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Experimenting with triangles

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1. Introduction

1.1 The problem and the strategy to tackle it

What are thought-experiments? Do they differ from real experiments? And can we consider them as good evidence or as reliable arguments for some claim?

These and other similar questions have been recently explored in the literature, in an on-going debate about the role of thought-experiments (henceforth TEs) in natural science and partly in philosophy (for a survey, see Stuart *et al.* (eds), 2018). Several TEs and some of their features have been investigated so far and, to make a long story short, the replies have clustered around two poles. On the one hand, following the Platonist Brown, the nature of TEs would be *imagistic*: in TEs, scientists use their intuition to reason “visually” (Brown, 1991); on the other hand, the empiricist Norton suggests instead that the nature of TEs is *linguistic*: TEs would correspond to “arguments” allowing for new inferences (Norton, 1996).

It is interesting to note that the case of mathematics and mathematical TEs has attracted less attention¹. One exception is Anapolitanos (1991), who considered TEs in mathematics as related to “extra” logic considerations constituting the *creative* part of mathematics, and proposed to divide them in six groups based on the different theoretical context². An important point is that TEs have to respect certain boundaries that he defines as “*conceivability conditions or constraints in mathematics*” (henceforth CCC). CCC do not correspond to the rules of logic, which are independent from the scientific discipline, and can be arranged in the following “more or less open-ended” list: (a) *simplicity* conditions or constraints, (b) *familiarity* constraints, (c) *plausibility* conditions or constraints, (d) *efficiency* constraints, (e) conditions or constraints related to the *success* of a proposed *reconceptualization*, (f) conditions or constraints induced by *specific philosophical ideas and positions*.

More recently, some new ideas have been put forward. Starikova & Giaquinto (2018) for example highlight the importance of TEs in mathematics as “visual” heuristics, thus aligning with one of the two poles described above for empirical science. Buzzoni (2021) compare mathematical TEs with TEs in the natural science

¹ This is not new for mathematics from the point of view of philosophy of science, which rarely addresses the specificities of this discipline compared to other scientific practices.

² Here are the groups of TEs as defined by Anapolitanos (1991): TEs (i) “usually performed in a conceptual framework which has not been yet theoretically solidified”; (2) “performed in a conceptual framework provided by a fixed mathematical theory; (3) “performed fervently during and immediately after a foundational crisis”; (4) “spurred by the intrinsic possibility of proving or disproving a postulate or a sentence, which mathematicians under certain circumstances come to think as basic, from the axioms of a more or less given mathematical theory”; (5) “triggered and influenced – among other things – by philosophical considerations of a quite revolutionary nature”; (6) “induced by the need to specify a new conceptual framework easier to work with in a specific area”.

and distinguishes between TEs in applied and pure mathematics; his conclusion is that in pure mathematics there is no distinction in principle between real experiments (henceforth REs) and TEs; for this reason, and differently from what he suggested in previous views (see Buzzoni 2011), even if visualization may play an important role, mathematical TEs end up being more similar to proofs (henceforth Ps) than to TEs in the natural sciences. Another possibility is to assume a framework based on the practice of mathematics – including the results obtained now with computers – and define TEs as a *subset* of other kinds of mathematical experiments (Jean-Paul van Bendegem, forthcoming).³

To sum up, several proposals are available but most importantly they disagree on the very meanings of the terms at play: TEs are intended in turn as visual images, as quasi-proofs, or as a particular kind of experiment. The aim of the present article will not be to present these views in details; my strategy will rather be to start from scratch. Without comparing TEs in empirical science and philosophy to TEs in mathematics, I will ask the question directly: What – if anything – are TEs in mathematics? Such a question, however, needs to be specified more, by breaking it down into two related questions: Is there anything like REs in mathematics? And if this is the case, should all/some/none of them be considered TEs?

Let me point out that when I speak in this context of mathematics, I mean “pure” mathematics. There are several reasons to focus on pure mathematics only, despite the fact that this distinction may be too strict and artificial. First, talking of pure mathematics allows considering TEs *for* and *from within* mathematics, and not as performed to the aim of learning something – mathematically – about reality. Second, pure mathematics clearly does not have an empirical counterpart, at least not in the sense of the natural sciences, and therefore it sounds counterintuitive to think that its practice might include experiments: can there really be something authentically “experimental” in pure mathematics? Moreover, if pure mathematics does not deal with the empirical reality, every move in it could be considered – to some extent – precisely a TE.

Before going further into these questions, in the next section I will present in turn three assumptions that will be in the background of my treatment of TEs in mathematics.

1.2 Three background assumptions

My first assumption is that I will consider mathematics as an *activity* including *experiments* and a *technology*, that is, some form of – *public* – representations. By making this assumption, of course I answer in part to some of the questions above: yes, mathematics – even pure mathematics – can be – under some particular circumstances to specify – an “experimental” activity.

To better clarify my point, I will focus on Rav (2005)’s suggestion, who argues that mathematics as well as empirical science live in the constant interaction among three poles supporting one another, in a continuous flux that is typical of scientific

³ A different question is how experimental methods coming from the social sciences can be used to study the practice of mathematics, analogously to what happens in so-called “experimental philosophy”. On this point, see Löwe & van Kerkhove (2019).

research: *Theory, Experiment and Technology* (henceforth, T-E-TC). However, the case of mathematics is peculiar, since in mathematics these poles happen to be one and the same, that is, they “*coalesce* to a single point” (Rav, 2005, p. 50). In fact, in their creative role, mathematicians are *at once* theoreticians, experimenters and toolmakers inventing useful instruments. As Rav sums up, they are problem-solvers.

The picture proposed by Rav gives a non-standard image of mathematics, if we consider as “standard” the “received view” as based on the work on the foundations of mathematics at the beginning of the 20th century, at least for two reasons: first, in Rav’s view, mathematics is a research activity that is analogous to other sciences, because it proceeds by formulating hypotheses that then are put to the test⁴; second, Rav argues that since mathematics makes use of different technologies and formats, the choice of one particular technology or format to display information makes a difference in the way the information is treated: for this reason, the definition of new problems and the use and creation of tools to solve them are in continuous interaction, in a dialectic process. Given the T-E-TC framework, only someone who knows the theory and masters the technology, that is, who is able to deal with the representations that are used in the practice, would be able to perform experiments no matter – for the moment – if REs or TEs.

My second assumption is that, compared to science, an element that is specific to mathematics is P; in fact, we cannot talk about evidence only – admitting that we can have some form of evidence in mathematics⁵ – because we also have “validity”. Therefore, when considering TEs in mathematics, it will be necessary to evaluate their relationship with Ps or, in other words, the epistemic value of TEs. However, two caveats are important to consider here. First, as Buzzoni (2011) sums up, “what is to be accepted as a “proof” is a *historical* question, to be decided by working mathematicians” (p. 78). Second, proof might still be seen as the main objective of the mathematical activity, as the received view maintains; however, it should also be acknowledged that at least in a first phase proof involves *dynamic* and *informal* reasoning, which means that proofs *in the practice* are not necessarily formal, but proceed in ways that are defined by the mathematical context, sometimes extending beyond what the formalism at play allows for.

On this point, Lakatos (1976) warns us against the danger of considering mathematics only from the point of view of formalism. As he explains in an important footnote, “formal philosophy” proceeds in an ill-posed way. First, it states something that is in fact right about formal systems; second, it makes the claim that this something applies to all mathematics, which is acceptable if one identifies mathematics and formal systems; finally, “with a surreptitious shift in meaning” (Lakatos, 1976, footnote 1, p. 4), the term “mathematics” is used in the ordinary sense. Lakatos quotes here Quine (1951, p. 87): “this reflects the characteristic mathematical situation; the mathematician hits upon his proof by unregimented insight and good fortune, but afterwards other mathematicians can check his proof”. However, according to Lakatos, the checking of an *ordinary* (informal) proof is in

⁴ Ferreiros (2016) as well defends the idea that advanced mathematics – differently from elementary mathematics – is based on hypotheses, that is, on hypothetical assumptions. As he explains, this refers to a long tradition of reflection on mathematical knowledge, including Riemann, Peirce, Poincaré, Weyl, Quine, and Putnam.

⁵ On this point, see Gowers (forthcoming).

most cases a very delicate enterprise, and “to hit on a ‘mistake’ requires as much insight and luck as to hit on a proof: the discovery of ‘mistakes’ in informal proofs may sometimes take decades – if not centuries.” In this spirit, I will endorse a “genetic-reconstructive” approach to the logical moment of justification, in line with Buzzoni’s view (2011, p. 75). In particular, I will focus on the possibility for a proof to be *reproduced*, that is, on the human ability to re-appropriate proofs in the first person.⁶ This point is crucial: if the present aim is to study the relationship between experiments and proofs, in analogy with empirical science, then replicability is an issue. TEs can offer an insight into a new proof, or means to understand a proof (see van Bendegem 2003). Such a genetic-reconstructive approach allows bypassing the dichotomy proposed by the standard view between the context of discovery and the context of justification.

My third assumption is that when it comes to TEs in mathematics, I will consider human reasoning only, without taking into account the possible contributions of computers. It is important to clarify this point since the term “experimental” mathematics (henceforth EM) has been commonly used to refer to mathematics that is done “computationally” (see van Bendegem 1998). However, I will follow once again Buzzoni in excluding this form of mathematics from the practice relating to TEs since, when using computers to do mathematics, there is “no interaction between our body and mathematical objects” (Buzzoni, 2011, p. 70). In the course of the paper, it will become clearer what I intend here by interaction and mathematical “objects”. Of course, I am not claiming that the contribution of computers is not relevant for contemporary mathematics; by contrast, EM will be included in the general framework proposed at the end.

To sum up, these three assumptions about experiments in mathematics and in particular TEs define a background that is close to “non-standard” views of mathematics such as the ones proposed by Lakatos, Rav and van Bendegem.⁷

1.3 Some complications: what general framework to endorse?

There is a sense in which the main objective of the present article is terminological, that is, the definition of the terms in question in such a way that it will be possible to conceptually distinguish TEs from REs, EM and Ps, precisely for the reason that the literature is at present confounding them.

There are even more complications. Rav (2005) argues in a footnote that traditionally experiments in mathematics are TEs for searching connections and possible deductive paths; the use of computers, which is a recent innovation, can help “in exploring and formulating conjectures or as an aid in searching for proofs.” (footnote 1, p. 50); however, this does not make mathematics a “quasi-empirical science *in its totality*”. In his view, “quasi-experiments” (henceforth QEs) in mathematics are experiments that are performed for searching connections and possible deductive path, and therefore they seem to be closer to inferential views of TEs such as Norton’s, considering TEs as arguments. If we go back to *Proofs and Refutations* (Lakatos 1976), from which many of the views presented so far -

⁶ Discussing Greek mathematics, Netz (1999) introduces the principle “repeatability” of proof, of which the generalizability of result is a derivation.

⁷ Despite being a Platonist, also Brown is very close to Lakatos’ view. See on this point Brown (2022).

including Rav's – take inspiration⁸, the confusion increases. In a passage, the teacher proposes to retain the term “proof” for “a *thought-experiment* – or ‘*quasi-experiment*’ – which *suggests a decomposition of the original conjecture into subconjectures or lemmas*, thus *embedding* it in a possibly quite distant body of knowledge” (Lakatos 1976, p. 9). For example, the proof of the Descartes-Euler conjecture, which is the subject of the book, embeds the original conjecture – about crystals, that is, solids – into the theory of rubber sheets, which is something that Descartes or Euler, who conceived the original conjecture, could not predict. Lakatos seems to suggest here that all – or just some? – Ps are TEs or QEs. However, this does not necessarily imply that all TEs are proofs. Or does it? Are there mathematical TEs that are not proofs? If we consider pure mathematics, what would the difference between Ps and TEs or QEs be, if experiments in this field do not seem to define a domain or method that is “qualitatively distinct from other mathematical domains or methods or proofs” (Buzzoni, 2011, p. 77)?

There are also other two important points to consider. First, mathematical counterexamples do not arise from experimental practices (Buzzoni 2011, p. 71); this is also recognized by Putnam (1975), who speaks of a “quasi-empirical” method analogous to methods in the empirical science, with the important difference that in mathematics it results in the product of a calculation and not in the report of an observation. Second, it looks like visual reasoning and picture proofs are everywhere in mathematics. What is their relationship with TEs? Are they analogous to them? Or are they a case of REs in mathematics? Or just of Ps?

Let us try to sum up what we have so far. I assumed that there are experiments in mathematics. However, differently from the empirical scientist, the mathematician is at once an experimenter, a theoretician, and an engineer: following Rav's suggestion, every experiment is in a continuous – neither trivial nor “mechanical” – interaction with hypotheses to test, theoretical assumptions, and the available technology. This last aspect is crucial: mathematics – as a culture – has seen the creation of representational systems, which are in the course of time selected and accepted as reliable by the community and whose correct functioning has to be learned. These tools – these physical objects – are ready to be used to study the objects of mathematics. Mathematics is therefore an experimental science but of a particular nature: to test their hypotheses, mathematicians use their mind as a laboratory but they also enrich their reasoning by referring to the representational systems that mathematics has selected in time. Moreover, as Anapolitanos points out, the activity of the mathematicians is subject to CCC: if natural science is constrained by what exists, mathematics cannot go beyond what could *possibly* exist.

We can thus start from defining the domain of the mathematical activity, which is characterized by T-E-TC and CCC and includes TEs and Ps (see Figure 1). However, what is their relationship? Some TEs might be Ps, but are all of them as such? And are all Ps TEs? Where should experiments in general be put? Are TEs and Ps both kinds of experiments? And where to put REs and QEs? Is there something outside the circle that we are missing?

⁸ Several possible taxonomies such as the ones presented in Anapolitanos (1991) or van Bendegem (forthcoming) are indeed “extensions” of Lakatos' ideas.

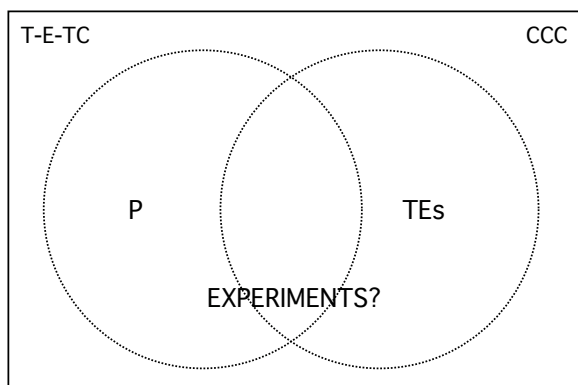


Figure 1. A first schema of the framework.

In the remainder of the paper I will try to disambiguate the different terms at play: the desideratum will be to clarify what a TE is in mathematics, whether it is the same as a RE or not, and what role technology has, an aspect in line with the T-E-TC framework that has been partly neglected by the standard approaches. As Buzzoni (2011, p. 66) claims, “we need no starting point other than our everyday, conceptually mediated interactions with the world”; for this reason, I will endorse as well an *operationalist-interventionist* view of mathematical experience. For this reason, in Section 2, I will present three possible examples of mathematical “experiments” with triangles – the simpler geometric figure that can be drawn: (i) a “Meno-inspired” example, (ii) a “Klein-inspired” example, and finally (iii) an experimental study in mathematics education. In the Section 3, I will present the general framework and discuss what these three examples are example of. Finally, in Section 4, I will list the advantages of this general framework.

2. Three examples of experimenting with triangles

2.1 “Fold the triangles and tell me what happens!”

Example 1 is taken from Giaquinto (2007) and is of a clear Platonic inspiration, since it recalls the task of doubling the area of a given square, that is, the problem that the slave of the *Meno* dialogue is asked to solve.

Giaquinto asks the reader to imagine a square:

“Each of its four sides has a midpoint. Now visualize the square whose corner-points coincide with these four midpoints. If you visualize the original square with a horizontal base, the new square should seem to be tilted, standing on one of its corners, “like a diamond” some people say.” (p. 51)

In the book, a figure illustrates this procedure, but there is no need of a physical figure: our imagination and our background knowledge about squares and diamonds is enough to follow the instructions in our mind. As we will see, the general framework that I shall propose at the end will take this issue into account. For now, the question is: is the inner square smaller than the outer square? And what is the relation between their areas? The original square is clearly bigger, but how much bigger? Giaquinto continues by giving new instructions:

“One can now visualize the corner triangles folding over, with creases along the sides of the tilted square. [...] Assuming that this leads you to the belief that the corner triangles can be arranged to cover the inner square exactly, you will infer that the area of the original square is twice the size of the tilted inner square”. (pp. 51-2)

In his reconstruction, by imagining performing the folding action described above, one gets to the belief B:

[B] If c_i (“the inner square”) is the square whose vertices are midpoints of the sides of a square c (“the original square”), then the parts of c beyond c_i (“the corner triangles”) can be arranged to fit exactly into c_i , without overlap or gap, without change of size or shape.”

How do we arrive at B? What kind of evidence do we have for B? Are we entitled to hold it true and why?

In Giaquinto’s reconstruction, B cannot be an inference coming from *sense experience*, since sense experience would not exclude counterexamples. We can fold napkins over and over again and never find that they exactly fit into the inner square. It cannot either be an *inner* experience, since such an experience would exclude the phenomenology of seeing and comparing and evaluating. In his view, the role of visualizing may be precisely to bring to mind prior beliefs and to activate prior inferential dispositions: in this case as well as in other analogous cases in geometry, visualizing has a non-evidential role, since it is part of an *a priori* means of acquiring belief or, in other words, of an *a priori* means of discovery. For this reason, despite being non-evidential, the visual element is non-superfluous.

According to Giaquinto, one may deny Plato’s claim, that is, that the subject’s prior cognitive state includes believing B, and accept that it accommodates *cognitive resources* that are sufficient to produce B upon visualizing; he argues for the existence of stored category representations for the visual recognition of particular figures, in this case a square, which are activated not only when we see a physical square, but also when we generate a square in our imagination. Giaquinto bases his view on cognitive science work and in particular on Kosslyn (1980), where it is claimed that such stored visual category representations exist and they are activated in both perceptual recognition of shapes, also in unfavorable conditions such as poor light or partial occlusion, and in the generation of visual images in our imagination. For this reason, and this is where Giaquinto goes back to Kant, the example is a case of *synthetic a priori* knowledge.

In the present article, my aim is neither to present Giaquinto’s view in its details, which include the appeal to restricted universal quantification, nor to endorse his “neo-Kantian” posture. By contrast, I will just take into account this example in relation to my question about the role of TEs in mathematics. As Giaquinto claims, by visualizing such a procedure, one gets to the belief B. Is this an experiment in the end? And if this were the case, would it be a RE or a TE? What is its epistemic value? What if instead it is not an experiment but a P, based on cognitive science findings? We will see what place Example 1 will take in the general framework proposed in the next section. For the moment, let us consider the second example.

2.2 “Look at triangles, they are in all cases isosceles!”

Example 2 is the famous (wrong) proof presented by Klein in 1908 that all triangles are isosceles. Here is how Klein asks the reader to proceed. Consider an arbitrary triangle ABC and draw the bisector line from the angle A and the perpendicular to side BC which goes to its middle point D . If these two lines were parallel, the bisector would also be the altitude of the triangle and the triangle would obviously be isosceles. Assume instead that these two lines meet. Two cases are logically possible: the meeting point O may be inside or outside the triangle. In both cases, draw the segments OE and OF that are perpendicular to AC and AB , respectively. Finally, join O to B and to C .

Figure 2 presents the case where O is *inside* the triangle. The right triangles AOE and AOF are congruent, because they have the hypotenuse AO in common, the angles in A are equal by definition and also the two right angles are equal. We can then conclude that AF is equal to AE . Analogously, the two right triangles OCD and OBD are congruent, since they have OD in common, DB is equal to DC by definition, and the right angles are equal. We can then conclude that $OB = OC$.

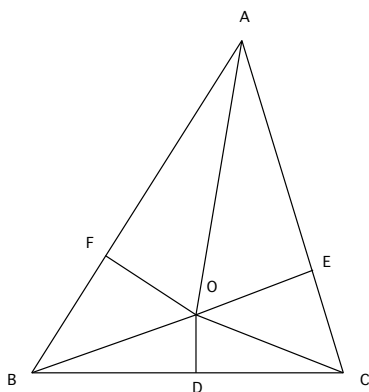


Figure 2. O is *inside* the triangle

Now, because of the first congruence, OE is equal OF ; then, we derive the congruence of triangles OEC and OFB . Hence, FB is equal to EC . But if we add equals to equals, then we get to the conclusion that AB is equal to AC . Therefore, the triangle ABC is isosceles.

What about the case where O is *outside* the triangle, as in Figure 3? Analogously to the case where O is inside the triangle, it can be inferred that the pairs of triangles OFA and OEA , OBD and OCD , OFB and OCE are all congruent. Therefore, AF is equal to AE , and FB to EC . If we subtract equals to equals, then we get to the conclusion that AB is equal AC . Once again, the triangle ABC is isosceles.

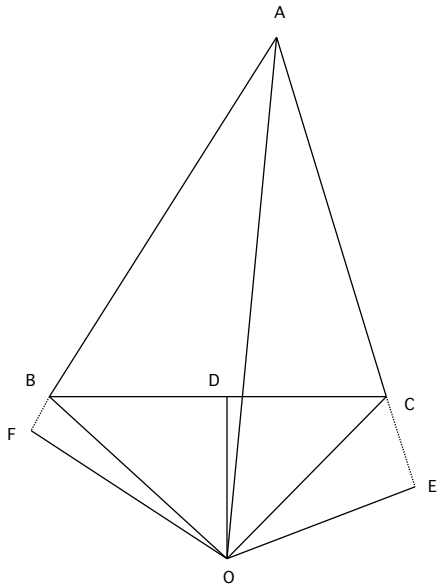


Figure 3. O is *outside* the triangle.

The proof took into account the two possible cases, and it showed that the triangle ABC is isosceles for both. Something has clearly gone wrong, but what?

According to Klein, “the only thing in this proof that is false is the figure”, because “the argument is always based upon inaccurate figures, with perverted order of points and lines” (Klein, 2004, p. 202). The truth is that O can *never* fall inside the triangle, and as a consequence the locations of this point can *never* be as drawn in the two figures above. Figure 4 is the *correct* figure that “clears up this sophism completely” (*ibidem*). Of the two feet E and F , one *must* in fact lie inside the side on which it lies and the other outside, as shown. If this is the case, then $AC = AE - CE$, and $AB = AF + BF = AE + CE$.

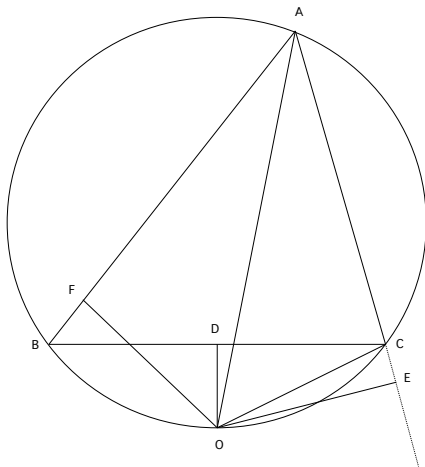


Figure 4. The correct figure.

As for Example 1, what kind of reasoning do we find here? Is Example 2 an experiment? To be true, in comparison to Example 1, Example 2 looks more like a RE than a TE: the reader is not (only) asked to imagine things, but to look at physical figures that were (wrongly) drawn on a piece of paper. It might also be that Example

2 is indeed a P, or better a bad P for reasons that we have to specify, but still a P. We will see below where Example 2 should be placed in the general framework.

2.3 “What do you get when you add more and more and more triangles?”

Example 3 is the report of a very interesting experimental work in mathematics education (Bråting & Pejlaré 2008). The authors’ aim was to establish what role visualizations might have in learning – and teaching – mathematics and in mathematical understanding.⁹ As a case study, they take von Koch’s “snowflake” (von Koch 1906, pp. 145–146).

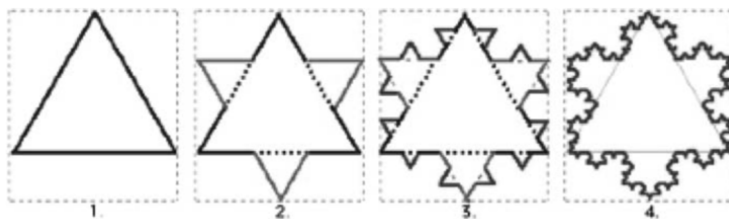
The snowflake is a visualization that was elaborated by the Swedish mathematician von Koch to represent an everywhere continuous but nowhere differentiable function. Weierstrass had already given a formalization of this kind of function, but von Koch was not satisfied by this analytic version of it because, according to him, it did not reflect the intrinsic geometrical nature of the function: to “see” it, it would be very different. von Koch’s idea is that in the snowflake figure the mathematician *sees* in the figure an everywhere continuous but nowhere differentiable function, that is, that it is not possible to draw the tangent in any point of the curve.

The authors of the study have doubts: to recognize this impossibility is a matter of possessing the necessary expertise and not of seeing the right visualization. In order to figure out whether they are right, they gave 31 university students in mathematics the following task.

Consider the construction that follows:

- *Start with an equilateral triangle where each side has length 1.*
- *On the middle third of each of the three sides, build an equilateral triangle with sides of length $1/3$. Erase the base of each of the three new triangles.*
- *On the middle third of each of the twelve sides, build an equilateral triangle with sides of length $1/9$. Erase the base of each of the twelve new triangles.*

Repeat the process with this 48-sided figure.



Please answer the following questions as carefully as you can!

1. *For how long can you repeat the process?*

⁹ The role and possibly the limits of the use of visualizations is still a core issue for mathematics education: more experimental work should be done on this point. In a more recent paper, Bråting (2012) discusses other examples in the light of the historical development of mathematics to show the importance of interpretation when it comes to visualizations, by arguing, *contra* Giaquinto, that a possible division between “visible mathematics” and “less visible mathematics” is misleading.

2. What figure will you get at the end? Is it continuous? Is it differentiable?

According to their results, 16 students think that the figure at the limit is uniform everywhere except in a certain finite number of points. Seven among these 16 students believe that at the end the figure will be a circle or a square, and 9 of them a "flower". One student in this last group comments that at the limit the triangles will become so small that the figure will transform into a uniform curve, that is, it will become continuous and differentiable. Some of these possibilities are illustrated in Figure 5. 14 students do not think that the figure at the end will be everywhere uniform, and therefore they show they are familiar with the flake curve. Nine students do not give pertinent answers. It seems therefore that intuition alone, if not "enhanced" by the right background knowledge, would lead people astray.



Figure 5. Some of the students' proposal of how the snowflake will look like at the limit.

What where the students asked for in this example? Was the task describing an experiment? And if this is the case, was it a RE or a TE? Moreover, in Example 3, and contrary to Example 2, we are confronted with a procedure that does not seem to have the character of a P. Example 3 as well will find a place in the general framework that I will propose in the following section.

3. Back to the framework

3.1 Three regions in the experimenting activity

In the previous section, I gave three examples of procedures that could be considered as experiments in mathematics. Of course, in order to be sure about it, we need to stipulate what mathematical experiments are, if there are any, and whether they are all TEs or it is possible to distinguish between RE and TEs also in mathematics.

In the Introduction, I assumed that mathematical activity is characterized by a dialectic process involving three elements: T-E-TC. A common feature of the three examples that I presented is the appeal to background knowledge and to the consequent theoretical assumptions one needs to be aware of. For this reason, whatever the nature of the reasoning process in the three cases, it should not be "naïve" but subject to *constraints*. In fact, as suggested by Anapolitanos, all over mathematics there are CCC to consider, at the point that experimenting in mathematics is defined by what mathematicians are allowed or not to conceive. Therefore, experimenting in mathematics is characterized by T-E-TC and CCC. However, what kinds of constraints are we talking about? Which constraints characterize REs and TEs in mathematics? And what about Ps?

In order to reply to these questions, let us focus on the nature of these constraints in turn. I assumed that in mathematical activity T-E-TC coalesce into a single point and cannot be considered in isolation one from the other; therefore, I will define as

notified the constraints that are all over the mathematical practice, which depend on background theoretical assumptions and are assimilated through learning. However, I will distinguish between different “regions” in the experimenting activity based on the constraints and on the technology involved in turn. Instead of drawing circles as I did in the first outline of my framework in Figure 1, I will divide the conceptual space into three regions. The first region is inhabited by TEs, which are subject to so-called *internalized* constraints, that is, to spatio-motoric constraints on the mental representations that get assimilated in perception. The second region is the REs’ region, where constraints are *built-in* in physical – public – representations and are recognized when these representations are correctly used. Buzzoni claims that there is a sense in which there cannot be REs in pure mathematics, since there is no external reality. However, as we will see, I will introduce a way to distinguish in pure mathematics between TEs and REs: in a mathematical RE, there is an interaction with some physical representation that has an epistemic value. The third region includes QEs, that is, a kind of experimentation that is based more on propositional knowledge and on “playing with” the formalism. In the next paragraphs, I will describe the three regions in turn.

3.2 TEs and mental models

The first region of the experimenting activity is occupied by TEs, which demand the activation of a *mental model*.

To define what a mental model is in this context, I will refer to the work of Nersessian (2018), who describes TEs in science as experiments on internal models of physical situations, that is, as a form of simulative model-based reasoning. Nersessian considers these models as “structural, behavioural or functional analogs to a real-world phenomenon” (p. 311). The important point is that the model preserves some constraints that are inherent to what it represents. These constraints are assimilated during perception and thus *internalized*: for this reason, such mental models are often visual or tactile in nature and get manipulated in real time; moreover, they embed a specific and personal point of view into the model (Nersessian, 1992, p. 294). Nersessian’s view moves against Norton’s proposal, since a model represents demonstratively as opposed to descriptively; however, Brown’s proposal is inappropriate to accommodate this view as well, since TEs are here characterized not only by visual processing but also by motor processing: Nersessian’s hypothesis is that TEs are based on the ability to imagine and to re-experience from memory.

Of course, one could be skeptical about the idea that in the case of pure mathematics, perception and action can play a role at all. However, this is the case when we consider how TEs are based on imaginative thinking. All mathematical activity is subject to notified constraints. Mathematicians know that they are doing mathematics and that their perception and their action have to respect the properties and the “behavior” of the mathematical objects that are in question each time: their manipulations are legitimated by the practice. For this reason, when considering the constraints that are used in constructing and manipulating mental models, they are conditioned by experience, including *mathematical* experience and understanding. For this reason, as for TEs in science, a considerable metacognitive control is needed: perception, action and conceptual knowledge play together in the

creation and the manipulation of specific mental models that are created to deal the mathematical objects.¹⁰

According to this definition of TEs, Example 1 described above is indeed a TE, since it is subject to internalized constraints that are assimilated through perception and involve not only vision but also action and manipulation as well as background knowledge about the properties of squares – we imagine folding four parts of a “mental” square. It is important to insist here that there is no need to physically draw the diagram, which does not have to be material: it suffices to refer to our imagination and memory to create a mental model analog to some physical situation – the one of folding a napkin – but applied in a new context to an abstract object of which we have learnt the properties.¹¹ TEs offer a heuristic to make hypotheses about the mathematical objects and to define strategies – to which we could not have access by relying on formalism only – on how to prove some mathematical claim.

Other examples of TEs in mathematics according to this definition are found in the literature. For example, it is a TE to imagine a cyclic graph as a necklace (Starikova & Giaquinto, 2018), which is not that far from Stevin’s chain example that has been discussed a lot in the literature on TEs in philosophy of science (see Brown, 2011; Norton, 1996, pp. 349–51 and Mach, 1905). Similarly, imagine holding a knot diagram up from one of its threads and let it fall to see if the knot is knotted or not; this would be another TEs.¹²

Let us now consider another example of TE according to my definition, that is, the narrative provided by Brown on how to prove the Intermediate Value Problem (Brown, 1999, Ch. 3). Brown asks the reader to imagine a mountain climber starting at the base of the mountain at noon and reaching the top at 6 p.m., where she sleeps that night. Then, the day after at noon, she follows the same path to go back. Of course, there has been a time at which she was at the same point of the mountain both days, no matter how quickly she walks. Imagine then an equivalent situation where the hikers are two, one starting at the bottom, the other at the top; for sure they will meet somewhere on the path.

According to Brown, these two visualizations give an illustration to the Intermediate Zero theorem and the Intermediate Value theorem respectively. A “purely analytic proof” in “Bolzano’s style”, in Brown’s view, will just confirm what was already *independently-known-to-be-true* thanks to the picture. This example nicely fits in the TEs region of my general framework: it is subject to internalized constraints – we

¹⁰ Nersessian (2018) goes on by attributing an important role to TEs-*narratives*, which lead to the creation and the manipulation of the mental models. In this paper, I will not focus on this element; however, I agree that narratives are an interesting topic to work on also for mathematics, not only for experiments but also for proofs. This is matter of further research.

¹¹ Rittberg & Freidman (2021) discuss paper folding as a form of “material reasoning”, that is, as a more reliable process that what I am describing here, having a strong epistemic value. In their view, material folding constitutes a third kind of mathematical reasoning besides propositional and diagrammatic reasoning. In order to show that paper folding is indeed a mathematical practice, they present a brief survey of the history of paper folding, concentrating mainly on the first third of the 20th century.

¹² I thank Uriah Kriegel for this example.

know what climbing a mountain is – and of course there are notified constraints as well we need to be aware of about the features and the behavior of a function.

However, there are two important points to make. First, I will not follow Brown in all the consequences of considering – from his point of view – this reasoning as a TE. In fact, here I side with Norton and deny that thanks to this kind of TEs one *immediately see the truth* of some mathematical statements; of course, imagination here is “constrained” by background knowledge that one needs to already possess, but would that be sufficient to conclude that the statement is true? Second, despite what Brown says, we do not really need the picture of the function to follow his narratives: it suffices to have an idea of what the picture would look like. In my view, he conflates the use of imagination and the use of diagrams and visualizations; one of the advantages of my general framework, as I will show, will be to conceptually distinguish between the two forms of experiments.

3.3 REs and material anchors

The second region of the experimenting activity is inhabited by mathematical REs; it is important to note that in my definition, by performing a RE one does not (only) refer to a “mental” model but to a “material” one, that is, to an “anchor” for reasoning.

Following the work of Hutchins (1995), “material anchors” are to be intended as physical artifacts that serve as “repositories of knowledge”. Hutchins takes the example of the instruments that are used in Western navigation, and explains their functioning: computational constraints are *built-in* the material anchors, which are constructed in durable media and as a consequence they can be shared and become part of a culture. To clarify, consider as an example the wind rose (Frake, 1985), where, in Hutchins’ reconstruction, the conceptual structure of time is blended with the material structure of the compass rose, thus allowing the navigator to experience direction as an expression of time (Hutchins, 2005, p. 1569). According to Hutchins, knowledge gets “crystallized” in such artifacts, by building computational constraints of the task into their physical structure; for this reason, material anchors are not only mnemonic devices but also *dynamic* tools that are supposed to be manipulated. Therefore, what gets crystallized really is the practice of using a physical artifact to solve some problem.

Hutchins’ material anchors are physical devices whose structure captures regularities in the world of phenomena in such a way that computations can be performed by manipulating them: the syntax of a particular symbolic world gets represented in the physical constraints of the very artifact. What about material anchors in pure mathematics? Similarly to the case of TEs and mental models, the structure of the physical object in a RE should capture regularities in the “behavior” of the mathematical objects that is designed to represent. As a consequence, the constraints that characterize the region of REs in the experimenting activity zone of mathematics are defined as *built-in* constraints, that is, constraints that are embodied *in* the structure of the material tools that are introduced by the practice. Here the interaction between T, E and TC becomes even more evident than in the case of TEs.

In this framework, Example 2 and Example 3 above are both cases of REs. To be true, they are both “limit” cases, in a sense that will be explained below. Assuming a

distinction that have been proposed in the literature (see Brown, 1991), they can be considered as “destructive” REs, since they “go wrong”, and this shows that some constraints have not been respected.

For the sake of my argument, let me start from Example 3. Its nature of experiment is “mixed”: notified constraints are not respected – students do not possess the right understanding of the function – and also they do not recognize the constraints that are built-in the representations that they are given in the text of the task. For this reason, it is a RE. However, the figures drawn by the students are illustrations of their mental model of the sides of the original representation becoming more and more; for this reason, Example 3 is also a TE. In fact, the lines in the framework are blurred, and this is because the framework allows for mixed cases. Once the terms have been specified, the categories can be considered as flexible.

Think for example of Freiling (1986)’s “philosophical proof” of the negation of Cantor’s hypothesis, where he imagines throwing darts at the real number line and that is how he gets to his axiom of symmetry.¹³ Freiling warns against arguing that such an experiment can be simulated by a physical experiment; by contrast, he believes that “such a process should be possible in the world of mathematics, which is perhaps an idealized real world.” This is an interesting mixed case of experiment, where some mental model – presenting internalized constraints we have assimilated from the action of physically throwing a dart – is applied to what in our definition is a material anchor, with built-in constraints, that is, the real number line. And its result looks like a P. I will go back to this point below when I will discuss the place for Ps in the general framework. For the moment, as I will discuss in the conclusions, I want to point out that the fact that the framework presents mixed cases is not a flaw but an advantage, since it shows that it offers a good grid to accommodate several cases and analyze their nature in its details.

What about Example 2? Notified and consequently built-in constraints are not respected: it is not the figure that is not correct, but the reasoning behind the incorrect way of drawing it. In other words, the figures were triggered by a series of uncorrected – and propositional – hypotheses; it is like having a flawless identikit obtained by a description that is considered as reliable, despite the fact that it is unfortunately given by an unreliable witness.¹⁴ To sum up, there is something wrong about the informal reasoning behind the construction of these figures and not in the figures themselves or in the possibility of putting them to the test. If they were drawn correctly, they would have not led us astray. However, Example 2 is a “limit” case. In fact, there is a sense in which it looks more like a P than Example 1 and Example 3. I will go back to this point below when I will talk about Ps.

Before describing the third region of the experimenting activity zone, let me discuss two important points. First, in both TEs and REs, some kind of “bodily” interaction with the objects of the experiments is acknowledge: perception and action can be intended as fully imaginative in the case of TEs or as carried out in interaction with external representations in REs. Moreover, in the distinction between internalized and built-in constraints, it is possible to see two different ways of proceedings in the

¹³ I thank Jean Paul van Bendegem and Colin Rittberg for having pointed out to me this illustrative example.

¹⁴ I thank Mario Piazza for this metaphor, which we presented in our book (Giardino & Piazza, 2008).

experimenting activity characterizing these two regions. In TEs, the mathematician happens to be more free, while in REs the anchors offer some “resistance” to their imagination: this is very clear in Example 2, where experts is not allowed to do whatever they want with the figures, which have to respect notified and built-in constraints. For this reason, we could think in terms of the two regions as expressing a more “creative” and a more “stipulative” way of reasoning¹⁵ in the experimenting activity: TEs offer the possibility of opening new horizons and identify novel strategies for Ps, for example by describing images that reveal some properties that are recognized for the first time thus exploring the hidden aspect of the conceptual structure at play; REs are instead constrained by the rules of the available representational system: however, since perception and action still play a role in REs, there is still a marge – even if a narrower one – to apply also in this region a mental model assimilated through the experience in the world, which is what happens in mixed cases such as throwing darts on the real number line.

In the next paragraph, I will describe the third region of the experimenting activity, where we find other forms of reasoning based on experimenting this times based on the appeal to the mathematical formalism.

3.4 QEs and material supports

The third region is inhabited by forms of reasoning in mathematics that were left out from the first schema of the framework, that is, by the mathematical QEs Lakatos talks about. In my framework, a QE can be of two kinds: (i) the decomposition of some conjecture into lemmas, by dissecting a problem into sub-problems and (2) the search for connections and deductive paths. This region takes in counterfactual reasoning, as for example when a mathematician imagines other axioms as an alternative to the accepted ones. According to my framework, this is the form of experimentation that relies more on the language of mathematics, that is, on the available formalism. However, in this case, the formalism does not serve as an anchor – if that were the case, we would have a RE – but just as a material “support” to the reasoning. As a consequence, the interaction with this support does not involve perception and action in the same relevant way as in the other two regions; of course one has to “see” the formal sentences to read and understand them, but it suffices to apply syntactic abstract rules in order to manipulate them. To some extent, QEs are closer to mathematical “arguments” as Norton defines them.¹⁶

None of the examples presented above can be put into this part of the framework, which is not surprising, given the intrinsically geometrical nature of the three of them. However, several of the cases of experiments discussed in the literature, in particular in van Bendegem (1998, 2003, forthcoming), are QEs according to my framework. Moreover, this region allows me to accommodate also EM: computers have now a role as “quasi-experimenters”, since they cannot rely neither on mental models nor on material anchors, but only on the mathematical formalism.

¹⁵ I thank Anouk Barberousse for having pointed out this issue to me.

¹⁶ Larvor (2013) argues that some arguments in mathematics are not in principle verbal but *inscribed*, that is, a form of syntactic reasoning exploiting mathematical notation. Since he focuses on the written formalisms, I would suggest that what he refers to is a limit case between QEs and REs, where the formalism is more than a simple material support but becomes a material anchor.

Let me then sum up what we have so far: mathematics as an experiment activity is divided into three regions, all subjects to *notified* constraints: (i) TEs, implying imagination and mental models, are subject to *internalized* constraints; (ii) REs make use of material anchors, whose structure is designed as having *built-in* constraints; (iii) QEs allow exploring counterfactual situations based on mathematical formalism as a material support; the category of QEs includes the appeal to computers and thus EM (see Figure 6).

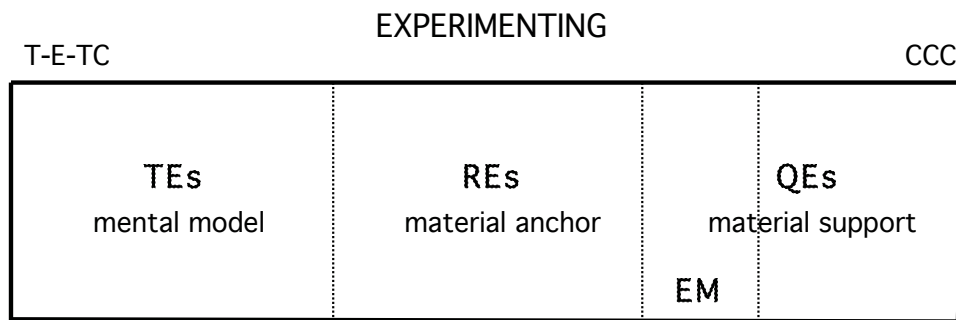


Figure 6. A schema of the three regions in the experimenting activity

Is this all? As I pointed out in the Introduction, Ps are still an important element of the practice of mathematics. In the next paragraph, I will show how my framework allows accommodating them too in the general picture.

3.5 The proving activity

P is part of mathematics and for this reason and in contrast with radical non-standard views, a specific place will be given in the framework to the *proving* activity as distinguished – at least conceptually – from the experimenting practice. However, remember that all the lines in the framework are blurred, thus allowing for mixed and limit cases as already seen for the experimenting activity; moreover, my aim is to try to be orthogonal to the – in my view – misleading distinction between context of discovery and context of justification.

First of all, in my framework, the proving activity is an extension only of the REs and QEs regions, since it is possible that in the two cases a mathematician will get from an experiment to a proof. This is not the case for TEs, which have lesser epistemic strength. As discussed above, in TEs mathematicians play with their imagination and “see” possibilities; however, the internalized constraints have to be carefully checked, and the natural way to go would be to start from a TE and then find ways to define a RE or a QE based on the insight obtained in the TE.

If REs become proofs, I define them “experimental” proofs (henceforth, EP), where anchors and their built-in constraints are still part of the reasoning; if QEs become proofs, I define them “formal” proofs (henceforth, FP), that is, proofs that are characterized by the use of some particular formalism and considered as rigorous. However, let me point out again that what is considered as an accepted P might change across time and among different communities, so the scope of the EP region can vary according to the mathematical cultures considered each time. For example, think again of Freiling’s axiom of symmetry. Depending on how much its mixed

nature is considered as a flaw, his argument can be considered as an EP or not. However, the FP region is by nature the one that is more rigorous and more rigorously defined. When it comes to FPs, disagreements among mathematicians are normally solved, and there is finally epistemic peace: mathematicians agree that what has been provided is indeed a proof; this is the only region in the framework where there is no point of view.¹⁷ Thus, FPs solve disagreements; despite the fact that this does not mean that they are necessary for mathematicians to believe in some particular result, it is a fact that when a FP is given, even if it does not respect the way in which the original informal proof was given and is a totally different proof, the result is considered as certain.¹⁸ According to the framework then, FPs are defined in opposition to “informal” proofs, that is, EPs but also some cases of QEs and REs; despite the fact that FPs are not necessarily the most creative part of mathematics – the strategy behind a FPs is normally identified in a REs or a QEs – they are still, and with good reason, a crucial part of the mathematical activity.¹⁹

Figure 7 illustrates the final general framework: this includes an experimenting zone, which is divided in the regions of TEs, REs and QEs, and a proving zone, which is divided in the regions of EPs and FPs and confines respectively with the REs and the QEs regions. It should not be forgotten that all internal lines in the framework are blurred, and that limit and mixed cases are allowed.

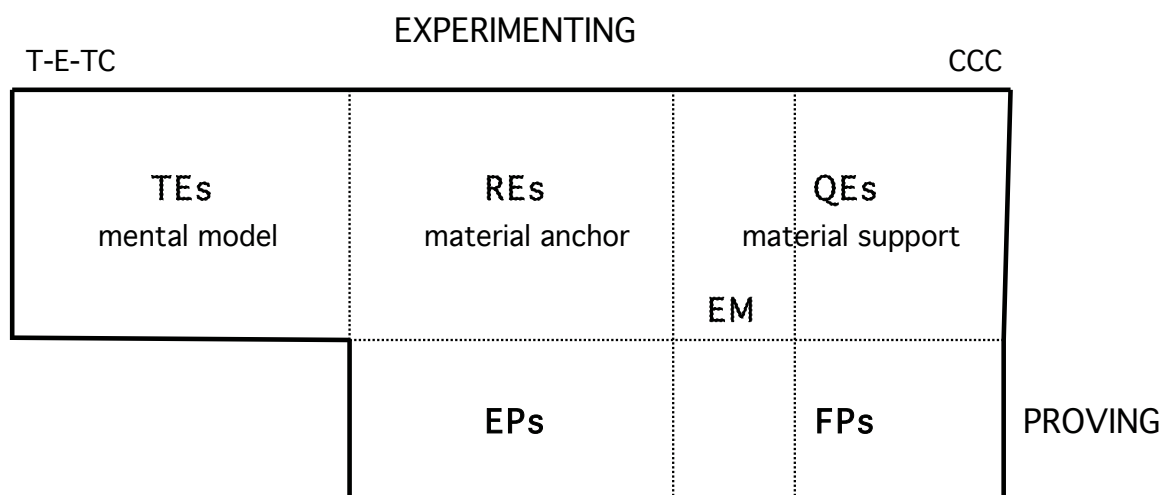


Figure 7. A schema of the general framework

4. Conclusions: the advantages of the framework

¹⁷ De Toffoli, S. (2021) argues that the main function of mathematical justification deriving from rigorous proofs is to guarantee that the mathematical community can correct errors that inevitable arising from our human fallible practices. Hamami (forthcoming) recognize that there are non-deductive reliable methods, but that it is thanks to deductive methods proofs that contain repairable mistakes can be corrected, and proofs that cannot be repaired will be rejected.

¹⁸ Rittberg *et al.* (2020) aims to show that epistemic injustice can manifest itself also in the practice of mathematics, and to do so the authors discuss a case where a FP is subject to doubt, since it proves a mathematical statement that is considered by a part of the community so obvious – in relation to “the existing body of knowledge” of the field – that it would need no proof.

¹⁹ On the relationship between rigorous and formal proofs, considering the practice of proving, see Hamami (2022).

I started with questions about the existence of TEs in mathematics and their relationship with other kinds of experiments. The framework I propose here presents several advantages, by dividing the activity of mathematics into different regions. If notified constraints characterize the mathematical activity as a whole, some constraints are specific to regions of it.²⁰

A first advantage is that this framework is based on the practice of mathematics and it is to some extent orthogonal to the opposition that we find in the discussion in philosophy of science between TEs as visualizations versus TEs as arguments. More importantly, the framework highlights the importance of visualizations but distinguishes between a kind of visualization that is more “free” and related to imagination and internalized constrained – as in TEs – and another kind of visualization relying on diagrams and more in general on systems of representations and their built-in constraints. This allows us distinguishing between so-called picture-proofs that are TEs or REs and to evaluate when and whether they reach the status of EPs.

A second advantage is that the reasoning that results from these forms of experimentation is not only of a visual nature, but includes the action on some representations – either mental or material. For this reason, the framework offers a way to distinguish between TEs and REs in pure mathematics, despite the fact that there is no empirical counterpart to take into account; however, this helps define an analogy with what happens in the laboratory, since also in the natural sciences there exist both mental TEs and material REs using the available tools.

A third advantage is that by specifying the role of technology, the framework offers a reasonable way to think of at least a part of mathematics as “embodied”. In TEs, REs, and EPs our embodied knowledge and experience is relevant; this is less so in the case of QEs, EM and FP.

A fourth advantage is that the framework rehabilitates QEs in mathematics beside visualization and representational systems, thus including counterfactual thinking based on the expressive power of formal languages and allowing fitting EM – which is a reality in contemporary mathematics – into the framework.

A fifth and last advantage is that the framework is very flexible; as I have already pointed out, the internal lines are blurred, so as to allow for accommodating limit or mixed cases. This responds to the objective of the article, which was to offer some grid to make order in the conceptual space between experimenting and proving, from the point of view of the mathematical practice.

To conclude, in this article I argued that mathematical practice is subject to notified constraints as a whole; one possible objection could be that I did not explain how notified constraints are established from the beginning. However, this issue is orthogonal to the framework, and is indeed one of the central questions for a philosophy of the mathematical practice. For this reason, I believe that my

²⁰ The aim of this article is in spirit very close to van Bendegem (2022), which presents an alternative general and complex scheme to accommodate mathematical thought experiments.

framework is for the moment safe from this objection, and that the question about where notified constraints come from can be matter of further research.

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