Adaptive Precision Sparse Matrix-Vector Product and its application to Krylov Solvers

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Adaptive Precision Sparse Matrix–Vector Product

and its Application to Krylov Solvers

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Joint work with

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Today's floating-point landscape

		Bits Signif. (t)	Exp.	Range	$u = 2^{-t}$
bfloat16	B	8	8	$10^{\pm 38}$	4×10^{-3}
fp16	H	11	5	$10^{\pm 5}$	5×10^{-4}
fp32	S	24	8	$10^{\pm 38}$	6×10^{-8}
fp64	D	53	11	$10^{\pm 308}$	1×10^{-16}
fp128	Q	113	15	$10^{\pm 4932}$	1×10^{-34}

- Low precision increasingly supported by hardware
- **Great benefits:**
  - Reduced **storage**, data movement, and communications
  - Reduced **energy** consumption ($5\times$ with fp16, $9\times$ with bfloat16)
  - Increased **speed** on emerging hardware ($16\times$ on A100 from fp32 to fp16/bfloat16)
- **Some limitations too:**
  - Low accuracy (large u)
  - Narrow range

Mixed precision algorithms

Mix several precisions in the same code with the goal of

- Getting the performance benefits of low precisions
- While preserving the accuracy and stability of high precision

Terminology varies: Mixed precision, Multiprecision, Adaptive precision, Variable precision, Transprecision, Dynamic precision, ...

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How to select the right precision for the right variable/operation

- **Precision tuning:** autotuning based on the source code, my thesis area: CADNA / PROMISE...
  - ▲ Does not need any understanding of what the code does
  - ▼ Does not have any understanding of what the code does
- **This work:** another point of view, **exploit as much as possible the knowledge we have about the code**

Adaptive precision algorithms

- Given an algorithm and a prescribed accuracy ε, adaptively select the minimal precision for each computation

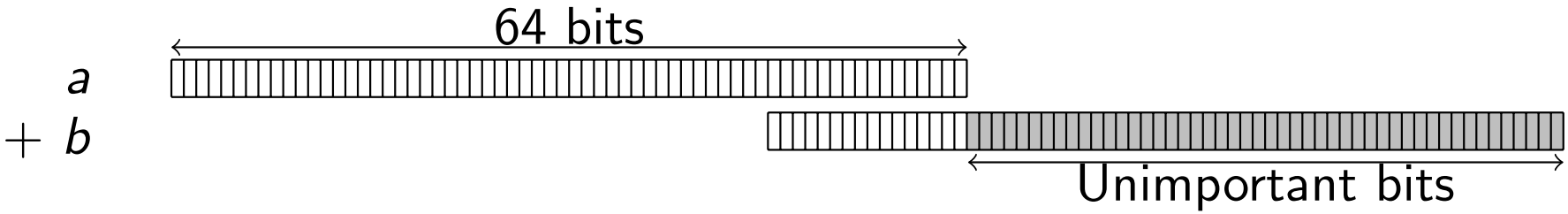
$\Rightarrow$ **Why does it make sense to make the precision vary?**

Adaptive precision algorithms

- Given an algorithm and a prescribed accuracy ε, adaptively select the minimal precision for each computation

$\Rightarrow$ **Why does it make sense to make the precision vary?**

- Because not all computations are equally "important"!
  Example:



and small elements produce small errors :

$$|\,\mathrm{fl}(a\,\mathrm{op}\,b) - a\,\mathrm{op}\,b| \leq u|a\,\mathrm{op}\,b|, \qquad \mathrm{op} \in \{+, -, *, \div\}$$

$\Rightarrow$ **Opportunity for mixed precision:** adapt the precisions to the data at hand by storing and computing "less important" (usually smaller) data in lower precision

Adaptive precision at the variable level?

- Pushing adaptive precision to the extreme: can we benefit from storing **each variable** in a (possibly) different precision?
- Example: $Ax = b$ with adaptive precision for each A_{ij}
  - **Is it worth it?**
    Need to have elements of widely different magnitudes
  - **Is it practical?**
    Probably not for compute-bound applications, but could it work for memory-bound ones?

  $\Rightarrow$ Natural candidate: **sparse matrices**

Sparse matrix–vector product (SpMV)

$$y = Ax,\ A \in \mathbb{R}^{m \times n}$$

```
for i = 1 : m do
    y_i = 0
    for j ∈ nnz_i(A) do
        y_i = y_i + a_ij x_j
    end for
end for
```

- Standard error analysis for $y = Ax$ performed in a uniform precision ε gives,

$$|\widehat{y}_i - y_i| \leq n_i \varepsilon \sum_{j \in nnz_i(A)} |a_{ij} x_j|$$

- **Idea:** store elements of A in a precision inversely proportional to their magnitude (**smaller elements in lower precision**)

Adaptive precision SpMV

for $i = 1 : m$ **do**
  $y_i = 0$
  for $k = 1 : p$ **do**
    $y_i^{(k)} = 0$
    for $j \in nnz_i(A)$ **do**
      if $a_{ij}x_j \in B_{ik}$ **then**
        $y_i^{(k)} = y_i^{(k)} + a_{ij}x_j$ **at precision** u_k
      end if
    end for
    $y_i = y_i + y_i^{(k)}$
  end for
end for

- Split row i of A into p buckets B_{ik} and sum elements of B_{ik} in precision u_k
- Error analysis: $|\widehat{y}_i^{(k)} - y_i^{(k)}| \leq n_i^{(k)} u_k \sum_{a_{ij}x_j \in B_{ik}} |a_{ij}x_j|$

Building the buckets

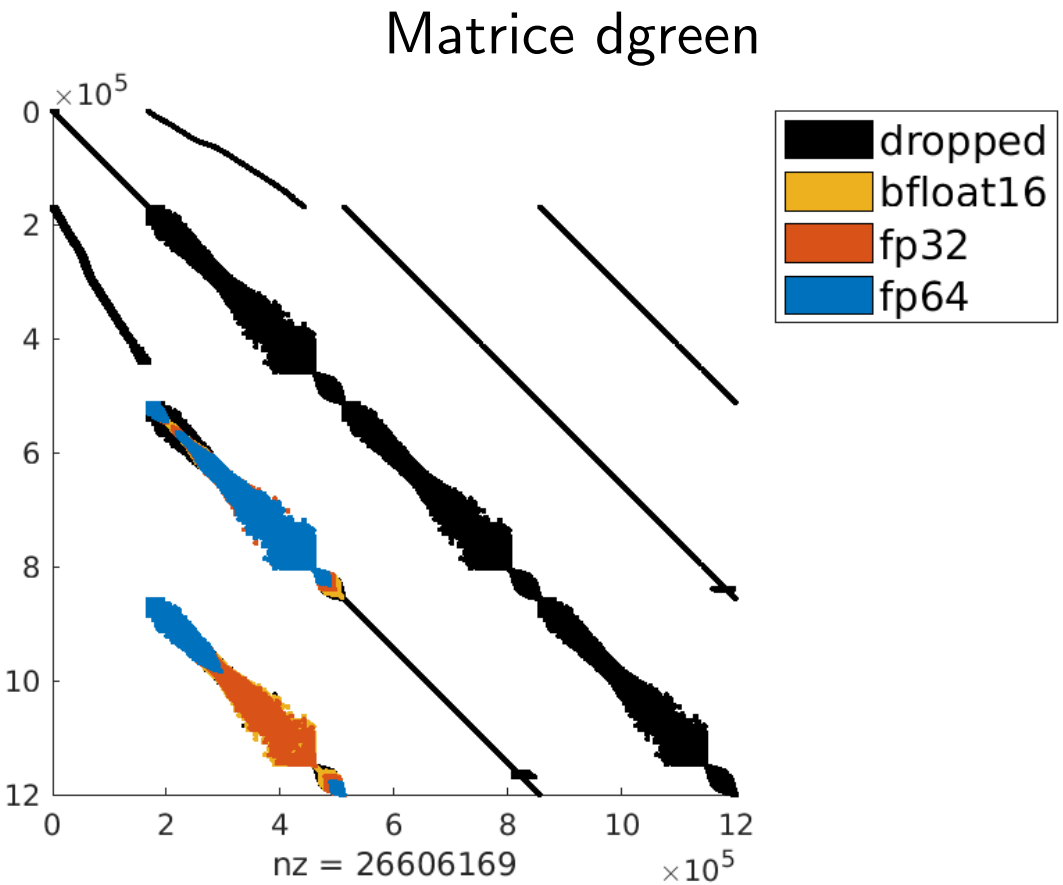
- $|\widehat{y}_i^{(k)} - y_i^{(k)}| \leq n_i^{(k)} u_k \sum_{a_{ij}x_j \in B_{ik}} |a_{ij}x_j|$

$\Rightarrow$ Build the buckets such that $u_k \sum_{a_{ij}x_j \in B_{ik}} |a_{ij}x_j| \approx \varepsilon \sum_j |a_{ij}x_j|$

- By setting B_{ik} to the interval $(\varepsilon\beta_i/u_{k+1}, \varepsilon\beta_i/u_k]$, we obtain $|\widehat{y}_i^{(k)} - y_i^{(k)}| \leq n_i^{(k)} \varepsilon\beta_i$ and so $|\widehat{y}_i - y_i| \leq n_i \varepsilon\beta_i$

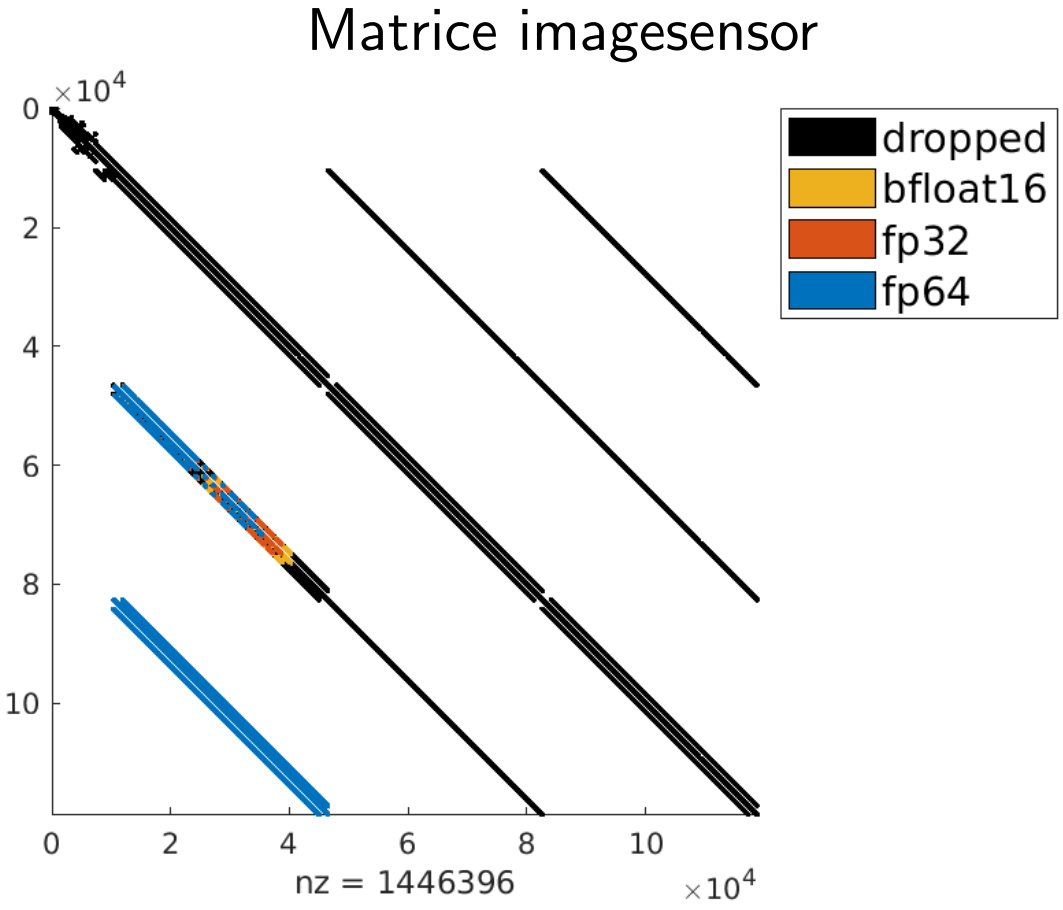
- Two possible choices for β_i:
  - $\beta_i = \sum_j |a_{ij}x_j| \Rightarrow$ guarantees $O(\varepsilon)$ componentwise error: $|\widehat{y}_i - y_i| \leq n\epsilon \sum_j |a_{ij}x_j| \quad \forall i \in \{1, ..., n\}$
  - $\beta_i = \|A\|\|x\| \Rightarrow$ guarantees $O(\varepsilon)$ normwise error: $||\widehat{y}_i - y_i|| \leq n\epsilon \|A\|\|x\|$
- Smallest elements are simply dropped

Visualise mixed-precision gains



For some matrices, many elements can be dropped that leads to major gains.

Visualise mixed-precision gains



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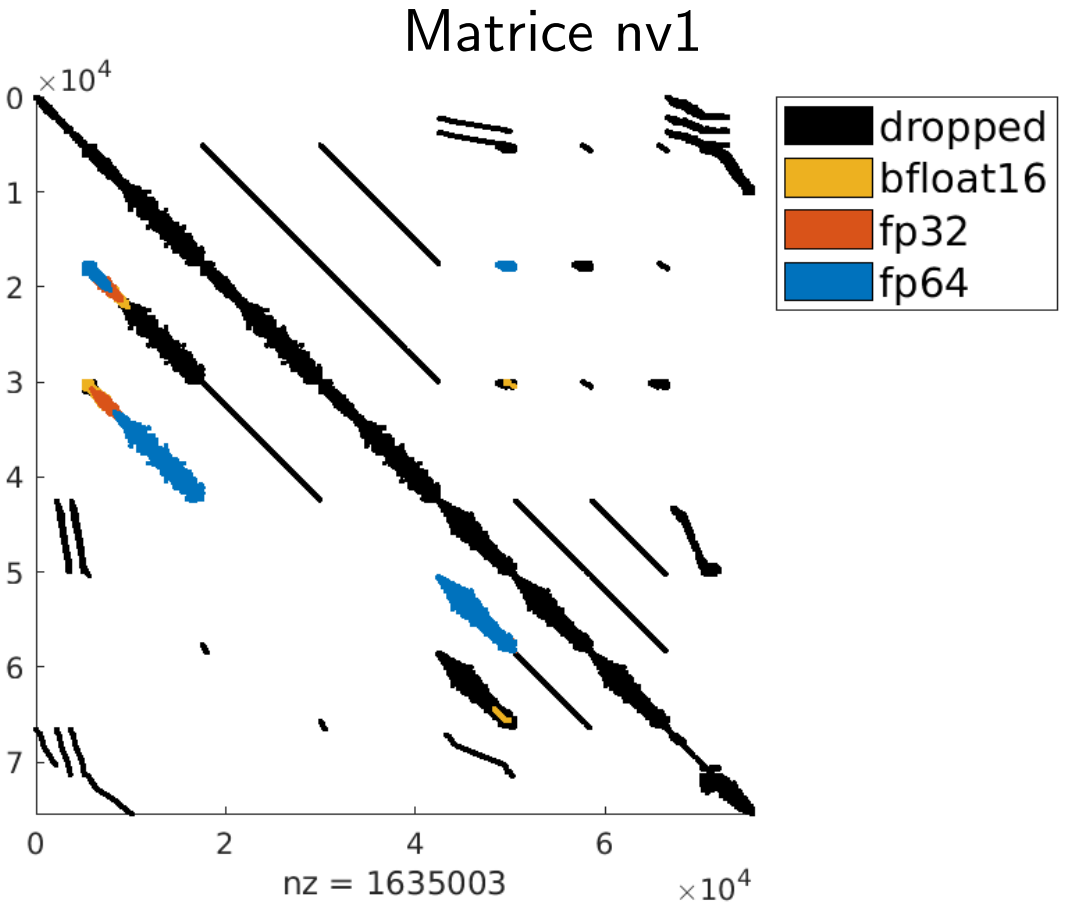

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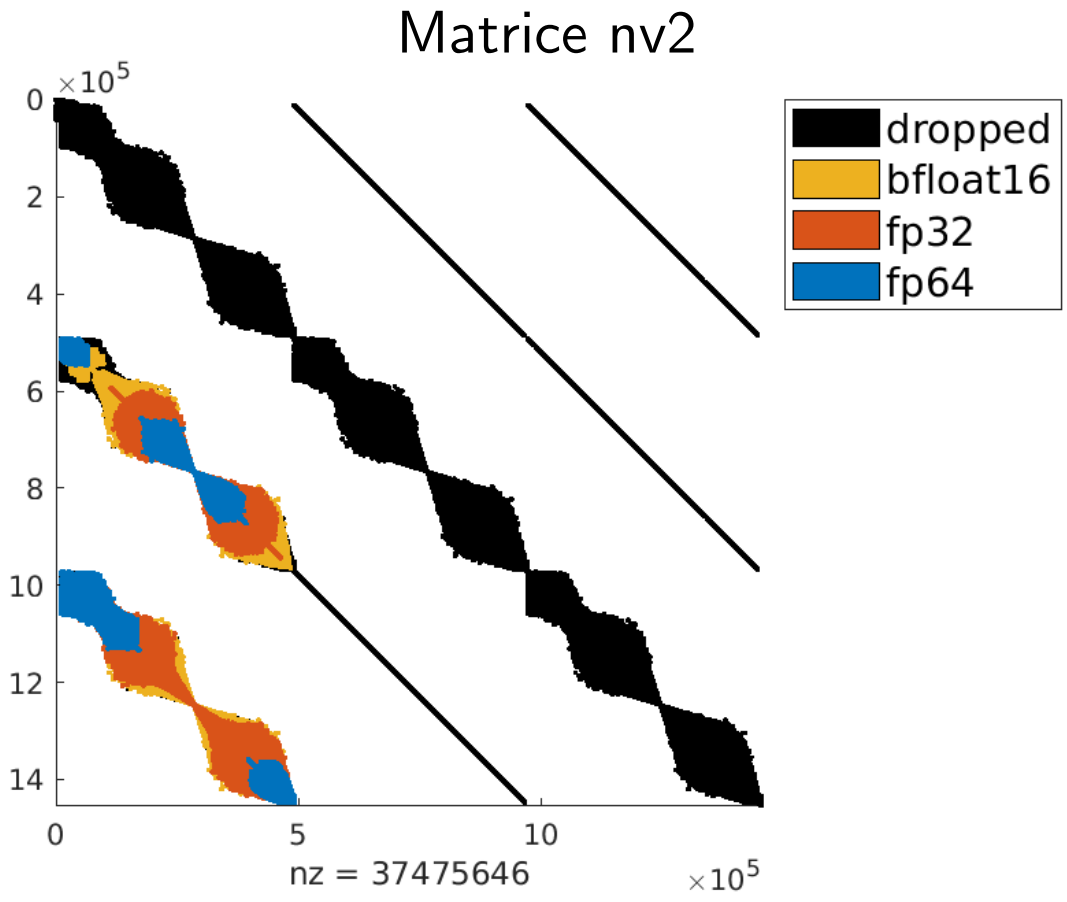

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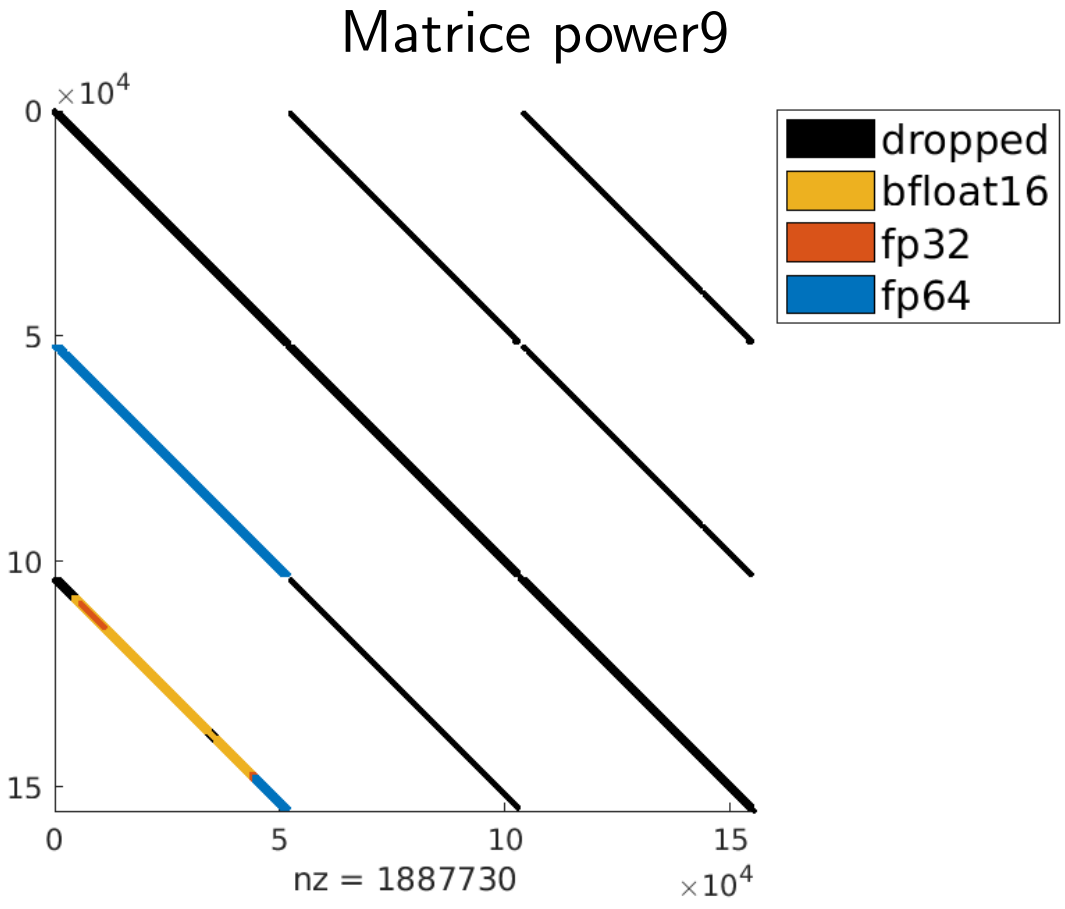
Visualise mixed-precision gains



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SpMV experimental settings

- 33 matrices coming from SuiteSparse collection and industrial partners with at most 166M non-zeros

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- 33 matrices coming from SuiteSparse collection and industrial partners with at most 166M non-zeros
- 3 different **accuracy targets**:
  - $\epsilon = 2^{-24}$
  - $\epsilon = 2^{-37}$
  - $\epsilon = 2^{-53}$

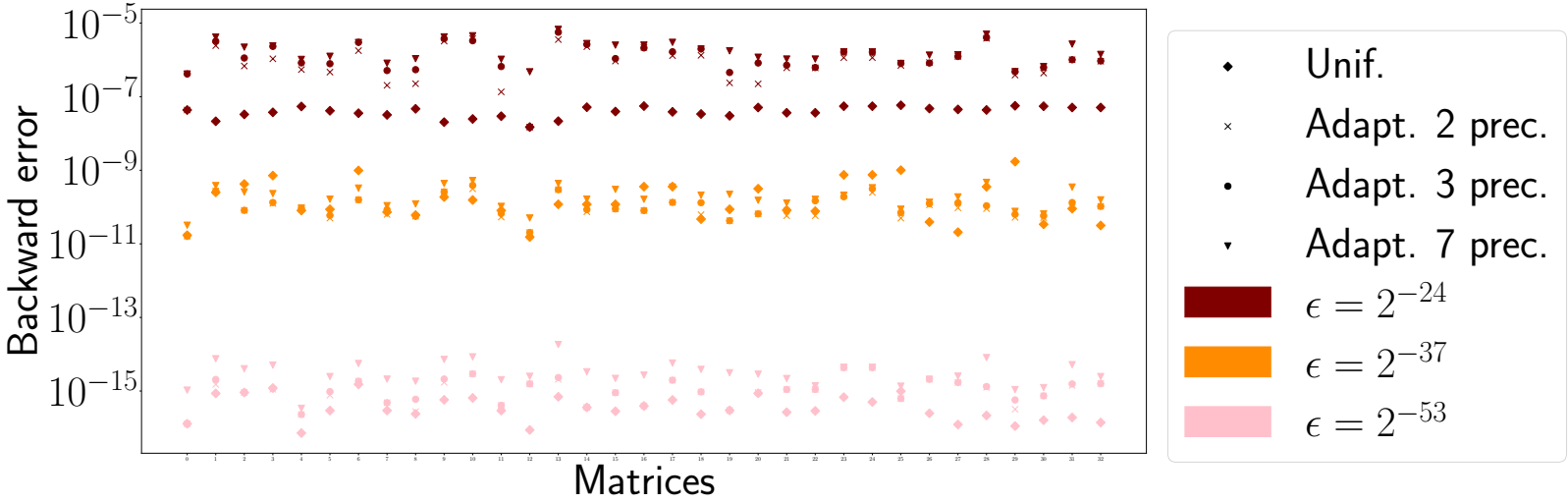
SpMV experimental settings

Various sets of precision formats:

- 2 **precisions**: fp32, fp64
- 3 **precisions**: bfloat16, fp32, fp64
- 7 **precisions**: bfloat16, "fp24", fp32, fp64, "fp40", "fp48", "fp56"

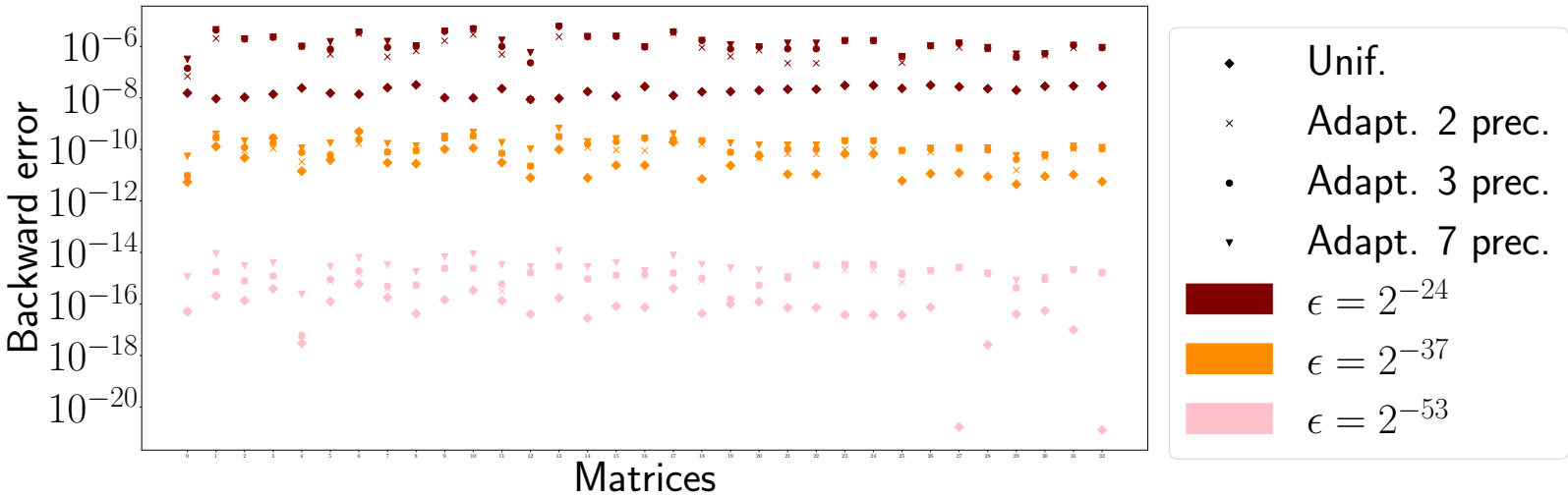
	Bits	
	Mantissa	Exponent
bfloat16	8	8
"fp24"	8	8
fp32	24	8
"fp40"	29	11
"fp48"	37	11
"fp56"	45	11
fp64	53	11

Maintaining componentwise accuracy



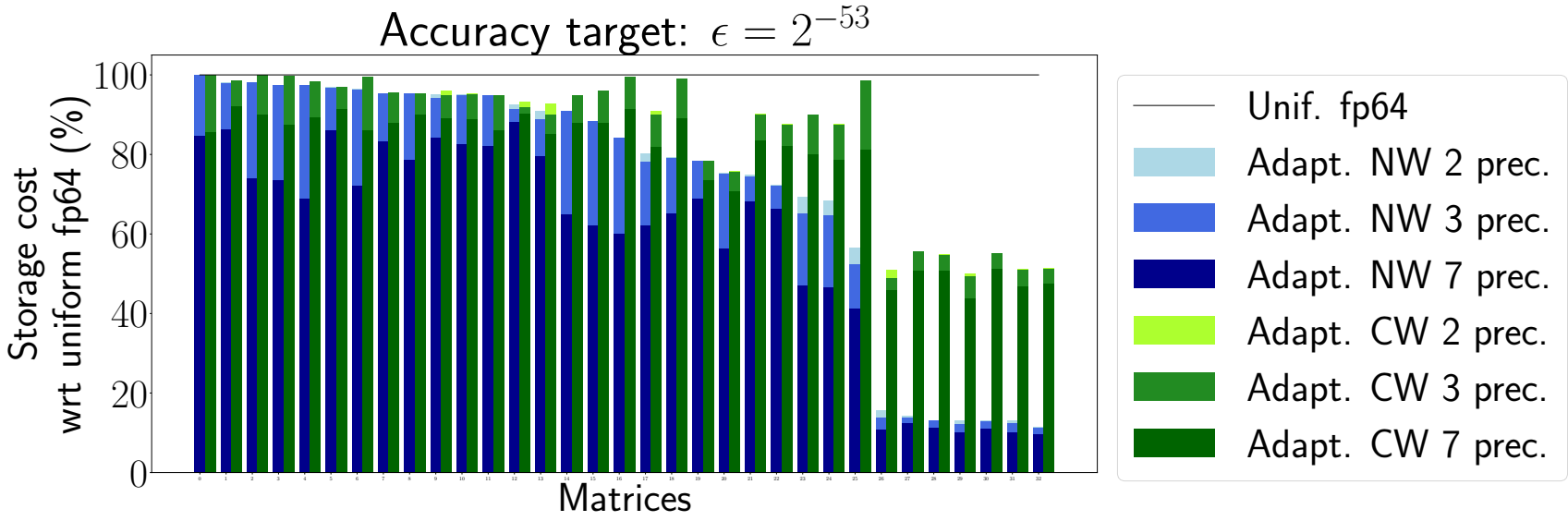
Adaptive methods preserve an accuracy close to the accuracy of uniform methods.

Maintaining normwise accuracy



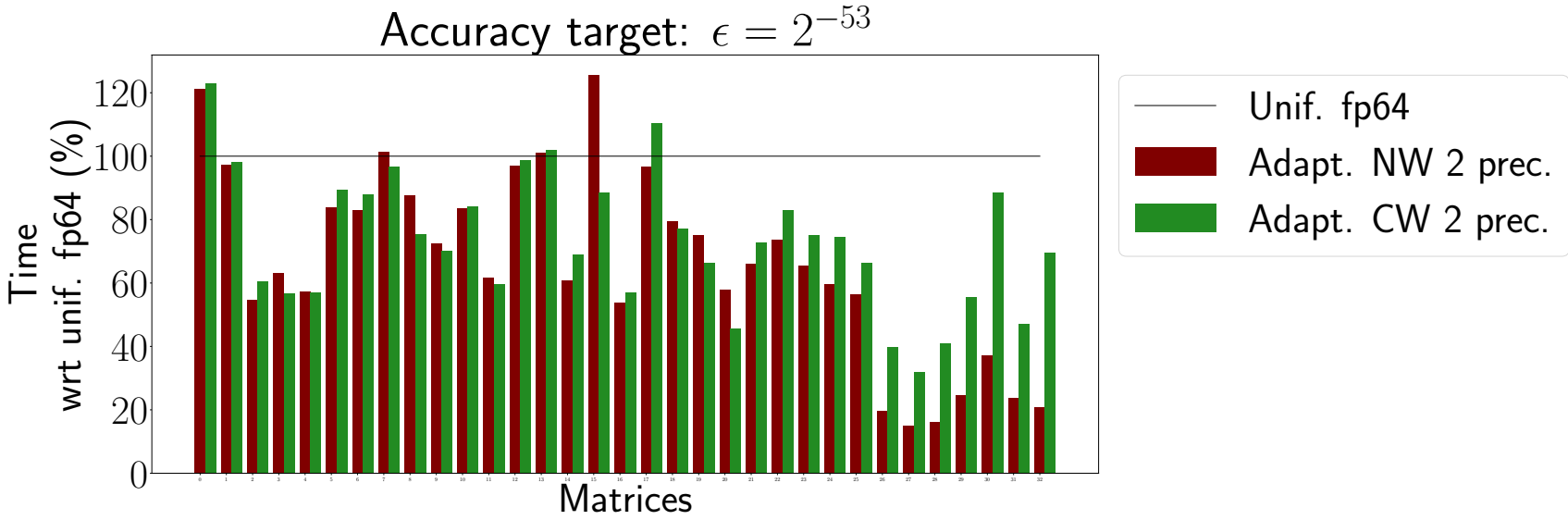
Adaptive methods preserve an accuracy close to the accuracy of uniform methods.

Theoretical storage gains targetting $\epsilon = 2^{-53}$ accuracy (fp64)



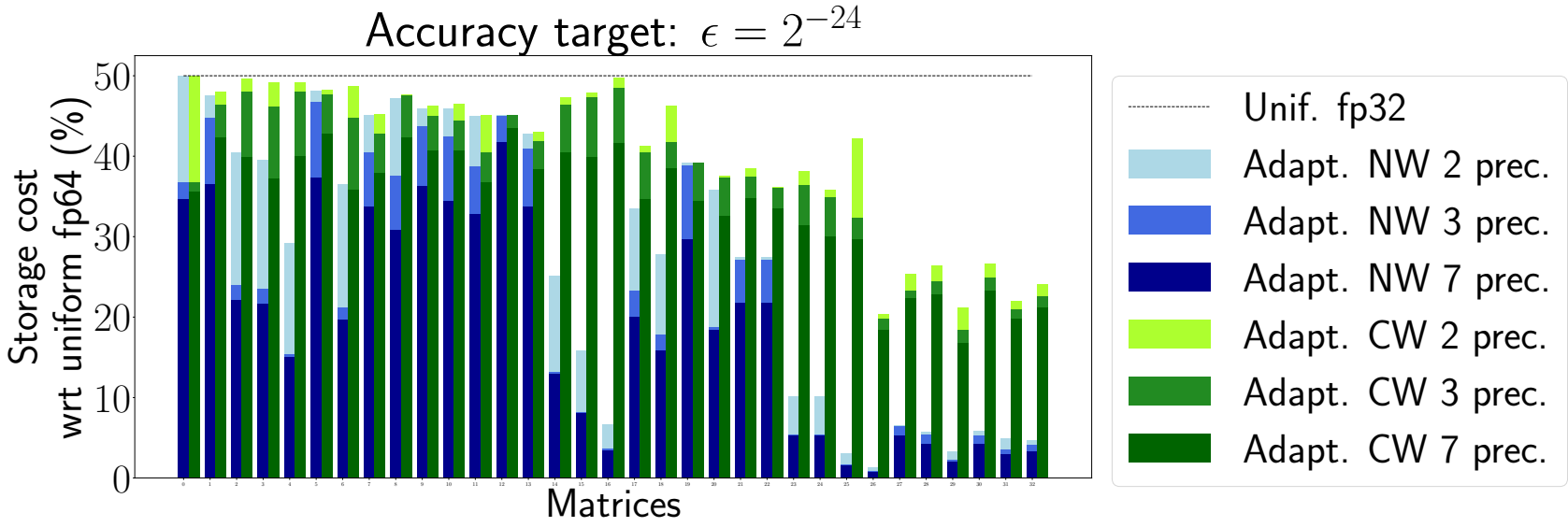
Up to **88%** of storage reduction

Actual time gains targetting $\epsilon = 2^{-53}$ accuracy (fp64)



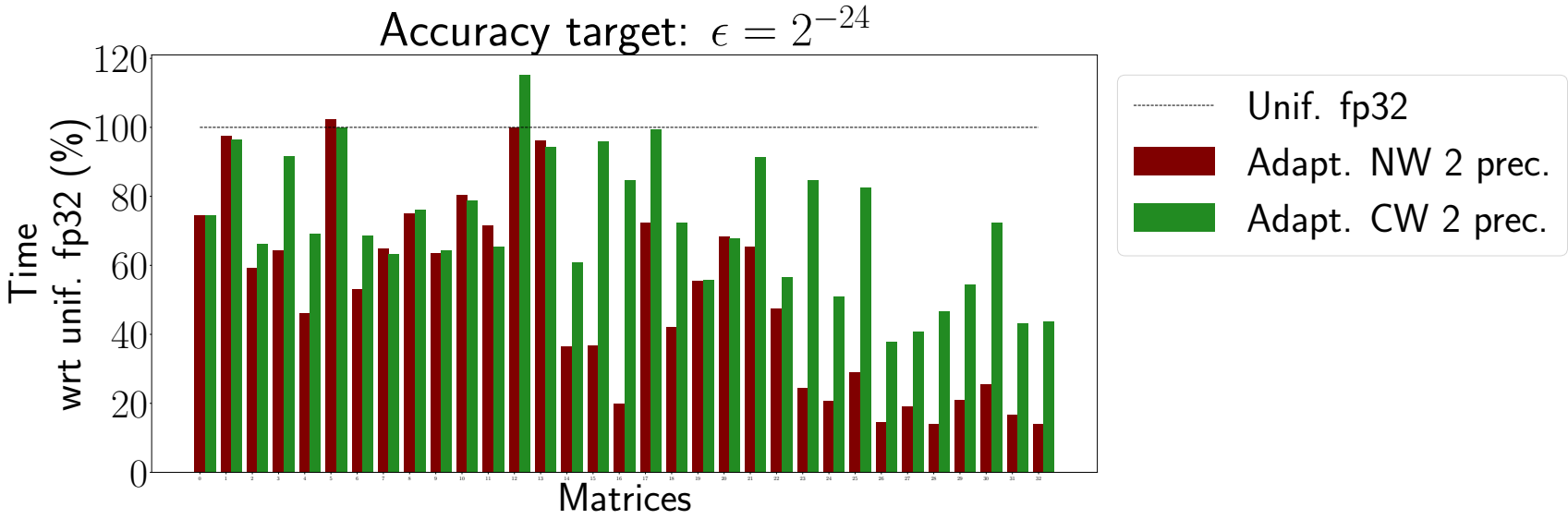
Up to **85%** of time reduction

Theoretical storage gains targetting $\epsilon = 2^{-24}$ accuracy (fp32)



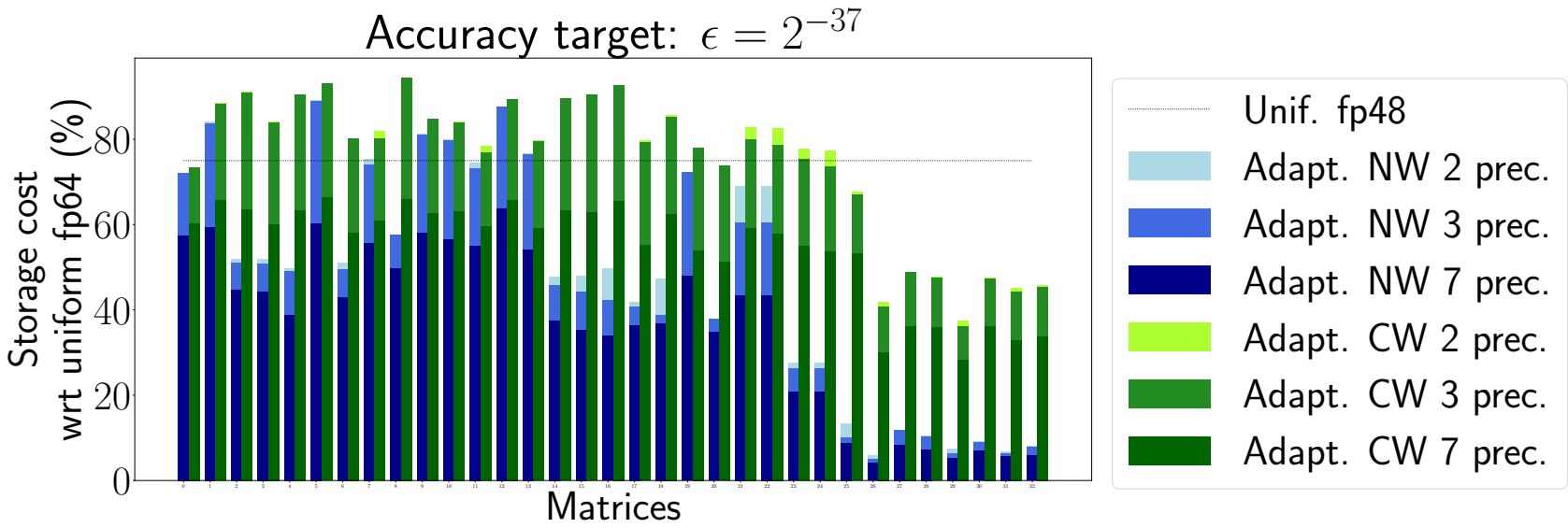
Up to **97%** of storage reduction

Actual time gains targetting $\epsilon = 2^{-24}$ accuracy (fp32)



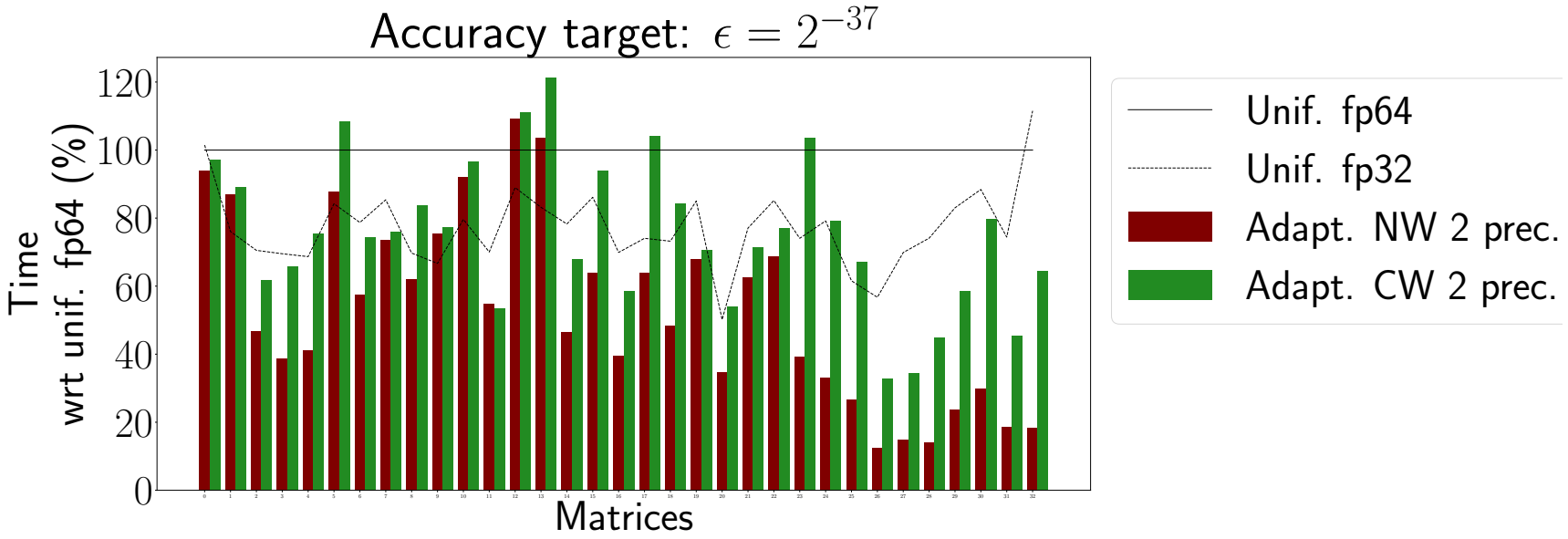
Up to **88%** of time reduction

Theoretical storage gains targetting $\epsilon = 2^{-37}$ accuracy (unavailable in hardware!)



We are able to target any kind of accuracy with only natively supported precisions.

Actual time gains targetting $\epsilon = 2^{-37}$ accuracy (unavailable in hardware!)



We are able to target any kind of accuracy with only natively supported precisions.

Performance of GMRES rely on SpMV

$r = b - Ax_0$
$\beta = \|r\|_2$
$q_1 = r/\beta$
for $k = 1, 2, \ldots$ **do**
  $y = Aq_k$
  for $j = 1 : k$ **do**
    $h_{jk} = q_j^T y$
    $y = y - h_{jk} q_j$
  end for
  $h_{k+1,k} = \|y\|_2$
  $q_{k+1} = y/h_{k+1,k}$
  Solve the least squares problem $\min_{c_k} \|Hc_k - \beta e_1\|_2$
  $x_k = x_0 + Q_k c_k$
end for

How does the adaptive method affect the convergence?

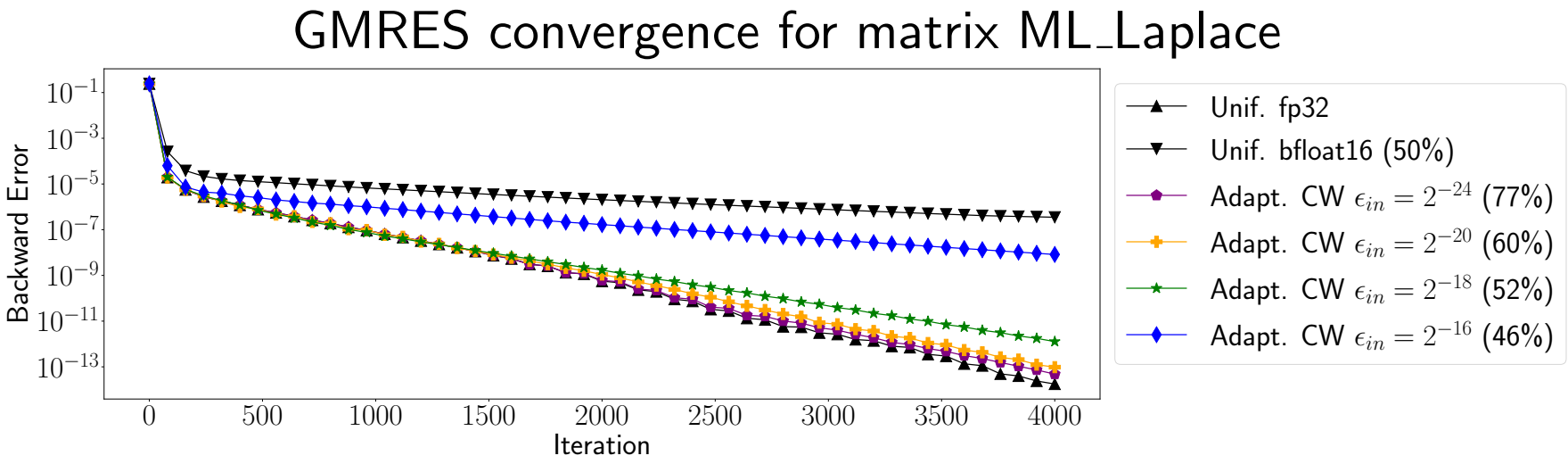
GMRES-IR experimental settings

- Adaptive SpMV achieve larger speedups for lower accuracy targets
- GMRES-based iterative refinement particularly attractive

```
1: for i = 1, 2, ... do
2:     r_i = b − A x_{i−1}
3:     Solve A d_i = r_i by GMRES
4:     x_i = x_{i−1} + d_i
5: end for
```

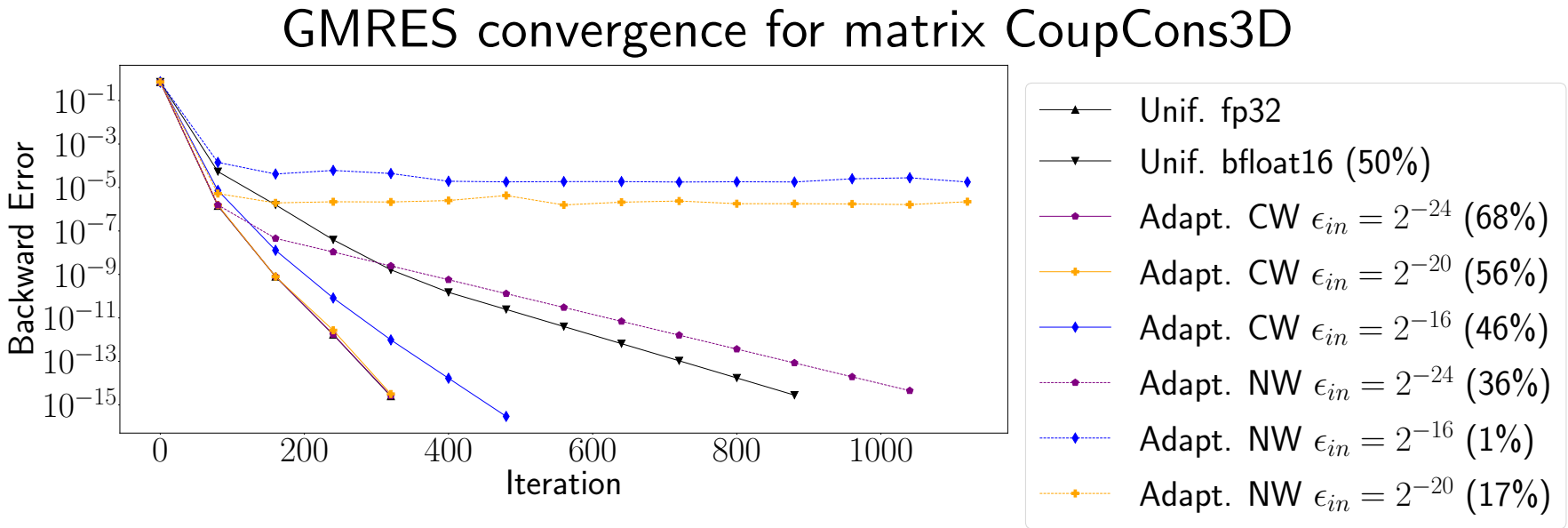
- SpMV line 2 in CW adaptive, target $\epsilon = 2^{-53}$
- GMRES line 3 experimented with multiple variants
- Use of simple Jacobi preconditioner
- Fixed outside threshold $\epsilon_{out} = 2^{-53}$
- Varying inside threshold $\epsilon_{in} = 2^{-24}$
- Fixed inside restart to 80 iterations

Application to GMRES: convergence scheme experiments



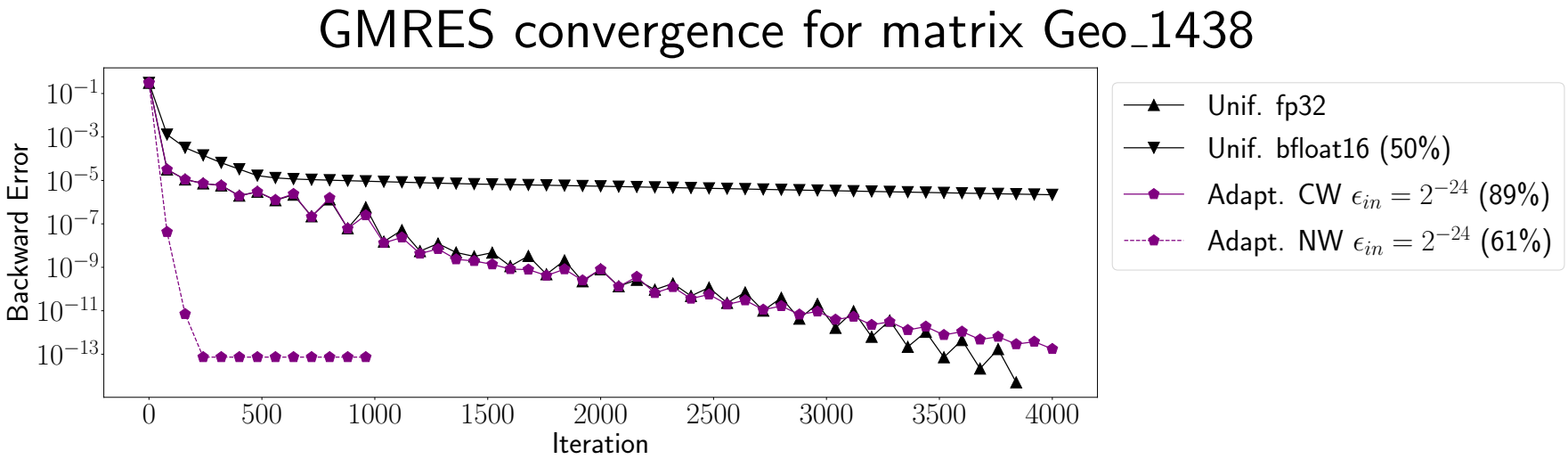
- With reasonable accuracy targets, adaptive SpMV, does not affect the confergence scheme
- Choose the best compromise between the iteration cost and the number of iterations

Application to GMRES: convergence scheme experiments



- For a given ϵ_{in} value, NW variants achieve a lower cost but a slower convergence than CW ones
- $\epsilon_{in} = 2^{-24}$ leads to the best NW variant, which converges in 1040 iterations with an SpMV cost of 36% of the fp32 uniform one
- $\epsilon_{in} = 2^{-20}$ leads to the best CW variant, which converges in 320 iterations with a corresponding SpMV cost of 56%
- Both options should be considered.

Application to GMRES: convergence scheme experiments



- Surprising behavior
- Consistently reproduced and occurs for several other matrices in our set
- Aggressive dropping of small coefficients might lead to a "nicer" matrix for which GMRES can converge quickly.

Conclusion: take-home messages

- **Adaptive precision SpMV algorithm**
  - Ruilds computing buckets according to the elements magnitude
  - Targets any accuracy
  - Matrix-dependent gains
- **Application to Krylov solvers**
  - Reasonable accuracy targets preserves convergence scheme

More info

Thank you! Any questions?