



**HAL**  
open science

# Zero degree of derivation for multi-lump solutions to the KPI equation

Pierre Gaillard

► **To cite this version:**

Pierre Gaillard. Zero degree of derivation for multi-lump solutions to the KPI equation. 2023. hal-03928824

**HAL Id: hal-03928824**

**<https://hal.science/hal-03928824>**

Preprint submitted on 7 Jan 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Zero degree of derivation for multi-lump solutions to the KPI equation

**Pierre Gaillard,**

Université de Bourgogne Franche Comté,  
Institut de mathématiques de Bourgogne,  
9 avenue Alain Savary BP 47870  
21078 Dijon Cedex, France :  
E-mail : Pierre.Gaillard@u-bourgogne.fr

## Abstract

We construct solutions to the Kadomtsev-Petviashvili equation (KPI) by using an extended Darboux transform. From elementary functions we give a method that provides different types of solutions in terms of wronskians of order  $N$ . For a given order, these solutions depend on the degree of summation and the degree of derivation of the generating functions.

In this study, we restrict ourselves to the case where the degree of derivation is equal to 0. In this case, we obtain multi-lump solutions and we study the patterns of their modulus in the plane  $(x, y)$  and their evolution according time and parameters.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1 Introduction

We consider the Kadomtsev-Petviashvili equation (KPI) in the following form

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

where subscripts  $x$ ,  $y$  and  $t$  denote partial derivatives.

Kadomtsev and Petviashvili [1] proposed first this equation in 1970. This equation is a universal model describing weakly nonlinear waves in media with dispersion of velocity. For example, it is used for surface and internal water waves [2, 3], and in nonlinear optics [4]. Zakharov extended the inverse scattering transform (IST) to this KPI equation, and obtained several exact solutions.

A lump solution of the KPI equation was first constructed by Petviashvili [5] in 1976. The first rational solutions were found in 1977 by Manakov, Zakharov, Bordag and Matveev [6]. Krichever [7, 8] showed that the dynamics of the lumps

of the KPI equation is governed by the Calogero-Moser system. Other researches were led and more general solutions of the KPI equation were obtained. We can mention the following works of Satsuma and Ablowitz in 1979 [9], Matveev in 1979 [10], Freeman and Nimmo in 1983 [11, 12], Ablowitz and Villarroel [13, 14] in 1997-1999, Biondini and Kodama [15, 16, 17] in 2003-2007.

We construct solutions of the KPI equation in terms of a second derivative with respect to  $x$  of a logarithm of a wronskian of order  $N$ . For any order  $N$  the solutions depend in the degree of summation  $S$  and the degree of derivation  $D$ . So these  $N$  order solutions depend in general on  $N(S(D+5)+1)$  real parameters. In the following, we restrict ourselves to the case where the order  $N$  is equal to 1 and the degree of derivation  $D$  equal to 0. In this case, we get lines of multi lump solutions to the KPI equation.

We construct some explicit solutions of order 1, depending on several real parameters, and give the representations of their modulus in the plane of the coordinates  $(x, y)$  according to parameters and time  $t$ .

## 2 Multiparametric solutions of order $N$ to the KPI equation

We consider the Kadomtsev-Petviashvili (KPI) equation

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

We will use in the following the wronskian of order  $N$  of the functions  $f_1, \dots, f_N$  which is the determinant denoted  $W(f_1, \dots, f_N)$ , defined by  $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$ ,  $\partial_x^i$  being the partial derivative of order  $i$  with respect to  $x$  and  $\partial_x^0 f_j$  being the function  $f_j$ .

We consider  $a_j, b_{j_s d}, d_{j_s}, e_{j_s}, g_{j_s}, h_{j_s}$ , arbitrary real numbers with  $c_{j_s} = g_{j_s} + ih_{j_s}$ .

We consider the following elementary functions  $f_{j_s}$  defined by

$$f_{j_s}(x, y, t) = e^{ic_{j_s}x - ic_{j_s}^2 y + ic_{j_s}^3 t + d_{j_s} + ie_{j_s}} = e^{i(g_{j_s} + ih_{j_s})x - i(g_{j_s} + ih_{j_s})^2 y + i(g_{j_s} + ih_{j_s})^3 t + d_{j_s} + ie_{j_s}}. \quad (2)$$

Then, we have the following statement :

**Theorem 2.1** *Let  $\psi_j$ , be the functions defined by*

$$\psi_j(x, y, t) = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j_s d} \partial_{c_{j_s}}^d e^{ic_{j_s}x - ic_{j_s}^2 y + ic_{j_s}^3 t + d_{j_s} + ie_{j_s}} = a_j + \sum_{s=1}^S \sum_{d=0}^D b_{j_s d} \partial_{c_{j_s}}^d f_{j_s}(x, y, t). \quad (3)$$

*Then the function  $u$  defined by*

$$u(x, y, t) = -2\partial_x^2 \ln(W(\psi_1(x, y, t), \dots, \psi_N(x, y, t))) \quad (4)$$

is a solution to the (KPI) equation (1) depending on real  $N(S(D + 5) + 1)$  parameters  $a_j, b_{j,s,d}, g_{j,s}, h_{j,s}, d_{j,s}, e_{j,s}, 1 \leq j \leq N, 1 \leq s \leq S, 0 \leq d \leq D$ .

We will call the order  $N$  of the wronskian, the order of the solution. The number  $S$  of terms of the summation will be call the degree of the summation of the solution and  $D$  the degree of derivation of the solution.

For each order  $N$  of the solution there is a lot of choices concerning the degrees of the summation and derivation.

In the following, we restrict ourselves to the study where  $D = 0$ .

### 3 Solutions of order 1 with a degree of summation equal to 1 ( $S = 1$ ) depending on 6 real parameters

In this case, we observe lines of lumps whose intensities and directions depend on 6 real parameters. These structures are more sensitive to  $g_1$  and  $h_1$  parameters than to others. If  $a_1$  or  $b_1$  is equal to 0, we get the trivial solution equal to 0. Solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

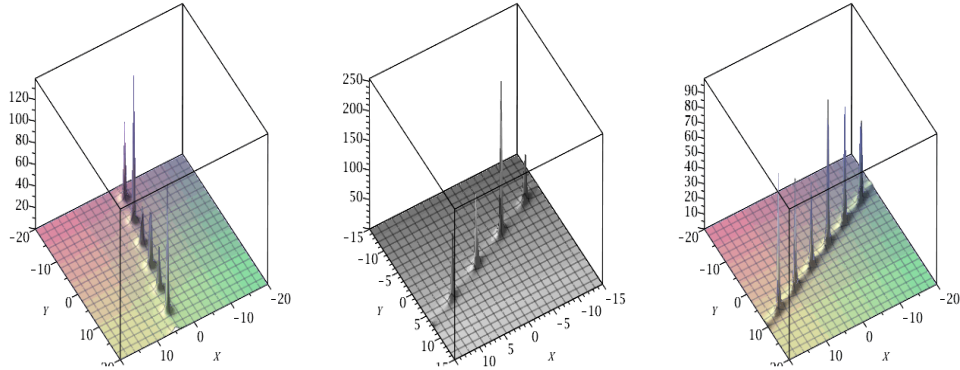
with

$$n(x, y, t) = b_{1,1,0}(ig_{1,1} - h_{1,1})^2 \times e^{ig_{1,1}x - h_{1,1}x - iyg_{1,1}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + itg_{1,1}^3 - 3tg_{1,1}^2h_{1,1} - 3itg_{1,1}h_{1,1}^2 + th_{1,1}^3 + d_{1,1} + ie_{1,1}} a_1$$

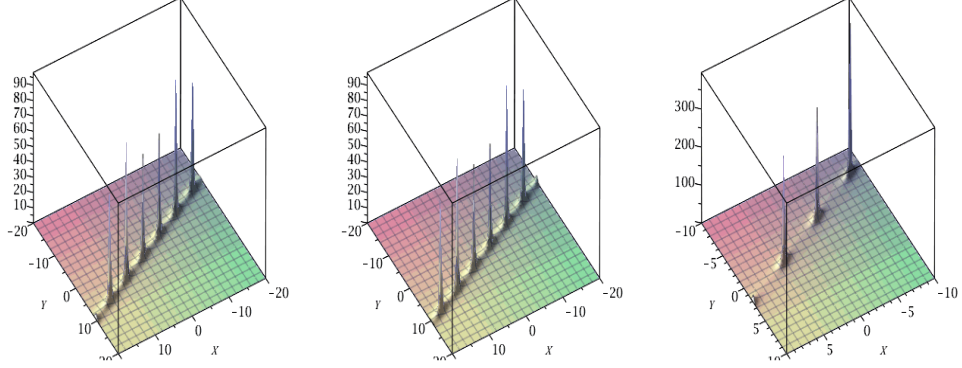
and

$$d(x, y, t) = b_{1,1,0} \times e^{-h_{1,1}x + ig_{1,1}x - 3tg_{1,1}^2h_{1,1} - 3itg_{1,1}h_{1,1}^2 + itg_{1,1}^3 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + th_{1,1}^3 - iyg_{1,1}^2 + d_{1,1} + ie_{1,1}} + a_1$$

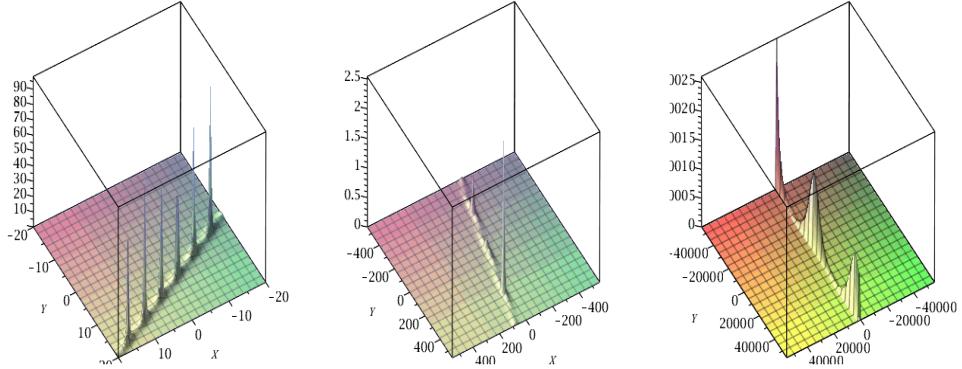
We give some figures in the  $(x, y)$  plane of coordinates. If  $a_1 = 0$  or  $b_{110} = 0$ , we get the trivial solution  $u = 0$ . Otherwise, we remark that parameters  $a_1$  or  $b_{110}$  are negligible and plays very little role in the structure of solutions. So if these parameters are not equal to 0 we fix them equal to 1.



**Figure 1.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 0.1$ ,  $h_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; in the center for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 1$ ,  $h_{1,1} = 0.9$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; on the right for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 1$ ,  $h_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 0, 1$ .



**Figure 2.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 1$ ,  $h_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 1$ ,  $h_{1,1} = 0.9$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; on the right for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 1$ ,  $h_{1,1} = 1$ ,  $d_{1,1} = 10$ ,  $e_{1,1} = 10$ .



**Figure 3.** Solution of order 1 to KPI, on the left for  $t = 10$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 0.1$ ,  $h_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; in the center for  $t = 10^2$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 1$ ,  $h_{1,1} = 0.9$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 1$ ; on the right for  $t = 10^3$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $g_{1,1} = 1$ ,  $h_{1,1} = 1$ ,  $d_{1,1} = 1$ ,  $e_{1,1} = 0, 1$ .

## 4 Solutions of order 1 with a degree of summation equal to 2 ( $S = 2$ ) depending on 11 real parameters

In this case, we observe lines of lumps but contrary to the order 1, with a bifurcation in two branches whose intensities and directions depend on 11 real parameters. These structures are more sensitive to  $g_i$  and  $h_i$  parameters than to others. A generic solution can be considered as two lumps chains meeting in a double point; we get also a whole set of degenerated configurations. In this case, we take  $N = 1$ ,  $S = 2$  and the solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

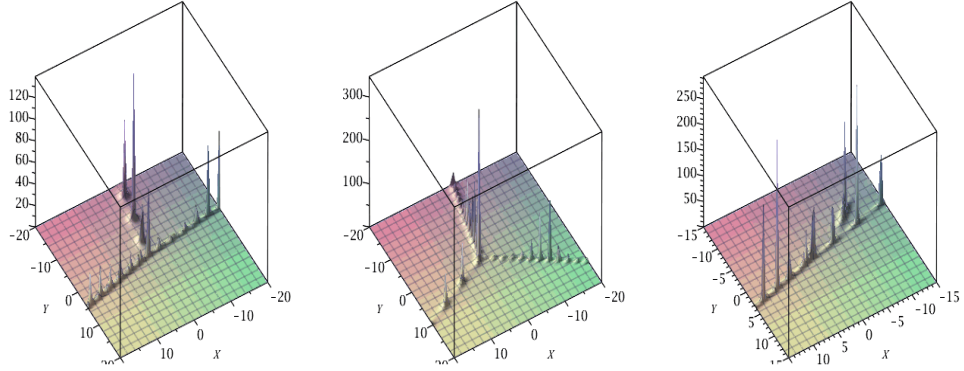
$$\begin{aligned} n(x, y, t) = & -b_{1,1,0} e^{ig_{1,1}x - h_{1,1}x - iyg_{1,1}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + itg_{1,1}^3 - 3tg_{1,1}^2h_{1,1} - 3itg_{1,1}h_{1,1}^2 + th_{1,1}^3 + d_{1,1} + ie_{1,1}} g_{1,1}^2 a_1 \\ & + b_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} h_{1,1}^2 b_{1,2,0} \\ & - b_{1,2,0} e^{ig_{1,2}x - h_{1,2}x - iyg_{1,2}^2 + 2yg_{1,2}h_{1,2} + iyh_{1,2}^2 + itg_{1,2}^3 - 3tg_{1,2}^2h_{1,2} - 3itg_{1,2}h_{1,2}^2 + th_{1,2}^3 + d_{1,2} + ie_{1,2}} g_{1,2}^2 a_1 \\ & + b_{1,2,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} h_{1,1}^2 b_{1,1,0} \\ & - 2ib_{1,2,0} e^{ig_{1,2}x - h_{1,2}x - iyg_{1,2}^2 + 2yg_{1,2}h_{1,2} + iyh_{1,2}^2 + itg_{1,2}^3 - 3tg_{1,2}^2h_{1,2} - 3itg_{1,2}h_{1,2}^2 + th_{1,2}^3 + d_{1,2} + ie_{1,2}} g_{1,2}h_{1,2}a_1 \\ & + 2ib_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} h_{1,1} b_{1,2,0} g_{1,2} \\ & + 2b_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} g_{1,1} b_{1,2,0} g_{1,2} \\ & - 2ib_{1,1,0} e^{ig_{1,1}x - h_{1,1}x - iyg_{1,1}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + itg_{1,1}^3 - 3tg_{1,1}^2h_{1,1} - 3itg_{1,1}h_{1,1}^2 + th_{1,1}^3 + d_{1,1} + ie_{1,1}} g_{1,1}h_{1,1}a_1 \\ & - 2ib_{1,2,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} g_{1,2}h_{1,2}b_{1,1,0} \\ & - b_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} g_{1,1}^2 b_{1,2,0} \\ & + 2ib_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} g_{1,1} b_{1,2,0} h_{1,2} \\ & - b_{1,2,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} g_{1,1}^2 b_{1,1,0} \\ & - 2ib_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} g_{1,1} h_{1,1} b_{1,2,0} \\ & - 2b_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}^2 - iyg_{1,2}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}^2 + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2h_{1,1}} \\ & \times e^{+iyh_{1,2}^2 + th_{1,1}^3 + itg_{1,2}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1} - 3tg_{1,2}^2h_{1,2} + itg_{1,1}^3 + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2} h_{1,1} b_{1,2,0} h_{1,2} \\ & + b_{1,1,0} e^{ig_{1,1}x - h_{1,1}x - iyg_{1,1}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + itg_{1,1}^3 - 3tg_{1,1}^2h_{1,1} - 3itg_{1,1}h_{1,1}^2 + th_{1,1}^3 + d_{1,1} + ie_{1,1}} h_{1,1}^2 a_1 \\ & + b_{1,2,0} e^{ig_{1,2}x - h_{1,2}x - iyg_{1,2}^2 + 2yg_{1,2}h_{1,2} + iyh_{1,2}^2 + itg_{1,2}^3 - 3tg_{1,2}^2h_{1,2} - 3itg_{1,2}h_{1,2}^2 + th_{1,2}^3 + d_{1,2} + ie_{1,2}} h_{1,2}^2 a_1 \end{aligned}$$

and

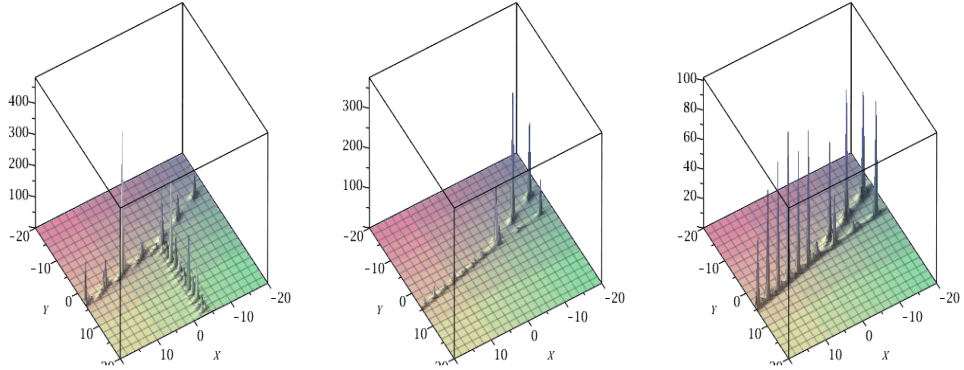
$$d(x, y, t) = b_{1,1,0}e^{-h_{1,1}x+ig_{1,1}x-3tg_{1,1}^2h_{1,1}-3itg_{1,1}h_{1,1}^2+itg_{1,1}^3+2yg_{1,1}h_{1,1}+iyh_{1,1}^2+th_{1,1}^3-iyg_{1,1}^2+d_{1,1}+ie_{1,1}} + b_{1,2,0}e^{-h_{1,2}x+ig_{1,2}x-3tg_{1,2}^2h_{1,2}-3itg_{1,2}h_{1,2}^2+itg_{1,2}^3+2yg_{1,2}h_{1,2}+iyh_{1,2}^2+th_{1,2}^3-iyg_{1,2}^2+d_{1,2}+ie_{1,2}} + a_1$$

We give some figures in the  $(x, y)$  plane of coordinates.

There is a formation of multi-lines of lumps meeting in a same point which varies according to the values of the parameters. A maximum of three lines has been observed.

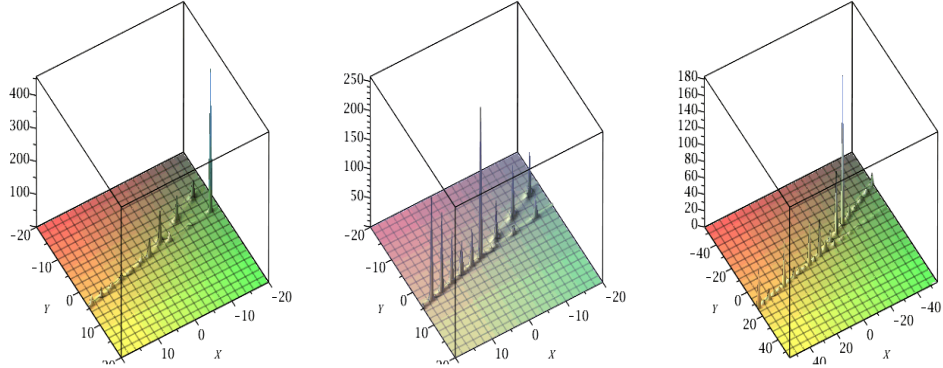


**Figure 4.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $g_{1,1} = 0.1$ ,  $g_{1,2} = 2$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; in the center for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 0.1$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; on the right for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $h_{1,1} = 0.5$ ,  $h_{1,2} = 2$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ .

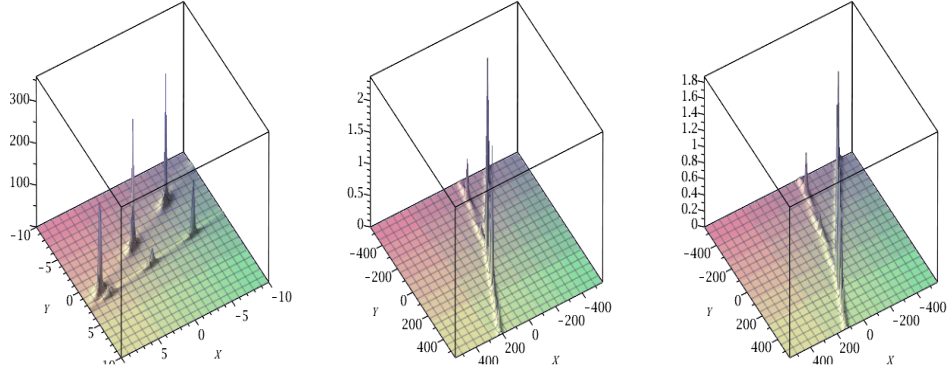


**Figure 5.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 0.5$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; in the center for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $d_{1,1} = 0.1$ ,  $d_{1,2} = 2$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ; on the right for  $t = 0$ ,

$$a_1 = 1, b_{1,1,0} = 1, b_{1,2,0} = 2, g_{1,1} = 1, g_{1,2} = 2, h_{1,1} = 1, h_{1,2} = 2, d_{1,1} = 1, \\ d_{1,2} = 0.5, e_{1,1} = 1, e_{1,2} = 2.$$



**Figure 6.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1, b_{1,1,0} = 1, b_{1,2,0} = 2, g_{1,1} = 1, g_{1,2} = 2, h_{1,1} = 1, h_{1,2} = 2, d_{1,1} = 1, d_{1,2} = 2, e_{1,1} = 0.1, e_{1,2} = 2$ ; in the center for  $t = 0$ ,  $a_1 = 1, b_{1,1,0} = 1, b_{1,2,0} = 2, g_{1,1} = 1, g_{1,2} = 2, h_{1,1} = 1, h_{1,2} = 2, d_{1,1} = 1, d_{1,2} = 2, e_{1,1} = 1, e_{1,2} = 0.1$ ; on the right for  $t = 0$ ,  $a_1 = 1, b_{1,1,0} = 1, b_{1,2,0} = 2, g_{1,1} = 1, g_{1,2} = 2, h_{1,1} = 1, h_{1,2} = 2, d_{1,1} = 1, d_{1,2} = 2, e_{1,1} = 1, e_{1,2} = 2$ .



**Figure 7.** Solution of order 1 to KPI, on the left for  $t = 1, a_1 = 10^2, b_{1,1,0} = 1, b_{1,2,0} = 2, g_{1,1} = 1, g_{1,2} = 2, h_{1,1} = 1, h_{1,2} = 2, d_{1,1} = 10, d_{1,2} = 20, e_{1,1} = 10, e_{1,2} = 20$ ; in the center for  $t = 10^2, a_1 = 10^2, b_{1,1,0} = 10, b_{1,2,0} = 20, g_{1,1} = 0.1, g_{1,2} = 0.2, h_{1,1} = 0.1, h_{1,2} = 0.2, d_{1,1} = 1, d_{1,2} = 2, e_{1,1} = 1, e_{1,2} = 2$ ; on the right for  $t = 10^3, a_1 = 10^3, b_{1,1,0} = 10, b_{1,2,0} = 20, g_{1,1} = 0.1, g_{1,2} = 0.2, h_{1,1} = 0.1, h_{1,2} = 0.2, d_{1,1} = 1, d_{1,2} = 2, e_{1,1} = 1, e_{1,2} = 2$ .

**Remark**

**Sub-particle case of solutions of order 1 without derivative ( $D = 0$ ) with a degree of summation equal to 2 ( $S=2$ )**

We can choose for the functions  $f_j$ , the functions defined by  $f_j = e^{ic_jx - ic_j^2y + ic_j^3t + ie_j}$



with  $c_j, e_j$  real numbers for  $j = 1$  and  $j = 2$ ,  $a_1 = 0$  and  $b_{110} = b_{120} = b_1$ . then the function  $\psi_1(x, y, t)$  can be written as

$$\begin{aligned}\psi_1(x, y, t) &= b_1(e^{ic_1x-ic_1^2y+ic_1^3t+ie_1} + e^{ic_2x-ic_2^2y+ic_2^3t+ie_2}) \\ &= 2b_1 e^{i(c_2+c_1)x-i(c_2^2+c_1^2)y+(c_2^3+ic_1^3)t+i(e_2+e_1)} \times \cosh(i(c_2-c_1)x-i(c_2^2-c_1^2)y+i(c_2^3-c_1^3)t+i(e_2-e_1))\end{aligned}$$

So the solution to the KPI equation can be written as

$$a(x, y, t) = -2(\ln \psi_1(x, y, t)) = \frac{-2((c_2 - c_1)^2)}{\cosh(i(c_2 - c_1)x - i(c_2^2 - c_1^2)y + i(c_2^3 - c_1^3)t + i(e_2 - e_1))}$$

## 5 Solutions of order 1 with a degree of summation equal to 3 ( $S = 3$ ) depending on 16 real parameters

In this case, we observe lines of lumps but contrary to order 1 and 2, with a bifurcation in three branches whose intensities and directions depend on the 16 real parameters. These structures are more sensitive to  $g_i$  and  $h_i$  parameters than to others. A generic solution can be considered as three lumps chains meeting in a triple point; we get also a whole set of degenerated configurations. In this section, we take  $N$  equal to 1 and  $S = 3$ . The general expression with all the parameters being too complex is given in appendix. We give here only this with some parameters values to simplify. For example we choose :

$$a_1 = 1, b_{1,1,0} = 1, b_{1,2,0} = 2, b_{1,3,0} = 3, g_{1,1} = 1, g_{1,2} = 2, g_{1,3} = 3, h_{1,1} = 1, h_{1,2} = 2, h_{1,3} = 3, d_{1,1} = 1, d_{1,2} = 2, d_{1,3} = 3, e_{1,1} = 1, e_{1,2} = 2, e_{1,3} = 3.$$

So, the solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}.$$

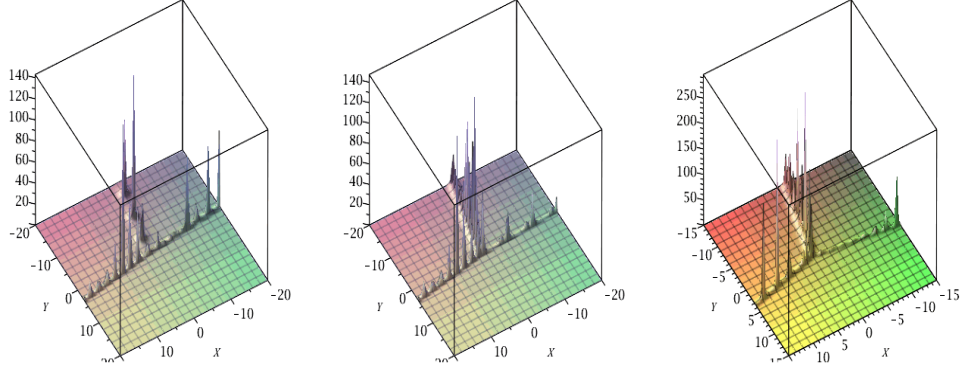
with

$$\begin{aligned}n(x, y, t) &= -2i(12e^{4ix-4x+20y-56it+4+4i-56t} + 27e^{3ix-3x+18y-54it-54t+3+3i} + \\ &e^{ix-x+2y-2it-2t+1+i} + 8e^{2ix-2x+8y-16it-16t+2+2i} + 6e^{5ix-5x+26y-70it+5+5i-70t} + \\ &2e^{3ix-3x+10y-18it+3+3i-18t})\end{aligned}$$

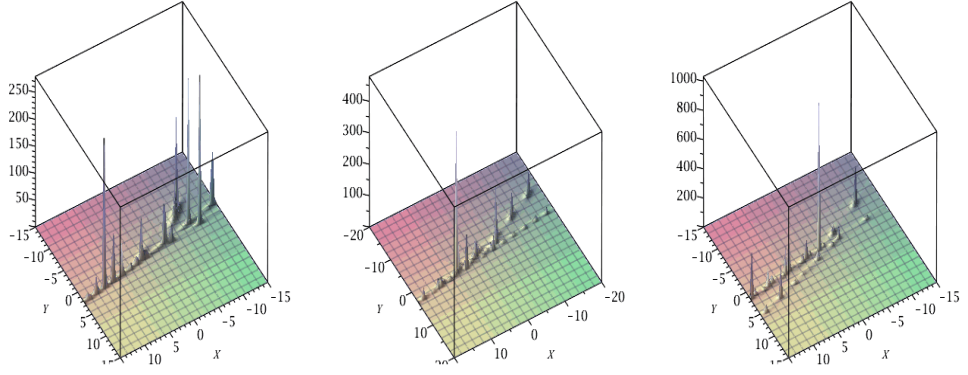
and

$$d(x, y, t) = e^{-x+ix-2t-2it+2y+1+i} + 2e^{-2x+2ix-16t-16it+8y+2+2i} + 3e^{-3x+3ix-54t-54it+18y+3+3i} + 1.$$

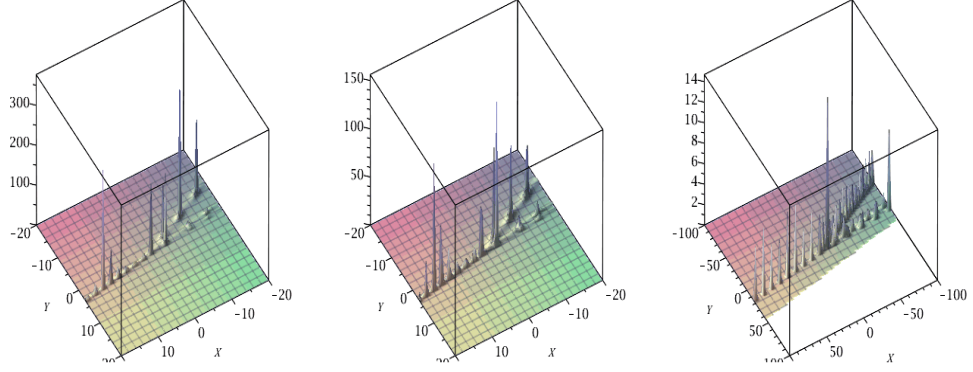
We give some figures in the  $(x, y)$  plane of coordinates.



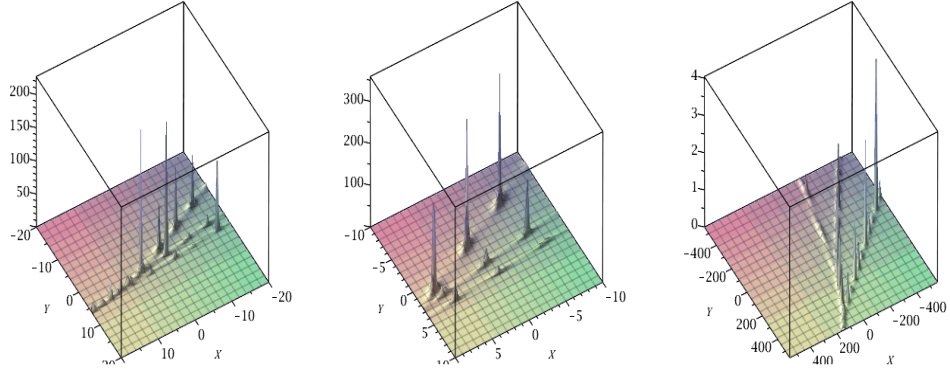
**Figure 8.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 0.1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; in the center for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 0.1$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; on the right for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 0.1$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ .



**Figure 9.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 0.5$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; in the center for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 0.5$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; on the right for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 0.7$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ .



**Figure 10.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 0.5$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 0.1$ ,  $d_{1,2} = 0.2$ ,  $d_{1,3} = 0.3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; in the center for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 0.5$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 0.1$ ,  $e_{1,2} = 0.2$ ,  $e_{1,3} = 0.3$ ; on the right for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 0.7$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ .



**Figure 11.** Solution of order 1 to KPI, on the left for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 0.5$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; in the center for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 0.5$ ,  $h_{1,3} = 3$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ; on the right for  $t = 10^3$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 0.7$ ,  $d_{1,1} = 10$ ,  $d_{1,2} = 20$ ,  $d_{1,3} = 30$ ,  $e_{1,1} = 10$ ,  $e_{1,2} = 20$ ,  $e_{1,3} = 30$ .

## 6 Solutions of order 1 with a degree of summation equal to 4 ( $S = 4$ ) depending on 21 real parameters

In this case, we observe generically lines of lumps, with a bifurcation in four branches whose intensities and directions depend on the 21 real parameters. These structures are more sensitive to  $g_i$  and  $h_i$  parameters than to others. We get also a whole set of degenerated configurations.

In the case  $S = 4$ , the expression of the solutions with all the parameters being too long, we present only one of them with particular values of parameters. For example we choose :

$$a_1 = 1, b_{1,1,0} = 1, b_{1,2,0} = 2, b_{1,3,0} = 3, b_{1,4,0} = 4, g_{1,1} = 1, g_{1,2} = 2, g_{1,3} = 3, g_{1,4} = 4, h_{1,1} = 1, h_{1,2} = 2, h_{1,3} = 3, h_{1,4} = 4, d_{1,1} = 1, d_{1,2} = 2, d_{1,3} = 3, d_{1,4} = 4, e_{1,1} = 1, e_{1,2} = 2, e_{1,3} = 3, e_{1,4} = 4.$$

So, the solution to the KPI equation, with these choices of parameters, can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

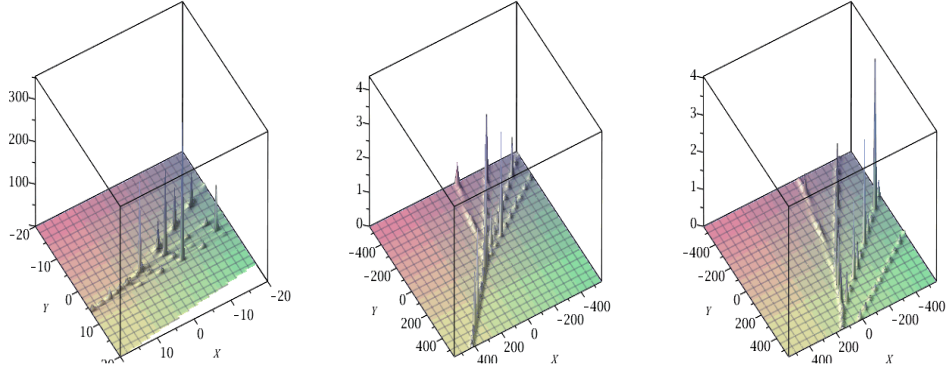
with

$$\begin{aligned} n(x, y, t) = & 2i(27e^{3ix-3x+18y-54it-54t+3+3i} + 12e^{-7x+7ix+7+7i+50y-182it-182t} + \\ & 8e^{2ix-2x+8y-16it-16t+2+2i} + 2e^{-3x+3ix+3+3i+10y-18it-18t} + 125e^{5ix-5x+50y-250it-250t+5+5i} + \\ & 60e^{-8x+8ix+8+8i+68y-304it-304t} + 90e^{-7x+7ix+7+7i+58y-266it-266t} + 6e^{-5x+5ix+5+5i+26y-70it-70t} + \\ & 32e^{-6x+6ix+6+6i+40y-144it-144t} + e^{ix-x+2y-2it-2t+1+i} + 64e^{4ix-4x+32y-128it-128t+4+4i} + \\ & 36e^{-5x+5ix+5+5i+34y-130it-130t} + 12e^{-4x+4ix+4+4i+20y-56it-56t} + 20e^{-9x+9ix+9+9i+82y-378it-378t} + \\ & 80e^{-6x+6ix+6+6i+52y-252it-252t}) \end{aligned}$$

and

$$\begin{aligned} d(x, y, t) = & e^{-x+ix-2t-2it+2y+1+i} + 2e^{-2x+2ix-16t-16it+8y+2+2i} + 3e^{-3x+3ix-54t-54it+18y+3+3i} + \\ & 4e^{-4x+4ix-128t-128it+32y+4+4i} + 5e^{-5x+5ix-250t-250it+50y+5+5i} + 1. \end{aligned}$$

Generically, we get five lines of lumps with degenerate situations. We present some figures in the  $(x, y)$  plane :



**Figure 12.** Solution of order 1 to KPI, on the left for  $t = 1$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $b_{1,4,0} = 4$ ,  $g_{1,1} = 1$ ,  $g_{1,2} = 2$ ,  $g_{1,3} = 3$ ,  $g_{1,4} = 4$ ,  $h_{1,1} = 1$ ,  $h_{1,2} = 2$ ,  $h_{1,3} = 3$ ,  $h_{1,4} = 4$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $d_{1,4} = 4$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ,  $e_{1,4} = 4$ ; in the center for  $t = 10^2$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $b_{1,4,0} = 4$ ,  $g_{1,1} = 0.1$ ,  $g_{1,2} = 0.2$ ,  $g_{1,3} = 0.3$ ,  $g_{1,4} = 0.4$ ,  $h_{1,1} = 0.1$ ,  $h_{1,2} = 0.2$ ,  $h_{1,3} = 0.3$ ,  $h_{1,4} = 0.4$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $d_{1,4} = 4$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ,  $e_{1,4} = 4$ ; on the right for  $t = 10^3$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $b_{1,4,0} = 4$ ,  $g_{1,1} = 0.1$ ,  $g_{1,2} = 0.2$ ,  $g_{1,3} = 0.3$ ,  $g_{1,4} = 0.4$ ,  $h_{1,1} = 0.1$ ,  $h_{1,2} = 0.2$ ,  $h_{1,3} = 0.3$ ,  $h_{1,4} = 0.4$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $d_{1,4} = 4$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ,  $e_{1,4} = 4$ .

## 7 Solutions of order 1 with a degree of summation equal to 5 (S=5) depending on 26 real parameters

In this case, we observe generically five lines of lumps and a whole set of degenerated configurations. These structures are more sensitive to  $g_i$  and  $h_i$  parameters than to others.

In the case  $S = 5$ , the expression of the solutions with all the parameters being too long, we present only one of them with particular values of parameters. For example we choose :

$$a_1 = 1, b_{1,1,0} = 1, b_{1,2,0} = 2, b_{1,3,0} = 3, b_{1,4,0} = 4, b_{1,5,0} = 5, g_{1,1} = 1, g_{1,2} = 2, g_{1,3} = 3, g_{1,4} = 4, g_{1,5} = 5, h_{1,1} = 1, h_{1,2} = 2, h_{1,3} = 3, h_{1,4} = 4, h_{1,5} = 5, d_{1,1} = 1, d_{1,2} = 2, d_{1,3} = 3, d_{1,4} = 4, d_{1,5} = 4, e_{1,1} = 1, e_{1,2} = 2, e_{1,3} = 3, e_{1,4} = 4, e_{1,5} = 5.$$

So, in this case, the solution to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

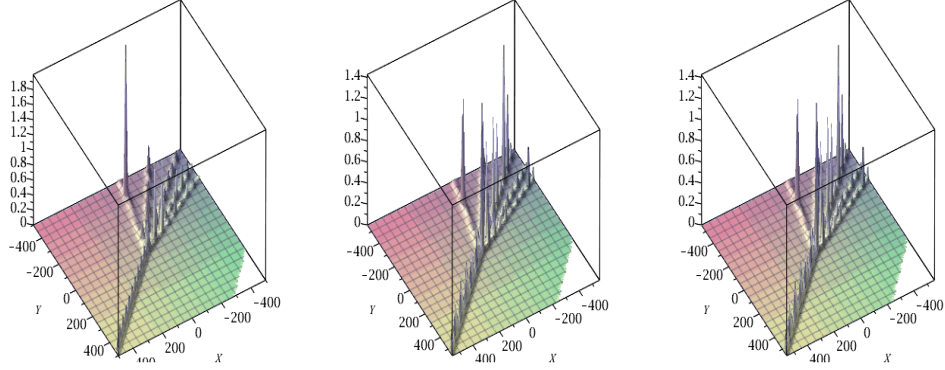
with

$$n(x, y, t) = -2i(27e^{3ix-3x+18y-54it-54t+3+3i} + 8e^{2ix-2x+8y-16it-16t+2+2i} + 6e^{-5x+5ix+5+5i+26y-70it-70t} + 32e^{-6x+6ix+6+6i+40y-144it-144t} + 64e^{4ix-4x+32y-128it-128t+4+4i} + 12e^{-4x+4ix+4+4i+20y-56it-56t} + 12e^{-7x+7ix+7+7i+50y-182it-182t} + e^{ix-x+2y-2it-2t+1+i} + 2e^{-3x+3ix+3+3i+10y-18it-18t} + 36e^{-5x+5ix+5+5i+34y-130it-130t})$$

and

$$d(x, y, t) = e^{-x+ix-2t-2it+2y+1+i} + 2e^{-2x+2ix-16t-16it+8y+2+2i} + 3e^{-3x+3ix-54t-54it+18y+3+3i} + 4e^{-4x+4ix-128t-128it+32y+4+4i} + 1.$$

Generically, we get four lines of lumps with degenerate situations. We present some figures in the  $(x, y)$  plane :



**Figure 13.** Solution of order 1 to KPI, on the left for  $t = 0$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $b_{1,4,0} = 4$ ,  $b_{1,5,0} = 5$ ,  $g_{1,1} = 0.1$ ,  $g_{1,2} = 0.2$ ,  $g_{1,3} = 0.3$ ,  $g_{1,4} = 0.4$ ,  $g_{1,5} = 0.5$ ,  $h_{1,1} = 0.1$ ,  $h_{1,2} = 0.2$ ,  $h_{1,3} = 0.3$ ,  $h_{1,4} = 0.4$ ,  $h_{1,5} = 0.5$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $d_{1,4} = 4$ ,  $d_{1,5} = 5$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ,  $e_{1,4} = 4$ ,  $e_{1,5} = 5$ ; in the center for  $t = 10$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $b_{1,4,0} = 4$ ,  $b_{1,5,0} = 5$ ,  $g_{1,1} = 0.1$ ,  $g_{1,2} = 0.2$ ,  $g_{1,3} = 0.3$ ,  $g_{1,4} = 0.4$ ,  $g_{1,5} = 0.5$ ,  $h_{1,1} = 0.1$ ,  $h_{1,2} = 0.2$ ,  $h_{1,3} = 0.3$ ,  $h_{1,4} = 0.4$ ,  $h_{1,5} = 0.5$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $d_{1,4} = 4$ ,  $d_{1,5} = 5$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ,  $e_{1,4} = 4$ ,  $e_{1,5} = 5$ ; on the right for  $t = 10^3$ ,  $a_1 = 1$ ,  $b_{1,1,0} = 1$ ,  $b_{1,2,0} = 2$ ,  $b_{1,3,0} = 3$ ,  $b_{1,4,0} = 4$ ,  $b_{1,5,0} = 5$ ,  $g_{1,1} = 0.1$ ,  $g_{1,2} = 0.2$ ,  $g_{1,3} = 0.3$ ,  $g_{1,4} = 0.4$ ,  $g_{1,5} = 0.5$ ,  $h_{1,1} = 0.1$ ,  $h_{1,2} = 0.2$ ,  $h_{1,3} = 0.3$ ,  $h_{1,4} = 0.4$ ,  $h_{1,5} = 0.5$ ,  $d_{1,1} = 1$ ,  $d_{1,2} = 2$ ,  $d_{1,3} = 3$ ,  $d_{1,4} = 4$ ,  $d_{1,5} = 5$ ,  $e_{1,1} = 1$ ,  $e_{1,2} = 2$ ,  $e_{1,3} = 3$ ,  $e_{1,4} = 4$ ,  $e_{1,5} = 5$ .

## 8 Conclusion

Using an extended Darboux transform, we succeed to construct multi-parametric solutions to the KPI equation. These solutions depend on the order of determinant  $N$ , the degree of summation  $S$  and the degree of derivation  $D$ . In the case general case the solutions to the KPI equation can be expressed as a second derivative with respect to  $x$  of a logarithm of a determinant of order  $N$ , we obtain solutions depending on  $N(S(D + 5) + 1)$  real parameters.

We only give the expressions of the solution in the simple case of order  $N = 1$ . All these solutions are different from these given in another approach given par the present author [50, 51, 52, 54, 56, 62, 64].

In the frame of plasmas, the KP equation is derived [76] to describe certain particular solutions called dust acoustic waves. The KP equation in [76] could be renormalized to use the classical KP equation. The solutions presented in [76] are different from those built in this study. Multi-lum solutions with 0 degree derivation are exact theoretical solutions to the KP equation. It will be interesting to observe them in plasmas.

It would important to study in more details this first order with the degree of derivation non equal to 0 and try to classify these solutions.

We postpone to another publication the study of the solutions with a non zero

degree of derivation.

It would be also relevant to study the cases of order greater or equal to 2 and to realize an exhaustive classification of the solutions to the KPI equation.

## References

- [1] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Sov. Phys. Dokl.*, V. 15, N. 6, 539-541, 1970
- [2] M.J. Ablowitz, H. Segur On the evolution of packets of water waves, *J. Fluid Mech.*, V. 92, 691-715, 1979
- [3] M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, 1991
- [4] D.E. Pelinovsky, Y.A. Stepanyants, Y.A. Kivshar, Self-focusing of plane dark solitons in nonlinear defocusing media, *Phys. Rev. E*, V. 51, 5016-5026, 1995
- [5] V.I. Petviashvili, Equation of an extraordinary soliton, *Plasma Phys.*, V. 2, 469-472, 1976
- [6] S.V. Manakov, V.E. Zakharov, L.A. Bordag, V.B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Letters*, V. 63A, N. 3, 205-206, 1977
- [7] I. Krichever, Rational solutions of the Kadomtcev-Petviashvili equation and integrable systems of n particles on a line, *Funct. Anal. and Appl.*, V. 12, N. 1 : 76-78, 1978
- [8] I. Krichever, S.P. Novikov Holomorphic bundles over Riemann surfaces and the KPI equation, *Funkt. Ana. E Pril.*, V. 12, 41-52, 1979
- [9] J. Satsuma, M.J. Ablowitz, Two-dimensional lumps in nonlinear dispersive systems, *J. Math. Phys.*, V. 20, 1496-1503, 1979
- [10] V.B. Matveev, Darboux transformation and explicit solutions of the Kadomtcev-Petviashvili equation depending on functional parameters, *Letters in Mathematical Physics*, V. 3, 213-216, 1979
- [11] N. C Freeman, J.J.C. Nimmo Rational solutions of the KdV equation in wronskian form, *Phys. Letters*, V. 96 A, N. 9, 443-446, 1983
- [12] N. C Freeman, J.J.C. Nimmo The use of Bäcklund transformations in obtaining N-soliton solutions in wronskian form, *J. Phys. A : Math. Gen.*, V. 17 A, 1415-1424, 1984

- [13] M.J Ablowitz, J. Villarroel, Solutions to the time dependent schrödinger and the Kadomtsev-Petviashvili equations, *Phys. Rev. Lett.*, V. 78, 570 , 1997
- [14] J. Villarroel, M.J Ablowitz, On the discrete spectrum of the nonstationary Schrödinger equation and multipole lumps of the Kadomtsev-Petviashvili I equation, *Commun. Math. Phys.*, V. 207, 1-42, 1999
- [15] G. Biondini, Y. Kodama, On a family of solutions of the Kadomtsev-Petviashvili equation which also satisfy the Toda lattice hierarchy, *J. Phys. A: Math. Gen.*, V. 36, 1051910536, 2003
- [16] Y. Kodama, Young diagrams and N solitons solutions to the KP equation, *J. Phys. A: Math. Gen.*, V. 37, 1116911190, 2004
- [17] G. Biondini, Line Soliton Interactions of the Kadomtsev-Petviashvili Equation, *PRL*, V. 99, 064103-1-4, 2007
- [18] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. **161**, 2002
- [19] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. **68**, 77-90, 2004
- [20] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. **89**, 1-12, 2009
- [21] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. **42**, 404409-1-16, 2009
- [22] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. **185**, 247-258, 2010
- [23] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 435204-1-15, 2011
- [24] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. **13**, N. 2, 71-153, 2012
- [25] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. **54**, 013504-1-32, 2013
- [26] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. **4**, N. 4, 246-266, 2013
- [27] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. **2013**, ID 645752, 1-10, 2013



- [28] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. **7**, N. **45**, 1-6, 2013
- [29] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. **88**, 042903-1-9, 2013
- [30] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. **2013**, 1-111, 2013
- [31] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. **54**, 073519-1-22, 2013
- [32] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. **61**, 365-369, 2014
- [33] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. **89**, 015004-1-7, 2014
- [34] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. **482**, 012016-1-7, 2014
- [35] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. **26**, N. 2, 1550016-1-14, 2014
- [36] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. **5**, 093506-1-12, 2014
- [37] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. **574**, 012031-1-5, 2015
- [38] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. **355**, 293-298, 2015
- [39] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. **379**, 1309-1313, 2015
- [40] P. Gaillard, Other  $2N-2$  parameters solutions to the NLS equation and  $2N+1$  highest amplitude of the modulus of the  $N$ -th order AP breather, *J. Phys. A: Math. Theor.*, V. **48**, 145203-1-23, 2015
- [41] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order  $N$  solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. **4**, N. **5**, 346-364, 2015

- [42] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. **633**, 012106-1-6, 2016
- [43] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. **10**,1-7, 2016
- [44] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. **2**, 136-144, 2016
- [45] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. **367**, 1-5, 2016
- [46] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. **10**, 83-89, 2016
- [47] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016
- [48] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [49] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. **3**, 130-139, 2017
- [50] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on  $2N^2$  parameters, *Int. Jour. of Appl. Sci. And Math.*, V. **4**, N. **3**, 60-70, 2017
- [51] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. **1**, N. **1**, 26-31, 2017
- [52] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. **2**, 92-98, 2017
- [53] P. Gaillard,  $N$ -Order rational solutions to the Johnson equation depending on  $2N - 2$  parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. **9**, 19-37, 2017
- [54] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. **11**, 24-30, 2017
- [55] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. **6**, N. **1**, 14-19, 2018

- [56] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. **196**, N. 2, 1174-1199, 2018
- [57] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. **2018**, 1-18, 2018
- [58] P. Gaillard, Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters, *NHIMP*, V. **2**, N. 4, 53-61, 2018
- [59] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. **1141**, 012102-1-10, 2018
- [60] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. , 109-112, 2019
- [61] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. **2**, N. 4, 1-4, 2019
- [62] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. **3**, N. 2, 44-52, 2020
- [63] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. **100**, 102667-1-9, 2021
- [64] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. **161**, 104059-1-12, 2021
- [65] P. Gaillard, 2N parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. **1**, N. **1**, 1-6, 2021
- [66] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. **3**, N. 3, 14-21, 2021
- [67] P. Gaillard, Multiparametric solutions to the Gardner equation and the degenerate rational case, *Jour. Of Appl. Ana. And Appl.*, V. **11**, N. 4, 2102-2113, 2021
- [68] P. Gaillard, Other families of rational solutions to the KPI equation, *Asi. Res. Jour Of Math.*, V. **17**, N. 6, 27-34, 2021
- [69] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, *Wave Motion*, V. **107**, 102824-1-11, 2021
- [70] P. Gaillard, Multiparametric Rational Solutions of Order N to the KPI Equation and the Explicit Case of Order 3, *Arch. Of cur. Res. Int.*, V. **21**, N . 6, 58-71, 2021

- [71] P. Gaillard, Rational solutions to the KPI equation from particular polynomials, *Wave Motion*, V. **108**, 102828-1-9, 2021
- [72] P. Gaillard, Rogue Waves of the Lakshmanan Porsezian Daniel Equation Depending on Multi-parameters, *As. Jour. Of Adv. Res. And Rep.*, V 16, N. 3, 32-40, 2022
- [73] P. Gaillard, Rational Solutions to the KdV Equation in Terms of Particular Polynomials, *Jour. Of Nonlin. Mod. And Anal.*, V 4, N. 4, 615-626, x2022
- [74] P. Gaillard, Quasi-rational and rational solutions to the defocusing NLS equation, *Jour. Of Math. Sci. And Mod. Res. And Rep.*, V 5, N. 1, 24-34, 2022
- [75] P. Gaillard, Rational solutions to the KdV equation in terms of particular polynomials, *Jour. Of Nonlin. Mod. and Ana.*, V. 4, N. 4, 615-626, 2022
- [76] D. Dorrnian, A. Sabetkar, Dust acoustic solitary waves in a dusty plasma with two kinds of nonthermal ions at different temperatures, *Phys. Of Plas.*, V. 19, 0137021-6, 2012

**Solution to the KPI equation for  $N = 1, S = 3, D = 0$  depending on 16 real parameters**

The solutions to the KPI equation can be written as

$$v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}$$

with

$$\begin{aligned} n(x, y, t) = & -b_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,2}x + ig_{1,1}x + ie_{1,1} + itg_{1,2}^3 - iyg_{1,2}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2 h_{1,1}} \\ & \times e^{-3itg_{1,1}h_{1,1}^2 + th_{1,1}^3 - 3itg_{1,2}h_{1,2}^2 + 2yg_{1,2}h_{1,2} + itg_{1,1}^3 + iyh_{1,2}^2 - 3tg_{1,2}^2 h_{1,2} + ie_{1,2} + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2 g_{1,1}^2 b_{1,2,0}} \\ & - b_{1,1,0} e^{ig_{1,3}x - h_{1,1}x + ig_{1,1}x - h_{1,3}x + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + ie_{1,1} + ie_{1,3} - 3itg_{1,3}h_{1,3}^2 + d_{1,1} - 3tg_{1,1}^2 h_{1,1} - 3itg_{1,1}h_{1,1}^2} \\ & \times e^{+th_{1,1}^3 + itg_{1,1}^3 + 2yg_{1,3}h_{1,3} - iyg_{1,1}^2 + iyh_{1,3}^2 - 3tg_{1,3}^2 h_{1,3} - iyg_{1,3}^2 + th_{1,3}^3 + d_{1,3} + itg_{1,3}^3 g_{1,1}^2 b_{1,3,0}} \\ & - b_{1,2,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,2}x + ig_{1,1}x + ie_{1,1} + itg_{1,2}^3 - iyg_{1,2}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2 h_{1,1} - 3itg_{1,1}h_{1,1}^2} \\ & \times e^{+th_{1,1}^3 - 3itg_{1,2}h_{1,2}^2 + 2yg_{1,2}h_{1,2} + itg_{1,1}^3 + iyh_{1,2}^2 - 3tg_{1,2}^2 h_{1,2} + ie_{1,2} + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2 g_{1,2}^2 b_{1,1,0}} \\ & - b_{1,2,0} e^{-h_{1,3}x - h_{1,2}x + ig_{1,3}x + ig_{1,2}x + 2yg_{1,2}h_{1,2} - 3itg_{1,3}h_{1,3}^2 + ie_{1,3} + itg_{1,2}^3 - iyg_{1,2}^2 + d_{1,2} - 3tg_{1,2}^2 h_{1,2}} \\ & \times e^{-3itg_{1,2}h_{1,2}^2 + th_{1,2}^3 + iyh_{1,2}^2 + 2yg_{1,3}h_{1,3} + ie_{1,2} + iyh_{1,3}^2 - 3tg_{1,3}^2 h_{1,3} - iyg_{1,3}^2 + th_{1,3}^3 + d_{1,3} + itg_{1,3}^3 g_{1,2}^2 b_{1,3,0}} \\ & - b_{1,3,0} e^{ig_{1,3}x - h_{1,1}x + ig_{1,1}x - h_{1,3}x + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + ie_{1,1} + ie_{1,3} - 3itg_{1,3}h_{1,3}^2 + d_{1,1} - 3tg_{1,1}^2 h_{1,1} - 3itg_{1,1}h_{1,1}^2} \\ & \times e^{+th_{1,1}^3 + itg_{1,1}^3 + 2yg_{1,3}h_{1,3} - iyg_{1,1}^2 + iyh_{1,3}^2 - 3tg_{1,3}^2 h_{1,3} - iyg_{1,3}^2 + th_{1,3}^3 + d_{1,3} + itg_{1,3}^3 g_{1,3}^2 b_{1,1,0}} \\ & - b_{1,3,0} e^{-h_{1,3}x - h_{1,2}x + ig_{1,3}x + ig_{1,2}x + 2yg_{1,2}h_{1,2} - 3itg_{1,3}h_{1,3}^2 + ie_{1,3} + itg_{1,2}^3 - iyg_{1,2}^2 + d_{1,2} - 3tg_{1,2}^2 h_{1,2} - 3itg_{1,2}h_{1,2}^2} \\ & \times e^{+th_{1,2}^3 + iyh_{1,2}^2 + 2yg_{1,3}h_{1,3} + ie_{1,2} + iyh_{1,3}^2 - 3tg_{1,3}^2 h_{1,3} - iyg_{1,3}^2 + th_{1,3}^3 + d_{1,3} + itg_{1,3}^3 g_{1,3}^2 b_{1,2,0}} \\ & + b_{1,1,0} e^{-h_{1,2}x - h_{1,1}x + ig_{1,2}x + ig_{1,1}x + ie_{1,1} + itg_{1,2}^3 - iyg_{1,2}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + d_{1,1} - 3tg_{1,1}^2 h_{1,1} - 3itg_{1,1}h_{1,1}^2} \\ & \times e^{+th_{1,1}^3 - 3itg_{1,2}h_{1,2}^2 + 2yg_{1,2}h_{1,2} + itg_{1,1}^3 + iyh_{1,2}^2 - 3tg_{1,2}^2 h_{1,2} + ie_{1,2} + th_{1,2}^3 + d_{1,2} - iyg_{1,1}^2 h_{1,1}^2 b_{1,2,0}} \\ & + b_{1,1,0} e^{ig_{1,3}x - h_{1,1}x + ig_{1,1}x - h_{1,3}x + 2yg_{1,1}h_{1,1} + iyh_{1,1}^2 + ie_{1,1} + ie_{1,3} - 3itg_{1,3}h_{1,3}^2 + d_{1,1} - 3tg_{1,1}^2 h_{1,1} - 3itg_{1,1}h_{1,1}^2} \\ & \times e^{+th_{1,1}^3 + itg_{1,1}^3 + 2yg_{1,3}h_{1,3} - iyg_{1,1}^2 + iyh_{1,3}^2 - 3tg_{1,3}^2 h_{1,3} - iyg_{1,3}^2 + th_{1,3}^3 + d_{1,3} + itg_{1,3}^3 h_{1,1}^2 b_{1,3,0}} \end{aligned}$$



$$\begin{aligned}
& +2ib_{1,1,0}e^{-h_{1,2}x-h_{1,1}x+ig_{1,2}x+ig_{1,1}x+ie_{1,1}+itg_{1,2}^3-iyg_{1,2}^2+2yg_{1,1}h_{1,1}+iyh_{1,1}^2+d_{1,1}-3tg_{1,1}^2h_{1,1}-3itg_{1,1}h_{1,1}^2} \\
& \times e^{+th_{1,1}^3-3itg_{1,2}h_{1,2}^2+2yg_{1,2}h_{1,2}+itg_{1,1}^3+iyh_{1,2}^2-3tg_{1,2}^2h_{1,2}+ie_{1,2}+th_{1,2}^3+d_{1,2}-iyg_{1,1}^2h_{1,1}b_{1,2,0}g_{1,2}} \\
& +2ib_{1,1,0}e^{ig_{1,3}x-h_{1,1}x+ig_{1,1}x-h_{1,3}x+2yg_{1,1}h_{1,1}+iyh_{1,1}^2+ie_{1,1}+ie_{1,3}-3itg_{1,3}h_{1,3}^2+d_{1,1}-3tg_{1,1}^2h_{1,1}} \\
& \times e^{-3itg_{1,1}h_{1,1}^2+th_{1,1}^3+itg_{1,1}^3+2yg_{1,3}h_{1,3}-iyg_{1,1}^2+iyh_{1,3}^2-3tg_{1,3}^2h_{1,3}-iyg_{1,3}^2+th_{1,3}^3+d_{1,3}+itg_{1,3}^3h_{1,1}b_{1,3,0}g_{1,3}} \\
& +2ib_{1,2,0}e^{-h_{1,3}x-h_{1,2}x+ig_{1,3}x+ig_{1,2}x+2yg_{1,2}h_{1,2}-3itg_{1,3}h_{1,3}^2+ie_{1,3}+itg_{1,2}^3-iyg_{1,2}^2+d_{1,2}-3tg_{1,2}^2h_{1,2}} \\
& \times e^{-3itg_{1,2}h_{1,2}^2+th_{1,2}^3+iyh_{1,2}^2+2yg_{1,3}h_{1,3}+ie_{1,2}+iyh_{1,3}^2-3tg_{1,3}^2h_{1,3}-iyg_{1,3}^2+th_{1,3}^3+d_{1,3}+itg_{1,3}^3g_{1,2}b_{1,3,0}h_{1,3}} \\
& +2ib_{1,2,0}e^{-h_{1,3}x-h_{1,2}x+ig_{1,3}x+ig_{1,2}x+2yg_{1,2}h_{1,2}-3itg_{1,3}h_{1,3}^2+ie_{1,3}+itg_{1,2}^3-iyg_{1,2}^2+d_{1,2}-3tg_{1,2}^2h_{1,2}} \\
& \times e^{-3itg_{1,2}h_{1,2}^2+th_{1,2}^3+iyh_{1,2}^2+2yg_{1,3}h_{1,3}+ie_{1,2}+iyh_{1,3}^2-3tg_{1,3}^2h_{1,3}-iyg_{1,3}^2+th_{1,3}^3+d_{1,3}+itg_{1,3}^3h_{1,2}b_{1,3,0}g_{1,3}} \\
& -2ib_{1,1,0}e^{ig_{1,1}x-h_{1,1}x-iyg_{1,1}^2+2yg_{1,1}h_{1,1}+iyh_{1,1}^2+itg_{1,1}^3-3tg_{1,1}^2h_{1,1}-3itg_{1,1}h_{1,1}^2+th_{1,1}^3+d_{1,1}+ie_{1,1}g_{1,1}h_{1,1}a_1} \\
& -2ib_{1,2,0}e^{ig_{1,2}x-h_{1,2}x-iyg_{1,2}^2+2yg_{1,2}h_{1,2}+iyh_{1,2}^2+itg_{1,2}^3-3tg_{1,2}^2h_{1,2}-3itg_{1,2}h_{1,2}^2+th_{1,2}^3+d_{1,2}+ie_{1,2}g_{1,2}h_{1,2}a_1} \\
& -2ib_{1,3,0}e^{ig_{1,3}x-h_{1,3}x-iyg_{1,3}^2+2yg_{1,3}h_{1,3}+iyh_{1,3}^2+itg_{1,3}^3-3tg_{1,3}^2h_{1,3}-3itg_{1,3}h_{1,3}^2+th_{1,3}^3+d_{1,3}+ie_{1,3}g_{1,3}h_{1,3}a_1} \\
& -2ib_{1,1,0}e^{-h_{1,2}x-h_{1,1}x+ig_{1,2}x+ig_{1,1}x+ie_{1,1}+itg_{1,2}^3-iyg_{1,2}^2+2yg_{1,1}h_{1,1}+iyh_{1,1}^2+d_{1,1}-3tg_{1,1}^2h_{1,1}} \\
& \times e^{-3itg_{1,1}h_{1,1}^2+th_{1,1}^3-3itg_{1,2}h_{1,2}^2+2yg_{1,2}h_{1,2}+itg_{1,1}^3+iyh_{1,2}^2-3tg_{1,2}^2h_{1,2}+ie_{1,2}+th_{1,2}^3+d_{1,2}-iyg_{1,1}^2g_{1,1}h_{1,1}b_{1,2,0}}
\end{aligned}$$

and

$$\begin{aligned}
d(x, y, t) & = b_{1,1,0}e^{-h_{1,1}x+ig_{1,1}x-3tg_{1,1}^2h_{1,1}-3itg_{1,1}h_{1,1}^2+itg_{1,1}^3+2yg_{1,1}h_{1,1}+iyh_{1,1}^2+th_{1,1}^3-iyg_{1,1}^2+d_{1,1}+ie_{1,1}} \\
& + b_{1,2,0}e^{-h_{1,2}x+ig_{1,2}x-3tg_{1,2}^2h_{1,2}-3itg_{1,2}h_{1,2}^2+itg_{1,2}^3+2yg_{1,2}h_{1,2}+iyh_{1,2}^2+th_{1,2}^3-iyg_{1,2}^2+d_{1,2}+ie_{1,2}} \\
& + b_{1,3,0}e^{-h_{1,3}x+ig_{1,3}x-3tg_{1,3}^2h_{1,3}-3itg_{1,3}h_{1,3}^2+itg_{1,3}^3+2yg_{1,3}h_{1,3}+iyh_{1,3}^2+th_{1,3}^3-iyg_{1,3}^2+d_{1,3}+ie_{1,3}} + \\
& a_1
\end{aligned}$$