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# Two families of strongly walk regular graphs from three-weight codes over $\mathbb{Z}_4^*$

Minjia Shi<sup>†</sup>, Wenjun Xu<sup>‡</sup>, Yue Cheng<sup>§</sup>, Huazhang Wu,<sup>¶</sup> Patrick Solé<sup>||</sup>

## Abstract

A necessary condition for a  $\mathbb{Z}_4$ -code to be a three-weight code for the Lee weight is given. Two special constructions of three-weight codes over  $\mathbb{Z}_4$  are derived. The coset graphs of their duals are shown to be strongly 3-walk-regular, a generalization of strongly regular graphs.

**Keywords:** Three-weight codes; Projective codes; Strongly walk-regular graphs; Gray map, Coset graph

**AMS Classification:** 94B 05, 05E 30

## 1 Introduction

Strongly walk-regular graphs (SWRG) were introduced in [2] as a generalization of strongly regular graphs. Recently, a simple numerical condition bearing on the homogeneous weights of three-weight codes over rings was introduced to check if the coset graph of their dual codes is SWRG [4, 6]. In parallel, in [5, 7, 8, 10, 11, 12], Shi *et al.* determined the algebraic structures of one-weight codes, two-weight and three-weight codes over different finite rings. Especially, Shi *et al.* considered the construction of one-Lee weight and two-Lee weight projective codes over  $\mathbb{Z}_4$  in [8]. Later, the authors also determined the linearity of these codes completely in [11].

Inspired by the above works, we revisit  $\mathbb{Z}_4$  codes. The alphabet  $\mathbb{Z}_4$  has been an important and popular example of ring alphabet ever since the paper [3], where, already the coset graph of the Preparata code is used to construct a distance regular graph of diameter three.

We give two special constructions of projective three-weight codes over  $\mathbb{Z}_4$ , with explicit weight distributions. Using the weight information, the spectrum of the coset

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graph of the dual codes are determined. Using a spectral condition of [2], these graphs are shown to be 3-SWRG.

This work is organized as follows. In Section 2, we recall some background and notations about linear codes over  $\mathbb{Z}_4$ , and their coset graphs. In Section 3, a necessary condition for a linear code to have three-Lee weight over  $\mathbb{Z}_4$  is given. The structures of three-Lee weight projective linear codes are discussed in Section 4, and a first construction method of three-Lee weight projective codes over  $\mathbb{Z}_4$  is given. Moreover, we also give some examples to illustrate the results. In section 5, we discuss the linearity of the Gray images of these codes. Section 6 derives similar results for a second construction. Section 7 concludes the article.

## 2 Definitions and Notation

### 2.1 $\mathbb{Z}_4$ codes

In this section, we first recall the codes over  $\mathbb{Z}_4$  in [4]. Let  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ , and  $\mathbb{Z}_4^n = \{(x_1, x_2, \dots, x_n) | x_i \in \mathbb{Z}_4, 1 \leq i \leq n\}$ . A linear code  $C$  over  $\mathbb{Z}_4$  of length  $n$  is a  $\mathbb{Z}_4$ -submodule of  $\mathbb{Z}_4^n$ . The Lee weights of  $0, 1, 2, 3 \in \mathbb{Z}_4$  are  $0, 1, 2, 1$  respectively. For any  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_4^n$ , we define  $W_L(\mathbf{x}) = \sum_{i=1}^n W_L(x_i)$ . Each element  $x \in \mathbb{Z}_4$  has a 2-adic expansion  $x = \alpha(x) + 2\beta(x)$ , where  $\alpha(x), \beta(x) \in \mathbb{F}_2$ , the Gray map from  $\mathbb{Z}_4$  to  $\mathbb{F}_2^2$  is given by  $\Phi(x) = (\beta(x), \alpha(x) + \beta(x))$ . Define  $\alpha(\mathbf{x}) = (\alpha(x_1), \dots, \alpha(x_n))$ , where  $\alpha(0) = \alpha(2) = 0$  and  $\alpha(1) = \alpha(3) = 1$ . This map can be extended to  $\mathbb{Z}_4^n$  naturally.  $\Phi$  is a weight-preserving map from  $(\mathbb{Z}_4^n, \text{Lee weight})$  to  $(\mathbb{F}_2^{2n}, \text{Hamming weight})$ , that is,  $W_L(\mathbf{x}) = W_H(\Phi(\mathbf{x}))$ .

Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  be two elements of  $\mathbb{Z}_4^n$ , the inner product of  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{Z}_4^n$  is defined by  $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$ , and the componentwise multiplication  $*$  of  $\mathbf{x}$  and  $\mathbf{y}$  is  $\mathbf{x} * \mathbf{y} = (x_1y_1, x_2y_2, \dots, x_ny_n)$ , where the operation is performed in  $\mathbb{Z}_4$ . The dual code of  $C$  is defined as  $C^\perp = \{\mathbf{x} \in \mathbb{Z}_4^n | \mathbf{x} \cdot \mathbf{y} = 0, \forall \mathbf{y} \in C\}$ . A Lee weight projective code  $C$  of length  $n$  over  $\mathbb{Z}_4$  is a linear code such that the minimum Lee weight of its dual code is at least three.

It is well known that a nonzero linear code  $C$  over  $\mathbb{Z}_4$  has a generator matrix which after a suitable permutation of the coordinates can be written in the form (see [3])

$$G_{k_1, k_2} = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2D \end{pmatrix} \quad (1)$$

where  $I_{k_1}$  and  $I_{k_2}$  denote the  $k_1 \times k_1$  and  $k_2 \times k_2$  identity matrices, respectively,  $A$  and  $D$  are  $\mathbb{Z}_2$ -matrices,  $B$  is a  $\mathbb{Z}_4$ -matrix, and  $|C| = 4^{k_1}2^{k_2}$ .

### 2.2 Graphs

An *eigenvalue* of a graph  $\Gamma$  (i.e., an eigenvalue of its adjacency matrix) is called a *restricted eigenvalue* if there is a corresponding eigenvector which is not a multiple of the all-one

vector  $\mathbf{1}$ . Note that for an  $\eta$ -regular connected graph, the restricted eigenvalues are simply the eigenvalues different from  $\eta$ .

**Definition 1.** Let  $T$  be a finite abelian group and  $S \subseteq T$  a subset satisfying  $S = -S$  and  $0_T \notin S$ . The corresponding *Cayley graph*  $C(T, S)$  has vertex set equal to  $T$ ; two vertices  $g, h \in T$  are adjacent in  $C(T, S)$  iff  $g - h \in S$ .

For a precise definition of a Cayley graph attached to a code in a canonical way we refer to [9]. Now, we recall the relation between the weight distribution of a linear code over  $\mathbb{Z}_4$  and the eigenvalues of the syndrome graph of its dual code. This extension of Lemma 3.4 in [1] was derived in [13, Theorem 4.1] and Theorem 3.4 in [9], thus its proof is omitted here.

**Theorem 1.** *Suppose that  $C$  is a regular, projective linear code over  $\mathbb{Z}_4$  with Lee weights  $w_i$  and corresponding weight distribution  $A_i = |\{x \in C; W_L(x) = w_i\}|$ . Then the eigenvalues of  $\Gamma(C^\perp)$  are  $2n - 2w_i$  with multiplicity  $A_i$ .*

### 3 A necessary condition

We motivate the constructions of the next section by two necessary conditions for a  $\mathbb{Z}_4$ -code to be a three-weight code.

**Lemma 1.** *Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$ . There is no three-Lee weight projective linear code when  $k_1 = 1$ .*

*Proof.* Suppose there is a three-Lee weight projective linear code when  $k_1 = 1$ . Let  $\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_{k_2}$  be the rows of generator matrix  $G_{1, k_2}$ . Let  $\mathbf{r}_0 = (r_{00}, r_{01}, \dots, r_{0, n-1})$ . We claim that  $r_{0j}$  can only take 1 or 3 for  $0 \leq j \leq n-1$ . Otherwise, if  $r_{0j} = 0$  or 2, there is  $(0, \dots, 2, \dots, 0)$  in its dual code  $C^\perp$ , this is a contradiction since  $C$  is projective. Thus  $\omega_1 = w_L(\mathbf{r}_0) = n$ ,  $\omega_2 = w_L(2\mathbf{r}_0) = 2n$ . We assume that  $\omega_3 = 2n_2$ , where  $n_2$  is the number of 2 in the row  $\mathbf{r}_i$  for  $1 \leq i \leq k_2$ , thus  $w_L(2\mathbf{r}_0 + \mathbf{r}_i) = 2n - 2n_2$  it is easy to check that  $2n - 2n_2$  can't equal to  $\omega_1, \omega_2$  and  $\omega_3$ . This completes the proof.  $\square$

We can generalize Lemma 1 to the general case by the similar proof, thus we omit the proof here.

**Lemma 2.** *Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$ , where  $k_1 \geq 1$  and  $k_2 \geq 1$ . Let  $\mathbf{r}_1, \dots, \mathbf{r}_{k_1}, \mathbf{r}_{k_1+1}, \dots, \mathbf{r}_{k_1+k_2}$  be the rows of the generator matrix  $G_{k_1, k_2}$ . If  $\mathbf{r} = (r_{00}, r_{01}, \dots, r_{0, n-1})$  is a linear combination of the first  $k_1$  rows and  $r_{0j}$  can only take 1 or 3 for  $0 \leq j \leq n-1$ , then there is no three-Lee weight projective linear code.*

### 4 First Construction

In this section, we will give the constructions of three-weight codes over  $\mathbb{Z}_4$ .

**Theorem 2.** Let  $C_1$  be the linear code over  $\mathbb{Z}_4$  with type  $4^2 2^{k_2}$  of length  $n = 3 \cdot 2^{k_2}$  with generator matrix

$$G_{k_2+2,n} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ G_{0,k_2} & G_{0,k_2} & G_{0,k_2} \end{pmatrix} \quad (2)$$

where  $\mathbf{i}$  is the row vector  $(\underbrace{i, i, \dots, i}_{2^{k_2}})$ ,  $i = 1$  or  $2$  and  $(c_{3,j}, \dots, c_{k_2+2,j})^\top$ ,  $c_{i,j} = 0$  or  $2$ ,  $3 \leq i \leq k_2 + 2$ ,  $1 \leq j \leq 2^{k_2}$  runs over all distinct column vectors of  $G_{0,k_2}$ . Then  $C_1$  is a three-Lee weight projective code with the nonzero weights  $\omega_1 = 2^{k_2+1}$ ,  $\omega_2 = 3 \cdot 2^{k_2}$ ,  $\omega_3 = 2^{k_2+2}$ , of respective frequencies  $A_{\omega_1} = 6$ ,  $A_{\omega_2} = 2^{k_2+4} - 16$ ,  $A_{\omega_3} = 9$ .

*Proof.* Let  $\mathbf{r}_{01}, \mathbf{r}_{02}, \mathbf{r}_1, \dots, \mathbf{r}_{k_2}$  be the rows of generator matrix  $G_{k_2+2,n}$ , where

$$\mathbf{r}_{01} = (\underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{2, 2, \dots, 2}_{2^{k_2}}), \mathbf{r}_{02} = (\underbrace{2, 2, \dots, 2}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}).$$

It is easy to check that  $W_L(p_{01}\mathbf{r}_{01} + p_{02}\mathbf{r}_{02}) = 2^{k_2+1}$  or  $2^{k_2+2}$  unless  $p_{01} = p_{02} = 0$  for any  $p_{0i} \in \mathbb{Z}_4$ ,  $1 \leq i \leq 2$ . Moreover, we can also prove that

$$W_L(p_{01}\mathbf{r}_{01} + p_{02}\mathbf{r}_{02} + p_1\mathbf{r}_1 + \dots + p_{k_2}\mathbf{r}_{k_2}) = 3 \cdot 2^{k_2}$$

unless  $p_1 = \dots = p_{k_2} = 0$  for any  $p_{0i} \in \mathbb{Z}_4$ ,  $1 \leq i \leq 2$ ,  $p_j \in \mathbb{F}_2$ ,  $1 \leq j \leq k_2$ . Hence  $C_1$  contains 9 codewords with weight  $2^{k_2+2}$ , 6 codewords of weight  $2^{k_2+1}$  and  $2^{k_2+4} - 16$  codewords with weight  $3 \cdot 2^{k_2}$ .

To prove the code is projective, it is equivalent to prove every pair of distinct columns being linearly independent, which can be checked easily upon noticing that the matrices  $G_{0,k_2}$  already have this property.  $\square$

**Example 1.** If  $k_1 = 2$  and  $k_2 = 2$ , then  $n = 12$  and  $\omega_1 = 8$ ,  $\omega_2 = 12$ ,  $\omega_3 = 16$ , according to Theorem 2, there exists a three-Lee weight code with the generator matrix:

$$G_{2,2} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

**Corollary 1.** The coset graph of  $C_1^\perp$  is a 3-SWRG.

*Proof.* From [2] we know that a graph with three restricted eigenvalues is 3-SWRG iff they add up to zero. Translating in terms of weights of  $C_1$  by Theorem 1 we see that this equivalent to check that  $\omega_1 + \omega_2 + \omega_3 = 3n$ . This is easy to check by the explicit formulas of Theorem 2.  $\square$

In the above theorem, if we assume

$$G_{2,k_2} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ G_{0,k_2} & G_{0,k_2} & G_{0,k_2} \end{pmatrix} = (G_1^2, G_2^2, G_3^2), \quad (3)$$

then we have the following result.

**Theorem 3.** Let  $C_2$  be the linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n = 3 \cdot 2^{2k_1+k_2-4}$  with a generator matrix  $G_{k_1,k_2}$ , defined inductively as follows.  $G_{2,k_2}$  is as above. Assume  $G_{m,k_2} = (G_1^m, G_2^m, G_3^m)$ , and if  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$ ,  $i \in \mathbb{Z}_4$ ,  $m \geq 2$  define  $G_{m+1,k_2}$  as

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ G_1^m & G_1^m & G_1^m & G_1^m & G_2^m & G_2^m & G_2^m & G_2^m & G_3^m & G_3^m & G_3^m & G_3^m \end{pmatrix}. \quad (4)$$

Then  $C_2$  is a three-Lee weight projective code with the nonzero weights  $\omega_1 = 2^{2k_1+k_2-3}$ ,  $\omega_2 = 3 \cdot 2^{2k_1+k_2-4}$ ,  $\omega_3 = 2^{2k_1+k_2-2}$ , of respective frequencies  $A_{\omega_1} = 6$ ,  $A_{\omega_2} = 2^{2k_1+k_2} - 16$ , and  $A_{\omega_3} = 9$ , where  $k_1 \geq 2$  and  $k_2 \geq 1$ .

*Proof.* If  $k_1 = 2$ , the result is proved by Theorem 2. Assume the result is valid for any  $k_1 = m$ . Namely, if  $G_{m,k_2} = (G_1^m, G_2^m, G_3^m)$  is the generator matrix of a three-Lee weight code with the nonzero weights  $\omega_1 = 2^{2m+k_2-3}$ ,  $\omega_2 = 3 \cdot 2^{2m+k_2-4}$  and  $\omega_3 = 2^{2m+k_2-2}$ . Now we consider the case when  $k_1 = m + 1$ , the generator matrix of the linear code  $C$  is (4), it is easily seen that  $C$  is a three-Lee weight projective code with the nonzero weight  $\bar{\omega}_1 = 2^{2m+k_2-1} = 4\omega_1$ ,  $\bar{\omega}_2 = 3 \times 2^{2m+k_2-2} = 4\omega_2$ ,  $\bar{\omega}_3 = 2^{2m+k_2} = 4\omega_3$ . By induction hypothesis,  $C$  is a three-Lee weight code over  $\mathbb{Z}_4$ . Similar to the discussion of Theorem 2, we can prove that  $C_2$  is projective.  $\square$

**Example 2.** If  $k_1 = 3, k_2 = 1$ , then  $n = 24$ ,  $\omega_1 = 16$ ,  $\omega_2 = 24$ ,  $\omega_3 = 32$ , according to Theorem 3, there is a three-Lee weight code with the generator matrix  $G_{3,1}$ :

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}$$

where

$$G_{2,1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

The analogue of Corollary 1 is the following result, the proof of which is omitted.

**Corollary 2.** The coset graph of  $C_2^\perp$  is a 3-SWRG.

## 5 Linearity of Gray Images of first construction

For our purpose, we first require the following classic lemma of [3, Th.4].

**Lemma 3.** If  $C$  be a linear code over  $\mathbb{Z}_4$ , with generators  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ , then  $\Phi(C)$  is linear if and only if  $2\alpha(\mathbf{x}_i) * \alpha(\mathbf{x}_j)$  in  $C$  for all  $i, j$ , satisfying  $1 \leq i \leq j \leq m$ .

We can now state and prove the following result.

**Theorem 4.** *The Gray image  $\phi(C_2)$  of  $C_2$  is nonlinear.*

*Proof.* It is enough to prove the result is valid for  $k_1 = 2$ . Let  $\mathbf{x}_1$  and  $\mathbf{x}_2$  be two row vectors of  $G_{k_2+2,n}$ . Clearly, if either  $\mathbf{x}_1$  or  $\mathbf{x}_2$  is from the last  $k_2$  rows, it is easy to check that  $2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2) = 0 \in C$ . When

$$\mathbf{x}_1 = (\underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{2, 2, \dots, 2}_{2^{k_2}}), \mathbf{x}_2 = (\underbrace{2, 2, \dots, 2}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}, \underbrace{1, 1, \dots, 1}_{2^{k_2}}).$$

We have

$$2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2) = (\underbrace{0, 0, \dots, 0}_{2^{k_2}}, \underbrace{2, 2, \dots, 2}_{2^{k_2}}, \underbrace{0, 0, \dots, 0}_{2^{k_2}})$$

The weight of  $2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2)$  is  $w_1$ . According to Theorem 2,  $A_{w_1} = 6$ . However, we can easily check that  $2\alpha(\mathbf{x}_1) * \alpha(\mathbf{x}_2)$  is not in  $C_2$ , because it is not the linear combination of the first two rows. Therefore  $C_2$  is nonlinear.  $\square$

## 6 Second Construction

**Lemma 4.** *Let  $C$  be an  $N$ -Lee weight linear code of length  $n$  with the nonzero Lee weights  $w_1, w_2, \dots, w_N$  over  $\mathbb{Z}_4$  and  $G$  be its generator matrix. If  $C'$  is generated by*

$$G' = \begin{pmatrix} G & G & G & G \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix},$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $n$ ,  $i \in \mathbb{Z}_4$ , then  $C'$  is an  $(N+1)$ -Lee weight linear code of length  $n' = 4n$  with the nonzero Lee weights  $w'_1 = 4w_1, w'_2 = 4w_2, \dots, w'_N = 4w_N$  and  $w'_{N+1} = 4n$ . Moreover,  $A'_{w_1} = A_{w_1}, A'_{w_2} = A_{w_2}, \dots, A'_{w_N} = A_{w_N}$ , and  $A'_{w_{N+1}} = 3|C|$ , where  $A'_{w_i}$  is the number of the codewords of weight  $w'_i$  in  $C'$ ,  $1 \leq i \leq N+1$ , and  $A_{w_j}$  is the number of the codewords of weight  $w_j$  in  $C$ ,  $1 \leq j \leq N$ .

*Proof.* It is easily seen that  $n' = 4n$ ,  $w'_i = 4w_i$  and  $A'_{w_i} = A_{w_i}$ ,  $1 \leq i \leq N$ . Without loss of generality, let

$$\begin{aligned} \mathbf{c}' &= (\mathbf{c} \ \mathbf{c} \ \mathbf{c} \ \mathbf{c}) \\ \mathbf{r}_0 &= (\mathbf{0} \ \mathbf{1} \ \mathbf{2} \ \mathbf{3}) \end{aligned}$$

where  $\mathbf{c} \in C$  and  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $n$ ,  $i \in \mathbb{Z}_4$ . It is easy to check that

$$W_L(\mathbf{r}_0) = W_L(2\mathbf{r}_0) = W_L(3\mathbf{r}_0) = 4n.$$

Denote the number of  $i$  in  $\mathbf{c}$  by  $n_i$ ,  $i \in \mathbb{Z}_4$ .

(1) If the order of  $\mathbf{c}$  is 2, then  $n = n_0 + n_2$ , and

$$\begin{aligned} W_L(\mathbf{r}_0 + \mathbf{c}') &= W_L(\mathbf{c}) + W_L(\mathbf{c} + \mathbf{1}) + W_L(\mathbf{c} + \mathbf{2}) + W_L(\mathbf{c} + \mathbf{3}) \\ &= 2n_2 + n + 2(n - n_2) + n = 4n. \end{aligned}$$

(2) If the order of  $\mathbf{c}$  is 4, then  $n = n_0 + n_1 + n_2 + n_3$ , and

$$\begin{aligned} W_L(\mathbf{r}_0 + \mathbf{c}') &= W_L(\mathbf{c}) + W_L(\mathbf{c} + \mathbf{1}) + W_L(\mathbf{c} + \mathbf{2}) + W_L(\mathbf{c} + \mathbf{3}) \\ &= (n_1 + 2n_2 + n_3) + (n_0 + 2n_1 + n_2) \\ &\quad + (2n_0 + n_1 + n_3) + (n_0 + n_2 + 2n_3) \\ &= 4(n_1 + n_3 + n_2 + n_0) = 4n. \end{aligned}$$

Similarly, we can also prove  $W_L(2\mathbf{r}_0 + \mathbf{c}') = W_L(3\mathbf{r}_0 + \mathbf{c}') = 4n = w'_{N+1}$ . Therefore,  $A'_{w_{N+1}} = 3(|C| - 1) + 3 = 3|C|$ .  $\square$

**Remark** Obviously, if  $w_i = n$  for some  $i$ , then  $C'$  is still an  $N$ -Lee weight code and  $A'_{w_i} = A_{w_i} + 3|C|$ .

Similar to the discussion of 4, we have the following Lemma.

**Lemma 5.** *Let  $C$  be an  $N$ -Lee weight linear code of length  $n$  with the nonzero Lee weights  $w_1, w_2, \dots, w_N$  over  $\mathbb{Z}_4$  and  $G$  be its generator matrix. If  $C'$  is generated by*

$$G' = \begin{pmatrix} G & G \\ \mathbf{0} & \mathbf{2} \end{pmatrix},$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $n$ ,  $i \in 2\mathbb{Z}_4$ , then  $C'$  is an  $(N+1)$ -Lee weight linear code of length  $n' = 2n$  with the nonzero Lee weights  $w'_1 = 2w_1, w'_2 = 2w_2, \dots, w'_N = 2w_N$  and  $w'_{N+1} = 2n$ . Moreover,  $A'_{w_1} = A_{w_1}, A'_{w_2} = A_{w_2}, \dots, A'_{w_N} = A_{w_N}$ , and  $A'_{w_{N+1}} = |C|$ , where  $A'_{w'_i}$  is the number of the codewords of weight  $w'_i$  in  $C'$ ,  $1 \leq i \leq N+1$ , and  $A_{w_j}$  is the number of the codewords of weight  $w_j$  in  $C$ ,  $1 \leq j \leq N$ .

*Remark 1.* Obviously, if  $w_i = n$  for some  $i$ , then  $C'$  is still an  $N$ -Lee weight code and  $A'_{w_i} = A_{w_i} + |C|$ .

By the propositions mentioned above, we can construct a three-Lee weight code with type  $4^{k_1}2^{k_2}$  based on a special two-Lee weight code or a special three-Lee weight code. Here is a special case, and we get a three-Lee weight code by this example.

**Example** Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^32^0$  of length 6 with the following generator matrix:

$$G_0 = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}.$$

It is easy to check that  $C$  is a three-Lee weight linear code with the nonzero Lee weights  $w_1 = 4, w_2 = 8$ , and  $w_3 = 6$ . Moreover,  $A_{w_1} = 18, A_{w_2} = 21$ , and  $A_{w_3} = 24$ .

**Theorem 5.** *Let  $C$  be a linear code over  $\mathbb{Z}_4$  with type  $4^{k_1}2^{k_2}$  of length  $n$ . If the columns of the generator matrix  $G_{(k_1+k_2) \times n}$  are all distinct nonzero vectors  $(c_1, c_2, \dots, c_{k_1}, c_{k_1+1}, \dots, c_{k_1+k_2})^T$ , where  $(c_1, c_2, \dots, c_{k_1})^T$  is one column of  $G_0$  defined above,  $c_i \in \mathbb{Z}_4, 4 \leq i \leq k_1$  when*

$k_1 \geq 4$ , and  $c_j = 0$  or  $2$ ,  $k_1 + 1 \leq j \leq k_1 + k_2$ , then  $C$  is a three-Lee weight projective code of length  $n = 3 \times 2^{2k_1+k_2-5}$  with the nonzero Lee weights  $w_1 = 2^{2k_1+k_2-4}$ ,  $w_2 = 2^{2k_1+k_2-3}$  and  $w_3 = 3 \times 2^{2k_1+k_2-5}$ . Moreover,  $A_{w_1} = 18$ ,  $A_{w_2} = 21$ , and  $A_{w_3} = 4^{k_1}2^{k_2} - 40$ , respectively.

*Proof.* Denote the generator matrix of  $C$  by  $G_{k_1, k_2}$ .

(1) If  $k_1 = 3$ ,  $k_2 = 0$ , then  $G_{3,0} = G_0$ , and the result follows from Example 6.5. Assume the result is valid for any  $k_2 = m$  and we denote the nonzero weights by  $w'_1$ ,  $w'_2$  and  $w'_3$ . Since

$$G_{3, m+1} = \begin{pmatrix} G_{3, m} & G_{3, m} \\ \mathbf{0} & \mathbf{2} \end{pmatrix},$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $3 \times 2^{m+1}$ ,  $i \in 2\mathbb{Z}_4$ , according to Lemma 6.3 and Remark 1,  $C$  is a three-Lee weight code with the nonzero Lee weights  $w_1 = 2w'_1$ ,  $w_2 = 2w'_2$  and  $w_3 = 2w'_3$ . By induction hypothesis, the results are valid.

It is not difficult to check that

$$G_{3, k_2} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ B_{k_2} & B_{k_2} & B_{k_2} & B_{k_2} & B_{k_2} & B_{k_2} \end{pmatrix},$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $2^{k_2}$ ,  $i \in \mathbb{Z}_4$  and  $B_{k_2}$  is a  $k_2 \times 2^{k_2}$  matrix over  $2\mathbb{Z}_4$  with all different columns.

To prove that the code is projective, it is equivalent to prove every pair of distinct columns being linearly independent, which can be checked easily upon noticing that  $B_{k_2}$  has already has this property.

(2) It is easy to check that

$$G_{k_1+1, k_2} = \begin{pmatrix} G_{k_1, k_2} & G_{k_1, k_2} & G_{k_1, k_2} & G_{k_1, k_2} \\ \mathbf{0}' & \mathbf{1}' & \mathbf{2}' & \mathbf{3}' \end{pmatrix},$$

where  $\mathbf{i}'$  is the row vector  $(i, i, \dots, i)$  of length  $3 \times 2^{2k_1+k_2-5}$ ,  $i \in \mathbb{Z}_4$ . Similar to the proof above, we can prove  $C$  is a three-Lee weight code by Lemma 4 and the following Remark.

Similar to the discussion of (1), we can prove that the code is projective.  $\square$

**Example** If  $k_1 = 3$ ,  $k_2 = 1$ , then  $w_1 = 8$ ,  $w_2 = 16$ , and  $w_3 = 12$ , then, according to Theorem 5, there is a three-Lee weight code with the generator matrix:

$$G_1 = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5)$$

Like in the Section 4 we have the following graphical consequence, whose proof is omitted.

**Corollary 3.** *The coset graph of  $C^\perp$  is 3-SWRG.*

Like in Section 5 we can study the linearity of the Gray image.

**Theorem 6.** *If the columns of the generator matrix of  $C$  satisfy the conditions of Theorem 5, then  $\Phi(C)$  is nonlinear when  $k_1 \geq 3$ .*

*Proof.* It is easy to check that the first three rows of  $G_{k_1, k_2}$  can be written as:

$$\underbrace{G_0 \ G_0 \ \cdots \ G_0}_{4^{k_1-3}2^{k_2}},$$

which is permutation-equivalent to the matrix

$$G'_0 = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix}, \quad (6)$$

where  $\mathbf{i}$  is the row vector  $(i, i, \dots, i)$  of length  $4^{k_1-3}2^{k_2}$ ,  $i \in \mathbb{Z}_4$ . Let  $\mathbf{r}_1$  and  $\mathbf{r}_3$  be the first and third rows of  $G'_0$ , respectively, then

$$W_L(2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3)) = 2 \times 4^{k_1-3}2^{k_2} = 2^{2k_1+k_2-5}.$$

By Theorem 5,  $C$  is a three-Lee weight code with the nonzero Lee weights  $w_1 = 2^{2k_1+k_2-4}$ ,  $w_2 = 2^{2k_1+k_2-3}$  and  $w_3 = 3 \times 2^{2k_1+k_2-5}$ . Thus,  $2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3) \notin C$ .

By Lemma 3,  $\Phi(C)$  is nonlinear.  $\square$

**Example** Consider the code defined in Example 6.7. Let  $\mathbf{r}_1$  and  $\mathbf{r}_3$  denote the first and third rows of  $G_1$  in (18), respectively. Then the Lee weight of

$$\begin{aligned} 2\alpha(\mathbf{r}_1) * \alpha(\mathbf{r}_3) &= 2\alpha(123000123000) * \alpha(112112112112) \\ &= (200000200000) \end{aligned}$$

is 4 which can not be obtained in  $C$  since  $C$  is a three-Lee weight code with  $w_1 = 8$ ,  $w_2 = 16$ , and  $w_3 = 12$ .

## 7 Conclusion and open problems

In this paper we have constructed two infinite families of three-weight projective codes over  $\mathbb{Z}_4$ . From there strongly walk-regular graphs have been built. There are two main directions of enquiry from that point on. One is to look at other families of three-weight  $\mathbb{Z}_4$ -codes. The other is to extend this research to other rings. In particular, chain rings are the natural candidates to replace  $\mathbb{Z}_4$  as an alphabet for our three-weight codes.

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