (\mathbf{t}_n, \widehat{\theta}_{\mathbf{t}_n}))_{n \geq 1}$. Now on the second step, using these procedures we will construct a sequential aggregation estimation procedure which will be defined as weighted sum of the estimators (3.6). To this end, first we set the random weight coefficient as

$$\mathbf{b}_n = \frac{1}{|G_{\mathbf{t}_n}^{-1}| \kappa_n} \mathbf{1}_{\{\lambda_{\min}(G_{\mathbf{t}_n}^{-1}) > 0\}}, \quad (3.7)$$

where $|G|$ is the euclidean norm of the matrix G , i.e. $|G|^2 = \text{tr} GG^\top$ and $\lambda_{\min}(G)$ is the minimal eigenvalue of the matrix G , i.e.

$$\lambda_{\min}(G) = \min_{z \in \mathbb{R}^2, z \neq 0} \frac{z^\top G z}{|z|^2}.$$

Now using these coefficients, we define the stopping time as

$$v_H^* = \inf \left\{ k \geq 1 : \sum_{n=1}^k \mathbf{b}_n^2 \geq H \right\}, \quad (3.8)$$

where $\inf\{\emptyset\} = +\infty$ and $H > 0$ is a positive non random threshold which will define the non-asymptotic mean square estimation accuracy as for the procedures (2.2) and (2.12). Now on the set $\{v_H^* < +\infty\}$ we define the sequential estimator as

$$\theta_H^* = \left(\sum_{n=1}^{v_H^*} \mathbf{b}_n^2 \right)^{-1} \sum_{n=1}^{v_H^*} \mathbf{b}_n^2 \widehat{\theta}_{\mathbf{t}_n}. \quad (3.9)$$

So, we obtain aggregated two-step sequential procedure

$$\delta_H^* = (\tau_H^*, \theta_H^*) \quad \text{and} \quad \tau_H^* = \mathbf{t}_{v_H^*}. \quad (3.10)$$

First, we study this procedure in non-asymptotic setting, i.e. for any fixed $H > 0$.

Theorem 5. *For any $b > 0$ and $a > \sigma/2$ and for any $H > 0$ the procedure (3.10) has the following properties*

$$\tau_H^* < +\infty \quad \mathbf{P}_\theta - \text{a.s.} \quad (3.11)$$

and

$$\mathbf{E}_\theta |\theta_H^* - \theta|^2 \leq \rho \frac{\sigma}{H}, \quad (3.12)$$

where the coefficient ρ is defined in (3.4).

Proof. Note, that the stopping time (3.5) can be rewritten as

$$\mathbf{t}_n = \inf \{ t \geq 0 : \text{tr} G_t \geq \kappa_n \}. \quad (3.13)$$

Therefore, the property (3.3) implies immediately, that $\mathbf{t}_n < \infty$ a.s. for any $n \geq 1$ and

$$\lim_{n \rightarrow \infty} \frac{\mathbf{t}_n}{\kappa_n} = \frac{1}{\text{tr} F} = \frac{(2a - \sigma)b}{2a^2 + 2b^2 - a\sigma} \quad \mathbf{P}_\theta - \text{a.s.} \quad (3.14)$$

Moreover, from this and the definition (3.7) it follows, that

$$\lim_{n \rightarrow \infty} \mathbf{b}_n = \mathbf{b}_* \quad \mathbf{P}_\theta - \text{a.s.}, \quad (3.15)$$

where

$$\mathbf{b}_* = \frac{1}{|F^{-1}| \text{tr} F} = \left(\frac{a}{\varkappa b} + \frac{2b}{\sigma} \right)^{-1} \left(\frac{a^2}{b^2} + 2 + \frac{4b^2}{\sigma^2} \varkappa^2 \right)^{-1/2} > 0 \quad \text{and} \quad \varkappa = \frac{\sigma}{2a - \sigma}. \quad (3.16)$$

i.e. $\sum_{n \geq 1} \mathbf{b}_n^2 = +\infty$ and, therefore, the stopping time (3.8) is finite, i.e. $v_H^* < \infty$ \mathbf{P}_θ a.s. for any $H > 0$ which implies immediately the property (3.11). Now we study the estimation accuracy for the procedure (3.10). To this end note, that on the set $\{\det G_n > 0\}$ the sequential MLE (3.6) can be represented as

$$\widehat{\theta}_{\mathbf{t}_n} = G_{\mathbf{t}_n}^{-1} \int_0^{\mathbf{t}_n} X_s^{-1} \mathbf{g}_s dX_s = \theta + \sqrt{\sigma} G_{\mathbf{t}_n}^{-1} \eta_{\mathbf{t}_n} \quad \text{and} \quad \eta_{\mathbf{t}_n} = \int_0^{\mathbf{t}_n} X_s^{-1/2} \mathbf{g}_s dW_s. \quad (3.17)$$

It should be noted here, that the definition (3.5) and the properties of the stochastic integrals yield

$$\mathbf{E}_\theta |\eta_{t_n}|^2 = \mathbf{E}_\theta \int_0^{t_n} X_s^{-1} |\mathbf{g}_s|^2 ds = \kappa_n. \quad (3.18)$$

Furthermore, in view of (3.17) we can represent the estimator (3.9) in the following form

$$\theta_H^* = \frac{\sum_{n=1}^{v_H^*} \mathbf{b}_n^2 \widehat{\theta}_{t_n}}{\sum_{n=1}^{v_H^*} \mathbf{b}_n^2} = \theta + \sqrt{\sigma} \frac{\sum_{n=1}^{v_H^*} \mathbf{b}_n \xi_n}{\sum_{n=1}^{v_H^*} \mathbf{b}_n^2} \quad \text{and} \quad \xi_n = \mathbf{b}_n G_{t_n}^{-1} \eta_{t_n}. \quad (3.19)$$

Taking into account here the definition (3.7) and the the property (3.18), we get, that

$$\mathbf{E}_\theta |\xi_n|^2 \leq \frac{1}{\kappa_n^2} \mathbf{E}_\theta |\eta_{t_n}|^2 = \frac{1}{\kappa_n}.$$

Finally, from here through the Cauchy - Schwarz - Bunyakovsky inequality and the definition (3.8), we find

$$\mathbf{E}_\theta |\theta_H^* - \theta|^2 \leq \sigma \mathbf{E}_\theta \frac{\sum_{n=1}^{v_H^*} |\xi_n|^2}{\sum_{n=1}^{v_H^*} \mathbf{b}_n^2} \leq \sigma \frac{1}{H} \sum_{n \geq 1} \mathbf{E}_\theta |\xi_n|^2 \leq \sigma \frac{1}{H} \sum_{n \geq 1} \frac{1}{\kappa_n} = \rho \frac{\sigma}{H}.$$

This implies Theorem 5. \square

Now, to study asymptotic properties for the sequential procedure (3.10) we chose the sequence $(\kappa_n)_{n \geq 1}$ as

$$\kappa_n = \begin{cases} H, & \text{for } n \leq \mathbf{n}_H^*; \\ \kappa_n^*, & \text{for } n > \mathbf{n}_H^*, \end{cases} \quad (3.20)$$

where $\mathbf{n}_H^* = L_H H$ and $L_H \geq 1$ is slowly increasing function, i.e.

$$\lim_{H \rightarrow \infty} L_H = +\infty \quad \text{and} \quad \lim_{H \rightarrow \infty} \frac{L_H}{H^\delta} = 0 \quad \text{for any } \delta > 0. \quad (3.21)$$

Moreover, $(\kappa_n^*)_{n \geq 1}$ is a increasing sequence such, that for all n it is bounded from below as $\kappa_n^* \geq n$ and for some $\mu > 1$ and $0 < \varrho < 1$,

$$\limsup_{n \rightarrow \infty} n^{-\mu} \kappa_n^* < \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} n^{-\varrho} \sum_{k=1}^n \frac{1}{\sqrt{\kappa_k^*}} < \infty. \quad (3.22)$$

For example, we can take $\mathbf{n}_H^* = H \ln H$ and $\kappa_n^* = n^\mu$ for some $\mu > 1$.

Theorem 6. For any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$ for the duration time in the sequential procedure (3.10) defined through the sequence (3.20) - (3.22) for any $r > 0$ the following limit property holds true

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left| \frac{\tau_H^*}{H} - \frac{1}{\text{tr}F} \right|^r = 0, \quad (3.23)$$

where the matrix F is defined in (3.3).

Proof. First of all, we prove that for any $\varepsilon > 0$,

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - \frac{1}{\text{tr}F} \right| > \varepsilon \right) = 0. \quad (3.24)$$

To do so, we see that

$$\mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - \frac{1}{\text{tr}F} \right| > \varepsilon \right) = \mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - \frac{1}{\text{tr}F} \right| > \varepsilon, v_H^* \leq \mathbf{n}_H^* \right) + \mathbf{P}_\theta \left(\left| \frac{\tau_H^*}{H} - \frac{1}{\text{tr}F} \right| > \varepsilon, v_H^* > \mathbf{n}_H^* \right)$$

$$\leq \mathbf{P}_\theta \left(\left| \frac{\mathbf{t}_H}{H} - \frac{1}{\text{tr}F} \right| > \varepsilon \right) + \mathbf{P}_\theta (v_H^* > \mathbf{n}_H^*), \quad (3.25)$$

where $\mathbf{t}_H = \inf \{ t \geq 0 : \int_0^t X_s^{-1} |\mathbf{g}_s|^2 ds \geq H \}$. Considering the first probability in this inequality, we have

$$\mathbf{P}_\theta \left(\left| \frac{\mathbf{t}_H}{H} - \frac{1}{\text{tr}F} \right| > \varepsilon \right) = \mathbf{P}_\theta (\mathbf{t}_H > T_1(H)) + \mathbf{P}_\theta (\mathbf{t}_H < T_2(H)),$$

where $T_1(H) = ((\text{tr}F)^{-1} + \varepsilon)H$, $T_2(H) = ((\text{tr}F)^{-1} - \varepsilon)H$ and $0 < \varepsilon < (\text{tr}F)^{-1}$. On the one hand, we have

$$\mathbf{P}_\theta (\mathbf{t}_H > T_1(H)) = \mathbf{P}_\theta \left(\text{tr} \left(\frac{G_{T_1(H)}}{T_1(H)} - F \right) < -\frac{\varepsilon(\text{tr}F)^2}{1 + \varepsilon\text{tr}F} \right) \leq \mathbf{P}_\theta \left(\left| \text{tr} \left(\frac{G_{T_1(H)}}{T_1(H)} - F \right) \right| > \varepsilon_* \right)$$

where $\varepsilon_* = \inf_{\theta \in \Theta} \varepsilon(\text{tr}F)^2 / (1 + \varepsilon\text{tr}F)$. Using here the definition of the matrix F in (3.3), we get, that

$$\mathbf{P}_\theta (\mathbf{t}_H > T_1(H)) \leq \mathbf{P}_\theta \left(\left| \int_0^{T_1(H)} (X_s^{-1} - f_1) ds \right| > T_1(H)\varepsilon_*/2 \right) + \mathbf{P}_\theta \left(\left| \int_0^{T_1(H)} (X_s - f_2) ds \right| > T_1(H)\varepsilon_*/2 \right).$$

Now the upper bounds (2.9) and (2.20) implies directly $\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{P}_\theta (\mathbf{t}_H > T_1(H)) = 0$. Similarly, we can deduce, that $\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{P}_\theta (\mathbf{t}_H < T_2(H)) = 0$. Therefore,

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{P}_\theta \left(\left| \frac{\mathbf{t}_H}{H} - \frac{1}{\text{tr}F} \right| > \varepsilon \right) = 0. \quad (3.26)$$

For the last probability in (3.25), note that the property (3.21) and Lemma 8 implies directly, that for any $r > 1$

$$\lim_{H \rightarrow \infty} H^r \sup_{\theta \in \Theta} \mathbf{P}_\theta (v_H^* > \mathbf{n}_H^*) = 0. \quad (3.27)$$

From (3.26) and (3.27), we obtain (3.24). Now, one needs to show, that for any $r > 0$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta \frac{(\tau_H^*)^r}{H^r} < +\infty. \quad (3.28)$$

To do this, note, that

$$\mathbf{E}_\theta (\tau_H^*)^r \leq \mathbf{E}_\theta \mathbf{t}_H^r + \mathbf{E}_\theta (\tau_H^*)^r \mathbf{1}_{\{v_H^* > \mathbf{n}_H^*\}}. \quad (3.29)$$

Now, to estimate the first expectation we set

$$M(H) = \gamma_\theta^{-1} H + H^{3/4} \quad \text{and} \quad \gamma_\theta = \mathbf{m}_\theta(\phi) + \frac{a}{b}, \quad (3.30)$$

where the term $\mathbf{m}_\theta(\phi)$ is given in (2.24) and $\phi(x) = (1+x)^{-1}$. Now note, that

$$\text{tr} G_t \geq \int_0^t \phi(X_s) ds + \int_0^t X_s ds = \gamma_\theta t + \Delta_t(\phi) + \mathbf{D}_t,$$

where $\mathbf{D}_t = \int_0^t (X_s - a/b) ds$ and $\Delta_t(\phi)$ is defined in (2.24). So,

$$\begin{aligned} \mathbf{E}_\theta \mathbf{t}_H^r &= r \int_0^\infty t^{r-1} \mathbf{P}_\theta (\mathbf{t}_H > t) dt = r \int_0^\infty t^{r-1} \mathbf{P}_\theta (\text{tr} G_t < H) dt \\ &\leq M(H)^r + r \int_{M(H)}^\infty t^{r-1} \mathbf{P}_\theta (|\Delta_t(\phi)| + |\mathbf{D}_t| > \gamma_\theta t - H) dt. \end{aligned}$$

Using here the upper bounds (2.8) and (4.2) with $q > 2r$, we obtain that for some positive constant \mathbf{c}_q and for $H > 1$

$$\begin{aligned} \mathbf{E}_\theta \mathbf{t}_H^r &\leq M(H)^r + \mathbf{c}_q \int_{M(H)}^\infty \frac{t^{q+r-1}}{(t - \gamma_\theta^{-1}H)^{2q}} dt \leq M(H)^r + \mathbf{c}_q \int_{H^{3/4}}^\infty \frac{x^{q+r-1} + H^{q+r-1}}{x^{2q}} dx \\ &= M(H)^r + \mathbf{c}_q \frac{1}{H^{3(q-r)/4}} + \mathbf{c}_q \frac{1}{H^{q/2-r+1/4}}. \end{aligned}$$

This implies directly, that for any $r > 0$

$$\sup_{H \geq 1} \sup_{\theta \in \Theta} \frac{\mathbf{E}_\theta \mathbf{t}_H^r}{H^r} < \infty. \quad (3.31)$$

Moreover, to study the last expectation in (3.29) note, that

$$\mathbf{E}_\theta (\tau_H^*)^r \mathbf{1}_{\{v_H^* > \mathbf{n}_H^*\}} = \sum_{n \geq \mathbf{n}_H^*} \mathbf{E}_\theta \mathbf{t}_{n+1}^r \mathbf{1}_{\{v_H^* = n+1\}} \leq \sum_{n \geq \mathbf{n}_H^*} \sqrt{\mathbf{E}_\theta \mathbf{t}_{n+1}^{2r}} \sqrt{\mathbf{P}_\theta(v_H^* > n)}.$$

Using here the bound (3.31) and the first condition in (3.22) in view of Lemma 8 we obtain, that for any $q > 1$ and sufficiently large H , for which $L_H > \mathbf{u}_*$

$$\sup_{\theta \in \Theta} \mathbf{E}_\theta (\tau_H^*)^r \mathbf{1}_{\{v_H^* > \mathbf{n}_H^*\}} \leq \mathbf{c} \sum_{n \geq \mathbf{n}_H^*} n^{r\mu} \frac{L_H^q H^{q/2} + n^{oq}}{(n - \mathbf{u}_* H)^q}. \quad (3.32)$$

Note here, that for any $0 < B < q - 1$

$$\sum_{n \geq \mathbf{n}_H^*} \frac{n^B}{(n - \mathbf{u}_* H)^q} \leq \mathbf{c} \left(\frac{1}{H^{q-B-1} (L_H - \mathbf{u}_*)^{q-B-1}} + \frac{1}{H^{q-B-1} (L_H - \mathbf{u}_*)^{q-1}} \right).$$

Therefore, choosing in (3.32) $q > (2r\mu + 2)/(1 - \varrho)$ we obtain, that

$$\limsup_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta (\tau_H^*)^r \mathbf{1}_{\{v_H^* > \mathbf{n}_H^*\}} < \infty.$$

Hence, Theorem 6. \square

Theorem 7. For any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$ the sequential procedure (3.10) defined through the sequence (3.20) - (3.22)

$$\limsup_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left| \Upsilon_H(\theta) (\theta_H^* - \theta) \right|^2 \leq 2, \quad (3.33)$$

where $\Upsilon_H(\theta) = I^{1/2}(\theta) \sqrt{\mathbf{E}_\theta \tau_H^*}$, $I(\theta) = F/\sigma$ and the matrix F is defined in (3.3).

Proof. First of all we represent the normalized deviation $\Psi_H = \Upsilon_H(\theta) (\theta_H^* - \theta)$ in the following form

$$\Psi_H = \Psi_H \mathbf{1}_{\{v_H^* \leq \mathbf{n}_H^*\}} + \Psi_H \mathbf{1}_{\{v_H^* > \mathbf{n}_H^*\}} := \Psi_{1,H} + \Psi_{2,H}. \quad (3.34)$$

Note, that on the set $\{v_H^* \leq \mathbf{n}_H^*\}$ the first term in this equality can be represented as

$$\Psi_{1,H} = \sqrt{\mathbf{E}_\theta \tau_H^*} F^{1/2} G_{\mathbf{t}_H}^{-1} \eta_{\mathbf{t}_H}, \quad \eta_{\mathbf{t}_H} = \int_0^{\mathbf{t}_H} X_s^{-1/2} \mathbf{g}_s dW_s$$

and the stopping time \mathbf{t}_H is defined in (3.5). Moreover, note that on this set

$$\Psi_{1,H} = \sqrt{\frac{\text{tr} F \mathbf{E}_\theta \tau_H^*}{H}} \tilde{F}^{-1/2} \tilde{\eta}_H + \sqrt{\frac{\mathbf{E}_\theta \tau_H^*}{H}} F^{1/2} (\tilde{G}_H^{-1} - \tilde{F}^{-1}) \tilde{\eta}_H,$$

where $\widetilde{G}_H = G_{t_H}/H$, $\widetilde{F} = F/\text{tr}F$ and $\widetilde{\eta}_H = \eta_{t_H}/\sqrt{H}$. Note here, that $\mathbf{E}_\theta \widetilde{\eta}_H \widetilde{\eta}_H^\top = \mathbf{E}_\theta \widetilde{G}_H$, and, moreover, using here the Burkholder - Gandy inequality (see, for example, in [27], p.75) and the definition (3.5) we obtain, that for any $r > 2$

$$\eta_r^* = \sup_{H \geq 1} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\widetilde{\eta}_H|^r < \infty. \quad (3.35)$$

It should be noted also here, that on the set $\{v_H^* \leq \mathbf{n}_H^*\}$ in view of the definitions (3.8) and (3.20) we obtain, that

$$H \leq \sum_{j=1}^{\mathbf{n}_H^*} b_j^2 = \frac{\mathbf{n}_H^*}{|\widetilde{G}_H^{-1}|^2},$$

i.e. the norm $|\widetilde{G}_H^{-1}|^2 \leq L_H$ and, therefore,

$$|\widetilde{G}_H^{-1} - \widetilde{F}^{-1}| \leq \sqrt{L_H} |\widetilde{F}^{-1}| |\widetilde{G}_H - \widetilde{F}|.$$

On the set $\{v_H^* \leq \mathbf{n}_H^*\}$ we can write

$$\Psi_{1,H} = \widetilde{F}^{-1/2} \widetilde{\eta}_H + \Delta_{1,H},$$

where in view of Lemma 7 and the upper bound (3.35) the last term is such, that for any $r > 1$

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\Delta_{1,H}|^r = 0.$$

Moreover, taking into account, that $\mathbf{E}_\theta |\widetilde{F}^{-1/2} \widetilde{\eta}_H|^2 = \text{tr}(\widetilde{F}^{-1} \mathbf{E}_\theta \widetilde{G}_H)$, we obtain, that

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\Psi_{1,H}|^2 \leq \lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\widetilde{F}^{-1/2} \widetilde{\eta}_H|^2 = 2.$$

Finally, to study the last term in (3.34) note, that from (3.19) by Cauchy–Bunyakovsky–Schwarz inequality

$$|\theta_H^* - \theta|^4 \leq \sigma^2 \frac{\left(\sum_{n=1}^{v_H^*} \mathbf{b}_n |\eta_{t_n}| / \kappa_n \right)^4}{\left(\sum_{n=1}^{v_H^*} \mathbf{b}_n^2 \right)^4} \leq \frac{\sigma^2}{H^2} \left(\sum_{n \geq 1} \frac{|\widetilde{\eta}_{t_n}|^2}{\kappa_n} \right)^2,$$

where $\widetilde{\eta}_{t_n} = \eta_{t_n} / \sqrt{\kappa_n}$. Therefore, using (3.35) we can obtain, that

$$\mathbf{E}_\theta |\theta_H^* - \theta|^4 \leq \frac{\sigma^2}{H^2} \sum_{n, l \geq 1} \frac{\mathbf{E}_\theta |\widetilde{\eta}_{t_n}|^2 |\widetilde{\eta}_{t_l}|^2}{\kappa_n \kappa_l} \leq \frac{\sigma^2}{H^2} \sum_{n, l \geq 1} \frac{\sqrt{\mathbf{E}_\theta |\widetilde{\eta}_{t_n}|^4} \sqrt{\mathbf{E}_\theta |\widetilde{\eta}_{t_l}|^4}}{\kappa_n \kappa_l} \leq \frac{\sigma^2 \eta_4^*}{H^2} \rho^2.$$

From this and Theorem 6 it follows immediately, that

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \frac{\mathbf{E}_\theta |\Psi_H|^4}{\rho^2} < \infty,$$

where in this case in view of the definition (3.20)

$$\rho = \rho(H) = \sum_{n \geq 1} \frac{1}{\kappa_n} \leq L_H + \sum_{n \geq 1} \frac{1}{\kappa_n^*}.$$

As to the last term in (3.34) we note, that

$$\mathbf{E}_\theta |\Psi_{2,H}|^2 \leq \sqrt{\mathbf{E}_\theta |\Psi_H|^4} \sqrt{\mathbf{P}_\theta(v_H^* > \mathbf{n}_H^*)}.$$

Now, the property (3.27) and the last condition in (3.21) yield immediately

$$\lim_{H \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\Psi_{2,H}|^2 = 0.$$

Hence, Theorem 7. \square

4. Concentration inequalities for the CIR models.

In this section we study the properties of the deviation in the ergodic theorem for the process (1.1). To this end for any continuous and bounded $\mathbb{R}_+ \rightarrow \mathbb{R}$ function ϕ we set

$$\Delta_T(\phi) = \int_0^T (\phi(X_t) - \mathbf{m}_\theta(\phi)) dt, \quad (4.1)$$

where $\mathbf{m}_\theta(\phi) = \int_{\mathbb{R}_+} \phi(z) \mathbf{q}_\theta(z) dz$, the ergodic density is defined in (2.25).

Theorem 8. For any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$, for any $m > 1$ and any continuous and bounded $\mathbb{R}_+ \rightarrow \mathbb{R}$ function ϕ

$$\sup_{T \geq 0} \frac{\sup_{\theta \in \Theta} \mathbf{E}_\theta |\Delta_T(\phi)|^{2m}}{1 + T^m} < \infty. \quad (4.2)$$

Proof. We use the method proposed in [9]. According to this method we need to find a bounded solution $y(x)$ of the differential equation

$$\frac{\sigma}{2} x \dot{y}(x) + (a - bx)y(x) = \tilde{\phi}(x) \quad \text{and} \quad \tilde{\phi}(x) = \phi(x) - \mathbf{m}_\theta(\phi). \quad (4.3)$$

One can check directly by taking derivative of the Leibniz rule, that in this case a solution can be represented as

$$y(x) = -\frac{2}{\sigma x^\alpha} \int_x^{+\infty} \tilde{\phi}(u) u^{\alpha-1} e^{-\beta(u-x)} du, \quad \alpha = \frac{2a}{\sigma} \quad \text{and} \quad \beta = \frac{2b}{\sigma}. \quad (4.4)$$

It is clear, that the function $\tilde{\phi}(u)$ is bounded, i.e.

$$\sup_{u \in \mathbb{R}_+} |\tilde{\phi}(u)| \leq 2\phi_* \quad \text{and} \quad \phi_* = \sup_{u \in \mathbb{R}_+} |\phi(u)|.$$

we can obtain, that there exists some positive constant $\mathbf{c} > 0$ such that for all $x \geq 1$

$$|y(x)| \leq \frac{4\phi_*}{\sigma x^\alpha} \int_x^{+\infty} u^{\alpha-1} e^{-\beta(u-x)} du \leq \frac{2^\alpha \phi_*}{\sigma x^\alpha} \int_0^{+\infty} z^{\alpha-1} e^{-\beta z} dz + \frac{2^\alpha \phi_*}{\sigma x} \int_0^{+\infty} e^{-\beta z} dz,$$

i.e. $\sup_{x \geq 1} |y(x)| < \infty$. In the case, when $0 < x < 1$ taking into account, that

$$\int_0^{+\infty} \tilde{\phi}(u) u^{\alpha-1} e^{-\beta u} du = \int_0^{+\infty} \phi(u) u^{\alpha-1} e^{-\beta u} du - \mathbf{m}_\theta(\phi) \int_0^{+\infty} u^{\alpha-1} e^{-\beta u} du = 0,$$

we can rewrite the solution (4.4) as

$$y(x) = \frac{2e^{\beta x}}{\sigma x^\alpha} \int_0^x \tilde{\phi}(u) u^{\alpha-1} e^{-\beta u} du.$$

So, for $0 < x < 1$

$$|y(x)| \leq \frac{4\phi_* e^\beta}{\sigma x^\alpha} \int_0^x u^{\alpha-1} e^{-\beta u} du \leq \frac{4\phi_* e^\beta}{\sigma \alpha},$$

i.e. $\sup_{0 \leq x \leq 1} |y(x)| < \infty$ and, therefore,

$$y_* = \sup_{x \in \mathbb{R}_+} |y(x)| < \infty.$$

In view of the Ito formula for the function $V(u) = \int_0^u y(x) dx$ and the equation (4.3), we obtain, that

$$\Delta_T(\phi) = V(X_T) - V(X_0) - \sqrt{\sigma} \int_0^T y(X_t) \sqrt{X_t} dW_t.$$

Using now the moment inequality (A.6), we get, that for any $m \geq 1$

$$\sup_{T>0} \sup_{\theta \in \Theta} \mathbf{E}_\theta |V(X_T)|^{2m} \leq y_*^{2m} \sup_{T>0} \sup_{\theta \in \Theta} \mathbf{E}_\theta X_T^{2m} < \infty.$$

Moreover, from the moment bound for the stochastic integrals [25, Lemma 4.12] we obtain, that for any $\theta \in \Theta$

$$\mathbf{E}_\theta \left(\int_0^T y(X_t) \sqrt{X_t} dW_t \right)^{2m} \leq (m(2m-1))^m T^{m-1} y_*^{2m} \int_0^T \mathbf{E}_\theta X_t^m dt.$$

Using here again the upper bound (A.6) we get the property (4.2), hence Theorem 8. \square

Remark 2. It should be noted, that we can't obtain the exponential bound as in [9].

5. LAN properties for the CIR models.

Let \mathbf{P}_θ be the probability measure induced by the CIR process (1.1) on the canonical space $(\mathbf{C}, \mathcal{B})$, where $\mathbf{C} = \mathbf{C}(\mathbb{R}_+, \mathbb{R})$ denotes the space of continuous $\mathbb{R}_+ \rightarrow \mathbb{R}$ functions and \mathcal{B} is the cylinder σ -field in this space. Moreover, for any $T > 0$, we denote by $\mathbf{P}_{\theta,T}$ restriction of the measure \mathbf{P}_θ on the on the cylinder field \mathcal{B}_T in space of continuous $[0, T] \rightarrow \mathbb{R}$ functions $\mathbf{C}[0, T]$, i.e. this is the distribution of the process $(X_t)_{t \in [0, T]}$. First we recall, that a family of probability measures $(\mathbf{P}_{\theta,T})_{\theta \in \Theta, T > 0}$ with $\Theta \subseteq \mathbb{R}^k$ is called to satisfy the Local Asymptotic Normality condition (LAN) at a point $\theta_0 \in \Theta$ if there exist scale $k \times k$ matrix φ_T going to zero as $T \rightarrow \infty$ such that for any $u \in \mathbb{R}^k$ for which the point $\tilde{\theta} = \theta_0 + \varphi_T u$ belongs to Θ , the Radon - Nikodym derivative has the following asymptotic representation

$$\ln \frac{d\mathbf{P}_{\tilde{\theta},T}}{d\mathbf{P}_{\theta_0,T}} = u^\top \xi_T - \frac{|u|^2}{2} + \mathbf{r}_T(u), \quad (5.1)$$

where $|\cdot|$ is the euclidean norm in \mathbb{R}^k ,

$$\xi_T \xrightarrow[T \rightarrow \infty]{\mathcal{L}(\mathbf{P}_{\theta_0,T})} \mathcal{N}(0, 1_k) \quad \text{and} \quad \sup_{|u| \leq \mathbf{u}_*} |\mathbf{r}_T(u)| \xrightarrow[T \rightarrow \infty]{\mathbf{P}_{\theta_0,T}} 0 \quad \text{for any} \quad \mathbf{u}_* > 0.$$

Here, 1_k is the identity matrix of order k . It should be noted, that for the process (1.1) (see, for example, in [25] p. 297) in $\mathbf{C}[0, T]$ the logarithm of the Radon -Nikodym derivative is given as

$$\ln \frac{d\mathbf{P}_{\theta,T}}{d\mathbf{P}_{\theta_0,T}} = \ln \frac{d\mathbf{P}_{\theta,T}}{d\mathbf{P}_{\theta_0,T}}(X) = \int_0^T \frac{a - a_0 - (b - b_0)X_t}{\sigma X_t} dX_t - \frac{1}{2\sigma} \int_0^T \frac{(a - bX_t)^2 - (a_0 - b_0X_t)^2}{X_t} dt,$$

where $\theta = (a, b)^\top$ and $\theta_0 = (a_0, b_0)^\top$. Note, that on the process (1.1) with $\theta = \theta_0$ this derivative can be represented as

$$\ln \frac{d\mathbf{P}_{\theta,T}}{d\mathbf{P}_{\theta_0,T}} = \int_0^T \frac{a - a_0 - (b - b_0)X_t}{\sqrt{\sigma X_t}} dW_t - \frac{1}{2\sigma} \int_0^T \frac{(a - a_0 - (b - b_0)X_t)^2}{X_t} dt. \quad (5.2)$$

Setting here the vector $z_t = (X_t^{-1/2}, -X_t^{1/2})^\top$, we obtain, that

$$\ln \frac{d\mathbf{P}_{\theta,T}}{d\mathbf{P}_{\theta_0,T}} = (\theta - \theta_0)^\top U_T - \frac{1}{2\sigma} (\theta - \theta_0)^\top G_T (\theta - \theta_0), \quad (5.3)$$

where $U_T = \sigma^{-1/2} \int_0^T X_t^{-1/2} \mathbf{g}_t dW_t$, the matrix G_T and the vector \mathbf{g}_t are given in (3.2).

Theorem 9. The family of probability measures $(\mathbf{P}_{\theta,T})_{\theta \in \Theta, T > 0}$ given by (5.5) with the parameter set

$$\Theta \subseteq \{(a, b) : a > \sigma/2, b > 0\} =]\sigma/2, +\infty[\times]0, +\infty[$$

satisfies the LAN condition at any interior point θ_0 from Θ with scaling factor

$$\varphi_T = \frac{1}{\sqrt{T}} I^{-1/2}(\theta_0) \quad \text{and} \quad I(\theta_0) = \sigma^{-1} F, \quad (5.4)$$

where the matrix F is defined in (3.3).

Proof. First, note, that the property (3.3) through the central limit theorem given by Y.A. Kutoyants (see [22, Theorem 1.19]) implies, that

$$\frac{1}{\sqrt{T}} U_T \xrightarrow[T \rightarrow \infty]{\mathcal{L}(\mathbf{P}_{\theta_0,T})} \mathcal{N}(0, I(\theta_0)).$$

Using this and the property (3.3) in (5.3), we obtain the LAN condition with the normalized 2×2 matrix φ_T defined in (5.4). \square

If in (5.2) $a = a_0$, then

$$\ln \frac{d\mathbf{P}_{\theta,T}}{d\mathbf{P}_{\theta_0,T}} = -(b - b_0) U_T - \frac{(b - b_0)^2}{2} \langle U \rangle_T, \quad (5.5)$$

where

$$U_T = \sigma^{-1/2} \int_0^T \sqrt{X_t} dW_t \quad \text{and} \quad \langle U \rangle_T = \sigma^{-1} \int_0^T X_t dt.$$

Now we study the properties of the density (5.5) as $T \rightarrow \infty$.

Theorem 10. For any fixed $a_0 > \sigma/2$ the family of probability measures $(\mathbf{P}_{\theta,T})_{\theta \in \Theta, T > 0}$ with

$$\Theta \subseteq \{(a_0, b) : b > 0\} = \{a_0\} \times]0, +\infty[$$

given by (5.5) satisfies the LAN condition at any interior point θ_0 in Θ with scaling factor

$$\varphi_T = \frac{1}{\sqrt{I(\theta_0)T}} \quad \text{and} \quad I(\theta_0) = \frac{a_0}{\sigma b_0}. \quad (5.6)$$

Proof. The proof is straightforward by [2, Proposition 3] and the central limit theorem given by Y.A. Kutoyants (see [22, Theorem 1.19]), that

$$\frac{1}{T} \int_0^T X_t dt \xrightarrow[T \rightarrow \infty]{\mathbf{P}_{\theta_0,T}} \frac{a_0}{b_0} \quad \text{and} \quad \frac{1}{\sqrt{T}} \int_0^T \sqrt{X_t} dW_t \xrightarrow[T \rightarrow \infty]{\mathcal{L}(\mathbf{P}_{\theta_0,T})} \mathcal{N}\left(0, \frac{a_0}{b_0}\right).$$

Using these properties in (5.5), we obtain the LAN condition with the scalar coefficient φ_T defined in (5.6). \square

Furthermore, we note, that if in (5.2) the parameter $b = b_0$, we get

$$\ln \frac{d\mathbf{P}_{\theta,T}}{d\mathbf{P}_{\theta_0,T}} = (a - a_0) U_T - \frac{(a - a_0)^2}{2} \langle U \rangle_T, \quad (5.7)$$

where

$$U_T = \sigma^{-1/2} \int_0^T X_t^{-1/2} dW_t \quad \text{and} \quad \langle U \rangle_T = \sigma^{-1} \int_0^T \frac{1}{X_t} dt.$$

We need to study the asymptotic properties of the density (5.7) as $T \rightarrow \infty$.

Theorem 11. For any fixed $b_0 > 0$ the family of probability measures $(\mathbf{P}_{\theta,T})_{\theta \in \Theta, T > 0}$ with

$$\Theta \subseteq \{(a, b_0) : a > \sigma/2\} =]\sigma/2, +\infty[\times \{b_0\}$$

given by (5.7) satisfies the LAN condition at any interior point θ_0 in Θ with scaling factor

$$\varphi_T = \frac{1}{\sqrt{I(\theta_0)T}} \quad \text{and} \quad I(\theta_0) = \frac{2b_0}{\sigma(2a_0 - \sigma)}. \quad (5.8)$$

Proof. The proof is straightforward by [2, Proposition 4] and the central limit theorem given by Y.A. Kutoyants (see [22, Theorem 1.19]) that

$$\frac{1}{T} \int_0^T \frac{1}{X_t} dt \xrightarrow{T \rightarrow \infty} \frac{2b_0}{2a_0 - \sigma} \quad \text{and} \quad \frac{1}{\sqrt{T}} \int_0^T X_t^{-1/2} dW_t \xrightarrow{T \rightarrow \infty} \mathcal{N}\left(0, \frac{2b_0}{2a_0 - \sigma}\right).$$

Using these properties in (5.7), we obtain the LAN condition with the scalar coefficient φ_T defined in (5.8). \square

Remark 3. Note here, that as we will see below the functions $I(\theta_0)$ defined in (5.4), (5.6) and (5.8) define the lower bounds for the quadratic corresponding risk. According to the general optimal statistic decision theory such functions are called the Fisher information (see, for example, in [11], Theorem 12.1 for quadratic loss functions).

6. Optimality properties for sequential procedures

Let now $\theta_0 \in \Theta \subset \mathbb{R}^k$ and $\gamma > 0$ such that $\{\|\theta - \theta_0\| \leq \gamma\} \subseteq \Theta$. We denote by $\mathcal{H}_T(\theta_0, \gamma)$ the local class of sequential procedures $\delta_T = (\tau, \widehat{\theta}_\tau)$ such that

$$\sup_{\|\theta - \theta_0\| < \gamma} \mathbf{E}_\theta \tau \leq T. \quad (6.1)$$

Inspired by the ideas from [7, Corollary 2], we prove the following theorem.

Proposition 3. Assume that, LAN holds for θ_0 from Θ with the scale matrix of the form $\varphi_T = (I(\theta_0)T)^{-1/2}$, where $I(\theta_0)$ is some positive defined matrix. Then, for any $\gamma > 0$ for which $\{\|\theta - \theta_0\| \leq \gamma\} \subseteq \Theta$,

$$\lim_{T \rightarrow \infty} \inf_{\delta \in \mathcal{H}_T(\theta_0, \gamma)} \sup_{\|\theta - \theta_0\| < \gamma} \mathbf{E}_\theta |\varphi_T^{-1}(\widehat{\theta}_\tau - \theta)|^2 \geq k. \quad (6.2)$$

Proof. Suppose that we are given the positive constants ε, δ and γ arbitrarily small. Let us denote

$$J_T(\theta) = \mathbf{E}_\theta |\varphi_T^{-1}(\widehat{\theta}_\tau - \theta)|^2$$

and $\widetilde{\theta} = \theta_0 + \varphi_T u$ where $u = (u_1, \dots, u_k)^\top$. Since φ_T tends to zero matrix as T tends to infinity, then for arbitrary fixed constant $\mathbf{b} > 1$ there exists $T_0 = T_0(\gamma, \mathbf{b}) > 0$ such that $\widetilde{\theta} \in \{\|\theta - \theta_0\| \leq \gamma\}$ for all $T > T_0$ and $|u_\ell| \leq \mathbf{b}$ for $\ell \in \{1, \dots, k\}$. Therefore,

$$\begin{aligned} J_T^* &= \sup_{\|\theta - \theta_0\| < \gamma} J_T(\theta) \geq \sup_{u: |u_\ell| \leq \mathbf{b}, \forall \ell} J_T(\widetilde{\theta}) \geq \frac{1}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} J_T(\widetilde{\theta}) d^k u \\ &= \frac{1}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{E}_{\widetilde{\theta}} |u + \eta_\tau|^2 d^k u \quad \text{and} \quad \eta_\tau = \varphi_T^{-1}(\theta_0 - \widehat{\theta}_\tau). \end{aligned} \quad (6.3)$$

Let now $K \geq 2$ and $0 < \varepsilon < 1$ be some fixed integers. Then, setting $t_i = (i\varepsilon)^2 T$, we can estimate J_T^* from below as

$$J_T^* \geq \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{E}_{\widetilde{\theta}} |u + \eta_\tau|^2 \mathbf{1}_{\mathbf{A}_i} d^k u \quad \text{and} \quad \mathbf{A}_i = \{t_{i-1} < \tau \leq t_i\}.$$

From LAN property (5.1), the Randon-Nikodym derivative is given by

$$\rho_T(u) = \frac{d\mathbf{P}_{\theta,T}}{d\mathbf{P}_{\theta_0,T}} = e^{u^\top \xi_T - \frac{|u|^2}{2} + \mathbf{r}_T(u)},$$

where ξ_T weakly under the distribution $\mathcal{L}(\mathbf{P}_{\theta_0,T})$ tends to $\mathcal{N}(0, \mathbf{1}_k)$ as $T \rightarrow \infty$ and for any $\mathbf{u}_* > 0$

$$\sup_{|u| \leq \mathbf{u}_*} |\mathbf{r}_T(u)| \xrightarrow[T \rightarrow \infty]{\mathbf{P}_{\theta_0,T}} 0. \quad (6.4)$$

It is clear, that

$$\frac{d\mathbf{P}_{\theta,t_i}}{d\mathbf{P}_{\theta_0,t_i}} = \rho_{t_i}(i\epsilon u). \quad (6.5)$$

So,

$$\mathbf{E}_{\theta} |u + \eta_\tau|^2 \mathbf{1}_{A_i} = \mathbf{E}_{\theta_0} |u + \eta_\tau|^2 \mathbf{1}_{A_i} \rho_{t_i}(i\epsilon u).$$

Moreover, setting

$$\rho_t^*(u) = e^{u^\top \xi_t - \frac{|u|^2}{2}} = e^{\frac{|\xi_t|^2}{2} - \frac{|u - \xi_t|^2}{2}}, \quad (6.6)$$

we obtain, that

$$\begin{aligned} J_T^* &\geq \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k} \mathbf{E}_{\theta_0} \mathbf{1}_{A_i} e^{-r_i^*} \int_{[-\mathbf{b}, \mathbf{b}]^k} |u + \eta_\tau|^2 \rho_t^*(i\epsilon u) d^k u \\ &= \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k} \mathbf{E}_{\theta_0} \mathbf{1}_{A_i} e^{-r_i^* + |\xi_{t_i}|^2/2} \int_{[-\mathbf{b}, \mathbf{b}]^k} |u + \eta_\tau|^2 e^{-\frac{|i\epsilon u - \xi_{t_i}|^2}{2}} d^k u \\ &= \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k} \mathbf{E}_{\theta_0} \mathbf{1}_{A_i} e^{-r_i^* + |\xi_{t_i}|^2/2} \int_{\mathcal{D}_i} |v + \tilde{\eta}_\tau|^2 e^{-\frac{(i\epsilon|v|)^2}{2}} d^k v, \end{aligned} \quad (6.7)$$

where $r_i^* = \sup_{|u| \leq \mathbf{b}k} |\mathbf{r}_i(u)|$, $\tilde{\eta}_\tau = \eta_\tau + \xi_{t_i}/i\epsilon$ and

$$\mathcal{D}_i = [-\mathbf{b} - \xi_{t_i,1}/i\epsilon, \mathbf{b} - \xi_{t_i,1}/i\epsilon] \times \cdots \times [-\mathbf{b} - \xi_{t_i,k}/i\epsilon, \mathbf{b} - \xi_{t_i,k}/i\epsilon].$$

Here $(\xi_{t_i,l})_{1 \leq l \leq k}$ are the components of the vector ξ_{t_i} , i.e. $\xi_{t_i} = (\xi_{t_i,1}, \dots, \xi_{t_i,k})^\top$. Now we note that on the set

$$\mathbf{D}_i = \cap_{l=1}^k \{|\xi_{t_i,l}| \leq i\epsilon(\mathbf{b} - \sqrt{\mathbf{b}})\} \quad (6.8)$$

the last integral in (6.7) can be estimated from below as

$$\int_{\mathcal{D}_i} |u + \tilde{\eta}_\tau|^2 e^{-\frac{(i\epsilon|u|)^2}{2}} d^k u \geq \int_{[-\sqrt{\mathbf{b}}, \sqrt{\mathbf{b}}]^k} |u + \tilde{\eta}_\tau|^2 e^{-\frac{(i\epsilon|u|)^2}{2}} d^k u. \quad (6.9)$$

Using here the Anderson inequality (see, for example, in [11], p.155), we get

$$\int_{[-\sqrt{\mathbf{b}}, \sqrt{\mathbf{b}}]^k} |u + \tilde{\eta}_\tau|^2 e^{-\frac{(i\epsilon|u|)^2}{2}} d^k u \geq \int_{[-\sqrt{\mathbf{b}}, \sqrt{\mathbf{b}}]^k} |u|^2 e^{-\frac{(i\epsilon|u|)^2}{2}} d^k u.$$

Taking into account, that for $\mathbf{b} \rightarrow \infty$

$$\int_{[-\sqrt{\mathbf{b}}, \sqrt{\mathbf{b}}]^k} |u|^2 e^{-\frac{(i\epsilon|u|)^2}{2}} d^k u \rightarrow \int_{\mathbb{R}^k} |u|^2 e^{-\frac{(i\epsilon|u|)^2}{2}} d^k u = \frac{k(\sqrt{2\pi})^k}{(i\epsilon)^{2+k}} = \frac{k}{(i\epsilon)^2} \int_{\mathbb{R}^k} e^{-\frac{(i\epsilon|u|)^2}{2}} d^k u, \quad (6.10)$$

we obtain, that for any $0 < \nu < 1$ there exists some $\mathbf{b}_0 = \mathbf{b}_0(\nu)$ such that for any $\mathbf{b} \geq \mathbf{b}_0$

$$J_T^* \geq k(1-\nu) \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k (i\varepsilon)^2} \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{A}_i \cap \mathbf{D}_i} e^{-\mathbf{r}_{t_i}^* + |\xi_{t_i}|^2/2} \int_{\mathbb{R}^k} e^{-\frac{(i\varepsilon|u|)^2}{2}} \mathrm{d}^k u.$$

Moreover, taking into account here, that in $\mathbf{P}_{\theta_0, t}$ probability $\mathbf{r}_t^* \rightarrow 0$ as $t \rightarrow \infty$, we deduce, that for any $1 \leq i \leq K$

$$\lim_{T \rightarrow \infty} \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{A}_i \cap \mathbf{D}_i} e^{|\xi_{t_i}|^2/2} \left(1 - e^{-\mathbf{r}_{t_i}^*}\right) = 0.$$

Therefore, there exists some $T_0 = T_0(\nu, b)$ such that for $T \geq T_0$

$$J_T^* \geq k(1-\nu) \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k (i\varepsilon)^2} \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{A}_i \cap \mathbf{D}_i} e^{|\xi_{t_i}|^2/2} \int_{\mathbb{R}^k} e^{-\frac{(i\varepsilon|u|)^2}{2}} \mathrm{d}^k u - \nu.$$

Moreover, we can estimate this term from below as

$$\begin{aligned} J_T^* &\geq k(1-\nu) \sum_{i=1}^K \frac{!}{(2\mathbf{b})^k (i\varepsilon)^2} \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{A}_i \cap \mathbf{D}_i} e^{|\xi_{t_i}|^2/2} \int_{\mathcal{D}_i} e^{-\frac{(i\varepsilon|u|)^2}{2}} \mathrm{d}^k u - \nu \\ &= k(1-\nu) \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k (i\varepsilon)^2} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{Q}_{i,u}(\mathbf{A}_i \cap \mathbf{D}_i) \mathrm{d}^k u - \nu. \end{aligned} \quad (6.11)$$

where the measure $\mathbf{Q}_{i,u}$ is defined as

$$\mathbf{Q}_{i,u}(\mathbf{A}) = \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{A}} \rho_{t_i}^*(i\varepsilon u). \quad (6.12)$$

It should be noted that generally speaking this measure is not probabilistic, but it can be approximated by a probabilistic one in the following sens.

Lemma 1. For any set $\mathbf{A} \in \mathcal{F}_{t_i}$, $1 \leq i \leq K$ and $\mathbf{b} > 1$

$$\lim_{T \rightarrow \infty} \int_{[-\mathbf{b}, \mathbf{b}]^k} \left| \mathbf{P}_{\theta, t_i}(\mathbf{A} \cap \mathbf{D}_i) - \mathbf{Q}_{i,u}(\mathbf{A} \cap \mathbf{D}_i) \right| \mathrm{d}^k u = 0. \quad (6.13)$$

Proof. Taking into account, that for any set \mathbf{A} the measure $\mathbf{Q}_{i,u}(\mathbf{A} \cap \mathbf{D}_i)$ is bounded, it suffices to show that for any $u \in [-\mathbf{b}, \mathbf{b}]^k$

$$\lim_{T \rightarrow \infty} \left| \mathbf{P}_{\theta, t_i}(\mathbf{A} \cap \mathbf{D}_i) - \mathbf{Q}_{i,u}(\mathbf{A} \cap \mathbf{D}_i) \right| = 0. \quad (6.14)$$

Indeed, let now $\mathbf{G}_{t_i} = \{\mathbf{r}_{t_i}^* \leq \alpha\}$ for some $\alpha > 0$. Note, that the property (6.4) implies, that for any $\alpha > 0$ and any $u \in [-\mathbf{b}, \mathbf{b}]^k$

$$\lim_{T \rightarrow \infty} \mathbf{Q}_{i,u}(\mathbf{D}_i \cap \mathbf{G}_{t_i}^c) = 0.$$

Moreover, from (6.5) we get, that for any $L > 1$

$$\mathbf{P}_{\theta, t_i}(\mathbf{G}_{t_i}^c) = \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{G}_{t_i}^c} \rho_{t_i}(i\varepsilon u) \leq L \mathbf{P}_{\theta_0, t_i}(\mathbf{G}_{t_i}^c) + \mathbf{E}_{\theta_0} \mathbf{1}_{\{\rho_{t_i}(i\varepsilon u) > L\}} \rho_{t_i}(i\varepsilon u) = L \mathbf{P}_{\theta_0, t_i}(\mathbf{G}_{t_i}^c) + 1 - \mathbf{E}_{\theta_0} \mathbf{1}_{\{\rho_{t_i}(i\varepsilon u) \leq L\}} \rho_{t_i}(i\varepsilon u).$$

Note here, that for any $u \in \mathbb{R}^k$

$$\rho_{t_i}(i\varepsilon u) \xrightarrow[T \rightarrow \infty]{\mathcal{L}(\mathbf{P}_{\theta_0, T})} \rho_{*,i} = e^{i\varepsilon u^\top \xi_* - \frac{(i\varepsilon)^2 |u|^2}{2}} \quad \text{and} \quad \xi_* \sim \mathcal{N}(0, 1_k).$$

Therefore, taking into account, that $\mathbf{E} \rho_{*,i} = 1$, we obtain, that

$$\limsup_{T \rightarrow \infty} \mathbf{P}_{\theta, t_i}(\mathbf{G}_{t_i}^c) \leq 1 - \mathbf{E}_{\theta_0} \mathbf{1}_{\{\rho_{*,i} \leq L\}} \rho_{*,i} \rightarrow 0 \quad \text{as} \quad L \rightarrow \infty,$$

i.e. $\lim_{T \rightarrow \infty} \mathbf{P}_{\bar{\theta}_i}(\mathbf{D}_i \cap \mathbf{G}_i^c) = 0$. Therefore, for (6.14) it suffices to show, that

$$\lim_{\alpha \rightarrow 0} \limsup_{T \rightarrow \infty} \left| \mathbf{P}_{\bar{\theta}_i}(\mathbf{A} \cap \mathbf{D}_i \cap \mathbf{G}_i) - \mathbf{Q}_{i,u}(\mathbf{A} \cap \mathbf{D}_i \cap \mathbf{G}_i) \right| = 0.$$

This follows from the fact that on the intersection $\mathbf{D}_i \cap \mathbf{G}_i$ the difference between the densities (6.5) and (6.6) can be estimated as

$$|\rho_{i_i}(i\varepsilon u) - \rho_{i_i}^*(i\varepsilon u)| \leq e^{(i\varepsilon)^2 \mathbf{b}^2} (e^\alpha - 1) \rightarrow 0 \quad \text{as } \alpha \rightarrow 0.$$

This implies the limit 6.14. \square

Therefore, the lower bound (6.11) and Lemma 1 yield, that there exists $T_1 = \mathbf{T}_1(\nu, \mathbf{b}) \geq T_0$ such that for any $\mathbf{b} \geq \mathbf{b}_0$ and $T \geq T_1$

$$J_T^* \geq k(1 - \nu) \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k (i\varepsilon)^2} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{P}_{\bar{\theta}_i}(\mathbf{A}_i \cap \mathbf{D}_i) d^k u - \nu \quad (6.15)$$

Now we study the last probability.

Lemma 2. For any $1 \leq i \leq K$

$$\lim_{\mathbf{b} \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{P}_{\bar{\theta}_i}(\mathbf{D}_i^c) d^k u = 0. \quad (6.16)$$

Proof. To this end we show, that for any $\mathbf{b} > 1$ and $1 \leq i \leq K$

$$\liminf_{T \rightarrow \infty} \frac{1}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{P}_{\bar{\theta}_i}(\mathbf{D}_i) d^k u \geq \left(\frac{\mathbf{b} - \sqrt{\mathbf{b}}}{\mathbf{b}} \frac{i\varepsilon}{\sqrt{2\pi}} \right)^k \int_{[-\sqrt{\mathbf{b}}, \sqrt{\mathbf{b}}]^k} e^{-\frac{(i\varepsilon)^2 |u|^2}{2}} d^k u. \quad (6.17)$$

To do this, first note that for any u s.t. $|u_l| \leq \mathbf{b}, \forall l$

$$\mathbf{P}_{\bar{\theta}_i}(\mathbf{D}_i) \geq \mathbf{E}_{\theta_0} e^{-\mathbf{r}_i^*} \rho_{i_i}^*(i\varepsilon u) \mathbf{1}_{\mathbf{D}_i}.$$

Taking into account, that in view of the dominated convergence theorem

$$\lim_{T \rightarrow \infty} \mathbf{E}_{\theta_0} (1 - e^{-\mathbf{r}_i^*}) \rho_{i_i}^*(i\varepsilon u) \mathbf{1}_{\mathbf{D}_i} = 0,$$

we obtain, that for any $\nu > 0$ and sufficiently large T

$$\frac{1}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{P}_{\bar{\theta}_i}(\mathbf{D}_i) d^k u \geq \frac{1}{(2\mathbf{b})^k} \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{D}_i} \int_{[-\mathbf{b}, \mathbf{b}]^k} \rho_{i_i}^*(i\varepsilon u) d^k u - \nu.$$

Using here the representation (6.6) in the same way as in (6.9) we get, that

$$\frac{1}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{P}_{\bar{\theta}_i}(\mathbf{D}_i) d^k u \geq \frac{1}{(2\mathbf{b})^k} \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{D}_i} e^{\frac{|\xi_{i_i}|^2}{2}} \int_{[-\sqrt{\mathbf{b}}, \sqrt{\mathbf{b}}]^k} e^{-\frac{(i\varepsilon)^2 |u|^2}{2}} d^k u - \nu.$$

Taking into account here, that ξ_{i_i} converges in law to a standard normal vector then

$$\lim_{T \rightarrow \infty} \frac{1}{(2\mathbf{b})^k} \mathbf{E}_{\theta_0} \mathbf{1}_{\mathbf{D}_i} e^{\frac{|\xi_{i_i}|^2}{2}} = \left(\frac{i\varepsilon(\mathbf{b} - \sqrt{\mathbf{b}})}{\sqrt{2\pi\mathbf{b}}} \right)^k,$$

we obtain the bound (6.17). Taking into account, that the right side of this inequality tends to 1 as $\mathbf{b} \rightarrow \infty$ we obtain Lemma 2. \square

Note now, that using this lemma in (6.15) we deduce, that there exists $\mathbf{b}_1 = \mathbf{b}_1(\nu)$ such that for any $\mathbf{b} \geq \mathbf{b}_1$ there exists $T_2 = \mathbf{T}_2(\nu, \mathbf{b}) \geq T_1$ such that for any $T \geq T_2$

$$J_T^* \geq k(1 - 2\nu) \sum_{i=1}^K \frac{1}{(2\mathbf{b})^k (i\varepsilon)^2} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{P}_{\bar{\theta}_i}(\mathbf{A}_i) d^k u - \nu$$

$$\begin{aligned}
&\geq \frac{k(1-2\nu)}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \sum_{i=1}^K \mathbf{E}_{\bar{\theta}, t_i} \frac{1}{(\sqrt{\tau}/\sqrt{T} + \varepsilon)^2} \mathbf{1}_{A_i} d^k u - \nu \\
&= \frac{k(1-2\nu)}{(2\mathbf{b})^k} \int_{[-\mathbf{b}, \mathbf{b}]^k} \mathbf{E}_{\bar{\theta}} \frac{1}{(\sqrt{\tau}/\sqrt{T} + \varepsilon)^2} \mathbf{1}_{\{\tau \leq (\varepsilon K)^2 T\}} d^k u - \nu.
\end{aligned} \tag{6.18}$$

Note, here, that

$$\mathbf{E}_{\bar{\theta}} \frac{1}{(\sqrt{\tau}/\sqrt{T} + \varepsilon)^2} \mathbf{1}_{\{\tau \leq (\varepsilon K)^2 T\}} \geq \mathbf{E}_{\bar{\theta}} \frac{1}{(\sqrt{\tau}/\sqrt{T} + \varepsilon)^2} - \frac{1}{\varepsilon^2} \mathbf{P}_{\bar{\theta}}(\tau > (\varepsilon K)^2 T).$$

Using here the Jensen and the Chebychev inequalities and the condition (6.1), we obtain, that

$$\mathbf{E}_{\bar{\theta}} \frac{1}{(\sqrt{\tau}/\sqrt{T} + \varepsilon)^2} \mathbf{1}_{\{\tau \leq (\varepsilon K)^2 T\}} \geq \frac{1}{(\mathbf{E}_{\bar{\theta}} \sqrt{\tau}/\sqrt{T} + \varepsilon)^2} - \frac{1}{\varepsilon^4 K^2} \geq \frac{1}{(1 + \varepsilon)^2} - \frac{1}{\varepsilon^4 K^2}.$$

Using this estimate in (6.18), we obtain, that

$$J_T^* \geq \frac{k(1-2\nu)}{(1 + \varepsilon)^2} - \frac{k}{\varepsilon^4 K^2} - k\nu.$$

Taking here the limits $\lim_{\nu \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{K \rightarrow \infty} \liminf_{T \rightarrow \infty}$, we get the lower bound (6.2). \square

Note, that to study asymptotic optimality properties for some sequential procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ usually one considers the following class

$$\Xi_H = \left\{ \delta = (\tau, \widehat{\theta}_\tau) : \sup_{\theta \in \Theta} \frac{\mathbf{E}_\theta \tau}{\mathbf{E}_{\theta_0} \tau_H^*} \leq 1 \right\}, \tag{6.19}$$

where $H > 0$ is some non random procedure indexing parameter such that for any $\theta \in \Theta$ the expectation $\mathbf{E}_\theta \tau_H^* \rightarrow +\infty$ as $H \rightarrow \infty$. For example, for the sequential parameter estimation procedures for the CIR model (1.1), this is the threshold $H > 0$ that determines the mean square estimation accuracy.

A₁) There exists $\theta_0 \in \Theta$, such that $\{|\theta - \theta_0| < \gamma\} \subset \Theta$ for all sufficiently small $\gamma > 0$ and

$$\lim_{\theta \rightarrow \theta_0} \limsup_{H \rightarrow \infty} \left| \frac{\mathbf{E}_\theta \tau_H^*}{\mathbf{E}_{\theta_0} \tau_H^*} - 1 \right| = 0. \tag{6.20}$$

A₂) There exists $\theta_0 \in \Theta$ for which the LAN condition holds true for the scale matrix of the form $\varphi_T = I^{-1/2}(\theta_0)T^{-1/2}$ in which $I(\theta)$ is positive defined and continuous matrix for any θ from some neighborhood of the point θ_0 in Θ .

Now we obtain a lower bound for this class.

Theorem 12. Assume that, the conditions **A₁)** – **A₂)** hold true for some θ_0 from Θ . Then,

$$\lim_{H \rightarrow \infty} \inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\Upsilon_H(\theta)(\widehat{\theta}_\tau - \theta)|^2 \geq k, \tag{6.21}$$

where $\Upsilon_H(\theta) = I^{1/2}(\theta) \sqrt{\mathbf{E}_{\theta_0} \tau_H^*}$.

Proof. First note, that, by the condition **A₁)**, for any $\varepsilon > 0$ there exist $\gamma_0 = \gamma_0(\varepsilon) > 0$ for which $\{|\theta - \theta_0| < \gamma\} \subset \Theta$ and $H_0 = H_0(\varepsilon) > 0$ such that, for any $\gamma < \gamma_0$ and $H > H_0$

$$\sup_{|\theta - \theta_0| < \gamma} \mathbf{E}_\theta \tau_H^* \leq (1 + \varepsilon) \mathbf{E}_{\theta_0} \tau_H^*$$

and, therefore, $\Xi_H \subset \mathcal{H}_T(\theta_0, \gamma)$ with $T = (1 + \varepsilon) \mathbf{E}_{\theta_0} \tau_H^*$. Moreover, using the property (6.20) and, taking into account, that $I(\theta) \rightarrow I(\theta_0)$ as $\theta \rightarrow \theta_0$, it is easy to deduce, that

$$\lim_{\theta \rightarrow \theta_0} \limsup_{H \rightarrow \infty} |\Upsilon_H(\theta) \Upsilon_H^{-1}(\theta_0) - 1_k| = 0, \tag{6.22}$$

where 1_k is the identity matrix of the size k . Therefore, for any $\epsilon > 0$ we can obtain, that for sufficiently small γ and sufficiently large $H > 0$

$$\inf_{|\theta-\theta_0|<\gamma} \inf_{z \in \mathbb{R}^k, |z|=1} z^\top (\Upsilon_H^{-1}(\theta_0))^\top \Upsilon_H^\top(\theta) \Upsilon_H(\theta) \Upsilon_H^{-1}(\theta_0) z \geq (1 - \epsilon).$$

So, for such γ and H

$$\inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\Upsilon_H(\theta)(\widehat{\theta}_\tau - \theta)|^2 \geq \frac{(1 - \epsilon)}{(1 + \epsilon)} \inf_{\delta \in \mathcal{H}_T(\theta_0, \gamma)} \sup_{|\theta-\theta_0|<\gamma} \mathbf{E}_\theta |\varphi_T^{-1}(\widehat{\theta}_\tau - \theta)|^2, \quad (6.23)$$

where $\varphi_T^{-1} = (1 + \epsilon)^{-1/2} \Upsilon_H(\theta_0) = (I(\theta_0)T)^{1/2}$. Therefore, using the bound (6.2) and taking the limit $\liminf_{\epsilon \rightarrow 0} \liminf_{H \rightarrow \infty}$, we obtain the lower bound (6.21). Hence, Theorem 12. \square

Now we apply this theorem to study the procedure (3.10). To this end, first note, that Theorem 6 implies, that uniformly over $\theta \in \Theta$ for any compact $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$

$$\frac{\Upsilon_H(\theta)}{\sqrt{H}} \rightarrow \frac{\widetilde{F}^{1/2}}{\sqrt{\sigma}} \quad \text{as } H \rightarrow \infty,$$

where $\widetilde{F} = F/\text{tr}F$ and the matrix F is defined in (3.3). Now, Theorem 12 for $k = 2$ and Theorem 7 imply the following results.

Theorem 13. *For any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$ the sequential procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ defined in (3.10) for the process (1.1) is asymptotically optimal in the minimax setting, i.e.*

$$\lim_{H \rightarrow \infty} H \inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta (\widehat{\theta}_\tau - \theta)^\top \widetilde{F} (\widehat{\theta}_\tau - \theta) = \lim_{H \rightarrow \infty} H \sup_{\theta \in \Theta} \mathbf{E}_\theta (\theta_H^* - \theta)^\top \widetilde{F} (\theta_H^* - \theta) = 2\sigma$$

and

$$\lim_{H \rightarrow \infty} \frac{\inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta (\widehat{\theta}_\tau - \theta)^\top \widetilde{F} (\widehat{\theta}_\tau - \theta)}{\sup_{\theta \in \Theta} \mathbf{E}_\theta (\theta_H^* - \theta)^\top \widetilde{F} (\theta_H^* - \theta)} = 2.$$

As to the scalar sequential procedures (2.2) and (2.12), first we note, that Theorem 2 and Theorem 4 imply, that for these cases

$$\lim_{H \rightarrow 0} \sup_{\theta \in \Theta} \left| \frac{\Upsilon_H(\theta)}{\sqrt{H}} - \sigma^{-1/2} \right| = 0.$$

Therefore, Theorem 12 for $k = 1$ and Corollaries 1 and 2 imply the following results.

Theorem 14. *For any $a > \sigma/2$ and any compact set $\Theta \subset]0, +\infty[$ the sequential procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ defined in (2.2) for the process (1.1) is asymptotically optimal in the minimax setting, i.e.*

$$\lim_{H \rightarrow \infty} H \inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta (\widehat{\theta}_\tau - \theta)^2 = \lim_{H \rightarrow \infty} H \sup_{\theta \in \Theta} \mathbf{E}_\theta (\theta_H^* - \theta)^2 = \sigma \quad \text{and} \quad \lim_{H \rightarrow \infty} \frac{\inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta (\widehat{\theta}_\tau - \theta)^2}{\sup_{\theta \in \Theta} \mathbf{E}_\theta (\theta_H^* - \theta)^2} = 1. \quad (6.24)$$

Theorem 15. *For any fixed $b > 0$ and any compact set $\Theta \subset]\sigma/2, +\infty[$ the sequential procedure (2.12) for the process (1.1) is asymptotically optimal in the minimax setting, i.e.*

$$\lim_{H \rightarrow \infty} H \inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta (\widehat{\theta}_\tau - \theta)^2 = \lim_{H \rightarrow \infty} H \sup_{\theta \in \Theta} \mathbf{E}_\theta (\theta_H^* - \theta)^2 = \sigma \quad \text{and} \quad \lim_{H \rightarrow \infty} \frac{\inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta (\widehat{\theta}_\tau - \theta)^2}{\sup_{\theta \in \Theta} \mathbf{E}_\theta (\theta_H^* - \theta)^2} = 1. \quad (6.25)$$

Remark 4. It should be noted, that the optimality properties are shown in the class of all possible sequential procedures, with arbitrary fixed stopping times that determine the duration of the observation for the estimation.

7. Conclusion

In the conclusion we emphasise, that

- The sequential estimation procedures are constructed and non asymptotic mean square accuracies are obtained in (2.3), (2.13) and (3.12). It is useful to note here, that Theorem 3 is shown for any $b \geq 0, \theta > 0$, but the classical maximum likelihood estimator can be defined only for $\theta \geq \sigma/2$ (see, Remark 1). The properties for the mean time of observations are studied (Theorem 2, Theorem 4 and Theorem 6).
- Based on the LAN property the minimax sequential estimation theory for the model (1.1) was developed: the conditions provided the sharp lower bounds are obtained (Theorem 12), the procedures for which the upper bounds coincide with the lower ones are constructed (Theorem 13, Theorem 14 and Theorem 15).
- For the first time for continuous time statistical models, the minimax properties for the sequential estimation procedures are established in the class of all possible sequential procedures with arbitrary stopping times determining the duration of the observation.

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A. Appendix

A.1. Properties of the CIR process

We start with studying the Laplace transformation of the integral $\int_0^t X_s ds$. To this end for any $a, b, \mu > 0$ and $t > 0$ we set the following functions

$$\phi_\mu(t) = -\frac{2}{\sigma} \log\left(\frac{2\rho e^{t(b-\rho)/2}}{(\rho-b)e^{-\rho t} + \rho + b}\right) \quad \text{and} \quad \psi_\mu(t) = \frac{2\mu(1 - e^{-\rho t})}{(\rho-b)e^{-\rho t} + \rho + b}, \quad (\text{A.1})$$

where $\rho = \sqrt{b^2 + 2\sigma\mu}$. Note, that for $b = 0$ these function can be rewritten as

$$\phi_\mu(t) = \frac{2}{\sigma} \log\left(\cosh(t\sqrt{\sigma\mu/2})\right) \quad \text{and} \quad \psi_\mu(t) = \frac{2\mu \sinh(t\sqrt{\sigma\mu/2})}{\rho \cosh(t\sqrt{\sigma\mu/2})}.$$

Now using the Lemmas 1 - 2 from [2] we can deduce the following form for the Laplace transformation of the integral $\int_0^t X_s ds$ for any $t > 0$.

Lemma 3. For any $\mu > 0$, the Laplace transform of $\int_0^t X_s ds$ is given by

$$\Lambda(t, \mu) = \mathbf{E}_\theta e^{-\mu \int_0^t X_s ds} = e^{-a\phi_\mu(t) - x\psi_\mu(t)}. \quad (\text{A.2})$$

Now using this lemma we study the behaviour of the integral $\int_0^t X_s ds$ as $t \rightarrow \infty$.

Proposition 4. For any $a > 0$ and $b \in \mathbb{R}$ the following property hold

$$\int_0^\infty X_s ds = +\infty \quad a.s.$$

Proof. In order to prove this property, we show that for any $\mu > 0$, the Laplace transform $\mathbf{E}_\theta e^{-\mu \int_0^t X_s ds} \xrightarrow[t \rightarrow \infty]{} 0$. This is straightforward using the Lemma 3 since $\phi_\mu(t)$ converges to plus infinity and $\psi_\mu(t)$ converges to some constant as t tends to infinity in both cases $b \neq 0$ and $b = 0$. \square

Now we obtain the Laplace transformation for the integral $\int_0^t X_s^{-1} ds$. To this end, for any u, v and z from \mathbb{R} we set the confluent hypergeometric function (see, for example, in [32]) as

$$\mathbf{B}(u, v, z) = 1 + \sum_{n=1}^{\infty} \frac{z^n \prod_{k=0}^{n-1} (u+k)}{n! \prod_{k=0}^{n-1} (v+k)}. \quad (\text{A.3})$$

Moreover, for any a, b and $t > 0$ we introduce the following functions

$$\alpha = \frac{2bx e^{bt}}{\sigma(e^{bt} - 1)}, \quad \beta = \frac{2bx}{\sigma(e^{bt} - 1)} \quad \text{and} \quad \gamma = -\frac{bat}{\sigma} + \frac{2bx}{\sigma(e^{bt} - 1)}.$$

It is clear, that $\alpha = \beta = \gamma = 2x/(\sigma t)$ for $b = 0$. Now, using these functions we obtain through Lemmas 1 - 2 from [2] the following result.

Lemma 4. For any $a > 0$, $b \geq 0$, $t > 0$ and $\mu > 0$, the Laplace transformation for the integral $\int_0^t X_s^{-1} ds$ is given as

$$\Upsilon(t, \mu) = \mathbf{E} e^{-\mu \int_0^t X_s^{-1} ds} = \frac{\Gamma(v_1)}{\Gamma(v+1)} \frac{\beta^{\frac{\gamma}{2} + \frac{1}{2}}}{\alpha^\kappa} e^{-\gamma} \mathbf{B}(v_1, v+1, \beta), \quad (\text{A.4})$$

where $\Gamma(v) = \int_0^\infty z^{v-1} e^{-z} dz$, $\kappa = a/\sigma$,

$$v = \frac{2}{\sigma} \sqrt{(a - \sigma/2)^2 + 2\mu\sigma} \quad \text{and} \quad v_1 = \frac{a}{\sigma} + \frac{v}{2} + \frac{1}{2}.$$

Proposition 5. For any $a > 0$ and $b \geq 0$ the following property hold

$$\int_0^\infty X_s^{-1} ds = +\infty \quad a.s.$$

Proof. Indeed, from Lemma 4 it follows directly, that for any $\mu > 0$, the Laplace transform $\Upsilon(t, \mu) \rightarrow 0$ as $t \rightarrow \infty$ and, therefore, we get Proposition 5. \square

We recall the results from [24, Proposition 6.2.5], [2, Lemmas 1-2] and [1, Propositions 3-4].

First, note that, when $a > \sigma/2$ and $b > 0$ the CIR process is ergodic and the stationary distribution is a Gamma law with shape $2a/\sigma$ and scale $\sigma/2b$. Moreover, by the ergodic theorem, in this case

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_s ds = \frac{a}{b} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_s^{-1} ds = \frac{2b}{2a - \sigma} \quad a.s. \quad (\text{A.5})$$

Now we study the moment properties of the stable CIR processes

Lemma 5. For any $q > -2a/\sigma$ and compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$

$$\sup_{t \geq 0} \sup_{\theta \in \Theta} \mathbf{E}_\theta X_t^q < \infty. \quad (\text{A.6})$$

The proof is given in Proposition 3 from [1].

A.2. Auxiliary Lemmas

Lemma 6. Let τ be a stopping time such that for any $q > 1$ and compact set $\Theta \subset]\sigma/2, +\infty[\times \mathbb{R}_+$ the expectation $\mathbf{m}_q = \sup_{\theta \in \Theta} \mathbf{E}_\theta \tau^q < \infty$. Moreover, let $(\eta_t)_{t \geq 0}$ be a non-negative process such that

$$\mathbf{u}_\epsilon = \sup_{t \geq 0} \sup_{\theta \in \Theta} \left(\mathbf{E}_\theta \eta_t^{1+\epsilon} \right)^{1/(1+\epsilon)} < \infty$$

for some $\epsilon > 0$. Then for any $q > (1 + \epsilon)/\epsilon := r$

$$\sup_{\theta \in \Theta} \mathbf{E}_\theta \int_0^\tau \eta_t dt \leq \mathbf{u}_\epsilon \left(1 + \frac{r \mathbf{m}_q^{1/r}}{q - r} \right). \quad (\text{A.7})$$

Proof. First note, that by Fatou's lemma for any $\theta \in \Theta$ we obtain, that

$$\mathbf{E}_\theta \int_0^\tau \eta_t dt \leq \lim_{T \rightarrow \infty} \int_0^T \mathbf{E}_\theta \eta_t \mathbf{1}_{\{t \leq \tau\}} dt.$$

Moreover, the $1 + \epsilon$ and $r = (1 + \epsilon)/\epsilon$ Hölder and Chebyshev inequalities imply, that for any $q > r$

$$\mathbf{E}_\theta \eta_t \mathbf{1}_{\{t \leq \tau\}} \leq \mathbf{u}_\epsilon \mathbf{P}_\theta^{1/r}(\tau \geq t) \leq \frac{\mathbf{u}_\epsilon \mathbf{m}_q^{1/r}}{t^{q/r}}.$$

Therefore,

$$\int_0^T \mathbf{E}_\theta \eta_t \mathbf{1}_{\{t \leq \tau\}} dt \leq \mathbf{u}_\epsilon \left(1 + \int_1^T \mathbf{P}_\theta^{1/r}(\tau \geq t) dt \right) \leq \mathbf{u}_\epsilon \left(1 + \mathbf{m}_q^{1/r} \int_1^T \frac{1}{t^{q/r}} dt \right).$$

This implies the upper bound (A.7). Hence, Lemma 6. \square

Lemma 7. For any $r > 2$ and any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$ for the matrices (3.4) and (3.3) the following property holds true

$$\sup_{z \geq 1} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left(\sqrt{z} \left| \frac{G_{t_z}}{z} - \frac{F}{trF} \right| \right)^r < \infty. \quad (\text{A.8})$$

Proof. First, setting $\mathbf{D}_t = G_t - Ft$ and using the definition (3.5) we can obtain, that

$$\frac{G_{t_z}}{z} - \frac{F}{\text{tr}F} = \frac{\mathbf{D}_{t_z}}{z} + F \left(\frac{t_z}{z} - \frac{1}{\text{tr}F} \right) = \frac{\mathbf{D}_{t_z}}{z} - \frac{F \text{tr} \mathbf{D}_{t_z}}{\text{tr}F z}$$

and, therefore, taking into account, that $|\text{tr} \mathbf{D}_{t_z}| \leq \sqrt{2} |\mathbf{D}_{t_z}|$ and $|F| = |F^{1/2} F^{1/2}| \leq |F^{1/2}|^2 = \text{tr} F$, we obtain, that

$$\sqrt{z} \left| \frac{G_{t_z}}{z} - \frac{F}{\text{tr}F} \right| \leq (1 + \sqrt{2}) \frac{|\mathbf{D}_{t_z}|}{\sqrt{z}} \leq (1 + \sqrt{2}) \frac{|\mathbf{D}_{1,t_z}| + |\mathbf{D}_{2,t_z}|}{\sqrt{z}},$$

where

$$\mathbf{D}_{1,t} = \int_0^t (X_s^{-1} - f_1) ds \quad \text{and} \quad \mathbf{D}_{2,t} = \int_0^t (X_s - f_2) ds.$$

We recall, that the coefficients f_1 and f_2 are defined in (3.3). To prove this lemma, it suffices to show, that for any $r > 2$

$$\sup_{z \geq 1} \frac{\sup_{\theta \in \Theta} \mathbf{E}_\theta |\mathbf{D}_{1,t_z}|^r}{z^{r/2}} < \infty \quad (\text{A.9})$$

and

$$\sup_{z \geq 1} \frac{\sup_{\theta \in \Theta} \mathbf{E}_\theta |\mathbf{D}_{2,t_z}|^r}{z^{r/2}} < \infty. \quad (\text{A.10})$$

Note, that from (2.17) it follows, that for some positive constant \mathbf{c}

$$|\mathbf{D}_{1,t_z}|^r \leq \mathbf{c} \left(1 + \left| \int_0^{t_z} X_s^{-1/2} dW_s \right|^r + |\ln X_{t_z}|^r \right). \quad (\text{A.11})$$

Taking into account the Burkholder - Gandy inequality (see, for example, in [27], p.75) and the definition of the stopping time in (3.5), we can estimate the martingale term in this inequality as

$$\mathbf{E}_\theta \left| \int_0^{t_z} X_s^{-1/2} dW_s \right|^r \leq \mathbf{c} \mathbf{E}_\theta \left| \int_0^{t_z} X_s^{-1} ds \right|^{r/2} \leq \mathbf{c} z^{r/2}.$$

From (2.16) through the Ito formule we can deduce, that for any integer $m \geq 2$

$$(\ln X_t)^{2m} = (\ln x)^{2m} + \int_0^t \alpha_u du + \int_0^t \beta_u dW_u, \quad (\text{A.12})$$

where

$$\alpha_u = 2m(\ln X_t)^{2m-1} \left(\frac{2a - \sigma}{2X_u} - b \right) + m(2m - 1)\sigma \frac{(\ln X_u)^{2m-2}}{X_u} \quad \text{and} \quad \beta_u = 2m \sqrt{\sigma} \frac{(\ln X_u)^{2m-1}}{X_u^{1/2}}.$$

Note here, that from the upper bounds (A.6) and (2.18) one can conclude, that there exists some $\epsilon > 0$ such that

$$\sup_{u \geq 0} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left(|\alpha_u|^{1+\epsilon} + |\beta_u|^{2(1+\epsilon)} \right) < \infty.$$

Therefore, using the property (3.31) and Lemma 6 we obtain, that for any $\theta \in \Theta$ the expectation $\mathbf{E}_\theta \int_0^{t_z} \beta_u^2 du < \infty$ and, therefore, $\mathbf{E}_\theta \int_0^{t_z} \beta_u dW_u = 0$. Using this in (A.12) we obtain, that for any $\theta \in \Theta$

$$\mathbf{E}_\theta \left(\ln X_{t_z} \right)^{2m} = (\ln x)^{2m} + \mathbf{E}_\theta \int_0^{t_z} \alpha_u du \leq (\ln x)^{2m} + \mathbf{E}_\theta \int_0^{t_z} |\alpha_u| du.$$

Therefore, using here Lemma 6 with $q = 2r$ and the property (3.31) we obtain that for any $m \geq 2$

$$\sup_{z \geq 1} \frac{\sup_{\theta \in \Theta} \mathbf{E}_\theta \left(\ln X_{t_z} \right)^{2m}}{z^m} < \infty. \quad (\text{A.13})$$

This implies immediately the upper bound (A.9). Moreover, to show (A.10) note, that from (2.7) and the Burkholder - Gandy inequality it follows that for some constant $\mathbf{c} > 0$ for any $\theta \in \Theta$

$$\mathbf{E}_\theta |\mathbf{D}_{2,t_z}|^r \leq \mathbf{c} \left(1 + \mathbf{E}_\theta |X_{t_z}|^r + \mathbf{E}_\theta \left| \int_0^{t_z} X_s ds \right|^{r/2} \right) \leq \mathbf{c} \left(1 + \mathbf{E}_\theta |X_{t_z}|^r + z^{r/2} \right).$$

Similarly to (A.13) through the Itô formula and Lemma 6 we can obtain, that for any integer $m \geq 2$

$$\sup_{z \geq 1} \frac{\sup_{\theta \in \Theta} \mathbf{E}_\theta X_{t_z}^{2m}}{z^m} < \infty.$$

Hence, Lemma 7. \square

Now we study the distribution properties for the moment (3.8).

Lemma 8. *Assume, that the stopping time ν_H^* is defined in (3.8) through the sequence (3.20) - (3.22). Then, for any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$, for any $r > 1$, $H > 1$ and $n > \mathbf{u}_* H$*

$$\sup_{\theta \in \Theta} \mathbf{P}_\theta \left(\nu_H^* > n \right) \leq \mathbf{c} \frac{L_H^r H^{r/2} + n^{or}}{(n - \mathbf{u}_* H)^r}, \quad (\text{A.14})$$

where $\mathbf{u}_* = \max_{\theta \in \Theta} \mathbf{b}_*^{-2} = \max_{\theta \in \Theta} (|F^{-1}| \text{tr} F)^2$.

Proof. First of all, note, that from the definitions (3.7) and (3.16) we can deduce directly, that $\mathbf{b}_n \leq 1$ and $\mathbf{b}_* \leq 1$. Therefore,

$$|\mathbf{b}_n^2 - \mathbf{b}_*^2| \leq 2 |\mathbf{b}_n - \mathbf{b}_*| \leq 2 |\mathbf{b}_n - \mathbf{b}_*| \mathbf{1}_{\{\lambda_{\min}(G_{t_n}) > 0\}} + 2 \mathbf{1}_{\{\lambda_{\min}(G_{t_n}) = 0\}}.$$

Note here, that on the set $\{\lambda_{\min}(G_{t_n}) > 0\}$ the first difference can be estimated as

$$|\mathbf{b}_n - \mathbf{b}_*| \leq |\varpi_n| \quad \text{and} \quad \varpi_n = \frac{G_{t_n}}{\kappa_n} - \frac{F}{\text{tr} F}.$$

Moreover, note, that for any $\theta \in \Theta$

$$\mathbf{P}_\theta \left(\lambda_{\min}(G_{t_n}) = 0 \right) = \mathbf{P}_\theta \left(\lambda_{\min} \left(\frac{G_{t_n}}{\kappa_n} \right) = 0 \right) \leq \mathbf{P}_\theta \left(|\varpi_n| \geq \mathbf{I}_* \right),$$

where $\mathbf{I}_* = \min_{\theta \in \Theta} \lambda_{\min}(F)/\text{tr} F > 0$. Using here Chebyshev's inequality and Lemma 7, we obtain, that for any $r > 1$

$$\sup_{\theta \in \Theta} \mathbf{P}_\theta \left(\lambda_{\min}(G_{t_n}) = 0 \right) \leq \frac{\sup_{\theta \in \Theta} \mathbf{E}_\theta |\varpi_n|^r}{\mathbf{I}_*^r} \leq \mathbf{c} \kappa_n^{-r/2},$$

where $\mathbf{c} > 0$ is some constant. So, denoting by $\eta_n = \sqrt{\kappa_n}(\mathbf{b}_n^2 - \mathbf{b}_*^2)$, we obtain, that for any $r > 1$

$$\sup_{n \geq 1} \sup_{\theta \in \Theta} \mathbf{E}_\theta |\eta_n|^r < \infty.$$

Now, this and the definition (3.8) imply, that for any $n > \mathbf{b}_*^{-2} H$ and $r > 1$

$$\mathbf{P}_\theta \left(\nu_H^* > n \right) = \mathbf{P}_\theta \left(\sum_{k=1}^n \mathbf{b}_k^2 < H \right) \leq \mathbf{P}_\theta \left(\sum_{k=1}^n \frac{|\eta_k|}{\sqrt{\kappa_k}} > \mathbf{b}_*^2 n - H \right) \leq \frac{\mathbf{E}_\theta \left(\sum_{k=1}^n \frac{|\eta_k|}{\sqrt{\kappa_k}} \right)^r}{(\mathbf{b}_*^2 n - H)^r}.$$

Using here, the $(r/(r-1), r)$ - Hölder inequality we obtain, that

$$\left(\sum_{k=1}^n \frac{|\eta_k|}{\sqrt{K_k}} \right)^r \leq \left(\sum_{k=1}^n \frac{1}{\sqrt{K_k}} \right)^{r-1} \left(\sum_{k=1}^n \frac{|\eta_k|^r}{\sqrt{K_k}} \right)$$

Therefore, there exists some positive constant $\mathbf{c} > 0$ such that $n > \mathbf{u}_* H$

$$\sup_{\theta \in \Theta} \mathbf{P}_\theta \left(v_H^* > n \right) \leq \frac{\mathbf{c}}{(n - \mathbf{u}_* H)^r} \left(\sum_{k=1}^n \frac{1}{\sqrt{K_k}} \right)^r.$$

Using here the conditions (3.20) - (3.22), we obtain the upper bound (A.14). Hence, Lemma 8. \square

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