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Numerical simulation of flows in unsaturated porous media

by an adaptive discontinuous Galerkin method: application to La Verne dam

Flows in unsaturated porous media: Richards' equation

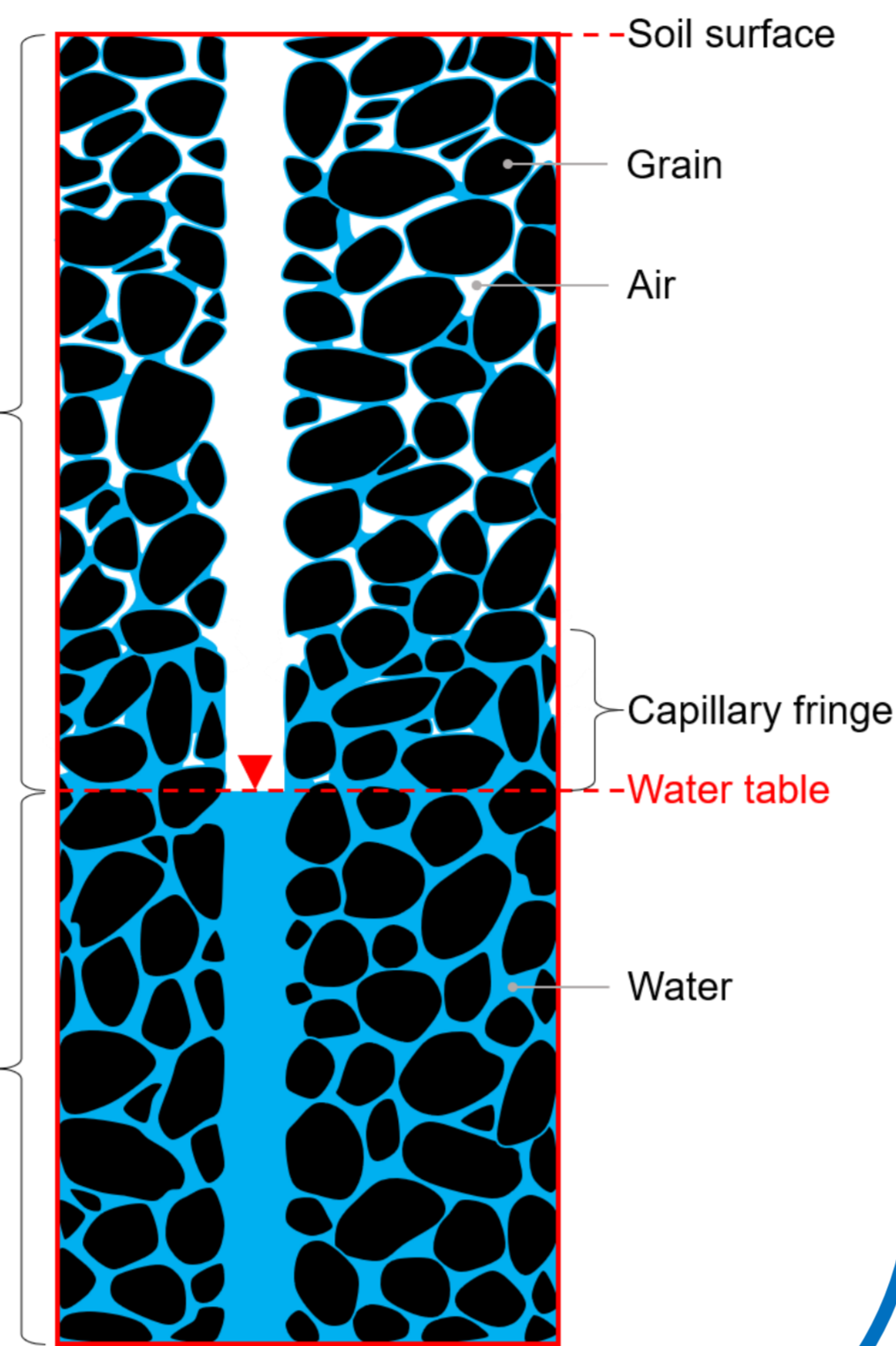
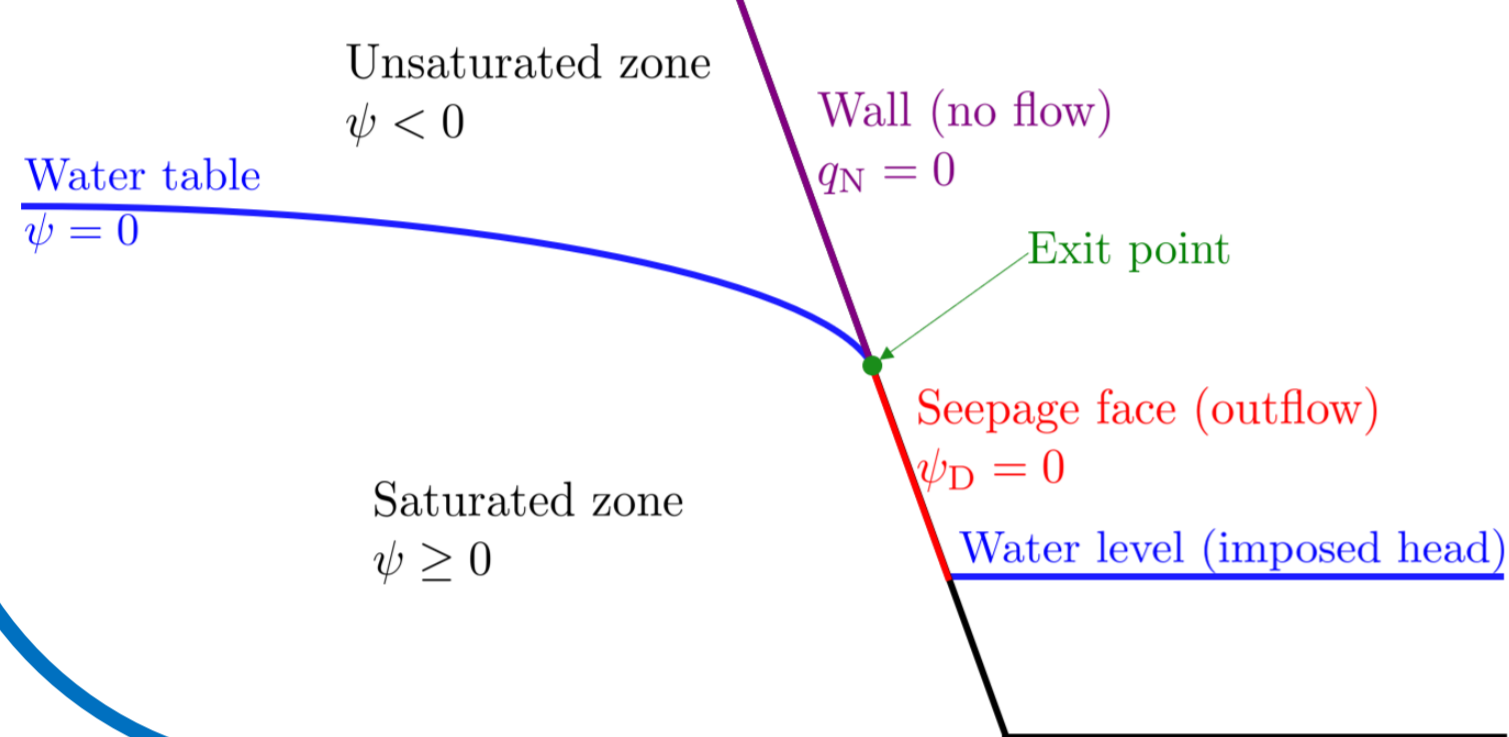
Richards' equation is a **degenerate nonlinear parabolic** equation to describe flow in variably-saturated porous media.

$$\partial_t(\theta(\psi)) - \nabla \cdot (\mathbb{K}(\psi)\nabla(\psi + z)) = 0$$

Mixed form of Richards' equation:

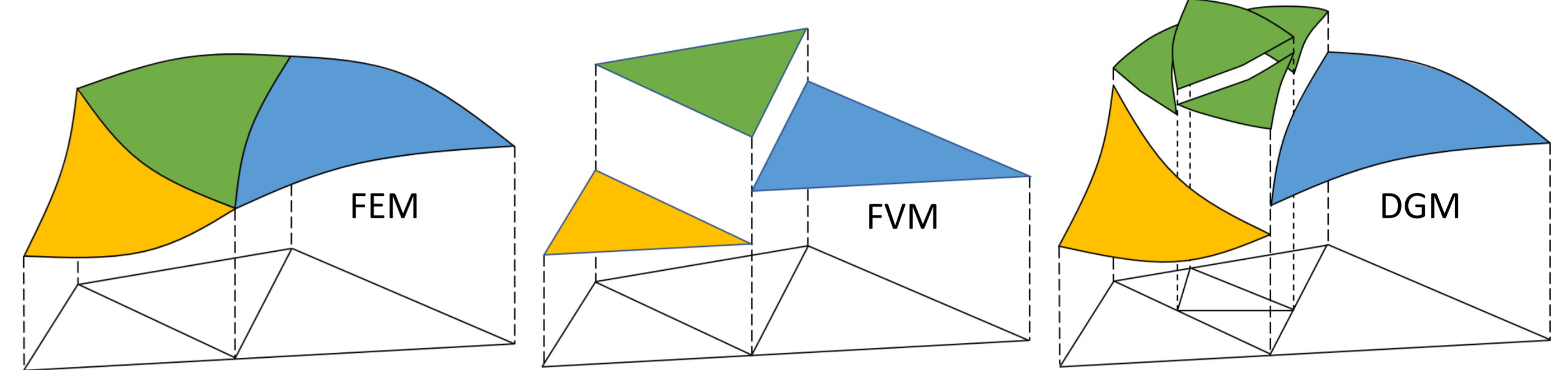
- Pressure head ψ [m]
- Water content θ [-]
- Elevation z [m]
- Hydraulic conductivity tensor \mathbb{K} [m/s]
- Hydraulic head $h = \psi + z$ [m]
- Darcy flux $q = -\mathbb{K}(\psi)\nabla h$ [m/s]

Seepage (nonlinear Robin) boundary condition:



Numerical tools: Discontinuous Galerkin (DG) methods

- Based on **variational formulation** as in Finite Element Methods (FEM)
- Designed in an **element-wise fashion** as in Finite Volume Methods (FVM)



Advantages:

- **High-order** approximation
- Element-wise formulation
 - Possible discontinuity
 - Unstructured, **non-conforming** and hybrid mesh
→ **mesh adaptation**
- **Weak** formulation for **boundary condition**
- **Local mass balance**
- Sparse structure by blocks allowing **parallelization**

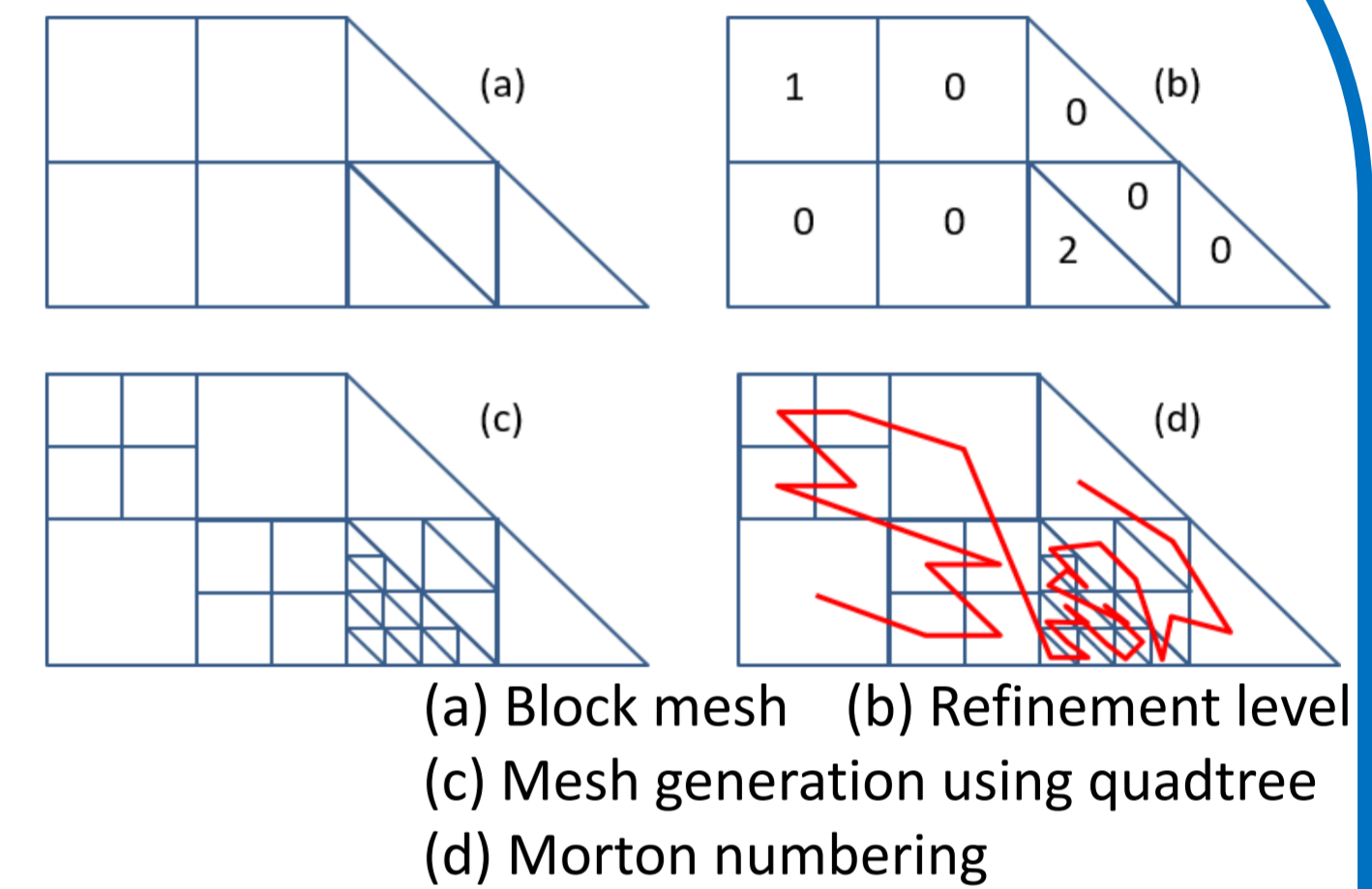
Drawbacks:

- Stability by penalty parameters
- **Computational cost**
- **Implementation** may be **difficult**
- Still **gaps in theoretical analysis**

Adaptive strategy: Adaptive Mesh Refinement (AMR) Weighted DG framework (WDG)

Adaptivity:

- Block-based AMR
- Sequential process
- Threshold values for refinement selection
- Projection by local DG problem
- **Error indicator inspired from a posteriori estimation**



$$(\eta_E^n)^2 = (\eta_{E,R}^n)^2 + (\eta_{E,F}^n)^2 + (\eta_{E,J}^n)^2$$

$$\text{Volume residual } (\eta_{E,R}^n)^2 = \frac{h_E^2}{p_E^2 \lambda_m(\mathbb{K})} \left\| \frac{\theta(u_b^{n+1}) - \theta(u_b^n)}{\tau^n} - \nabla \cdot (\mathbb{K}(u_b^{n+1}) \nabla u_b^{n+1}) \right\|_{L^2(E)}^2$$

$$\text{Face residual (flux jump)} (\eta_{E,F}^n)^2 = \sum_{F \in \partial E \cap \mathcal{F}_b^n} \frac{h_F}{2p_F \kappa_m} \left\| \mathbb{K}(u_b^{n+1}) \nabla u_b^{n+1} \cdot \mathbf{n} \right\|_{L^2(F)}^2$$

$$+ \sum_{F \in \partial E \cap \mathcal{F}_b^n} \frac{h_F}{p_F \kappa_l} \left\| q_N - \mathbb{K}(u_b^{n+1}) \nabla u_b^{n+1} \cdot \mathbf{n} \right\|_{L^2(F)}^2$$

$$\text{Solution jump } (\eta_{E,J}^n)^2 = \sum_{F \in \partial E \cap \mathcal{F}_b^n} \frac{1}{2} \left(\varrho_F^I + \frac{h_F}{p_F \kappa_m} \right) \left\| [u_b^{n+1}] \right\|_{L^2(F)}^2$$

$$+ \sum_{F \in \partial E \cap \mathcal{F}_b^n} \left(\varrho_F^D + \frac{h_F}{p_F \kappa_l} \right) \left\| u_D - u_b^{n+1} \right\|_{L^2(F)}^2$$

WDG to manage diffusivity:

- Weighted averages $\langle u \rangle_\omega := \omega_l u_l + \omega_r u_r$

$$\begin{cases} \omega_l = \frac{\kappa_r}{\kappa_l + \kappa_r}, & \omega_r = \frac{\kappa_l}{\kappa_l + \kappa_r} & \text{if } \kappa_l + \kappa_r \neq 0, \\ \omega_l = \omega_r = \frac{1}{2} & & \text{otherwise.} \end{cases}$$
- **Relaxation of penalty** $\gamma_F = \frac{2\kappa_l \kappa_r}{\kappa_l + \kappa_r}$

Discretization: DG for space BDF for time

Find a sequence of $(h^n)_{n \in \mathbb{N}_+} \in S_p(\mathcal{E}_h^n)$ such that

$$\begin{cases} h^0 = h_0 \\ \forall v \in S_p(\mathcal{E}_h^n), m_{b,n} \left(\sum_{k=0}^q \frac{\alpha_{q,k}}{\tau^n} \theta(h^{n+1-k} - z), v \right) + a_{b,n}(h^{n+1}, v) = l_{b,n}(v, t^{n+1}) \end{cases}$$

BDF methods:

- Multistep **implicit schemes** with **high-order** (up to 6)
- Divided differences for adaptive time stepping

$$m_{b,n}(\partial_t \theta(h - z), v) = \sum_{E \in \mathcal{E}_b} \int_E \partial_t \theta(h - z) v \, dE$$

$$a_{b,n}(h, v) = \sum_{E \in \mathcal{E}_b} \int_E \mathbb{K}(h - z) \nabla h \cdot \nabla v \, dE - \sum_{F \in \mathcal{F}_b^I} \int_F \mathbb{K}(h - z) \nabla h \cdot \mathbf{n}_F [v] \, dF + \sum_{F \in \mathcal{F}_b^I} \int_F \varrho_F^I [h] [v] \, dF$$

$$- \sum_{F \in \mathcal{F}_b^D} \int_F \mathbb{K}(h - z) \nabla h \cdot \mathbf{n}_F v \, dF - \sum_{F \in \mathcal{F}_b^D} \int_F \mathbb{1}_S(h) \mathbb{K}(h - z) \nabla h \cdot \mathbf{n}_F v \, dF$$

Penalty for continuity constraint

$$+ \sum_{F \in \mathcal{F}_b^D} \int_F \varrho_F^D h v \, dF + \sum_{F \in \mathcal{F}_b^S} \int_F \varrho_F^D \mathbb{1}_S(h) h v \, dF$$

$$l_{b,n}(v) = - \sum_{F \in \mathcal{F}_b^N} \int_F q_N v \, dF$$

Penalty for Dirichlet and seepage boundary conditions

$$+ \sum_{F \in \mathcal{F}_b^D} \int_F \varrho_F^D h_D v \, dF + \sum_{F \in \mathcal{F}_b^S} \int_F \varrho_F^D \mathbb{1}_S(h) z v \, dF$$

$$\varrho_F^I := \frac{\sigma_F^I \gamma_F}{\mu_F}$$

$$\mu_F := \frac{h_F^\beta}{p_F^2}$$

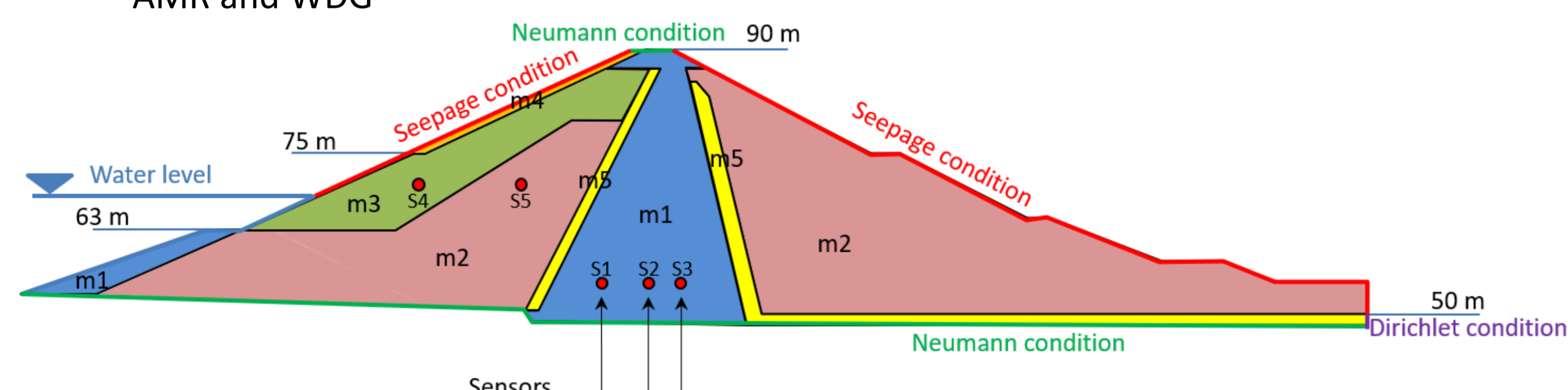
Wetting of La Verne dam

La Verne dam is an earth-filled dam:

- 40 days of reservoir impoundment (**dynamic forcing**)
- Experimental data available
- **Multi-materials** configuration (Vachaud's and Van Genuchten-Mualem relations for hydraulic properties θ and \mathbb{K})
- **Seepage**

Demanding simulation to get robust, accurate and cost-effective computation:

- 1-order BDF (implicit Euler) and quadratic approximation
- AMR and WDG



AMR and WDG are working in **synergy** through the **estimation-based error indicator**:

- **AMR** is used as a **capturing technique** for sharp internal layers/wetting fronts
- **WDG** adjusts the **local numerical smoothness** to reduce spurious oscillations by introducing jumps into the solution

