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Congruences of Modal μ -algebras

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If $f(y)$ is an order preserving endofunction of a closed ordered monoid, then the two inequations

$$f(g(x)) \otimes x \leq g(x) \qquad g(f(x) \multimap x) \leq x$$

determine an order preserving endofunction $g(x)$ as $\mu_y.(f(y) \otimes x)$, the parameterized least prefixed point of $f(y) \otimes x$. This observation is the core of a method to axiomatize by equations the least prefixed point [7]. When this method is applied to the propositional Boolean modal μ -calculus, it shows that Kozen's axiomatization [4] can be turned into an equivalent equational axiomatization that is complete with respect to Kripke frames by Walukiewicz's theorem [9].

Being able to equationally axiomatize the propositional Boolean modal μ -calculus implies that its algebraic models form a variety of algebras, in the usual sense [3]. We call these models modal μ -algebras. The main characteristic that distinguishes a variety from a quasivariety – i.e. a class of models that can be axiomatized by equational implications – is that such a class is closed under homomorphic images defined from congruences. Thus, in a variety of algebras, there is always a bijection between quotients of an algebra (categorically, strong epimorphisms) and congruences on the algebra.

Modal μ -algebras are therefore closed under homomorphic images; we have investigated congruences of modal μ -algebras and found a characterization suitable for algebraic computations. In this talk we will present this characterization and illustrate its use by lifting some classical results from modal logic and modal algebras [5] to the setting of modal μ -algebras.

The characterization of congruences can also be exploited in the context of verification of systems, as these results can be reformulated in a logical framework. For example, it could be useful to analyze a restricted class of transition systems where given subsets of states are known to satisfy a finite number of algebraic relations. Some propositions of the modal μ -calculus could hold true in these models, and it would be desirable to know whether their truth is a consequence of Kozen's axiomatization and of those algebraic relations. This is a particular instance of a common problem in logic: is a proposition derivable from a finite set of propositions that are assumed as axioms? Among the results that we present is a deduction theorem for the propositional Boolean modal μ -calculus; this theorem allows to reduce derivability from a finite set of axioms to derivability from no axioms. As the latter problem is well known to be decidable for the modal μ -calculus, then derivability from a finite set of propositions becomes decidable as well.

1 Definitions

In the following Σ is a finite non empty alphabet of actions and X is a countable set of variables.

Definition 1.1. *We define by induction a set $\mathcal{T}(Y)$ of μ -terms in context Y , for each finite subset Y of X .*

- If $x \in Y$, then $x \in \mathcal{T}(Y)$.
- $\top \in \mathcal{T}(Y)$, and if $t_1, t_2 \in \mathcal{T}$, then $t_1 \wedge t_2 \in \mathcal{T}(Y)$.
- If $t \in \mathcal{T}(Y)$, then $\neg t \in \mathcal{T}(Y)$,

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- If $t \in \mathcal{T}(Y)$ and $\sigma \in \Sigma$, then $[\sigma]t \in \mathcal{T}(Y)$.
- If $t \in \mathcal{T}(Y)$ and $x \in Y$ is a variable that occurs in t in the scope of an even number of symbols \neg , then $\mu_x.t \in \mathcal{T}(Y \setminus \{x\})$.

Definition 1.2. A modal μ -algebra is a pair $\langle \mathcal{A}, |-\rangle$, where $\mathcal{A} = \langle A, \top, \wedge, \neg, \{[\sigma]\}_{\sigma \in \Sigma} \rangle$ is a modal algebra, see [5, §2.2.2], and $|-\rangle$ is an interpretation of terms $t \in \mathcal{T}(Y)$ as functions $|t| : A^Y \rightarrow A$ with the following properties:

- $|x|$ is the projection on the x -coordinate.
- $|-\rangle$ agrees with the structure of the modal algebra \mathcal{A} , that is, $|\top| = \top$, $|t_1 \wedge t_2| = |t_1| \wedge |t_2|$, $|\neg t| = \neg |t|$, and $|[\sigma]t| = [\sigma]|t|$.
- The function

$$|\mu_x.t| : A^{Y \setminus \{x\}} \rightarrow A$$

is the parameterized least prefixed point of

$$|t| : A^{\{x\}} \times A^{Y \setminus \{x\}} \rightarrow A.$$

Note that if \mathcal{A} is a modal algebra, then there exists at most one interpretation $|-\rangle$ such that $\langle \mathcal{A}, |-\rangle$ is a modal μ -algebra. Thus we can leave the interpretation $|-\rangle$ in the background and, by saying that \mathcal{A} is a modal μ -algebra, we mean that \mathcal{A} is a modal algebra and that such interpretation $|-\rangle$ can be found. Accordingly, if $t \in \mathcal{T}(Y)$, we simply write t for the function $|t|$. An *homomorphism* of modal μ -algebras from \mathcal{A} to \mathcal{B} is a function f between the underlying sets such that for all $t \in \mathcal{T}(Y)$ $f \circ t = t \circ f^Y$. Other operations on modal μ -algebras can be defined in terms of the given operations, for example:

$$\begin{aligned} x \rightarrow y &= \neg(x \wedge \neg y) & x \leftrightarrow y &= (x \rightarrow y) \wedge (y \rightarrow x) \\ \langle \sigma \rangle x &= \neg[\sigma]\neg x & \nu_x.t(x, y) &= \neg\mu_x.\neg t(\neg x, y). \end{aligned}$$

Definition 1.3. A congruence on a modal μ -algebra \mathcal{A} is an equivalence relation $\sim \subseteq A \times A$ such that, for every μ -term $t \in \mathcal{T}(Y)$ and every pair $a, b \in A^Y$, if $a_y \sim b_y$ for each $y \in Y$, then $t(a) \sim t(b)$ as well.

Given a modal μ -algebra \mathcal{A} and a congruence \sim on \mathcal{A} , we can define the homomorphic image \mathcal{A}/\sim as follows. Its elements are equivalence classes under \sim ; we denote by $[a]$ the equivalence class of $a \in A$. If $t \in \mathcal{T}(Y)$ and $\{[a_y]\}_{y \in Y}$ is a collection of equivalence classes, then we can define

$$t(\{[a_y]\}_{y \in Y}) = [t(\{a_y\}_{y \in Y})].$$

This is well defined, as usual, and \mathcal{A}/\sim is certainly a modal algebra. By the results in [7] this is also a modal μ -algebra.

2 The Characterization

On every modal μ -algebra we define the operation

$$[\Sigma^*]x = \nu_y.(x \wedge \bigwedge_{\sigma \in \Sigma} [\sigma]y).$$

Recall that a *filter* on a modal μ -algebra \mathcal{A} is a subset F of A such that $\top \in F$ and $a \wedge b \in F$ if and only if $a, b \in F$. We say that a filter F is *open* if $f \in F$ implies that $[\Sigma^*]f \in F$.

Theorem 2.1. *There is a bijection between congruences on a modal μ -algebra \mathcal{A} and open filters of \mathcal{A} .*

The bijection is as follows: a congruence $\sim \subseteq A \times A$ is sent to the filter F_{\sim} defined as:

$$x \in F_{\sim} \text{ iff } x \sim \top.$$

The filter F_{\sim} is open, since if $x \sim \top$, then $[\Sigma^*]x \sim [\Sigma^*]\top = \top$. Conversely, given an open filter F , we define the relation \sim_F as follows:

$$a \sim_F b \text{ iff } a \leftrightarrow b \in F.$$

This is an equivalence relation, as it is usual in the context of Boolean algebras. To prove that this is also a congruence we use the following lemma:

Lemma 2.2. *For all $c \in A$, $t \in \mathcal{T}(Y)$ and $a, b \in A^Y$, if for all $x \in Y$ $[\Sigma^*]c \leq a_x \leftrightarrow b_x$, then $[\Sigma^*]c \leq t(a) \leftrightarrow t(b)$ as well.*

A filter F is *principal* if it is of the form $\{b \mid a \leq b\}$ for some a ; in this case we say that F is generated by a and write $F = F_a$. Remark that a principal filter F_a is open if and only if it is generated by some element of the form $[\Sigma^*]b$. Indeed, if F_a is open, then $a = [\Sigma^*]a$ since $[\Sigma^*]a \leq a$ and $[\Sigma^*]a \in F_a$. On the other hand, a filter of the form $F_{[\Sigma^*]b}$ is open since $[\Sigma^*]b$ is a fixed point of $[\Sigma^*]$.

Let now \sim be the principal congruence generated by the pair (a, b) . From the above discussion, it follows that $c \sim d$ if and only if $c \leftrightarrow d \in F_{[\Sigma^*](a \leftrightarrow b)}$, i.e. if and only if

$$[\Sigma^*](a \leftrightarrow b) \leq c \leftrightarrow d.$$

Thus, we have observed that:

Corollary 2.3. *The variety of modal μ -algebras has equationally definable principal congruences.*

See [1], for a general study of varieties with this property. As noted there, this property corresponds to a deduction theorem for Kozen's axiomatization of the Boolean modal μ -calculus. Indeed, write $\Gamma \vdash p$ if it is possible to derive the proposition p by adding to the set of axioms all the propositions in Γ . Then, from $\Gamma \cup \{q\} \vdash p$, it follows that $\Gamma \vdash [\Sigma^*]q \rightarrow p$.

As $F_{[\Sigma^*](a \wedge b)}$ is the least filter containing both $F_{[\Sigma^*]a}$ and $F_{[\Sigma^*]b}$, we can argue that every finitely generated congruence is principal. It follows that if \sim is the congruence generated by the pairs (a_i, b_i) , $i = 1, \dots, n$, then $[c] \leq [d]$ in \mathcal{A}/\sim if and only if $[\Sigma^*](\bigwedge_{i=1, \dots, n} a_i \leftrightarrow b_i) \leq c \leftrightarrow d$ in \mathcal{A} . Thus:

Corollary 2.4. *If \sim is a finitely generated congruence on \mathcal{A} , then the order relation of \mathcal{A}/\sim is decidable, provided the one of \mathcal{A} is decidable.*

Since it is easily argued that $[\Sigma^*]\perp = \perp$ and that the greatest open filter contained in both $F_{[\Sigma^*]a}$ and $F_{[\Sigma^*]b}$ is the principal filter generated by $[\Sigma^*](\bigvee([\Sigma^*]a \vee [\Sigma^*]b))$, it follows that:

Corollary 2.5. *The set of finitely generated congruences of \mathcal{A} is a sublattice of the lattice of congruences on \mathcal{A} . Moreover it is a dual Heyting algebra.*

The last statement can be seen as follows. The signature of modal μ -algebras extends the one of Boolean algebras, thus the variety of modal μ -algebras is congruence distributive. Moreover it was shown in [1] that in a variety with equationally definable principal congruences the lattice of finitely generated congruences of an algebra is a dual Brouwerian semilattice.

3 Subdirectly Irreducible Algebras

Recall that an algebra is *subdirectly irreducible* if it has a least proper congruence. Subdirectly irreducible algebras are interesting as any algebra can be represented as a subdirect product of subdirectly irreducible algebras. The following proposition is easily derived:

Proposition 3.1. *A modal μ -algebra \mathcal{A} is subdirectly irreducible if and only if there exists an element $a \in A$, distinct from \top , such that if $b \in A$ and $b \neq \top$ then $[\Sigma^*]b \leq a$.*

A frame \mathcal{F} is a pair $\langle X, \{R_\sigma\}_{\sigma \in \Sigma} \rangle$, where X is a set and, for each $\sigma \in \Sigma$, R_σ is a relation on X . The powerset $\mathcal{P}(X)$, with the modal operators

$$[\sigma]S = \{x \in X \mid \forall y \in X (xR_\sigma y \Rightarrow y \in S)\},$$

is then a modal μ -algebra. The graph of a frame \mathcal{F} is the pair $\langle X, \bigcup_{\sigma \in \Sigma} R_\sigma \rangle$. A *source* in \mathcal{F} is an element $x \in X$ such that every $y \in X$ is reachable from x in the graph of \mathcal{F} . A frame is *connected* if every element is a source.

Proposition 3.2. *The algebra of a frame is subdirectly irreducible if and only if the frame has a source. The algebra of a frame is simple if and only if the frame is connected.*

The above statements are easily derived when the characterization of congruences is dualized in terms of ideals. Observe that the analogous statements fail for modal algebras. They are true for modal μ -algebras since reachability is algebraically expressible.

Corollary 3.3. *The variety of modal μ -algebras is not residually small.*

See [8] for the definition of a residually small variety. To prove the corollary it is enough to construct connected frames of arbitrary cardinality.

4 Congruence Extension and Amalgamation

Among the consequences of corollary 2.3 is that the variety of modal μ -algebras has the *congruence extension property*. This property also holds for modal algebras and states that if (i, \mathcal{B}) is an extension of a modal μ -algebra \mathcal{A} and (p, \mathcal{C}) is an homomorphic image of \mathcal{A} , then it is possible to find an extension j of \mathcal{C} and an homomorphic image q of \mathcal{B} such that $q \circ i = j \circ p$:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathcal{B} \\ \downarrow p & & \downarrow q \\ \mathcal{C} & \xrightarrow{j} & \mathcal{D} \end{array}$$

Using the finite model theorem for the modal μ -calculus, the fact that the variety of modal μ -algebras has equationally definable principal congruences, and the congruence extension property, the following fact can easily be proved:

Proposition 4.1. *Let \mathcal{V} be the variety of modal μ -algebras with additional constants subject to a finite number of given algebraic relations. Then \mathcal{V} is residually finite, that is, if an equation holds in every finite member of \mathcal{V} , then it holds in every member of \mathcal{V} .*

The congruence extension property equivalently states that the pushout of a monomorphism along a quotient is again a monomorphism. As monomorphisms and homomorphic images form a factorization system in any category of algebras, it becomes interesting to investigate the analogous property:

Definition 4.2. *A variety of algebras \mathcal{V} has the amalgamation property, if for any algebra \mathcal{A} in \mathcal{V} and any two extensions (i_1, \mathcal{B}_1) and (i_2, \mathcal{B}_2) of \mathcal{A} , there exists extensions j_1, j_2 such that*

$$j_1 \circ i_1 = j_2 \circ i_2:$$

$$\begin{array}{ccc}
 A & \xrightarrow{i_2} & B_1 \\
 \downarrow i_1 & & \downarrow j_2 \\
 B_1 & \xrightarrow{j_1} & C
 \end{array}$$

Equivalently, the amalgamation property states that the pushout of a monomorphism along a monomorphism is again a monomorphism. The congruence extension property and the amalgamation property together imply the *transferability property* which states that the pushout of a monomorphism along any homomorphism is again a monomorphism.

The characterization of congruences given here is of help in relating amalgamation to interpolation, a property of a syntactical nature. This has been done for modal algebras and modal logic [6] and can be generalized to modal μ -algebras, as follows.

Definition 4.3. *A variety of modal μ -algebras \mathcal{V} has the strong amalgamation property if it has the amalgamation property as in 4.2 and moreover there exists $a \in A$ such that $b_1 \leq i_1(a)$ and $i_2(a) \leq b_2$, whenever $b_1 \in B_1$, $b_2 \in B_2$ and $j_1(b_1) \leq j_2(b_2)$.*

In the following definition $F_{\mathcal{V}}(X)$ denotes the modal μ -algebra in the variety \mathcal{V} freely generated by the set X .

Definition 4.4. *A variety of modal μ -algebras \mathcal{V} has the interpolation property if and only if, whenever $p \in F_{\mathcal{V}}(X)$, $q \in F_{\mathcal{V}}(Y)$ and $p \leq q$ in $F_{\mathcal{V}}(X \cup Y)$, then there exists $r \in F_{\mathcal{V}}(X \cap Y)$ such that $p \leq r$ in $F_{\mathcal{V}}(X)$ and $r \leq q$ in $F_{\mathcal{V}}(Y)$.*

Thus:

Proposition 4.5. *A variety of modal μ -algebras has the interpolation property if and only if it has the strong amalgamation property.*

Recently the interpolation property of modal logic has been extended to the propositional Boolean modal μ -calculus [2]. It follows that the variety of modal μ -algebras has the interpolation property, as we have defined it in 4.4, and therefore the amalgamation property.

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