



HAL
open science

Estimation of Sobol indices with a multi-element polynomial chaos model: application to Flight Dynamics

Nicolas Vauchel, Eric Garnier, Thomas Gomez

► To cite this version:

Nicolas Vauchel, Eric Garnier, Thomas Gomez. Estimation of Sobol indices with a multi-element polynomial chaos model: application to Flight Dynamics. 25ème Congrès Français de Mécanique, Aug 2022, NANTES, France. hal-03932174

HAL Id: hal-03932174

<https://hal.science/hal-03932174v1>

Submitted on 10 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Estimation of Sobol indices with a multi-element polynomial chaos model : application to Flight Dynamics

N. VAUCHEL^{a*}, E. GARNIER^a, T. GOMEZ^a

a. Univ. Lille, CNRS, ONERA, Arts et Metiers Institute of Technology, Centrale Lille, UMR 9014 - LMFL - Laboratoire de Mécanique des fluides de Lille - Kampé de Fériet

* nicolas.vauchel@onera.fr

Abstract :

Flight Dynamics aims at studying the motion of an aircraft modelled as a solid. The equations of motion are gathered in a dynamical system of coupled Ordinary Differential Equations. Classically, behaviours of the aircraft are linked to equilibria of this system, and bifurcation theory is used. In the dynamical system, the forces and the moments from the fluid on the aircraft rely on models with a chosen structure, constructed with data coming from experiments. These models are therefore uncertain. It had be shown that the uncertainties of these models can have influence on some equilibria, notably on their stability. In this paper, a joint Uncertainty Quantification and Global Sensitivity Analysis study is conducted on the temporal evolution of a given predicted behaviour of a light aircraft. As different combinations of the uncertain parameters can lead to instability of the behaviour and other combinations to stability, the state variables of the dynamical system can present some irregularities or some discontinuities with respect to the uncertain parameters at a given instant. In this paper, a method to estimate Sobol indices of a solution of an Ordinary Differential Equation approximated with a multi-element polynomial chaos model is therefore used. It relies on two upcoming papers. The first paper presents a method relying on Machine learning methods to get a multi-element polynomial chaos model for a non-time dependent Quantity of Interest. The second one extends the analytical formula between the Sobol indices and the coefficients of a polynomial chaos surrogate model to the case of a multi-element polynomial chaos model. In the present paper, the time-dependent extension of the multi-element model is introduced. The analytical formula permits to find Sobol indices from the coefficients of the local polynomial models at every considered instants. Conversely to standard polynomial chaos models, the use of a multi-element model permits to accurately approximate solutions of Ordinary Differential Equation with bifurcations, being irregular or discontinuous with respect to the input parameters at some instants. This approach is then used to conduct the sensitivity study on the state variables of an aircraft behaviour predicted by the Flight Dynamics system.

Key words : Uncertainty Quantification, Sobol indices, Multi-element polynomial chaos, Flight Dynamics, Machine learning.

1 Introduction

Global Sensitivity Analysis (GSA) aims at finding the impact of uncertain parameters on a Quantity of Interest (QOI). In the present study, the chosen parameters are called the input parameters, are supposed to be random, independent and to follow a stochastic distribution of known probability density function (pdf). Sobol indices, relying on ANOVA decomposition [3], rank the actions of the input parameters and the interactions between them according to their impacts on the QOI. The indices also give a quantitative number reflecting these impacts. Sobol indices can be estimated with Monte-Carlo methods, but it usually requires a large number of simulations. Approximating the relation between the QOI and the input parameters using a surrogate model in the form of Polynomial Chaos (PC) model permits to establish an analytical link between the chaos coefficients and the Sobol indices [4, 5]. Once the model is learned, all the Sobol indices are directly available. However, PC models are not accurate in presence of irregularities or discontinuities. It may need a high truncation degree to achieve accuracy, which becomes problematic with the famous curse of dimensionality. Moreover, Gibbs phenomenon amplifies by increasing the degree of the model in presence of discontinuities.

Multi-elements (ME) models [6, 7, 8] were developed to address this issue. The abstract parameter space, containing all possible combinations the input parameters, is partitioned into subsets, called elements, where the QOI is locally smoother. Local PC models are trained on every element. The global model is then a piecewise PC model and contains a method to determine which local model to use when an evaluation is needed. However, with this piecewise model, the classical analytical formula with Sobol indices and PC models does not hold anymore, as the formula relies on orthogonal properties on all the parameter space. In an upcoming paper [2], the analytical link between PC models and Sobol indices is extended in the case of ME models with non time-dependent QOI's.

In the present paper, the approach is extended to the case of a ME model approximating a time-dependent QOI, solution of an Ordinary Differential Equation (ODE). The time-dependent version of a new ME method, called ME-ACD, developed in another upcoming paper [1], is used, although any ME polynomial model able to handle time-dependent QOI's would have been appropriate. At every considered instants, the values of all the Sobol indices are obtained with the analytical formula linking them to the coefficients of the local models. The new approach is then able to give an estimation of the temporal evolution of Sobol indices with a relatively low number of runs, even if at a given instant, the QOI is irregular or discontinuous according to the input parameters (trading the presence of bifurcations).

The method is used in practical applications of Flight Dynamics. Flight Dynamics aims at studying the motion of an aircraft modelled as a solid. The forces and the moments of the fluid on the aircraft rely on a model with a chosen structure and come from data, which usually are data from wind-tunnel experiments. These models are therefore uncertain.

Traditionally, the bifurcation theory is used on the dynamical system and equilibria are linked to behaviours of the aircraft. At low angle of attack, equilibria are usually safe trajectories but in the post-stall domain, equilibria are usually dangerous behaviours, as spins. It has been shown in [12] that the uncertainties of the model of the aerodynamic forces and moments can have some influence on some predicted

spins, notably on their stability. For these behaviours, some combinations of the uncertain parameters can lead to instability and others to stability, which implies that, at a given instant, state variables can present irregularities and/or discontinuities with respect to these input parameters. The method presented in this paper is then suitable to conduct a joint Uncertainty Quantification (UQ) and GSA study on the temporal evolution of the state variables of this system.

2 Notations

The QOI is denoted \hat{y} . \hat{y} depends on $d \in \mathbb{N}^*$ input parameters $\mathbf{p} = (p_1, \dots, p_d)$. Every p_m follows a stochastic distribution law of known pdf ρ_m and all the input parameters are supposed independent. To stay close to the original definition of Sobol indices [3], every p_m is mapped to a corresponding x_m following an uniform law in the interval $[0, 1]$. The mapping takes into account the distribution law of the input parameter

$$x_m = \mathcal{C}_m(p_m) \quad (1)$$

where \mathcal{C}_m is the cumulative distribution function of p_m . If $\rho_m > 0$ (and not only $\rho_m \geq 0$), the mapping is bijective. This hypothesis is supposed verified in the scope of this study. From now on, "input parameters" denotes $\mathbf{x} = (x_1, \dots, x_d)$ and "QOI" denotes $y(\mathbf{x}) = \hat{y}(\mathbf{p})$.

GSA aims at ranking the impact of the actions and of the interactions between the input parameters on the QOI. A multidimensional indexing is used to denote these actions/interactions. The subscript $\mathbf{i} \in \mathbb{R}^j$, $j \in \mathbb{R}$ denotes all items referring to the interactions between the j input parameters x_{i_1}, \dots, x_{i_j} . $\mathcal{I}_j^{(d)}$ is the set containing all the possible indices \mathbf{i} indexing interactions of order j . It is the set of combinations without replacement of j elements of $[[1, d]]$. By convention, $\mathcal{I}_0^{(d)} = \{0\}$ contains the index denoting the part of the QOI which does not depend on any input parameter.

For a given vector $\mathbf{x} \in \mathbb{R}^d$, and a given $\mathbf{i} \in \mathcal{I}_j^{(d)}$, $\mathbf{x}_{\mathbf{i}} = (x_{i_1}, \dots, x_{i_j})$ is the vector composed of the components of \mathbf{x} indexed by the components of \mathbf{i} and $\mathbf{x}_{\sim \mathbf{i}}$ is the vector composed of the $d - j$ other components.

To denote the terms of multivariate polynomial series depending on d variables, another multidimensional indexing logic is used. $\mathbf{k} = (k_1, \dots, k_d)$ denotes the term being a coefficient times a product of d univariate polynomials of degree k_m depending of x_m . Series of polynomials of degree $\gamma \in \mathbb{R}$ are then series containing terms denoted by an index \mathbf{k} with $\sum_{m=1}^d k_m \leq \gamma$. The set containing all these possible indices is denoted $\mathcal{K}_\gamma^{(d)}$. It is the set of combinations with replacement of d elements of $[[1, \gamma]]$ following the previous inequality.

3 Sobol indices

In [3], the so-called ANOVA decomposition is introduced

$$y(\mathbf{x}) = \sum_{j=0}^d \sum_{\mathbf{i} \in \mathcal{I}_j^{(d)}} y_{\mathbf{i}}^{(j)}(\mathbf{x}_{\mathbf{i}}) \quad (2)$$

with

$$y_0^{(0)} = \int_{\mathcal{H}_d} y(\mathbf{x}) d\mathbf{x} \quad (3)$$

and, $\forall j \in [1, d]$, $\forall \mathbf{i} \in \mathcal{I}_j^{(d)}$, $\forall m \in \{i_1, \dots, i_d\}$

$$\int_0^1 y_{\mathbf{i}}^{(j)}(\mathbf{x}_{\mathbf{i}}) dx_m = 0 \quad (4)$$

The variance of the QOI is denoted D . With ANOVA decomposition (2), D can be decomposed as a sum of terms called "partial variances"

$$D = \sum_{j=1}^d \sum_{\mathbf{i} \in \mathcal{I}_j^{(d)}} D_{\mathbf{i}}^{(j)} \quad (5)$$

Every $D_{\mathbf{i}}^{(j)}$ represents the impact of interaction \mathbf{i} of order j on the total variance D . Sobol indices are then defined as

$$S_{\mathbf{i}}^{(j)} = \frac{D_{\mathbf{i}}^{(j)}}{D} \quad (6)$$

Partial variances can be estimated with Monte-Carlo methods [3], using random or pseudo-random samplings as Sobol sequences. To estimate all the first order partial variances, $(d + 1)N$ computations of the QOI are needed. To estimate all the second order partial variances, $\binom{d}{2}N$ additional computations of the QOI are needed and so on. To estimate all the partial variances, $2^d N$ computations are needed. In [3], a method to estimate the total Sobol indices (being the sum of all the sobol indices involving a given parameter appears) is also available.

4 Multi-element Polynomials Chaos models

First introduced by Wiener [9] to approximate random variables as series of Hermite polynomials depending on Gaussian variables, PC models were later extended in [10] as series of polynomials depending on random variables having another distribution and having orthogonal properties corresponding to this distribution. PC models' coefficients can be obtained with Galerkin methods or Non Intrusive (NI) methods [11]. In the scope of this article, only NI methods are considered. With NI methods, a given number of deterministic computations of the QOI are realised with different combinations of input parameters. These computations are used to get the chaos coefficients. In this paper, the computations are called "training observations". An observation can be seen as a point in an abstract space. The abstract space where the observations are seen as points of coordinates the values of the input parameters is called the parameter space \mathcal{H}_d . It is the unit hypercube of dimension d .

With the mapped input parameters, PC expansion is

$$y(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{K}_\gamma^{(d)}} \alpha_{\mathbf{k}} \Phi_{\mathbf{k}}(\mathbf{x}) \quad (7)$$

$$\int_{\mathcal{H}_d} \Phi_{\mathbf{a}}(\mathbf{x}) \Phi_{\mathbf{b}}(\mathbf{x}) d\mathbf{x} = \prod_{m=1}^d \delta_{a_m, b_m} \quad (8)$$

where $(\Phi_{\mathbf{k}})$ is the normalised shifted multivariate Legendre polynomials basis, products of univariate polynomials orthonormal with respect to the scalar product associated with the uniform law in $[0, 1]$. $\delta_{m,n}$ is the standard Kroenecker symbol.

An analytical formula [4, 5] links Sobol indices of a QOI with the coefficients of its PC model. Conversely to the Monte-Carlo method to get the Sobol indices, all the training observations are used to get the surrogate model and then all Sobol indices are available. A better exploration of the parameter space is therefore possible. However, PC models suffer from the curse of dimensionality, the number of terms of the chaos series being equal to $\binom{\gamma+d}{\gamma}$. More than else, accuracy of PC models deteriorates in presence of irregularities and/or discontinuities. ME methods permit to get piecewise polynomial models. The parameter space \mathcal{H}_d , is partitioned into N_E subsets E_e called elements where the QOI is locally smoother. ME-ACD, developed in an submitted paper [1], permits to get ME polynomials models expressed as

$$y(\mathbf{x}) = \sum_e^{N_E} \left[\sum_{\mathbf{k} \in \mathcal{K}_{\gamma_e}^{(d)}} c_{e,\mathbf{k}} \Psi_{\mathbf{k}}(\mathbf{x}) \right] \mathcal{I}_{E_e}(\mathbf{x}) \quad (9)$$

where $\Psi_{\mathbf{k}}(\mathbf{x}) = \prod_{m=1}^d x_m^{k_m}$ are the polynomials of the canonical basis of multivariate polynomials depending on d variables and \mathcal{I}_{E_e} is the indicator function returning 1 if $\mathbf{x} \in E_e$ or 0 otherwise.

With a non time-dependent QOI, ME-ACD algorithm uses agglomerative clustering on the training observations represented as points in a abstract space taking into account the values of the QOI and the values of its derivatives with respect to the input parameters. It permits to partition the parameter space in presence of discontinuities and irregularities with respect to the input parameters. The algorithm tests partitions having between $N_{E_{min}}$ and $N_{E_{max}}$ elements, and for all these partitions, local polynomial models are trained using Ordinary Least Square method with the training observations located in the elements. Local models of degree between n_{min} and n_{max} are tested, and the optimal degree is selected as the one minimising the predicted residual error sum of squares (PRESS) error. The partition is then selected as the one minimising the maximum of PRESS errors of the local models with optimal degree. This maximum is denoted below "partition error" δ_z . If, during the search, $\delta_z \leq \delta_{min}$, the partition is kept and there is no need to look for another partition with more elements. If during the search, an element has a number of training observations lower or equal than $N_{outlier}$, the observations are considered as outliers and are removed from the training set. The training is then conducted once again from the beginning. Computations of the derivatives are done with the use of OLS and first order Taylor theory on every training observations with its nearest neighbours in the space taking into account the values of the input parameters and the value of the QOI. Regressions with a number of neighbours from k_{min} to

k_{max} are tested and the one minimising the PRESS error is kept. During the agglomerative clustering, the training observations are connected with their $k_{connect}$ closest neighbours in the parameter space. Once the local models are obtained, a neural network classifier is trained to know which local model to use when an evaluation is needed. In the present paper, a brute force search on a set of learning rates is realised to find the one leading the best accuracy of the classifier on the training set itself.

In ME-ACD expansion (9), the canonical basis is used instead of a polynomial basis having orthogonal properties corresponding to the pdf of the input parameters like in classical PC models. The orthogonal properties need information on all the parameter space to be used, and not only on local elements. This explains why the formula linking standard PC models' coefficients to Sobol indices [4, 5] cannot be used in the case of a ME model. In a second upcoming paper [2], an analytical formula linking the coefficients of the local polynomial models of a ME expansion to partial variances is found. In order to establish this link, the elements are first approximated as an union of hyperrectangles. To find the set of hyperrectangles, the parameter space (the d -dimensional unit hypercube) is split in a grid of N_h hypercubes in every dimensions (thus being a grid of $(N_h)^d$ hypercubes in total). Every hypercube is then labelled with the index of the element it is located in with the use of the classifier of ME method on N_{random} points sampled in the hypercube. Once the hypercubes are labelled, a merging operation with identical hypercolumns is effectuated for every direction. This leads to $N_r \leq (N_h)^d$ hyperrectangles. For every hyperrectangle, a change of basis is realised for the local model located in this hyperrectangle from the canonical basis to a multivariate polynomial basis being orthonormal with respect to a scalar product proper to the bounds of the hyperrectangle. Therefore, the ME model is now a piecewise polynomial model over the hyperrectangles with polynomials having orthonormal properties proper to their corresponding local hyperrectangle. The coefficients in these new basis are then linked with partial variances with thanks to these orthonormal properties. This formula is therefore an extension of the analytical formula linking chaos coefficients of standard PC models to Sobol indices in the case of a ME model.

In the present paper, the QOI is supposed to be the solution of an ODE. It is then time-dependent. The ME model (9) is then extended

$$y(\mathbf{x}, t) = \sum_e^{N_E(t)} \left[\sum_{\mathbf{k} \in \mathcal{K}_{\gamma_e(t)}^{(d)}} c_{e,\mathbf{k}}(t) \Psi_{\mathbf{k}}(\mathbf{x}) \right] \mathcal{I}_{E_e(t)}(\mathbf{x}) \quad (10)$$

A discrete set of N_t instants $\{t_1, \dots, t_{N_t}\}$ (indexed by increasing order) is considered. Taking advantage of the potential similarities of the QOI from one instant to the next one, for every instant t_z with $z \geq 2$, the partition obtained at the previous instant t_{z-1} is kept and the local models corresponding to its elements are trained with the information of the QOI at instant z . The partition error δ_z is calculated. This error is then a metric of accuracy of the global piecewise model at instant t_z with the partition corresponding to the previous instant. If $\delta_z \leq \delta_{max}$, δ_{max} being a threshold to choose, the partition and the local models are kept for this instant. In this case, the partition is the same for t_{z-1} and t_z , but the coefficients of the local models differ. For the first instant or if $\delta_z > \delta_{max}$ for $z > 1$, a new suitable partition is obtained with the clustering method of ME-ACD algorithm and local models are found at this instant with the values of the QOI at instant t_z . A classifier is then trained for the new partition. Time-dependent ME-ACD leads to $N_p \leq N_t$ partitions with their corresponding classifiers. A list \mathcal{T} of

size N_t links the instants to the corresponding partitions. At every instant, a set of coefficients for every elements of the partition considered for this instant is available.

For a dynamical system composed of several coupled ODE's, this algorithm can be repeated for every of the state variables of the system, all state variables being treated independently. As variables can have different scales, a normalised partition error δ_z can be used to be compared with the chosen δ_{min} and δ_{max} , these two parameters being chosen for all the variables. In the present study, δ_z is normalised by the maximum of the absolute value the considered variable can take in all the scenarios corresponding to training observations, at the considered instant t_z .

Once every state variable has its temporal ME-ACD model, temporal evolution of all Sobol indices of all the state variables can be estimated with the method described above.

5 Applications in Flight Dynamics

5.1 Presentation of the system

In this section, the described approach is used to study the influence of some parameters of a Flight Dynamics system. The forces and moments of the fluid on the aircraft are traditionally built on data coming from wind tunnel experiments. In this study, the mathematical models of the forces and moments are built on data coming from experiments conducted at the vertical wind tunnel of ONERA Lille on a light aircraft scale model.

The motion of the aircraft in the terrestrial reference frame, supposed as an inertial reference frame, is modelled by a dynamical system of 8 coupled ODE's. The 8 state variables are :

- V : Norm of the velocity of the aircraft
- α : Angle of attack
- β : Angle of sideslip
- p : roll rate
- q : pitch rate
- r : yaw rate
- Θ : pitch angle
- Φ : yaw angle

and the parameters are :

- m : mass of the aircraft
- ρ : density of surrounding air
- S : reference surface of a wing
- c : reference chord of a wing
- A, B, C : Inertial moments of the aircraft
- $\delta_l, \delta_m, \delta_n$: angular deflections of control surfaces, respectively aileron deflection, elevator deflection and rudder deflection.

Forces applying on the aircraft are

- Its weight. Its mass is supposed constant in the scope of the study.
- Propulsion. In the scope of the study, propulsion is not considered.
- Aerodynamic forces from the fluid on the aircraft. In the scope of this study, the density of air is supposed constant.

The global aerodynamic coefficients are the dimensionless aerodynamic forces and moments. In this paper, the model has the following chosen structure

$$C_x = C_{x0}(\alpha) \quad (11)$$

$$C_y = C_{y0}(\alpha, \beta) + C_{y\delta_l}(\alpha, \delta_l) + C_{y\delta_n}(\alpha, \delta_n) + C_{y\hat{p}_a}(\alpha, \beta, \hat{p}_a) \quad (12)$$

$$C_z = C_{z0}(\alpha, \beta) + C_{z\delta_m}(\alpha, \delta_m) + C_{z\hat{p}_a}(\alpha, \beta, \hat{p}_a) \quad (13)$$

$$C_l = C_{l0}(\alpha, \beta) + C_{l\delta_l}(\alpha, \delta_l) + C_{l\delta_n}(\alpha, \delta_n) + C_{l\hat{p}_a}(\alpha, \beta, \hat{p}_a) \quad (14)$$

$$C_m = C_{m0}(\alpha, \beta) + C_{m\delta_m}(\alpha, \delta_m) + C_{m\hat{p}_a}(\alpha, \beta, \hat{p}_a) \quad (15)$$

$$C_n = C_{n0}(\alpha, \beta) + C_{n\delta_l}(\alpha, \delta_l) + C_{n\delta_n}(\alpha, \delta_n) + C_{n\hat{p}_a}(\alpha, \beta, \hat{p}_a) \quad (16)$$

Every coefficient takes the form of a sum of non-linear subfunctions depending of a reduced number of variables. For example, $C_{m\delta_m}$ represents the non-linear effects of the elevator angular deflection on the dimensionless pitch moment. Every subfunction has a corresponding database, data coming from wind tunnel experiments conducted at Onera Lille on a light aircraft scale model. During numerical integration of the dynamical system, evaluations of the subfunctions are done by linear interpolations in the corresponding databases.

$$\hat{p}_a = \frac{c}{V} (p \cos(\alpha) \cos(\beta) + q \sin(\beta) + r \sin(\alpha) \cos(\beta)) \quad (17)$$

is the dimensionless form of the aerodynamic roll, component of the rotation vector along the wind axis (along the velocity of the aircraft). Databases corresponding to subfunctions dependent of \hat{p}_a come from experiments conducted in the vertical wind tunnel of ONERA Lille where rotary balance permits to get a conical motion of the aircraft around the wind axis (which is vertical in these experiments).

The equations of the system are



FIGURE 1 – Picture of the light aircraft scale model fixed on the rotary balance in the vertical wind tunnel of ONERA Lille. ©Onera

$$\dot{V} = g(-\sin \Theta \cos \alpha \cos \beta + \cos \Theta \sin \Phi \sin \beta + \cos \Theta \cos \Phi \sin \alpha \cos \beta) - \frac{1}{2m} \rho S V^2 C_{x_a} \quad (18)$$

$$\dot{\alpha} = \frac{q_a}{\cos \beta} + \frac{g}{V \cos \beta} (\sin \Theta \sin \alpha + \cos \Theta \cos \Phi \cos \alpha) - \frac{1}{2m \cos \beta} \rho S V C_{z_a} \quad (19)$$

$$\dot{\beta} = -r_a + \frac{g}{V} (\sin \Theta \cos \alpha \sin \beta + \cos \Theta \sin \Phi \cos \beta - \cos \Theta \cos \Phi \sin \alpha \sin \beta) + \frac{1}{2m} \rho S V C_{y_a} \quad (20)$$

$$\dot{p} = \frac{B-C}{A} r q + \frac{1}{2A} \rho S c V^2 C_l \quad (21)$$

$$\dot{q} = \frac{C-A}{B} p r + \frac{1}{2B} \rho S c V^2 C_m \quad (22)$$

$$\dot{r} = \frac{A-B}{C} p q + \frac{1}{2C} \rho S c V^2 C_n \quad (23)$$

$$\dot{\Theta} = q \cos \Phi - r \sin \Phi \quad (24)$$

$$\dot{\Phi} = p + \tan \Theta (q \sin \Phi + r \cos \Phi) \quad (25)$$

$$(26)$$

with

$$q_a = p \cos \alpha \sin \beta + q \cos \beta + r \sin \alpha \sin \beta \quad (27)$$

$$r_a = -p \sin \alpha + r \cos \alpha \quad (28)$$

$$C_{x_a} = C_x \cos \alpha \cos \beta - C_y \sin \beta + C_z \sin \alpha \cos \beta \quad (29)$$

$$C_{y_a} = C_x \cos \alpha \sin \beta + C_y \cos \beta + C_z \sin \alpha \sin \beta \quad (30)$$

$$C_{z_a} = -C_x \sin \alpha + C_z \cos \alpha \quad (31)$$

Traditionally, bifurcation diagrams are drawn with the equilibria of this system [14]. Equilibria represent safe trajectories in the linear domain (at low angles of attacks), but represent potentially dangerous

behaviours, as spins, in the post-stall domain (at high angles of attack). In the present paper, bifurcation theory is not used, but the joint UQ/GSA study is conducted on the temporal evolution of the state variables during the first instants. To integrate the system, a Runge-kutta method of order 2 is employed. The system has potentially several bifurcations, which is motivating the use of temporal ME-ACD. Every state variable is a QOI \hat{y} depending on time and on the selected input parameters. Every of these QOIs is approximated by ME-ACD at every considered instant and partial variances are found with the analytical formula linking them to the coefficients of the local polynomial models.

5.2 Influence of damping parameters

With a model taking into account rotation effects with data only coming from rotary balance, a lot of equilibria of the system in the post stall region are not stable [12]. It is shown that it can come from the lack of knowledge on the dependence of the aerodynamic coefficients in terms of q_a and r_a , components traducing the misalignment between rotation and velocity vectors. Experiments on a vertical wind-tunnel using a rotary balance only give information on dependences of the aerodynamic coefficients on p_a , the projection of the rotation vector on the velocity direction. In [12], authors add the so-called damping terms with information coming from forced oscillations along each of the 3 body axis of a similar aircraft scale model. Damping terms are obtained using a linear hypothesis, as usually $q_a \ll p_a$ and $r_a \ll p_a$ in spin motions. In this application, damping terms are added artificially in the Onera aerodynamic model. The subfunctions depending on these variables are also supposed linear, and only the dimensionless moments, having more impact, are modified.

$$C_l = C_l^{standard} + C_{l\hat{q}_a} \hat{q}_a + C_{l\hat{r}_a} \hat{r}_a \quad (32)$$

$$C_m = C_m^{standard} + C_{m\hat{q}_a} \hat{q}_a \quad (33)$$

$$C_n = C_n^{standard} + C_{n\hat{q}_a} \hat{q}_a + C_{n\hat{r}_a} \hat{r}_a \quad (34)$$

where $C_l^{standard}$, $C_m^{standard}$ and $C_n^{standard}$ are the structures of the coefficients defined at (14), (15) and (16).

The added damping terms $C_{l\hat{q}_a}$, $C_{l\hat{r}_a}$, $C_{m\hat{q}_a}$, $C_{n\hat{q}_a}$ and $C_{n\hat{r}_a}$ are the input parameters of the study. In [12], the results of the forced oscillatory experiments are added with a change of frame to get the damping terms, as functions depending on α . In the considered approach, they are supposed constant, but taking random values following uniform distribution laws.

- $C_{l\hat{q}_a} \sim \mathcal{U}(-0.1, 0.1)$
- $C_{l\hat{r}_a} \sim \mathcal{U}(-0.1, 0.1)$
- $C_{m\hat{q}_a} \sim \mathcal{U}(-10, 0)$
- $C_{n\hat{q}_a} \sim \mathcal{U}(-0.1, 0.1)$
- $C_{n\hat{r}_a} \sim \mathcal{U}(-0.1, 0.1)$

The intervals of the distribution laws are estimated with the extrema of values of the forced oscillation data provided in [12] with a wider range for an in-depth theoretical study of the impact of the damping terms.

The chosen initial conditions are the one corresponding to a spin, unstable without the damping terms

Hyperparameters	ME-ACD
N	10000
$N_{E_{min}}$	1
$N_{E_{max}}$	10
k_{min}	30
k_{max}	60
n_{min}	0
n_{max}	5
δ_{min}	10^{-3}
δ_{max}	10^{-3}
$k_{connect}$	100
$N_{outliers}$	5
tested learning rates for the classifier	$[10^{-m}, 2 \times 10^{-m}, 5 \times 10^{-m}], \forall m \in [1, 7]$
N_h	5
N_{random}	100

TABLE 1 – Selected hyperparameters for the study of the impact of damping parameters for an unstable spin.

- $V_0 = 27 \text{ m.s}^{-1}$
- $\alpha_0 = 59^\circ$
- $\beta_0 = 5.9^\circ$
- $p_0 = 1.8 \text{ rad.s}^{-1}$, $q_0 = 0.58 \text{ rad.s}^{-1}$, $r_0 = 3.03 \text{ rad.s}^{-1}$, leading to a rotation velocity of $\Omega = 3.6 \text{ rad.s}^{-1} = 0.57 \text{ tr.s}^{-1}$
- $\Theta_0 = -32^\circ$
- $\Phi_0 = 13^\circ$

and control surfaces angular deflections are set to :

- $\delta_l = -100\%$
- $\delta_m = -20^\circ$
- $\delta_n = -20^\circ$

Table 1 sums-up the selected hyperparameters for temporal ME-ACD and Figure 2 shows the temporal evolutions of the variance and of the partial variances. High variances at the last considered instants are due to the instability of the equilibrium with some combinations of input parameters and stability with another combinations. As it can be seen on the figure, $C_{m\hat{q}_a}$ and $C_{n\hat{q}_a}$ are the key parameters for the stability of this spin. Figure 3 is a plot of α at instant $t = 20\text{s}$ vs the two key parameters. It underlines that for low values of $C_{m\hat{q}_a}$ and $C_{n\hat{q}_a}$ (mapped in $[0, 1]$ on the figure), α keeps the initial value (59°), thus a proof of stability, and for higher values (mapped $C_{m\hat{q}_a}$ close to 1 and then real $C_{m\hat{q}_a}$ close to 0), α is not equal to the initial value at $t = 20\text{s}$. Figure 2 underlines the apparition of oscillations with different amplitudes dependent of the unstable scenarios and with pretty similar frequencies, as variances have pseudo oscillations with low minima. The approach permits to verify, with a relatively low number of run, that adding damping terms gives some stability to the studied spin. Moreover, the approach permits to identify that only 2 of the 5 suspected parameters have actually some impact on the considered spin. The limit values of the input parameters changing the stability of the spin are also available with the ME model. Some elements of the ME model at a given instant can be interpreted. For example, the orange element of Figure 3 represents the different combinations of values of the input parameters which lead

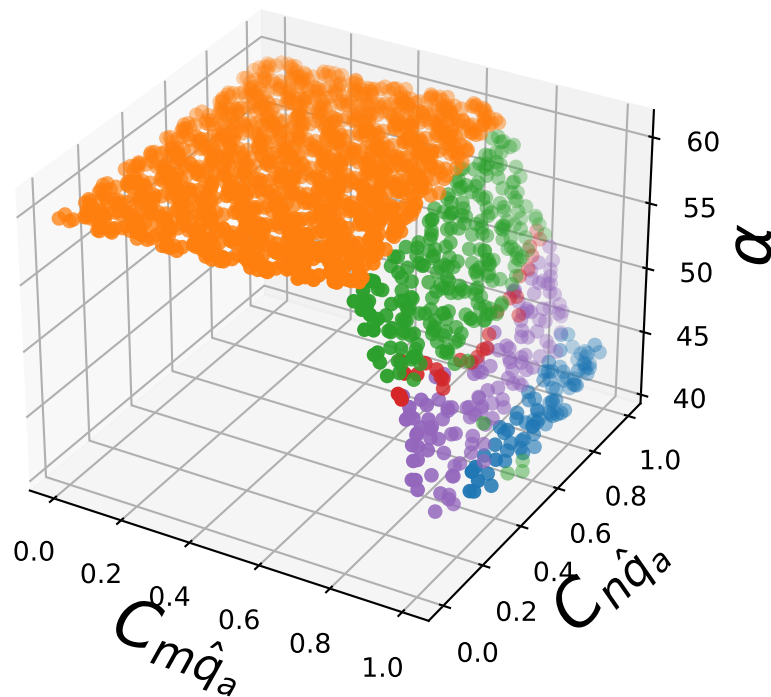


FIGURE 3 – 2000 evaluations of the temporal ME-ACD model of α for $t = 20s$. The different colors refer to different elements and then different local models.

Initial conditions are the same from previous section, and damping terms $C_{m\hat{q}_a} = -10$ and $C_{n\hat{q}_a} = -0.1$ are added according to the previous study to give some stability to the spin.

Selected input parameters are ΔC_l , ΔC_m and ΔC_n and are supposed to follow a normal distribution law :

- ΔC_l follows a normal distribution of mean 0 and of standard deviation $\frac{0.04}{3}$.
- ΔC_m follows a normal distribution of mean 0 and of standard deviation $\frac{0.08}{3}$.
- ΔC_n follows a normal distribution of mean 0 and of standard deviation 0.005.

Figure 4 underlines that the considered aircraft behaviour is not robust to experimental uncertainties, even with the added damping terms. Uncertainties on the coefficients C_n make the long-time stability of this spin uncertain. More experimental data or more accurate experimental data on this component of the aerodynamic moment are then necessary to be more sure of the actual long time stability of the spin (with the theoretically added damping terms). However, the state variables are constant during the first 10s, which means that it can be ensured that the spin has a short-time stability, and is therefore dangerous, even with the current uncertainties.

6 Conclusion and discussion

In this paper, an upcoming joint UQ/GSA method to estimate Sobol indices with a ME model is extended to solutions of ODE's. Considered ME method is the time-extension of ME-ACD algorithm [1] and the

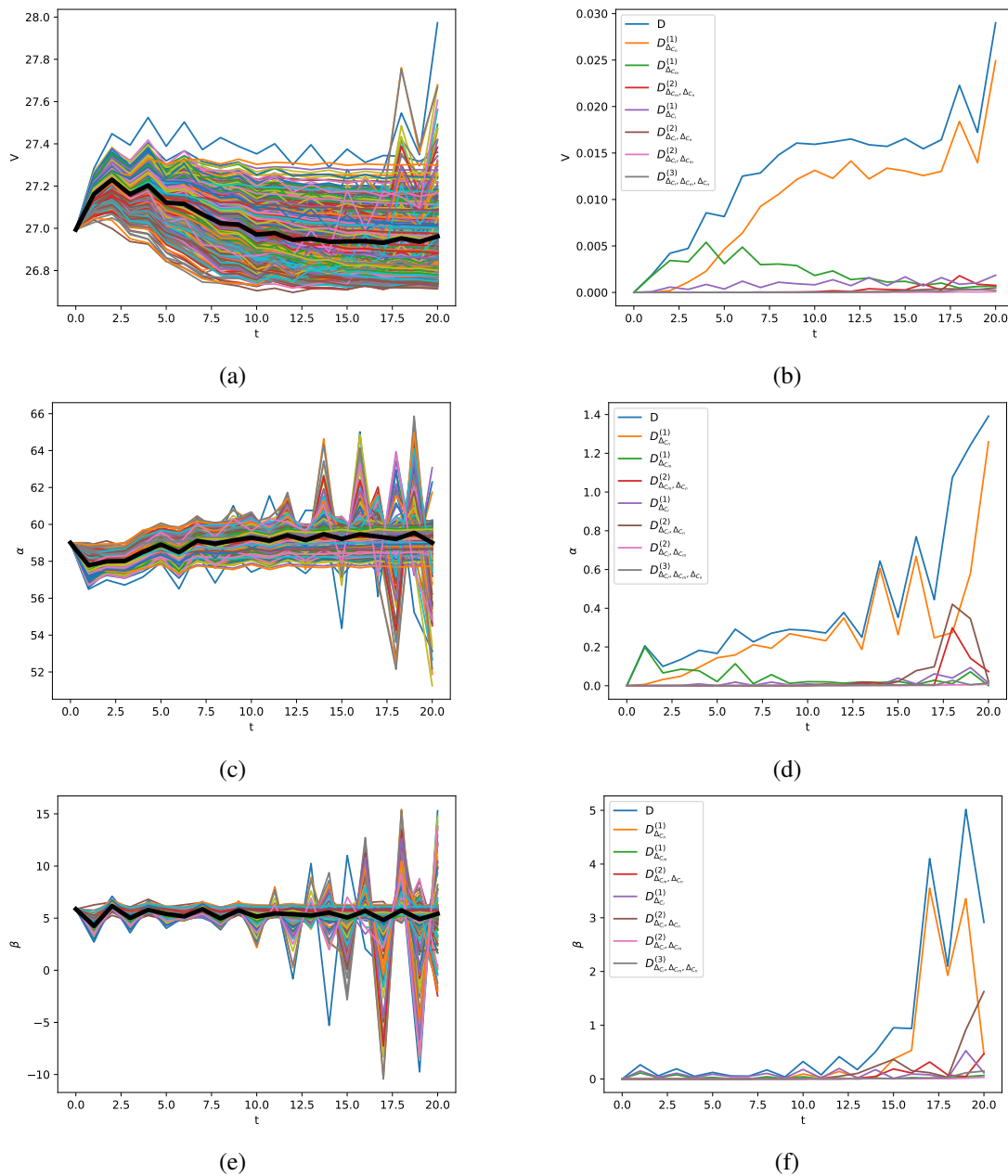


FIGURE 4 – On the left, temporal evolutions of the variables with all the combinations of input parameters taken as training observations and evolution of the expectation (a) V , (c) α and (e) β . On the right, temporal evolution of total and partial variances of (b) V , (d) α and (f) β

use of a formula linking coefficients of the local polynomial models to Sobol indices [2] is used at every considered instant. Taking advantage of the ME approach, a pretty accurate model can be obtained at every instant with a relatively low number of runs, handling both Gibbs phenomenon and the curse of dimensionality.

The approach is used to study the impact of the uncertainties of the parameters of a Flight Dynamics system. As predicted aircraft behaviours rely on models of forces and moments with a chosen structure and evaluated through interpolation in experimental databases, they suffer from uncertainties. UQ and GSA were conducted on the system and the impact of a modification of the model structure and the

impact of the uncertainties of experimental data on a spin behaviour were studied. It permitted to underline, as in [12], that the behaviour is highly impacted by these uncertainties. The approach permitted to determine that the added damping terms $C_{m\dot{q}_a}$ and $C_{n\dot{q}_a}$ had the most important impact on the stability of the studied spin.

ME method leads to elements which can be interpreted. In the above example, the element where the QOI is locally approximated by a constant polynomial highlights the combinations of values leading to stability of the spin.

The applications highlight that interpretation of some predictions of behaviours must be taken with precaution, as it is not robust to uncertainties and to lack of knowledge on the model of the aerodynamic coefficients. A temporal evolution is considered in this paper. The study of attraction domains [13] of spins, coupled with UQ and GSA, could be another step forward in the study of dangerous aircraft's behaviours with an uncertain model.

ME-ACD is under development and may lack of robustness in some practical applications. For example, in the applications of the paper, ME-ACD failed to find a suitable partition at some instants. ME-ACD thus preferred a standard PC model with no partition of the parameter space, thus suffering of a lack of accuracy with the irregularity at these problematic instants. Ameliorations of the algorithm to be more robust with practical applications are under development. Moreover, the approach can currently handle only a moderate number of input parameters (less than 9).

Références

- [1] N. Vauchel, É. Garnier, T. Gomez, A multi-element non-intrusive Polynomial Chaos method using agglomerative clustering based on the derivatives to study irregular and discontinuous Quantities of Interest, submitted to Journal Of Computational Physics
- [2] N. Vauchel, É. Garnier, T. Gomez, Estimation of Sobol indices using a multi-element Polynomial Chaos model, in process
- [3] I.M. Sobol, Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates, *Mathematics and computers in simulation* 55.1-3 (2001) 271-280.
- [4] T. Crestaux, O. Le Maitre, J.M. Martinez, Polynomial chaos expansion for sensitivity analysis, *Engineering & System Safety* (2009).
- [5] B. Sudret, Global sensitivity analysis using polynomial chaos expansions, *Reliability engineering & system safety*, 93(7) (2008), 964-979.
- [6] J. D.Jakeman, A. Narayan, D. Xiu, Minimal multi-element stochastic collocation for uncertainty quantification of discontinuous functions. *Journal of Computational Physics*, 242 (2013), 790-808.
- [7] K. Sargsyan, C. Safta, B. Debusschere, Najm, H. . Uncertainty quantification given discontinuous model response and a limited number of model runs. *SIAM Journal on Scientific Computing*, 34(1), B44-B64, 2012.

- [8] Y. V. Halder, B. Sanderson, B. Koren, An adaptive minimum spanning tree multielement method for uncertainty quantification of smooth and discontinuous responses, *SIAM Journal on Scientific Computing*, 41(6), A3624-A3648, 2019.
- [9] N. Wiener, The homogeneous chaos. *American Journal of Mathematics*, 60(4) (1938), 897-936.
- [10] D. Xiu, G. E. Karniadakis, The Wiener–Askey polynomial chaos for stochastic differential equations, *SIAM journal on scientific computing*, 24(2) (2002), 619-644.
- [11] D. Xiu, Fast numerical methods for stochastic computations : a review, *Communications in computational physics*, 5(2-4) (2009), 242-272.
- [12] D. Farcy, A. Khrabrov, M. Sidoryuk, Sensitivity of Spin Parameters to Uncertainties of the Aerodynamic Model of a Light Aircraft, In *Proceedings of the 8th European Conference for Aerospace Sciences (EUCASS)*, 2019.
- [13] U. Topcu, A. K. Packard, P. Seiler, G. J. Balas, Robust region-of-attraction estimation, *IEEE Transactions on Automatic Control*, 55(1) (2009), 137-142.
- [14] J. V. Carroll, R. K. Mehra, Bifurcation analysis of nonlinear aircraft dynamics, *Journal of Guidance, Control, and Dynamics*, 5(5) (1982), 529-536.