



HAL
open science

Analogues of hyperlogarithm functions on affine complex curves

Benjamin Enriquez, Federico Zerbini

► **To cite this version:**

Benjamin Enriquez, Federico Zerbini. Analogues of hyperlogarithm functions on affine complex curves. 2023. <hal-03930304>

HAL Id: hal-03930304

<https://hal.science/hal-03930304v1>

Preprint submitted on 9 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

ANALOGUES OF HYPERLOGARITHM FUNCTIONS ON AFFINE COMPLEX CURVES

BENJAMIN ENRIQUEZ AND FEDERICO ZERBINI

ABSTRACT. For C a smooth affine complex curve, we show the existence and uniqueness of a minimal (for the inclusion) subalgebra A_C of the algebra $\mathcal{O}_{hol}(\tilde{C})$ of holomorphic functions on its universal cover \tilde{C} , containing the algebra $\mathcal{O}(C)$ of regular functions on C and stable under all the operations $f \mapsto \int f\omega$, for ω in the space $\Omega(C)$ of regular differentials on C . We identify A_C with the image of the iterated integration map $I_{x_0} : \text{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{hol}(\tilde{C})$ based at any point x_0 of \tilde{C} (here $\text{Sh}(-)$ denotes the shuffle algebra of a vector space), as well as with the unipotent part, with respect to the action of $\text{Aut}(\tilde{C}/C)$, of a subalgebra of $\mathcal{O}_{hol}(\tilde{C})$ of moderate growth functions. We show that a section of the projection $\Omega(C) \rightarrow H_{dR}^1(C)$ gives rise to an isomorphism of A_C with $\mathcal{O}(C) \otimes \text{Sh}(H_{dR}^1(C))$, leading to the assignment of a subalgebra $\mathcal{H}(C, \sigma)$ of A_C to any such section σ . When C has genus 0, we exhibit a particular section σ_0 and identify $\mathcal{H}(C, \sigma_0)$ with the algebra of hyperlogarithm functions (Poincaré, Lappo-Danilevsky).

CONTENTS

1. Introduction	2
1.1. The context	2
1.2. Presentation of the results	3
2. Moderate growth functions and iterated integrals	6
2.1. Moderate growth functions on \tilde{C}	6
2.2. The module $\Omega_{mod}(\tilde{C})$ of moderate growth differentials	9
2.3. Iterated integrals	11
3. The isomorphism of filtered algebras $f_{\sigma, x_0} : F_\bullet \text{Sh}(H_C^{dR}) \otimes \mathcal{O}(C) \rightarrow F_\bullet \mathcal{O}_{mod}(\tilde{C})$	14
3.1. Background on filtrations	14
3.2. Computation of $(\mathbb{C}\Gamma_C)'$	15
3.3. Computation of $\text{Sh}(V)^\heartsuit$	16
3.4. A Hopf pairing $p_{\sigma, x_0} : \text{Sh}(H_C^{dR}) \otimes \mathbb{C}\Gamma_C \rightarrow \mathbb{C}$	17
3.5. Proof that $\nu(p_{\sigma, x_0}) : \text{Sh}(H_C^{dR})^\heartsuit \rightarrow (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism	17
3.6. The isomorphism of filtered algebras $f_{\sigma, x_0} : F_\bullet \text{Sh}(H_C^{dR}) \otimes \mathcal{O}(C) \rightarrow F_\bullet \mathcal{O}_{mod}(\tilde{C})$	19
3.7. Filtered formality of the HACA $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{mod}(\tilde{C}))$	21
4. Local expansion of the elements of $F_\infty \mathcal{O}_{mod}(\tilde{C})$	21
5. Filtrations on $\mathcal{O}_{hol}(\tilde{C})$ and the minimal stable extension of $\mathcal{O}(C)$	23
5.1. An algebra filtration of $\mathcal{O}_{hol}(\tilde{C})$ defined by I_{x_0}	23
5.2. The filtrations $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$ and $F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C})$	25
5.3. Relation with the minimal stable extension of $\mathcal{O}(C)$	28
5.4. Relation with minimal acyclic extensions of dgas	28
5.5. Relation with Chen's work	30
6. Connections for HACAs	31
6.1. Connections for HACAs	31
6.2. A connection for $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{mod}(\tilde{C}))$	31
6.3. A connection for $(\text{Sh}(H_C^{dR}), \text{Sh}(H_C^{dR}) \otimes \mathcal{O}(C))$	33
6.4. The genus 0 case	34
7. Relation with hyperlogarithm functions	34

7.1. The algebra $\mathcal{H}(C, \sigma)$	34
7.2. The algebra $\mathcal{H}(C, \sigma)$ in the genus 0 case	36
8. Computation of $\text{Ker}(I_{x_0})$	37
Appendix A. Background on Hopf algebras	42
A.1. An endofunctor $O \mapsto O^\heartsuit$ of \mathbf{HA}	42
A.2. A duality functor $\mathbf{HA}_{fd} \rightarrow \mathbf{HA}^{op}$, $H \mapsto H'$	44
A.3. Hopf algebra pairings and Hopf algebra morphisms	46
Appendix B. Background on Hopf algebras with (co)actions on algebras	47
B.1. An endofunctor $(O, A) \mapsto (O^\heartsuit, F_\infty A)$ of \mathbf{HACA}	47
B.2. A category \mathbf{HAMA} and a functor $\mathbf{HAMA}_{fd} \rightarrow \mathbf{HACA}$	48
B.3. Pairing-morphisms from a HACA to a HAMA and HACA morphisms	51
B.4. Two functors $\mathbf{HACA} \rightarrow \mathbf{Alg}_{gr}$ and a natural transformation	53
B.5. Isomorphisms in \mathbf{HACA}	54
Appendix C. Filtered formality for Hopf algebras and HACAs	56
C.1. Filtered formality for Hopf algebras	56
C.2. Filtered formality for HACAs	57
Appendix D. Hopf algebra duality and prounipotent completions	58
D.1. Completion and duality of Hopf algebras	59
D.2. Isomorphism $\mathbb{C}\Gamma' \simeq \mathcal{O}(\Gamma_{unip})$	61
D.3. Relation between the filtered formalities of Γ and $(\mathbb{C}\Gamma)'$	62
References	64

1. INTRODUCTION

1.1. The context. For $S \subset \mathbb{P}_{\mathbb{C}}^1$ a finite subset with $S \ni \infty$, a natural question is to construct an algebra of multivalued functions on $\mathbb{P}_{\mathbb{C}}^1 \setminus S$, containing the algebra $\mathcal{O}(\mathbb{P}_{\mathbb{C}}^1 \setminus S) := \mathbb{C}[z, 1/(z - s), s \in S_\infty]$ (where $S_\infty := S \setminus \{\infty\}$) and stable under the antiderivation operation $f \mapsto (z \mapsto \int_{z_0}^z f(u)du)$ (here z_0 is a fixed point in $\mathbb{C} \setminus S$). Such an algebra necessarily contains all the iterated integrals of the differentials $dz/(z - s)$, $s \in S_\infty$, which are the hyperlogarithm (HL) functions $L_w(z)$, $w \in S_\infty^*$ (where $S_\infty^* := \sqcup_{n \geq 0} S_\infty^n$ is the set of words in S_∞). It was proved in [Br], Cor. 5.6, that the algebra $A_{\mathbb{P}_{\mathbb{C}}^1 \setminus S} := \mathcal{O}(\mathbb{P}_{\mathbb{C}}^1 \setminus S)[L_w, w \in S_\infty^*]$ is stable under antiderivation, so that $A_{\mathbb{P}_{\mathbb{C}}^1 \setminus S}$ is the smallest (for the inclusion) extension of $\mathcal{O}(\mathbb{P}_{\mathbb{C}}^1 \setminus S)$ with this property.

The HL functions, and hence all the functions of $A_{\mathbb{P}_{\mathbb{C}}^1 \setminus S}$, have unipotent monodromies along the paths encircling the points of S , and one can show that $A_{\mathbb{P}_{\mathbb{C}}^1 \setminus S}$ is a union of unipotent modules (i.e., iterated extensions of the trivial module) over $\pi_1(\mathbb{P}_{\mathbb{C}}^1 \setminus S)$.

The theory of HL functions has been notably applied in [Br] to the identification of a set of periods arising from the moduli space of marked stable genus-zero curves with the set of multiple zeta values (Goncharov-Manin conjecture). The HL techniques of [Br] led in [Pa] to an algorithm for expressing a large class of Feynman integrals computing scattering amplitudes¹ in quantum field theory in terms of HLs; this was implemented in the software program `HyperInt`.

¹Scattering amplitudes predict the probability of interactions of elementary particles. Feynman integrals are a useful tool to obtain approximations of the amplitudes.

Similar questions were studied in genus one. To an elliptic curve E , one attaches an algebra \mathcal{A}_3 containing the function field of E (see [BDDT1], three lines before (3.35)) using iterated integration. In *loc. cit.*, §6, it is proved that \mathcal{A}_3 is stable under $f \mapsto \int f\omega$, where ω is a non-zero regular differential over E . One can derive from this the construction, for any finite subset S of E , of an algebra containing $\mathcal{O}(E \setminus S)$, which is stable under $f \mapsto \int f\omega$. Similarly to the genus-zero case, the functions from \mathcal{A}_3 arise naturally in the computation of Feynman integrals (see [BDDT2]).

It is a natural question to construct analogues of the HL functions associated to an arbitrary affine curve C . Such functions are likely to find an application in the computation of amplitudes, such as for instance the genus-two contribution to string theory amplitudes (see [DGP]).

In order to treat this problem, it seems convenient to introduce the notion of *minimal stable extension* (MSE) of the algebra of regular functions $\mathcal{O}(C)$ of C . We fix a universal cover $p : \tilde{C} \rightarrow C$ and call *stable extension (SE)* of $\mathcal{O}(C)$ a subalgebra A of the algebra $\mathcal{O}_{hol}(\tilde{C})$ of holomorphic functions on \tilde{C} which contains $p^*\mathcal{O}(C)$, and such that for any $f \in A$, regular differential ω on C and $z_0 \in \tilde{C}$, the function $z \mapsto \int_{z_0}^z f \cdot p^*\omega$ belongs to A . The intersection A_C of all SEs of $\mathcal{O}(C)$ is again a SE which is minimal for the inclusion, and which we call the MSE of $\mathcal{O}(C)$.

The present paper is devoted to the study of A_C . We show that A_C is a union of unipotent $\pi_1(C)$ -modules, contained in an algebra of moderate growth functions over \tilde{C} , and is maximal with respect to this property (see Thm. 1.1). This leads to the isomorphism of A_C with an explicit algebra (see Thm. 1.2). We also relate A_C to the minimal acyclic extension of the differential graded algebra (dga) $\Omega^\bullet(C) := (\mathcal{O}(C) \oplus \Omega(C), d)$ (see Prop. 5.14). As in the cases of genera zero and one, iterated integration plays a key role in the construction of A_C .

1.2. Presentation of the results.

1.2.1. *The main results.* Let C be a smooth complex affine algebraic curve. Let $p : \tilde{C} \rightarrow C$ be a universal cover of the analytic curve underlying C . Let $\mathcal{O}(C) \subset \mathcal{O}_{hol}(\tilde{C})$ be the algebras of regular functions on C and holomorphic functions on \tilde{C} , and let $\Omega(C)$ be the space of regular differentials on C .

We define a subalgebra $\mathcal{O}_{mod}(\tilde{C}) \subset \mathcal{O}_{hol}(\tilde{C})$ of *moderate growth functions* on \tilde{C} . For this, we consider the smooth compactification \bar{C} of C , and denote by $S := \bar{C} \setminus C$ the complement of C in \bar{C} , which is a finite set; then \tilde{C} is a universal cover of $\bar{C} \setminus p^{-1}(S)$. The elements of $\mathcal{O}_{mod}(\tilde{C})$ are the functions of $\mathcal{O}_{hol}(\tilde{C})$ whose growth at the neighborhood of $p^{-1}(S)$ is moderate (cf. §2.1 and [Ph]). Let $\Gamma_C := \text{Aut}(\tilde{C}/C)$; the group Γ_C acts from the right on $\mathcal{O}_{mod}(\tilde{C}) \subset \mathcal{O}_{hol}(\tilde{C})$, by $f|_\gamma := f \circ \gamma$. We then set

$$\forall n \geq 0, \quad F_n \mathcal{O}_{mod}(\tilde{C}) := \{f \in \mathcal{O}_{mod}(\tilde{C}) \mid \forall \gamma_1, \dots, \gamma_{n+1} \in \Gamma_C, f|_{(\gamma_1^{-1}) \dots (\gamma_{n+1}^{-1})} = 0\},$$

$$F_\infty \mathcal{O}_{mod}(\tilde{C}) := \cup_{n \geq 0} F_n \mathcal{O}_{mod}(\tilde{C}).$$

Inspired by [Ch], we also set

(1.2.1)

$$F_0^\delta \mathcal{O}_{hol}(\tilde{C}) = \mathbb{C}, \quad F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C}) := \{f \in \mathcal{O}_{hol}(\tilde{C}) \mid df \in F_n^\delta \mathcal{O}_{hol}(\tilde{C}) \cdot p^* \Omega(C)\} \text{ for } n \geq 0,$$

(1.2.2)

$$F_n^\mu \mathcal{O}_{hol}(\tilde{C}) := F_n^\delta \mathcal{O}_{hol}(\tilde{C}) \cdot p^* \mathcal{O}(C) \text{ for } n \geq 0,$$

as well as

(1.2.3)

$$F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) := \cup_{n \geq 0} F_n^\delta \mathcal{O}_{hol}(\tilde{C}), \quad F_\infty^\mu \mathcal{O}_{hol}(\tilde{C}) := \cup_{n \geq 0} F_n^\mu \mathcal{O}_{hol}(\tilde{C}).$$

Moreover, iterated integration gives rise, for any $x_0 \in \tilde{C}$, to an algebra morphism

$$I_{x_0} : \text{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{hol}(\tilde{C}),$$

where $\text{Sh}(\Omega(C))$ is the (commutative, associative) shuffle algebra over $\Omega(C)$ (see §2.3).

Denoting by $F_n \text{Sh}(\Omega(C))$ the degree $\leq n$ part of $\text{Sh}(\Omega(C))$, one then has:

Theorem 1.1. *One has the equalities*

(1.2.4)

$$F_\bullet \mathcal{O}_{mod}(\tilde{C}) = F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C}), \quad I_{x_0}(F_\bullet \text{Sh}(\Omega(C))) = F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$$

of filtrations of $\mathcal{O}_{hol}(\tilde{C})$ as well as the equalities

(1.2.5)

$$A_C = I_{x_0}(\text{Sh}(\Omega(C))) = F_\infty \mathcal{O}_{mod}(\tilde{C}) = F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) = F_\infty^\mu \mathcal{O}_{hol}(\tilde{C})$$

of subalgebras of $\mathcal{O}_{hol}(\tilde{C})$ (see §1.1 for the definition of A_C).

The first equality of (1.2.4) follows from the combination of Props. 5.6 and 3.17(b), its second equality follows from Prop. 5.5, and (1.2.5) follows from the combination of Cor. 5.4(c) and of the equalities between total spaces of filtrations derived from (1.2.4), and from Prop. 5.9.

The proof of Thm. 1.1 relies on results on the action of Γ_C on the involved spaces, which might be of independent interest. In order to formulate these results, we denote by $\mathbb{C}\Gamma$ the group algebra of a finitely generated group Γ and for $n \geq 0$, by $F_n^\heartsuit(\mathbb{C}\Gamma)'$ the set of linear forms $\ell \in (\mathbb{C}\Gamma)^*$ such that $\ell((\gamma_1 - 1) \cdots (\gamma_{n+1} - 1)) = 0$ for all $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$. Then $(\mathbb{C}\Gamma)' := \cup_{n \geq 0} F_n^\heartsuit(\mathbb{C}\Gamma)'$ is a commutative Hopf algebra (cf. Lem. A.8), which is the algebra of regular functions on the pro-unipotent completion of Γ (cf. §D.2).

Following Def. B.1, let us call *Hopf algebra with comodule-algebra* (HACA) a pair (O, A) , where O is a Hopf algebra, A is an associative algebra, equipped with an algebra morphism $\Delta_A : A \rightarrow O \otimes A$, which turns A into a comodule over the coalgebra O .

The right Γ_C -module structure of $\mathcal{O}_{mod}(\tilde{C})$ induces a Γ_C -module structure on $F_\infty \mathcal{O}_{mod}(\tilde{C})$, which by Lems. 3.15 and B.8 gives rise to a HACA structure on $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{mod}(\tilde{C}))$.

The proof of Thm. 1.1 is based on the following result. To state it, one needs to choose a section $\sigma : H_C^{\text{dR}} \rightarrow \Omega(C)$ of the canonical projection, where $H_C^{\text{dR}} := \Omega(C)/d(\mathcal{O}(C))$ and $d : \mathcal{O}(C) \rightarrow \Omega(C)$ is the differential (as C is affine, one has $H_C^{\text{dR}} \simeq H_{\text{dR}}^1(C)$).

Theorem 1.2. *Let $\sigma : \mathbf{H}_C^{\mathrm{dR}} \rightarrow \Omega(C)$ be a section of the canonical projection.*

(a) *The bilinear form*

$$p_{\sigma, x_0} : \mathbb{C}\Gamma_C \times \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \rightarrow \mathbb{C}, \quad \gamma \otimes a \mapsto I_{x_0}(\sigma(a))(\gamma \cdot x_0)$$

is a Hopf algebra pairing; it induces an isomorphism of commutative Hopf algebras

$$\nu(p_{\sigma, x_0}) : \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \rightarrow (\mathbb{C}\Gamma_C)'.$$

(b) *The map*

$$f_{\sigma, x_0} : \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C) \rightarrow F_\infty \mathcal{O}_{\mathrm{mod}}(\tilde{C}), \quad a \otimes g \mapsto I_{x_0}(\sigma(a)) \cdot p^*(g)$$

is an isomorphism of commutative algebras; moreover, the following equalities hold

(1.2.6)

$$F_\bullet \mathcal{O}_{\mathrm{mod}}(\tilde{C}) = f_{\sigma, x_0}(F_\bullet \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C)), \quad F_\bullet^\delta \mathcal{O}_{\mathrm{hol}}(\tilde{C}) = f_{\sigma, x_0}(F_\bullet \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes F_\bullet^{\mathrm{triv}} \mathcal{O}(C)),$$

where $F_\bullet^{\mathrm{triv}} \mathcal{O}(C)$ is given by $F_0^{\mathrm{triv}} \mathcal{O}(C) = \mathbb{C}$ and $F_1^{\mathrm{triv}} \mathcal{O}(C) = \mathcal{O}(C)$ (see §3.1).

(c) *($\nu(p_{\sigma, x_0}), f_{\sigma, x_0}$) : ($\mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}), \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C)$) \rightarrow ($(\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{\mathrm{mod}}(\tilde{C})$) is a HACA isomorphism, where the source is equipped with the HACA structure given by the coproduct $\Delta_{\mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}})} \otimes \mathrm{id}_{\mathcal{O}(C)} : \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \rightarrow \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes (\mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C))$, and $\Delta_{\mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}})}$ is the (deconcatenation) coproduct of the Hopf algebra structure of $\mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}})$.*

(a) is proved in Prop. 3.12, the first sentence in (b) is proved in Prop. 3.17(a), the first (resp. second) equality in (1.2.6) is proved in Prop. 3.17(b) (resp. follows from the combination of Props. 5.2 and 5.5), (c) is proved in Prop. 3.17(a). Note that the second equality of (1.2.6) may be expressed as $F_\bullet^\delta \mathcal{O}_{\mathrm{hol}}(\tilde{C}) = f_{\sigma, x_0}(F_\bullet \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes \mathbb{C} + F_{\bullet-1} \mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C))$.

1.2.2. *Further results on A_C .* In §4, we give precisions on the local behavior of elements of A_C , which may be viewed, since $A_C \subset \mathcal{O}_{\mathrm{mod}}(\tilde{C})$, as functions on a universal cover of $\tilde{C} \setminus p^{-1}(T)$ with moderate growth near each element of $p^{-1}(T)$ (see Prop. 4.2); in particular, the germs of the functions of A_C near each such element are *Nilsson-class functions* (in the sense of [Ph], p. 154) of a particular kind.

In §7, we further study the algebra structure of A_C . We show that $f_{\sigma, x_0}(\mathrm{Sh}(\mathbf{H}_C^{\mathrm{dR}}) \otimes \mathbb{C})$ is a subalgebra of A_C , which is independent of x_0 (§7.1); we denote it by $\mathcal{H}(C, \sigma)$. In Prop. 7.2, we exhibit a vector space basis of $\mathcal{H}(C, \sigma)$, and show it to be a basis of A_C as an $\mathcal{O}(C)$ -module. For $S \subset \mathbb{C}$ any finite subset and $C = \mathbb{C} \setminus S$, we exhibit (§7.2) a section σ_0 such that $\mathcal{H}(\mathbb{C} \setminus S, \sigma_0)$ coincides with the usual algebra of hyperlogarithm functions ([Po, LD]), so that Prop. 7.2 is a generalization of Cor. 5.6 in [Br].

1.2.3. *Further results on HACA structures.* The result of Thm. 1.2(c) leads to the following extension of the notion of filtered formality ([SW1]) to the setting of HACAs. Any Hopf algebra O is equipped with a Hopf algebra filtration $F_\bullet^\heartsuit O$ generalizing the already introduced filtration when $O = (\mathbb{C}\Gamma)'$ (see Lem. A.2, [BGF], §3.3.4 and [Fr] §7.2), which gives rise to a

graded Hopf algebra $\text{gr}^\heartsuit(O)$. We say that O is *filtered formal* if there exists an isomorphism of filtered Hopf algebras $O \rightarrow \text{gr}^\heartsuit O$, compatible with the filtrations and whose associated graded is the identity (see Def. C.2); we show that if a finitely generated group Γ is filtered formal in the sense of [SW1], then the Hopf algebra $(\mathbb{C}\Gamma)'$ is filtered formal (see Prop. D.18). These notions extend to HACAs as follows. Any HACA (O, A) is equipped with a HACA filtration $(F_\bullet^\heartsuit O, F_\bullet A)$, whose first term is the above filtration of O (see Lem. B.3), and therefore gives rise to a graded HACA $(\text{gr}^\heartsuit O, \text{gr} A)$ (see Lem. C.3). We say that the HACA (O, A) is *filtered formal* if there is an isomorphism of filtered HACAs $(O, A) \rightarrow (\text{gr}^\heartsuit O, \text{gr} A)$, whose associated graded is the identity (see Def. C.4).

Proposition 1.3. *(see Prop. 3.18) The HACA $((\mathbb{C}\Gamma)', F_\infty \mathcal{O}_{\text{mod}}(\tilde{C}))$ is filtered formal; the associated graded HACA is $(\text{gr}^\heartsuit((\mathbb{C}\Gamma)'), \text{gr} \mathcal{O}_{\text{mod}}(\tilde{C}))$, where $\text{gr}^\heartsuit, \text{gr}$ are graded spaces associated to the filtrations F_\bullet^\heartsuit and F_\bullet of $(\mathbb{C}\Gamma)'$ and $F_\infty \mathcal{O}_{\text{mod}}(\tilde{C})$.*

Moreover, in §6 we introduce the notion of connection of a HACA (Def. 6.1); we show that the HACA $((\mathbb{C}\Gamma)', F_\infty \mathcal{O}_{\text{mod}}(\tilde{C}))$ is equipped with a natural connection (Prop. 6.3), and we compute its pull-back under the HACA isomorphism $(\nu(p_{\sigma, x_0}), f_{\sigma, x_0})$ (Prop. 6.5).

1.2.4. *Further results on I_{x_0} .* In §8, we obtain an explicit description of $\text{Ker}(I_{x_0})$ (Thm. 8.7(a)), and associate to each section $\sigma : H_C^{\text{dR}} \rightarrow \Omega(C)$ a complement Sub_σ of $\text{Ker}(I_{x_0})$ in $\text{Sh}(\Omega(C))$ (Thm. 8.7(b)).

1.2.5. *The main tools.* The proof of Thm. 1.2 and Prop. 1.3 rely mostly on a collection of general results on Hopf algebras and HACAs, which are the subject of Appendices A to D. The main of these results are Lem. B.17 on the construction and injectivity of a natural transformation $(O, A) \mapsto \text{nat}_{(O, A)}$ relating two functors $\mathbf{HACA} \rightarrow \mathbf{Alg}_{gr}$, where \mathbf{Alg}_{gr} is the category of graded associative algebras, and Prop. B.18 giving criteria for the construction of isomorphisms of HACAs.

2. MODERATE GROWTH FUNCTIONS AND ITERATED INTEGRALS

We introduce in §2.1 the algebra $\mathcal{O}_{\text{mod}}(\tilde{C})$ of moderate growth functions on \tilde{C} , and in §2.2 the space of moderate growth differentials $\Omega_{\text{mod}}(\tilde{C})$, which is a module over it. In §2.3, we define the iterated integral morphism I_{x_0} , review its standard properties, and relate it to moderate growth functions.

2.1. **Moderate growth functions on \tilde{C} .** Let C be a nonsingular affine algebraic curve defined over \mathbb{C} ; we also denote by C the underlying Riemann surface. We fix a universal cover $p : \tilde{C} \rightarrow C$, and set $\Gamma_C := \text{Aut}(\tilde{C}/C)$. We denote by $\mathcal{O}(C) := H^0(C, \mathcal{O}_C)$ the commutative algebra of algebraic functions on C and by $\Omega(C) := H^0(C, \Omega_C^1)$ the $\mathcal{O}(C)$ -module of regular differentials on C . We also denote by $\mathcal{O}_{\text{hol}}(\tilde{C})$ the algebra of holomorphic functions on \tilde{C} .

2.1.1. *Moderate growth functions on a disc.* Set $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$, $D^\times := D \setminus \{0\}$, $\tilde{D}^\times := \{u \in \mathbb{C} \mid \Im(u) \geq 0\}$, $e : \tilde{D}^\times \rightarrow D^\times$ be the map defined by $e(u) := \exp(2\pi i u)$ (we set $i := \sqrt{-1}$), θ be the automorphism of \tilde{D}^\times given by $\theta(u) = u + 1$. For $f : M \rightarrow N$ a morphism in the category of complex manifolds, we denote by $f^* : \mathcal{O}_{hol}(N) \rightarrow \mathcal{O}_{hol}(M)$ the induced morphism between algebras of holomorphic functions. For $F = D, D^\times, \tilde{D}^\times$, we denote by $\mathcal{O}_{hol}(F)$ the algebra of continuous complex functions on F which are holomorphic on its interior.

Definition 2.1. (see [Ph], Ch. VIII, Def. 1.2) Define $\mathcal{O}_{mod}(\tilde{D}^\times) \subset \mathcal{O}_{hol}(\tilde{D}^\times)$ as the set of functions f such that there exists an integer $n > 0$ and a function $\{(a, b) \in \mathbb{R}^2 \mid a \leq b\} \ni (a, b) \mapsto C_{a,b} \in \mathbb{R}_+$ such that $|f(x + iy)| \leq C_{a,b} e^{2\pi n y}$ for $(x, y) \in [a, b] \times \mathbb{R}_+$.

Remark 2.2. If f satisfies these conditions for the pair $(n, (a, b) \mapsto C_{a,b})$, then it satisfies it also for $(n + 1, (a, b) \mapsto C_{a,b})$.

Lemma 2.3. $\mathcal{O}_{mod}(\tilde{D}^\times)$ is a subalgebra of $\mathcal{O}_{hol}(\tilde{D}^\times)$, equipped with an action of \mathbb{Z} where 1 acts by θ^* .

Proof. The constant function c satisfies the conditions from Def. 2.1 with $n = 0$ and $(a, b) \mapsto C_{a,b} = |c|$; if the function f (resp. g) satisfies them with $(n, (a, b) \mapsto C_{a,b})$ (resp. $(m, (a, b) \mapsto D_{a,b})$), then the function $f + g$ satisfies them with $(\max(n, m), (a, b) \mapsto C_{a,b} + D_{a,b})$, the function fg satisfies them with $(n + m, (a, b) \mapsto C_{a,b} D_{a,b})$, and the function $\theta^* f$ satisfies them with $(n, (a, b) \mapsto C_{a+1, b+1})$. \square

Definition 2.4. Set $\mathcal{O}(D^\times) := \{f \in \mathcal{O}_{hol}(D^\times) \mid \exists n \geq 0, z^n f \in \mathcal{O}_{hol}(D)\}$.

Then $\mathcal{O}(D^\times)$ is the algebra of meromorphic functions on D^\times with only possible poles at 0.

Lemma 2.5. $\mathcal{O}_{mod}(\tilde{D}^\times)^{\mathbb{Z}} = \mathcal{O}(D^\times)$.

Proof. The inclusion $\mathcal{O}_{mod}(\tilde{D}^\times)^{\mathbb{Z}} \supset \mathcal{O}(D^\times)$ is evident, let us show the opposite inclusion. Let $f \in \mathcal{O}_{mod}(\tilde{D}^\times)^{\mathbb{Z}}$. Then there exists $n > 0$ and $C_{0,1} \in \mathbb{R}_+$, such that $|f(x + iy)| \leq C_{0,1} e^{2\pi n y}$ for any $(x, y) \in [0, 1] \times \mathbb{R}_+$. Since f is \mathbb{Z} -invariant, $f \in \mathcal{O}_{hol}(D^\times)$, therefore $z^n f \in \mathcal{O}_{hol}(D^\times)$. For $z \in D^\times$, there exists a unique $(x, y) \in [0, 1] \times \mathbb{R}_+$ such that $z = e(x + iy)$. Then $|z^n f(z)| \leq C_{0,1} e^{-2\pi n y} e^{2\pi n y} = C_{0,1}$. By the removable singularity theorem (see [L], Thm. 3.1), $z^n f$ is the restriction to D^\times of a function of $\mathcal{O}_{hol}(D)$, therefore $f \in \mathcal{O}(D^\times)$. \square

2.1.2. *The algebra $\mathcal{O}_{mod}(\tilde{C})$ of moderate growth functions.*

Definition 2.6. (a) (\bar{C}, S) is the pair of a nonsingular projective algebraic curve \bar{C} and a finite set S of complex points of \bar{C} such that $C = \bar{C} \setminus S$.

(b) For $s \in S$, $\varphi_s : D \rightarrow \bar{C}$ is an injective holomorphic map, such that $0 \mapsto s$ and $\varphi_s(D) \cap S = \{s\}$; we set $U_s := \varphi_s(D) \subset \bar{C}$, so that φ_s corestricts to a biholomorphic map $D \xrightarrow{\sim} U_s$.

(c) For $s \in S$, $\varphi_s^\times : D^\times \rightarrow C$ is the injective holomorphic map obtained by restriction of φ_s ; we set $U_s^\times := U_s \setminus \{s\}$, so that φ_s^\times corestricts to a biholomorphic map $D^\times \xrightarrow{\sim} U_s^\times$.

(d) For $s \in S$, we set $\tilde{U}_s^\times := p^{-1}(U_s^\times)$ and $X_s := \pi_0(\tilde{U}_s^\times)$; for $x \in X_s$, we define $\tilde{U}_{s,x}^\times$ as the connected component of \tilde{U}_s^\times corresponding to x .

(e) For $s \in S$ and $x \in X_s$, $\tilde{\varphi}_{s,x}^\times : \tilde{D}^\times \rightarrow \tilde{C}$ is a holomorphic map with image contained in $\tilde{U}_{s,x}^\times$, such that $p \circ \tilde{\varphi}_{s,x}^\times = \varphi_s^\times \circ e$. Then $\tilde{\varphi}_{s,x}^\times$ corestricts to a biholomorphic map $\tilde{D}^\times \rightarrow \tilde{U}_{s,x}^\times$.

The choice of $(\tilde{\varphi}_{s,x}^\times)_{s \in S, x \in X_s}$ is not unique, but any two choices are related by $\tilde{\psi}_{s,x}^\times = \tilde{\varphi}_{s,x}^\times \circ \theta^{a_{s,x}}$, where $(a_{s,x})_{s \in S, x \in X_s}$ is in $\bigoplus_{s \in S} \mathbb{Z}^{X_s}$.

For any $s \in S$, the group Γ_C acts on \tilde{U}_s^\times , and therefore also on the set X_s . The latter action is transitive, and the stabilizer of any element $x \in X_s$ is a cyclic group, generated by an element $\theta_{s,x} \in \Gamma_C$ which restricts to an automorphism of $\tilde{U}_{s,x}^\times$, equal to the conjugation of θ by the corestriction of $\tilde{\varphi}_{s,x}^\times$ to an isomorphism $\tilde{D}^\times \rightarrow \tilde{U}_{s,x}^\times$.

Lemma 2.7. *There exists a map $c : \Gamma_C \times X_s \rightarrow \mathbb{Z}$ satisfying the identity $c(\gamma'\gamma, x) = c(\gamma', \gamma x) + c(\gamma, x)$, such that for any $(\gamma, x) \in \Gamma_C \times X_s$, one has $\tilde{\varphi}_{s,\gamma x}^\times \circ \theta^{c(\gamma,x)} = \gamma \circ \tilde{\varphi}_{s,x}^\times$ (equality of holomorphic maps $\tilde{D}^\times \rightarrow \tilde{C}$).*

Proof. The existence of a map c satisfying the identity $\tilde{\varphi}_{s,\gamma x}^\times \circ \theta^{c(\gamma,x)} = \gamma \circ \tilde{\varphi}_{s,x}^\times$ follows from the fact that for any pair of holomorphic maps $\alpha, \beta : \tilde{D}^\times \rightarrow \tilde{C}$ such that $p \circ \alpha = \varphi_s^\times \circ e = p \circ \beta$, there exists $n \in \mathbb{Z}$ such that $\beta = \alpha \circ \theta^n$. Then $\tilde{\varphi}_{s,\gamma'\gamma x}^\times \circ \theta^{c(\gamma'\gamma, x)} = \gamma'\gamma \circ \tilde{\varphi}_{s,x}^\times = \gamma' \circ \tilde{\varphi}_{s,\gamma x}^\times \circ \theta^{c(\gamma, x)} = \tilde{\varphi}_{s,\gamma'\gamma x}^\times \circ \theta^{c(\gamma', \gamma x)} \circ \theta^{c(\gamma, x)}$ which implies the identity satisfied by c . \square

Lemma 2.8. (a) *For any $s \in S$, the space $X_s \times \tilde{D}^\times$ is equipped with an action of Γ_C given by $\gamma \cdot (x, d) := (\gamma x, \theta^{c(\gamma, x)} d)$. The induced right action of Γ_C on $\prod_{x \in X_s} \mathcal{O}_{hol}(\tilde{D}^\times)$ is given by $(f_x)_{x \in X_s} \cdot \gamma = (g_x)_{x \in X_s}$, where $g_x := (\theta^{c(\gamma, x)})^*(f_{\gamma x})$.*

(b) *The map $\prod_{s \in S} X_s \times \tilde{D}^\times \rightarrow \tilde{C}$, $(s, x, d) \mapsto \tilde{\varphi}_{s,x}^\times(d)$ is Γ_C -equivariant, its source being equipped with the direct sum of the actions of Γ_C defined in (a). It induces a Γ_C -equivariant algebra morphism*

$$(2.1.1) \quad \bigoplus_{s \in S} \prod_{x \in X_s} (\tilde{\varphi}_{s,x}^\times)^* : \mathcal{O}_{hol}(\tilde{C}) \rightarrow \bigoplus_{s \in S} \prod_{x \in X_s} \mathcal{O}_{hol}(\tilde{D}^\times),$$

in which the target is equipped with the direct sum over $s \in S$ of the right actions from (a).

Proof. This follows from Lem. 2.7. \square

Definition 2.9. $\mathcal{O}_{mod}(\tilde{C})$ is the subset of $\mathcal{O}_{hol}(\tilde{C})$ of all functions f such that for any $s \in S$ and $x \in X_s$, $(\tilde{\varphi}_{s,x}^\times)^*(f) \in \mathcal{O}_{mod}(\tilde{D}^\times)$.

Proposition 2.10. $\mathcal{O}_{mod}(\tilde{C})$ is a subalgebra of $\mathcal{O}_{hol}(\tilde{C})$, stable under the action of Γ_C .

Proof. This follows from the equality of $\mathcal{O}_{mod}(\tilde{C})$ with the preimage of $\bigoplus_{s \in S} \prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times)$ under (2.1.1), and from the fact that this is a Γ_C -stable subalgebra of the target of (2.1.1). \square

2.1.3. *Computation of $\mathcal{O}_{mod}(\tilde{C})^{\Gamma_C}$.*

Lemma 2.11. *For any $s \in S$, the diagonal embedding $\mathcal{O}(D^\times) \hookrightarrow \prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times)$ gives rise to an isomorphism $\mathcal{O}(D^\times) \simeq (\prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times))^{\Gamma_C}$.*

Proof. The inclusion of the image of the diagonal embedding in $(\prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times))^{\Gamma_C}$ is evident, let us prove the opposite inclusion. Let $y \in X_s$. The stabilizer of y for the action of Γ_C on X_s is a cyclic group, generated by an element $\theta_y \in \Gamma_C$ such that $c(\theta_y, y) = 1$. If now $\underline{f} := (f_x)_{x \in X_s} \in (\prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times))^{\Gamma_C}$, then $\underline{f} = \underline{f} \cdot \theta_y$, therefore for any $x \in X_s$, $f_x = (\theta^{c(\theta_y, x)})^*(f_{\theta_y x})$, which for $x = y$ implies $f_x = \theta^* f_x$. By Lem. 2.5(b), this implies $f_x \in \mathcal{O}(D^\times)$. For any $\gamma \in \Gamma$, one has $\underline{f} = \underline{f} \cdot \gamma$, which given the θ -invariance of each f_x , $x \in X_s$ implies $f_x = f_{\gamma x}$ for any $x \in X_s$. Since the action of Γ_C on X_s is transitive, this implies that the map $x \mapsto f_x$ is constant. \square

Proposition 2.12. $\mathcal{O}_{mod}(\tilde{C})^{\Gamma_C} = \mathcal{O}(C)$.

Proof. Let $n \geq 0$. It follows from Prop. 2.10 that (2.1.1) induces a linear and Γ_C -equivariant algebra morphism

$$(2.1.2) \quad \mathcal{O}_{mod}(\tilde{C}) \rightarrow \bigoplus_{s \in S} \prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times).$$

This map restricts to a linear map $\mathcal{O}_{mod}(\tilde{C})^{\Gamma_C} \rightarrow (\bigoplus_{s \in S} \prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times))^{\Gamma_C}$. The target is equal to $\bigoplus_{s \in S} (\prod_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times))^{\Gamma_C}$, which by Lem. 2.11 is equal to $\bigoplus_{s \in S} \mathcal{O}(D^\times)$.

On the other hand, $\mathcal{O}_{mod}(\tilde{C})^{\Gamma_C} \subset \mathcal{O}_{hol}(\tilde{C})^{\Gamma_C} = \mathcal{O}_{hol}(C)$. All this implies that $\mathcal{O}_{mod}(\tilde{C})^{\Gamma_C}$ is the preimage of $\bigoplus_{s \in S} \mathcal{O}(D^\times)$ by the map

$$\bigoplus_{s \in S} (\varphi_s^\times)^* : \mathcal{O}_{hol}(C) \rightarrow \bigoplus_{s \in S} \mathcal{O}_{hol}(D^\times),$$

which is equal to $\mathcal{O}(C)$. \square

One checks that the subalgebra $\mathcal{O}_{mod}(\tilde{D}) \subset \mathcal{O}_{hol}(\tilde{C})$ is independent of the choice of the family $(\varphi_s)_{s \in S}$.

2.2. The module $\Omega_{mod}(\tilde{C})$ of moderate growth differentials. If M is a complex manifold, let $\Omega_{hol}^\bullet(M)$ be the dga of holomorphic differential forms on M . Then $\Omega_{hol}^0(M) = \mathcal{O}_{hol}(M)$. The assignment $M \mapsto \Omega_{hol}^\bullet(M)$ is a contravariant functor, so a morphism $f : M \rightarrow N$ of complex manifolds gives rise to a dga morphism $f^* : \Omega_{hol}^\bullet(N) \rightarrow \Omega_{hol}^\bullet(M)$. If M is 1-dimensional, we set $\Omega_{hol}(M) := \Omega_{hol}^1(M)$; then $\Omega_{hol}(M)$ is a $\mathcal{O}_{hol}(M)$ -module, equipped with a derivation $d : \mathcal{O}_{hol}(M) \rightarrow \Omega_{hol}(M)$.

Lemma 2.13. $\Omega_{hol}(\tilde{D}^\times)$ is a free rank 1 module over $\mathcal{O}_{hol}(\tilde{D}^\times)$ generated by $e^*(dz/z)$, so $f \mapsto f \cdot e^*(dz/z)$ gives rise to a $\mathcal{O}_{hol}(\tilde{D}^\times)$ -module isomorphism $\mathcal{O}_{hol}(\tilde{D}^\times) \xrightarrow{\sim} \Omega_{hol}(\tilde{D}^\times)$.

Proof. This follows from the fact that dz/z is an invertible differential in $\Omega_{hol}(D^\times)$, which implies the same about its pull-back by $\tilde{D}^\times \rightarrow D^\times$. \square

Definition 2.14. Define $\Omega_{mod}(\tilde{D}^\times) := \mathcal{O}_{mod}(\tilde{D}^\times) \cdot e^*(dz/z)$ as the image of $\mathcal{O}_{mod}(\tilde{D}^\times)$ under the isomorphism from Lem. 2.13.

Lemma 2.15. $\Omega_{mod}(\tilde{D}^\times)$ is a free rank 1 module over $\mathcal{O}_{mod}(\tilde{D}^\times)$.

Proof. The follows from the injectivity of the map $\mathcal{O}_{hol}(\tilde{D}^\times) \rightarrow \Omega_{hol}(\tilde{D}^\times)$, $f \mapsto f \cdot e^*(dz/z)$. \square

The morphism $\prod_{s \in S} \prod_{x \in X_s} \varphi_{s,x} : \prod_{s \in S} \prod_{x \in X_s} \tilde{D}^\times \rightarrow \tilde{C}$ gives rise to a morphism

$$(2.2.1) \quad \bigoplus_{s \in S} \prod_{x \in X_s} (\tilde{\varphi}_{s,x}^\times)^* : \Omega_{hol}(\tilde{C}) \rightarrow \bigoplus_{s \in S} \prod_{x \in X_s} \Omega_{hol}(\tilde{D}^\times).$$

Definition 2.16. Define $\Omega_{mod}(\tilde{C})$ as the preimage of $\bigoplus_{s \in S} \prod_{x \in X_s} \Omega_{mod}(\tilde{D}^\times)$ under (2.2.1).

Lemma 2.17. $\Omega_{mod}(\tilde{C})$ is a module over $\mathcal{O}_{mod}(\tilde{C})$.

Proof. If $f \in \mathcal{O}_{mod}(\tilde{C})$, $\omega \in \Omega_{mod}(\tilde{C})$, then $f\omega \in \Omega_{hol}(\tilde{C})$. If $s \in S$ and $x \in X_s$, then $i_{s,x}^*(f\omega) = i_{s,x}^*(f)i_{s,x}^*(\omega) \in \Omega_{mod}(\tilde{D}^\times)$ where the last relation follows from $i_{s,x}^*(f) \in \mathcal{O}_{mod}(\tilde{D}^\times)$, $i_{s,x}^*(\omega) \in \Omega_{mod}(\tilde{D}^\times)$. Therefore $f\omega \in \Omega_{mod}(\tilde{C})$. \square

Since \tilde{D}^\times is simply-connected, the assignment $\omega \mapsto (z \mapsto \int_i^z \omega)$ is a well-defined linear map $\text{int}_i : \Omega_{hol}(\tilde{D}^\times) \rightarrow \mathcal{O}_{hol}(\tilde{D}^\times)$.

Lemma 2.18. The map int_i takes $\Omega_{mod}(\tilde{D}^\times)$ to $\mathcal{O}_{mod}(\tilde{D}^\times)$.

Proof. Let $\omega \in \Omega_{mod}(\tilde{D}^\times)$. Then there exists $f \in \mathcal{O}_{mod}(\tilde{D}^\times)$ such that $\omega = f \cdot e^*(dz/z)$. Let $n > 0$ and $(a, b) \mapsto C_{a,b}$ be the integer and function associated with f (see Def. 2.1). Then $\text{int}_i(\omega) \in \mathcal{O}_{hol}(\tilde{D}^\times)$ is the function $u \mapsto 2\pi i \int_i^u f(u') du'$. Let $A \geq 0$ and $u = x + iy$ with $x \in [-A, A]$; set $C_A := C_{-A,A}$. Then the path of integration may be chosen as the sequence of paths $i \rightarrow iy \rightarrow u = x + iy$, therefore $\int_i^u f(u') du' = \int_i^{iy} f(u') du' + \int_{iy}^{x+iy} f(u') du' = i \int_1^y f(it) dt + \int_0^x f(t+iy) dt$.

One has $|f(it)| \leq C_A e^{2\pi n t}$ for $t \in [1, y]$, while $|f(t+iy)| \leq C_A e^{2\pi n y}$ for $y \in [0, x]$. Therefore $|i \int_1^y f(it) dt| \leq C_A |\int_1^y e^{2\pi n t} dt| = C_A |e^{2\pi n y} - e^{2\pi n}| / (2\pi n)$, and $|\int_0^x f(t+iy) e^{2\pi i t} dt| \leq AC_A e^{2\pi n y}$. Therefore

$$(2.2.2) \quad \left| i \int_i^u f(u') du' \right| \leq C_A \frac{|e^{2\pi n y} - e^{2\pi n}|}{2\pi n} + AC_A e^{2\pi n y}.$$

If $y \geq 1$, the right-hand side of (2.2.2) is $\leq C_A(A + 1/(2\pi n))e^{2\pi n y}$ (expressing the absolute value as its argument as the latter is ≥ 0 and bounding the resulting expression from above by removing the negative term); if $y < 1$, the right-hand side of (2.2.2) is $\leq C_A e^{2\pi n} / (2\pi n) + AC_A e^{2\pi n y}$ (expressing the absolute value as the negative of its argument as the latter is ≤ 0 and bounding the resulting expression from above by removing the negative term) which is $\leq C_A (e^{2\pi n} / (2\pi n)) e^{2\pi n y} + AC_A e^{2\pi n y}$ (as $e^{2\pi n y} \geq 1$), therefore the right-hand side of (2.2.2) is $\leq (A + (e^{2\pi n} / (2\pi n))) C_A e^{2\pi n y}$.

Set $D_A := (A + (e^{2\pi n}/(2\pi n)))C_A$, then one obtains $|\int_i^u f(u')du'| \leq D_A e^{2\pi ny}$ for every $u \in [-A, A] + i\mathbb{R}_+^\times$, and therefore

$$\left| \int_i^u \omega \right| \leq 2\pi D_A e^{2\pi ny}$$

for any $u \in [-A, A] + i\mathbb{R}_+^\times$. This shows that $u \mapsto \int_i^u \omega$ satisfies the condition of Def. 2.1 with the pair $(n, (a, b) \mapsto 2\pi D_{\max(|a|, |b|)})$, and therefore belongs to $\mathcal{O}_{mod}(\tilde{D}^\times)$. \square

Fix a point $x_0 \in \tilde{C}$. Since \tilde{C} is simply-connected, the assignment $\omega \mapsto (z \mapsto \int_{x_0}^z \omega)$ is a well-defined linear map $\text{int}_{x_0} : \Omega_{hol}(\tilde{C}) \rightarrow \mathcal{O}_{hol}(\tilde{C})$.

Lemma 2.19. *The map int_{x_0} takes $\Omega_{mod}(\tilde{C})$ to $\mathcal{O}_{mod}(\tilde{C})$.*

Proof. Let $\omega \in \Omega_{mod}(\tilde{C})$. Let $s \in S$ and $x \in X_s$. By additivity of the integral w.r.t. the composition of paths, one has

$$(2.2.3) \quad (\tilde{\varphi}_{s,x}^\times)^*(\text{int}_{x_0}(\omega)) = \int_{x_0}^{\tilde{\varphi}_{s,x}^\times(i)} \omega + \text{int}_i((\tilde{\varphi}_{s,x}^\times)^*\omega)$$

(equality in $\mathcal{O}_{hol}(\tilde{D}^\times)$), where $\int_{x_0}^{\tilde{\varphi}_{s,x}^\times(i)} \omega$ belongs to \mathbb{C} . Since $\omega \in \Omega_{mod}(\tilde{C})$, one has $(\tilde{\varphi}_{s,x}^\times)^*\omega \in \Omega_{mod}(\tilde{D}^\times)$; it then follows from Lem. 2.18 that $\text{int}_i((\tilde{\varphi}_{s,x}^\times)^*\omega) \in \mathcal{O}_{mod}(\tilde{D}^\times)$. Since $\mathcal{O}_{mod}(\tilde{D}^\times)$ is an algebra containing \mathbb{C} , it follows that the right-hand side of (2.2.3) belongs to $\mathcal{O}_{mod}(\tilde{D}^\times)$. Eq. (2.2.3) then implies that $(\tilde{\varphi}_{s,x}^\times)^*(\text{int}_{x_0}(\omega)) \in \mathcal{O}_{mod}(\tilde{D}^\times)$. As this holds for any $s \in S$ and $x \in X_s$, one derives $\text{int}_{x_0}(\omega) \in \mathcal{O}_{mod}(\tilde{C})$. \square

2.3. Iterated integrals. Let $(\text{Sh}(V), \sqcup)$ be the shuffle algebra associated with a vector space V ; it is isomorphic to the tensor algebra $\bigoplus_{d \geq 0} V^{\otimes d}$ as a vector space and is equipped with the shuffle product, which is commutative. It has the decomposition $\text{Sh}(V) = \bigoplus_{d \geq 0} \text{Sh}_d(V)$, where $\text{Sh}_d(V)$ is the degree d component $V^{\otimes d}$, which is an algebra grading; an element $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$ will be denoted $[v_1 | \cdots | v_d] \in \text{Sh}_d(V)$.

For $n \geq 0$, let $\partial_n^V : \text{Sh}_n(V) \rightarrow \text{Sh}_{n-1}(V) \otimes V$ given by $[v_1 | \cdots | v_n] \mapsto [v_1 | \cdots | v_{n-1}] \otimes v_n$ if $n > 0$, $1 \mapsto 0$ if $n = 0$. Let $\partial^V : \text{Sh}(V) \rightarrow \text{Sh}(V) \otimes V$ be the direct sum of the maps ∂_n^V for $n \geq 0$. Then ∂^V is a derivation of the commutative and associative algebra $\text{Sh}(V)$ with values in the free module $\text{Sh}(V) \otimes V$ (with action $a \sqcup (b \otimes v) := (a \sqcup b) \otimes v$), namely

$$(2.3.1) \quad \partial^V(a \sqcup b) = a \sqcup \partial^V(b) + b \sqcup \partial^V(a)$$

for any $a, b \in \text{Sh}(V)$.

Definition 2.20. (a) For $x_0 \in \tilde{C}$ and $n \geq 0$, $I_{x_0}^{(n)} : \text{Sh}_n(\Omega(C)) \rightarrow \mathcal{O}_{hol}(\tilde{C})$ is the linear map inductively defined as follows:

- for $n = 0$, the composition of $I_{x_0}^{(0)}$ with the equality $\text{Sh}_0(\Omega(C)) = \mathbb{C}$ is the map $\mathbb{C} \rightarrow \mathcal{O}_{hol}(\tilde{C})$, $1 \mapsto 1$;

- for $n > 0$, $I_{x_0}^{(n)}$ is the composition

$$\mathrm{Sh}_n(\Omega(C)) \xrightarrow{\partial_n^{\Omega(C)}} \mathrm{Sh}_{n-1}(\Omega(C)) \otimes \Omega(C) \xrightarrow{I_{x_0}^{(n-1)} \otimes p^*} \mathcal{O}_{hol}(\tilde{C}) \otimes \Omega_{hol}(\tilde{C}) \xrightarrow{act} \Omega_{hol}(\tilde{C}) \xrightarrow{\mathrm{int}_{x_0}} \mathcal{O}_{hol}(\tilde{C}),$$

where the map act is the action map of $\mathcal{O}_{hol}(\tilde{C})$ on $\Omega_{hol}(\tilde{C})$.

- (b) The linear map $I_{x_0} : \mathrm{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{hol}(\tilde{C})$ is the direct sum $\bigoplus_{n \geq 0} I_{x_0}^{(n)}$.

One has therefore for $n \geq 1$ and $\omega_1, \dots, \omega_n \in \Omega(C)$ the equality (in $\mathcal{O}_{hol}(\tilde{C})$)

$$(2.3.2) \quad I_{x_0}([\omega_1 | \dots | \omega_n]) = \mathrm{int}_{x_0}(I_{x_0}([\omega_1 | \dots | \omega_{n-1}]) \cdot p^* \omega_n).$$

Lemma 2.21. $I_{x_0} : \mathrm{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{hol}(\tilde{C})$ is an algebra morphism.

Proof. It follows from Def. 2.20 that for any $a \in \mathrm{Sh}(\Omega(C))$, one has

$$(2.3.3) \quad d(I_{x_0}(a)) = act \circ (I_{x_0} \otimes p^*)(\partial^{\Omega(C)}(a)).$$

Let us show inductively on $n := \deg(a) + \deg(b)$ that for $a, b \in \mathrm{Sh}(\Omega(C))$ homogeneous, one has $I_{x_0}(a \sqcup b) = I_{x_0}(a) \cdot I_{x_0}(b)$.

This is obvious if $n = 0$. Assume that this holds for total degree $< n$ and let $a, b \in \mathrm{Sh}(\Omega(C))$ be homogeneous, such that $\deg(a) + \deg(b) = n$. Then

$$\begin{aligned} d(I_{x_0}(a \sqcup b)) &= act \circ (I_{x_0} \otimes p^*)(\partial^{\Omega(C)}(a \sqcup b)) = act \circ (I_{x_0} \otimes p^*)(a \sqcup \partial^{\Omega(C)}(b) + b \sqcup \partial^{\Omega(C)}(a)) \\ &= act \left(I_{x_0}(a) \cdot (I_{x_0} \otimes p^*)(\partial^{\Omega(C)}(b) + I_{x_0}(b) \cdot (I_{x_0} \otimes p^*)(\partial^{\Omega(C)}(a)) \right) \\ &= I_{x_0}(a) \cdot act((I_{x_0} \otimes p^*)(\partial^{\Omega(C)}(b) + I_{x_0}(b) \cdot act((I_{x_0} \otimes p^*)(\partial^{\Omega(C)}(a))) \\ &= I_{x_0}(a) \cdot d(I_{x_0}(b)) + I_{x_0}(b) \cdot d(I_{x_0}(a)) = d(I_{x_0}(a) \cdot I_{x_0}(b)). \end{aligned}$$

where the first and fifth equalities follow from (2.3.3), the second equality follows from (2.3.1), the third equality follows from the induction hypothesis, the fourth equality follows from the fact that act is a $\mathcal{O}_{hol}(\tilde{C})$ -module morphism. Since the functions $I_{x_0}(a \sqcup b)$ and $I_{x_0}(a) \cdot I_{x_0}(b)$ both vanish at x_0 , it follows that $I_{x_0}(a \sqcup b) = I_{x_0}(a) \cdot I_{x_0}(b)$ \square

Lemma 2.22. For any $x_0, x_1 \in \tilde{C}$, $\gamma \in \Gamma_C$ and $a \in \mathrm{Sh}(\Omega(C))$, $I_{x_0}(a)(x_1) = I_{\gamma x_0}(a)(\gamma x_1)$.

Proof. It suffices to prove this for a homogeneous. One argues by induction on $\deg(a)$. The identity is obvious for $\deg(a) = 0$, and the identity for degree n follows from the identity for degree $n - 1$, (2.3.2) and the invariance of $p^* \omega$ for $\omega \in \Omega(C)$. \square

Lemma 2.23. (a) The maps $e^* : \mathcal{O}(D^\times) \rightarrow \mathcal{O}_{hol}(\tilde{D}^\times)$ and $e^* : \Omega(D^\times) \rightarrow \Omega_{hol}(\tilde{D}^\times)$ have their images respectively contained in $\mathcal{O}_{mod}(\tilde{D}^\times)$ and $\Omega_{mod}(\tilde{D}^\times)$.

(b) The maps $p^* : \mathcal{O}(C) \rightarrow \mathcal{O}_{hol}(\tilde{C})$ and $p^* : \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})$ have their images respectively contained in $\mathcal{O}_{mod}(\tilde{C})$ and $\Omega_{mod}(\tilde{C})$.

Proof. (a) Let $f \in \mathcal{O}(D^\times)$. There there exist $n \geq 1$ and $g \in \mathcal{O}(D)$ such that $f = g/z^n$. Then $e^* f = e^* g / (u \mapsto e(nu))$. The function g is bounded on D , therefore there exists $C \in \mathbb{R}_+$ such

that for any $u \in \tilde{D}^\times$, $|(e^*f)(u)| \leq C|1/e(nu)| = Ce^{2\pi ny}$. So e^*f satisfies the condition of Def. 2.1 with the pair $(n, (a, b) \mapsto C)$, so $e^*f \in \mathcal{O}_{\text{mod}}(D^\times)$.

Let $\omega \in \Omega(D^\times)$. There exists $f \in \mathcal{O}(D^\times)$ such that $\omega = f \cdot (dz/z)$. Then $e^*\omega = e^*f \cdot e^*(dz/z) \in \mathcal{O}_{\text{mod}}(\tilde{D}^\times) \cdot e^*(dz/z) = \Omega_{\text{mod}}(\tilde{D}^\times)$ (see Def. 2.14).

(b) Let $f \in \mathcal{O}(C)$ and $s \in S$, $x \in X_t$. Then it follows from $\varphi_s^\times \circ e = p \circ \tilde{\varphi}_{s,x}^\times$ that $(\tilde{\varphi}_{s,x}^\times)^*p^*f = e^*(\varphi_s^\times)^*f$. Then $(\varphi_s^\times)^*f \in \mathcal{O}(D^\times)$, and (a) then implies $e^*(\varphi_s^\times)^*f \in \mathcal{O}_{\text{mod}}(\tilde{D}^\times)$. It follows that $(\tilde{\varphi}_{s,x}^\times)^*p^*f \in \mathcal{O}_{\text{mod}}(\tilde{D}^\times)$ for any pair (s, x) , therefore by Def. 2.9, that $p^*f \in \mathcal{O}_{\text{mod}}(\tilde{C})$.

Similarly, let $\omega \in \Omega(C)$ and $s \in S$, $x \in X_s$. Then $(\tilde{\varphi}_{s,x}^\times)^*p^*\omega = e^*(\varphi_s^\times)^*\omega$, and since $(\varphi_s^\times)^*\omega \in \Omega(D^\times)$, (a) implies $e^*(\varphi_s^\times)^*\omega \in \Omega_{\text{mod}}(\tilde{D}^\times)$. It follows that $(\tilde{\varphi}_{s,x}^\times)^*p^*\omega \in \Omega_{\text{mod}}(\tilde{D}^\times)$ for any pair (s, x) , therefore by Def. 2.16, that $p^*f \in \Omega_{\text{mod}}(\tilde{C})$. \square

Proposition 2.24. *The algebra morphism $I_{x_0} : \text{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$ has its image contained in $\mathcal{O}_{\text{mod}}(\tilde{C})$, therefore it induces an algebra morphism $I_{x_0} : \text{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{\text{mod}}(\tilde{C})$.*

Proof. We prove inductively on $n \geq 0$ that $\text{im}(I_{x_0}^{(n)}) \subset \mathcal{O}_{\text{mod}}(\tilde{C})$. This is obvious if $n = 0$. Assume that $\text{im}(I_{x_0}^{(n-1)}) \subset \mathcal{O}_{\text{mod}}(\tilde{C})$. Let $\omega_1, \dots, \omega_n \in \Omega(C)$. By the induction hypothesis, $I_{x_0}^{(n-1)}([\omega_1 | \dots | \omega_{n-1}]) \in \mathcal{O}_{\text{mod}}(\tilde{C})$, and by Lem. 2.23(b), $p^*\omega_n \in \Omega_{\text{mod}}(\tilde{C})$; since $\Omega_{\text{mod}}(\tilde{C})$ is a module over $\mathcal{O}_{\text{mod}}(\tilde{C})$ (see Lem. 2.17), then $I_{x_0}^{(n-1)}([\omega_1 | \dots | \omega_{n-1}]) \cdot p^*\omega_n \in \Omega_{\text{mod}}(\tilde{C})$. Lem. 2.19 then implies $\text{int}_{x_0}(I_{x_0}^{(n-1)}([\omega_1 | \dots | \omega_{n-1}]) \cdot p^*\omega_n) \in \mathcal{O}_{\text{mod}}(\tilde{C})$. By (2.3.2), the latter term is $I_{x_0}^{(n)}([\omega_1 | \dots | \omega_n])$, therefore $I_{x_0}^{(n)}([\omega_1 | \dots | \omega_n]) \in \mathcal{O}_{\text{mod}}(\tilde{C})$, so $\text{im}(I_{x_0}^{(n)}) \subset \mathcal{O}_{\text{mod}}(\tilde{C})$. \square

Definition 2.25. If V is a vector space, then $\Delta_{\text{Sh}(V)}$ is the deconcatenation coproduct on $\text{Sh}(V)$, defined by $[v_1 | \dots | v_n] \mapsto \sum_{k=0}^n [v_1 | \dots | v_k] \otimes [v_{k+1} | \dots | v_n]$ for $v_1, \dots, v_n \in V$; it equips $\text{Sh}(V)$ with a commutative Hopf algebra structure.

Lemma 2.26. *For $x_0, x_1 \in \tilde{C}$ and $a \in \text{Sh}(\Omega(C))$, one has*

$$(2.3.4) \quad I_{x_0}(a) = I_{x_0}(a^{(1)})(x_1)I_{x_1}(a^{(2)}),$$

(equality in $\mathcal{O}_{\text{mod}}(\tilde{C})$), where $\Delta_{\text{Sh}(\Omega(C))}(a) = a^{(1)} \otimes a^{(2)}$.

Proof. The fact that the two sides of this equality belong to $\mathcal{O}_{\text{mod}}(\tilde{C})$ follows from Prop. 2.24. Let us prove (2.3.4) by induction on the degree of a homogeneous element a in $\text{Sh}(\Omega(C))$. (2.3.4) is obvious if a has degree 0. Assume that (2.3.4) is proved for any a of degree $< n$ and let us prove it in degree n . Let $\omega_1, \dots, \omega_n \in \Omega(C)$. Then

$$\begin{aligned} dI_{x_0}([\omega_1 | \dots | \omega_n]) &= I_{x_0}([\omega_1 | \dots | \omega_{n-1}]) \cdot p^*\omega_n = \sum_{k=0}^{n-1} I_{x_0}([\omega_1 | \dots | \omega_k])(x_1)I_{x_1}([\omega_{k+1} | \dots | \omega_{n-1}]) \cdot p^*\omega_n \\ &= \sum_{k=0}^{n-1} I_{x_0}([\omega_1 | \dots | \omega_k])(x_1)dI_{x_1}([\omega_{k+1} | \dots | \omega_n]) = d\left(\sum_{k=0}^{n-1} I_{x_0}([\omega_1 | \dots | \omega_k])(x_1)I_{x_1}([\omega_{k+1} | \dots | \omega_n])\right). \end{aligned}$$

where the first and third equalities follow from (2.3.2), and the second equality from (2.3.4) in degree $n-1$.

It follows that $I_{x_0}([\omega_1 | \dots | \omega_n]) - \sum_{k=0}^{n-1} I_{x_0}([\omega_1 | \dots | \omega_k])(x_1) I_{x_1}([\omega_{k+1} | \dots | \omega_n]) \in \mathcal{O}_{hol}(\tilde{C})$ is a constant function. Its value at x_1 is $I_{x_0}([\omega_1 | \dots | \omega_n])(x_1)$, therefore $I_{x_0}([\omega_1 | \dots | \omega_n]) = \sum_{k=0}^n I_{x_0}([\omega_1 | \dots | \omega_k])(x_1) I_{x_1}([\omega_{k+1} | \dots | \omega_n])$, proving (2.3.4) in degree n . \square

3. THE ISOMORPHISM OF FILTERED ALGEBRAS $f_{\sigma, x_0} : F_{\bullet} \text{Sh}(\mathbb{H}_C^{\text{dR}}) \otimes \mathcal{O}(C) \rightarrow F_{\bullet} \mathcal{O}_{mod}(\tilde{C})$

We start this section by reminders on filtrations (§3.1). We compute the restricted duals $(\mathbb{C}\Gamma_C)'$ of the Hopf algebra $\mathbb{C}\Gamma_C$ in §3.2 and $\text{Sh}(\mathbb{H}_C^{\text{dR}})^{\vee}$ of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$ in §3.3. We define a pairing p_{σ, x_0} between these Hopf algebras in §3.4 and prove in §3.5 that the induced Hopf algebra morphism $\nu(p_{\sigma, x_0})$ is an isomorphism (Prop. 3.12). We define a filtered algebra morphism f_{σ, x_0} in §3.6 and use Prop. 3.12 and Prop. B.18(b) to show in Prop. 3.17 that it is an isomorphism. In §3.7, we use the material of this proof together with Prop. B.18(c) to prove a filtered formality statement. .

3.1. Background on filtrations. A *vector space filtration* of a \mathbb{C} -vector space M is an increasing collection $F_{\bullet} M = (F_n M)_{n \geq 0}$ of vector subspaces of M . The filtration $F_{\bullet} M$ is called *exhaustive* if and only if $\cup_{n \geq 0} F_n M = M$. If $F_{\bullet} M$ is a filtration of M , then $F_{\infty} M := \cup_{n \geq 0} F_n M$ is a vector subspace of M , called the *total vector space* of the filtration; then $F_{\bullet} M$ is an exhaustive filtration of this vector subspace. The *associated graded* of $F_{\bullet} M$ is the graded \mathbb{C} -vector space $\text{gr}_{\bullet}(M) := \oplus_{n \geq 0} \text{gr}_n(M)$, where $\text{gr}_n(M) = F_n M / F_{n-1} M$ (with $F_{-1} M := 0$).

Let $f : M \rightarrow N$ be a morphism of \mathbb{C} -vector spaces and $F_{\bullet} M$ be a filtration of M . Then f is said to be *compatible* with a filtration $F_{\bullet} N$ on N if and only if $f(F_n M) \subset F_n N$ for any $n \geq 0$. This is in particular the case if $F_{\bullet} N$ is the *image of $F_{\bullet} M$ by f* (denoted $f(F_{\bullet} M)$), defined by $F_n N := f(F_n M)$ for any $n \geq 0$; one then has $F_{\infty} N = f(F_{\infty} M)$.

Lemma 3.1. *If M, N are filtered vector spaces and $f : M \rightarrow N$ is a linear map compatible with the filtrations, and such that $\text{gr}_{\bullet}(f) : \text{gr}_{\bullet}(M) \rightarrow \text{gr}_{\bullet}(N)$ is an isomorphism of graded vector spaces, then the maps $F_n f : F_n M \rightarrow F_n N$ for any $n \geq 0$, as well as $F_{\infty} f : F_{\infty} M \rightarrow F_{\infty} N$, are linear isomorphisms.*

Proof. Let us prove the first statement by induction on $n \geq 0$. The map $F_0 f : F_0 M \rightarrow F_0 N$ is the composition of the isomorphisms $F_0 M \simeq \text{gr}_0 M$, $\text{gr}_0 N \simeq F_0 N$ and $\text{gr}_0(f)$, which is an isomorphism, therefore $F_0 f : F_0 M \rightarrow F_0 N$ is an isomorphism of vector spaces. Let $n \geq 0$ and assume that $F_n f$ is an isomorphism of vector spaces. The image of $\text{Ker}(F_{n+1} f) \subset F_{n+1} M \rightarrow \text{gr}_{n+1} M$ is contained in $\text{Ker}(\text{gr}_{n+1} f)$ which is 0 by assumption, so this image is 0, which implies that $\text{Ker}(F_{n+1} f) \subset F_n M$; the restriction of $F_{n+1} f$ to $F_n M$ coincides with $F_n f$, which by the induction hypothesis is injective, therefore $\text{Ker}(F_{n+1} f) = 0$, so $F_{n+1} f$ is injective. For $y \in F_{n+1} N$, let \bar{y} be its image in $\text{gr}_{n+1} N$. By the surjectivity of $\text{gr}_{n+1}(f)$, there exists $\alpha \in \text{gr}_{n+1} M$ with image \bar{y} by $\text{gr}_{n+1}(f)$. Then if $x \in F_{n+1} M$ is any lift of α , one has $F_{n+1}(x) \equiv y \pmod{F_n N}$. Then $y - F_{n+1}(x) \in F_n N$. Since $F_n f : F_n M \rightarrow F_n N$ is surjective, there exists

$x_0 \in F_n M$ such that $F_n(x_0) = y - F_{n+1}(x)$. Then $y = F_n(x + x_0)$, which implies the surjectivity of $F_{n+1}f$. It follows that $F_{n+1}f$ is an isomorphism, proving the induction.

One has $\text{Ker}(F_\infty f) = \cup_{n \geq 0} (\text{Ker}(F_\infty f) \cap F_n M) = \cup_{n \geq 0} \text{Ker}(F_n f) = 0$, where the first equality follows from $F_\infty M = \cup_{n \geq 0} F_n M$ and the last equality follows from the injectivity of $F_n f$ for $n \geq 0$; this shows the injectivity of $F_\infty f$.

For any $n \geq 0$, one has $\text{Im}(F_\infty M) \supset \text{Im}(F_n M)$, and $\text{Im}(F_n M) = F_n N$ by the surjectivity of $F_n f$. Then $\text{Im}(F_\infty M) \supset \cup_{n \geq 0} F_n N = F_\infty N$, which shows the surjectivity of $F_\infty f$. It follows that $F_\infty f$ is a linear isomorphism. \square

If $F_\bullet M$ and $F_\bullet N$ are filtrations of \mathbb{C} -vector spaces M and N , then a filtration $F_\bullet(M \otimes N)$ of their tensor product $M \otimes N$ is defined by $F_n(M \otimes N) := \sum_{p+q=n} F_p M \otimes F_q N$; we denote it by $F_\bullet M \otimes F_\bullet N$, and we call it the *tensor product of $F_\bullet M$ and $F_\bullet N$* .

An *algebra filtration* of a \mathbb{C} -algebra A is a vector space filtration $F_\bullet A$ of A , such that $F_n A \cdot F_m A \subset F_{n+m} A$ for $n, m \geq 0$. Then $F_\infty A$ is a subalgebra of A , called the *total algebra* of the filtration; $\text{gr}_\bullet(A)$ is then a graded algebra. If $f : A \rightarrow B$ is an algebra morphism and $F_\bullet A$ is a filtration of A , then $f(F_\bullet A)$ is an algebra filtration of B . If $F_\bullet A$ and $F_\bullet B$ are filtrations of \mathbb{C} -algebras A and B , then $F_\bullet A \otimes F_\bullet B$ is an algebra filtration of $A \otimes B$.

An example of a filtration of an algebra A is the *trivial filtration* $F_\bullet^{\text{triv}} A$ defined by $F_n^{\text{triv}} A = A$ for any $n \geq 0$. If A has a unit, another example is the *unit filtration* $F_\bullet^{\text{unit}} A$ defined by $F_0^{\text{unit}} A = \mathbb{C}1$ and $F_n^{\text{unit}} A = A$ for any $n > 0$.

Similarly, a Hopf algebra filtration of a Hopf algebra H with coproduct Δ_H is a vector space filtration $F_\bullet H$ of H , which is an algebra filtration and such that $\Delta_H(F_n H) \subset \sum_{p+q=n} F_p H \otimes F_q H$ for any $n \geq 0$. Then $F_\infty H$ is a Hopf subalgebra of H and $\text{gr}_\bullet(H)$ is a graded Hopf algebra.

3.2. Computation of $(\mathbb{C}\Gamma_C)'$. In §D.1, we recall the category **CHA** of complete Hopf algebras (CHAs) and the functor $\mathbf{HA}_{\text{coco}} \rightarrow \mathbf{CHA}$, $H \mapsto H^\wedge$ with source the category $\mathbf{HA}_{\text{coco}}$ of cocommutative Hopf algebras.

Lemma 3.2. *If Γ is a free group, there is an isomorphism $(\mathbb{C}\Gamma)^\wedge \simeq \hat{T}(\Gamma^{ab} \otimes \mathbb{C})$ of CHAs, where for V a vector space, $\hat{T}(V)$ is the CHA defined as the degree completion of the tensor algebra of V , where the elements of V are primitive.*

Proof. By assumption, Γ is the free group over a set X . Let $(\gamma_x)_{x \in X}$ be the corresponding generating family. Then $\Gamma^{ab} \otimes \mathbb{C} = \mathbb{C}X$; let $(v_x)_{x \in X}$ be the canonical generating family of $\mathbb{C}X$. The assignment $\gamma_x \mapsto \exp(v_x)$ (defined as $\sum_{n \geq 0} v_x^{\otimes n} / n!$) for $x \in X$ defines a group morphism $\Gamma \rightarrow \hat{T}(\mathbb{C}X)^\times$, therefore an algebra morphism $\mathbb{C}\Gamma \rightarrow \hat{T}(\mathbb{C}X)$, which is checked to be compatible with coproducts. It is compatible with augmentations, therefore gives rise to a CHA morphism $(\mathbb{C}\Gamma)^\wedge \rightarrow \hat{T}(\mathbb{C}X)$. The assignment $v_x \mapsto \log(\gamma_x)$ (defined as $\sum_{n \geq 1} (-1)^{n+1} (\gamma_x - 1)^n / n$) for $x \in X$ defines a linear map $\mathbb{C}X \rightarrow (\mathbb{C}\Gamma)^\wedge$, therefore an algebra morphism $T(\mathbb{C}X) \rightarrow (\mathbb{C}\Gamma)^\wedge$, which is checked to be compatible with coproducts. It is compatible with augmentations, therefore

gives rise to a CHA morphism $\hat{T}(\mathbb{C}X) \rightarrow (\mathbb{C}\Gamma)^\wedge$. The two constructed CHA morphisms are checked to be inverse to one another. \square

In §D.1, we define a subcategory \mathbf{HA}_{fd} of finite-dimensional Hopf algebras of the category \mathbf{HA} of complex Hopf algebras, and a duality functor $\mathbf{HA}_{fd} \rightarrow \mathbf{HA}$, $H \mapsto H'$. When H is the group algebra of a finitely generated group, H' is as in §1.2.1 (definition following Thm. 1.1).

Lemma 3.3. *Let Γ be the free group over a finite set of generators. Then $\mathbb{C}\Gamma$ is an object in \mathbf{HA}_{fd} and there is a Hopf algebra isomorphism $(\mathbb{C}\Gamma)' \simeq \text{Sh}((\Gamma^{ab} \otimes \mathbb{C})^*)$ (we denote by V^* the dual of a vector space V).*

Proof. In this proof, we set $V := \Gamma^{ab} \otimes \mathbb{C}$. The first statement follows from the finite generation of Γ . It implies that $(\mathbb{C}\Gamma)^\wedge$ is an object in \mathbf{CHA}_{fd} , and by Lem. D.7, the duals $(\mathbb{C}\Gamma)'$ and $((\mathbb{C}\Gamma)^\wedge)'$ are well-defined isomorphic objects in \mathbf{HA}_{coco} .

By Lem. 3.2, the CHAs $(\mathbb{C}\Gamma)^\wedge$ and $\hat{T}(V)$ are isomorphic. Since $(\mathbb{C}\Gamma)^\wedge$ is an object in \mathbf{CHA}_{fd} , so is $\hat{T}(V)$, so $(\mathbb{C}\Gamma)^\wedge$ and $\hat{T}(V)$ are isomorphic objects in \mathbf{CHA}_{fd} . By Lem. D.5(b), this gives rise to an isomorphism $((\mathbb{C}\Gamma)^\wedge)' \simeq \hat{T}(V)'$ in \mathbf{HA}_{coco} . Since V is finite dimensional, there is an isomorphism $\hat{T}(V)' \simeq \text{Sh}(V^*)$.

The result follows by composition of these isomorphisms. \square

Remark 3.4. Lem. 3.3 is proved in [BGF], §3.125 when $|X| = 2$.

3.3. Computation of $\text{Sh}(V)^\heartsuit$. In §A.1, we equip each Hopf algebra H with a Hopf algebra filtration $F_\bullet^\heartsuit H$; this construction gives rise to an endofunctor $H \mapsto H^\heartsuit$ of \mathbf{HA} .

For V a vector space, let $F_\bullet \text{Sh}(V)$, be the algebra filtration corresponding to the grading of $\text{Sh}(V)$; one has $F_n \text{Sh}(V) := \bigoplus_{k=0}^n \text{Sh}_d(V)$ for $n \geq 0$.

Lemma 3.5. *Let V be a vector space.*

- (a) *For $n \geq 0$, the subspace $F_n^\heartsuit \text{Sh}(V)$ of $\text{Sh}(V)$ (see Def. A.1) is equal to $F_n \text{Sh}(V)$.*
- (b) *One has $\text{Sh}(V)^\heartsuit = \text{Sh}(V)$.*

Proof. (a) Recall that $F_n^\heartsuit \text{Sh}(V)$ is the kernel of the map $(id - \epsilon\eta)^{\otimes n+1} \circ \Delta^{(n+1)} : \text{Sh}(V) \rightarrow \text{Sh}(V)^{\otimes n+1}$, where we denote by ϵ, η, Δ the structure maps of $\text{Sh}(V)$. Since $(id - \epsilon\eta)^{\otimes n+1} \circ \Delta^{(n+1)}$ is graded, its kernel is the direct sum of the kernels of its restrictions to the various homogeneous components of $\text{Sh}(V)$. For $d \geq 0$, let $\text{Sh}_d(V)$ be the degree d component of $\text{Sh}(V)$. One checks that the restriction of $(id - \epsilon\eta)^{\otimes n+1} \circ \Delta^{(n+1)}$ to $\text{Sh}_d(V)$ is zero if $d \leq n$. On the other hand, for $k \geq 0$, let $pr_k^V : \text{Sh}(V) \rightarrow \text{Sh}_k(V)$ be the canonical projection. Then one checks that for $d \geq n+1$, the composition

$$\text{Sh}_d(V) \hookrightarrow \text{Sh}(V) \xrightarrow{(id - \epsilon\eta)^{\otimes n+1} \circ \Delta^{(n+1)}} \text{Sh}(V)^{\otimes n+1} \xrightarrow{pr_{d-n}^V \otimes (pr_1^V)^{\otimes n}} \text{Sh}_{d-n}(V) \otimes V^{\otimes n}$$

is the deconcatenation map corresponding to the partition $(d-n, 1, \dots, 1)$ of d , which is injective. This implies that the restriction of $(id - \epsilon\eta)^{\otimes n+1} \circ \Delta^{(n+1)}$ to $\text{Sh}_d(V)$ is injective. So $F_n^\heartsuit \text{Sh}(V) = \bigoplus_{d=0}^n \text{Sh}_d(V) = F_n \text{Sh}(V)$.

(b) follows from (a) and from $\bigcup_{n \geq 0} F_n \text{Sh}(V) = \text{Sh}(V)$. \square

3.4. A Hopf pairing $p_{\sigma, x_0} : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \otimes \mathbb{C}\Gamma_C \rightarrow \mathbb{C}$.

Definition 3.6. p_{σ, x_0} is the linear map $\mathbb{C}\Gamma_C \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathbb{C}$ such that $\gamma \otimes a \mapsto I_{x_0}(\sigma(a))(\gamma x_0)$ for $\gamma \in \Gamma_C$, $a \in \text{Sh}(\mathbb{H}_C^{\text{dR}})$.

Lemma 3.7. *The pairing $p_{\sigma, x_0} : \mathbb{C}\Gamma_C \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathbb{C}$ is a Hopf pairing (in the sense of §A.3).*

Proof. Let $p_{\sigma, x_0} : \mathbb{C}\Gamma_C \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathbb{C}$ be the map defined in Def. 3.7. For any $\gamma, \gamma' \in \Gamma_C$ and $a \in \text{Sh}(\mathbb{H}_C^{\text{dR}})$, one has

$$(3.4.1) \quad \begin{aligned} p_{\sigma, x_0}(\gamma\gamma', a) &= I_{x_0}(\sigma(a))(\gamma\gamma'x_0) = I_{x_0}(\sigma(a^{(1)}))(\gamma x_0) I_{\gamma x_0}(\sigma(a^{(2)}))(\gamma'x_0) \\ &= I_{x_0}(\sigma(a^{(1)}))(\gamma x_0) I_{x_0}(\sigma(a^{(2)}))(\gamma'x_0) = p_{\sigma, x_0}(\gamma, a^{(1)}) p_{\sigma, x_0}(\gamma', a^{(2)}), \end{aligned}$$

where the second equality follows from Lem. 2.26 and the fact that $\sigma : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \text{Sh}(\Omega(C))$ is a Hopf algebra morphism, the third equality follows from the invariance of the image of I_{x_0} by the diagonal action of Γ_C (see Lem. 2.22), and the first and last equalities follow from definitions.

Denote by \sqcup the product in the algebra $\text{Sh}(\mathbb{H}_C^{\text{dR}})$. Let $\gamma \in \Gamma_C$ and $a, a' \in \text{Sh}(\mathbb{H}_C^{\text{dR}})$. Then

$$(3.4.2) \quad \begin{aligned} p_{\sigma, x_0}(\gamma, a \sqcup a') &= I_{x_0}(\sigma(a \sqcup a'))(\gamma x_0) = I_{x_0}(\sigma(a))(\gamma x_0) I_{x_0}(\sigma(a'))(\gamma x_0) \\ &= p_{\sigma, x_0}(\gamma, a) p_{\sigma, x_0}(\gamma, a') = p_{\sigma, x_0}(\gamma^{(1)}, a) p_{\sigma, x_0}(\gamma^{(2)}, a') \end{aligned}$$

where the first and third equalities follow from definitions, the second equality follows from the facts that $\sigma : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \text{Sh}(\Omega(C))$ and $I_{x_0} : \text{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C}^2)^{\Gamma_C}$ are algebra morphisms, and the last equality follows from the group-likeness of γ for the coproduct of $\mathbb{C}\Gamma_C$.

The equalities (3.4.1) and (3.4.2) imply the statement. \square

3.5. Proof that $\nu(p_{\sigma, x_0}) : \text{Sh}(\mathbb{H}_C^{\text{dR}})^\heartsuit \rightarrow (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism.

3.5.1. *Construction of a Hopf algebra morphism $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \text{Sh}((\mathbb{H}_C^{\text{B}})^*)$.* Denote $\text{H}_1(C, \mathbb{Z})$ the first singular homology group of C with integer coefficients and set $\mathbb{H}_C^{\text{B}} := \text{H}_1(C, \mathbb{Z}) \otimes \mathbb{C}$.

Lemma 3.8. *There is a Hopf algebra isomorphism $(\mathbb{C}\Gamma_C)' \simeq \text{Sh}((\mathbb{H}_C^{\text{B}})^*)$.*

Proof. Since C is an affine curve, the group Γ_C is free. Lem. 3.3 then implies that $(\mathbb{C}\Gamma_C)'$ is isomorphic to $\text{Sh}((\Gamma_C^{ab} \otimes \mathbb{C})^*)$. The choice of a point x_0 in \tilde{C} induces an isomorphism $\Gamma_C \simeq \pi_1(C, x_0)$, whose conjugation class is independent on this choice; this isomorphism induces an isomorphism $\Gamma_C^{ab} \simeq \pi_1(C, x_0)^{ab} = \text{H}_1^{\text{B}}(C, \mathbb{Z})$ also independent on this choice, from which one derives an isomorphism $\Gamma_C^{ab} \otimes \mathbb{C} \simeq \mathbb{H}_C^{\text{B}}$. \square

Definition 3.9. $q_{\sigma, x_0} : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \text{Sh}((\mathbb{H}_C^{\text{B}})^*)$ is the Hopf algebra morphism obtained by composition of

- (a) the Hopf algebra isomorphism $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \xrightarrow{\sim} \text{Sh}(\mathbb{H}_C^{\text{dR}})^{\heartsuit}$ (see Lem. 3.5),
- (b) the Hopf algebra morphism $\nu(p_{\sigma, x_0}) : \text{Sh}(\mathbb{H}_C^{\text{dR}})^{\heartsuit} \rightarrow (\text{C}\Gamma_C)'$ arising by Lem. A.11 from the Hopf algebra pairing p_{σ, x_0} (see Lem. 3.7),
- (c) the Hopf algebra isomorphism $(\text{C}\Gamma_C)' \simeq \text{Sh}((\mathbb{H}_C^{\text{B}})^*)$ (see Lem. 3.8).

3.5.2. *A criterion for a Hopf algebra morphism $\text{Sh}(V) \rightarrow \text{Sh}(W)$ being an isomorphism.*

Lemma 3.10. *Let V, W be vector spaces and let $f : \text{Sh}(V) \rightarrow \text{Sh}(W)$ be a Hopf algebra morphism. Then $f(V) \subset W$ where V, W are the degree 1 subspaces of $\text{Sh}(V), \text{Sh}(W)$. Denote by $\text{gr}_1^{\heartsuit}(f) : V \rightarrow W$ the corresponding linear map. Then f is a Hopf algebra isomorphism if and only if $\text{gr}_1^{\heartsuit}(f)$ is a vector space isomorphism.*

Proof. Since f is an Hopf algebra morphism, Lem. A.4(c) implies that it induces a linear map $F_n^{\heartsuit} f : F_n^{\heartsuit} \text{Sh}(V) \rightarrow F_n^{\heartsuit} \text{Sh}(W)$ for any $n \geq 0$. When $n = 1$, $F_1^{\heartsuit} f$ is a linear map $\mathbb{C} \oplus V \rightarrow \mathbb{C} \oplus W$. The compatibility of $F_1^{\heartsuit} f$ with the units and counits on both sides implies that $F_1^{\heartsuit} f$ is the direct sum of $\text{id}_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$ and a linear map $V \rightarrow W$, which can be identified with the associated graded of f for the filtration F_{\bullet}^{\heartsuit} .

If f is an Hopf algebra morphism, then for each $n \geq 0$, $F_n^{\heartsuit} f : F_n^{\heartsuit} \text{Sh}(V) \rightarrow F_n^{\heartsuit} \text{Sh}(W)$ is a linear isomorphism, which when $n = 1$ implies the same for $\text{id}_{\mathbb{C}} \oplus \text{gr}_1^{\heartsuit}(f)$, which implies that $\text{gr}_1^{\heartsuit}(f)$ is a linear isomorphism.

Assume now that $f : \text{Sh}(V) \rightarrow \text{Sh}(W)$ is a Hopf algebra morphism such that $\text{gr}_1^{\heartsuit}(f)$ is a linear isomorphism. The associated graded map $\text{gr}_{\bullet}^{\heartsuit} f : \text{gr}_{\bullet}^{\heartsuit} \text{Sh}(V) \rightarrow \text{gr}_{\bullet}^{\heartsuit} \text{Sh}(W)$ can be identified, under the canonical isomorphisms $\text{gr}_{\bullet}^{\heartsuit} \text{Sh}(X) \simeq \text{Sh}(X)$ for $X = V, W$ (see Lem. 3.5(a)), with $\text{Sh}(\text{gr}_1^{\heartsuit}(f))$, which is an isomorphism of graded vector spaces. Since the filtrations F_{\bullet}^{\heartsuit} in the source and target are exhaustive and by Lem. 3.1, this implies that f is an isomorphism. \square

3.5.3. *Isomorphism status of the linear map $\text{gr}_1^{\heartsuit}(q_{\sigma, x_0}) : \mathbb{H}_C^{\text{dR}} \rightarrow (\mathbb{H}_C^{\text{B}})^*$.*

Lemma 3.11. (a) *Let $x_0 \in \tilde{C}$. The pairing $J_{\text{C}\Gamma_C} \otimes \Omega(C) \rightarrow \mathbb{C}$ given by $(\gamma - 1) \otimes \omega \mapsto \int_{x_0}^{\gamma x_0} \omega$ for $\gamma \in \Gamma_C$, $\omega \in \Omega$ is independent on x_0 . It factors through a pairing*

$$(3.5.1) \quad \mathbb{H}_C^{\text{B}} \otimes \mathbb{H}_C^{\text{dR}} = J_{\text{C}\Gamma_C} / J_{\text{C}\Gamma_C}^2 \otimes (\Omega(C) / d\mathcal{O}(C)) \rightarrow \mathbb{C}.$$

(b) *The linear map $\mathbb{H}_C^{\text{dR}} \rightarrow (\mathbb{H}_C^{\text{B}})^*$ induced by (3.5.1) is equal to $\text{gr}_1^{\heartsuit}(q_{\sigma, x_0}) : \mathbb{H}_C^{\text{dR}} \rightarrow (\mathbb{H}_C^{\text{B}})^*$.*

(c) *The linear map $\text{gr}_1^{\heartsuit}(q_{\sigma, x_0}) : \mathbb{H}_C^{\text{dR}} \rightarrow (\mathbb{H}_C^{\text{B}})^*$ is an isomorphism.*

Proof. (a) If $x_0, x_1 \in \tilde{C}$, $\gamma \in \Gamma_C$ and $\omega \in \Omega(C)$, then $\int_{x_1}^{\gamma x_1} \omega - \int_{x_0}^{\gamma x_0} \omega = (\int_{x_1}^{x_0} - \int_{x_1}^{\gamma x_1}) \omega = 0$ by the Γ_C -invariance of ω ; this implies the claimed independence. If $\gamma, \gamma' \in \Gamma_C$ and $\omega \in \Omega(C)$, then $(\int_{x_0}^{\gamma \gamma' x_0} - \int_{x_0}^{\gamma x_0} - \int_{x_0}^{\gamma' x_0}) \omega = (\int_{\gamma x_0}^{\gamma \gamma' x_0} - \int_{x_0}^{\gamma' x_0}) \omega = 0$ by the same reason; since the elements

$(\gamma - 1)(\gamma' - 1)$ generate $J_{\mathbb{C}\Gamma_C}^2$, this implies the claimed factorization. The equality follows from $\text{gr}^1(\mathbb{C}\Gamma_C) = \Gamma_C^{ab} \otimes \mathbb{C} = \mathbb{H}_C^{\mathbb{B}}$ (see [Q2]).

(b) Let O, H be Hopf algebras with $\text{gr}^1(H)$ finite dimensional and let $p : O \otimes H \rightarrow \mathbb{C}$ be a Hopf algebra pairing. By Lem. A.11, p gives rise to a Hopf algebra morphism $\nu(p) : O^\heartsuit \rightarrow H'$, and by Prop. A.4(c), this morphism is compatible with the filtrations F_\bullet^\heartsuit on both sides. For any $n \geq 0$, the restriction of p to $F_n^\heartsuit O \otimes H$ induces a pairing $F_n^\heartsuit O \otimes (H/F^{n+1}H) \rightarrow \mathbb{C}$, which gives rise to a linear map $F_n^\heartsuit O \rightarrow (H/F^{n+1}H)^* = F_n H^*$; composing this linear map with the identification from Lem. A.9 gives rise to a linear map $F_n^\heartsuit O \rightarrow F_n^\heartsuit H'$, which is equal to $F_n^\heartsuit \nu(p)$. The morphism $\nu(p)$ is compatible the augmentation maps $\epsilon_O, \epsilon_{H'}$, therefore gives rise to a linear map $F_n^\heartsuit O \cap J_O \rightarrow F_n^\heartsuit H' \cap J_{H'}$, which for $n = 1$ coincides with $\text{gr}_1^\heartsuit(\nu(p))$. It follows that $\text{gr}_1^\heartsuit(\nu(p))$ may be constructed as follows: the restriction of p to $(F_1^\heartsuit O \cap J_O) \otimes F^1 H$ induces a pairing $(F_1^\heartsuit O \cap J_O) \otimes (F^1 H/F^2 H) \rightarrow \mathbb{C}$; then $\text{gr}_1^\heartsuit(\nu(p))$ is the induced map $\text{gr}_1^\heartsuit(O) = F_1^\heartsuit O \cap J_O \rightarrow (F^1 H/F^2 H)^* = \text{gr}_1^\heartsuit(H')$.

It follows that $\text{gr}_1^\heartsuit(\nu(p)) : \mathbb{H}_C^{\text{dR}} = \text{gr}_1^\heartsuit(\text{Sh}(\mathbb{H}_C^{\text{dR}})) \rightarrow \text{gr}_1^\heartsuit((\mathbb{C}\Gamma)') = (\Gamma_C^{ab} \otimes \mathbb{C})^*$ is induced by the restriction of p_{σ, x_0} to $\mathbb{H}_C^{\text{dR}} \otimes J_{\mathbb{C}\Gamma}$. This restriction coincides with the lift of (3.5.1) since σ is a lift of the projection $\Omega(C) \rightarrow \mathbb{H}_C^{\text{dR}}$. The statement follows.

(c) The pairing (3.5.1) coincides with the period pairing for C , which is nondegenerate. It follows that the map $\mathbb{H}_C^{\text{dR}} \rightarrow (\mathbb{H}_C^{\mathbb{B}})^*$ induced by (3.5.1) is an isomorphism. The statement then follows from (b). \square

3.5.4. *Proof that $\nu(p_{\sigma, x_0}) : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism.*

Proposition 3.12. *$\nu(p_{\sigma, x_0}) : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism.*

Proof. By Lem. 3.11(c) and Lem. 3.10, q_{σ, x_0} is a Hopf algebra isomorphism. By Def. 3.9, $\nu(p_{\sigma, x_0})$ is obtained from q_{σ, x_0} by pre- and post-composition with Hopf algebra isomorphisms, which implies that it is a Hopf algebra isomorphism. \square

Remark 3.13. Prop. 3.12 may be related to Chen's π_1 theorem as follows. Let $(\mathbb{H}_{\text{dR}}^\bullet(C), \epsilon_{\text{triv}})$ be the augmented dga with $\mathbb{H}_{\text{dR}}^\bullet(C) := \mathbb{C} \oplus \mathbb{H}_C^{\text{dR}}$ with zero differential and $\epsilon_{\text{triv}} : \mathbb{H}_{\text{dR}}^\bullet(C) \rightarrow \mathbb{C}$ the projection in degree 0, and let $(\mathcal{E}^\bullet(C), \epsilon_{x_0})$ be the augmented dga of smooth differential forms on C , with ϵ_{x_0} given by evaluation at x_0 . Then the dga morphism $\mathbb{H}_{\text{dR}}^\bullet(C) \rightarrow \mathcal{E}^\bullet(C)$ induced by σ is compatible with the augmentations, therefore it induces an isomorphism of commutative Hopf algebras $H^0(B(\mathbb{H}_{\text{dR}}^\bullet(C), \epsilon_{\text{triv}})) \rightarrow H^0(B(\mathcal{E}^\bullet(C), \epsilon_{x_0}))$, where $H^0(B(-))$ is the zero-th cohomology of the bar-construction of an augmented dga. The combination of this isomorphism with the Hopf algebra isomorphisms $H^0(B(\mathbb{H}_{\text{dR}}^\bullet(C), \epsilon_{\text{triv}})) \simeq \text{Sh}(\mathbb{H}_C^{\text{dR}})$ and $H^0(B(\mathcal{E}^\bullet(C), \epsilon_{x_0})) \simeq (\mathbb{C}\Gamma_C)'$ (by [BGF], Thm. 3.150) is the Hopf algebra isomorphism from (a).

3.6. **The isomorphism of filtered algebras** $f_{\sigma, x_0} : F_\bullet \text{Sh}(\mathbb{H}_C^{\text{dR}}) \otimes \mathcal{O}(C) \rightarrow F_\bullet \mathcal{O}_{\text{mod}}(\tilde{C})$.

Lemma 3.14. *The map $f_{\sigma, x_0} : \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C) \rightarrow \mathcal{O}_{\mathrm{mod}}(\tilde{C})$ such that $f_{\sigma, x_0}(a \otimes f) := I_{x_0}(a) \cdot p^* f$ is an algebra morphism.*

Proof. Follows from the decomposition of f_{σ, x_0} as $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C) \xrightarrow{I_{x_0} \otimes p^*} \mathcal{O}_{\mathrm{mod}}(\tilde{C})^{\otimes 2} \xrightarrow{m_{\mathcal{O}_{\mathrm{mod}}(\tilde{C})}} \mathcal{O}_{\mathrm{mod}}(\tilde{C})$, where $m_{\mathcal{O}_{\mathrm{mod}}(\tilde{C})}$ is the product map of $\mathcal{O}_{\mathrm{mod}}(\tilde{C})$ and from the algebra morphism status of I_{x_0}, p^* and $m_{\mathcal{O}_{\mathrm{mod}}(\tilde{C})}$ (the latter coming from the commutativity of $\mathcal{O}_{\mathrm{mod}}(\tilde{C})$). \square

In App. B, we introduce the notions of Hopf algebras with comodule algebra (HACA) and Hopf algebra with module algebra (HAMA). Then by Prop. B.18(a), the algebra $\mathcal{O}(C)$ and Hopf algebra $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})$ give rise to a HACA structure $(\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}), \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C))$; on the other hand, a HAMA structure is constructed as follows.

Lemma 3.15. *The pair $(\mathcal{O}_{\mathrm{mod}}(\tilde{C}), \mathbb{C}\Gamma_C)$ is equipped with a HAMA structure.*

Proof. This structure is induced by the right Γ_C -structure of $\mathcal{O}_{\mathrm{mod}}(\tilde{C})$ (see Prop. 2.10). \square

In App. B (see Def. B.12), we also introduce the notion of pairing-morphism from a HACA (O, A) to a HAMA (B, H) , and denote by $\mathbf{PM}((O, A), (B, H))$ the set of such structures.

Recall the Hopf algebra pairing $p_{\sigma, x_0} : \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathbb{C}\Gamma_C \rightarrow \mathbb{C}$ (see Lem. 3.7).

Lemma 3.16. $(p_{\sigma, x_0}, f_{\sigma, x_0}) \in \mathbf{PM}((\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}), \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C)), (\mathcal{O}_{\mathrm{mod}}(\tilde{C}), \mathbb{C}\Gamma_C))$.

Proof. By Def. B.12, the identity to check is

$$(I_{x_0}(a)p^*(f))|_{\gamma} = I_{x_0}(a^{(2)})p^*(f)p_{\sigma, x_0}(\gamma \otimes a^{(1)})$$

for any $a \in \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})$, $f \in \mathcal{O}(C)$ and $\gamma \in \Gamma_C$, using the notation $\Delta_{\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})}(a) = a^{(1)} \otimes a^{(2)}$. This follows from the invariance of p^*f under the action of Γ_C and from the identity $I_{x_0}(a)|_{\gamma} = I_{x_0}(a^{(2)})p_{\sigma, x_0}(\gamma \otimes a^{(1)})$ which is proved as follows: for any $x \in \tilde{C}$, one has

$$I_{x_0}(a)|_{\gamma}(x) = I_{x_0}(a)(\gamma x) = I_{x_0}(a^{(1)})(\gamma x_0)I_{\gamma x_0}(a^{(2)})(\gamma x) = p_{\sigma, x_0}(\gamma \otimes a^{(1)})I_{x_0}(a^{(2)})(x).$$

where the second identity follows from Lem. 2.26 and the third identity follows from the definition of p_{σ, x_0} and Lem. 2.22. \square

By Lem. B.8, the HAMA structure $(\mathcal{O}_{\mathrm{mod}}(\tilde{C}), \mathbb{C}\Gamma_C)$ gives rise to an algebra filtration $F_{\bullet}\mathcal{O}_{\mathrm{mod}}(\tilde{C})$ of $\mathcal{O}_{\mathrm{mod}}(\tilde{C})$.

Proposition 3.17. (a) $(\nu(p_{\sigma, x_0}), f_{\sigma, x_0})$ induces an isomorphism of HACAs $(\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}), \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C)) \rightarrow ((\mathbb{C}\Gamma_C)', F_{\infty}\mathcal{O}_{\mathrm{mod}}(\tilde{C}))$.

(b) f_{σ, x_0} induces an isomorphism of algebra filtrations $F_{\bullet}\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C) \rightarrow F_{\bullet}\mathcal{O}_{\mathrm{mod}}(\tilde{C})$.

Proof. Set $\mathbf{a} := \mathcal{O}(C)$, $O := \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})$, $B := \mathcal{O}_{\mathrm{mod}}(\tilde{C})$, $H := \mathbb{C}\Gamma_C$, (B, H) be the HAMA structure induced by the action of Γ_C on \tilde{C} ; it is an object in \mathbf{HAMA}_{fd} since Γ_C is finitely generated. Set $p := p_{\sigma, x_0} \in \mathbf{Pair}(O, H)$, $f := f_{\sigma, x_0}$. By Lem. 3.16, $(p, f) \in \mathbf{PM}((O, O \otimes \mathbf{a}), (B, H))$. By Prop. 3.12, $\nu(p_{\sigma, x_0}) : \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})^{\heartsuit} \rightarrow (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism

and by Prop. 2.12, f_{σ, x_0} induces an algebra isomorphism $\mathbb{C} \otimes \mathcal{O}(C) \rightarrow \mathcal{O}_{mod}(\tilde{C})^{\mathbb{C}\Gamma_C}$. The assumptions of Prop. B.18(b) are therefore satisfied. The conclusion follows by applying this lemma. \square

3.7. Filtered formality of the HACA $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{mod}(\tilde{C}))$. In §C.2, we introduce the definition of a filtered formal HACA.

Proposition 3.18. *The pair $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{mod}(\tilde{C}))$ is filtered formal.*

Proof. It follows from the proof of Lem. 3.17 that the data

$$(3.7.1) \quad (\mathcal{O}, \mathbf{a}) := (\mathrm{Sh}(\mathbb{H}_C^{\mathrm{dR}}), \mathcal{O}(C)), \quad (B, H) = (\mathcal{O}_{mod}(\tilde{C}), \mathbb{C}\Gamma_C), \quad (p, f) = (p_{\sigma, x_0}, f_{\sigma, x_0}).$$

satisfy the hypotheses of Prop. B.18(b),(c). The statement is then a consequence of Prop. C.5. \square

4. LOCAL EXPANSION OF THE ELEMENTS OF $F_\infty \mathcal{O}_{mod}(\tilde{C})$

By §2.1.1, one may view u, z as elements of $\mathcal{O}_{hol}(\tilde{D}^\times)$ and $\mathcal{O}_{hol}(D^\times)$ respectively, such that $e^*z = e(u)$. We therefore use the notation $e^*\mathrm{log}z$ for $2\pi iu$. Recall the action of \mathbb{Z} on $\mathcal{O}_{mod}(\tilde{D}^\times)$, where 1 acts by θ^* (cf. Lem. 2.3). Let us denote by $F_\bullet \mathcal{O}_{mod}(\tilde{D}^\times)$ the algebra filtration of $\mathcal{O}_{mod}(\tilde{D}^\times)$ induced by this action according to Def. B.7 and Lem. B.8.

Lemma 4.1. *The algebra morphism $\mathcal{O}(D^\times)[X] \rightarrow \mathcal{O}_{mod}(\tilde{D}^\times)$ given by $f \mapsto f$ for $f \in \mathcal{O}(D^\times)$ and $X \mapsto e^*\mathrm{log}z$ induces an isomorphism between the algebra filtrations $F_\bullet \mathcal{O}(D^\times)[X]$ of $\mathcal{O}(D^\times)[X]$ given by $F_n \mathcal{O}(D^\times)[X] := \mathcal{O}(D^\times)[X]_{\leq n}$ for $n \geq 0$ and $F_\bullet \mathcal{O}_{mod}(\tilde{D}^\times)$ of $\mathcal{O}_{mod}(\tilde{D}^\times)$. In particular, one has $F_\infty \mathcal{O}_{mod}(\tilde{D}^\times) = \mathcal{O}(D^\times)[e^*\mathrm{log}z]$ (equality of subalgebras of $\mathcal{O}_{mod}(\tilde{D}^\times)$).*

Proof. Let us denote by $\mathrm{can} : \mathcal{O}(D^\times)[X] \rightarrow \mathcal{O}_{mod}(\tilde{D}^\times)$ the algebra morphism from the statement. The image of $\mathcal{O}(D^\times)$ by can is contained in $F_0 \mathcal{O}_{mod}(\tilde{D}^\times)$ and the image of $e^*\mathrm{log}z$ by can is contained in $F_1 \mathcal{O}_{mod}(\tilde{D}^\times)$ since $(\theta^* - 1)^2(e^*\mathrm{log}z) = (\theta^* - 1)(1) = 0$, therefore for any $n \geq 0$, $\mathrm{can}(F_n \mathcal{O}(D^\times)[X]) = \mathrm{can}(\mathcal{O}(D^\times)[X]_{\leq n}) \subset F_n \mathcal{O}_{mod}(\tilde{D}^\times)$. Therefore can is compatible with the algebra filtrations in its source and target. Let

$$\mathrm{gr}(\mathrm{can}) : \mathcal{O}(D^\times)[X] \rightarrow \mathrm{gr}(\mathcal{O}_{mod}(\tilde{D}^\times))$$

be the corresponding graded algebra morphism. Its restriction to the degree 0 part of its source is given by $\mathcal{O}(D^\times) \ni f \mapsto f \in \mathcal{O}(D^\times) = F_0 \mathcal{O}_{mod}(\tilde{D}^\times) = \mathrm{gr}_0 \mathcal{O}_{mod}(\tilde{D}^\times)$ and it is such that $\mathcal{O}(D^\times)[X] \ni X \mapsto [e^*\mathrm{log}z] \in \mathrm{gr}_1 \mathcal{O}_{mod}(\tilde{D}^\times)$ (degree 1 elements).

Let $n \geq 0$. By Lem. B.8(b), the linear map $\mu_n : F_n \mathcal{O}_{mod}(\tilde{D}^\times) \rightarrow \mathcal{O}_{mod}(\tilde{D}^\times)$ given by $f \mapsto (\theta^* - 1)^n(f)/n!$ has its image contained in $F_0 \mathcal{O}_{mod}(\tilde{D}^\times)$, whereas the image of the subspace $F_{n-1} \mathcal{O}_{mod}(\tilde{D}^\times)$ by this map is zero.

Under the identification $\mathbb{C}\mathbb{Z} \simeq \mathbb{C}[X, X^{-1}]$ with X group-like, the ideal $F_n \mathbb{C}\mathbb{Z}$ is identified with $((X - 1)^n)$, therefore $\mathrm{Ker}(\mu_n) = \mathrm{Ann}((\theta^* - 1)^n) = \mathrm{Ann}(F_n \mathbb{C}\mathbb{Z}) = F_{n-1} \mathcal{O}_{mod}(\tilde{D}^\times)$. It

follows that μ_n induces an injective linear map $\mu'_n : \text{gr}_n(\mathcal{O}_{\text{mod}}(\tilde{D}^\times)) \rightarrow F_0\mathcal{O}_{\text{mod}}(\tilde{D}^\times)$. where by Lem. 2.5, the target space is equal to $\mathcal{O}(D^\times)$.

Define a graded linear map

$$\mu : \text{gr}(\mathcal{O}_{\text{mod}}(\tilde{D}^\times)) \rightarrow \mathcal{O}(D^\times)[X]$$

by $\mu(a) := \mu'_n(a)X^n$ for $a \in \text{gr}_n(\mathcal{O}_{\text{mod}}(\tilde{D}^\times))$ and $n \geq 0$. As μ is a direct sum of injective maps, μ is injective.

Let us show that for $n, m \geq 0$, the diagram

$$(4.0.1) \quad \begin{array}{ccc} F_n\mathcal{O}_{\text{mod}}(\tilde{D}^\times) \otimes F_m\mathcal{O}_{\text{mod}}(\tilde{D}^\times) & \xrightarrow{\mu'_n \otimes \mu'_m} & \mathcal{O}(D^\times) \otimes \mathcal{O}(D^\times) \\ \downarrow & & \downarrow \\ F_n\mathcal{O}_{\text{mod}}(\tilde{D}^\times) & \xrightarrow{\mu'_{n+m}} & \mathcal{O}(D^\times) \end{array}$$

is commutative, where the vertical maps are given by multiplication. The relation

$$(4.0.2) \quad \frac{(XY-1)^{n+m}}{(n+m)!} \in \frac{(X-1)^n}{n!} \otimes \frac{(X-1)^m}{m!} + ((X-1)^{n+1}) \otimes \mathbb{C}[X, X^{-1}] + \mathbb{C}[X, X^{-1}] \otimes ((X-1)^{m+1})$$

in $\mathbb{C}[X, X^{-1}]^{\otimes 2}$ is a consequence of the relation

$$\frac{(XY-1)^{n+m}}{(n+m)!} = \frac{((X-1)Y+(Y-1))^{n+m}}{(n+m)!} \in \frac{((X-1)Y)^n Y^m}{n!m!} + I = \frac{(X-1)^n}{n!} \frac{(Y-1)^m}{m!} + I$$

in $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$, where $I := ((X-1)^{n+1}) + ((Y-1)^{m+1})$. Then $f \in F_n\mathcal{O}_{\text{mod}}(\tilde{D}^\times)$, $g \in F_m\mathcal{O}_{\text{mod}}(\tilde{D}^\times)$, one has

$$\begin{aligned} \mu'_{n+m}(fg) &= \frac{(\theta^*-1)^{n+m}(fg)}{(n+m)!} = \frac{(X-1)^{n+m}}{(n+m)!} \cdot (f \cdot g) = \Delta\left(\frac{(X-1)^{n+m}}{(n+m)!}\right) \cdot (f \otimes g) \\ &\in \frac{(X-1)^n}{n!} \otimes \frac{(X-1)^m}{m!} + ((X-1)^{n+1}) \otimes \mathbb{C}[X, X^{-1}] + \mathbb{C}[X, X^{-1}] \otimes ((X-1)^{m+1}) \cdot (f \otimes g) \\ &= \frac{(X-1)^n}{n!} \cdot f \otimes \frac{(X-1)^m}{m!} \cdot g = \mu'_n(f) \cdot \mu'_m(g), \end{aligned}$$

where the third equality follows from the Hopf algebra action properties, the inclusion relation follows from (4.0.2) and the group-likeness of X , and the last equality follows from $(\theta^* - 1)^{n+1}f = (\theta^* - 1)^{m+1}g = 0$ as $f \in F_n\mathcal{O}_{\text{mod}}(\tilde{D}^\times)$, $g \in F_m\mathcal{O}_{\text{mod}}(\tilde{D}^\times)$. The commutativity of (4.0.1) implies that $\mu : \text{gr}(\mathcal{O}_{\text{mod}}(\tilde{D}^\times)) \rightarrow \mathcal{O}(D^\times)[X]$ is a morphism of graded algebras.

Then $\mu \circ \text{gr}(\text{can})$ is a graded algebra endomorphism of $\mathcal{O}(D^\times)[X]$. For $f \in \mathcal{O}(D^\times)$, one has $\mu \circ \text{gr}(\text{can})(f) = \mu(f) = f$. One also has $\mu \circ \text{gr}(\text{can})(X) = \mu([e^* \log z]) = \mu'_1([e^* \log z])X = X$ as $\mu'_1([e^* \log z]) = \mu_1(e^* \log z) = 1$. It follows that $\mu \circ \text{gr}(\text{can})$ is the identity of $\mathcal{O}(D^\times)[X]$, therefore that μ is surjective. It is therefore an isomorphism, which using again $\mu \circ \text{gr}(\text{can}) = \text{id}$ implies that $\text{gr}(\text{can})$ is an isomorphism. Lem. 3.1 then implies the statement. \square

Proposition 4.2. *The algebra morphism $\bigoplus_{s \in S} \prod_{x \in X_s} (\tilde{\varphi}_{s,x}^\times)^*$ (see (2.1.1)) is such that*

$$\left(\bigoplus_{s \in S} \prod_{x \in X_s} (\tilde{\varphi}_{s,x}^\times)^* \right) (F_\infty \mathcal{O}_{\text{mod}}(\tilde{C})) \subset \bigoplus_{s \in S} \prod_{x \in X_s} \mathcal{O}(D^\times)[e^* \log z].$$

Proof. It follows from the proof of Prop. 2.12 that (2.1.2) is a Γ_C -equivariant algebra morphism, where the action on the target is the direct sum over $s \in S$ of the actions of Lem. 2.8(a). For any $s \in S$, the composition of (2.1.2) with the canonical projection is a Γ_C -equivariant algebra morphism $\bigoplus_{x \in X_s} (\tilde{\varphi}_{s,x}^\times)^* : \mathcal{O}_{mod}(\tilde{C}) \rightarrow \bigoplus_{x \in X_s} \mathcal{O}_{mod}(\tilde{D}^\times)$. For $x \in X_s$, the decomposition of the target as $\mathcal{O}_{mod}(\tilde{D}^\times) \oplus (\bigoplus_{x' \in X_s \setminus \{x\}} \mathcal{O}_{mod}(\tilde{D}^\times))$ is preserved by the action of the stabilizer subgroup $\text{Stab}_{\Gamma_C}(x) \subset \Gamma_C$ of $x \in X_s$ under the action of Γ_C . The map

$$(4.0.3) \quad (\tilde{\varphi}_{s,x}^\times)^* : \mathcal{O}_{mod}(\tilde{C}) \rightarrow \mathcal{O}_{mod}(\tilde{D}^\times)$$

is therefore equivariant under the action of $\text{Stab}_{\Gamma_C}(x)$.

By §2.1.2, there is a group isomorphism $\mathbb{Z} \simeq \text{Stab}_{\Gamma_C}(x)$ given by $1 \mapsto \theta_{s,x}$, and (4.0.3) is \mathbb{Z} -equivariant, the action of \mathbb{Z} on the target being as in Lem. 4.1. Then

$$\begin{aligned} F_\infty \mathcal{O}_{mod}(\tilde{C}) &= F_\infty^{\mathbb{C}\Gamma_C} \mathcal{O}_{mod}(\tilde{C}) \subset F_\infty^{\mathbb{C}\mathbb{Z}} \mathcal{O}_{mod}(\tilde{C}) \subset (\varphi_{s,x}^*)^{-1}(F_\infty^{\mathbb{C}\mathbb{Z}} \mathcal{O}_{mod}(\tilde{D}^\times)) \\ &= (\varphi_{s,x}^*)^{-1}(\mathcal{O}_{mod}(D^\times)[e^* \log z]) \end{aligned}$$

where we use the notation of Lem. B.11, the first equality is the definition of $F_\infty \mathcal{O}_{mod}(\tilde{C})$, the first inclusion follows from Lem. B.11(a), the second inclusion follows from Lem. B.11(b), and the last equality follows from Lem. 4.1. \square

Remark 4.3. Prop. 4.2 implies that the elements of $F_\infty \mathcal{O}_{mod}(\tilde{C})$ are Nilsson class functions on C in the sense of [Ph], p. 154. Indeed, for any $f \in \mathcal{O}(D^\times)[e^* \log z]$, there exists $\alpha_0 \in \mathbb{Z}$ such that $f = z^{\alpha_0} P_0(\log z)$ with P_0 as in [Ph], eq. (1.4), p. 151. The Nilsson class functions obtained in this way are not of the most general form, as the class in \mathbb{C}/\mathbb{Z} of the α_i in their expansion from [Ph], eq. (1.4), p. 151, is always 0.

5. FILTRATIONS ON $\mathcal{O}_{hol}(\tilde{C})$ AND THE MINIMAL STABLE EXTENSION OF $\mathcal{O}(C)$

In this section, we first study the filtration of $\mathcal{O}_{hol}(\tilde{C})$ given by the image by I_{x_0} of the filtration $F_\bullet(\text{Sh}(\Omega(C)))$ of $\text{Sh}(\Omega(C))$ (§5.1), and identify it with the image by f_{σ, x_0} of the filtration $F_\bullet^{triv} \mathcal{O}(C) \otimes F_\bullet \text{Sh}(\mathbb{H}_C^{\text{dR}})$ of $\mathcal{O}(C) \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}})$ (Prop. 5.2). In §5.2, we introduce and study the filtrations $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$ and $F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C})$ of $\mathcal{O}_{hol}(\tilde{C})$ inspired by the work of [Ch] on Picard-Vessiot extensions of the function algebra of a smooth manifold. We study the relation of the total space of these filtrations with the MSE A_C of $\mathcal{O}(C)$ in §5.3, and with the maximal acyclic extension E_C^\bullet of the dga $(\Omega^\bullet(C), d)$ in §5.4. In §5.5, we discuss the relation of this material with [Ch].

5.1. An algebra filtration of $\mathcal{O}_{hol}(\tilde{C})$ defined by I_{x_0} . Recall the algebra morphism $I_{x_0} : \text{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{hol}(\tilde{C})$ (Def. 2.20 and Lem. 2.21) and the algebra filtration $F_\bullet \text{Sh}(V)$ for V an arbitrary vector space (see §2.3). By §3.1, these data give rise to an algebra filtration $I_{x_0}(F_\bullet \text{Sh}(\Omega(C)))$ of $\mathcal{O}_{hol}(\tilde{C})$, which we study in Prop. 5.2.

Lemma 5.1. *One has*

$$(5.1.1) \quad \forall p, q > 0, \quad I_{x_0}([\mathrm{Sh}_p(\Omega(C))|d\mathcal{O}(C)|\mathrm{Sh}_q(\Omega(C))]) \subset I_{x_0}(\mathrm{Sh}_{p+q}(\Omega(C))),$$

$$(5.1.2) \quad \forall n > 0, \quad I_{x_0}([d\mathcal{O}(C)|\mathrm{Sh}_n(\Omega(C))]) \subset I_{x_0}(\mathrm{Sh}_n(\Omega(C))),$$

$$(5.1.3) \quad \forall n > 0, \quad I_{x_0}([\mathrm{Sh}_n(\Omega(C))|d\mathcal{O}(C)]) \subset p^*\mathcal{O}(C) \cdot I_{x_0}(\mathrm{Sh}_n(\Omega(C))).$$

Proof. Let us prove (5.1.1). The space $[\mathrm{Sh}_p(\Omega(C))|d\mathcal{O}(C)|\mathrm{Sh}_q(\Omega(C))]$ is linearly spanned by the elements $[\alpha_1|\dots|\alpha_p|df|\beta_1|\dots|\beta_q]$, where $\alpha_i, \beta_j \in \Omega(C)$ for any i, j and $f \in \mathcal{O}(C)$. Then $I_{x_0}([\alpha_1|\dots|\alpha_p|df|\beta_1|\dots|\beta_q]) = I_{x_0}([\alpha_1|\dots|\alpha_p|f \cdot \beta_1|\dots|\beta_q] - [\alpha_1|\dots|\alpha_p \cdot f|\beta_1|\dots|\beta_q]) \in I_{x_0}(\mathrm{Sh}_{p+q}(\Omega(C)))$.

Let us prove (5.1.2). The space $[d\mathcal{O}(C)|\mathrm{Sh}_n(\Omega(C))]$ is linearly spanned by the elements $[df|\alpha_1|\dots|\alpha_n]$, where $f \in \mathcal{O}(C)$ and $\alpha_i \in \Omega(C)$ for any i . Then $I_{x_0}([df|\alpha_1|\dots|\alpha_n]) = I_{x_0}([f \cdot \alpha_1|\dots|\alpha_n]) - f(x_0) \cdot I_{x_0}([\alpha_1|\dots|\alpha_n]) \in I_{x_0}(\mathrm{Sh}_n(\Omega(C)))$. (5.1.3) similarly follows from $I_{x_0}([\alpha_1|\dots|\alpha_n|df]) = p^*f \cdot I_{x_0}([\alpha_1|\dots|\alpha_n]) - I_{x_0}([\alpha_1|\dots|\alpha_n \cdot f]) \in p^*\mathcal{O}(C) \cdot I_{x_0}(\mathrm{Sh}_n(\Omega(C)))$. \square

Proposition 5.2. *One has the equality*

$$I_{x_0}(F_\bullet \mathrm{Sh}(\Omega(C))) = f_{\sigma, x_0}(\mathbb{C}1 \otimes F_\bullet \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) + \mathcal{O}(C) \otimes F_{\bullet-1} \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}))$$

of filtrations of $\mathcal{O}_{\mathrm{hol}}(\tilde{C})$, where f_{σ, x_0} is the algebra morphism from Lem. 3.14.

Proof. Let us prove

$$(5.1.4) \quad \forall n \geq 0, \quad I_{x_0}(F_n \mathrm{Sh}(\Omega(C))) = f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) + \mathcal{O}(C) \otimes F_{n-1} \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})).$$

For any $n \geq 0$, one has $f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})) = I_{x_0}(F_n \mathrm{Sh}(\sigma(\mathrm{H}_C^{\mathrm{dR}}))) \subset I_{x_0}(F_n \mathrm{Sh}(\Omega(C)))$. For $f \in \mathcal{O}(C)$, one has $p^*(f) = f(x_0) + I_{x_0}([df])$, and therefore

$$(5.1.5) \quad p^*(\mathcal{O}(C)) = \mathbb{C} + I_{x_0}([d\mathcal{O}(C)]).$$

Then

$$\begin{aligned} f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_{n-1} \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})) &= p^*\mathcal{O}(C) \cdot I_{x_0}(F_{n-1} \mathrm{Sh}(\sigma(\mathrm{H}_C^{\mathrm{dR}}))) \\ &= (\mathbb{C} + I_{x_0}([d\mathcal{O}(C)])) \cdot I_{x_0}(F_{n-1} \mathrm{Sh}(\sigma(\mathrm{H}_C^{\mathrm{dR}}))) \\ &= I_{x_0}(F_{n-1} \mathrm{Sh}(\sigma(\mathrm{H}_C^{\mathrm{dR}}))) + I_{x_0}([d\mathcal{O}(C)] \sqcup F_{n-1} \mathrm{Sh}(\sigma(\mathrm{H}_C^{\mathrm{dR}}))) \subset I_{x_0}(F_n \mathrm{Sh}(\Omega(C))), \end{aligned}$$

where the second equality follows from (5.1.5), the third equality follows from the algebra morphism status of I_{x_0} , and the last equality follows from the inclusions of $F_{n-1} \mathrm{Sh}(\sigma(\mathrm{H}_C^{\mathrm{dR}}))$ and $[d\mathcal{O}(C)] \sqcup F_{n-1} \mathrm{Sh}(\sigma(\mathrm{H}_C^{\mathrm{dR}}))$ in $F_n \mathrm{Sh}(\Omega(C))$. This proves that

$$\forall n \geq 0, \quad I_{x_0}(F_n \mathrm{Sh}(\Omega(C))) \supset f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) + \mathcal{O}(C) \otimes F_{n-1} \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})).$$

Let us now show, by induction on $n \geq 0$, the opposite inclusion

$$\forall n \geq 0, \quad I_{x_0}(F_n \mathrm{Sh}(\Omega(C))) \subset f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) + \mathcal{O}(C) \otimes F_{n-1} \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})).$$

For $n = 0$, the equality is obvious as both sides are equal to \mathbb{C} . Let $n > 0$, assume the equality for all steps $\leq n - 1$ and let us prove it at step n . By the induction hypothesis, it suffices to prove the inclusion $I_{x_0}(\text{Sh}_n(\Omega(C))) \subset f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \text{Sh}(\mathbb{H}_C^{\text{dR}}) + \mathcal{O}(C) \otimes F_{n-1} \text{Sh}(\mathbb{H}_C^{\text{dR}}))$, i.e.

$$(5.1.6) \quad I_{x_0}(\text{Sh}_n(\Omega(C))) \subset I_{x_0}(F_n \text{Sh}(\sigma(\mathbb{H}_C^{\text{dR}}))) + p^* \mathcal{O}(C) \cdot I_{x_0}(F_{n-1} \text{Sh}(\mathbb{H}_C^{\text{dR}})).$$

The space $\text{Sh}_n(\Omega(C))$ is linearly spanned by the elements $[\omega_1 | \dots | \omega_n]$ with $\omega_1, \dots, \omega_n \in \Omega(C)$. For all i , let $h_i \in \mathbb{H}_C^{\text{dR}}$ be the projection of ω_i and choose $f_i \in \mathcal{O}(C)$ such that $\omega_i = \sigma(h_i) + df_i$. Then

$$\begin{aligned} [\omega_1 | \dots | \omega_n] &\in [\sigma(h_1) | \dots | \sigma(h_n)] + [df_1 | \text{Sh}_{n-1}(\Omega(C))] + [\text{Sh}_{n-1}(\Omega(C)) | df_{n-1}] \\ &+ \sum_{i=1}^{n-1} [\text{Sh}_{i-1}(\Omega(C)) | df_i | \text{Sh}_{n-i}(\Omega(C))] \subset \text{Sh}_n(\sigma(\mathbb{H}_C^{\text{dR}})) + [d\mathcal{O}(C) | \text{Sh}_{n-1}(\Omega(C))] \\ &+ [\text{Sh}_{n-1}(\Omega(C)) | d\mathcal{O}(C)] + \sum_{i=1}^{n-1} [\text{Sh}_{i-1}(\Omega(C)) | d\mathcal{O}(C) | \text{Sh}_{n-i}(\Omega(C))]. \end{aligned}$$

Lem. 5.1 then implies that

$$(5.1.7) \quad \begin{aligned} I_{x_0}([\omega_1 | \dots | \omega_n]) &\in I_{x_0}(\text{Sh}_n(\sigma(\mathbb{H}_C^{\text{dR}}))) + p^* \mathcal{O}(C) \cdot I_{x_0}(\text{Sh}_{n-1}(\Omega(C))) + p^* \mathcal{O}(C) \cdot I_{x_0}(\text{Sh}_{n-1}(\Omega(C))) \\ &+ \sum_{i=1}^{n-1} I_{x_0}(\text{Sh}_{n-1}(\Omega(C))) = I_{x_0}(\text{Sh}_n(\sigma(\mathbb{H}_C^{\text{dR}}))) + p^* \mathcal{O}(C) \cdot I_{x_0}(\text{Sh}_{n-1}(\Omega(C))). \end{aligned}$$

Moreover,

$$p^* \mathcal{O}(C) \cdot I_{x_0}(\text{Sh}_{n-1}(\Omega(C))) \subset p^* \mathcal{O}(C) \cdot I_{x_0}(F_{n-1} \text{Sh}(\Omega(C))) \subset p^* \mathcal{O}(C) \cdot I_{x_0}(F_{n-1} \text{Sh}(\sigma(\mathbb{H}_C^{\text{dR}}))),$$

where the first inclusion follows from $\text{Sh}_{n-1}(\Omega(C)) \subset F_{n-1} \text{Sh}(\Omega(C))$ and the second inclusion from the induction hypothesis, i.e. (5.1.6) at step $n-1$ by multiplication by $p^* \mathcal{O}(C)$. Combining this inclusion with (5.1.7), one obtains

$$I_{x_0}([\omega_1 | \dots | \omega_n]) \in I_{x_0}(\text{Sh}_n(\sigma(\mathbb{H}_C^{\text{dR}}))) + p^* \mathcal{O}(C) \cdot I_{x_0}(F_{n-1} \text{Sh}(\sigma(\mathbb{H}_C^{\text{dR}}))),$$

which is (5.1.6) at step n . \square

5.2. The filtrations $F_{\bullet}^{\delta} \mathcal{O}_{\text{hol}}(\tilde{C})$ and $F_{\bullet}^{\mu} \mathcal{O}_{\text{hol}}(\tilde{C})$.

Lemma 5.3. *Let $f^{\bullet} : (A^{\bullet}, d_A) \rightarrow (B^{\bullet}, d_B)$ be a morphism of \mathbb{C} -dgas. Define $F_{\bullet}^{\delta} B^0$ and $F_{\bullet}^{\mu} B^0$ as follows:*

- $F_0^{\delta} B^0 := \mathbb{C}$ and for $n \geq 0$, $F_{n+1}^{\delta} B^0 := \{b \in B^0 \mid d_B(b) \in f^1(A^1) \cdot F_n^{\delta} B^0\}$,
- for any $n \geq 0$, $F_n^{\mu} B^0 := f^0(A^0) \cdot F_n^{\delta} B^0$.

Then:

- (a) Both $F_{\bullet}^{\delta} B^0$ and $F_{\bullet}^{\mu} B^0$ are algebra filtrations of B^0 .
- (b) For any $n \geq 0$, one has $F_n^{\delta} B^0 \subset F_n^{\mu} B^0 \subset F_{n+1}^{\delta} B^0$.
- (c) One has $F_{\infty}^{\delta} B^0 = F_{\infty}^{\mu} B^0$ (equality of subalgebras of B^0).

Proof. (a) Let us prove by induction on $n \geq 0$ that $F_n^\delta B^0 \subset F_{n+1}^\delta B^0$. One has $F_0^\delta B^0 \subset F_1^\delta B^0$ since $d_B(1) = 0$. Assume that $n \geq 1$ and that $F_{n-1}^\delta B^0 \subset F_n^\delta B^0$. Then for $f \in F_n^\delta B^0$, $d_B(f) \in f^1(A^1) \cdot F_{n-1}^\delta B^0 \subset f^1(A^1) \cdot F_n^\delta B^0$, where the inclusion follows from the induction hypothesis, and therefore $f \in F_{n+1}^\delta B^0$. This proves that $F_n^\delta B^0 \subset F_{n+1}^\delta B^0$ for any $n \geq 0$.

Let us prove by induction on $n \geq 0$ that for² any $p \in \llbracket 0, n \rrbracket$, one has $F_p^\delta B^0 \cdot F_{n-p}^\delta B^0 \subset F_n^\delta B^0$. For $n = 0$, this follows from the algebra structure of \mathbb{C} . Assume that $n \geq 0$ and that $F_p^\delta B^0 \cdot F_{n-p}^\delta B^0 \subset F_n^\delta B^0$ for any $p \in \llbracket 0, n \rrbracket$. Let $p' \in \llbracket 0, n+1 \rrbracket$. The inclusion $F_{p'}^\delta B^0 \cdot F_{n+1-p'}^\delta B^0 \subset F_{n+1}^\delta B^0$ is obvious if $p' = 0$ or $n+1$. If $p' \in \llbracket 1, n \rrbracket$, let $f \in F_{p'}^\delta B^0$, $g \in F_{n+1-p'}^\delta B^0$. Then $d_B(fg) = f \cdot d_B g + g \cdot d_B f \in F_{p'}^\delta B^0 \cdot f^1(A^1) \cdot F_{n-p'}^\delta B^0 + F_{n+1-p'}^\delta B^0 \cdot f^1(A^1) \cdot F_{p'-1}^\delta B^0 \subset f^1(A^1) \cdot F_n^\delta B^0$, where the first relation follows from the definitions of $F_{p'}^\delta B^0$ and $F_{n+1-p'}^\delta B^0$, and the second relation follows from the induction hypothesis. Therefore $fg \in F_{n+1}^\delta B^0$. This proves the inclusion $F_{p'}^\delta B^0 \cdot F_{n+1-p'}^\delta B^0 \subset F_{n+1}^\delta B^0$, in all cases, hence the induction step. It follows that

$$(5.2.1) \quad \forall p, q \geq 0, \quad F_p^\delta B^0 \cdot F_q^\delta B^0 \subset F_{p+q}^\delta B^0.$$

All this implies that $F_\bullet^\delta B^0$ is an algebra filtration of B^0 .

For $n \geq 0$, one has $F_n^\mu B^0 = f^0(A^0) \cdot F_n^\delta B^0 \subset f^0(A^0) \cdot F_{n+1}^\delta B^0 = F_{n+1}^\mu B^0$, where the equalities follow from the definitions and the inclusion follows from the already proved statement $F_n^\delta B^0 \subset F_{n+1}^\delta B^0$. For $p, q \geq 0$, one has $F_p^\mu B^0 \cdot F_q^\mu B^0 = f^0(A^0) \cdot F_p^\delta B^0 \cdot f^0(A^0) \cdot F_q^\delta B^0 = f^0(A^0) \cdot F_p^\delta B^0 \cdot F_q^\delta B^0 \subset f^0(A^0) \cdot F_{p+q}^\delta B^0 = F_{p+q}^\mu B^0$, where the first and last equalities follow from the definitions, the second equality follows by commutativity and by the unital algebra status of $f^0(A^0)$, the inclusion follows from (5.2.1). All this implies that $F_\bullet^\mu B^0$ is an algebra filtration of B^0 , and therefore (a).

(b) Let $n \geq 0$. One has $F_n^\delta B^0 = \mathbb{C} \cdot F_n^\delta B^0 \subset f^0(A^0) \cdot F_n^\delta B^0 = F_n^\delta B^0$, where the inclusion follows from $1 \in f^0(A^0)$.

For $a \in A^0$, $d_B(f^0(a)) = f^1(d_A(a)) = f^1(d_A(a)) \cdot 1 \in f^1(A^1) \cdot F_0^\delta B^0$, where the first equality follows from the compatibility of f^\bullet with differentials, and the last relation follows from $F_0^\delta B^0 = \mathbb{C}$. It follows that $f^0(a) \in F_1^\delta B^0$, so

$$(5.2.2) \quad f^0(A^0) \subset F_1^\delta B^0.$$

Let again $n \geq 0$. Then $F_n^\mu B^0 = f^0(A^0) \cdot F_n^\delta B^0 \subset F_1^\delta B^0 \cdot F_n^\delta B^0 \subset F_{n+1}^\delta B^0$, where the equality follows by definition, the first inclusion follows from (5.2.2), and the second inclusion follows from (5.2.1). All this proves (b).

Statement (c) follows from (a) and (b). \square

Recall the definition of the dga $(\Omega_{hol}^\bullet(\tilde{C}), d)$ from §2.2. Denote by $(\Omega^\bullet(C), d)$ the dga of algebraic differential forms on C ; it is concentrated in degrees 0 and 1, with $\Omega^0(C) = \mathcal{O}(C)$, $\Omega^1(C) = \Omega(C)$, algebra structure given by the algebra structure of $\mathcal{O}(C)$ and the module

²For a, b integers, $\llbracket a, b \rrbracket$ is the discrete interval $[a, b] \cap \mathbb{Z}$ if $a \leq b$ and \emptyset otherwise.

structure of $\Omega(C)$ over it, and differential given by d . The pull-back of $p : \tilde{C} \rightarrow C$ gives rise to an injective dga morphism $p^* : (\Omega^\bullet(C), d) \hookrightarrow (\Omega_{hol}^\bullet(C), d)$.

In (1.2.1), (1.2.2), (1.2.3), we defined $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$, $F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C})$, $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$ and $F_\infty^\mu \mathcal{O}_{hol}(\tilde{C})$.

Corollary 5.4. (a) Both $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$ and $F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C})$ are algebra filtrations of $\mathcal{O}_{hol}(\tilde{C})$.

(b) For any $n \geq 0$, one has $F_n^\delta \mathcal{O}_{hol}(\tilde{C}) \subset F_n^\mu \mathcal{O}_{hol}(\tilde{C}) \subset F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C})$.

(c) One has $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) = F_\infty^\mu \mathcal{O}_{hol}(\tilde{C})$ (equality of subalgebras of $\mathcal{O}_{hol}(\tilde{C})$).

Proof. $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$ (resp. $F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C})$, $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$, $F_\infty^\mu \mathcal{O}_{hol}(\tilde{C})$) is equal to $F_\bullet^\delta B^0$ (resp. $F_\bullet^\mu B^0$, $F_\infty^\delta B^0$, $F_\infty^\mu B^0$) from Lem. 5.3 in the case of the dga morphism p^* ; the statement then follows from this lemma. \square

Proposition 5.5. One has the equality

$$F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C}) = I_{x_0}(F_\bullet \text{Sh}(\Omega(C)))$$

of filtrations of $\mathcal{O}_{hol}(\tilde{C})$.

Proof. Let us prove

$$(5.2.3) \quad \forall n \geq 0, \quad I_{x_0}(F_n \text{Sh}(\Omega(C))) = F_n^\delta \mathcal{O}_{hol}(\tilde{C})$$

by induction on n . For $n = 0$, the equality holds since both sides are equal to \mathbb{C} . Assume the equality at step $n \geq 0$ and let us show it at step $n + 1$.

Let us first show the inclusion $I_{x_0}(F_{n+1} \text{Sh}(\Omega(C))) \subset F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C})$. For this, in view of the induction hypothesis, it suffices to prove $I_{x_0}(\text{Sh}_{n+1}(\Omega(C))) \subset F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C})$. The space is linearly spanned by the elements $[\omega_1 | \dots | \omega_{n+1}]$, where $\omega_1, \dots, \omega_{n+1} \in \Omega(C)$. Then $d(I_{x_0}([\omega_1 | \dots | \omega_{n+1}])) = I_{x_0}([\omega_1 | \dots | \omega_n] \cdot \omega_{n+1})$, and $I_{x_0}([\omega_1 | \dots | \omega_n]) \in I_{x_0}(\text{Sh}_n(\Omega(C))) \subset F_n^\delta \mathcal{O}_{hol}(\tilde{C})$ where the last inclusion follows from the induction hypothesis. This shows that $I_{x_0}([\omega_1 | \dots | \omega_{n+1}]) \in F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C})$, therefore $I_{x_0}(\text{Sh}_{n+1}(\Omega(C))) \subset F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C})$ as wanted.

Let us now show the inclusion $F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C}) \subset I_{x_0}(F_{n+1} \text{Sh}(\Omega(C)))$. Let $f \in F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C})$, then there exist elements $f_1, \dots, f_k \in F_n^\delta \mathcal{O}_{hol}(\tilde{C})$ and $\omega_1, \dots, \omega_k \in \Omega(C)$ such that $df = \sum_i f_i \cdot p^* \omega_i$. By the induction hypothesis, there exist $t_1, \dots, t_k \in F_n \text{Sh}(\Omega(C))$, such that $f_i = I_{x_0}(t_i)$ for any i . Then $df = \sum_i I_{x_0}(t_i) \cdot p^* \omega_i$. Integration gives $f = f(x_0) + \sum_i I_{x_0}([t_i | \omega_i]) = I_{x_0}(f(x_0) + \sum_i [t_i | \omega_i]) \in I_{x_0}(F_{n+1}(\text{Sh}(\Omega(C))))$, which proves the claimed inclusion. \square

Proposition 5.6. One has the equality

$$F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C}) = f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_\bullet \text{Sh}(\mathbb{H}_C^{\text{dR}}))$$

of filtrations of $\mathcal{O}_{hol}(\tilde{C})$.

Proof. Let $n \geq 0$. In Prop. 5.2, we proved the equality $I_{x_0}(F_n \text{Sh}(\Omega(C))) = I_{x_0}(F_n \text{Sh}(\mathbb{H}_C^{\text{dR}})) + p^* \mathcal{O}(C) \cdot I_{x_0}(F_{n-1} \text{Sh}(\mathbb{H}_C^{\text{dR}}))$. Multiplying it by $p^* \mathcal{O}(C)$, we obtain $p^* \mathcal{O}(C) \cdot I_{x_0}(F_n \text{Sh}(\Omega(C))) = p^* \mathcal{O}(C) \cdot I_{x_0}(F_n \text{Sh}(\mathbb{H}_C^{\text{dR}}))$. By Prop. 5.5, $I_{x_0}(F_n \text{Sh}(\Omega(C))) = F_n^\delta \mathcal{O}_{hol}(\tilde{C})$ therefore $p^* \mathcal{O}(C) \cdot$

$I_{x_0}(F_n \text{Sh}(\Omega(C))) = p^* \mathcal{O}(C) \cdot F_n^\delta \mathcal{O}_{hol}(\tilde{C}) = F_n^\mu \mathcal{O}_{hol}(\tilde{C})$. It follows that $F_n^\mu \mathcal{O}_{hol}(\tilde{C}) = p^* \mathcal{O}(C) \cdot I_{x_0}(F_n \text{Sh}(\mathbb{H}_C^{\text{dR}}))$; the right-hand side of this equality is equal to $f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbb{H}_C^{\text{dR}}))$. \square

5.3. Relation with the minimal stable extension of $\mathcal{O}(C)$. Recall from §1.1:

Definition 5.7. (a) A *stable extension* (SE) of $\mathcal{O}(C)$ is a unital subalgebra A of $\mathcal{O}_{hol}(\tilde{C})$, such that for any $f \in A$ and $\omega \in \Omega(C)$, on has $(z \mapsto \int_{x_0}^z f \cdot p^* \omega) \in A$.

(b) $A_C := \bigcap_{A \text{ a SE of } \mathcal{O}(C)} A$.

Lemma-Definition 5.8. A_C is a SE of $\mathcal{O}(C)$, contained in any SE of $\mathcal{O}(C)$; we therefore call A_C the *minimal stable extension* (MSE) of $\mathcal{O}(C)$.

Proof. This follows from the fact that an intersection of two SEs of $\mathcal{O}(C)$ is a SE of $\mathcal{O}(C)$. \square

Proposition 5.9. $A_C = F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$.

Proof. Let us show that $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$ is a SE of $\mathcal{O}(C)$. By Cor. 5.4(c), $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$ is equal to $F_\infty^\mu \mathcal{O}_{hol}(\tilde{C})$ and is a unital subalgebra of $\mathcal{O}_{hol}(\tilde{C})$; the algebra $F_\infty^\mu \mathcal{O}_{hol}(\tilde{C})$ also contains $F_0^\mu \mathcal{O}_{hol}(\tilde{C}) = \mathcal{O}(C)$, therefore $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$ is a unital subalgebra of $\mathcal{O}_{hol}(\tilde{C})$ containing $\mathcal{O}(C)$. Let $\omega \in \Omega(C)$ and $f \in F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$. Then for some $n \geq 0$, one has $f \in F_n^\delta \mathcal{O}_{hol}(\tilde{C})$. Let $F := (z \mapsto \int_{x_0}^z f \cdot p^* \omega)$, then $dF = f \cdot p^* \omega$ which implies $F \in F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C}) \subset F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$. It follows that $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$ is a SE, therefore that $A_C \subset F_\infty^\delta \mathcal{O}_{hol}(\tilde{C})$.

Let us show that if A is a SE of $\mathcal{O}(C)$, and $n \geq 0$, then $F_n^\delta \mathcal{O}_{hol}(\tilde{C}) \subset A$. For $n = 0$, $F_0^\delta \mathcal{O}_{hol}(\tilde{C}) = \mathbb{C} \subset A$. Assume that $n \geq 0$ and $F_n^\delta \mathcal{O}_{hol}(\tilde{C}) \subset A$. Let $f \in F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C})$, then there exists a finite set I and maps $I \rightarrow \Omega(C)$, $I \rightarrow F_n^\delta \mathcal{O}_{hol}(\tilde{C})$, denoted $i \mapsto \omega_i$ and $i \mapsto f_i$, such that $df = \sum_{i \in I} f_i \cdot p^* \omega_i$. Then $f = f(x_0) + \sum_{i \in I} (z \mapsto \int_{x_0}^z f_i \cdot p^* \omega_i)$. By the SE property of A , and since $f_i \in A$ for any $i \in I$, one has $f \in A$, therefore $F_{n+1}^\delta \mathcal{O}_{hol}(\tilde{C}) \subset A$. It follows that for any $n \geq 0$, one has $F_n^\delta \mathcal{O}_{hol}(\tilde{C}) \subset A$, therefore $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) \subset A$. As A_C is a SE of $\mathcal{O}(C)$, it follows that $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) \subset A_C$. \square

5.4. Relation with minimal acyclic extensions of dgas. Recall the inclusion of dgas $(\Omega^\bullet(C), d) \hookrightarrow (\Omega_{hol}^\bullet(C), d)$ from §5.2. One has $H^1(\Omega^\bullet(C)) \simeq H_C^{\text{dR}}$, while $H^1(\Omega_{hol}^\bullet(\tilde{C})) = 0$.

Definition 5.10. (a) An *acyclic extension* (AE) of $\Omega^\bullet(C)$ is a dga (E^\bullet, d) with $\Omega^\bullet(C) \subset E^\bullet \subset \Omega_{hol}^\bullet(\tilde{C})$ and $H^1(E^\bullet) = 0$.

(b) $E_C^\bullet := \bigcap_{E^\bullet \text{ an AE of } \Omega^\bullet(C)} E^\bullet$.

Lemma-Definition 5.11. E_C^\bullet is an AE of $\Omega^\bullet(C)$, which is contained in any AE of $\Omega^\bullet(C)$; we therefore call it the *minimal AE* of $\Omega^\bullet(C)$.

Proof. Let us show that if E^\bullet, F^\bullet are AEs of $\Omega^\bullet(C)$, then so is $E^\bullet \cap F^\bullet$. The intersection $E^\bullet \cap F^\bullet$ is obviously a dga containing $\Omega^\bullet(C)$. If $x \in E^1 \cap F^1$, there exists $e \in E^0$ such that $x = d(e)$ since $H^1(E^\bullet) = 0$, and $f \in F^0$ such that $x = d(f)$ since $H^1(F^\bullet) = 0$. Then $d(e - f) = 0$, therefore

$e - f \in \mathbb{C}$, therefore $f \in e + \mathbb{C} \subset E^0 + \mathbb{C} = E^0$, so $f \in E^0 \cap F^0$. So $E^1 \cap F^1 \subset d(E^0 \cap F^0)$, which implies $H^1(E^\bullet \cap F^\bullet) = 0$. It follows that $E^\bullet \cap F^\bullet$ is an AE of $\Omega^\bullet(C)$. This fact implies the statement. \square

Lemma 5.12. *The maps $E^\bullet \mapsto E^0$ and $A \mapsto (A \oplus dA, d)$ define inverse bijections between the sets $\{\text{AEs of } \Omega^\bullet(C)\}$ and $\text{Alg}_C := \{\text{algebras } A \text{ with } \mathcal{O}(C) \subset A \subset \mathcal{O}_{hol}(\tilde{C}) \text{ such that } d(A) \supset \Omega(C) \text{ and } A \cdot d(A) = dA \text{ (equality of subspaces of } \Omega_{hol}(\tilde{C}))\}$.*

Proof. Let E^\bullet be an AE of $\Omega^\bullet(C)$. Then $\mathcal{O}(C) \subset E^0 \subset \Omega_{hol}(\tilde{C})$ since $\Omega^\bullet(C) \subset E^\bullet \subset \Omega_{hol}^\bullet(\tilde{C})$. Since $H^1(E^\bullet) = 0$, one has $E^1 = d(E^0)$ and since $\Omega^\bullet(C) \subset E^\bullet$, one has $E^1 \supset \Omega(C)$, therefore $d(E^0) \supset \Omega(C)$. The equality $E^1 = d(E^0)$ implies the two extreme equalities in $E^0 \cdot d(E^0) = E^0 \cdot E^1 = E^1 = d(E^0)$, while the middle equality follows from the fact that E^\bullet is a dga with unit. It follows that E^0 belongs to Alg_C .

Let A belong to Alg_C . Since $A \cdot d(A) = d(A)$, the pair $(A \oplus d(A), d)$ is a sub-dga of $\Omega_{hol}^\bullet(\tilde{C})$, and since $d(A) \supset \Omega(C)$ and $A \supset \mathcal{O}(C)$, the dga $(A \oplus d(A), d)$ contains $\Omega^\bullet(C)$ as a sub-dga. One has clearly $H^1(A \oplus d(A), d) = 0$. It follows that $(A \oplus d(A), d)$ is an AE of $\Omega^\bullet(C)$.

The composed map $\text{Alg}_C \rightarrow \{\text{AEs of } \Omega^\bullet(C)\} \rightarrow \text{Alg}_C$ is obviously the identity, and the fact that the composed map $\{\text{AEs of } \Omega^\bullet(C)\} \rightarrow \text{Alg}_C \rightarrow \{\text{AEs of } \Omega^\bullet(C)\}$ is the identity follows from the fact that if E^\bullet is an AE of $\Omega^\bullet(C)$, then $E^1 = d(E^0)$ due to $H^1(E^\bullet) = 0$. \square

Lemma 5.13. (a) $\text{Alg}_C \subset \{\text{SEs of } \mathcal{O}(C)\}$.

(b) $A_C \in \text{Alg}_C$.

(c) $\cap_{A \in \text{Alg}_C} A = A_C$.

Proof. (a) Let $A \in \text{Alg}_C$. Then $A \supset \mathcal{O}(C)$. Let $f \in A$, $\omega \in \Omega(C)$ and set $F := (z \mapsto \int_{x_0}^z f p^* \omega)$. Since $\Omega(C) \subset d(A)$ (by $A \in \text{Alg}_C$), there exists $g \in A$ with $dg = \omega$. Then $dF = f \cdot p^* \omega = f dg \in A \cdot dA = dA$, where the last equality follows from $A \in \text{Alg}_C$. It follows that $F \in A$, therefore A is a SE of $\mathcal{O}(C)$.

(b) By Lem. 5.8, A_C is a SE of $\mathcal{O}(C)$, so it is a subalgebra of $\mathcal{O}_{hol}(\tilde{C})$ containing $\mathcal{O}(C)$. Let $\omega \in \Omega(C)$ and set $F_\omega := (z \mapsto \int_{x_0}^z p^* \omega) \in \mathcal{O}_{hol}(\tilde{C})$. Then $F_\omega \in A_C$ by the SE properties of A_C . Then $d(A_C) \ni d(F_\omega) = \omega$, which implies $d(A_C) \supset \Omega(C)$.

Let us show that $A_C \cdot dA_C = dA_C$. It suffices to prove that, for any $f, g \in A_C$, the element $h := (z \mapsto \int_{x_0}^z f \cdot dg) \in \mathcal{O}_{hol}(\tilde{C})$ belongs to A_C . By Prop. 5.9 and $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) = \cup_{n \geq 0} F_n^\delta \mathcal{O}_{hol}(\tilde{C})$, there exist $n, m \geq 0$ such that $f \in F_n^\delta \mathcal{O}_{hol}(\tilde{C})$ and $g \in F_m^\delta \mathcal{O}_{hol}(\tilde{C})$. Then there exists a finite set I and maps $I \rightarrow \Omega(C)$, $I \rightarrow F_{m-1}^\delta \mathcal{O}_{hol}(\tilde{C})$ denoted $i \mapsto \omega_i$, $i \mapsto k_i$ such that $dg = \sum_{i \in I} k_i \cdot p^* \omega_i$. Then $dh = \sum_{i \in I} f k_i \cdot p^* \omega_i$. For any $i \in I$, $f k_i \in F_n^\delta \mathcal{O}_{hol}(\tilde{C}) \cdot F_{m-1}^\delta \mathcal{O}_{hol}(\tilde{C}) \subset F_{n+m-1}^\delta \mathcal{O}_{hol}(\tilde{C})$, where the last inclusion follows from Cor. 5.4(a). It follows that $h \in F_{n+m}^\delta \mathcal{O}_{hol}(\tilde{C})$, and therefore $h \in A_C$ by $F_\infty^\delta \mathcal{O}_{hol}(\tilde{C}) = \cup_{n \geq 0} F_n^\delta \mathcal{O}_{hol}(\tilde{C})$ and Prop. 5.9.

(c) By (a), one has $\cap_{A \in \text{Alg}_C} A \supset \cap_{A \text{ a SE of } \mathcal{O}(C)} A = A_C$. On the other hand, $\cap_{A \in \text{Alg}_C} A \subset A$ for any $A \in \text{Alg}_C$, so by (b) $\cap_{A \in \text{Alg}_C} A \subset A_C$. \square

Proposition 5.14. $E_C^\bullet = (A_C \oplus d(A_C), d)$.

Proof. Set $X := \cap_{A \in \text{Alg}_C} A$. One has

$$E_C^\bullet = \cap_{E^\bullet \text{ a AE of } \Omega^\bullet(C)} E^\bullet = \cap_{A \in \text{Alg}_C} (A \oplus d(A), d) = (X \oplus d(X), d) = (A_C \oplus d(A_C), d)$$

where the first equality follows from the definition of E_C^\bullet , the second equality follows from Lem. 5.12, the third equality follows from $(A \oplus d(A)) \cap (B \oplus d(B)) = ((A \cap B) \oplus d(A \cap B))$ (equality of subspaces of $\Omega^\bullet(\tilde{C})$) for A, B any pair of subspaces of $\mathcal{O}_{hol}(\tilde{C})$ containing 1, and the last equality follows from Lem. 5.13(c). \square

5.5. Relation with Chen's work. In the Introduction of [Ch], K.-T. Chen defines an algebra filtration $\tilde{F}_\bullet C^\infty(\tilde{M})$ of $C^\infty(\tilde{M})$ for any smooth manifold M , where $\tilde{M} \rightarrow M$ is the universal cover; it has the additional property that $\tilde{F}_n C^\infty(\tilde{M})$ is a subalgebra of $C^\infty(\tilde{M})$ for any $n \geq 0$. If X is a nonsingular complex algebraic variety, and \tilde{X} is its universal cover, equipped with its natural structure of complex manifold, one can similarly define a filtration $\tilde{F}_\bullet \mathcal{O}_{hol}(\tilde{X})$ of the algebra $\mathcal{O}_{hol}(\tilde{X})$, by replacing in the definition of [Ch] the spaces of smooth functions and 1-forms (denoted there $\Lambda^0(M)$ and $\Lambda^1(M)$) by the spaces of regular functions and differentials on X . When $X = C$, the explicit definition of $\tilde{F}_\bullet \mathcal{O}_{hol}(\tilde{C})$ is as follows: $\tilde{F}_0 \mathcal{O}_{hol}(\tilde{C}) := p^* \mathcal{O}(C)$ and $\tilde{F}_{r+1} \mathcal{O}_{hol}(\tilde{C}) := \mathbb{C}[f, \text{int}_{x_0}(g \cdot p^* \omega) \mid f, g \in \tilde{F}_r \mathcal{O}_{hol}(\tilde{C}), \omega \in \Omega(C)]$ for $r \geq 0$, where $\mathbb{C}[-]$ means the subalgebra generated by a family.

Lemma 5.15. (a) For any $r \geq 0$, $\tilde{F}_r \mathcal{O}_{hol}(\tilde{C}) \subset I_{x_0}(\text{Sh}(\Omega(C)))$.

(b) For any $r \geq 0$, $I_{x_0}(\text{Sh}_r(\Omega(C))) \subset \tilde{F}_r \mathcal{O}_{hol}(\tilde{C})$.

(c) One has $\tilde{F}_\infty \mathcal{O}_{hol}(\tilde{C}) = I_{x_0}(\text{Sh}(\Omega(C)))$ (equality of subalgebras of $\mathcal{O}_{hol}(\tilde{C})$).

Proof. (a) $\tilde{F}_0 \mathcal{O}_{hol}(\tilde{C}) = p^* \mathcal{O}(C) = I_{x_0}(\mathbb{C} \oplus [d\mathcal{O}(C)]) \subset I_{x_0}(\text{Sh}(\Omega(C)))$. Let $r \geq 0$ and assume that $\tilde{F}_r \mathcal{O}_{hol}(\tilde{C}) \subset I_{x_0}(\text{Sh}(\Omega(C)))$. Let $g \in \tilde{F}_r \mathcal{O}_{hol}(\tilde{C})$, $\omega \in \Omega(C)$. One knows that for some $a \in \text{Sh}(\Omega(C))$, $\omega = I_{x_0}(a)$. Then $\text{int}_{x_0}(g \cdot p^* \omega) = \text{int}_{x_0}(I_{x_0}(a) \cdot p^* \omega) = I_{x_0}([a|\omega]) \in I_{x_0}(\text{Sh}(\Omega(C)))$. Then $\tilde{F}_{r+1} \mathcal{O}_{hol}(\tilde{C}) = \mathbb{C}[f, \text{int}_{x_0}(g \cdot p^* \omega) \mid f, g \in \tilde{F}_r \mathcal{O}_{hol}(\tilde{C}), \omega \in \Omega(C)] \subset I_{x_0}(\text{Sh}(\Omega(C)))$ by $\tilde{F}_r \mathcal{O}_{hol}(\tilde{C}) \subset I_{x_0}(\text{Sh}(\Omega(C)))$ and the fact that $I_{x_0}(\text{Sh}(\Omega(C)))$ is an algebra.

(b) $I_{x_0}(\text{Sh}_0(\Omega(C))) = \mathbb{C} \subset \tilde{F}_0 \mathcal{O}_{hol}(\tilde{C})$. Let $r \geq 0$ and assume that $I_{x_0}(\text{Sh}_r(\Omega(C))) \subset \tilde{F}_r \mathcal{O}_{hol}(\tilde{C})$. Let $a \in \text{Sh}_{r+1}(\Omega(C))$. There exist elements $(a_i)_{i=1, \dots, k}$, $(\omega_i)_{i=1, \dots, k}$ where $a_i \in \text{Sh}_r(\Omega(C))$, $\omega_i \in \Omega(C)$ such that $a = \sum_{i=1}^k [a_i|\omega_i]$. Then

$$I_{x_0}(a) = \sum_{i=1}^k I_{x_0}([a_i|\omega_i]) = \sum_{i=1}^k \text{int}_{x_0}(I_{x_0}(a_i) \cdot p^* \omega_i).$$

One has $I_{x_0}(a_i) \in \tilde{F}_r \mathcal{O}_{hol}(\tilde{C})$ by assumption, therefore the final term of this equality belongs to $\tilde{F}_{r+1} \mathcal{O}_{hol}(\tilde{C})$. Therefore $I_{x_0}(\text{Sh}_{r+1}(\Omega(C))) \subset \tilde{F}_{r+1} \mathcal{O}_{hol}(\tilde{C})$. The statement follows by induction.

(c) By (a), we know that $\tilde{F}_\infty \mathcal{O}_{hol}(\tilde{C}) \subset I_{x_0}(\text{Sh}(\Omega(C)))$. Moreover, (b) implies that, for any $r \geq 0$, $I_{x_0}(\text{Sh}_r(\Omega(C))) \subset \tilde{F}_\infty \mathcal{O}_{hol}(\tilde{C})$, and so $I_{x_0}(\text{Sh}(\Omega(C))) \subset \tilde{F}_\infty \mathcal{O}_{hol}(\tilde{C})$. (c) follows \square

Remark 5.16. §2.3 of [Ch] contains the definition of another filtration $F_\bullet C^\infty(\tilde{M})$. This definition is both an analogue of that of $F_\bullet^\mu \mathcal{O}_{hol}(\tilde{C})$ (as both definitions give analogous values for the degree 0 term of the filtration) and of $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$ (as both definitions share the same induction step). However, the statement “ $F_r C^\infty(\tilde{M}) \cdot F_s C^\infty(\tilde{M}) \subset F_{r+s} C^\infty(\tilde{M})$ for any $r, s \geq 0$ ” from Prop. 2.3.1 in [Ch] appears to be wrong. Indeed, if $r = 0, s = 1$, $F_0 C^\infty(\tilde{M}) = p^* C^\infty(M)$ while $F_1 C^\infty(\tilde{M}) = \{f \in C^\infty(\tilde{M}) \mid df \in F_0 C^\infty(\tilde{M}) \cdot p^* \Lambda^1(M)\}$; since $F_0 C^\infty(\tilde{M}) \cdot p^* \Lambda^1(M) = p^* \Lambda^1(M)$, the set $F_1 C^\infty(\tilde{M})$ is the set of functions on \tilde{M} of the form $x \mapsto c + \int_{x_0}^x p^* \omega$, where $c \in \mathbb{C}$ and $\omega \in \Lambda^1(M)$; this set is not stable under multiplication by $p^* C^\infty(M)$ (the mistake can be traced to the proof of Prop. 2.3.1, which overlooks the fact that the inclusion $dF_r C^\infty(\tilde{M}) \subset F_{r-1} C^\infty(\tilde{M}) \cdot p^* \Lambda^1(M)$ is valid in general only if one introduces $F_{-1} C^\infty(\tilde{M}) = \mathbb{C}$).

6. CONNECTIONS FOR HACAs

We introduce the notion of connection on a HACA in §6.1. In §6.2, we construct a natural connection on the HACA $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{mod}(\tilde{C}))$. We compute its pull-back under the HACA isomorphism $(\nu(p_{\sigma, x_0}), f_{\sigma, x_0})$ from Prop. 3.17 in §6.3, and we relate it in §6.4 to the KZ connection when C is of genus 0.

6.1. Connections for HACAs. Let \mathbf{a} be a commutative algebra. Recall that the \mathbf{a} -module of Kähler differentials of \mathbf{a} is the quotient $\Omega_{\mathbf{a}} := \text{Ker}(m)/\text{Ker}(m)^2$, where m is the product map $\mathbf{a} \otimes \mathbf{a} \rightarrow \mathbf{a}$; the derivation $d : \mathbf{a} \rightarrow \Omega_{\mathbf{a}}$ is defined by $d(a) = a \otimes 1 - 1 \otimes a + \text{Ker}(m)^2$. One has $\Omega_{\mathcal{O}(C)} = \Omega(C)$.

Let (O, A) be a HACA where A is a commutative algebra; denote by $\Delta_A : A \rightarrow O \otimes A$ its coaction morphism. Then $A^O = \{a \in A \mid \Delta_A(a) = 1 \otimes a\} = F_0 A$ and $A \otimes_{A^O} \Omega_{A^O}$ is a left A -module.

Definition 6.1. A connection for (O, A) is a map $\nabla_A : A \rightarrow A \otimes_{A^O} \Omega_{A^O}$, which: (a) is a derivation, i.e. $\nabla_A(aa') = a\nabla_A(a') + a'\nabla_A(a)$ for any $a, a' \in A$; (b) is O -equivariant, i.e. $(\Delta_A \otimes id_{\Omega_{A^O}}) \circ \nabla_A = (id_O \otimes \nabla_A) \circ \Delta_A$; (c) is such that $\nabla_A(a) = 1 \otimes da$ for $a \in A^O$.

If (O, A) is a HACA with connection ∇_A , then one defines the pull-back of ∇_A by a HACA isomorphism $(O', A') \rightarrow (O, A)$, which is a connection for (O', A') .

Remark 6.2. If G is an algebraic group and P is a principal G -bundle over an affine base, and (O, A) is the pair of regular functions on these spaces, then a connection for (O, A) is an algebraic version of a G -invariant Ehresmann connection on the bundle $P \rightarrow P/G$.

6.2. A connection for $((\mathbb{C}\Gamma)', F_\infty \mathcal{O}_{mod}(\tilde{C}))$.

Proposition 6.3. (a) The map $F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})$ given by $f \otimes \omega \mapsto f \cdot p^* \omega$ is injective.

(b) There exists a unique map $\nabla : F_\infty \mathcal{O}_{mod}(\tilde{C}) \rightarrow F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C)$ such that the diagram

$$(6.2.1) \quad \begin{array}{ccc} F_\infty \mathcal{O}_{mod}(\tilde{C}) & \xrightarrow{\nabla} & F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \\ \downarrow & & \downarrow \\ \mathcal{O}_{hol}(\tilde{C}) & \xrightarrow{d} & \Omega_{hol}(\tilde{C}) \end{array}$$

commutes.

(c) ∇ is a connection for the HACCA $((\mathbb{C}\Gamma)^\vee, F_\infty \mathcal{O}_{mod}(\tilde{C}))$.

Proof. (a) By Lem. 3.17, $f_{\sigma, x_0} : \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C) \rightarrow F_\infty \mathcal{O}_{mod}(\tilde{C})$ is an isomorphism of filtered $\mathcal{O}(C)$ -modules. Its image by the functor $- \otimes_{\mathcal{O}(C)} \Omega(C)$ is an isomorphism of filtered vector spaces $\varphi_{\sigma, x_0} : \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \Omega(C) \rightarrow F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C)$. The natural morphism $\mathrm{can} : F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})$ is Γ_C -equivariant, therefore by Lem. B.5(b) is compatible with the filtrations induced by the action of Γ_C . The composed morphism $\mathrm{can} \circ \varphi_{\sigma, x_0} : \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})$, given by $\varphi_{\sigma, x_0}(a \otimes \omega) = I_{x_0}(\sigma(a)) \cdot p^*(\omega)$, is therefore compatible with the filtrations $F_\bullet \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \Omega(C)$ of the source and $F_\bullet \Omega_{hol}(\tilde{C})$ of the target (see Def. B.4), therefore it gives rise to an associated graded map $\mathrm{gr}_\bullet(\mathrm{can} \circ \varphi_{\sigma, x_0}) : \mathrm{Sh}_\bullet(\mathrm{H}_C^{\mathrm{dR}}) \otimes \Omega(C) \rightarrow \mathrm{gr}_\bullet \Omega_{hol}(\tilde{C})$. By Lem. B.5(c), one attaches to the Γ_C -module $\Omega_{hol}(\tilde{C})$ the injective graded map $\mathrm{gr}_\bullet \Omega_{hol}(\tilde{C}) \hookrightarrow \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C), \Omega_{hol}(\tilde{C})^{\Gamma_C})$, where $\Omega_{hol}(\tilde{C})^{\Gamma_C}$ can be identified with the space of holomorphic differentials on C ; it contains $\Omega(C)$ as a subspace.

For any $n \geq 0$, the composition of $\mathrm{gr}_n \Omega_{hol}(\tilde{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C), \Omega_{hol}(\tilde{C})^{\Gamma_C})$ with $\mathrm{gr}_n(\mathrm{can} \circ \varphi_{\sigma, x_0})$ is given by the composition of the isomorphism $\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C), \mathbb{C}) \otimes \Omega_{hol}(\tilde{C})^{\Gamma_C} \simeq \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C), \Omega_{hol}(\tilde{C})^{\Gamma_C})$ (due to the finite dimensionality of $\mathrm{gr}^n(\mathbb{C}\Gamma_C)$) with the tensor product of the injection $\Omega(C) \hookrightarrow \Omega_{hol}(\tilde{C})^{\Gamma_C}$ with the map $\mathrm{Sh}_n(\mathrm{H}_C^{\mathrm{dR}}) \rightarrow \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C), \mathbb{C})$, which is injective by Lem. 3.8. It follows that $\mathrm{gr}_\bullet(\mathrm{can} \circ \varphi_{\sigma, x_0})$ is injective, which as the filtration of the source of $\mathrm{can} \circ \varphi_{\sigma, x_0}$ is exhaustive, implies the injectivity of $\mathrm{can} \circ \varphi_{\sigma, x_0}$, which as φ_{σ, x_0} is an isomorphism implies the injectivity of can .

(b) In this proof, we abbreviate $F_\bullet^\delta \mathcal{O}_{hol}(\tilde{C})$ into F_\bullet^δ . For any $n \geq 0$, the inclusion $F_n^\delta \subset F_\infty \mathcal{O}_{mod}(\tilde{C})$ gives rise to the inclusion of subspaces $\mathrm{im}(F_n^\delta \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})) \subset \mathrm{im}(F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})) \subset \Omega_{hol}(\tilde{C})$. By the definition of F_\bullet^δ , one has $d(F_{n+1}^\delta) \subset \mathrm{im}(F_n^\delta \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C}))$, therefore $d(F_{n+1}^\delta) \subset \mathrm{im}(F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C}))$. This holds for any $n \geq 0$ and $F_\infty \mathcal{O}_{mod}(\tilde{C}) = \bigcup_{n \geq 0} F_n^\delta$ (see Prop. 3.17, Cor. 5.4 and Prop. 5.6), therefore $d(F_\infty \mathcal{O}_{mod}(\tilde{C})) \subset \mathrm{im}(F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C}))$. The linear map $F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})$ admits a factorization $F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes \Omega(C) \rightarrow F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})$ where the first map is surjective, therefore $\mathrm{im}(F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes \Omega(C) \rightarrow \Omega_{hol}(\tilde{C})) = \mathrm{im}(F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \rightarrow \Omega_{hol}(\tilde{C}))$, which implies $dF_\infty \mathcal{O}_{mod}(\tilde{C}) \subset \mathrm{im}(F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \rightarrow \Omega_{hol}(\tilde{C}))$. The claim then follows from (a).

(c) The derivation property of ∇ follows from the derivation property of d and of the injectivity of the right vertical map of (6.2.1). The equivariance of ∇ follows from the same injectivity and from the Γ_C -equivariance of d ; the identity $\nabla(f) = 1 \otimes df$ for $f \in \mathcal{O}(C)$ follows from the same injectivity. \square

6.3. A connection for $(\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}), \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C))$.

Definition 6.4. For V a vector space and $\xi \in V^*$, let R_ξ be the vector space endomorphism of $\mathrm{Sh}(V)$ given by $[v_1 | \dots | v_n] \mapsto [v_1 | \dots | v_{n-1}] \xi(v_n)$ for $n \geq 1$ and $v_1, \dots, v_n \in V$ and by $1 \mapsto 0$.

One has therefore $R_\xi = (id \otimes \xi) \circ \partial^V$, where ∂^V is as in §2.3.

Proposition 6.5. (a) The map $\nabla_\sigma : \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C) \rightarrow \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \Omega(C)$ given by $a \otimes f \mapsto a \otimes df + \sum_i R_{h^i}(a) \otimes \sigma(h_i)f$ defines a connection for the HACA $(\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}), \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C))$ (here $(h_i)_i, (h^i)_i$ are dual bases of $\mathrm{H}_C^{\mathrm{dR}}$ and $(\mathrm{H}_C^{\mathrm{dR}})^*$).

(b) The connection ∇_σ is the pull-back of ∇ under the HACA isomorphism $(\nu(p_{\sigma, x_0}), f_{\sigma, x_0})$ (see Prop. 3.17).

Proof. (a) Each of the maps $a \otimes f \mapsto a \otimes df$ and $a \otimes f \mapsto R_h(a) \otimes \sigma(h')f$ (where $h \in (\mathrm{H}_C^{\mathrm{dR}})^*$, $h' \in \mathrm{H}_C^{\mathrm{dR}}$) is a derivation of $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathcal{O}(C)$ with values in $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \Omega(C)$, it follows that the same is true of ∇_σ . For V a vector space and $\xi \in V^*$, it follows from $R_\xi = (id \otimes \xi) \circ \partial^V$ that $\Delta_{\mathrm{Sh}(V)} \circ R_\xi = (id_{\mathrm{Sh}(V)} \otimes R_\xi) \circ \Delta_{\mathrm{Sh}(V)}$ (equality of linear maps $\mathrm{Sh}(V) \rightarrow \mathrm{Sh}(V)^{\otimes 2}$). This implies the $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})$ -equivariance of $\sum_i \sigma(h_i) \otimes R_{h^i}$. The equivariance of $d \otimes id$, as well as the identity $\nabla_\sigma(1 \otimes f) = 1 \otimes df$ for $f \in \mathcal{O}(C)$ are trivial. All this implies the $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})$ -equivariance of $d \otimes id + \sum_i \sigma(h_i) \otimes R_{h^i}$.

(b) For $a_1, \dots, a_n \in \Omega(C)$, one has $dI_{x_0}([a_1 | \dots | a_n]) = p^*(a_n)I_{x_0}([a_1 | \dots | a_{n-1}])$, therefore for $a \in \mathrm{Sh}(\Omega(C))$ one has $\nabla \circ I_{x_0} = (I_{x_0} \otimes id) \circ \partial^{\Omega(C)}$ (equality of maps $\mathrm{Sh}(\Omega(C)) \rightarrow \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C)$). On the other hand, $\partial^{\Omega(C)} \circ \sigma = (\sigma \otimes \sigma) \circ \partial^{\mathrm{H}_C^{\mathrm{dR}}}$. It follows that

$$(6.3.1) \quad \nabla \circ I_{x_0} \circ \sigma = ((I_{x_0} \circ \sigma) \otimes \sigma) \circ \partial^{\mathrm{H}_C^{\mathrm{dR}}}$$

(equality of maps $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \rightarrow \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C)$).

One has $\nabla_\sigma \circ (id \otimes 1) = (id \otimes \sigma) \circ \partial^{\mathrm{H}_C^{\mathrm{dR}}}$ (equality of maps $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \rightarrow \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \Omega(C)$), $(I_{x_0} \circ \sigma) \otimes \sigma = \varphi_{\sigma, x_0} \circ (id \otimes \sigma)$ (equality of maps $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \otimes \mathrm{H}_C^{\mathrm{dR}} \rightarrow F_\infty \mathcal{O}_{mod}(\tilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C)$) and $I_{x_0} \circ \sigma = f_{\sigma, x_0} \circ (id \otimes 1)$ (equality of maps $\mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}}) \rightarrow F_\infty \mathcal{O}_{mod}(\tilde{C})$) which together with (6.3.1) imply

$$(6.3.2) \quad (\nabla \circ f_{\sigma, x_0})(a \otimes 1) = (\varphi_{\sigma, x_0} \circ \nabla_\sigma)(a \otimes 1)$$

for any $a \in \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})$.

Then for $a \in \mathrm{Sh}(\mathrm{H}_C^{\mathrm{dR}})$, $f \in \mathcal{O}(C)$, one has $\nabla_\sigma(a \otimes f) = \nabla_\sigma(a)(1 \otimes f) + a \otimes df$, which implies

$$(6.3.3) \quad \varphi_{\sigma, x_0} \circ \nabla_\sigma(a \otimes f) = \varphi_{\sigma, x_0} \circ \nabla_\sigma(a)(1 \otimes f) + I_{x_0}(\sigma(a)) \otimes df.$$

On the other hand, for any $f \in \mathcal{O}(C)$ and $g \in \mathcal{O}_{hol}(\tilde{C})$, one has $\nabla(p^*f \cdot g) = (p^*f \otimes 1)\nabla(g) + g \otimes df$, which implies for $a \in \text{Sh}(\mathbf{H}_C^{\text{dR}})$ the equality

$$(6.3.4) \quad \nabla(f_{\sigma, x_0}(a \otimes f)) = (p^*f \otimes 1)\nabla(I_{x_0}(\sigma(a))) + I_{x_0}(\sigma(a)) \otimes df.$$

$$(6.3.3), (6.3.4) \text{ and } (6.3.2) \text{ imply } \nabla \circ f_{\sigma, x_0} = \varphi_{\sigma, x_0} \circ \nabla_{\sigma}. \quad \square$$

The HACA $(\text{Sh}(\mathbf{H}_C^{\text{dR}}), \text{Sh}(\mathbf{H}_C^{\text{dR}}) \otimes \mathcal{O}(C))$ corresponds to the trivial principal bundle over C with group $\text{Spec}(\text{Sh}(\mathbf{H}_C^{\text{dR}}))$, and ∇_{σ} is the flat connection $d + A_{\sigma}$ over it, where $A_{\sigma} := \sum_i \sigma(h_i) \otimes h^i \in \Omega(C) \otimes \mathbb{L}((\mathbf{H}_C^{\text{dR}})^*)$, where $\mathbb{L}(-)$ is the free Lie algebra over a vector space; one has $\mathbb{L}((\mathbf{H}_C^{\text{dR}})^*) = \text{LieSpec}(\text{Sh}(\mathbf{H}_C^{\text{dR}}))$.

6.4. The genus 0 case. Let $S_{\infty} \subset \mathbb{C}$ and $C := \mathbb{P}_{\mathbb{C}}^1 \setminus S$, where $S := S_{\infty} \cup \{\infty\}$. Then $\mathcal{O}(C) = \mathbb{C}[z, 1/(z-s), s \in S_{\infty}]$ and $\Omega(C) = \mathcal{O}(C) \cdot dz$. Then $\mathbf{H}_C^{\text{dR}} = \bigoplus_{s \in S_{\infty}} \mathbb{C}h_s$, where h_s is the class of $dz/(z-s)$ in $\mathbf{H}_C^{\text{dR}} = \Omega(C)/d\mathcal{O}(C)$. Let $\sigma_0 : \mathbf{H}_C^{\text{dR}} \rightarrow \Omega(C)$ be the map given by $h_s \mapsto dz/(z-s)$ for $s \in S_{\infty}$. Then the 1-form for the flat $\text{Spec}(\text{Sh}(\mathbf{H}_C^{\text{dR}}))$ -connection ∇_{σ_0} on C is $A_{\sigma_0} = \sum_{s \in S_{\infty}} dz/(z-s) \otimes h^s$.

This connection is related to the Knizhnik-Zamolodchikov (KZ) connection as follows. Recall that this connection, denoted $\nabla_{\text{KZ}} = d + A_{\text{KZ}}$, is a $\exp(\mathfrak{t}_n)$ -connection over the configuration space $C_n(\mathbb{C})$ of n points in \mathbb{C} , where \mathfrak{t}_n is the topological Lie algebra with generators t_{ij} , $i \neq j \in \llbracket 1, n \rrbracket$ and relations $t_{ji} = t_{ij}$ for $|\{i, j\}| = 2$, $[t_{ik} + t_{jk}, t_{ij}] = 0$ for $|\{i, j, k\}| = 3$ and $[t_{ij}, t_{kl}] = 0$ for $|\{i, j, k, l\}| = 4$, and that $A_{\text{KZ}} = \sum_{i \neq j} t_{ij} \otimes d\log(z_i - z_j)$. Let $(s_1, \dots, s_n) \in C_n(\mathbb{C})$ and $S_{\infty} := \{s_1, \dots, s_n\}$; then $\mathbb{C} \setminus S_{\infty}$ is the preimage of (s_1, \dots, s_n) by the projection $C_{n+1}(\mathbb{C}) \rightarrow C_n(\mathbb{C})$. Let $\text{inj}_{S_{\infty}} : \mathbb{C} \setminus S_{\infty} \hookrightarrow C_{n+1}(\mathbb{C})$ be the canonical injection, and let $\iota : \mathbb{L}((\mathbf{H}_C^{\text{dR}})^*) \rightarrow \mathfrak{t}_{n+1}$ be the Lie algebra morphism induced by $h^i \mapsto t_{i, n+1}$ for $i = 1, \dots, n$. Then

$$\iota_*(A_{\sigma_0}) = (\text{inj}_{S_{\infty}})^*(A_{\text{KZ}}).$$

7. RELATION WITH HYPERLOGARITHM FUNCTIONS

In §7.1, we attach to each pair (C, σ) a subalgebra $\mathcal{H}(C, \sigma)$ of $\mathcal{O}_{hol}(\tilde{C})$, and we show that it is a free $\mathcal{O}(C)$ -module with an explicit basis (Prop. 7.2). In §7.2, we consider the case where $C = \mathbb{C} \setminus S_{\infty}$, and we exhibit a section σ_0 such that $\mathcal{H}(\mathbb{C} \setminus S_{\infty}, \sigma_0)$ coincides with the algebra of hyperlogarithm functions (Prop. 7.7).

7.1. The algebra $\mathcal{H}(C, \sigma)$.

Lemma 7.1. (a) For any (σ, x_0) , the algebra morphism $I_{x_0} \circ \sigma : \text{Sh}(\mathbf{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{hol}(\tilde{C})$ is injective.

(b) For any $\sigma \in \Sigma$, the image $I_{x_0} \circ \sigma(\text{Sh}(\mathbf{H}_C^{\text{dR}}))$ is independent of $x_0 \in \tilde{C}$. We denote by $\mathcal{H}(C, \sigma) \subset \mathcal{O}_{hol}(\tilde{C})$ this subalgebra.

Proof. (a) follows from Prop. 3.17(a).

(b) Let x_0, x_1 in \tilde{C} . Let $\mu_{x_0}^{x_1} : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathbb{C}$ be the map $t \mapsto I_{x_0}(\sigma(t))(x_1)$. This is an algebra morphism, as it is the composition of algebra morphisms $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \xrightarrow{\sigma} \text{Sh}(\Omega(C)) \xrightarrow{I_{x_0}} \mathcal{O}_{\text{mod}}(\tilde{C}) \xrightarrow{\text{ev}_{x_1}} \mathbb{C}$, the last one being the evaluation at x_1 . Set $a_{x_0}^{x_1} := (\mu_{x_0}^{x_1} \otimes \text{id}) \circ \Delta_{\text{Sh}(\mathbb{H}_C^{\text{dR}})}$; this is an algebra endomorphism of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$ as it is a composition of algebra morphisms. It is also a vector space automorphism of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$ since it is compatible with the filtration and the associated graded endomorphism of $\text{grSh}(\mathbb{H}_C^{\text{dR}})$ is the identity. Therefore $a_{x_0}^{x_1}$ is an algebra automorphism of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$.

Equation (2.3.4) implies the identity $I_{x_1}(\sigma(t)) = I_{x_1}(\sigma(t^{(1)}))(x_0)I_{x_0}(\sigma(t^{(2)}))$ (equality in $\mathcal{O}_{\text{hol}}(\tilde{C})$) for any $t \in \text{Sh}(\mathbb{H}_C^{\text{dR}})$, therefore the equality $I_{x_1} \circ \sigma = I_{x_0} \circ \sigma \circ a_{x_0}^{x_1}$ (equality of maps $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$), which together with the bijectivity of $a_{x_0}^{x_1}$ implies (b). \square

Let $d := \dim(\mathbb{H}_C^{\text{dR}})$ and let $(h_i)_{i \in \llbracket 1, d \rrbracket}$ be a basis of \mathbb{H}_C^{dR} . Then

$$(I_{x_0}([\sigma(h_{i_1}) | \dots | \sigma(h_{i_k})]))_{k \geq 0, i_1, \dots, i_k \in \llbracket 1, d \rrbracket}$$

is a family of elements of $\mathcal{H}(C, \sigma)$. The following statement is a generalization of Cor. 5.6 in [Br].

Proposition 7.2. (a) *The family $(I_{x_0}([\sigma(h_{i_1}) | \dots | \sigma(h_{i_k})]))_{k \geq 0, i_1, \dots, i_k \in \llbracket 1, d \rrbracket}$ is a basis of the vector space $\mathcal{H}(C, \sigma)$.*

(b) *It is free over $\mathcal{O}(C)$, namely for any family $(f^{i_1, \dots, i_k})_{k \geq 0, i_1, \dots, i_k \in \llbracket 1, d \rrbracket}$ in $\mathcal{O}(C)$, the relation*

$$\sum_{k \geq 0, i_1, \dots, i_k \in \llbracket 1, d \rrbracket} p^*(f^{i_1, \dots, i_k}) I_{x_0}([\sigma(h_{i_1}) | \dots | \sigma(h_{i_k})]) = 0$$

implies the vanishing of $(f^{i_1, \dots, i_k})_{k \geq 0, i_1, \dots, i_k \in \llbracket 1, d \rrbracket}$.

Proof. (a) follows from Lem. 7.1 (a). (b) follows from (a) and Prop. 3.17(a). \square

For $(\sigma, x_0) \in \Sigma \times \tilde{C}$, set

$$J_{\sigma, x_0} := \sum_{k \geq 0, i_1, \dots, i_k \in \llbracket 1, d \rrbracket} I_{x_0}([\sigma(h_{i_1}) | \dots | \sigma(h_{i_k})]) \otimes (h^{i_1} \otimes \dots \otimes h^{i_k}) \in \mathcal{O}_{\text{hol}}(\tilde{C}) \hat{\otimes} \hat{T}((\mathbb{H}_C^{\text{dR}})^*),$$

where $(h^i)_{i \in \llbracket 1, d \rrbracket}$ is the basis of $(\mathbb{H}_C^{\text{dR}})^*$ dual to $(h_i)_{i \in \llbracket 1, d \rrbracket}$ (one checks that J_{σ, x_0} is independent of the choice of $(h_i)_{i \in \llbracket 1, d \rrbracket}$).

Proposition 7.3. *The element J_{σ, x_0} satisfies the equality*

$$(7.1.1) \quad (d \otimes \text{id})(J_{\sigma, x_0}) = J_{\sigma, x_0} \cdot \left(\sum_{i \in \llbracket 1, d \rrbracket} p^* \sigma(h_i) \otimes h^i \right)$$

(equality in $\Omega_{\text{hol}}(\tilde{C}) \hat{\otimes} \hat{T}((\mathbb{H}_C^{\text{dR}})^)$) as well as $J_{\sigma, x_0}(x_0) = 1$; it is the only element of the algebra $\mathcal{O}_{\text{hol}}(\tilde{C}) \hat{\otimes} \hat{T}((\mathbb{H}_C^{\text{dR}})^*)$ satisfying these conditions.*

Proof. (7.1.1) follows from $d(I_{x_0}([t_1|\dots|t_n])) = I_{x_0}([t_1|\dots|t_{n-1}]) \cdot p^*t_n$ for $n \geq 1$ and $t_1, \dots, t_n \in \Omega(C)$. The second statement follows from the fact that $t \mapsto I_{x_0}(t)(x_0)$ is the augmentation map $\text{Sh}(\Omega(C)) \rightarrow \mathbb{C}$. The uniqueness follows from the fact the ratio of two solutions must be equal to 1 at x_0 and be killed by d , hence be 1. \square

Equation (7.1.1) is a generalization of (5.1) in [Br], and J_{σ, x_0} is a generalization of the solution $L(z)$ of this equation constructed in *loc. cit.*, Prop. 5.1.

Remark 7.4. One can also derive (7.1.1) from the relation $J_{\sigma, x_0} = (f_{\sigma, x_0} \otimes \text{id})(J^{13})$, where J^{13} is the image of the canonical element J of $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \hat{\otimes} \hat{T}((\mathbb{H}_C^{\text{dR}})^*)$ in $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \otimes \mathcal{O}(C) \hat{\otimes} \hat{T}((\mathbb{H}_C^{\text{dR}})^*)$, from $\nabla([h_1|\dots|h_k]) = [h_1|\dots|h_{k-1}] \otimes h_k$, which implies $\nabla_\sigma(J^{13}) = J^{13} \cdot \sum_i 1 \otimes \sigma(h_i) \otimes h^i$, and from Prop. 6.5(b).

7.2. The algebra $\mathcal{H}(C, \sigma)$ in the genus 0 case. Let $S_\infty \subset \mathbb{C}$ be a finite set with $S_\infty \ni 0$ and let $C := \mathbb{C} \setminus S_\infty$. Recall the linear isomorphism $\mathbb{H}_C^{\text{dR}} \simeq \mathbb{C}S_\infty$ and $\sigma_0 \in \Sigma$ such that for any $s \in S_\infty$, $\sigma_0(s) = \text{dlog}(z - s)$. By Lem. 7.1 (b), one attaches to it the subalgebra $\mathcal{H}(\mathbb{C} \setminus S_\infty, \sigma_0) \subset \mathcal{O}_{\text{hol}}(\tilde{C})$.

Set $H'_C := \bigoplus_{s \in S_\infty \setminus \{0\}} \mathbb{C}s \subset \mathbb{H}_C^{\text{dR}}$ and $\text{Sh}'(\mathbb{H}_C^{\text{dR}}) := \mathbb{C} \oplus [H'_C | \text{Sh}(\mathbb{H}_C^{\text{dR}})]$ (recall that $[-| -]$ denotes the concatenation in $\text{Sh}(\mathbb{H}_C^{\text{dR}})$). Then $\text{Sh}'(\mathbb{H}_C^{\text{dR}})$ is a subalgebra of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$, and:

Lemma 7.5 (see [Pa], §3.3). (a) Denote by $\text{Sh}'(\mathbb{H}_C^{\text{dR}})[X]$ be the polynomial algebra in one variable over the algebra $\text{Sh}'(\mathbb{H}_C^{\text{dR}})$. The combination of the canonical injection $\text{Sh}'(\mathbb{H}_C^{\text{dR}}) \hookrightarrow \text{Sh}(\mathbb{H}_C^{\text{dR}})$ and of the assignment $X \mapsto [0]$ gives rise to an algebra isomorphism $\text{Sh}'(\mathbb{H}_C^{\text{dR}})[X] \rightarrow \text{Sh}(\mathbb{H}_C^{\text{dR}})$.

(b) For $t \in \text{Sh}'(\mathbb{H}_C^{\text{dR}})$ and $z \in \tilde{C}$, the limit $\lim_{\epsilon \rightarrow 0} I(\sigma_0(t))(\epsilon, z)$ exists; the function $(z \mapsto \lim_{\epsilon \rightarrow 0} I_\epsilon(\sigma_0(t))(z)) =: I'_0(t)$ belongs to $\mathcal{O}_{\text{hol}}(\tilde{C})$; the map $I'_0 : \text{Sh}'(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$, $t \mapsto I'_0(t)$ is an algebra morphism.

Definition 7.6 (see [Pa], §3.3). $I_0 : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$ is the composition with the inverse of the isomorphism of (a) of the morphism $\text{Sh}'(\mathbb{H}_C^{\text{dR}})[X] \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$ extending I'_0 by $X \mapsto \log$.

The map I_0 is an algebra morphism. For $w \in \text{Sh}(\mathbb{H}_C^{\text{dR}})$, the element $I_0(w) \in \mathcal{O}_{\text{hol}}(\tilde{C})$ is denoted L_w in [Pa], §3.3 and called the hyperlogarithm function associated to w .

Define an algebra morphism $f_{\sigma_0, 0} : \mathcal{O}(C) \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$ by $f \otimes t \mapsto p^*(f)I_0(t)$.

Proposition 7.7. The algebra morphism $f_{\sigma_0, 0}$ is injective. Moreover $I_0(\text{Sh}(\mathbb{H}_C^{\text{dR}})) = \mathcal{H}(\mathbb{C} \setminus S_\infty, \sigma_0)$.

Proof. Fix $x_0 \in \tilde{C}$. The subalgebra $\text{Sh}'(\mathbb{H}_C^{\text{dR}})$ of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$ is a right coideal for the coalgebra structure, so that $\Delta_{\text{Sh}(\mathbb{H}_C^{\text{dR}})}$ induces an algebra morphism $\Delta_{\text{Sh}'(\mathbb{H}_C^{\text{dR}})} : \text{Sh}'(\mathbb{H}_C^{\text{dR}}) \rightarrow \text{Sh}'(\mathbb{H}_C^{\text{dR}}) \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}})$. Composing with the tensor product of the composition $\text{Sh}'(\mathbb{H}_C^{\text{dR}}) \xrightarrow{I_0} \mathcal{O}_{\text{hol}}(\tilde{C}) \xrightarrow{\text{ev}_{x_0}} \mathbb{C}$ with the identity, one gets an algebra morphism $((\text{ev}_{x_0} \circ I_0) \otimes \text{id}) \circ \Delta_{\text{Sh}'(\mathbb{H}_C^{\text{dR}})} : \text{Sh}'(\mathbb{H}_C^{\text{dR}}) \rightarrow$

$\text{Sh}(\mathbb{H}_C^{\text{dR}})$. By Lem. 7.5, there exists a unique algebra endomorphism $a_0^{x_0}$ of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$, whose restriction to $\text{Sh}'(\mathbb{H}_C^{\text{dR}})$ coincides with $((\text{ev}_{x_0} \circ I_0) \otimes \text{id}) \circ \Delta_{\text{Sh}'(\mathbb{H}_C^{\text{dR}})}$ and such that $[0] \mapsto [0] + \log(x_0)$. One checks that $a_0^{x_0}$ is compatible with the filtration of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$ and that the associated graded endomorphism of $\text{grSh}(\mathbb{H}_C^{\text{dR}})$ is the identity, so that $a_0^{x_0}$ is an algebra automorphism of $\text{Sh}(\mathbb{H}_C^{\text{dR}})$.

Specializing (2.3.4) for $t \in \sigma_0(\text{Sh}'(\mathbb{H}_C^{\text{dR}}))$ and taking its limit for $z \rightarrow 0$, one obtains the equality $I_0(t)(z'') = I_0(t^{(1)})(z')I_{z'}(\sigma_0(t^{(2)}))(z'')$ for $t \in \text{Sh}'(\mathbb{H}_C^{\text{dR}})$ and any $z', z'' \in \tilde{C}$. Setting in this identity $z' := x_0$ and viewing both sides as a function of z'' , one obtains the identity

$$I_0(t) = I_0(t^{(1)})(x_0)I_{x_0}(\sigma_0(t^{(2)}))$$

(in $\mathcal{O}_{\text{hol}}(\tilde{C})$) which is equivalent to the statement that the restrictions to $\text{Sh}'(\mathbb{H}_C^{\text{dR}})$ of I_0 and $I_{x_0} \circ \sigma_0 \circ a_0^{x_0}$, which are algebra morphisms $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$ are equal. The images by these morphisms of $[0] \in \text{Sh}(\mathbb{H}_C^{\text{dR}})$ are also equal, since $I_0([0]) = (z \mapsto \log z)$ while $a_0^{x_0}([0]) = [0] + \log(x_0)$ and $I_{x_0} \circ \sigma_0([0]) = (z \mapsto \log z - \log x_0)$. As $\text{Sh}(\mathbb{H}_C^{\text{dR}})$ is generated by $\text{Sh}'(\mathbb{H}_C^{\text{dR}})$ and $[0]$, the algebra morphism status of both I_0 and $I_{x_0} \circ \sigma_0 \circ a_0^{x_0}$ implies

$$(7.2.1) \quad I_0 = I_{x_0} \circ \sigma_0 \circ a_0^{x_0}$$

(equality of algebra morphisms $\text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$). Since $f_{\sigma_0,0} = m \circ (p^* \otimes I_0)$ and $f_{\sigma_0,x_0} = m \circ (p^* \otimes (I_{x_0} \circ \sigma_0))$ (where $m : \mathcal{O}_{\text{hol}}(\tilde{C})^{\otimes 2} \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$ is the product map), this equality implies

$$(7.2.2) \quad f_{\sigma_0,0} = f_{\sigma_0,x_0} \circ (\text{id} \otimes a_0^{x_0})$$

(equality of algebra morphisms $\mathcal{O}(C) \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$).

The injectivity of f_{σ_0,x_0} (see Prop. 3.17(a)) and the automorphism status of $a_0^{x_0}$ then imply that $f_{\sigma_0,0}$ is injective. Then $I_0(\text{Sh}(\mathbb{H}_C^{\text{dR}})) = I_{x_0} \circ \sigma_0 \circ a_0^{x_0}(\text{Sh}(\mathbb{H}_C^{\text{dR}})) = I_{x_0} \circ \sigma_0(\text{Sh}(\mathbb{H}_C^{\text{dR}})) = \mathcal{H}(C \setminus \tilde{S}, \sigma_0)$, where the first equality follows from (7.2.1), the second follows from the automorphism status of $a_0^{x_0}$, and the last from Lem. 7.1(b). \square

Lemma 7.8 (see also [Br], Cor. 5.6). (a) *The map I_0 is injective.*

(b) *The family of functions $(L_{[\sigma_0(s_1)] \dots [\sigma_0(s_k)]})_{k \geq 0, s_1, \dots, s_k \in S_\infty}$ of $\mathcal{O}_{\text{hol}}(\tilde{C})$ (see notation after Def. 7.6) is a basis of $\mathcal{H}(C \setminus S_\infty, \sigma_0)$.*

(c) *This family of functions is free over $\mathcal{O}(C)$ (see Prop. 7.2 for an explicit formulation).*

Proof. Statement (a) follows from Prop. 7.7, as I_0 is the composition of $f_{\sigma_0,0}$ with the canonical injection $1 \otimes - : \text{Sh}(\mathbb{H}_C^{\text{dR}}) \rightarrow \mathcal{O}(C) \otimes \text{Sh}(\mathbb{H}_C^{\text{dR}})$. Statement (b) follows from (a) and from the second part of Prop. 7.7. Statement (c) follows from (b) and from the first part of Prop. 7.7. \square

8. COMPUTATION OF $\text{Ker}(I_{x_0})$

The main result of this section is Thm. 8.7, where we compute $\text{Ker}(I_{x_0})$ and exhibit a complement of this space in $\text{Sh}(\Omega(C))$. Fix an element $\sigma \in \Sigma$.

Lemma-Definition 8.1. (a) *The direct sum of the map $\text{Sh}(\sigma) : \text{Sh}(\mathbf{H}_C^{\text{dR}}) \rightarrow \text{Sh}(\Omega(C))$ induced by σ and of the concatenation of $\text{Sh}(\sigma)$ with the canonical inclusion $d\mathcal{O}(C) \hookrightarrow \Omega(C)$ is an injective map $\text{Sh}(\mathbf{H}_C^{\text{dR}}) \oplus [\text{Sh}(\mathbf{H}_C^{\text{dR}})|d\mathcal{O}(C)] \rightarrow \text{Sh}(\Omega(C))$.*

(b) *We define*

$$(8.0.1) \quad \text{Sub}_\sigma := \text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}})) \oplus [\text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}}))|d\mathcal{O}(C)] \hookrightarrow \text{Sh}(\Omega(C))$$

to be the image of this map.

Proof. (a) follows from $\Omega(C) = \sigma(\mathbf{H}_C^{\text{dR}}) \oplus d\mathcal{O}(C)$. \square

Define a vector space filtration $F_\bullet \text{Sub}_\sigma$ by $F_n \text{Sub}_\sigma := F_n \text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}})) \oplus [F_n \text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}}))|d\mathcal{O}(C)]$ for $n \geq 0$.

Lemma 8.2. *One has for any $n \geq 0$,*

$$(8.0.2) \quad I_{x_0}(F_n \text{Sub}_\sigma) + I_{x_0}(F_n \text{Sh}(\Omega(C))) = f_{\sigma, x_0}(F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}) \otimes \mathcal{O}(C)) + I_{x_0}(F_n \text{Sh}(\Omega(C))),$$

$$(8.0.3) \quad f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) \subset I_{x_0}(F_n \text{Sub}_\sigma).$$

Proof. (8.0.2) follows from the inclusion $F_{n-1} \text{Sub}_\sigma \subset F_n \text{Sh}(\Omega(C))$, from the fact that $\text{gr}_n \text{Sub}_\sigma$ is linearly spanned by the elements $[\sigma(h_1)|\dots|\sigma(h_n)|df] \oplus 0$ and $0 \oplus [\sigma(h_1)|\dots|\sigma(h_n)]$, where $h_1, \dots, h_n \in \mathbf{H}_C^{\text{dR}}$ and $f \in \mathcal{O}(C)$ and from the identities

$$(8.0.4) \quad \begin{aligned} I_{x_0}([\sigma(h_1)|\dots|\sigma(h_n)|df] \oplus 0) &= f_{\sigma, x_0}([h_1|\dots|h_n] \otimes f) - I_{x_0}([\sigma(h_1)|\dots|\sigma(h_{n-1})|\sigma(h_n)]p^*(f)). \\ I_{x_0}(0 \oplus [\sigma(h_1)|\dots|\sigma(h_n)]) &= f_{\sigma, x_0}([h_1|\dots|h_n] \otimes 1). \end{aligned}$$

for $h_1, \dots, h_n \in \mathbf{H}_C^{\text{dR}}$ and $f \in \mathcal{O}(C)$. (8.0.3) follows from the equality of the maps $F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$, $t \mapsto f_{\sigma, x_0}(1 \otimes t)$ and $t \mapsto I_{x_0}(\sigma(t) \oplus 0)$. \square

Lemma 8.3. *One has, for any $n \geq 0$,*

$$(8.0.5) \quad f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) = I_{x_0}(F_n \text{Sub}_\sigma).$$

Proof. For any $n \geq 0$, one has

$$\begin{aligned} f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) &\subset I_{x_0}(F_n \text{Sub}_\sigma) + I_{x_0}(F_n \text{Sh}(\Omega(C))) \\ &= I_{x_0}(F_n \text{Sub}_\sigma) + f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}) + \mathcal{O}(C) \otimes F_{n-1} \text{Sh}(\mathbf{H}_C^{\text{dR}})) \\ &= I_{x_0}(F_n \text{Sub}_\sigma) + f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) + f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_{n-1} \text{Sh}(\mathbf{H}_C^{\text{dR}})) \\ &= I_{x_0}(F_n \text{Sub}_\sigma) + f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_{n-1} \text{Sh}(\mathbf{H}_C^{\text{dR}})) \end{aligned}$$

where the first inclusion follows from (8.0.2), the first equality follows from Prop. 5.2, the third equality follows from the inclusion (8.0.3) applied to the two first summands of its left-hand side, therefore

$$f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) \subset I_{x_0}(F_n \text{Sub}_\sigma) + f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_{n-1} \text{Sh}(\mathbf{H}_C^{\text{dR}})).$$

Based on this inclusion, one proves inductively that, for any $n \geq 0$, one has

$$(8.0.6) \quad f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) \subset I_{x_0}(F_n \text{Sub}_\sigma).$$

On the other hand, for any $n \geq 0$, one has

$$\begin{aligned} I_{x_0}(F_n \text{Sub}_\sigma) &\subset f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) + I_{x_0}(F_n \text{Sh}(\Omega(C))). \\ &= f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) + f_{\sigma, x_0}(\mathbb{C}1 \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}) + \mathcal{O}(C) \otimes F_{n-1} \text{Sh}(\mathbf{H}_C^{\text{dR}})) \\ &= f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) \end{aligned}$$

where the inclusion follows from (8.0.2), the first equality follows from Prop. 5.2, and the last equality follows from the inclusion of the second summand of its left-hand side in its first one, therefore

$$I_{x_0}(F_n \text{Sub}_\sigma) \subset f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})).$$

This and (8.0.6) implies the statement. \square

It follows from Prop. 3.17(a) that the corestriction of the map f_{σ, x_0} defines a linear isomorphism $\underline{f}_{\sigma, x_0} : \mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}) \rightarrow f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}))$. Define

$$\text{map}_{\sigma, x_0} : \text{Sub}_\sigma \rightarrow \mathcal{O}(C) \otimes \text{Sh}(\mathbf{H}_C^{\text{dR}})$$

to be the composition

$$(8.0.7) \quad \text{Sub}_\sigma \xrightarrow{I_{x_0}} I_{x_0}(\text{Sub}_\sigma) = f_{\sigma, x_0}(\mathcal{O}(C) \otimes \text{Sh}(\mathbf{H}_C^{\text{dR}})) \xrightarrow{(\underline{f}_{\sigma, x_0})^{-1}} \mathcal{O}(C) \otimes \text{Sh}(\mathbf{H}_C^{\text{dR}}),$$

where the equality (equality of subspaces of $\mathcal{O}_{\text{hol}}(\tilde{C})$) follows from the collection of all equalities (8.0.5) for $n \geq 0$.

Lemma 8.4. *The map map_{σ, x_0} is an isomorphism of vector spaces.*

Proof. For any $n \geq 0$, the composition (8.0.7) restricts to a composition

$$F_n \text{Sub}_\sigma \xrightarrow{I_{x_0}} I_{x_0}(F_n \text{Sub}_\sigma) = f_{\sigma, x_0}(\mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})) \xrightarrow{(\underline{f}_{\sigma, x_0})^{-1}} \mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}),$$

where the equality follows from (8.0.5).

It follows that the map map_{σ, x_0} is compatible with the filtrations on both sides, and therefore induces, for any $n \geq 0$, a linear map

$$F_n \text{map}_{\sigma, x_0} : F_n \text{Sub}_\sigma \rightarrow \mathcal{O}(C) \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}}).$$

The composition of the associated graded map with the canonical isomorphisms is then

$$(8.0.8) \quad \begin{aligned} \text{map}_n : \text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}})) \oplus [\text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}}))|d\mathcal{O}(C)] &\simeq \text{gr}_n \text{Sub}_\sigma \xrightarrow{\text{gr}_n \text{map}_{\sigma, x_0}} \mathcal{O}(C) \otimes \text{gr}_n \text{Sh}(\mathbf{H}_C^{\text{dR}}) \\ &\simeq \text{Sh}_n(\mathbf{H}_C^{\text{dR}}) \otimes \mathcal{O}(C). \end{aligned}$$

Recall that a 2-step filtration of a vector space is the same as a vector subspace. The source and target of map_n are equipped with the 2-step filtrations associated respectively with the subspaces $\text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}}))$ and $\text{Sh}_n(\mathbf{H}_C^{\text{dR}}) \otimes \mathbb{C}$.

The composition (8.0.7) restricts to a composition

$$\text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}})) \xrightarrow{I_{x_0}} I_{x_0}(\text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}}))) = f_{\sigma, x_0}(\mathbb{C} \otimes \text{Sh}(\mathbf{H}_C^{\text{dR}})) \xrightarrow{(f_{\sigma, x_0})^{-1}} \mathbb{C} \otimes \text{Sh}(\mathbf{H}_C^{\text{dR}}),$$

where the middle equality follows from the equality of maps $\text{Sh}(\mathbf{H}_C^{\text{dR}}) \rightarrow \mathcal{O}_{\text{hol}}(\tilde{C})$, $t \mapsto I_{x_0}(\sigma(t))$ and $t \mapsto f_{\sigma, x_0}(1 \otimes t)$, therefore map_{σ, x_0} restricts to the isomorphism $\text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}})) \rightarrow \mathbb{C} \otimes \text{Sh}(\mathbf{H}_C^{\text{dR}})$ given by $\sigma(t) \mapsto 1 \otimes t$. This map is compatible with the filtrations, therefore it induces an isomorphism $F_n \text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}})) \rightarrow \mathbb{C} \otimes F_n \text{Sh}(\mathbf{H}_C^{\text{dR}})$ for any $n \geq 0$, which by passing to the associated graded implies that map_n is compatible to the 2-step filtrations on both sides. Its associated graded for this filtration is then a map

$$\oplus_{i=0}^1 \text{gr}_i \text{map}_n : \text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}})) \oplus [\text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}})) | d\mathcal{O}(C)] \rightarrow \text{Sh}_n(\mathbf{H}_C^{\text{dR}}) \otimes (\mathbb{C} \oplus (\mathcal{O}(C)/\mathbb{C})).$$

It follows from the fact that map_{σ, x_0} restricts to the isomorphism $\text{Sh}(\sigma(\mathbf{H}_C^{\text{dR}})) \rightarrow \mathbb{C} \otimes \text{Sh}(\mathbf{H}_C^{\text{dR}})$ given by $\sigma(t) \mapsto 1 \otimes t$ that $\text{gr}_0 \text{map}_n$ is the map $\text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}})) \rightarrow \mathbb{C} \otimes \text{Sh}_n(\mathbf{H}_C^{\text{dR}})$, $\sigma(t) \mapsto 1 \otimes t$. Moreover, (8.0.4) implies

$$\begin{aligned} I_{x_0}([\sigma(h_1) | \dots | \sigma(h_n) | df] \oplus 0) &\in f_{\sigma, x_0}([h_1 | \dots | h_n] \otimes f) + I_{x_0}(F_n \text{Sh}(\Omega(C))) \\ &\subset f_{\sigma, x_0}([h_1 | \dots | h_n] \otimes f) + f_{\sigma, x_0}(\mathbb{C}1 \otimes F_\bullet \text{Sh}(\mathbf{H}_C^{\text{dR}}) + \mathcal{O}(C) \otimes F_{\bullet-1} \text{Sh}(\mathbf{H}_C^{\text{dR}})) \end{aligned}$$

where the first relation follows from (8.0.4) and the second relation follows from Prop. 5.2, therefore $\text{gr}_1 \text{map}_n$ is the map $[\text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}})) | d\mathcal{O}(C)] \rightarrow (\mathcal{O}(C)/\mathbb{C}) \otimes \text{Sh}_n(\mathbf{H}_C^{\text{dR}})$, $[\sigma(t) | df] \mapsto [f] \otimes t$.

The maps $\text{gr}_i \text{map}_n$ are isomorphisms for $i = 0, 1$ and any $n \geq 0$, which implies that map_n is an isomorphism for any $n \geq 0$. This implies that $\text{gr}_n \text{map}_{\sigma, x_0}$ is an isomorphism for any $n \geq 0$, which as the filtrations on the source and target of map_{σ, x_0} are complete implies that this map is an isomorphism. \square

Lemma 8.5. *The restriction of I_{x_0} to Sub_σ is injective.*

Proof. This follows from the equality of this restriction with the composition $f_{\sigma, x_0} \circ \text{map}_{\sigma, x_0}$, which is injective by Prop. 3.17(a) and Lem. 8.4. \square

Proposition 8.6. *One has $\text{Sh}(\Omega(C)) = \text{Sub}_\sigma + \text{im}(D_{x_0})$.*

Proof. Let us prove the inclusion

$$(8.0.9) \quad \text{Sh}_n(\Omega(C)) \subset \text{Sub}_\sigma + \text{im}(D_{x_0}) + F_{n-1} \text{Sh}(\Omega(C))$$

for any $n \geq 0$. One has $\mathbb{C}1 \in \text{Sub}_\sigma$, which proves (8.0.9) for $n = 0$. One has $\Omega(C) = \sigma(\mathbf{H}_C^{\text{dR}}) \oplus d\mathcal{O}(C)$, therefore $\text{Sh}_1(\Omega(C)) \subset \text{Sub}_\sigma$; this proves (8.0.9) for $n = 1$. Let $n \geq 2$. It

follows from $\Omega(C) = d\mathcal{O}(C) + \sigma(\mathbf{H}_C^{\text{dR}})$ that

$$(8.0.10) \quad \text{Sh}_n(\Omega(C)) = [\text{Sh}_{n-1}(\sigma(\mathbf{H}_C^{\text{dR}}))|\Omega(C)] + \sum_{k=0}^{n-1} [\text{Sh}_{k-1}(\Omega(C))|d\mathcal{O}(C)|\text{Sh}_{n-k}(\Omega(C))].$$

One has

$$(8.0.11) \quad [\text{Sh}_n(\sigma(\mathbf{H}_C^{\text{dR}}))|\Omega(C)] = \text{Sh}_{n-1}(\sigma(\mathbf{H}_C^{\text{dR}})) \oplus [\text{Sh}_{n-1}(\sigma(\mathbf{H}_C^{\text{dR}}))|d\mathcal{O}(C)] \subset \text{Sub}_\sigma.$$

Moreover for $k \in \llbracket 0, n-1 \rrbracket$, $f \in \mathcal{O}(C)$ and $\omega_i \in \Omega(C)$, $i \in \llbracket 0, n \rrbracket \setminus \{k\}$, one has

$$\begin{aligned} & [\omega_1 | \dots | \omega_{k-1} | df | \omega_{k+1} | \dots | \omega_n] = D_{x_0}([\omega_1 | \dots | \omega_{k-1}] \otimes f \otimes [\omega_{k+1} | \dots | \omega_n]) \\ & + [\omega_1 | \dots | \omega_{k-1} | p^*(f) \omega_{k+1} | \dots | \omega_n] - [\omega_1 | \dots | \omega_{k-1} p^*(f) | \omega_{k+1} | \dots | \omega_n] \\ & \in \text{im}(D_{x_0}) + F_{n-1} \text{Sh}(\Omega(C)) \end{aligned}$$

if $k > 0$ and

$$\begin{aligned} & [df | \omega_2 | \dots | \omega_n] = D_{x_0}(1 \otimes f \otimes [\omega_2 | \dots | \omega_n]) + [p^*(f) \omega_2 | \dots | \omega_n] - f(x_0)[\omega_2 | \dots | \omega_n] \\ & \in \text{im}(D_{x_0}) + F_{n-1} \text{Sh}(\Omega(C)) \end{aligned}$$

if $k = 0$, which implies

$$(8.0.12) \quad [\text{Sh}_{k-1}(\Omega(C))|d\mathcal{O}(C)|\text{Sh}_{n-k}(\Omega(C))] \subset \text{im}(D_{x_0}) + F_{n-1} \text{Sh}(\Omega(C)).$$

Then (8.0.10), (8.0.11) and (8.0.12) imply (8.0.9), which in its turn can be shown to imply the statement by induction on n . \square

Let $\text{Sh}_+(\Omega(C))$ be the augmentation ideal of $\text{Sh}(\Omega(C))$. A left module structure of $\text{Sh}_+(\Omega(C))$ over $\mathcal{O}(C)$ is defined by $f \cdot [\omega_1 | \dots | \omega_k] := [f\omega_1 | \dots | \omega_k]$ and a right module structure of $\text{Sh}(\Omega(C))$ over $\mathcal{O}(C)$ is defined by $[\omega_1 | \dots | \omega_k] \cdot f := [\omega_1 | \dots | \omega_k f]$ if $k > 0$ and $1 \cdot f := f(x_0)1$, a tensor $v_1 \otimes \dots \otimes v_n$ in $T(V)$ being denoted $[v_1 | \dots | v_n]$ in $\text{Sh}(V)$. Define a map

$$D_{x_0} : \text{Sh}(\Omega(C)) \otimes \mathcal{O}(C) \otimes \text{Sh}_+(\Omega(C)) \rightarrow \text{Sh}(\Omega(C)), \quad s \otimes f \otimes s' \mapsto [s|df|s'] - [s|f \cdot s'] + [s \cdot f|s'].$$

Theorem 8.7. (a) *The sequence of maps*

$$\text{Sh}(\Omega(C)) \otimes \mathcal{O}(C) \otimes \text{Sh}_+(\Omega(C)) \xrightarrow{D_{x_0}} \text{Sh}(\Omega(C)) \xrightarrow{I_{x_0}} \mathcal{O}_{\text{hol}}(\tilde{C})$$

is an exact complex, so that $\text{Ker}(I_{x_0}) = \text{im}(D_{x_0})$.

(b) *There is a direct sum decomposition $\text{Sh}(\Omega(C)) = \text{Sub}_\sigma \oplus \text{Ker}(I_{x_0})$.*

Proof. (a) One checks that $I_{x_0} \circ D_{x_0} = 0$, therefore $\text{im}(D_{x_0}) \subset \text{Ker}(I_{x_0})$. Let us prove the opposite inclusion. The subspace $\text{Sub}_\sigma \subset \text{Sh}(\Omega(C))$ (see (8.0.1)) is such that (i) the restriction of I_{x_0} to Sub_σ is injective (see Lem. 8.5), (ii) $\text{Sh}(\Omega(C)) = \text{Sub}_\sigma + \text{im}(D_{x_0})$ (see Prop. 8.6). Then (i) implies $\text{Sub}_\sigma \cap \text{Ker}(I_{x_0}) = 0$, and therefore $\text{Sub}_\sigma \cap \text{im}(D_{x_0}) = 0$. Then (ii) implies that $\text{Sh}(\Omega(C)) = \text{Sub}_\sigma \oplus \text{im}(D_{x_0})$. The restriction of I_{x_0} to the two summands are then respectively injective (by Lem. 8.5) and zero (by $I_{x_0} \circ D_{x_0} = 0$), which implies $\text{im}(D_{x_0}) = \text{Ker}(I_{x_0})$.

(b) This follows from combining the already proved equalities $\text{Sh}(\Omega(C)) = \text{Sub}_\sigma \oplus \text{im}(D_{x_0})$ and $\text{im}(D_{x_0}) = \text{Ker}(I_{x_0})$. \square

APPENDIX A. BACKGROUND ON HOPF ALGEBRAS

This section is devoted to constructions on Hopf algebras. In §A.1, we define an endofunctor $O \mapsto O^\heartsuit$ of the category of Hopf algebras \mathbf{HA} , and in §A.2, a duality functor $\mathbf{HA} \supset \mathbf{HA}_{fd} \rightarrow \mathbf{HA}_{comm}$, $H \mapsto H'$. In §A.3, we show that a Hopf algebra pairing $p : O \otimes H \rightarrow \mathbb{C}$ gives rise, under a finite-dimensionality assumption, to a Hopf algebra morphism $\nu(p) : O^\heartsuit \rightarrow H'$ (see Lem. A.11).

A.1. An endofunctor $O \mapsto O^\heartsuit$ of \mathbf{HA} . Let O be a complex Hopf algebra with coproduct Δ_O . For $a \geq 1$, let $\Delta_O^{(a)} : O \rightarrow O^{\otimes a}$ be the morphism obtained by iteration of Δ_O . Let $pr_O : O \rightarrow O/\mathbb{C}$ be the canonical projection.

Definition A.1 (see [Q1] B3, [BGF], §3.3.4 or [Fr] §7.2). For $n \geq 0$, we define $F_n^\heartsuit O := \text{Ker}(pr_O^{\otimes n+1} \circ \Delta_O^{(n+1)})$.

We also set $F_{-1}^\heartsuit O = \{0\}$. Note that $F_0^\heartsuit O = \mathbb{C}1$.

Lemma A.2 (see also [Fr], §7.2.15). (a) For $n \geq 0$, $F_n^\heartsuit O \subset F_{n+1}^\heartsuit O$.

(b) For $n \geq 0$ and $k \in \llbracket 0, n+1 \rrbracket$, one has $\Delta_O(F_n^\heartsuit O) \subset F_{k-1}^\heartsuit O \otimes O + O \otimes F_{n-k}^\heartsuit O$.

(c) For $n \geq 0$, one has $\Delta_O(F_n^\heartsuit O) \subset \sum_{k=0}^n F_k^\heartsuit O \otimes F_{n-k}^\heartsuit O$.

(d) For $n, m \geq 0$, one has $(F_n^\heartsuit O) \cdot (F_m^\heartsuit O) \subset F_{n+m}^\heartsuit O$.

Proof. (a) Let η_O, ϵ_O be the unit and counit maps of O . One checks that $F_n^\heartsuit O = \text{Ker}((id - \eta_O \epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)})$. One has $(id - \eta_O \epsilon_O)^{\otimes 2} \circ \Delta_O = \Delta_O \circ (id - \eta_O \epsilon_O)$, which implies that $(id - \eta_O \epsilon_O)^{\otimes n+2} \circ \Delta_O^{(n+2)} = (\Delta_O \otimes id^{\otimes n}) \circ (id - \eta_O \epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)}$ (equality of linear maps $O \rightarrow O^{\otimes n+2}$). Therefore $F_n^\heartsuit O = \text{Ker}((id - \eta_O \epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)}) \subset \text{Ker}((id - \eta_O \epsilon_O)^{\otimes n+2} \circ \Delta_O^{(n+2)}) = F_{n+1}^\heartsuit O$.

(b) Assume that $k \in \llbracket 1, n \rrbracket$. One has $\Delta_O^{(n+1)} = (\Delta_O^{(k)} \otimes \Delta_O^{(n-k+1)}) \circ \Delta_O$, and so $pr_O^{\otimes n+1} \circ \Delta_O^{(n+1)} = ((pr_O^{\otimes k} \circ \Delta_O^{(k)}) \otimes (pr_O^{\otimes n-k+1} \circ \Delta_O^{(n-k+1)})) \circ \Delta_O$. Therefore $\Delta_O(F_n^\heartsuit O) \subset \text{Ker}((pr_O^{\otimes k} \circ \Delta_O^{(k)}) \otimes (pr_O^{\otimes n-k+1} \circ \Delta_O^{(n-k+1)})) = \text{Ker}(pr_O^{\otimes k} \circ \Delta_O^{(k)}) \otimes O + O \otimes \text{Ker}(pr_O^{\otimes n-k+1} \circ \Delta_O^{(n-k+1)}) = F_{k-1}^\heartsuit O \otimes O + O \otimes F_{n-k}^\heartsuit O$.

Assume that $k = 0$. It follows from the statement with $k = 1$ that $\Delta_O(F_n^\heartsuit O) \subset \mathbb{C} \otimes O + O \otimes F_{n-1}^\heartsuit O$. It follows that for $f \in F_n^\heartsuit O$, there exists $a \in O$ with $\Delta_O(f) \in 1 \otimes a + O \otimes F_{n-1}^\heartsuit O$. Applying $\epsilon_O \otimes id$, one derives $f \in a + F_{n-1}^\heartsuit O$, hence $a \in f + F_{n-1}^\heartsuit O \subset F_n^\heartsuit O$. Therefore $\Delta_O(F_n^\heartsuit O) \subset \mathbb{C} \otimes F_n^\heartsuit O + O \otimes F_{n-1}^\heartsuit O$. It follows that $\Delta_O(F_n^\heartsuit O) \subset O \otimes F_n^\heartsuit O$.

For $k = n+1$, the proof of the statement $\Delta_O(F_n^\heartsuit O) \subset F_n^\heartsuit O \otimes O$ is similar, based on the statement for $k = n$.

(c) It follows from the statements $\Delta_O(F_n^\heartsuit O) \subset O \otimes F_n^\heartsuit O$ and $\Delta_O(F_n^\heartsuit O) \subset F_n^\heartsuit O \otimes O$ ((b) for $k = 0, n$) that $\Delta_O(F_n^\heartsuit O) \subset F_n^\heartsuit O \otimes F_n^\heartsuit O$. Together with the statement (b) for $k \in \llbracket 1, n \rrbracket$, this implies that $\Delta_O(F_n^\heartsuit O) \subset F_{k-1}^\heartsuit O \otimes F_n^\heartsuit O + F_n^\heartsuit O \otimes F_{n-k}^\heartsuit O$ for any $k \in \llbracket 1, n \rrbracket$. The statement

then follows from

$$(A.1.1) \quad \bigcap_{k=1}^n (F_{k-1}^\heartsuit O \otimes F_n^\heartsuit O + F_n^\heartsuit O \otimes F_{n-k}^\heartsuit O) = \sum_{k=0}^n F_k^\heartsuit O \otimes F_{n-k}^\heartsuit O,$$

which we now prove. For $i \in \llbracket 0, n \rrbracket$, let A_i be a complement of $F_i^\heartsuit O$ in $F_{i-1}^\heartsuit O$. One has then $F_n^\heartsuit O = \bigoplus_{i=0}^n A_i$. Then $F_{k-1}^\heartsuit O \otimes F_n^\heartsuit O + F_n^\heartsuit O \otimes F_{n-k}^\heartsuit O = \bigoplus_{(i,j) \in \llbracket 0, n \rrbracket^2 | i \leq k-1 \text{ or } j \leq n-k} A_i \otimes A_j$. It follows that $\bigcap_{k=1}^n (F_{k-1}^\heartsuit O \otimes F_n^\heartsuit O + F_n^\heartsuit O \otimes F_{n-k}^\heartsuit O) = \sum_{(i,j) \in S} A_i \otimes A_j$, where $S := \{(i, j) \in \llbracket 0, n \rrbracket^2 | \forall k \in \llbracket 1, n \rrbracket, \text{ one has } i \leq k-1 \text{ or } j \leq n-k\}$. One checks that $S = \{(i, j) \in \llbracket 0, n \rrbracket^2 | i + j \leq n\}$, therefore the left-hand side of (A.1.1) is equal to $\bigoplus_{(i,j) \in \llbracket 0, n \rrbracket^2 | i+j \leq n} A_i \otimes A_j = \sum_{k=0}^n F_k^\heartsuit O \otimes F_{n-k}^\heartsuit O$, proving (A.1.1).

(d) It follows from (c) that

$$(A.1.2) \quad \Delta^{(n+m+1)}(F_n^\heartsuit O) \subset \sum_{\substack{(i_1, \dots, i_{n+m+1}) \in \mathbb{Z}_{\geq 0}^{n+m+1} \\ i_1 + \dots + i_{n+m+1} = n}} F_{i_1}^\heartsuit O \otimes \dots \otimes F_{i_{n+m+1}}^\heartsuit O.$$

For $a \geq 1$ and $L \subset \llbracket 1, a \rrbracket$, define $\varphi_L : \llbracket 1, a \rrbracket \rightarrow \{0, 1\}$ by $\varphi_L(x) = 0$ if $x \in L$ and $\varphi_L(x) = 1$ otherwise. Set then

$$(A.1.3) \quad O_L^{(a)} := \otimes_{i=1}^a F_{\varphi_L(i)}^{unit} O \subset O^{\otimes a},$$

where we recall $F_0^{unit} O = \mathbb{C}1$, $F_1^{unit} O = O$.

Then for any $(i_1, \dots, i_{n+m+1}) \in \mathbb{Z}_{\geq 0}^{n+m+1}$, one has $F_{i_1}^\heartsuit O \otimes \dots \otimes F_{i_{n+m+1}}^\heartsuit O \subset O_{\{j | i_j = 0\}}^{(n+m+1)}$. This and (A.1.2), together with the fact that $|\{j | i_j = 0\}| \geq m+1$ if $(i_1, \dots, i_{n+m+1}) \in \mathbb{Z}_{\geq 0}^{n+m+1}$ is such that $i_1 + \dots + i_{n+m+1} \geq m+1$, imply that

$$(A.1.4) \quad \Delta^{(n+m+1)}(F_n^\heartsuit O) \subset \sum_{\substack{J \subset \llbracket 1, n+m+1 \rrbracket \\ |J| \geq m+1}} O_J^{(n+m+1)}.$$

Then

$$\begin{aligned} & \Delta^{(n+m+1)}((F_n^\heartsuit O) \cdot (F_m^\heartsuit O)) \subset \Delta^{(n+m+1)}(F_n^\heartsuit O) \cdot \Delta^{(n+m+1)}(F_m^\heartsuit O) \\ & \subset \sum_{\substack{J, K \subset \llbracket 1, n+m+1 \rrbracket \\ |J| \geq m+1, |K| \geq n+1}} O_J^{(n+m+1)} \cdot O_K^{(n+m+1)} \subset \sum_{\substack{J, K \subset \llbracket 1, n+m+1 \rrbracket \\ |J| \geq m+1, |K| \geq n+1}} O_{J \cap K}^{(n+m+1)} \\ & \subset \sum_{\substack{L \subset \llbracket 1, n+m+1 \rrbracket \\ L \neq \emptyset}} O_L^{(n+m+1)} \subset \text{Ker}(pr_O^{\otimes n+m+1}) \end{aligned}$$

where the first inclusion follows from the fact that $\Delta^{(n+m+1)}$ is an algebra morphism, the second inclusion follows from (A.1.4), the third inclusion follows from $O_J^{(n+m+1)} \cdot O_K^{(n+m+1)} \subset O_{J \cap K}^{(n+m+1)}$ for $J, K \subset \llbracket 1, n+m+1 \rrbracket$, the fourth inclusion follows from the fact that if $J, K \subset \llbracket 1, n+m+1 \rrbracket$ are such that $|J| \geq m+1, |K| \geq n+1$, then $J \cap K \neq \emptyset$, the last inclusion follows from the vanishing of the restriction of $pr_O^{\otimes n+m+1}$ to any O_L where $\emptyset \neq L \subset \llbracket 1, n+m+1 \rrbracket$. The resulting inclusion $\Delta^{(n+m+1)}((F_n^\heartsuit O) \cdot (F_m^\heartsuit O)) \subset \text{Ker}(pr_O^{\otimes n+m+1})$ implies the statement. \square

Definition A.3. Set $O^\heartsuit := \bigcup_{n \geq 0} F_n^\heartsuit O$.

Proposition A.4. (a) $F_{\bullet}^{\heartsuit}O$ defines a Hopf algebra filtration of O .

(b) O^{\heartsuit} is a Hopf subalgebra of O . The assignment $O \mapsto O^{\heartsuit}$ is an endofunctor of the category \mathbf{HA} of complex Hopf algebras.

(c) If $f : O_1 \rightarrow O_2$ is a morphism in \mathbf{HA} , then f is compatible with the filtrations F_{\bullet}^{\heartsuit} on both sides.

Proof. (a) follows from Lem. A.2. (b) follows from the fact that O^{\heartsuit} is the total space of $F_{\bullet}^{\heartsuit}O$ and from (a). The functoriality statement is obvious. (c) follows from $f^{\otimes n+1} \circ (id - \eta_{O_1} \epsilon_{O_1})^{\otimes n+1} \circ \Delta_{O_1}^{(n+1)} = (id - \eta_{O_2} \epsilon_{O_2})^{\otimes n+1} \circ \Delta_{O_2}^{(n+1)} \circ f$ for any $n \geq 0$. \square

A.2. A duality functor $\mathbf{HA}_{fd} \rightarrow \mathbf{HA}^{op}$, $H \mapsto H'$. Let H be a complex Hopf algebra with coproduct Δ_H . Denote by J_H its augmentation ideal of H , and set $F^n H := H$ for $n = 0$, $F^n H := J_H^n$ for $n \geq 1$.

Lemma A.5 (see also [Fr], §8.1.1). *For $n, m \geq 0$, one has $F^n H \cdot F^m H \subset F^{n+m} H$ and $\Delta_H(F^n H) \subset \sum_{n'+n''=n} F^{n'} H \otimes F^{n''} H$.*

Proof. The decreasing character of $(F^n H)_{n \in \mathbb{Z}}$ is obvious. The inclusion $F^n H \cdot F^m H \subset F^{n+m} H$ follows from definitions. The last statement follows from $\Delta_H(J_H) \subset H \otimes J_H + J_H \otimes H$, which is itself a consequence of the compatibility of Δ_H with the augmentation of H . \square

The coalgebra structure of H induces an algebra structure on H^* . For $n \geq 0$, set $F_n H^* := (F^{n+1} H)^{\perp}$.

Lemma A.6. $F_{\bullet} H^*$ is an algebra filtration of H^* , and $H' := \cup_{n \geq 0} F_n H^*$ is a subalgebra of H^* .

Proof. Let $n, m \geq 0$, $\alpha \in (F^{n+1} H)^{\perp}$, $\beta \in (F^{m+1} H)^{\perp}$, then if $h \in F^{n+m+1} H$, one has $\Delta_H(h) \in F^n H \otimes H + H \otimes F^m H$ by the second statement of A.5, therefore $(\alpha \cdot \beta)(h) = (\alpha \otimes \beta)(\Delta_H(h)) = 0$, therefore $\alpha \cdot \beta \in (F^{n+m+1} H)^{\perp}$. This proves the first statement. The second statement follows from the first, as H' is the total subspace of an algebra filtration. \square

Definition A.7. \mathbf{HA}_{fd} is the full subcategory of \mathbf{HA} of Hopf algebras H such that $\text{gr}^1 H := F^1 H / F^2 H$ is finite dimensional.

Lemma A.8. *If H is an object of \mathbf{HA}_{fd} , then H' is equipped with a linear map $\Delta_{H'} : H' \rightarrow H' \otimes H'$, uniquely determined by the identity $\Delta_{H'}(\alpha)(h \otimes h') = \alpha(hh')$ for $\alpha \in H'$ and $h, h' \in H'$. Then $(H', \Delta_{H'})$ is a Hopf algebra. The assignment $H \mapsto H'$ is a functor $\mathbf{HA}_{fd} \rightarrow \mathbf{HA}^{op}$.*

Proof. It follows from Lem. A.5 that $F_{\bullet} H$ is a decreasing algebra filtration of H . The associated graded algebra $\text{gr}^{\bullet} H$ is such that $\text{gr}^n H = F^n H / F^{n+1} H$ for any $n \geq 0$. Then $\text{gr}^0 H = \mathbb{C}$, and $\text{gr}^{\bullet} H$ is generated by $\text{gr}^1 H$. As this space is finite dimensional, so is $\text{gr}^k H$ for any $k \geq 0$. It

follows that for any $n \geq 0$, $\bigoplus_{i=0}^{n-1} \text{gr}^i H$ is finite dimensional. As this space is non-canonically isomorphic to the quotient space $H/F^n H$, this quotient is finite dimensional as well.

It also follows from Lem. A.5 that for any $n \geq 0$, $F^n H$ is a two-sided ideal of H , therefore $H/F^n H$ is an algebra, and $H/F^{n+1} H \rightarrow H/F^n H$ is an algebra morphism.

Since $H/F^n H$ is finite-dimensional, its associative algebra structure gives rise to a coassociative coalgebra structure on its dual $(H/F^n H)^* = F_{n-1} H^*$ (with $F_{-1} H^* := 0$). It follows from the algebra morphism status of $H/F^{n+1} H \rightarrow H/F^n H$ that the canonical inclusion $i_{n,n+1} : F_{n-1} H^* \subset F_n H^*$ is a coalgebra morphism.

For $n \geq 0$, define then an algebra morphism $\Delta_{H',n} : F_n H^* \rightarrow (H')^{\otimes 2}$ to be the composition $i_n^{\otimes 2} \circ \Delta_{F_n H^*}$, where $\Delta_{F_n H^*}$ is the coproduct of the coalgebra structure of $F_n H^*$ and $i_n : F_n H^* \rightarrow H'$ is the canonical inclusion. One has

$$(A.2.1) \quad \Delta_{H',n+1} \circ i_{n,n+1} = \Delta_{H',n}.$$

Indeed, $\Delta_{H',n+1} \circ i_{n,n+1} = i_{n+1}^{\otimes 2} \circ \Delta_{F_{n+1} H^*} \circ i_{n,n+1} = i_{n+1}^{\otimes 2} \circ i_{n,n+1}^{\otimes 2} \circ \Delta_{F_n H^*} = i_n^{\otimes 2} \circ \Delta_{F_n H^*} = \Delta_{H',n}$, where the first and last equalities follow from the definitions of $\Delta_{H',n+1}$ and $\Delta_{H',n}$, the second equality follows from the coalgebra morphism status of $i_{n,n+1}$, and the third equality follows from $i_{n+1} \circ i_{n,n+1} = i_n$. It follows from (A.2.1) and from $H' = \bigcup_{n \geq 0} F_n H^*$ that there is a unique linear map $\Delta_{H'} : H' \rightarrow (H')^{\otimes 2}$, such that $\Delta_{H'} \circ i_n = \Delta_{H',n}$ for any $n \geq 0$. The identities relating $\Delta_{H',n}$ with the product implies that it satisfies the announced identity. The uniqueness statement follows from the fact that the annihilator of $H \otimes H$ in $F_n H^* \otimes F_m H^*$ is zero for any $n, m \geq 0$. One checks that $\Delta_{H'}$ satisfies the Hopf algebra axioms as well as the functoriality statement. \square

If H is an object of \mathbf{HA}_{fd} , it follows from Lem. A.8 and Lem. A.4 that H' is equipped with a Hopf algebra filtration $F_\bullet^\heartsuit H'$; it is also equipped with the vector space filtration $F_\bullet H^*$ used to define it, given by $F_n H^* = (F^{n+1} H)^\perp$ for any $n \geq 0$. One has:

Lemma A.9. *One has $F_\bullet^\heartsuit H' = F_\bullet H^*$.*

Proof. Let $n \geq 0$ and let us show that $F_n^\heartsuit H' = F_n H^*$. Recall that $F_n^\heartsuit H' = \text{Ker}((id - \eta_{H'} \epsilon_{H'})^{\otimes n+1} \circ \Delta_{H'}^{(n+1)})$, where $\Delta_{H'}$, $\eta_{H'}$ and $\epsilon_{H'}$ are the coproduct, unit and counit maps of H' . Recall that $H' = \bigcup_{m \geq 0} F_m H^* \subset H^*$ and that for each $m \geq 0$, $F_m H^*$ is a sub-coalgebra of H' ; denote by $\Delta_{F_m H^*} : F_m H^* \rightarrow (F_m H^*)^{\otimes 2}$ the corresponding coproduct. Let $\epsilon_{F_m H^*} : F_m H^* \rightarrow \mathbb{C}$ be the composition of $\epsilon_{H'}$ with the inclusion $F_m H^* \subset H'$. The unit of H' corresponds to the counit map of H , which as it vanishes on $F^{m+1} H$ defines an element in $(H/F^{m+1} H)^* = F_m H^*$; let $\eta_{F_m H^*}$ be the corresponding map $\mathbb{C} \rightarrow F_m H^*$. Then

$$F_n^\heartsuit H' \cap F_m H^* = \text{Ker} \left(F_m H^* \xrightarrow{\Delta_{F_m H^*}^{(n+1)}} (F_m H^*)^{\otimes n+1} \xrightarrow{(id - \eta_{F_m H^*} \epsilon_{F_m H^*})^{\otimes n+1}} (F_m H^*)^{\otimes n+1} \right)$$

(a subspace of $F_m H^*$). Using that if $f : E \rightarrow F$ is a linear map, then $\text{Ker}(f^* : F^* \rightarrow E^*) = \text{im}(f)^\perp$, that the duals of $\Delta_{F_m H^*} : F_m H^* \rightarrow (F_m H^*)^{\otimes 2}$, $\eta_{F_m H^*} : \mathbb{C} \rightarrow F_m H^*$ and $\epsilon_{F_m H^*} :$

$F_m H^* \rightarrow \mathbb{C}$ are respectively the product map $m_{H/F^{m+1}H} : (H/F^{m+1}H)^{\otimes 2} \rightarrow H/F^{m+1}H$, the map $\epsilon_{H/F^{m+1}H} : H/F^{m+1}H \rightarrow \mathbb{C}$ induced by the counit of H , and the map $\eta_{H/F^{m+1}H} : \mathbb{C} \rightarrow H/F^{m+1}H$ induced the unit of H , we see that $F_n^\heartsuit H' \cap F_m H^*$ is the annihilator (in $(H/F^{m+1}H)^*$) of the image of the map

$$(H/F^{m+1}H)^{\otimes n+1} \xrightarrow{(id - \eta_{H/F^{m+1}H} \epsilon_{H/F^{m+1}H})^{\otimes n+1}} (H/F^{m+1}H)^{\otimes n+1} \xrightarrow{m_{H/F^{m+1}H}^{(n+1)}} H/F^{m+1}H.$$

Since the image of $id - \eta_H \epsilon_H : H \rightarrow H$ is $F^1 H$, and since the image of $(F^1 H)^{\otimes n+1}$ by $m_H^{(n+1)} : H^{\otimes n+1} \rightarrow H$ is $F^{n+1} H$, the said image is $(F^{n+1} H + F^{m+1} H)/F^{m+1} H$. Therefore the subspace $F_n^\heartsuit H' \cap F_m H^*$ of $(H/F^{m+1}H)^*$ is the annihilator of $(F^{n+1} H + F^{m+1} H)/F^{m+1} H$. If $m \geq n$, this subspace is the annihilator of $F^{n+1} H/F^{m+1} H$, which is the kernel of the canonical map $(H/F^{m+1}H)^* \rightarrow (F^{n+1} H/F^{m+1} H)^*$, which is the image of the injection $F_n H^* \hookrightarrow F_m H^*$. Therefore $F_n^\heartsuit H' = F_n H^*$. \square

A.3. Hopf algebra pairings and Hopf algebra morphisms. Recall that if O, H are Hopf algebras with coproducts Δ_O, Δ_H and counits ϵ_O, ϵ_H , then a Hopf algebra pairing between O and H is a linear map $p : O \otimes H \rightarrow \mathbb{C}$, such that $p(o \otimes h) = (p \otimes p) \circ \tau_2(o \otimes o' \otimes \Delta_H(h))$, $p(o \otimes h') = (p \otimes p) \circ \tau_2(\Delta_O(o) \otimes h \otimes h')$ and $p(1 \otimes h) = \epsilon_H(h)$, $p(o \otimes 1) = \epsilon_O(o)$ for $o, o' \in O$ and $h, h' \in H$ (for any $n \geq 2$, τ_n is the canonical map $O^{\otimes n} \otimes H^{\otimes n} \rightarrow (O \otimes H)^{\otimes n}$).

Definition A.10. For O, H Hopf algebras, we denote by $\mathbf{Pair}(O, H)$ the set of Hopf algebra pairings between O and H .

Lemma A.11. For O an object of \mathbf{HA} and H an object of \mathbf{HA}_{fd} , there is a map

$$\nu : \mathbf{Pair}(O, H) \rightarrow \mathbf{HA}(O^\heartsuit, H').$$

For $p \in \mathbf{Pair}(O, H)$, $\nu(p)$ is such that the diagram

$$\begin{array}{ccc} O^\heartsuit \otimes H & \xrightarrow{p \otimes id_H} & H' \otimes H \\ i_O \otimes id_H \downarrow & & \downarrow p_H \\ O \otimes H & \xrightarrow{p} & \mathbb{C} \end{array}$$

commutes, $i_O : O^\heartsuit \rightarrow O$ being the canonical injection and $p_H : H \otimes H' \rightarrow \mathbb{C}$ being the composition $H \otimes H' \hookrightarrow H \otimes H^* \rightarrow \mathbb{C}$.

Proof. Let $p \in \mathbf{Pair}(O, H)$. Let $n \geq 0$, then $p(F_n^\heartsuit O \otimes F^{n+1} H) = p(F_n^\heartsuit O \otimes J_H^{n+1}) \subset p^{\otimes n+1} \circ \tau_{n+1}(\Delta_O^{(n+1)}(F_n^\heartsuit O) \otimes J_H^{\otimes n+1}) \subset \sum_{\emptyset \neq L \subset [1, n+1]} p^{\otimes n+1} \circ \tau_{n+1}(O_L^{(n+1)} \otimes J_H^{\otimes n+1}) = 0$, where the first inclusion follows from the behavior of p with respect to coproducts, the second inclusion follows from $\Delta_O^{(n+1)}(F_n^\heartsuit O) \subset \sum_{\emptyset \neq L \subset [1, n+1]} O_L^{(n+1)}$, where $O_L^{(n+1)}$ is as in (A.1.3) (a consequence of the definition of $F_n^\heartsuit O$), the last inclusion follows from $p(1 \otimes J_H) = 0$, itself a consequence of the behavior of p with respect to counits. It follows that $p(F_n^\heartsuit O \otimes F^{n+1} H) = 0$. This implies that the restriction $p|_{F_n^\heartsuit O \otimes H}$ of p to $F_n^\heartsuit O \otimes H$ induces a linear map $p_n : F_n^\heartsuit O \otimes (H/F^{n+1}H) \rightarrow \mathbb{C}$. As

$H/F^{n+1}H$ is finite dimensional, this gives rise to a linear map $\nu(p)_n : F_n^\heartsuit O \rightarrow (H/F^{n+1}H)^* = F_n H^*$. One checks that $i_{n,n+1}^{H'} \circ \nu(p)_n = \nu(p)_{n+1} \circ i_{n,n+1}^O$, where $i_{n,n+1}^O, i_{n,n+1}^{H'}$ the canonical maps $F_n^\heartsuit O \rightarrow F_{n+1}^\heartsuit O$ and $F_n H^* \rightarrow F_{n+1} H^*$. It follows that there exists a unique linear map $\nu(p) : O^\heartsuit \rightarrow H'$, such that $\nu(p) \circ i_n^O = i_n^{H'} \circ \nu(p)_n$ for any $n \geq 0$, where $i_n^O, i_n^{H'}$ are the canonical maps $F_n^\heartsuit O \rightarrow O^\heartsuit$ and $F_n H^* \rightarrow H'$. One checks that $\nu(p)$ defines a Hopf algebra morphism as well as the announced commutative diagram. \square

APPENDIX B. BACKGROUND ON HOPF ALGEBRAS WITH (CO)ACTIONS ON ALGEBRAS

In §B.1, we introduce a category **HACA** of Hopf algebras with comodule algebra (HACAs) and an endofunctor of that category extending that of §A.1. In §B.2, we introduce a category **HAMA** of Hopf algebras with module algebras (HAMAs), together with a diagram **HAMA** \supset **HAMA**_{fd} \rightarrow **HACA** extending that of §A.2. In §B.3, we introduce the notion of a pairing-morphism from a HAMA to a HACA and show that it gives rise, under a finite dimensionality assumption, to a HACA morphism extending the Hopf algebra morphism of §A.3. In §B.4, we make use of the natural filtration attached to each HACA to construct two functors $\text{gr}, \Phi : \mathbf{HACA} \rightarrow \mathbf{Alg}_{gr}$ and a natural transformation $\text{nat} : \text{gr} \Rightarrow \Phi$ between these functors. §B.5 is devoted to the main technical tool of this paper (Prop. B.18), based on nat and giving a criterion for the HACA morphisms arising from pairing-morphisms of a certain type by the construction of §B.3 to be isomorphisms.

B.1. An endofunctor $(O, A) \mapsto (O^\heartsuit, F_\infty A)$ of **HACA**.

Definition B.1. A Hopf algebra with comodule algebra (HACA) is a pair (O, A) where O is a Hopf algebra and A is an algebra, equipped with a left coaction of O ; the coproduct map of O being denoted Δ_O and the coaction by $\Delta_A : A \rightarrow O \otimes A$, in particular Δ_A is an algebra morphism and $(\Delta_O \otimes id_A) \circ \Delta_A = (id_O \otimes \Delta_A) \circ \Delta_A$. If (O, A) and (O', A') are HACAs, a morphism from (O, A) to (O', A') is the pair of (f_A, f_O) of an algebra morphism $f_A : A \rightarrow A'$ and a Hopf algebra morphism $f_O : O \rightarrow O'$, such that $(f_O \otimes f_A) \circ \Delta_A = \Delta_{A'} \circ f_A$. Denote by **HACA** the resulting category.

Let (O, A) be a HACA.

Definition B.2. For $n \geq 0$, define $F_n A$ to be the preimage of $F_n^\heartsuit O \otimes A$ by the map $\Delta_A : A \rightarrow O \otimes A$.

Lemma B.3. (a) $F_\bullet A$ is an algebra filtration of A .

(b) For $n \geq 0$, $\Delta_A(F_n A) \subset F_n^\heartsuit O \otimes F_n A$.

(c) For $n \geq 0$, $\Delta_A(F_n A) \subset \sum_{p+q=n} F_p^\heartsuit O \otimes F_q A$.

(d) The restriction of Δ_A to $F_\infty A$ corestricts to $O^\heartsuit \otimes F_\infty A$. Together with the structures of algebra of $F_\infty A$ and of Hopf algebra of O^\heartsuit , the resulting map $\Delta_{F_\infty A} : F_\infty A \rightarrow O^\heartsuit \otimes F_\infty A$ equips

$(O^\heartsuit, F_\infty A)$ with a HACA structure. The assignment $(O, A) \mapsto (O^\heartsuit, F_\infty A)$ is an endofunctor of **HACA**.

Proof. (a) $A \otimes F_\bullet^\heartsuit O$ is an algebra filtration of the algebra $O \otimes A$; as $\Delta_A : A \rightarrow O \otimes A$ is an algebra morphism, the preimage of this filtration is an algebra filtration of the source; as this preimage is $F_\bullet A$, the latter is an algebra filtration. (b) Let $a \in F_n A$, then $\Delta_A(a) \in F_n^\heartsuit O \otimes A$. There exist families $(o_s)_{s \in S}$, $(a_s)_{s \in S}$ of elements of O and A , indexed by a finite set S , such that $(o_s)_{s \in S}$ is free and $\Delta_A(a) = \sum_s o_s \otimes a_s$. Since $\Delta_O(F_n^\heartsuit O) \subset (F_n^\heartsuit O)^{\otimes 2}$, one has $(\Delta_O \otimes id_A) \circ \Delta_A(a) \in (F_n^\heartsuit O)^{\otimes 2} \otimes A$. By the coassociativity of the coaction, this term is equal to $(id_O \otimes \Delta_A) \circ \Delta_A(a)$, so $(id_O \otimes \Delta_A) \circ \Delta_A(a) \in (F_n^\heartsuit O)^{\otimes 2} \otimes A$. Therefore $\sum_{s \in S} o_s \otimes \Delta_A(a_s) \in (F_n^\heartsuit O)^{\otimes 2} \otimes A$. As $(o_s)_{s \in S}$ is free, this implies $\Delta_A(a_s) \in F_n^\heartsuit O \otimes A$ for any s , therefore $a_s \in F_n A$. The claim follows.

(c) Let $a \in F_n A$, then by (b) $\Delta_A(a) \in F_n^\heartsuit O \otimes F_n A$. Let U_i be a complement of $F_{i-1}^\heartsuit O$ in $F_i^\heartsuit O$, so $F_i^\heartsuit O = U_1 \oplus \cdots \oplus U_i$. Then $F_n^\heartsuit O \otimes F_n A = (U_0 \otimes F_n A) \oplus \cdots \oplus (U_n \otimes F_n A)$, giving rise to a decomposition $\Delta_A(a) = z_0 + \cdots + z_n$.

Since $\Delta_A(a) \in F_n^\heartsuit O \otimes F_n A$, Lem. A.2 (c) implies that $(\Delta_O \otimes id_A) \circ \Delta_A(a) \in \sum_{p+q=n} F_p^\heartsuit O \otimes F_q^\heartsuit O \otimes F_n A \subset (F_n^\heartsuit O)^{\otimes 2} \otimes F_n A$. By the coassociativity identity, this term is equal to $(id_O \otimes \Delta_A) \circ \Delta_A(a) = \sum_{i=0}^n (id_O \otimes \Delta_A)(z_i)$. It follows that $\sum_{i=0}^n (id_O \otimes \Delta_A)(z_i) \in \sum_{p+q=n} F_p^\heartsuit O \otimes F_q^\heartsuit O \otimes F_n A$. There is a direct sum decomposition $(F_n^\heartsuit O)^{\otimes 2} \otimes F_n A = \bigoplus_{i,j \in [0,n]} U_i \otimes U_j \otimes F_n A$; then $\sum_{p+q=n} F_p^\heartsuit O \otimes F_q^\heartsuit O \otimes F_n A = \bigoplus_{i,j \in [0,n], i+j \leq n} U_i \otimes U_j \otimes F_n A$, and $(id_O \otimes \Delta_A)(z_i) \in U_i \otimes F_n^\heartsuit O \otimes F_n A = \bigoplus_{j \in [0,n]} U_i \otimes U_j \otimes F_n A$. Decomposing $(id_O \otimes \Delta_A)(z_i) = \sum_{j \in [0,n]} t_{ij}$ for any $i \in [0, n]$, one obtains $\sum_{i,j \in [0,n]} t_{ij} \in \bigoplus_{i,j \in [0,n], i+j \leq n} U_i \otimes U_j \otimes F_n A$, therefore $t_{ij} = 0$ if $i+j > n$, therefore for any $i \in [0, n]$, one has $(id_O \otimes \Delta_A)(z_i) \in \bigoplus_{j=0}^{n-i} U_i \otimes U_j \otimes F_n A = U_i \otimes F_{n-i}^\heartsuit O \otimes F_n A$. There exist families $(o_s^i)_{s \in S_i}$, $(a_s^i)_{s \in S_i}$ of elements of U_i and $F_n A$, indexed by a finite set S_i , such that $(o_s^i)_{s \in S_i}$ is free and $z_i = \sum_{s \in S_i} o_s^i \otimes a_s^i$. Then $\sum_{s \in S_i} o_s^i \otimes \Delta_A(a_s^i) \in U_i \otimes F_{n-i}^\heartsuit O \otimes F_n A$. Since $(o_s^i)_{s \in S_i}$ is free, this implies $\Delta_A(a_s^i) \in F_{n-i}^\heartsuit O \otimes F_n A$ for any $s \in S_i$, therefore $a_s^i \in F_{n-i} A$. Therefore $z_i \in F_i^\heartsuit O \otimes F_{n-i} A$, proving the claim.

(d) By (b), $\Delta_A(F_n A) \subset O^\heartsuit \otimes F_\infty A$ for any $n \geq 0$, which implies $\Delta_A(F_\infty A) \subset O^\heartsuit \otimes F_\infty A$. One can then define the announced map $\Delta_{F_\infty A} : F_\infty A \rightarrow O^\heartsuit \otimes F_\infty A$ and check it to have the announced properties. The endofunctor statement is then immediate. \square

B.2. A category HAMA and a functor HAMA_{fd} → HACA. For M a right module over an algebra A and $V \subset A$, we use the notation $\text{Ann}(V)$ to denote $\{m \in M \mid m \cdot V = 0\}$.

Definition B.4. For H a Hopf algebra, M a right H -module, define $F_n M := \text{Ann}(F^{n+1} H, M)$ for $n \geq 0$.

Note that $F_0 M$ is equal to M^H , the submodule of M of invariants of the action of H .

Lemma B.5. *Let H be a Hopf algebra.*

(a) For M a right H -module, $F_\bullet M$ is an increasing H -module filtration of M .

(b) Any morphism $f : M \rightarrow N$ a morphism of right H -modules is compatible with the filtrations.

(c) If M is a right H -module and $n \geq 0$, then for any $m \in F_n M$, the map $F^n H \rightarrow M$, $h \mapsto m \cdot h$ takes its values in M^H . The resulting map $F_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(F^n H, M^H)$ induces a map $\text{gr}_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(\text{gr}^n H, M^H)$, which is injective.

Proof. (a) follows from the decreasing character of $n \mapsto F^{n+1} H$ and from the ideal nature of $F^{n+1} H$ for $n \geq 0$. (b) is immediate. (c) If $m \in F_n M$, $h \in F^n H$ and $h' \in F_1 H$, then $hh' \in F^{n+1} H$ so $m \cdot (hh') = 0$, therefore $(m \cdot h) \cdot h' = 0$, so $m \cdot h \in M^H$. The map $F_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(F^n H, M^H)$ factors through a map $F_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(\text{gr}^n H, M^H)$ as $m \cdot h = 0$ for any $m \in F_n M$, $h \in F^{n+1} H$. The restriction of this map to $F_{n-1} M$ is zero as $m \cdot h = 0$ for any $m \in F_{n-1} M$, $h \in F^n H$. This leads to the announced map $\text{gr}_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(\text{gr}^n H, M^H)$. As the map $\text{Hom}_{\mathbb{C}\text{-vec}}(\text{gr}^n H, M^H) \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(F^n H, M^H)$ is injective, the kernel of the map $F_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(\text{gr}^n H, M^H)$ is equal to the kernel of the map $F_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(F^n H, M^H)$, which is $\{m \in F_n M \mid m \cdot h = 0 \text{ for any } h \in F^n H\}$, that is $F^{n-1} H$. It follows that the map $\text{gr}_n M \rightarrow \text{Hom}_{\mathbb{C}\text{-vec}}(\text{gr}^n H, M^H)$ is injective. \square

Definition B.6. A Hopf algebra with module algebra (HAMA) is pair (B, H) where H is a Hopf algebra and B is an algebra equipped with a right action of H , i.e. a linear map $B \otimes H \rightarrow H$, $b \otimes h \mapsto b \cdot h$, such that $(b \cdot h) \cdot h' = b \cdot (hh')$ and $(bb') \cdot h = (b \cdot h^{(1)})(b' \cdot h^{(2)})$ for any $b, b' \in B$ and $h, h' \in H$, the coproduct of $h \in H$ being denoted $h^{(1)} \otimes h^{(2)}$ (Sweedler). If (B, H) and (B', H') are HAMAs, a morphism $(B, H) \rightarrow (B', H')$ is a pair (f_H, f_B) , where $f_H : H' \rightarrow H$ is a Hopf algebra morphism and $f_B : B \rightarrow B'$ is an algebra morphism, such that $f_B(b \cdot f_H(h')) = f_B(b) \cdot h'$ for any $b \in B$ and $h' \in H'$. Morphisms are composed as follows: $(g_H, g_B) \circ (f_H, f_B) := (f_H \circ g_H, g_B \circ f_B)$.

One checks that HAMAs build up a category, which will be denoted **HAMA**.

Definition B.7. Let (B, H) be a HAMA. For $n \geq 0$, set $F_n B := \text{Ann}(F^{n+1} H, B) = \{b \in B \mid b \cdot F^{n+1} H = 0\}$.

Lemma B.8. Let (B, H) be a HAMA.

(a) $F_\bullet B$ is an algebra filtration of B .

(b) Let $n, m \geq 0$. For $n \geq m$, then $(F_n B) \cdot (F^m H) \subset F_{n-m} B$. If $n < m$, then $(F_n B) \cdot (F^m H) = 0$.

(c) For any $n \geq 0$, the action map $B \otimes H \rightarrow B$ induces a linear map $F_n B \otimes H \rightarrow F_n B$, which factorizes through a linear map $F_n B \otimes (H/F^{n+1} H) \rightarrow F_n B$.

Proof. (a) Let $n, m \geq 0$. By the second part of Lem. A.5, one has $\Delta_H(F^{n+m+1} H) \subset F^{n+1} H \otimes H + H \otimes F^{m+1} H$. Together with the Hopf algebra action axiom, this implies the inclusion in

$((F_n B)(F_m B)) \cdot (F^{n+m+1} H) \subset ((F_n B) \cdot (F^{n+1} H))((F_m B) \cdot H) + ((F_n B) \cdot H)((F_m B) \cdot (F^{m+1} H)) = 0$, and the equality follows from $(F_n B) \cdot (F^{n+1} H) = (F_m B) \cdot (F^{m+1} H) = 0$. The resulting equality $(F_n B)(F_m B) \cdot (F^{n+m+1} H) = 0$ implies $(F_n B)(F_m B) \subset F_{n+m} B$. (b) Assume $n \geq m \geq 0$. Then $((F_n B) \cdot (F^m H)) \cdot (F^{n-m+1} H) \subset (F_n B) \cdot ((F^m H)(F^{n-m+1} H)) \subset (F_n B) \cdot (F^{n+1} H) = 0$, where the first inclusion follows from the right action axioms, the second inclusion follows from the first part of Lem. A.5, and the equality follows from the definition of $F_n B$. This implies $(F_n B) \cdot (F^m H) \subset F_{n-m} B$, as claimed. In particular, for any $n \geq 0$, $(F_n B) \cdot (F^n H) \subset F_0 B = B^H$. Then if $m > n$, one has $(F_n B) \cdot (F^m H) = (F_n B) \cdot ((F^n H)(F^{m-n} H)) \subset ((F_n B) \cdot (F^n H)) \cdot (F^{m-n} H) \subset B^H \cdot (F^{m-n} H) = 0$, where the first equality follows from $F^m H = (F^n H)(F^{m-n} H)$, the first inclusion follows from the right action axioms, the second inclusion follows from the inclusion $(F_n B) \cdot (F^n H) \subset B^H$, the last equality follows from $B^H \cdot J_H = 0$. This completes the proof of the claim. (c) The first statement follows from (b) for $m = 0$. The second statement follows from $F_n B \cdot (F^{n+1} H) = 0$, which follows from the definition of $F_n B$. \square

Definition B.9. \mathbf{HAMA}_{fd} is the full subcategory of \mathbf{HAMA} of objects (B, H) such that H is an object of \mathbf{HA}_{fd} .

Lemma B.10. *Let (B, H) be an object of \mathbf{HAMA}_{fd} .*

(a) *For any $n \geq 0$, the linear map $F_n B \otimes (H/F^{n+1} H) \rightarrow F_n B$ from Lem. B.8 (c) induces a linear map $\Delta_{F_n B} : F_n B \rightarrow (H/F^{n+1} H)^* \otimes F_n B = F_n H^* \otimes F_n B$.*

(b) *There is a linear map $\Delta_{F_\infty B} : F_\infty B \rightarrow H' \otimes F_\infty B$, uniquely determined by the condition that $\Delta_{F_\infty B} \circ i_n^{F_\infty B} = (i_n^{H'} \otimes i_n^{F_\infty B}) \circ \Delta_{F_n B}$ for any $n \geq 0$, where $\Delta_{F_n B}$ is as in (a) and $i_n^{H'} : F_n H^* \rightarrow H'$, $i_n^{F_\infty B} : F_n B \rightarrow F_\infty B$ are the canonical injections.*

(c) *Together with the algebra structure of $F_\infty B$ and Hopf algebra structure of H' , $\Delta_{F_\infty B}$ equips $(H', F_\infty B)$ with a HACA structure. The assignment $(B, H) \mapsto (H', F_\infty B)$ is a functor $\mathbf{HAMA}_{fd} \rightarrow \mathbf{HACA}$.*

Proof. (a) follows from the finite-dimensionality of $H/F^{n+1} H$, which has been established in the proof of Lem. A.8.

(b) Let $n \geq 0$ and $i_{n,n+1}^{F_\infty B} : F_n B \hookrightarrow F_{n+1} B$ and $i_{n,n+1}^{H'} : F_n H^* \hookrightarrow F_{n+1} H^*$ be the canonical inclusions. Recall that $i_{n,n+1}^{H'}$ is the dual of the projection map $p_{n+1,n}^H : H/F^{n+2} H \rightarrow H/F^{n+1} H$. The following diagram

$$\begin{array}{ccc}
 & F_n B \otimes H & \longrightarrow & F_n B \otimes (H/F^{n+1} H) \\
 & \swarrow & \downarrow & \downarrow \\
 F_{n+1} B \otimes H & \longrightarrow & B \otimes H & \\
 \downarrow & & \searrow & \downarrow \\
 F_{n+1} B \otimes (H/F^{n+2} H) & \longrightarrow & & B
 \end{array}$$

is built up of two commutative square and a commutative triangle, therefore is commutative; it is the composition of the square whose commutativity expresses the equality $\Delta_{F_\infty B} \circ i_{n,n+1}^{F_\infty B} = (i_{n,n+1}^{F_\infty B} \otimes i_{n,n+1}^{H'}) \circ \Delta_{F_n B}$ with the morphisms $F_{n+1}B \rightarrow B$ and $F_n B \otimes H \rightarrow F_n B \otimes (H/F^{n+2}H)$, which are respectively injective and surjective. This implies the commutativity of this square, therefore the said equality, from which one derives the statement.

(c) Let us show the coassociativity of $(H', F_\infty B)$. Recall that for $\xi \in H'$ and $b \in F_\infty B$, one has $\xi \cdot b = \langle id \otimes \xi, \Delta_B(b) \rangle$. Let $b \in F_\infty B$ and $\xi, \eta \in H'$, then $\xi \cdot (\eta \cdot b) = \langle id \otimes \xi, \Delta_B(\eta \cdot b) \rangle = \langle id \otimes \xi, \Delta_B(\langle id \otimes \eta, \Delta_B(b) \rangle) \rangle = \langle id \otimes \xi \otimes \eta, (\Delta_B \otimes id) \circ \Delta_B(b) \rangle = \langle id \otimes \xi \otimes \eta, (id \otimes \Delta_H) \circ \Delta_B(b) \rangle = \langle id \otimes (\xi \cdot \eta), \Delta_B(b) \rangle = (\xi \cdot \eta) \cdot b. \quad \square$

Lemma B.11. (a) Let (B, H) be a HAMA and let $H' \subset H$ be a Hopf subalgebra. Let us denote by $F_\bullet^{H'} B$, $F_\bullet^H B$ the algebra filtrations of B attached to the actions of H, H' . Then $F_\infty^H B \subset F_\infty^{H'} B$.

(b) Let $f : B \rightarrow C$ be an algebra morphism; let H be a Hopf algebra right acting on B and C ; assume that f is H -equivariant. Then $(f, id) : (B, H) \rightarrow (C, H)$ is a HAMA morphism, and $f(F_\infty B) \subset F_\infty C$.

Proof. (a) For $n \geq 0$, one has $F_n^H B = \{b \in B \mid h \cdot b = 0 \text{ for } h \in F^{n+1}H\} \subset \{b \in B \mid h \cdot b = 0 \text{ for } h \in F^{n+1}H'\} = F_n^{H'} B$, where the inclusion follows from $F^{n+1}H' \subset F^{n+1}H$. This implies $F_\infty^H B \subset F_\infty^{H'} B$.

(b) For $n \geq 0$, $b \in F_n B$ and $h \in F^{n+1}H$, one has $f(b) \cdot h = f(b \cdot h) = f(0) = 0$ where the first equality follows the equivariance of f and the second equality follows from $b \in F_n B$. Therefore $f(b) \in F_n C$. So $f(F_n B) \subset F_n C$, hence $f(F_\infty B) \subset F_\infty C. \quad \square$

B.3. Pairing-morphisms from a HACA to a HAMA and HACA morphisms.

Definition B.12. Let (O, A) be a HACA and (B, H) be a HAMA. A pairing-morphism from (O, A) to (B, H) is a pair (p, f) , where $p : O \otimes H \rightarrow \mathbb{C}$ is a Hopf algebra pairing, and $f : A \rightarrow B$ is an algebra morphism, such that the following diagram commutes

$$(B.3.1) \quad \begin{array}{ccc} A \otimes H & \xrightarrow{f \otimes id} & B \otimes H \\ \Delta_A \otimes id \downarrow & & \downarrow \\ O \otimes A \otimes H & \xrightarrow{\sigma_{O,A} \otimes id} A \otimes O \otimes H \xrightarrow{f \otimes p} & B \end{array}$$

where $\Delta_A : A \rightarrow O \otimes A$ is the coaction morphism of (O, A) , $\sigma_{O,A} : O \otimes A \rightarrow A \otimes O$ is the permutation isomorphism, and the right vertical map is the action map of (B, H) . Equivalently, one requests the identity

$$f(a) \cdot h = f(a^{(2)})p(a^{(1)} \otimes h)$$

to be satisfied for any $a \in A$ and $h \in H$, where one denotes $\Delta_A(a) := a^{(1)} \otimes a^{(2)}$.

We denote by $\mathbf{PM}((O, A), (B, H))$ the set of pairing-morphisms from (O, A) to (B, H) .

Lemma B.13. *Let (O, A) an object in **HACA** and (B, H) an object in **HAMA**_{fd}. If $(p, f) \in \mathbf{PM}((O, A), (B, H))$, then $f(F_\bullet A) \subset F_\bullet B$, so that f induces an algebra morphism $F_\infty f : F_\infty A \rightarrow F_\infty B$. The assignment $(p, f) \mapsto (\nu(p), F_\infty f)$ defines a map*

$$\tilde{\nu} : \mathbf{PM}((O, A), (B, H)) \rightarrow \mathbf{HACA}((O^\heartsuit, F_\infty A), (H', F_\infty B)).$$

Proof. Let $n \geq 0$ and $a \in F_n A$. Then $\Delta_A(a) \in F_n^\heartsuit O \otimes A$. So $(\Delta_O^{(n+1)} \otimes id_A) \circ \Delta_A(a) \in (\sum_{\emptyset \neq L \subset [1, n+1]} O_L^{(n+1)}) \otimes A \subset A \otimes O^{\otimes n+1}$. Then for $h_1, \dots, h_{n+1} \in J_H$, one has

$$\begin{aligned} f(a) \cdot (h_1 \cdots h_{n+1}) &= \langle (id \otimes f) \circ \Delta_A(a), (h_1 \cdots h_{n+1}) \otimes id \rangle \\ &= \langle (id^{\otimes n+1} \otimes f) \circ \Delta_A^{(n+1)}(a), (h_1 \otimes \cdots \otimes h_{n+1}) \otimes id \rangle = 0, \end{aligned}$$

where the first equality follows from Def. B.12, the second equality follows from the Hopf algebra pairing axiom, the last equality follows from $\langle O_L^{(n+1)}, J_L^{\otimes n+1} \rangle = 0$, which follows from $\langle 1, J_L \rangle = 0$. It follows that $f(a) \in F_n B$.

Let us prove that $(\nu(p), F_\infty f)$ is a morphism in **HACA**. Let $n \geq 0$. Since $f(F_n A) \subset F_n B$, f induces a linear map $F_n f : F_n A \rightarrow F_n B$.

By Lem. B.3(b), the coaction map $\Delta_A : A \rightarrow O \otimes A$ induces a map $\Delta_{F_n A} : F_n A \rightarrow F_n^\heartsuit O \otimes F_n A$. By Lem. B.8(c), the action map $\text{act}_B : B \otimes H \rightarrow B$ induces a map $\text{act}_{F_n B} : F_n B \otimes H \rightarrow F_n B$. Therefore (B.3.1) induces a commutative diagram

$$(B.3.2) \quad \begin{array}{ccc} F_n A \otimes H & \xrightarrow{F_n f \otimes id} & F_n B \otimes H \\ \Delta_{F_n A} \otimes id \downarrow & & \downarrow \text{act}_{F_n B} \\ F_n^\heartsuit O \otimes F_n A \otimes H & \xrightarrow{(F_n f \otimes p) \circ (\sigma_{O A} \otimes id)} & F_n B \end{array}$$

By the proof of Lem. A.11, the restriction of p to $F_n^\heartsuit O \otimes F^{n+1} H$ is zero, which induces a pairing $p_n : F_n^\heartsuit O \otimes (H/F^{n+1} H) \rightarrow \mathbb{C}$. By Lem. B.8(c), the restriction of $\text{act}_{F_n B}$ to $F_n B \otimes (H/F^{n+1} H)$ is zero, inducing a linear map $\underline{\text{act}}_{F_n B} : F_n B \otimes (H/F^{n+1} H) \rightarrow F_n B$. The above diagram therefore gives rise to a commutative diagram

$$\begin{array}{ccc} F_n A \otimes (H/F^{n+1} H) & \xrightarrow{F_n f \otimes id} & F_n B \otimes (H/F^{n+1} H) \\ \Delta_{F_n A} \otimes id \downarrow & & \downarrow \underline{\text{act}}_{F_n B} \\ F_n^\heartsuit O \otimes F_n A \otimes (H/F^{n+1} H) & \xrightarrow{(F_n f \otimes p_n) \circ (\sigma_{O A} \otimes id)} & F_n B \end{array}$$

Since $H/F^{n+1} H$ is finite dimensional, the map p_n gives rise to a linear map $\nu(p)_n : F_n^\heartsuit O \rightarrow F_n H^*$ (see proof of Lem. A.11) and the map $\underline{\text{act}}_{F_n B}$ gives rise to the map $\Delta_{F_n B} : F_n B \rightarrow$

$F_n H^* \otimes F_n B$ (see Lem. B.10(a)). The above commutative diagram then gives rise to a commutative diagram

$$\begin{array}{ccc} F_n A & \xrightarrow{F_n f} & F_n B \\ \Delta_{F_n A} \downarrow & & \downarrow \Delta_{F_n B} \\ F_n^\heartsuit O \otimes F_n A & \xrightarrow{\nu(p)_n \otimes F_n f} & F_n H^* \otimes F_n B \end{array}$$

This commutativity means that the restriction to $F_n A$ of the two composed maps of the diagram

$$\begin{array}{ccc} F_\infty A & \xrightarrow{F_\infty f} & F_\infty B \\ \Delta_{F_\infty A} \downarrow & & \downarrow \Delta_{F_\infty B} \\ O^\heartsuit \otimes F_\infty A & \xrightarrow{\nu(p) \otimes F_\infty f} & H' \otimes F_\infty B \end{array}$$

are equal. Since $F_\infty A = \cup_{n \geq 0} F_n A$, this diagram is commutative, therefore the pair $(\nu(p), F_\infty f)$ is a morphism in **HACA**. \square

B.4. Two functors **HACA** \rightarrow **Alg**_{gr} and a natural transformation.

Definition B.14. **Alg**_{gr} is the category of $\mathbb{Z}_{\geq 0}$ -graded associative algebras.

By Prop. A.4(a), a Hopf algebra O is naturally equipped with a filtration $F_\bullet O$. Set $\text{gr}(O) := \bigoplus_{i \geq 0} \text{gr}_i(O)$, where $\text{gr}_i(O) := F_i^\heartsuit O / F_{i-1}^\heartsuit O$ for $i \geq 0$, then $\text{gr}(O)$ is a $\mathbb{Z}_{\geq 0}$ -graded Hopf algebra, therefore also a $\mathbb{Z}_{\geq 0}$ -graded associative algebra. Recall that an object (O, A) of **HACA** gives rise to an algebra filtration $F_\bullet A$ on A (see Lem. B.3(a)). Then $\text{gr}(A) := \bigoplus_{i \geq 0} \text{gr}_i(A)$, where $\text{gr}_i(A) := F_i A / F_{i-1} A$ for $i \geq 0$ is a $\mathbb{Z}_{\geq 0}$ -graded associative algebra.

Lemma B.15. (a) The assignment $(O, A) \mapsto \text{gr}(A)$ defines a functor $\text{gr} : \mathbf{HACA} \rightarrow \mathbf{Alg}_{gr}$.

(b) For (O, A) an object in **HACA**, equip $\text{gr}(O) \otimes F_0 A$ with the tensor product $\mathbb{Z}_{\geq 0}$ -graded associative algebra structure, where $F_0 A$ is concentrated in degree 0. Then the assignment $(O, A) \mapsto \text{gr}(O) \otimes F_0 A$ defines a functor $\Phi : \mathbf{HACA} \rightarrow \mathbf{Alg}_{gr}$.

Proof. (a) The assignment $(O, A) \mapsto (A, F_\bullet A)$ defines a functor $\mathbf{HACA} \rightarrow \mathbf{Alg}_{fil}$ where \mathbf{Alg}_{fil} is the category of filtered algebras (in the sense of §3.1). The said assignment is the composition of this functor with the ‘‘associated-graded functor’’ $\mathbf{Alg}_{fil} \rightarrow \mathbf{Alg}_{gr}$. (b) The assignments $(O, A) \mapsto F_0 A$ and $(O, A) \mapsto \text{gr}(O)$ define functors $\mathbf{HACA} \rightarrow \mathbf{Alg}$ and $\mathbf{HACA} \rightarrow \mathbf{Alg}_{gr}$, where \mathbf{Alg} is the category of associative algebras. The said assignment is the composition of the product of these functors, of the ‘‘degree-0 functor’’ $\mathbf{Alg} \rightarrow \mathbf{Alg}_{gr}$, and of the tensor product functor $\mathbf{Alg}_{gr}^2 \rightarrow \mathbf{Alg}_{gr}$. \square

Let (O, A) be an object in **HACA**. By Lem. B.3(c), the coaction map $\Delta_A : A \rightarrow O \otimes A$ is compatible with the filtrations on both sides, therefore gives rise to a morphism of graded algebras $\text{gr}_\bullet(\Delta_A) : \text{gr}_\bullet(A) \rightarrow \text{gr}_\bullet(O \otimes A) = \text{gr}_\bullet^\heartsuit(O) \otimes \text{gr}_\bullet(A)$. There is a unique morphism

of graded algebras $pr_0 : \text{gr}_\bullet(A) \rightarrow F_0A$ given by the identity in degree 0 and 0 on all degree components of degree > 0 .

Definition B.16. $\text{nat}_{(O,A)} : \text{gr}_\bullet(A) \rightarrow \text{gr}_\bullet^\heartsuit(O) \otimes F_0A$ is the composition of $\text{gr}_\bullet(\Delta_A)$ with $id \otimes pr_0$.

Lemma B.17. (a) *The assignment $(O, A) \mapsto \text{nat}_{(O,A)} \in \mathbf{Alg}_{gr}(\text{gr}(A), \Phi(O, A))$ is a natural transformation relating the functors $\text{gr}, \Phi : \mathbf{HACA} \rightarrow \mathbf{Alg}_{gr}$.*

(b) *For any object (O, A) of \mathbf{HACA} , the morphism $\text{nat}_{(O,A)}$ is injective.*

Proof. (a) $\text{nat}_{(O,A)}$ is a morphism of graded algebras as it is a composition of such morphisms. The naturality is obvious. (b) Let us prove that for any $n \geq 0$, the degree n component $\text{nat}_{(O,A)}^n$ of $\text{nat}_{(O,A)}$ is injective. The double inclusion of vector spaces of $O \otimes A$

$$F_{n-1}(O \otimes A) \subset \sum_{p=1}^n F_{n-p}^\heartsuit O \otimes F_p A \subset F_n(O \otimes A)$$

where $F_k(O \otimes A) := \sum_{i=0}^k F_{k-i}^\heartsuit O \otimes F_i A$, gives rise to the map

$$F_n(O \otimes A)/F_{n-1}(O \otimes A) \rightarrow F_n(O \otimes A)/\left(\sum_{p=1}^n F_{n-p}^\heartsuit O \otimes F_p A\right),$$

which can be identified with the projection $id \otimes pr_0 : \text{gr}_n(O \otimes A) \rightarrow \text{gr}_n^\heartsuit(O) \otimes F_0A$. It follows that the map $\text{nat}_{(O,A)}^n$ is the vertical cokernel of the diagram

$$\begin{array}{ccc} F_n A & \xrightarrow{\Delta_A} & F_n(O \otimes A) \\ \uparrow & & \uparrow \\ F_{n-1} A & \longrightarrow & \sum_{p=1}^n F_{n-p}^\heartsuit O \otimes F_p A \end{array}$$

therefore that its kernel is the image in $F_n A/F_{n-1} A$ of the preimage by $\Delta_A : F_n A \rightarrow F_n(O \otimes A)$ of the subspace $\sum_{p=1}^n F_{n-p}^\heartsuit O \otimes F_p A$ of $F_n(O \otimes A)$. This subspace is contained $F_{n-1}^\heartsuit O \otimes A$, which together with Def. B.2 implies that this preimage is contained in $F_{n-1} A$. This implies the vanishing of the kernel of $\text{nat}_{(O,A)}^n$ and therefore the injectivity of $\text{nat}_{(O,A)}^n$. \square

B.5. Isomorphisms in \mathbf{HACA} .

Proposition B.18. (a) *A pair (O, \mathbf{a}) of a Hopf algebra O and an associative algebra \mathbf{a} gives rise to an object $(O, O \otimes \mathbf{a})$ of \mathbf{HACA} , with coaction morphism given by $\Delta_{O \otimes \mathbf{a}} := \Delta_O \otimes id_{\mathbf{a}}$ (where Δ_O is the coproduct of O). Its image by the endofunctor of \mathbf{HACA} from Lem. B.3(d) is the pair $(O^\heartsuit, O^\heartsuit \otimes \mathbf{a})$, where the coaction morphism is $\Delta_{O^\heartsuit \otimes \mathbf{a}} := \Delta_{O^\heartsuit} \otimes id_{\mathbf{a}}$ (where Δ_{O^\heartsuit} is the coproduct of O^\heartsuit).*

(b) *If (O, \mathbf{a}) is a pair of a Hopf algebra and an associative algebra, if (B, H) is an object of \mathbf{HAMA}_{fd} , and if $(p, f) \in \mathbf{PM}((O, O \otimes \mathbf{a}), (B, H))$ is a pairing-morphism such that $\nu(p) : O^\heartsuit \rightarrow H'$ is a Hopf algebra isomorphism and the algebra morphism $f : O \otimes \mathbf{a} \rightarrow B$ induces*

an algebra isomorphism $\mathbb{C} \otimes \mathbf{a} \xrightarrow{\sim} B^H$ between the subalgebras $\mathbb{C} \otimes \mathbf{a}$ and B^H of both sides, then the morphism $(\nu(p), F_\infty f) : (O^\heartsuit, O^\heartsuit \otimes \mathbf{a}) \rightarrow (H', F_\infty B)$ in **HACA** is an isomorphism; in particular, $F_\infty f : O^\heartsuit \otimes \mathbf{a} \rightarrow F_\infty B$ is an filtered algebra isomorphism.

(c) In the situation of (b), the algebra morphism $f : O \otimes \mathbf{a} \rightarrow B$ is compatible with the filtrations $F_\bullet^\heartsuit O \otimes \mathbf{a}$ and $F_\bullet B$, and induces an isomorphism $\text{gr}_\bullet^\heartsuit(O) \otimes \mathbf{a} \rightarrow \text{gr}_\bullet(B)$ in \mathbf{Alg}_{gr} .

Proof. (a) If $o \in O$ is such that $\Delta_O(o) \in F_n^\heartsuit O \otimes O$, then $o = (id \otimes \epsilon_O) \circ \Delta_O(o) \in F_n^\heartsuit O$, where ϵ_O is the counit of O , therefore $\{o \in O \mid \Delta_O(o) \in F_n^\heartsuit O \otimes O\} \subset F_n^\heartsuit O$. On the other hand, Lem. A.2(c) implies $F_n^\heartsuit O \subset \{o \in O \mid \Delta_O(o) \in F_n^\heartsuit O \otimes O\}$, therefore $F_n^\heartsuit O = \{o \in O \mid \Delta_O(o) \in F_n^\heartsuit O \otimes O\}$. Then $F_n(O \otimes \mathbf{a}) = \{o \in O \mid \Delta_O(o) \in F_n^\heartsuit O \otimes O\} \otimes \mathbf{a} = F_n^\heartsuit O \otimes \mathbf{a}$. It follows that $F_\infty(O \otimes \mathbf{a}) = \cup_{n \geq 0} F_n^\heartsuit O \otimes \mathbf{a} = O^\heartsuit \otimes \mathbf{a}$. The fact that $(O^\heartsuit, O^\heartsuit \otimes \mathbf{a}) \rightarrow (O, O \otimes \mathbf{a})$ is a morphism in **HACA** (see Lem. B.3(d)) implies that the coaction morphism of $O^\heartsuit \otimes \mathbf{a}$ has the claimed value.

(b) and (c) Lem. B.13 applied to the pairing-morphism (p, f) implies that $f : O \otimes \mathbf{a} \rightarrow B$ is a morphism of filtered algebras. By (a), there is an isomorphism $\text{gr}_\bullet(O \otimes \mathbf{a}) = \text{gr}_\bullet^\heartsuit(O) \otimes \mathbf{a}$. Composing it with the associated graded algebra morphism $\text{gr}_\bullet(f) : \text{gr}_\bullet(O \otimes \mathbf{a}) \rightarrow \text{gr}_\bullet(B)$, one obtains a morphism

$$\text{gr}_\bullet(f) : \text{gr}_\bullet^\heartsuit(O) \otimes \mathbf{a} \rightarrow \text{gr}_\bullet(B)$$

of graded algebras. Since (B, H) is an object of \mathbf{HAMA}_{fd} , one may apply to it the functor from Lem. B.10(c) to obtain the object $(H', F_\infty B)$ of **HACA**. By Lem. B.17, the latter object gives rise to an injective morphism of graded algebras

$$\text{gr}_\bullet(B) = \text{gr}_\bullet(F_\infty B) \xrightarrow{\text{nat}_{(H', F_\infty B)}} \Phi(H', F_\infty B) = \text{gr}_\bullet^\heartsuit(H') \otimes (F_\infty B)^H = \text{gr}_\bullet^\heartsuit(H') \otimes B^H.$$

The composition of these morphisms is the morphism

$$\text{nat}_{(H', F_\infty B)} \circ \text{gr}_\bullet(f) : \text{gr}_\bullet^\heartsuit(O) \otimes \mathbf{a} \rightarrow \text{gr}_\bullet^\heartsuit(H') \otimes B^H$$

of graded algebras. As $\nu(p) : O^\heartsuit \rightarrow H'$ is a Hopf algebra isomorphism, it induces an isomorphism between the filtrations F_\bullet^\heartsuit on both sides (see Prop. A.4(c)), and an associated graded isomorphism $\text{gr}_\bullet^\heartsuit(\nu(p)) : \text{gr}_\bullet^\heartsuit(O^\heartsuit) \rightarrow \text{gr}_\bullet^\heartsuit(H')$, which we identify with its composition with the equality $\text{gr}_\bullet^\heartsuit(O) = \text{gr}_\bullet^\heartsuit(O^\heartsuit)$. Let us show that

$$(B.5.1) \quad \text{nat}_{(H', F_\infty B)} \circ \text{gr}_\bullet(f) = \text{gr}_\bullet^\heartsuit(\nu(p)) \otimes f_0,$$

(equality of morphisms of graded algebras $\text{gr}_\bullet^\heartsuit(O) \otimes \mathbf{a} \rightarrow \text{gr}_\bullet^\heartsuit(H') \otimes B^H$), where f_0 is the isomorphism $f_0 : \mathbb{C} \otimes \mathbf{a} \rightarrow B^H$. The composition $\mathbb{C} \otimes \mathbf{a} = F_0(O \otimes \mathbf{a}) \xrightarrow{\text{gr}_0(f)} F_0 B = B^H$ is f_0 , which together with $\text{gr}_0^\heartsuit(\nu(p)) = 1$ proves the degree 0 part of (B.5.1). Let $n \geq 0$ and $o \in \text{gr}_n^\heartsuit(O)$. The image of o under $\text{nat}_{(H', F_\infty B)} \circ \text{gr}_n(f)$ is equal to the image of $\tilde{o} \otimes 1$ by the

horizontal composition

$$\begin{array}{ccc}
& & F_n^\heartsuit H' \otimes B \longrightarrow \mathrm{gr}_n^\heartsuit(H') \otimes B \\
& & \uparrow \qquad \qquad \qquad \uparrow \\
F_n^\heartsuit O \otimes \mathbf{a} & \xrightarrow{F_n f} & F_n B \xrightarrow{\Delta_{F_n B}} \sum_{k=0}^n F_{n-k}^\heartsuit H' \otimes F_k B \longrightarrow \mathrm{gr}_n^\heartsuit(H') \otimes F_0 B
\end{array}$$

where $\tilde{o} \in F_n^\heartsuit O$ is a lift of o . Since $F_n f$ is compatible with the coaction maps, the image of this element in $\sum_{k=0}^n F_{n-k}^\heartsuit H' \otimes F_k B$ is equal to $\nu(p)(\tilde{o}^{(1)}) \otimes f(\tilde{o}^{(2)} \otimes 1)$, where $\tilde{o}^{(1)} \otimes \tilde{o}^{(2)} = \Delta_O(\tilde{o}) \in \sum_{k=0}^n F_{n-k}^\heartsuit O \otimes F_k^\heartsuit O \subset O^{\otimes 2}$. One has $\tilde{o}^{(1)} \otimes \tilde{o}^{(2)} \in \tilde{o} \otimes 1 + F_{n-1}^\heartsuit O \otimes O$, therefore $\nu(p)(\tilde{o}^{(1)}) \otimes f(\tilde{o}^{(2)} \otimes 1) \in \nu(p)(\tilde{o}) \otimes 1 + F_{n-1}^\heartsuit H' \otimes B$ (inclusion in $F_n^\heartsuit H' \otimes B$). It follows that the image of $\tilde{o} \otimes 1$ in $\mathrm{gr}_n^\heartsuit(H') \otimes B$ is $\mathrm{im}(\nu(p)(\tilde{o}) \in F_n^\heartsuit H' \rightarrow \mathrm{gr}_n^\heartsuit(H')) \otimes 1 = \mathrm{gr}_n^\heartsuit(\nu(p))(o) \otimes 1$. This implies that both sides of (B.5.1) coincide when restricted to $\mathrm{gr}_n^\heartsuit(O) \otimes \mathbb{C}$ for any n , therefore to $\mathrm{gr}_\bullet^\heartsuit(O) \otimes \mathbb{C}$. (B.5.1) then follows from the fact that each of its sides is an algebra morphism, and that they agree on $\mathrm{gr}_\bullet^\heartsuit(O) \otimes \mathbb{C}$ and $\mathbb{C} \otimes \mathbf{a}$, which generate the source algebra.

It follows from (B.5.1) that $\mathrm{nat}_{(H', F_\infty B)} \circ \mathrm{gr}_\bullet(f)$ is an isomorphism of graded algebras. The injectivity of $\mathrm{nat}_{(H', F_\infty B)}$ then implies that both $\mathrm{nat}_{(H', F_\infty B)}$ and $\mathrm{gr}_\bullet(f)$ are isomorphisms of graded algebras. By Lem. B.13, f induces a morphism of filtered algebras $F_\infty f : O^\heartsuit \otimes \mathbf{a} \rightarrow F_\infty B$, and $\mathrm{gr}_\bullet(f) = \mathrm{gr}_\bullet(F_\infty f)$. It follows that $\mathrm{gr}_\bullet(F_\infty f)$ is an isomorphism of graded algebras, which together with the fact that the filtrations of the source and target of $F_\infty f$ are exhaustive, implies that $F_\infty f$ is an isomorphism of filtered algebras (see Lem. 3.1). \square

APPENDIX C. FILTERED FORMALITY FOR HOPF ALGEBRAS AND HACAS

In §C.1, we introduce a notion of filtered formality for Hopf algebras; this notion is related in §D to the similar notion for discrete groups, introduced in [SW1]. We extend this to a notion of filtered formality for HACAs in §C.2. The main result of this section is Prop. C.5, which shows that a HACA constructed in the context of Prop. B.18 is filtered formal.

C.1. Filtered formality for Hopf algebras. In Prop. A.4(a), we attach to a Hopf algebra O a filtration $F_\bullet^\heartsuit O$. Let $\mathrm{gr}^\heartsuit O$ be the associated graded vector space.

Lemma C.1. (a) *If O is a Hopf algebra, then $\mathrm{gr}^\heartsuit O$ is a graded Hopf algebra, which is commutative if O is, and such that for each $n \geq 0$, $F_n^\heartsuit(\mathrm{gr}^\heartsuit(O)) = \mathrm{gr}_{\leq n}^\heartsuit(O)$.*

(b) *The assignment $O \mapsto \mathrm{gr}^\heartsuit O$ defines an endofunctor of the category $\mathbf{HA}_{\mathrm{comm}}$ of commutative Hopf algebras.*

Proof. Let us show (a). The first statement follows from the fact that $F_\bullet^\heartsuit O$ is a Hopf algebra filtration (see Prop. A.4(a)). Let us show the second statement. Let $n \geq 0$. The inclusion $F_n^\heartsuit(\mathrm{gr}^\heartsuit O) \supset \mathrm{gr}_{\leq n}^\heartsuit O$ follows from the fact that $\mathrm{gr}^\heartsuit O$ is a graded and connected Hopf algebra. Let us show the opposite inclusion. Since $\mathrm{gr}^\heartsuit O$ is a graded Hopf algebra, $F_n^\heartsuit(\mathrm{gr}^\heartsuit O)$ is the direct sum of its intersection with the homogeneous components of $\mathrm{gr}^\heartsuit O$. If $k \geq 1$, the intersection

$F_n^\heartsuit(\mathrm{gr}^\heartsuit O) \cap \mathrm{gr}_{n+k}^\heartsuit O$ is contained in $F_n^\heartsuit(\mathrm{gr}^\heartsuit O)$, which is the kernel of $(id - \eta_{\mathrm{gr}^\heartsuit O} \epsilon_{\mathrm{gr}^\heartsuit O})^{\otimes n+1} \circ \Delta_{\mathrm{gr}^\heartsuit O}^{(n+1)} : \mathrm{gr}^\heartsuit O \rightarrow (\mathrm{gr}^\heartsuit O)^{\otimes n+1} = \mathrm{gr}^\heartsuit(O^{\otimes n+1})$, therefore $F_n^\heartsuit(\mathrm{gr}^\heartsuit O) \cap \mathrm{gr}_{n+k}^\heartsuit O$ is contained in the kernel of the map $\mathrm{gr}_{n+k}^\heartsuit O \rightarrow \mathrm{gr}_{n+k}^\heartsuit(O^{\otimes n+1})$ induced by $(id - \eta_{\mathrm{gr}^\heartsuit O} \epsilon_{\mathrm{gr}^\heartsuit O})^{\otimes n+1} \circ \Delta_{\mathrm{gr}^\heartsuit O}^{(n+1)}$. This map is the degree $n+k$ part of the associated graded of the map $(id - \eta_O \epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)} : O \rightarrow O^{\otimes n+1}$. It follows that if V is the preimage of $F_n^\heartsuit(\mathrm{gr}^\heartsuit O) \cap \mathrm{gr}_{n+k}^\heartsuit O$ in $F_{n+k}^\heartsuit O$, one has $(id - \eta_O \epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)}(V) \subset F_{n+k-1}^\heartsuit(O^{\otimes n+1})$. The map $((id - \eta_O \epsilon_O)^{\otimes k} \circ \Delta_O^{(k)}) \otimes id^{\otimes n}$ maps $F_{n+k-1}^\heartsuit(O^{\otimes n+1})$ to $F_{n+k-1}^\heartsuit(O^{\otimes n+k})$, therefore $(id - \eta_O \epsilon_O)^{\otimes n+k} \circ \Delta_O^{(n+k)}(V) \subset F_{n+k-1}^\heartsuit(O^{\otimes n+k})$. Now $O = \mathbb{C}1 \oplus F_1^\heartsuit O$; one has $(id - \eta_O \epsilon_O)^{\otimes n+k} \circ \Delta_O^{(n+k)}(V) \subset (F_1^\heartsuit O)^{\otimes n+k}$ while $F_{n+k-1}^\heartsuit(O^{\otimes n+k}) \subset \sum_{i=1}^{n+k} O^{\otimes i-1} \otimes \mathbb{C}1 \otimes O^{\otimes n+k-i}$. As the second terms of both inclusions have zero intersection in $O^{\otimes n+k}$, one has $(id - \eta_O \epsilon_O)^{\otimes n+k} \circ \Delta_O^{(n+k)}(V) = 0$. Therefore $V \subset F_{n+k-1}^\heartsuit O$, which implies $F_n^\heartsuit(\mathrm{gr}^\heartsuit O) \cap \mathrm{gr}_{n+k}^\heartsuit O = 0$. Therefore $F_n^\heartsuit(\mathrm{gr}^\heartsuit O) = \mathrm{gr}_{\leq n}^\heartsuit O$.

(b) follows from the naturality of the assignment $O \mapsto F_\bullet^\heartsuit O$. \square

Definition C.2. The Hopf algebra \mathcal{O} is called *filtered formal* if there is an isomorphism of Hopf algebra $O \rightarrow \mathrm{gr}^\heartsuit(O)$ whose associated graded for the grading gr^\heartsuit is the identity.

This terminology is justified by Prop. D.18.

C.2. Filtered formality for HACAs. Let \mathbf{HACA}_{gr} be the category of $\mathbb{Z}_{\geq 0}$ -graded HACAs.

Lemma C.3. *The assignment $(O, A) \mapsto (\mathrm{gr}^\heartsuit O, \mathrm{gr} A)$ is a functor $(\mathrm{gr}^\heartsuit, \mathrm{gr}) : \mathbf{HACA} \rightarrow \mathbf{HACA}_{gr}$.*

Proof. Follows from the fact that for (O, A) a HACA, $(F_\bullet^\heartsuit O, F_\bullet A)$ is a HACA filtration. \square

Definition C.4. The HACA (O, A) is called *filtered formal* if there exists a isomorphism $(O, A) \rightarrow (\mathrm{gr}^\heartsuit O, \mathrm{gr} A)$ in \mathbf{HACA} , which is compatible with the filtrations $(F_\bullet^\heartsuit O, F_\bullet A)$ in the source and induced by the grading in the target, and whose image by the functor $(\mathrm{gr}^\heartsuit, \mathrm{gr}) : \mathbf{HACA} \rightarrow \mathbf{HACA}_{gr}$ is the identity endomorphism of $(\mathrm{gr}^\heartsuit O, \mathrm{gr} A)$.

If the HACA (O, A) is filtered formal, then the Hopf algebra O is filtered formal in the sense of Def. C.2.

Proposition C.5. *Let $(O, \mathfrak{a}), (B, H), (p, f)$ be as in the hypothesis of Prop. B.18(b),(c). Then the HACA $(H', F_\infty B)$ is filtered formal.*

Proof. Recall from Lem. C.3 that $(\tilde{O}, A) \mapsto (\mathrm{gr}^\heartsuit \tilde{O}, \mathrm{gr} A)$ is a functor $\mathbf{HACA} \rightarrow \mathbf{HACA}_{gr}$; one checks that the same is true of $(\tilde{O}, A) \mapsto (\mathrm{gr}^\heartsuit \tilde{O}, \Phi(\tilde{O}, A))$. For (\tilde{O}, A) a HACA, $(id_{\tilde{O}}, \mathrm{nat}_{(\tilde{O}, A)})$ is a morphism $(\mathrm{gr}^\heartsuit \tilde{O}, \mathrm{gr} A) \rightarrow (\mathrm{gr}^\heartsuit \tilde{O}, \Phi(\tilde{O}, A))$ in \mathbf{HACA}_{gr} , which is an isomorphism if and only if $\mathrm{nat}_{(\tilde{O}, A)}$ is.

The proof of Prop. B.18(c) implies that $\mathrm{nat}_{(H', F_\infty B)}$ is an isomorphism in \mathbf{Alg}_{gr} , therefore

$$(C.2.1) \quad (id_{\mathrm{gr}^\heartsuit H'}, \mathrm{nat}_{(H', F_\infty B)}) : (\mathrm{gr}^\heartsuit H', \mathrm{gr} B) \rightarrow (\mathrm{gr}^\heartsuit H', \mathrm{gr}^\heartsuit H' \otimes B^H)$$

is an isomorphism in \mathbf{HACA}_{gr} .

By the assumptions of Prop. B.18(b), $\nu(p) : O^\heartsuit \rightarrow H'$ is an isomorphism in \mathbf{HA} . Prop. C.1 then implies that $\text{gr}^\heartsuit \nu(p) : \text{gr}^\heartsuit(O) \rightarrow \text{gr}^\heartsuit(H')$ is an isomorphism in \mathbf{HA}_{gr} . The assumptions of Prop. B.18(b) also imply that f restricts to an algebra isomorphism $\mathbb{C} \otimes \mathbf{a} \rightarrow B^H$, we denote by f_0 the composed isomorphism $\mathbf{a} \simeq \mathbb{C} \otimes \mathbf{a} \rightarrow B^H$. Both isomorphisms $\text{gr}^\heartsuit \nu(p)$ and f_0 induce an isomorphism

$$(C.2.2) \quad (\text{gr}^\heartsuit \nu(p)^{-1}, \text{gr}^\heartsuit \nu(p)^{-1} \otimes f_0^{-1}) : (\text{gr}^\heartsuit(H'), \text{gr}^\heartsuit(H') \otimes B^H) \rightarrow (\text{gr}^\heartsuit(O), \text{gr}^\heartsuit(O) \otimes \mathbf{a}).$$

in \mathbf{HACA}_{gr} . By Prop. B.18(b) and (c), the pair

$$(C.2.3) \quad (\nu(p), F_\infty f) : (O^\heartsuit, O^\heartsuit \otimes \mathbf{a}) \rightarrow (H', F_\infty B)$$

is an isomorphism in \mathbf{HACA} . The composition of (C.2.1), (C.2.2) and (C.2.3) gives rise to an isomorphism

$$(C.2.4) \quad (\nu(p) \circ (\text{gr}^\heartsuit \nu(p))^{-1}, F_\infty f \circ (\text{gr}^\heartsuit(\nu(p))^{-1} \otimes f_0^{-1}) \circ \text{nat}_{(\text{gr}^\heartsuit H', F_\infty B)}) : (\text{gr}^\heartsuit H', \text{gr} B) \rightarrow (H', F_\infty B)$$

in \mathbf{HACA} .

By Prop. A.4(c), $\nu(p)$ is compatible with the filtrations F_\bullet^\heartsuit ; it follows that $\nu(p) \circ \text{gr}^\heartsuit \nu(p)^{-1}$ is compatible with these filtrations as well, and one computes $\text{gr}^\heartsuit(\nu(p) \circ \text{gr}^\heartsuit \nu(p)^{-1}) = id$.

The algebra morphisms in (C.2.1) and (C.2.2) are compatible with the filtrations since they are graded, and the algebra morphism in (C.2.3) is compatible with the filtrations. It follows that the algebra morphism in (C.2.4) is compatible with the filtrations. Its associated graded is the composed morphism

$$(C.2.5) \quad \text{gr} B \xrightarrow{\text{nat}_{(H', F_\infty B)}} \text{gr}^\heartsuit H' \otimes B^H \xrightarrow{\text{gr}^\heartsuit(\nu(p))^{-1} \otimes f_0^{-1}} O^\heartsuit \otimes \mathbf{a} \xrightarrow{\text{gr} f} \text{gr} B.$$

It follows from the proof of Prop. B.18(b) that relation (B.5.1) holds, and that $\text{nat}_{(H', F_\infty B)}$ and $\text{gr} f$ are both isomorphisms. The combination of these facts then implies that the map (C.2.5) is the identity. Therefore the algebra morphism in (C.2.4) is compatible with the filtrations, and its associated graded is the identity. \square

APPENDIX D. HOPF ALGEBRA DUALITY AND PROUNIPOTENT COMPLETIONS

§D.1 relates the Hopf algebraic constructions of §A to complete Hopf algebras (CHAs) in the sense of [Q1]. This is applied in §D.2 to obtain an expression of the function algebra of the prounipotent completion of a finitely generated group in the terms of the functors of §A. The main result of §D.3 is Prop. D.18, which relates the notion of filtered formality for a finitely generated group ([SW1]) with the similar notion for Hopf algebras, introduced in §C.

D.1. Completion and duality of Hopf algebras. Recall from [Q1], A1 and A2 the categories **CAA** of complete augmented algebras (CAAs) and **CHA** of complete Hopf algebra (CHAs): **CAA** is the full subcategory, in the category of pairs $(A, F^\bullet A)$ of an augmented algebra A and a decreasing algebra filtration $A = F^0 A \supset F^1 A \supset \dots$, of CAAs, i.e. of pairs such that A is complete and Hausdorff for the topology of $F^\bullet A$, such that $F^1 A$ is the augmentation ideal of A and such that $\text{gr}(A)$ is generated by $\text{gr}^1(A)$; a monoidal structure is defined on **CAA**, given at the level of objects by $((A, F^\bullet A), (B, F^\bullet B)) \mapsto (A \hat{\otimes} B, F^\bullet(A \hat{\otimes} B))$, where $A \hat{\otimes} B := \varprojlim_n (A/F^n A) \otimes (B/F^n B)$ and $F^n(A \hat{\otimes} B) = \varprojlim_m \text{im}(\sum_{n'+n''=n} F^{n'} A \otimes F^{n''} B \rightarrow (A/F^m A) \otimes (B/F^m B))$; a CHA is a pair (A, Δ_A) , where A is a CAA, $\Delta_A : A \rightarrow A \hat{\otimes} A$ is a cocommutative and coassociative algebra morphism, which admits the augmentation of A as a counit, and a morphism in **CHA** is a morphism in **CAA** which is compatible with coproducts.

Lemma D.1. (a) Let H be a cocommutative Hopf algebra with coproduct Δ_H and counit ϵ_H . Let $J_H := \text{Ker}(\epsilon_H)$. Set $H^\wedge := \varprojlim_m H/J_H^m$ and for $n \geq 0$, $F^n H^\wedge := \varprojlim_m F^n H/F^{\max(n,m)} H$. Then H^\wedge is a complete augmented algebra. There is a unique continuous extension of Δ_H to a map $\hat{\Delta}_H : H^\wedge \rightarrow H^\wedge \hat{\otimes} H^\wedge$, which equips H^\wedge with the structure of a complete Hopf algebra. The assignment $H \mapsto H^\wedge$ is a functor $\mathbf{HA}_{coco} \rightarrow \mathbf{CHA}$, where \mathbf{HA}_{coco} is the category of cocommutative Hopf algebras.

(b) If V is a vector space and H is the cocommutative Hopf algebra $T(V)$ with coproduct defined by the condition that the elements of V are primitive, then $T(V)^\wedge = \hat{T}(V)$ (the degree completion of $T(V)$).

Proof. (a) The first statement follows from [Q1], A1, Example 1.2. One has $\Delta_H(J_H) \subset J_H \otimes H + H \otimes J_H$, which implies that for any $n \geq 0$, one has $\Delta_H(J_H^n) \subset \sum_{p,q|p+q=n} J_H^p \otimes J_H^q$. It follows that Δ_H defines a collection of maps $H/J_H^{2n} \rightarrow (H/J_H^n)^{\otimes 2}$ indexed by $n \geq 0$, which are compatible for various n and which therefore induce a map $\hat{\Delta}_H : H^\wedge \rightarrow H^\wedge \hat{\otimes} H^\wedge$, which is a continuous extension of Δ_H . The uniqueness of this extension follows from the Hausdorff property of H^\wedge . The verification of the other properties of $\hat{\Delta}_H$ is standard. (b) follows from the fact that $J_{T(V)}^n$ is the part of $T(V)$ of degree $\geq n$, and that $\hat{T}(V)$ is the completion w.r.t. the corresponding topology. \square

Definition D.2. (a) \mathbf{CHA}_{fd} is the full subcategory of **CHA** of CHAs A such that $\text{gr}^1 A$ is finite dimensional.

(b) For H a Hopf algebra, set $\text{gr}^1(H) := J_H/J_H^2$; where $J_H := \text{Ker}(\epsilon_H)$, and ϵ_H is the counit of H .

(c) $\mathbf{HA}_{coco,fd}$ is the full subcategory of \mathbf{HA}_{coco} of all cocommutative Hopf algebras H such that $\text{gr}^1 H$ is finite dimensional.

(d) \mathbf{Gp}_{fg} is the full subcategory of **Gp** of finitely generated groups.

Lemma D.3. (a) The functor $\mathbf{Gp} \rightarrow \mathbf{HA}_{coco}$ given by $\Gamma \mapsto \mathbb{C}\Gamma$ induces a functor $\mathbf{Gp}_{fg} \rightarrow \mathbf{HA}_{coco,fd}$.

(b) The functor $\mathbf{HA}_{coco} \rightarrow \mathbf{CHA}$, $H \mapsto H^\wedge$ from Lem. D.1 induces a functor $\mathbf{HA}_{coco,fd} \rightarrow \mathbf{CHA}_{fd}$.

Proof. (a) If Γ is a group, then $\mathrm{gr}^1(\mathbb{C}\Gamma) = \Gamma^{ab} \otimes \mathbb{C}$ (see [Q2]). Moreover, if Γ is finitely generated, then Γ^{ab} is a finitely generated abelian group, which together with the above equality implies the finite dimensionality of $\mathrm{gr}^1(\mathbb{C}\Gamma)$. (b) follows from the fact that the natural map $\mathrm{gr}^1 H \rightarrow \mathrm{gr}^1(H^\wedge)$ is a linear isomorphism. \square

Definition D.4. For H a CHA with associated filtration $H = F^0 H \supset \dots$, set $H' := \cup_{n \geq 0} F^n H^\perp \subset H^*$.

Lemma D.5. (a) If H is a CHA, then H' has an commutative and associative algebra structure.

(b) If H is an object in \mathbf{CHA}_{fd} , then the algebra structure on H' can be upgraded to a commutative Hopf algebra structure; the resulting assignment $H \mapsto H'$ is a contravariant functor $\mathbf{CHA}_{fd} \rightarrow \mathbf{HA}_{comm}$.

Proof. (a) Let $\xi, \eta \in H'$. Let $n \geq 0$ be such that $\xi, \eta \in F^n H^\perp$, then $\xi \otimes \eta$ is a linear form on $(H/F^n H)^{\otimes 2}$, which can be pulled back by the map $H^{\hat{\otimes} 2} \rightarrow (H/F^n H)^{\otimes 2}$ to define a linear form on $H^{\hat{\otimes} 2}$; it can be checked to be independent on the choice of n and will be denoted $\ell_{\xi, \eta}$. Then $\ell_{\xi, \eta}$ vanishes on $F^n H \otimes H + H \otimes F^n H$. The assignment $\xi \cdot \eta : h \mapsto \ell_{\xi, \eta} \circ \Delta_H(h)$ is then a linear form on H , i.e. an element of H^* . Since $\Delta_H(F^{2n} H) \subset F^n H \otimes H + H \otimes F^n H$, one has $\xi \cdot \eta \in F^{2n} H^\perp$, therefore $\xi \cdot \eta \in H'$. One checks that this defines a commutative and associative algebra structure on H' .

(b) If $n \geq 0$, then $H/F^n H$ is noncanonically isomorphic to $\oplus_{k=0}^n \mathrm{gr}^k(H)$, which is finite dimensional since it is generated by $\mathrm{gr}^1 H$ and the latter space is finite dimensional. The coproduct Δ_H defines a coproduct $\Delta_{F^n H} : H/F^n H \rightarrow (H/F^n H)^{\otimes 2}$ for any $n \geq 0$. Since $H/F^n H$ is finite dimensional, this coproduct can be dualized and defines a product $m_{(F^n H)^\perp} : ((F^n H)^\perp)^{\otimes 2} \rightarrow (F^n H)^\perp$. As in the proof of Lem. A.8, one checks that the collection of these products is compatible, which gives rise to a product $m_{H'} : H'^{\otimes 2} \rightarrow H'$, which can be shown to define a Hopf structure. \square

The following lemma is an topological analogue of Lem. A.9.

Lemma D.6. If H is a CHA, then for any $n \geq 0$ one has $F_n^\heartsuit(H') = (F^{n+1} H)^\perp$ (equality of subspaces of H^*).

Proof. Let us show that if H is a CHA and $k, n \geq 1$, then the image of the product map $(F^1 H/F^{n+k} H)^{\otimes n} \rightarrow H/F^{n+k} H$ is $F^n H/F^{n+k} H$. We proceed by induction on k . For $k = 1$, the statement follows from the surjectivity of the product map $(F^1 H/F^2 H)^{\otimes n} \rightarrow F^n H/F^{n+1} H$.

Assume the statement at order k and let us show it at order $k + 1$. The product map $(F^1 H/F^2 H)^{\otimes n+k} \rightarrow F^{n+k} H/F^{n+k+1} H$ is surjective, therefore $F^{n+k} H/F^{n+k+1} H$ is equal to the image of a subspace of $(F^1 H/F^{n+k+1} H)^{\otimes n}$, namely

$$(F^1 H/F^{n+k+1} H)^{\otimes n-1} \otimes \text{im}((F^1 H/F^{n+k+1} H)^{\otimes k+1} \rightarrow F^1 H/F^{n+k+1} H).$$

Combining this with the statement at order k , one obtains the statement at order $k + 1$ and therefore the induction. The proof of the lemma is then similar to that of Lem. A.9. \square

Lemma D.7. *The functors $\mathbf{HA}_{coco,fd} \rightarrow \mathbf{CHA}_{fd} \rightarrow \mathbf{HA}_{comm}$ (see Lem. D.3(b) and Lem. D.5(b)) and $\mathbf{HA}_{coco,fd} \rightarrow \mathbf{HA}_{comm}$ (see Lem. A.8) are naturally equivalent.*

Proof. For each $n \geq 0$, the algebra morphism $H \rightarrow H^\wedge$ induces an algebra isomorphism $H/F^n H \xrightarrow{\sim} H^\wedge/F^n H^\wedge$, which fits in a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & H^\wedge \\ \downarrow & & \downarrow \\ H/F^n H & \longrightarrow & H^\wedge/F^n H^\wedge \end{array}$$

Dualizing and using the equalities $(H/F^n H)^* = F^n H^\perp$, $(H^\wedge/F^n H^\wedge)^* = (F^n H^\wedge)^\perp$, one obtains a commutative diagram

$$(D.1.1) \quad \begin{array}{ccc} (F^n H^\wedge)^\perp & \xrightarrow{\sim} & (F^n H)^\perp \\ \downarrow & & \downarrow \\ (H^\wedge)^* & \longrightarrow & H^* \end{array}$$

If V, W are filtered vector spaces and $f : V \rightarrow W$ is a linear map such that for any $n \geq 0$, f induces a linear isomorphism $F_n V \rightarrow F_n W$, then f induces an isomorphism $F_\infty V \rightarrow F_\infty W$. This and (D.1.1) imply that the map $(H^\wedge)^* \rightarrow H^*$ restricts to a linear isomorphism $(H^\wedge)^\perp \rightarrow H^\perp$. One then checks this isomorphism to be compatible with the Hopf algebra structures. \square

D.2. Isomorphism $\mathbb{C}\Gamma' \simeq \mathcal{O}(\Gamma_{unip})$. Let Γ be a group. A pro-unipotent completion of a group Γ is the pair (U, c) of a prounipotent \mathbb{C} -group scheme U and a morphism $\Gamma \rightarrow U(\mathbb{C})$ satisfying a universal property (see [BGF], Def. 3.105).

If Γ is an object in \mathbf{Gp}_{fg} , then $\mathbb{C}\Gamma$ is an object in \mathbf{HA}_{fd} by Lem. D.3(a), therefore $(\mathbb{C}\Gamma)'$ is a commutative Hopf algebra (see Lem. A.8). Moreover, it was shown in [BGF], Thm. 3.109, that a pro-unipotent completion exists is unique up to isomorphism, and can be constructed as the pair $(\Gamma_{unip}, c_{unip})$, where Γ_{unip} is the spectrum of commutative Hopf algebra $\mathcal{O}(\Gamma_{unip}) := ((\mathbb{C}\Gamma)^\wedge)'$, which is isomorphic to $(\mathbb{C}\Gamma)'$ by Lem. D.7, and c_{unip} is induced by the commutative algebra morphism $\mathcal{O}(\Gamma_{unip}) = (\mathbb{C}\Gamma)' \hookrightarrow (\mathbb{C}\Gamma)^* = \mathbb{C}^\Gamma$, where \mathbb{C}^Γ is the algebra of all functions $\Gamma \rightarrow \mathbb{C}$.

D.3. Relation between the filtered formalities of Γ and $(\mathbb{C}\Gamma)'$.

Definition D.8 (see [Q1], §A.2). (a) A Malcev Lie algebra (MLA) is a Lie algebra \mathfrak{g} , equipped with a decreasing Lie algebra filtration $\mathfrak{g} = F^1\mathfrak{g} \supset F^2\mathfrak{g} \supset \dots$ (i.e., $[F^i\mathfrak{g}, F^j\mathfrak{g}] \subset F^{i+j}\mathfrak{g}$ for $i, j \geq 1$) for which it is complete and Hausdorff, and such that the associated graded Lie algebra $\text{gr}\mathfrak{g}$ is generated by $\text{gr}^1\mathfrak{g}$.

(b) A morphism between two MLAs is a Lie algebra morphism which is compatible with the filtrations. We denote by **MLA** the category of MLAs.

Lemma D.9 (see [Q1], Thm. 3.3). *The functors $\hat{U} : \mathbf{MLA} \leftrightarrow \mathbf{CHA} : \mathcal{P}$, where $\hat{U}(\mathfrak{g}) := \varprojlim_i U(\mathfrak{g}/F^i\mathfrak{g})$ and \mathcal{P} is the “primitive-elements functor”, taking (A, Δ_A) to $\mathcal{P}(A) := \{a \in A \mid \Delta_A(a) = a \otimes 1 + 1 \otimes a\}$, equipped with the filtration given by $F_n\mathcal{P}(A) := \mathcal{P}(A) \cap F_n A$ for $n \geq 0$, are quasi-inverse to one another.*

Definition D.10. The composed functor $\mathbf{Gp} \rightarrow \mathbf{CHA} \xrightarrow{\mathcal{P}} \mathbf{MLA}$, where the first functor is $\Gamma \mapsto (\mathbb{C}\Gamma)^\wedge$ (see Lem. D.1), is denoted $\Gamma \mapsto \text{Lie}(\Gamma)$.

Lemma-Definition D.11. *The assignment $\mathfrak{g} \mapsto (\hat{\text{gr}}(\mathfrak{g}), F^\bullet \hat{\text{gr}}(\mathfrak{g}))$, where $\hat{\text{gr}}(\mathfrak{g}) := \prod_{i \geq 1} \text{gr}^i(\mathfrak{g})$ and for $n \geq 1$, $F^n \hat{\text{gr}}(\mathfrak{g}) := \prod_{i \geq n} \text{gr}^i(\mathfrak{g})$, is an endofunctor of **MLA**. For \mathfrak{g} an object in **MLA**, one has $\text{gr}(\hat{\text{gr}}(\mathfrak{g})) \simeq \text{gr}(\mathfrak{g})$.*

Proof. Immediate. □

Definition D.12 (see [SW1]). A group Γ is called filtered formal if there exists an isomorphism $\text{Lie}(\Gamma) \rightarrow \hat{\text{gr}}\text{Lie}(\Gamma)$ in **MLA**, whose associated graded is the identity.

One checks that $A \mapsto \hat{\text{gr}}(A) := \prod_{i \geq 0} F^i A / F^{i+1} A$ is an endofunctor of **CHA**.

Lemma D.13. *The category equivalence $\mathbf{CHA} \leftrightarrow \mathbf{MLA}$ quasi-intertwines the endofunctors $\hat{\text{gr}}$ on both sides.*

Proof. Follows from the natural isomorphism of graded Lie algebras

$$(D.3.1) \quad \text{gr}(\mathcal{P}A) \simeq \mathcal{P}(\text{gr}(A))$$

for A on object of **CHA** from [Q1], A1, Thm. 2.14. □

Definition D.14. \mathbf{MLA}_{fd} is the full subcategory of **MLA** of MLAs \mathfrak{g} such that $\text{gr}^1\mathfrak{g}$ is finite dimensional.

Recall the full subcategory \mathbf{CHA}_{fd} of **CHA** (see Def. D.2).

Lemma D.15. (a) *The equivalence $\mathbf{CHA} \leftrightarrow \mathbf{MLA}$ induces an equivalence $\mathbf{CHA}_{fd} \leftrightarrow \mathbf{MLA}_{fd}$.*

(b) *The endofunctors $\hat{\text{gr}}$ of **CHA** and **MLA** induce endofunctors (still denoted $\hat{\text{gr}}$) of \mathbf{CHA}_{fd} and \mathbf{MLA}_{fd} .*

(c) *The category equivalence $\mathbf{CHA}_{fd} \leftrightarrow \mathbf{MLA}_{fd}$ quasi-intertwines the endofunctors $\hat{\text{gr}}$ on both sides.*

Proof. (a) For A an object in **CHA**, (D.3.1) implies the vector space isomorphism $\mathrm{gr}^1(\mathcal{P}A) \simeq \mathcal{P}(\mathrm{gr}(A)) \cap \mathrm{gr}^1 A$; moreover, $\mathrm{gr}(A)$ is a graded connected Hopf algebra, therefore $\mathrm{gr}^1(A) \subset \mathcal{P}(\mathrm{gr}(A))$; therefore $\mathrm{gr}^1(\mathcal{P}A) \simeq \mathrm{gr}^1 A$.

(b) follows from $\mathrm{gr}^1(\hat{\mathrm{gr}}(A)) = \mathrm{gr}^1(A)$ for A a CHA and $\mathrm{gr}^1(\hat{\mathrm{gr}}(\mathfrak{g})) = \mathrm{gr}^1(\mathfrak{g})$ for \mathfrak{g} an MLA.

(c) follows from Lemma D.13. \square

Lemma D.16. *Let H be an object in \mathbf{CHA}_{fd} .*

(a) *There is a natural vector space isomorphism $(\hat{\mathrm{gr}}H)' \simeq \bigoplus_{n \geq 0} \mathrm{gr}^n(H)^*$; it induces a natural isomorphism $F_n^\heartsuit((\hat{\mathrm{gr}}H)') \simeq \bigoplus_{k \leq n} \mathrm{gr}^k(H)^*$ for any $n \geq 0$;*

(b) *There is a natural isomorphism*

$$(D.3.2) \quad (\hat{\mathrm{gr}}H)' \simeq \mathrm{gr}^\heartsuit(H')$$

in **HA**.

(c) *The composed isomorphism $\bigoplus_{k \leq n} \mathrm{gr}^k(H)^* \simeq F_n^\heartsuit((\hat{\mathrm{gr}}H)') \simeq F_n^\heartsuit(\mathrm{gr}^\heartsuit(H')) \simeq \mathrm{gr}_{\leq n}^\heartsuit(H')$, where the first isomorphism arises from (a), the second from the image by F_n^\heartsuit of (D.3.2) and the third isomorphism arises from Lem. C.1(a), is the direct sum over $k \leq n$ of the isomorphisms $\mathrm{gr}^k(H)^* \simeq \mathrm{gr}_{\leq n}^\heartsuit(H')$ arising from Lem. D.6.*

Proof. (a) $\hat{\mathrm{gr}}(H)$ is $\prod_{n \geq 0} \mathrm{gr}^n H$, equipped with the filtration $(\prod_{i \geq n} \mathrm{gr}^i H)_{n \geq 0}$. It follows that $\hat{\mathrm{gr}}(H)' = \bigoplus_{n \geq 0} \mathrm{gr}^n(H)^*$. One then has $F_n^\heartsuit(\hat{\mathrm{gr}}(H)') = (F^{n+1} \hat{\mathrm{gr}}(H))^\perp \simeq \bigoplus_{k \leq n} \mathrm{gr}^k(H)^*$ for any $n \geq 0$, where the first equality follows from Lem. D.6.

(b) follows from the sequence of equalities

$$\begin{aligned} \mathrm{gr}^\heartsuit(H') &= \bigoplus_{n \geq 0} F_{n+1}^\heartsuit H' / F_n^\heartsuit H' = \bigoplus_{n \geq 0} (F^{n+1} H)^\perp / (F^n H)^\perp = \bigoplus_{n \geq 0} (F^n H / F^{n+1} H)^* \\ &= \bigoplus_{n \geq 0} \mathrm{gr}^n(H)^* = (\hat{\mathrm{gr}}H)', \end{aligned}$$

where the second equality follows from $F_\bullet^\heartsuit H' = (F^{\bullet+1} H)^\perp$ (see Lem. D.6) and the last equality follows from (a).

(c) follows from the identification of the said isomorphism with the degree $\leq k$ part of the above sequence of equalities. \square

Lemma D.17. *Let Γ be group. If Γ is filtered formal, then there exists an isomorphism*

$$(D.3.3) \quad \mathrm{iso}_\Gamma : (\mathbb{C}\Gamma)^\wedge \simeq \hat{\mathrm{gr}}((\mathbb{C}\Gamma)^\wedge)$$

in **CHA**, such that $\mathrm{gr}(\mathrm{iso}_\Gamma) = \mathrm{id}$.

Proof. Let Γ be a filtered formal group. There is an isomorphism

$$(D.3.4) \quad \mathrm{Lie}(\Gamma) \simeq \hat{\mathrm{gr}}\mathrm{Lie}(\Gamma)$$

in **MLA** with associated graded the identity. There is a sequence of isomorphisms in **CHA** given by

$$(\mathbb{C}\Gamma)^\wedge \simeq \hat{U}(\mathrm{Lie}(\Gamma)) \simeq \hat{U}(\hat{\mathrm{gr}}\mathrm{Lie}(\Gamma)) \simeq \hat{\mathrm{gr}}(\hat{U}(\mathrm{Lie}(\Gamma))) = \hat{\mathrm{gr}}((\mathbb{C}\Gamma)^\wedge)$$

where the first and last isomorphisms come from Lem. D.9 and Def. D.10, the second isomorphism arises from applying the functor \hat{U} to the isomorphism (D.3.4), and the third isomorphism arises from Lem. D.15(c). This results in an isomorphism with the claimed properties. \square

Proposition D.18. *If Γ is a filtered formal finitely generated group, then the Hopf algebra $(\mathbb{C}\Gamma)'$ is filtered formal (in the sense of Def. C.2).*

Proof. Let Γ be a filtered formal finitely generated group. Then (D.3.3) is an isomorphism in \mathbf{CHA}_{fd} . There is a sequence of isomorphisms in \mathbf{HA}

$$(D.3.5) \quad (\mathbb{C}\Gamma)' \simeq ((\mathbb{C}\Gamma)^\wedge)' \simeq (\hat{\text{gr}}((\mathbb{C}\Gamma)^\wedge))' \simeq \text{gr}^\heartsuit(((\mathbb{C}\Gamma)^\wedge)') \simeq \text{gr}^\heartsuit((\mathbb{C}\Gamma)')$$

where the first and last isomorphisms follow from Lem. D.7, the second isomorphism arises from applying the functor $\mathbf{CHA}_{fd} \rightarrow \mathbf{HA}$, $H \mapsto H'$ (see Def. D.4) to (D.3.3), and the third isomorphism comes from (D.3.2). It follows from $\text{gr}(\text{iso}_\Gamma) = \text{id}$ and Lem. D.6 that the second isomorphism in (D.3.5) is compatible with the filtration F_\bullet^\heartsuit and that the associated graded is the identity. The third isomorphism in (D.3.5) induces an isomorphism between the F_n^\heartsuit of both sides for any $n \geq 0$. By Lem. D.16(c), the associated graded morphism coincides with the composition of natural isomorphisms $\text{gr}_n^\heartsuit((\hat{\text{gr}}((\mathbb{C}\Gamma)^\wedge))') \simeq \text{gr}^n(\mathbb{C}\Gamma)^* \simeq \text{gr}_n^\heartsuit(\text{gr}^\heartsuit(((\mathbb{C}\Gamma)^\wedge)'))$. This implies that the image by gr_n^\heartsuit of the isomorphism $(\mathbb{C}\Gamma)' \simeq \text{gr}^\heartsuit((\mathbb{C}\Gamma)')$ induced by (D.3.5) is the identification $\text{gr}_n^\heartsuit((\mathbb{C}\Gamma)') \simeq \text{gr}_n^\heartsuit \text{gr}^\heartsuit((\mathbb{C}\Gamma)')$ induced by Lem. C.1(a). \square

Remark D.19. Lem. D.17 relates as follows to [SW2]: combining Cors. 6.2 and 2.7 of *loc. cit.*, one obtains the equivalence of the filtered formality of Γ with the conclusion of Lem. D.17 for any finitely generated group Γ .

Acknowledgements. The research of B.E. has been partially funded by ANR grant ‘‘Project HighAGT ANR20-CE40-0016’’. The research of F.Z. has been funded by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 843960 for the project ‘‘HIPSAM’’, and by the Royal Society, under the grant URF\R1\201473. Both authors express their thanks to F. Brown, J. Burgos Gil, J. Fresán, R. Hain, N. Matthes and E. Panzer for fruitful exchanges on various aspects of this work.

REFERENCES

- [Br] F. Brown, Multiple zeta values and periods of moduli spaces $\mathfrak{M}_{0,n}(\mathbb{R})$, *Annales scientifiques de l’École Normale Supérieure, Série 4, Tome 42 (2009) no. 3*, pp. 371-489.
- [BGF] J. Burgos Gil, J. Fresan, Multiple zeta values: from numbers to motives, *Clay Mathematics Proceedings*, to appear.
- [BDDT1] J. Broedel, C. Duhr, F. Dulat, L. Tancredi, Elliptic polylogarithms and iterated integrals on elliptic curves I: general formalism. *J. High Energy Phys.* 2018, no. 5, 093.
- [BDDT2] J. Broedel, C. Duhr, F. Dulat, L. Tancredi. Elliptic polylogarithms and iterated integrals on elliptic curves II. An application to the sunrise integral. *Physical Review D*, Vol. 97, no.11, p. 116009 (2018)
- [Ch] K.-T. Chen, Extension of C^∞ functions by integrals and Malcev completion of π_1 , *Adv. in Math.* 23 (1977), no. 2, 181–210.

- [DGP] E. D'Hoker, M.B. Green and B. Pioline. Asymptotics of the D^8R^4 genus-two string invariant. Commun. Num. Theor. Phys. 13, 351-462 (2019).
- [Fr] B. Fresse, Homotopy of operads and Grothendieck-Teichmüller groups. Part 1. The algebraic theory and its topological background. Mathematical Surveys and Monographs, 217. American Mathematical Society, Providence, RI, 2017.
- [L] S. Lang, Complex analysis. Fourth edition. Graduate Texts in Mathematics, 103. Springer-Verlag, New York, 1999.
- [LD] J.A. Lappo-Danilevsky, Mémoires sur la théorie des systèmes des équations différentielles linéaires. Chelsea Publishing Co., New York, N. Y., 1953.
- [Pa] E. Panzer, Feynman integrals and hyperlogarithms, PhD Thesis, arXiv:1506.07243v1, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2015
- [Ph] F. Pham, Singularities of integrals. Homology, hyperfunctions and microlocal analysis. Universitext. Springer, London; EDP Sciences, Les Ulis, 2011
- [Po] H. Poincaré, Sur les groupes des équations linéaires. Acta Math. 4 (1884), no. 1, 201–312.
- [Q1] D. Quillen, Rational homotopy theory. Ann. of Math. (2) 90 (1969), 205–295.
- [Q2] D. Quillen, On the associated graded ring of a group ring. J. Algebra 10 (1968), 411–418.
- [SW1] A. Suciuc, H. Wang, Formality properties of finitely generated groups and Lie algebras. Forum Math. 31 (2019), no. 4, 867–905.
- [SW2] A. Suciuc, H. Wang, Taylor expansions of groups and filtered-formality, European Journal of Mathematics, 6 (2020), 1073–1096.

IRMA (UMR 7501) ET DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ-DESCARTES, 67084 STRASBOURG (FRANCE)

Email address: `b.enriquez@math.unistra.fr`

UNIVERSITY OF OXFORD, MATHEMATICAL INSTITUTE, ANDREW WILES BUILDING, RADCLIFFE OBSERVATORY QUARTER (550), WOODSTOCK ROAD, OXFORD, OX2 6GG (UK)

Email address: `federico.zerbini@maths.ox.ac.uk`