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Ramin Imany, Mehri Javanian. External wiener index and saturated nodes of exponential recursive ternary trees. 2023. hal-03933754v3

HAL Id: hal-03933754

<https://hal.science/hal-03933754v3>

Preprint submitted on 9 Nov 2023

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External wiener index and saturated nodes of exponential recursive ternary trees

Ramin Imany Nabiyyi* Mehri Javanian†

September 29, 2023

Abstract

An exponential recursive ternary trees (ERTTs) have the property that each node has 3 possible places (external nodes) to which a child can be attached. At each step of growing of ERTTs, each external node is independently changed into a leaf with probability p , or fails to do so with probability $1 - p$. In this note, we investigate the external Wiener index, i.e. the sum of all distances between pairs of external nodes; and the number of saturated nodes (nodes with 3 children) in ERTTs. For these trees, we first calculate the expectation and second moment of the number of saturated nodes and Wiener index. Then, by martingale convergence theorem, we show the almost sure convergence and quadratic mean convergence of an appropriate scaled external Wiener index that is indicative the distance of two randomly chosen external nodes. Via an application of the contraction method, we also show the convergence in distribution for a scaled number of saturated nodes.

AMS 2000 Subject Classification : 05C05, 05C09, 60F25.

Key words: Exponential recursive ternary trees, Wiener index, saturated nodes, L^2 -norm, Wasserstein metric, contraction method, martingale convergence theorem.

1 Introduction and setting

A *random recursive tree* of size n is starting with a root, i.e. node 1, and node 2 connected to the root node. Afterwards, at step i ($i = 3, \dots, n$) node i is connected uniformly at random to one of the prior nodes $1, \dots, i - 1$; see Figure 1. Random recursive trees have been used as a model for various phenomena such as the stochastic growth of networks [3]; genealogy of ancient and medieval texts [15]; and the spread of epidemics [14]. For only a few of variations of recursive trees, see [7], [8], [9], [10], [12] and [20]; for fundamental properties of recursive trees, see [19].

*Corresponding Author: Department of Statistics, Faculty of Mathematical Sciences, University of Tabriz, Iran.
Email: imany@tabrizu.ac.ir

†Department of Statistics, Faculty of Sciences, University of Zanjan, Iran.

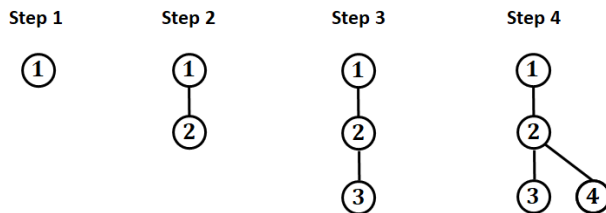


Figure 1: The first four steps of evolution of a recursive tree.

Since only one node is added at each step in the growth of a recursive tree, then these slow-growing models cannot be suitable for fast-growing phenomena such as the growing the Twitter and the spreading of infectious diseases. In [13], it has been proposed a fast-growing recursive tree model, i.e. the *exponential recursive tree* (ERT), where every node independently attracts a new node with probability p , or fails to do at each step in the growth of the tree. The *size* (the number of nodes), the *leaves* (the nodes with no children), the *depth* of a node v (the distance, i.e. the number of edges, of node v from tree's root node) and the *total path length* (the sum of depth of all nodes) have been studied for ERTs by [1].

In [6], an *exponential recursive k -ary tree* has been defined as an ERT in which every node has at most k children. The nodes that already exist in the tree are called *internal*. In order to identify the locations of possible insertions of a new node, an exponential recursive k -ary tree is extended by having $k - d$ *external* nodes joined as children of a node with d children. Internal and external nodes are shown by circles and squares, respectively (see, e.g., the trees in Figure 2).

An *exponential recursive k -ary tree of index p and age n* grows as follows: Initially (at age 0), there is an external node. For $n \geq 1$, at the n th step (at age n), every external node independently is changed into a leaf with probability $p \in (0, 1)$, or not to be changed with probability $q := 1 - p$. The parameter p is called the *index* of the ERT. The *age* of an ERT is the number of steps it takes to grow the tree. Figure 2 shows all exponential recursive 3-ary trees of ages 0, 1 and 2.

The *Wiener index*, a distance-based topological index, is defined as the sum of distances between all nodes in a graph. For recursive trees, Neininger [17] has obtained the expectations, the variances and a bivariate limit law for the Wiener index and the total path length. It also has been obtained a strong limit theorem for the Wiener index of recursive trees in [5].

By *saturated nodes*, we mean the nodes with k children in an exponential recursive k -ary tree.

The limiting distribution of depth and the expectation and variance of *external path length* (the sum of depth of all external nodes) in recursive k -ary trees have been obtained in [8] and [9], respectively. The size, leaves, external path length and depth of a randomly chosen external node in exponential recursive k -ary tree have been investigated; see [6]. The asymptotic behaviour of *profile* (number of nodes of the same depth) of exponential binary trees has been studied by [4].

In this paper, for the sake of simplicity, we consider the exponential recursive k -ary tree for which $k = 3$. The resulting tree is called *exponential recursive ternary tree* (ERTT). Our interest here is to study external Wiener index W_n and number of saturated nodes Y_n in an ERTT of age n . We give the first two moments of W_n and Y_n . Then we show convergence in distribution and in L^2 of $\frac{W_n}{n(2p+1)^{2n}}$ that is a martingale with finite second moment. By the contraction method, we also characterize limiting distribution of $\frac{Y_n}{(2p+1)^n}$ as the unique fixed point of a distributional equation.

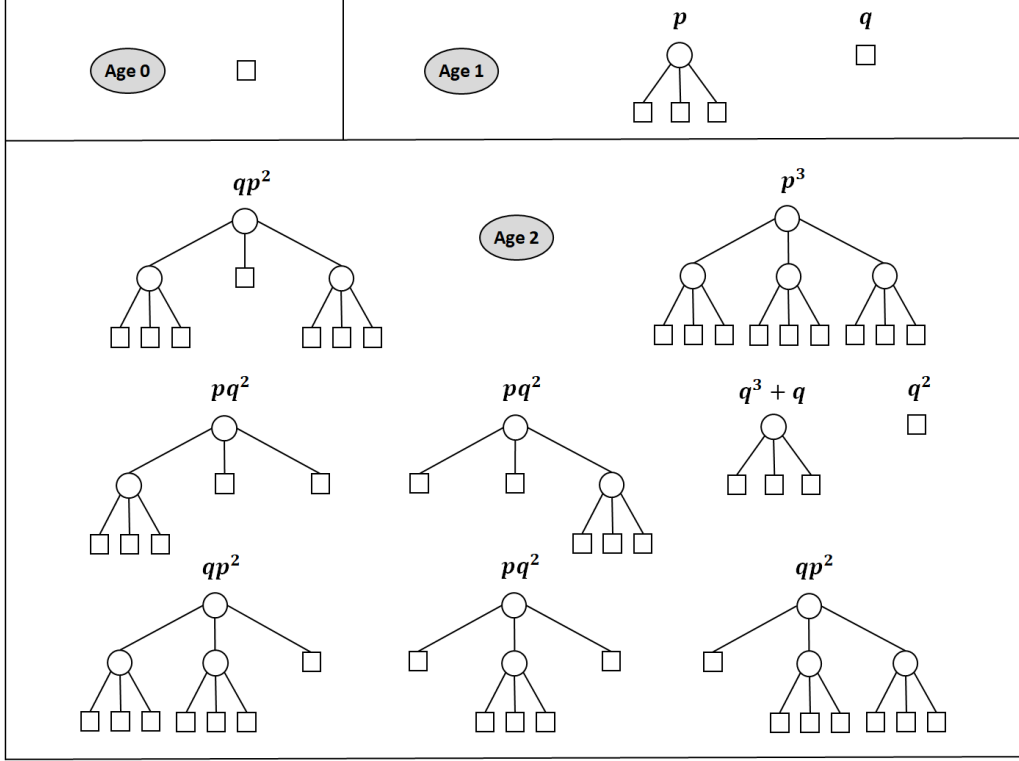


Figure 2: All exponential recursive ternary trees of index p and ages 0, 1 and 2. The probability of getting each tree appears above the root of that tree.

We use the recursive structure of the ERTTs under consideration in order to setup some recurrences for Wiener index and the number of saturated nodes. After one step of the evolution of an ERTT, the initial external node either succeeds to recruit a leaf (event \mathcal{R}) or not. We denote the indicator function of the event \mathcal{R} by \mathbb{I} . If the event \mathcal{R} does not occur, then after $n-1$ steps, the initial external node of the tree fathers an ERTT \tilde{T}_{n-1} . Alternatively, If \mathcal{R} occurs, the extended ERTT has an internal node with 3 external nodes. By time n , these 3 external nodes produce 3 independent ERTTs T'_{n-1} , T''_{n-1} and T'''_{n-1} . Now, we hook T'_{n-1} , T''_{n-1} and T'''_{n-1} to construct an ERTT of index p and age n . Let $(\tilde{X}_{n-1}, \tilde{Z}_{n-1})$, (X'_{n-1}, Z'_{n-1}) , (X''_{n-1}, Z''_{n-1}) and (X'''_{n-1}, Z'''_{n-1}) , be the pairs of the number of external nodes and the external path length in \tilde{T}_{n-1} , T'_{n-1} , T''_{n-1} and T'''_{n-1} , respectively. Then for an ERTT of index p at age n with the number of saturated nodes Y_n and the Wiener index W_n , we have

$$X_n \stackrel{d}{=} (1 - \mathbb{I})\tilde{X}_{n-1} + \mathbb{I}(X'_{n-1} + X''_{n-1} + X'''_{n-1}), \quad (1)$$

$$Z_n \stackrel{d}{=} (1 - \mathbb{I})\tilde{Z}_{n-1} + \mathbb{I}(Z'_{n-1} + Z''_{n-1} + Z'''_{n-1} + X'_{n-1} + X''_{n-1} + X'''_{n-1}), \quad (2)$$

where the symbol $\stackrel{d}{=}$ denotes the equality in distribution; the pairs $(\tilde{X}_{n-1}, \tilde{Z}_{n-1})$, (X'_{n-1}, Z'_{n-1}) , (X''_{n-1}, Z''_{n-1}) and (X'''_{n-1}, Z'''_{n-1}) are independent copies of (X_{n-1}, Z_{n-1}) and independent of \mathbb{I} . More-

over, the almost sure and L^2 convergence of the scaled X_n and Z_n have been shown in [6],

$$\frac{X_n}{(2p+1)^n} \xrightarrow{\text{a.s.}} X_*, \quad \frac{Z_n}{n(2p+1)^n} \xrightarrow{L^2 \text{ \& a.s.}} X_*, \quad \left(\mathbb{E}[X_*] = 1, \mathbb{E}[X_*^2] = \frac{3}{2p+1} \right), \quad (3)$$

as $n \rightarrow \infty$, for a limiting variable X_* characterized by inductively-constructed moments.

To setup the recurrences for the number of saturated nodes and Wiener index, let T_n be an ERTT of index p at age n , $(\tilde{Y}_{n-1}, \tilde{W}_{n-1})$, (Y'_{n-1}, W'_{n-1}) , (Y''_{n-1}, W''_{n-1}) and (Y'''_{n-1}, W'''_{n-1}) be the pairs of the number of saturated nodes and the Wiener index in \tilde{T}_{n-1} , T'_{n-1} , T''_{n-1} and T'''_{n-1} , respectively. Let \mathbb{J}_{n-1} , \mathbb{J}'_{n-1} , \mathbb{J}''_{n-1} and \mathbb{J}'''_{n-1} be the indicators of the events that, after $n-1$ extra steps, the initial external node succeeds to recruit a child in T_{n-1} , T'_{n-1} , T''_{n-1} and T'''_{n-1} , respectively. Thus for the tree T_n with the number of saturated nodes Y_n and the Wiener index W_n , by direct enumeration we obtain the recurrences

$$Y_n \stackrel{d}{=} (1 - \mathbb{I})\tilde{Y}_{n-1} + \mathbb{I}\left(Y'_{n-1} + Y''_{n-1} + Y'''_{n-1} + \mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1}\right), \quad (4)$$

$$W_n \stackrel{d}{=} (1 - \mathbb{I})\tilde{W}_{n-1} + \mathbb{I}\left(W'_{n-1} + W''_{n-1} + W'''_{n-1} + 2X'_{n-1}X''_{n-1} + 2X'_{n-1}X'''_{n-1} + 2X''_{n-1}X'''_{n-1} + Z'_{n-1}X''_{n-1} + Z''_{n-1}X'_{n-1} + Z'_{n-1}X'''_{n-1} + Z'''_{n-1}X'_{n-1} + Z''_{n-1}X'''_{n-1} + Z'''_{n-1}X''_{n-1}\right), \quad (5)$$

where the triples $(Y'_{n-1}, \mathbb{J}'_{n-1}, W'_{n-1})$, $(Y''_{n-1}, \mathbb{J}''_{n-1}, W''_{n-1})$ and $(Y'''_{n-1}, \mathbb{J}'''_{n-1}, W'''_{n-1})$ are independent copies of $(Y_{n-1}, \mathbb{J}_{n-1}, W_{n-1})$ and independent of \mathbb{I} .

In Section 2, we calculate the first two moments of Y_n and W_n . If S_n be the size of be an ERTT of index p at age n , in Section 3 and Section 4, we use the martingale convergence theorem and contraction method to show the following two types of convergence for W_n and Y_n , respectively,

$$\frac{\mathbb{E}[Y_n]}{\mathbb{E}[S_n]} \leq \frac{1}{3}, \quad \frac{Y_n}{(2p+1)^n} \xrightarrow{d} \hat{Y}, \quad \frac{W_n}{n(2p+1)^{2n}} \xrightarrow{L^2 \text{ \& a.s.}} \frac{3p}{2p+1} X_*^2,$$

where the random variable \hat{Y} is the unique distributional fixed-point of a distributional recurrence.

2 The first two moments

In this section, we calculate the first two moments of Y_n and W_n by the equations (1), (2), (4) and (5). We need some results that the authors of [6] have found

$$\mathbb{E}[X_n] = (2p+1)^n, \quad \mathbb{E}[X_n^2] = 3(2p+1)^{2n-1} - 2(1-p)(2p+1)^{n-1}, \quad \mathbb{E}[Z_n] = 3pn(2p+1)^{n-1}.$$

Theorem 1. *Let Y_n be the number of saturated nodes in an ERTT of age n and index p . We have*

$$\mathbb{E}[Y_n] = \frac{(p^3 - p^2 + p + 2)(2p+1)^n}{2(4-p)(p^2 - 3p + 5)} - \frac{1}{2} + (1-p)^n - \frac{3(1-p)^{2n}}{4-p} + \frac{(1-p)^{3n}}{p^2 - 3p + 5}, \quad n \geq 1,$$

$$\mathbb{E}[Y_n^2] = \frac{3(p^3 - p^2 + p + 2)^2(2p+1)^{2n-1}}{4(4-p)^2(p^2 - 3p + 5)^2} + \mathcal{O}((2p+1)^n), \quad \text{as } n \rightarrow \infty.$$

Proof. Taking the expectation of the equation (4), one gets

$$\mathbb{E}[Y_n] = (2p + 1)\mathbb{E}[Y_{n-1}] + p(1 - (1 - p)^{n-1})^3,$$

with initial condition $\mathbb{E}[Y_0] = 0$. Iterating this recurrence we find

$$\mathbb{E}[Y_n] = \sum_{i=1}^n (2p + 1)^{n-i} p(1 - (1 - p)^{i-1})^3.$$

This sum yields the explicit solution as stated in the theorem.

Raise both sides of the distributional equation (4) to the second power and take expectations:

$$\begin{aligned} \mathbb{E}[Y_n^2] = \mathbb{E} \left[(1 - \mathbb{I})\tilde{Y}_{n-1}^2 + \mathbb{I} \left((Y'_{n-1})^2 + (Y''_{n-1})^2 + (Y'''_{n-1})^2 + \mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1} \right. \right. \\ \left. \left. + 2Y'_{n-1}Y''_{n-1} + 2Y'_{n-1}Y'''_{n-1} + 2Y''_{n-1}Y'''_{n-1} + 2(Y'_{n-1}\mathbb{J}'_{n-1})\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1} \right. \right. \\ \left. \left. + 2(Y''_{n-1}\mathbb{J}''_{n-1})\mathbb{J}'_{n-1}\mathbb{J}'''_{n-1} + 2(Y'''_{n-1}\mathbb{J}'''_{n-1})\mathbb{J}'_{n-1}\mathbb{J}''_{n-1} \right) \right]. \end{aligned}$$

where $(Y'_{n-1}, \mathbb{J}'_{n-1})$, $(Y''_{n-1}, \mathbb{J}''_{n-1})$ and $(Y'''_{n-1}, \mathbb{J}'''_{n-1})$ are independent copies of $(Y_{n-1}, \mathbb{J}_{n-1})$ and independent of \mathbb{I} . Then

$$\begin{aligned} \mathbb{E}[Y_n^2] = (2p + 1)\mathbb{E}[Y_{n-1}^2] + 6p(\mathbb{E}[Y_{n-1}])^2 \\ + 6p(1 - (1 - p)^{n-1})^2\mathbb{E}[Y_{n-1}\mathbb{J}_{n-1}] + p(1 - (1 - p)^{n-1})^3. \end{aligned}$$

Since $Y_n\mathbb{J}_n = Y_n$, therefore we have

$$\begin{aligned} \mathbb{E}[Y_n^2] &= \sum_{i=1}^n (2p + 1)^{n-i} p \left(6(\mathbb{E}[Y_{i-1}])^2 + 6(1 - (1 - p)^{i-1})^2\mathbb{E}[Y_{i-1}] + (1 - (1 - p)^{i-1})^3 \right) \\ &= \frac{3(p^3 - p^2 + p + 2)^2(2p + 1)^{2n-1}}{4(4 - p)^2(p^2 - 3p + 5)^2} + \frac{6(2 + p)((2p + 1)^n - (1 - p)^{5n})}{(p - 4)(p^2 - 3p + 5)(p^4 - 5p^3 + 10p^2 - 10p + 7)} \\ &+ \frac{3(p^3 - p^2 + p + 2)(p^5 - 4p^4 + 5p^3 + 8p^2 - 21p + 14)(2p + 1)^n}{(p - 2)(2p + 1)(p - 4)^2(p^2 - 3p + 5)^2(p^2 - 3p + 3)} - (1 - p)^n + \frac{6(1 - p)^{2n}}{4 - p} \\ &+ \frac{(3p^7 - 18p^6 + 23p^5 - 12p^4 + 49p^3 - 284p^2 + 234p + 194)(2p + 1)^n}{2(2p + 1)(p - 4)^2(p^2 - 3p + 5)^2} + \frac{1}{4} - \frac{5(1 - p)^{3n}}{p^2 - 3p + 5} \\ &+ \frac{18(1 - p)((2p + 1)^n - (1 - p)^{4n})}{(p^3 - 4p^2 + 6p - 6)(p - 4)^2} - \frac{6((2p + 1)^n - (1 - p)^{6n})}{(p^5 - 6p^4 + 15p^3 - 20p^2 + 15p - 8)(p^2 - 3p + 5)^2} \\ &+ \frac{3(2 + p)(p^3 - p^2 + p + 2)(1 - p)^{2n}(2p + 1)^{n-1}}{(2 - p)(p - 4)^2(p^2 - 3p + 3)(p^2 - 3p + 5)} + \frac{6(p^3 - p^2 + p + 2)(1 - p)^{3n}(2p + 1)^{n-1}}{(p - 4)(p^2 - 3p + 3)(p^2 - 3p + 5)^2}, \end{aligned}$$

where the calculation has been done by MAPLE software. \square

Let S_n be the size of be an ERTT of index p at age n , then by the result $\mathbb{E}[S_n] \sim \frac{(2p+1)^n}{2}$ in [6] and Theorem 1, we have

$$\frac{\mathbb{E}[Y_n]}{\mathbb{E}[S_n]} \leq \frac{1}{3}, \quad \left(\because \frac{p^3 - p^2 + p + 2}{(4 - p)(p^2 - 3p + 5)} \leq \frac{1}{3} \right).$$

Theorem 2. Let W_n be the wiener index in an ERTT of age n and index p . We have

$$\begin{aligned}\mathbb{E}[W_n] &= \frac{27p^2n^2(2p+1)^{2n-2}}{3np+2p+1} + \frac{9pn(2p+1)^{2n-1}}{2(3np+2p+1)} - \frac{3(2p+1)^{2n}}{2(3np+2p+1)} + \frac{3}{2}(2p+1)^{n-1}, \quad n \geq 1, \\ \mathbb{E}[W_n^2] &= \frac{324p^2(12p^2+31p+35)}{4(p+1)(4p^2+6p+3)} n^2(2p+1)^{4n-5} + \mathcal{O}(n(2p+1)^{4n}), \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Proof. Take expectations both sides of the equation (5). By the identical distribution of W'_n, W''_n, W'''_n and W_n ; and the independence of \mathbb{I} , we get

$$\begin{aligned}\mathbb{E}[W_n] &= (2p+1)\mathbb{E}[W_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}] + 6p(\mathbb{E}[X_{n-1}])^2 \\ &= (2p+1)\mathbb{E}[W_{n-1}] + 18p^2n(p+1)^{2n-3} + 6p(1-p)(p+1)^{2n-3},\end{aligned}$$

with $\mathbb{E}[W_0] = 0$. By iterating, we have

$$\mathbb{E}[W_n] = \sum_{i=1}^n (p+1)^{n-i} \left(18p^2i(2p+1)^{2i-3} + 6p(1-p)(2p+1)^{2i-3} \right),$$

that is obtained as claimed in the theorem.

For obtaining $\mathbb{E}[W_n^2]$, raise both sides of (5) to the second power and take expectations. The second moment recurrence takes the form,

$$\begin{aligned}\mathbb{E}[W_n^2] &= (2p+1)\mathbb{E}[W_{n-1}^2] + 6p\mathbb{E}[Z_{n-1}^2]\mathbb{E}[X_{n-1}^2] + 12p(\mathbb{E}[X_{n-1}^2])^2 + 24p\mathbb{E}[Z_{n-1}X_{n-1}](\mathbb{E}[X_{n-1}])^2 \\ &\quad + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}]\mathbb{E}[W_{n-1}] + 12p\mathbb{E}[W_{n-1}X_{n-1}]\mathbb{E}[Z_{n-1}] + 24p\mathbb{E}[W_{n-1}X_{n-1}]\mathbb{E}[X_{n-1}] \\ &\quad + 12p\mathbb{E}[W_{n-1}Z_{n-1}]\mathbb{E}[X_{n-1}] + 24p\mathbb{E}[Z_{n-1}X_{n-1}]\mathbb{E}[X_{n-1}^2] + 24p\mathbb{E}[X_{n-1}]\mathbb{E}[X_{n-1}^2]\mathbb{E}[Z_{n-1}] \\ &\quad + 6p(\mathbb{E}[Z_{n-1}X_{n-1}])^2 + 12p\mathbb{E}[W_{n-1}](\mathbb{E}[X_{n-1}])^2 + 6p(\mathbb{E}[W_{n-1}])^2 + 24p\mathbb{E}[X_{n-1}^2](\mathbb{E}[X_{n-1}])^2 \\ &\quad + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[Z_{n-1}^2](\mathbb{E}[X_{n-1}])^2 + 6p\mathbb{E}[X_{n-1}^2](\mathbb{E}[Z_{n-1}])^2. \quad (6)\end{aligned}$$

In the recurrence (6), we need the expectations $\mathbb{E}[Z_n^2]$, $\mathbb{E}[W_nZ_n]$, $\mathbb{E}[W_nX_n]$ and $\mathbb{E}[Z_nX_n]$.

From (1), (2) and (5), we have the following recurrences, for $n \geq 1$,

$$\begin{aligned}\mathbb{E}[Z_nX_n] &= (2p+1)\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}] + 3p\mathbb{E}[X_{n-1}^2] + 6p(\mathbb{E}[X_{n-1}])^2 \\ \mathbb{E}[Z_n^2] &= (2p+1)\mathbb{E}[Z_{n-1}^2] + 3p\mathbb{E}[X_{n-1}^2] + 6p(\mathbb{E}[Z_{n-1}])^2 + 6p(\mathbb{E}[X_{n-1}])^2 \\ &\quad + 6p\mathbb{E}[Z_{n-1}X_{n-1}] + 12p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}] \\ \mathbb{E}[W_nX_n] &= (2p+1)\mathbb{E}[W_{n-1}X_{n-1}] + 6p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[Z_{n-1}](\mathbb{E}[X_{n-1}])^2 \\ &\quad + 6p\mathbb{E}[W_{n-1}]\mathbb{E}[X_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}^2] + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[X_{n-1}^2] + 6p(\mathbb{E}[X_{n-1}])^3 \\ \mathbb{E}[W_nZ_n] &= (2p+1)\mathbb{E}[W_{n-1}Z_{n-1}] + 18p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 3p\mathbb{E}[W_{n-1}X_{n-1}] + 6p(\mathbb{E}[X_{n-1}])^3 \\ &\quad + 6p\mathbb{E}[Z_{n-1}^2]\mathbb{E}[X_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[W_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[X_{n-1}](\mathbb{E}[Z_{n-1}])^2 \\ &\quad + 6p\mathbb{E}[W_{n-1}]\mathbb{E}[X_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}^2] + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[X_{n-1}^2] + 12p\mathbb{E}[Z_{n-1}](\mathbb{E}[X_{n-1}])^2,\end{aligned}$$

with initial conditions $\mathbb{E}[Z_0^2] = 0$, $\mathbb{E}[W_0 Z_0] = 0$, $\mathbb{E}[W_0 X_0] = 0$ and $\mathbb{E}[Z_0 X_0] = 0$.

The above recurrence equations are of the form $y_n = (p+1)y_{n-1} + h_n$, with solution

$$y_n = \sum_{i=1}^n (p+1)^{n-i} h_i. \quad (7)$$

Therefore, after some calculations, their solutions are obtained as follows

$$\begin{aligned} \mathbb{E}[Z_n X_n] &= 9pn(2p+1)^{2n-2} + 3(1-p)(2p+1)^{2n-2} - 6p(1-p)n(2p+1)^{n-2} - 3(1-p)(2p+1)^{n-2}, \\ \mathbb{E}[Z_n^2] &= 27p^2 n^2 (2p+1)^{2n-3} + 8p(1-p)n(2p+1)^{2n-3} + \frac{3}{2}(1-p)(5-2p)(2p+1)^{2n-3} \\ &\quad - 18(1-p)p^2 n^2 (2p+1)^{n-3} - 6p(1-p)(4-p)n(2p+1)^{n-3} - \frac{3}{2}(1-p)(5-2p)(2p+1)^{n-3}, \\ \mathbb{E}[W_n X_n] &= \frac{9p(p+5)}{p+1} n(2p+1)^{3n-3} - \frac{3(2p^3+22p^2+13p-1)}{2(p+1)^2} (2p+1)^{3n-3} - 36p(1-p)n(2p+1)^{2n-3} \\ &\quad - \frac{3}{2}(8p^2-16p-1)(2p+1)^{2n-3} - \frac{3(1-p)(4p^2+3p+2)}{2(p+1)^2} (2p+1)^{n-2}, \\ \mathbb{E}[W_n Z_n] &= \frac{27p^2(p+5)}{p+1} n^2(2p+1)^{3n-4} - \frac{9p(4p^3+38p^2+11p-17)}{2(p+2)^2} n(2p+1)^{3n-4} \\ &\quad - \frac{3(1-p)(4p^3+62p^2+17p-20)}{4(p+2)^2} (2p+1)^{3n-4} - 108p^2(1-p)n^2(2p+1)^{2n-4} \\ &\quad - \frac{9}{2}p(20p^2-52p+23)n(2p+1)^{2n-4} - 3(1-p)(10p^2-20p+1)(2p+1)^{2n-4} \\ &\quad - \frac{9p(1-p)(4p^2+3p+2)}{2(p+2)^2} n(2p+1)^{n-3} + \frac{3(1-p)(20p^3-8p^2-23p-16)}{4(p+2)^2} (2p+1)^{n-3}. \end{aligned}$$

Again, according to (7), the solution to the recurrence (6) is

$$\begin{aligned} \mathbb{E}[W_n^2] &= \frac{324p^2(12p^2+31p+35)}{4(p+1)(4p^2+6p+3)} n^2(2p+1)^{4n-5} \\ &\quad - \frac{27p(176p^6+880p^5+2056p^4+2222p^3+927p^2-66p-111)}{(p+1)^2(4p^2+6p+3)^2} n(2p+1)^{4n-5} + \mathcal{O}((2p+1)^{4n}), \end{aligned}$$

as $n \rightarrow \infty$. Hence we obtain the assertion. \square

3 Almost sure convergence for the external Wiener index

In T_n , an ERTT of age n and index p , assume we numbered the external nodes of the tree arbitrarily with the numbers $1, 2, \dots, X_n$. Let $D_{i,j}$ be the distance between external nodes i and j , for $i, j \in \{1, 2, \dots, X_n\}$. The external Wiener index of T_n is

$$W_n = \sum_{i=1}^{X_n} \sum_{\substack{j=1 \\ i < j}}^{X_n} D_{i,j}.$$

Theorem 3. Let W_n be the external Wiener index in an ERTT of age n and index p . As $n \rightarrow \infty$, we have almost sure convergence and in L^2 ,

$$\frac{W_n}{n(2p+1)^{2n}} \xrightarrow{L^2 \text{ \& a.s.}} \frac{3p}{2p+1} X_*^2.$$

Proof. Let \mathbb{I}_k be a sequence of independent Bernoulli(p) random variables, defined on the same probability space as the trees. Consider $\{\mathbb{I}_k\}_{k=1}^\infty$ to be independent of the structure of the tree at any age, too. If each of two external nodes i and j or both succeed in attracting a new node, then by direct enumeration, the contribution of the external nodes i and j of T_{n-1} into W_n , is

$$9\mathbb{I}_i\mathbb{I}_j(D_{i,j}+2)D_{i,j}\mathbb{I}_{i<j} + 3\mathbb{I}_i(1-\mathbb{I}_j)(D_{i,j}+1)\mathbb{I}_{i\neq j} + 6\mathbb{I}_i - \mathbb{I}_i\mathbb{I}_jD_{i,j}\mathbb{I}_{i<j} - \mathbb{I}_i(1-\mathbb{I}_j)D_{i,j}$$

for $i, j \in \{1, 2, \dots, X_{n-1}\}$, and we write

$$\begin{aligned} W_n &= W_{n-1} - \sum_{i=1}^{X_{n-1}} \sum_{\substack{j=1 \\ i \neq j}}^{X_{n-1}} \mathbb{I}_i(1-\mathbb{I}_j)D_{i,j} - \sum_{i=1}^{X_{n-1}} \sum_{\substack{j=1 \\ i < j}}^{X_{n-1}} \mathbb{I}_i\mathbb{I}_jD_{i,j} + 6 \sum_{i=1}^{X_{n-1}} \mathbb{I}_i \\ &+ 3 \sum_{i=1}^{X_{n-1}} \sum_{\substack{j=1 \\ i \neq j}}^{X_{n-1}} \mathbb{I}_i(1-\mathbb{I}_j)(D_{i,j}+1) + 9 \sum_{i=1}^{X_{n-1}} \sum_{\substack{j=1 \\ i < j}}^{X_{n-1}} \mathbb{I}_i\mathbb{I}_j(D_{i,j}+2). \end{aligned}$$

Let \mathcal{F}_n be the sigma field generated by the first n steps of the evolution of T_n . Then

$$\begin{aligned} \mathbb{E}[W_n|\mathcal{F}_{n-1}] &= W_{n-1} + 4p(1-p)W_{n-1} + 8p^2W_{n-1} + 6pX_{n-1} + 3p(2p+1)X_{n-1}(X_{n-1}-1) \\ &= (2p+1)^2W_{n-1} + 3p(2p+1)X_{n-1}^2 - 3p(2p-1)X_{n-1}. \end{aligned} \quad (8)$$

Let $M_n := \alpha_n W_n + \beta_n X_n^2 + \gamma_n X_n$, for specified factors α_n , β_n and γ_n that transformed M_n a martingale. So, by (8) and a martingalization procedure, we have

$$\begin{aligned} \mathbb{E}[M_n|\mathcal{F}_{n-1}] &= \alpha_n(2p+1)^2W_{n-1} + 3\alpha_np(2p+1)X_{n-1}^2 - 3\alpha_np(2p-1)X_{n-1} \\ &+ \beta_n\mathbb{E}[X_n^2|\mathcal{F}_{n-1}] + \gamma_n\mathbb{E}[X_n|\mathcal{F}_{n-1}] \\ &= \alpha_n(2p+1)^2W_{n-1} + 3\alpha_np(2p+1)X_{n-1}^2 - 3\alpha_np(2p-1)X_{n-1} \\ &+ \beta_n(2p+1)^2X_{n-1}^2 + 4\beta_np(1-p)X_{n-1} + \gamma_n(2p+1)X_{n-1} \\ &= \alpha_n(2p+1)^2W_{n-1} + (3\alpha_np(2p+1) + \beta_n(2p+1)^2)X_{n-1}^2 \\ &+ (\gamma_n(2p+1) + 4\beta_np(1-p) - 3\alpha_np(2p-1))X_{n-1} \\ &= M_{n-1} = \alpha_{n-1}W_{n-1} + \beta_{n-1}X_{n-1}^2 + \gamma_{n-1}X_{n-1}. \end{aligned}$$

This is possible, if we choose

$$\begin{aligned} \alpha_{n-1} &= \alpha_n(2p+1)^2, \\ \beta_{n-1} &= 3\alpha_np(2p+1) + \beta_n(2p+1)^2 \\ \gamma_{n-1} &= \gamma_n(2p+1) + 4\beta_np(1-p) - 3\alpha_np(2p-1). \end{aligned}$$

From these solvable recurrences, we obtain

$$\alpha_n = \frac{\alpha_0}{(2p+1)^{2n}}, \quad \beta_n = \frac{\beta_0}{(2p+1)^{2n}} - \frac{3\alpha_0 p n}{(2p+1)^{2n+1}}$$

$$\gamma_n = \frac{3\alpha_0 - 4(1-p)\beta_0 + 2(2p+1)\gamma_0}{2(2p+1)^{n+1}} - \frac{6p(1-p)\alpha_0 n}{(2p+1)^{2n+2}} - \frac{3\alpha_0 - 4(1-p)\beta_0}{2(2p+1)^{2n+1}}.$$

For simplicity, we take $\alpha_0 = 1$ and $\beta_0 = 0 = \gamma_0$. Thus,

$$M_n = \frac{1}{(2p+1)^{2n}} W_n - \frac{3pn}{(2p+1)^{2n+1}} X_n^2 + \left(\frac{3}{2(2p+1)^{n+1}} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{2n+2}} \right) X_n,$$

is a martingale with bounded moments

$$\begin{aligned} \mathbb{E}[M_n] &= \frac{\mathbb{E}[W_n]}{(2p+1)^{2n}} - \frac{3pn\mathbb{E}[X_n^2]}{(2p+1)^{2n+1}} + \left(\frac{3}{2(2p+1)^{n+1}} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{2n+2}} \right) \mathbb{E}[X_n] \\ &= \frac{27p^2 n^2 (2p+1)^{-2}}{3np + 2p + 1} + \frac{9pn(2p+1)^{-1}}{2(3np + 2p + 1)} - \frac{3}{2(3np + 2p + 1)} + \frac{3}{2(2p+1)^{n+1}} \\ &\quad - 3pn \left(\frac{3}{(2p+1)^2} - \frac{2(1-p)}{(2p+1)^{n+2}} \right) + \frac{3}{2(2p+1)} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{n+2}} = 0, \\ \mathbb{E}[M_n^2] &= \frac{\mathbb{E}[W_n^2]}{(2p+1)^{4n}} + \frac{9p^2 n^2 \mathbb{E}[X_n^4]}{(2p+1)^{4n+2}} + \left(\frac{3}{2(2p+1)^{n+1}} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{2n+2}} \right)^2 \mathbb{E}[X_n^2] \\ &\quad + \left(\frac{3}{(2p+1)^{3n+1}} - \frac{12p(1-p)n + 3(2p+1)}{(2p+1)^{4n+2}} \right) \left(\mathbb{E}[W_n X_n] - \frac{3pn\mathbb{E}[X_n^3]}{2p+1} \right) - \frac{6pn\mathbb{E}[W_n X_n^2]}{(2p+1)^{4n+1}} \\ &= \frac{1}{(2p+1)^{4n}} \mathbb{E}[W_n^2] + \frac{9p^2 n^2}{(2p+1)^{4n+2}} \mathbb{E}[X_n^4] - \frac{6pn}{(2p+1)^{4n+1}} \mathbb{E}[W_n X_n^2] + \mathcal{O}(1) \\ &= \frac{1}{(2p+1)^{4n}} \left(\frac{324p^2(12p^2 + 31p + 35)}{4(p+1)(4p^2 + 6p + 3)} n^2 (2p+1)^{4n-5} \right. \\ &\quad \left. - \frac{27p(176p^6 + 880p^5 + 2056p^4 + 2222p^3 + 927p^2 - 66p - 111)n(2p+1)^{4n-5}}{(p+1)^2(4p^2 + 6p + 3)^2} \right) \\ &\quad + \frac{9p^2 n^2}{(2p+1)^{4n+2}} \left(\frac{9(12p^2 + 31p + 35)(2p+1)^{4n-3}}{(p+1)(4p^2 + 6p + 3)} \right) \\ &\quad - \frac{6pn}{(2p+1)^{4n+1}} \left(\frac{27p(12p^2 + 31p + 35)n(2p+1)^{4n-4}}{(4p^2 + 6p + 3)(p+1)} \right. \\ &\quad \left. - \frac{9(176p^6 + 880p^5 + 2056p^4 + 2222p^3 + 927p^2 - 66p - 111)}{2(p+1)^2(4p^2 + 6p + 3)^2} (2p+1)^{4n-4} \right) + \mathcal{O}(1) \\ &= 0 + \mathcal{O}(1) < \infty, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where

$$\mathbb{E}[X_n^3] = \frac{3(p+5)(2p+1)^{3n-2}}{p+1} - 18(1-p)(2p+1)^{2n-2} + \frac{2(1-p)(2-p)(2p+1)^{n-1}}{p+1},$$

$$\begin{aligned}\mathbb{E}[X_n^4] &= \frac{9(12p^2 + 31p + 35)(2p + 1)^{4n-3}}{(p + 1)(4p^2 + 6p + 3)} - \frac{36(p + 5)(1 - p)(2p + 1)^{3n-3}}{p + 1} \\ &\quad - \frac{12(1 - p)(7p^2 - 6p - 7)(2p + 1)^{2n-3}}{p + 1} - \frac{8(1 - p)(2p^4 - 12p^3 + 7p^2 + 6p + 3)(2p + 1)^{n-2}}{(p + 1)(4p^2 + 6p + 3)},\end{aligned}$$

$$\begin{aligned}\mathbb{E}[W_n X_n^2] &= \frac{27p(12p^2 + 31p + 35)n(2p + 1)^{4n-4}}{(4p^2 + 6p + 3)(p + 1)} \\ &\quad - \frac{9(176p^6 + 880p^5 + 2056p^4 + 2222p^3 + 927p^2 - 66p - 111)}{2(p + 1)^2(4p^2 + 6p + 3)^2} (2p + 1)^{4n-4} \\ &\quad + \frac{3(60p^4 + 300p^3 - 60p^2 - 300p)n(2p + 1)^{3n-4}}{2(p + 1)^2} + \mathcal{O}((2p + 1)^{3n}), \quad \text{as } n \rightarrow \infty.\end{aligned}$$

So, by the martingale convergence theorem (Theorem 6.6.9 in [2]), M_n converges to some M_* almost surely and in L^2 . Consequently,

$$\begin{aligned}\frac{M_n}{n} &= \frac{W_n}{n(2p + 1)^{2n}} - \frac{3p}{2p + 1} \cdot \left(\frac{X_n}{(2p + 1)^n} \right)^2 \\ &\quad + \left(\frac{3}{2(2p + 1)} - \frac{12p(1 - p)n + 3(2p + 1)}{2(2p + 1)^{n+2}} \right) \cdot \frac{X_n}{n(2p + 1)^n} \xrightarrow{L^2 \text{ \& a.s.}} 0.\end{aligned}$$

Hence, by (3), the result follows. \square

Corollary 4. *Let D_n^* be the distance between two randomly chosen nodes in a random ERTT of age n and index p . Then we have*

$$\frac{\mathbb{E}[D_n^*]}{n} \longrightarrow \frac{6p}{2p + 1}.$$

Proof. In T_n , a random ERTT of age n and index p , we have

$$\mathbb{E}[D_n^* | T_n] = \frac{2 \sum_{i=1}^{X_n} \sum_{\substack{j=1 \\ i < j}}^{X_n} D_{i,j}}{X_n(X_n - 1)} = \frac{2W_n}{X_n(X_n - 1)}.$$

Therefore, by (3) and Theorem 3, we have

$$\frac{\mathbb{E}[D_n^* | T_n]}{n} = \frac{2W_n / (n(2p + 1)^{2n})}{X_n(X_n - 1) / ((2p + 1)^{2n})} \xrightarrow{\text{a.s.}} \frac{6p}{2p + 1}.$$

Since $W_n \leq X_n(X_n - 1)n$, then $\mathbb{E}[D_n^* | T_n] / n \leq 2$. Consequently, by dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[D_n^*]}{n} = \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{\mathbb{E}[D_n^* | T_n]}{n} \right] = \mathbb{E} \left[\lim_{n \rightarrow \infty} \frac{\mathbb{E}[D_n^* | T_n]}{n} \right].$$

This yields the result. \square

4 L^2 convergence for the number of saturated nodes

In this section we apply the contraction method in [18] or the multivariate contraction method in [16] to state the limiting distribution of $\hat{Y}_n := \frac{Y_n}{(2p+1)^n}$.

We first need to set up some notation as follows: For a random variable X , we write $X \sim \lambda$ if the distribution of X is λ , i.e. the law $\mathcal{L}(X)$ of X is λ . The symbol $\|X\|_2 := (\mathbb{E}[|X|^2])^{1/2}$ denotes the usual L_2 -norm of X . The Wasserstein-metric ℓ_2 is defined on the space of probability distributions with finite second moments by

$$\ell_2(X, Y) := \ell_2(\mathcal{L}(X), \mathcal{L}(Y)) := \ell_2(\lambda, \nu) := \inf\{\|X - Y\|_2 : X \sim \lambda, Y \sim \nu\}.$$

By \mathcal{M}_2 the space of all probability distributions λ with mean $\frac{p^3 - p^2 + p + 2}{2(4-p)(p^2 - 3p + 5)}$ (as in Theorem 1) and finite second moment is denoted. The metric space (\mathcal{M}_2, ℓ_2) is complete and convergence in ℓ_2 is equivalent to convergence in distribution plus convergence of the second moments.

By equation (4) we have

$$\begin{aligned} \frac{Y_n}{(2p+1)^n} &\stackrel{d}{=} \frac{\mathbb{I}}{2p+1} \left(\frac{Y'_{n-1}}{(2p+1)^{n-1}} + \frac{Y''_{n-1}}{(2p+1)^{n-1}} + \frac{Y'''_{n-1}}{(2p+1)^{n-1}} \right) \\ &\quad + \frac{1 - \mathbb{I}}{2p+1} \cdot \frac{\tilde{Y}_{n-1}}{(2p+1)^{n-1}} + \mathbb{I} \cdot \frac{\mathbb{J}'_{n-1} \mathbb{J}''_{n-1} \mathbb{J}'''_{n-1}}{(2p+1)^n}. \end{aligned} \quad (9)$$

Theorem 5. *Let Y_n be the saturated nodes of an ERTT of age n and index p . The normalized saturated nodes $\hat{Y}_n := \frac{Y_n}{(2p+1)^n}$ satisfies the distributional recursion*

$$\hat{Y}_n \stackrel{d}{=} \frac{\mathbb{I}}{2p+1} \cdot (\hat{Y}'_{n-1} + \hat{Y}''_{n-1} + \hat{Y}'''_{n-1}) + \frac{1 - \mathbb{I}}{2p+1} \cdot \bar{Y}_{n-1} + \mathbb{I} \cdot \frac{\mathbb{J}'_{n-1} \mathbb{J}''_{n-1} \mathbb{J}'''_{n-1}}{(2p+1)^n}, \quad (10)$$

where $\mathbb{J}'_{n-1} \stackrel{d}{=} \mathbb{J}''_{n-1} \stackrel{d}{=} \mathbb{J}'''_{n-1} \stackrel{d}{=} \text{Bernoulli}(1 - (1-p)^{n-1})$, \bar{Y}_n , $(\hat{Y}'_n, \mathbb{J}'_{n-1})$, $(\hat{Y}''_n, \mathbb{J}''_{n-1})$, $(\hat{Y}'''_n, \mathbb{J}'''_{n-1})$ and \mathbb{I} are independent, $\bar{Y}_n \stackrel{d}{=} \hat{Y}'_n \stackrel{d}{=} \hat{Y}''_n \stackrel{d}{=} \hat{Y}'''_n \stackrel{d}{=} \hat{Y}_n$, and \mathbb{I} is a Bernoulli random variable with success probability p , then $\hat{Y}_n \xrightarrow{d} \hat{Y}$, as $n \rightarrow \infty$, where the random variable \hat{Y} is the unique distributional fixed-point of

$$\hat{Y} \stackrel{d}{=} \frac{\mathbb{I}}{2p+1} (\hat{Y}' + \hat{Y}'' + \hat{Y}''') + \frac{1 - \mathbb{I}}{2p+1} \hat{Y}, \quad (11)$$

with \hat{Y}' , \hat{Y}'' and \hat{Y}''' are independent copies of \hat{Y} and independent of \mathbb{I} .

Proof. The equation (10) is an immediate consequence of the equation (9), where $\bar{Y}_n := \frac{\tilde{Y}_n}{(2p+1)^n}$, $\hat{Y}'_n := \frac{Y'_n}{(2p+1)^n}$, $\hat{Y}''_n := \frac{Y''_n}{(2p+1)^n}$ and $\hat{Y}'''_n := \frac{Y'''_n}{(2p+1)^n}$. Using Chebyshev's inequality and $\mathbb{E}[\mathbb{J}''_{n-1}] = \mathbb{E}[\mathbb{J}'''_{n-1}] = 1 - (1-p)^{n-1}$, we have the convergence in probability $\frac{\mathbb{J}'_{n-1} \mathbb{J}''_{n-1} \mathbb{J}'''_{n-1}}{(2p+1)^n} \xrightarrow{P} 0$. That is,

$$\hat{Y}_n \stackrel{d}{=} \frac{\mathbb{I}}{2p+1} \cdot (\hat{Y}'_{n-1} + \hat{Y}''_{n-1} + \hat{Y}'''_{n-1}) + \frac{1 - \mathbb{I}}{2p+1} \cdot \hat{Y}_{n-1} + o_p(1), \quad (12)$$

where the notation $o_p(1)$ will denote an asymptotically negligible random variable that converges to 0 in probability. For purposes of convergence we can, and hence will, ignore the $o_p(1)$ term.

Consider the well-defined transformation $T : \mathcal{M}_2 \rightarrow \mathcal{M}_2$,

$$T(\lambda) := \mathcal{L}\left(\frac{\mathbb{I}}{2p+1}(\hat{Y}'_\lambda + \hat{Y}''_\lambda + \hat{Y}'''_\lambda) + \frac{1-\mathbb{I}}{2p+1}\hat{Y}_\lambda\right), \quad (13)$$

where $\hat{Y}_\lambda \stackrel{d}{=} \hat{Y}$, $\hat{Y}'_\lambda \stackrel{d}{=} \hat{Y}'$, $\hat{Y}''_\lambda \stackrel{d}{=} \hat{Y}''$ and $\hat{Y}'''_\lambda \stackrel{d}{=} \hat{Y}'''$ have λ as distribution and are independent of \mathbb{I} .

At a first step, we have to prove that the transformation T has a unique fixed point with respect to the ℓ_2 -metric. Let $\lambda, \nu \in \mathcal{M}_2$ be given. By (13), we have

$$\begin{aligned} T(\lambda) &= \mathcal{L}\left(\frac{\mathbb{I}}{2p+1}(\hat{Y}'_\lambda + \hat{Y}''_\lambda + \hat{Y}'''_\lambda) + \frac{1-\mathbb{I}}{2p+1}\hat{Y}_\lambda\right), \\ T(\nu) &= \mathcal{L}\left(\frac{\mathbb{I}}{2p+1}(\hat{Y}'_\nu + \hat{Y}''_\nu + \hat{Y}'''_\nu) + \frac{1-\mathbb{I}}{2p+1}\hat{Y}_\nu\right), \end{aligned}$$

and

$$\begin{aligned} \ell_2^2(T(\lambda), T(\nu)) &\leq 3 \frac{\mathbb{E}[\mathbb{I}^2]}{(2p+1)^2} \mathbb{E}[|\hat{Y}_\lambda - \hat{Y}_\nu|^2] + \frac{\mathbb{E}[(1-\mathbb{I})^2]}{(2p+1)^2} \mathbb{E}[|\hat{Y}_\lambda - \hat{Y}_\nu|^2] \\ &= \frac{1}{2p+1} \mathbb{E}[|\hat{Y}_\lambda - \hat{Y}_\nu|^2]. \end{aligned}$$

Therefore, we have

$$\ell_2^2(T(\lambda), T(\nu)) \leq \frac{1}{2p+1} \ell_2^2(\lambda, \nu).$$

Since $\frac{1}{2p+1} < 1$, we deduce that T is a contraction with respect to the ℓ_2 -metric. Thus, Banach's fixed point theorem provides existence and uniqueness of solutions of the fixed point equation $T(\lambda) = \lambda$. By (12), if \hat{Y}_n converges in distribution to some random variable \hat{Y} , then \hat{Y} satisfies (11). Consequently, the distribution of \hat{Y} will be λ_0 , the unique fixed point of T , i.e. $\mathcal{L}(\hat{Y}) = T(\lambda_0) = \lambda_0$.

Therefore, we have to prove $\lim_{n \rightarrow \infty} \ell_2(\hat{Y}_n, \hat{Y}) = 0$ to conclude $\hat{Y}_n \xrightarrow{d} \hat{Y}$.

Since $\hat{Y}'_n \stackrel{d}{=} \hat{Y}'_n \stackrel{d}{=} \hat{Y}'''_n \stackrel{d}{=} \hat{Y}_n$ and $\hat{Y}' \stackrel{d}{=} \hat{Y}'' \stackrel{d}{=} \hat{Y}''' \stackrel{d}{=} \hat{Y}$, we deduce

$$\begin{aligned} \lim_{n \rightarrow \infty} \ell_2^2(\hat{Y}_n, \hat{Y}) &\leq \frac{p}{(2p+1)^2} \lim_{n \rightarrow \infty} \left(\|\hat{Y}'_{n-1} - \hat{Y}'\|_2^2 + \|\hat{Y}''_{n-1} - \hat{Y}''\|_2^2 + \|\hat{Y}'''_{n-1} - \hat{Y}'''\|_2^2 \right) \\ &\quad + \frac{1-p}{(2p+1)^2} \lim_{n \rightarrow \infty} \|\hat{Y}_{n-1} - \hat{Y}\|_2^2 \\ &\leq \frac{1}{2p+1} \lim_{n \rightarrow \infty} \|\hat{Y}_n - \hat{Y}\|_2. \end{aligned}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \ell_2^2(\hat{Y}_n, \hat{Y}) \leq \frac{1}{2p+1} \lim_{n \rightarrow \infty} \ell_2^2(\hat{Y}_n, \hat{Y}).$$

This is true only if $\lim_{n \rightarrow \infty} \ell_2(\hat{Y}_n, \hat{Y}) = 0$. □

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