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# Stochastic Primal Dual Hybrid Gradient Algorithm with Adaptive Step-Sizes

Antonin Chambolle\*      Claire Delplancke†      Matthias J. Ehrhardt‡  
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## Abstract

In this work we propose a new primal-dual algorithm with adaptive step-sizes. The stochastic primal-dual hybrid gradient (SPDHG) algorithm with constant step-sizes has become widely applied in large-scale convex optimization across many scientific fields due to its scalability. While the product of the primal and dual step-sizes is subject to an upper-bound in order to ensure convergence, the selection of the ratio of the step-sizes is critical in applications. Up-to-now there is no systematic and successful way of selecting the primal and dual step-sizes for SPDHG. In this work, we propose a general class of adaptive SPDHG (A-SPDHG) algorithms, and prove their convergence under weak assumptions. We also propose concrete parameters-updating strategies which satisfy the assumptions of our theory and thereby lead to convergent algorithms. Numerical examples on computed tomography demonstrate the effectiveness of the proposed schemes.

## 1 Introduction

The stochastic primal-dual hybrid gradient (SPDHG) algorithm introduced in [8] is a stochastic version of the primal-dual hybrid gradient (PDHG) algorithm, also known as Chambolle–Pock algorithm [9]. SPDHG has proved more efficient than PDHG for a variety of problems in the framework of large-scale non-smooth convex inverse problems [13, 22, 24, 27]. Indeed, SPDHG only uses a subset of the data at each iteration, hence reducing the computational cost of evaluating the forward operator and its adjoint; as a result, for the same computational burden, SPDHG attains convergence faster than PDHG. This is especially relevant in the context of medical imaging, where there is a need for algorithms whose convergence speed is compatible with clinical standards, and at the same time able to deal with convex, non-smooth priors like Total Variation (TV), which are well-suited to ill-posed imaging inverse problems, but preclude the recourse to scalable gradient-based methods.

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Like PDHG, SPDHG is provably convergent under the assumption that the product of its primal and dual step-sizes is bounded by a constant depending on the problem to solve. On the other hand, the ratio between the primal and dual step-sizes is a free parameter, whose value needs to be chosen by the user. The value of this parameter, which can be interpreted as a control on balance between primal and dual convergence, can have a severe impact on the convergence speed of PDHG, and the same also holds true for SPDHG [12]. This leads to an important challenge in practice, as there is no known theoretical or empirical rule to guide the choice of the parameter. Manual tuning is computationally expensive, as it would require running and comparing the algorithm on a range on values, and there is no guarantee that a value leading to fast convergence for one dataset would keep being a good choice for another dataset. For PDHG, [14] have proposed an online primal-dual balancing strategy to solve the issue, where the values of the step-sizes evolve along the iterations. More generally, adaptive step-sizes have been used for PDHG with backtracking in [14, 20], adapting to local smoothness in [25] and are widely used for a variety of other algorithms, namely gradient methods in [19], subgradient methods in [3] and splitting methods in [4, 6, 5, 7, 18] to improve convergence speed and bypass the need for explicit model constants, like Lipschitz constants or operator norms. For SPDHG, an empirical adaptive scheme has been used for Magnetic Particle Imaging but without convergence proof [27].

On the theoretical side, a standard procedure to prove the convergence of proximal-based algorithms for convex optimization is to use the notion of Féjer-monotonicity [2]. Constant step-sizes lead to a fixed metric setting, while adaptive step-sizes lead to a variable metric setting. Work [11] states the convergence of deterministic Féjer-monotone sequences in the variable metric setting, while work [10] is concerned by the convergence of random Féjer-monotone sequences in the fixed metric setting.

In this work, we introduce and study an adaptive version of SPDHG. More precisely:

- We introduce a broad class of strategies to adaptively choose the step-sizes of SPDHG. This class includes, but is not limited to, the adaptive primal-dual balancing strategy, where the ratio of the step-sizes, which controls the balance between convergence of the primal and dual variable, is tuned online.
- We prove the almost-sure convergence of SPDHG under the schemes of the class. In order to do that, we introduce the concept of  $C$ -stability, which generalizes the notion of Féjer-monotonicity, and we prove the convergence of random  $C$ -stable sequences in a variable metric setting, hence generalizing results from [11] and [10]. We then show that our proposed algorithm falls within this novel theoretical framework by following similar strategies than in the almost-sure convergence proofs of [16, 1].
- We compare the performance of SPDHG for various adaptive schemes and the known fixed step-size scheme on large-scale imaging inverse tasks (sparse-view CT, limited-angle CT, low-dose CT). We observe that the primal-dual balancing adaptive strategy is always as fast or faster than all the other strategies. In particular, it consistently leads to substantial gains in convergence speed over the fixed strategy if the fixed step-sizes, while in the theoretical convergence range, are badly chosen. This is especially relevant as it is impossible to know whether the fixed step-sizes are well or badly chosen without running expensive comparative tests. Even in the cases where the SPDHG's fixed step-sizes are well-tuned, meaning that they are in the range to which the adaptive step-sizes are observed to converge, we observe that our adaptive scheme still provides convergence acceleration over the standard SPDHG

after a certain number of iterations. Finally, we pay special attention to the hyperparameters used in the adaptive schemes. These hyperparameters are essentially controlling the degree of adaptivity for the algorithm and each of them has a clear interpretation and is easy to choose in practice. We observe in our extensive numerical tests that the convergence speed of our adaptive scheme is robust to the choices of these parameters within the empirical range we provide, hence can be applied directly to the problem at hand without fine-tuning, and solves the step-sizes choice challenge encountered by the user.

The rest of the paper is organized as follows. In Section 2, we introduce SPDHG with adaptive step-sizes, state the convergence theorem, and carry the proof. In Section 3, we propose concrete schemes to implement the adaptiveness, followed by numeric tests on CT data in Section 4. We conclude in Section 5. Finally, Section 6 collects some useful lemmas and proofs.

## 2 Theory

### 2.1 Convergence theorem

The variational problem to solve takes the form:

$$\min_{x \in X} \sum_{i=1}^n f_i(A_i x) + g(x),$$

where  $X$  and  $(Y_i)_{i \in \{1, \dots, n\}}$  are Hilbert spaces,  $A_i : X \rightarrow Y_i$  are bounded linear operators,  $f_i : Y_i \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$  are convex functions. We define  $Y = Y_1 \times \dots \times Y_n$  with elements  $y = (y_1, \dots, y_n)$  and  $A : X \rightarrow Y$  such that  $Ax = (A_1 x, \dots, A_n x)$ . Each iteration of SPDHG involves the selection of a random subset of  $\llbracket 1, n \rrbracket := \{1, \dots, n\}$ . In this article, we are interested in the serial sampling case, where the random subset is a singleton.

SPDHG with constant step-sizes is described in Algorithm 2.1. Under the condition

$$\tau \sigma_i < \frac{p_i}{\|A_i\|^2}, \quad i \in \llbracket 1, n \rrbracket, \quad (2.1)$$

SPDHG iterates converge almost surely to a solution of the saddle-point problem [1, 16]:

$$\min_{x \in X} \sup_{y \in Y} \sum_{i=1}^n \langle A_i x, y_i \rangle - f_i^*(y_i) + g(x). \quad (2.2)$$

The set of solution to (2.2) is denoted by  $\mathcal{C}$ . Elements  $(x^*, y^*)$  of  $\mathcal{C}$  are called saddle-points and characterized by

$$A_i x^* \in \partial f_i^*(y_i^*), \quad -A_i^* y^* \in \partial g(x^*), \quad i \in \llbracket 1, n \rrbracket. \quad (2.3)$$

We introduce the adaptive stochastic primal-dual hybrid gradient (A-SPDHG) algorithm in Algorithm 2.2. The main theorem, Theorem 2.1 below, gives conditions on the update rule under which A-SPDHG is provably convergent. Plainly speaking, these conditions are threefold:

- (i) the step-sizes for step  $k+1$ ,  $(\sigma_i^{k+1})_{i \in \llbracket 1, n \rrbracket}$  and  $\tau^{k+1}$ , depend only of the iterates up to step  $k$ ,
- (ii) the step-sizes satisfy to a uniform version of condition (2.1),

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**Algorithm 2.1:** SPDHG (constant step-sizes, serial sampling)

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Input: dual step-sizes  $(\sigma_i)_{i \in \llbracket 1, n \rrbracket}$ , primal step-size  $\tau$ ; probabilities  $(p_i)_{i \in \llbracket 1, n \rrbracket}$ ; primal variable  $x^0$ , dual variable  $y^0$   
Initialize  $\bar{y}^0 = y^0$   
**for**  $k \in \llbracket 0, K - 1 \rrbracket$  **do**  
     $x^{k+1} = \text{prox}_{\tau g}(x^k - \tau A^* \bar{y}^k)$   
    Randomly pick  $i \in \llbracket 1, n \rrbracket$  with probability  $p_i$   
     $y_j^{k+1} = \begin{cases} \text{prox}_{\sigma_i f_i^*}(y_i^k + \sigma_i A_i x^{k+1}) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
     $\bar{y}_j^{k+1} = \begin{cases} y_i^{k+1} + \frac{1}{p_i} (y_i^{k+1} - y_i^k) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
**end for**  
**return**  $x^K$

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**Algorithm 2.2:** A-SPDHG (variable step-sizes, serial sampling)

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Input: dual step-sizes  $(\sigma_i^0)_{i \in \llbracket 1, n \rrbracket}$ , primal step-size  $\tau^0$ , update rule; probabilities  $(p_i)_{i \in \llbracket 1, n \rrbracket}$ ; primal variable  $x^0$ , dual variable  $y^0$   
Initialize  $\bar{y}^0 = y^0$   
**for**  $k \in \llbracket 0, K - 1 \rrbracket$  **do**  
    Determine  $(\sigma_i^{k+1})_{i \in \llbracket 1, n \rrbracket}$ ,  $\tau^{k+1}$  according to the update rule  
     $x^{k+1} = \text{prox}_{\tau^{k+1} g}(x^k - \tau^{k+1} A^* \bar{y}^k)$   
    Randomly pick  $i \in \llbracket 1, n \rrbracket$  with probability  $p_i$   
     $y_j^{k+1} = \begin{cases} \text{prox}_{\sigma_i^{k+1} f_i^*}(y_i^k + \sigma_i^{k+1} A_i x^{k+1}) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
     $\bar{y}_j^{k+1} = \begin{cases} y_i^{k+1} + \frac{1}{p_i} (y_i^{k+1} - y_i^k) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
**end for**  
**return**  $x^K$

---

- (iii) the step-sizes sequences  $(\tau^k)_{k \geq 0}$  and  $(\sigma_i^k)_{k \geq 0}$  for  $i \in \llbracket 1, n \rrbracket$  do not decrease too fast. More precisely, they are uniformly almost surely quasi-increasing in the sense defined below.

In order to state the theorem rigorously, let us introduce some useful notation and definitions. The set of non-negative integers is denoted by  $\mathbb{N}$ . For all  $k \in \mathbb{N}$ , the  $\sigma$ -algebra generated by the iterates up to point  $k$ ,  $\mathcal{F}((x^l, y^l), l \in \llbracket 0, k \rrbracket)$ , is denoted by  $\mathcal{F}^k$ . We say that a sequence  $(u^k)_{k \in \mathbb{N}}$  is  $(\mathcal{F}^k)_{k \in \mathbb{N}}$ -adapted if for all  $k \in \mathbb{N}$ ,  $u^k$  is measurable with respect to  $\mathcal{F}^k$ .

A positive real sequence  $(u^k)_{k \in \mathbb{N}}$  is said to be *quasi-increasing* if there exists a sequence  $(\eta^k)_{k \in \mathbb{N}}$  with values in  $[0, 1)$ , called the control on  $(u^k)_{k \in \mathbb{N}}$ , such that  $\sum_{k=1}^{\infty} \eta_k < \infty$  and :

$$u^{k+1} \geq (1 - \eta^k)u^k, \quad k \in \mathbb{N}. \quad (2.4)$$

By extension, we call a random positive real sequence  $(u^k)_{k \in \mathbb{N}}$  *uniformly almost surely quasi-increasing* if there exists a deterministic sequence  $(\eta^k)_{k \in \mathbb{N}}$  with values in  $[0, 1)$  such that  $\sum_{k=1}^{\infty} \eta_k < \infty$  and equation (2.4) above holds almost surely (a.s.).

**Theorem 2.1** (Convergence of A-SPDHG). *Let  $X$  and  $Y$  be separable Hilbert spaces,  $A_i : X \rightarrow Y_i$  bounded linear operators,  $f_i : Y_i \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$  proper, convex and lower semi-continuous functions for all  $i \in \llbracket 1, n \rrbracket$ . Assume that the set of saddle-points  $\mathcal{C}$  is non-empty and the sampling is proper, that is to say  $p_i > 0$  for all  $i \in \llbracket 1, n \rrbracket$ . If the following conditions are met:*

- (i) *the step-size sequences  $(\tau^{k+1})_{k \in \mathbb{N}}$ ,  $(\sigma_i^{k+1})_{k \in \mathbb{N}}$ ,  $i \in \llbracket 1, n \rrbracket$  are  $(\mathcal{F}^k)_{k \in \mathbb{N}}$ -adapted,*
- (ii) *there exists  $\beta \in (0, 1)$  such that for all index  $i \in \llbracket 1, n \rrbracket$  and iterate  $k \in \mathbb{N}$ ,*

$$\tau^k \sigma_i^k \frac{\|A_i\|^2}{p_i} \leq \beta < 1, \quad (2.5)$$

- (iii) *the initial step-sizes  $\tau^0$  and  $\sigma_i^0$  for all index  $i \in \llbracket 1, n \rrbracket$  are positive and the step-sizes sequences  $(\tau^k)_{k \in \mathbb{N}}$  and  $(\sigma_i^k)_{k \in \mathbb{N}}$  for all index  $i \in \llbracket 1, n \rrbracket$  are uniformly almost surely quasi-increasing,*

*then the sequence of iterates  $(x^k, y^k)_{k \in \mathbb{N}}$  converges almost surely to an element of  $\mathcal{C}$ .*

While the conditions (i)-(iii) are general enough to cover a large range of step-sizes update rules, we will focus in practice on the primal-dual balancing strategy, which consists in scaling the primal and the dual step-sizes by an inverse factor at each iteration. In that case, the update rule depends on a random positive sequence  $(\gamma^k)_{k \in \mathbb{N}}$  and reads as:

$$\tau^{k+1} = \frac{\tau^k}{\gamma^k}, \quad \sigma_i^{k+1} = \gamma^k \sigma_i^k, \quad i \in \llbracket 1, n \rrbracket. \quad (2.6)$$

**Lemma 2.2** (Primal-dual balancing). *Let the step-sizes sequences satisfy to equation (2.6) and assume in addition that  $(\gamma^k)_{k \in \mathbb{N}}$  is  $(\mathcal{F}^k)_{k \in \mathbb{N}}$ -adapted, that the initial step-sizes satisfy to*

$$\tau^0 \sigma_i^0 \frac{\|A_i\|^2}{p_i} < 1, \quad i \in \llbracket 1, n \rrbracket,$$

and are positive, that there exists a deterministic sequence  $(\epsilon^k)_{k \in \mathbb{N}}$  with values in  $[0, 1)$  such that  $\sum \epsilon^k < \infty$  and for all  $k \in \mathbb{N}$  and  $i \in \llbracket 1, n \rrbracket$ ,

$$\min \{ \gamma^k, (\gamma^k)^{-1} \} \geq 1 - \epsilon^k. \quad (2.7)$$

Then, the step-sizes sequences satisfy to assumptions (i)-(iii) of Theorem 2.1.

Lemma 2.2 is proved in Section 6.

*Connection with the literature:*

- The primal-dual balancing strategy has been introduced in [14] for PDHG and indeed for  $n = 1$  we recover with Lemma 2.2 the non-backtracking algorithm presented in [14]. As a consequence, our theorem also implies the pointwise convergence of this algorithm, whose convergence was established in the sense of vanishing residuals in [14].
- Still for PDHG, [20] proposes without proof an update rule where the ratio of the step-sizes is either quasi non-increasing or quasi non-decreasing. This requirement is similar to but not directly connected with ours, where we ask the step-sizes themselves to be quasi non-increasing.
- For SPDHG, the angular constraint step-size rule proposed without convergence proof in [27] satisfies to assumptions (i)-(iii).

*Outline of the proof:* Theorem 2.1 is proved in the following sub-sections. We first define in Section 2.2 metrics related to the algorithm step-sizes on the primal-dual product space. As the step-sizes are adaptive, we obtain a sequence of metrics. The proof of Theorem 2.1 is then similar in strategy to those of [1] and [16] but requires novel elements to deal with the metrics variability. In Theorem 2.5, we state convergence conditions for an abstract random sequence in a Hilbert space equipped with random variable metrics. In Section 2.4 and Section 2.5 we show that A-SPDHG falls within the scope of Theorem 2.5. We collect all elements and conclude the proof in Section 2.6.

## 2.2 Variable metrics

For a Hilbert space  $H$ , we call  $\mathbb{S}(H)$  the set of bounded self-adjoint linear operators from  $H$  to  $H$ , and for all  $M \in \mathbb{S}(H)$  we introduce the notation:

$$\|u\|_M^2 = \langle Mu, u \rangle, \quad u \in H.$$

By an abuse of notation we write  $\|\cdot\|_\alpha^2 = \|\cdot\|_{\alpha \text{Id}}^2$  for a scalar  $\alpha \in \mathbb{R}$ . Notice that  $\|\cdot\|_M$  is a norm on  $H$  if  $M$  is positive definite. Furthermore, we introduce the partial order  $\preceq$  on  $\mathbb{S}(H)$  such that for  $M, N \in \mathbb{S}(H)$ ,

$$N \preceq M \quad \text{if} \quad \forall u \in H, \|u\|_N \leq \|u\|_M.$$

We call  $\mathbb{S}_\alpha(H)$  the subset of  $\mathbb{S}(H)$  comprised of  $M$  such that  $\alpha \text{Id} \preceq M$ . Furthermore a random sequence  $(M^k)_{k \in \mathbb{N}}$  in  $\mathbb{S}(H)$  is said to be *uniformly almost surely quasi-decreasing* if there exists a non-negative sequence  $(\eta^k)_{k \in \mathbb{N}}$  such that  $\sum_{k=1}^{\infty} \eta^k < \infty$  and a.s.

$$M^{k+1} \preceq (1 + \eta^k) M^k, \quad k \in \mathbb{N}.$$

Coming back to A-SPDHG, let us define for every iteration  $k \in \mathbb{N}$  and every index  $i \in \llbracket 1, n \rrbracket$  two block operators of  $\mathbb{S}(X \times Y_i)$  as:

$$M_i^k = \begin{pmatrix} \frac{1}{\tau^k} \text{Id} & -\frac{1}{p_i} A_i \\ -\frac{1}{p_i} A_i^* & \frac{1}{p_i \sigma_i^k} \text{Id} \end{pmatrix}, \quad N_i^k = \begin{pmatrix} \frac{1}{\tau^k} \text{Id} & 0 \\ 0 & \frac{1}{p_i \sigma_i^k} \text{Id} \end{pmatrix},$$

and a block operator of  $\mathbb{S}(X \times Y)$  as:

$$N^k = \begin{pmatrix} \frac{1}{\tau^k} \text{Id} & & & 0 \\ & \ddots & & \\ & & \frac{1}{p_i \sigma_i^k} \text{Id} & \\ 0 & & & \ddots \end{pmatrix}. \quad (2.8)$$

The following lemma translates assumptions (i)-(iii) of Theorem 2.1 on properties on the variable metric sequences.

**Lemma 2.3** (Variable metric properties). *(a) Assumption (i) of Theorem 2.1 implies that  $(M_i^k)_{k \in \mathbb{N}}, (N_i^k)_{k \in \mathbb{N}}, i \in \llbracket 1, n \rrbracket$  and  $(N^k)_{k \in \mathbb{N}}$  are  $(\mathcal{F}^k)_{k \in \mathbb{N}}$ -adapted.*

*(b) Assumption (ii) of Theorem 2.1 is equivalent to the the existence of  $\beta \in (0, 1)$  such that for all index  $i \in \llbracket 1, n \rrbracket$  and iterate  $k \in \mathbb{N}$ ,*

$$(1 - \beta)N_i^k \preceq M_i^k.$$

*(c) Assumption (ii) and (iii) of Theorem 2.1 implies that  $(M_i^k)_{k \in \mathbb{N}}, (N_i^k)_{k \in \mathbb{N}}, i \in \llbracket 1, n \rrbracket$  and  $(N^k)_{k \in \mathbb{N}}$  are uniformly a.s. quasi-decreasing.*

*(d) Assumption (ii) and (iii) of Theorem 2.1 imply that there exists  $\alpha > 0$  such that for all index  $i \in \llbracket 1, n \rrbracket$  and iterate  $k \in \mathbb{N}$ ,  $\tau^k > \alpha^{-1}$  and  $\sigma_i^k > \alpha^{-1}$ , or equivalently that  $N_i^k \in \mathbb{S}_\alpha(X \times Y_i)$  for all  $i \in \llbracket 1, n \rrbracket$  and  $k \in \mathbb{N}$ , or equivalently that  $N^k \in \mathbb{S}_\alpha(X \times Y)$  for all  $k \in \mathbb{N}$ .*

**Remark 2.4** (Step-sizes induced metrics on the primal-dual product space). *The lemma implies that  $M_i^k, N_i^k$  and  $N^k$  are positive definite, hence induce a metric on the corresponding spaces. If  $n = 1$  and for constant step-sizes,  $M_i^k$  corresponds to the metric used in [17], where PDHG is reformulated as a proximal point algorithm for a non-trivial metric on the primal-dual product space.*

*Proof of Lemma 2.3.* Assertion (a) of the lemma follows from the fact that for all iterate  $k \in \mathbb{N}$ , the operators  $M_i^k, N_i^k$  and  $N^k$  are in the  $\sigma$ -algebra generated by  $\{\tau^k, \sigma_i^k, i \in \llbracket 1, n \rrbracket\}$ . Assertion (b) follows from equation (6.2) of Lemma 6.1 to be found in the complementary material. The proof of assertion (c) is a bit more involved. Let us assume that assumption (iii) of Theorem 2.1 holds and let  $(\eta_0^k)_{k \in \mathbb{N}}$  and  $(\eta_i^k)_{k \in \mathbb{N}}$  be the controls of  $(\tau^k)_{k \in \mathbb{N}}$  and  $(\sigma_i^k)_{k \in \mathbb{N}}$  for  $i \in \llbracket 1, n \rrbracket$  respectively. We define a common control  $(\eta^k)_{k \in \mathbb{N}}$  by:

$$\eta^k = \max \{ \eta_i^k, i \in \llbracket 0, n \rrbracket \}, \quad k \in \mathbb{N}. \quad (2.9)$$



Let us fix  $k \in \mathbb{N}$  and  $i \in \llbracket 1, n \rrbracket$ . It holds almost surely that for all  $(x, y_i) \in X \times Y_i$ ,

$$\begin{aligned} \|x, y_i\|_{N_i^{k+1}}^2 &= \frac{1}{\tau^{k+1}} \|x\|^2 + \frac{1}{\sigma_i^{k+1}} \|y_i\|^2 \\ &\leq \frac{1}{1 - \eta^k} \left( \frac{1}{\tau^k} \|x\|^2 + \frac{1}{\sigma_i^k} \|y_i\|^2 \right) \\ &= \frac{1}{1 - \eta^k} \|x, y_i\|_{N_i^k}^2. \end{aligned}$$

Hence the sequence  $(N_i^k)_{k \in \mathbb{N}}$  is uniformly quasi-decreasing with control  $((1 - \eta^k)^{-1})_{k \in \mathbb{N}}$ , which is indeed a positive sequence with bounded sum. One can see by a similar proof that  $(N^k)_{k \in \mathbb{N}}$  is uniformly quasi-decreasing with the same control. To follow with the case of  $(M_i^k)_{k \in \mathbb{N}}$ , let us first reformulate the desired conclusion. By equation (6.2) of Lemma 6.1,  $(M_i^k)_{k \in \mathbb{N}}$  is uniformly quasi-decreasing with control  $(\epsilon^k)_{k \in \mathbb{N}}$  if and only if a.s. for all  $k \in \mathbb{N}$

$$\begin{aligned} &\begin{pmatrix} \frac{1}{\tau^{k+1}} \text{Id} & -\frac{1}{p_i} A_i \\ -\frac{1}{p_i} A_i^* & \frac{1}{p_i \sigma_i^{k+1}} \text{Id} \end{pmatrix} \preceq (1 + \epsilon^k) \begin{pmatrix} \frac{1}{\tau^k} \text{Id} & -\frac{1}{p_i} A_i \\ -\frac{1}{p_i} A_i^* & \frac{1}{p_i \sigma_i^k} \text{Id} \end{pmatrix} \\ \Leftrightarrow 0 &\preceq \begin{pmatrix} \left( \frac{1+\epsilon^k}{\tau^k} - \frac{1}{\tau^{k+1}} \right) \text{Id} & -\frac{\epsilon^k}{p_i} A_i \\ -\frac{\epsilon^k}{p_i} A_i^* & \left( \frac{1+\epsilon^k}{p_i \sigma_i^k} - \frac{1}{p_i \sigma_i^{k+1}} \right) \text{Id} \end{pmatrix} \\ \Leftrightarrow \tau^k \sigma_i^k \frac{\|A_i\|^2}{p_i} &\frac{(\epsilon^k)^2}{\left(1 + \epsilon^k - \frac{\tau^k}{\tau^{k+1}}\right) \left(1 + \epsilon^k - \frac{\sigma_i^k}{\sigma_i^{k+1}}\right)} \leq 1. \end{aligned} \tag{2.10}$$

Now, by assumption (ii), there exists  $\beta \in (0, 1)$  such that  $\tau^k \sigma_i^k \|A_i\|^2 p_i^{-1} \leq \beta$  for all  $k \in \mathbb{N}$ . Let us define

$$\epsilon^k = \lambda^{-1} \left( \frac{1}{1 - \eta^k} - 1 \right), \quad k \in \mathbb{N},$$

with  $\lambda$  a real number such that  $0 < \lambda \leq 1 - \sqrt{\beta}$ . Then,  $(\epsilon^k)_{k \in \mathbb{N}}$  is a positive sequence with bounded sum and it holds that for all  $k \in \mathbb{N}$

$$\frac{\tau^k}{\tau^{k+1}} \leq \frac{1}{1 - \eta^{k+1}}, \quad \frac{\sigma_i^k}{\sigma_i^{k+1}} \leq \frac{1}{1 - \eta^{k+1}}, \quad \frac{1}{1 - \eta^{k+1}} = \lambda \epsilon^k + 1.$$

As a consequence, the left-hand side of (2.10) is bounded by above by

$$\beta \frac{(\epsilon^k)^2}{(1 + \epsilon^k - (1 + \lambda \epsilon^k))^2} = \frac{\beta}{(1 - \lambda)^2} \leq 1,$$

hence (2.10) holds and  $(M_i^k)_{k \in \mathbb{N}}$  is uniformly quasi-decreasing with control  $(\epsilon^k)_{k \in \mathbb{N}}$ .

To conclude with the proof of assertion (c), observe that by assumption (ii), the product of the sequences  $(\tau^k)_{k \in \mathbb{N}}$  and  $(\sigma_i^k)_{k \in \mathbb{N}}$  is almost surely bounded by above. Furthermore, each sequence

$(\tau^k)_{k \in \mathbb{N}}$  and  $(\sigma_i^k)_{k \in \mathbb{N}}$  is uniformly a.s. quasi-increasing, hence is a.s. bounded by below by the deterministic constant  $C = \min \{ \tau^0, \sigma_i^0, i \in \llbracket 1, n \rrbracket \} \prod_{j=1}^{\infty} (1 - \eta^j)$  which is positive as the initial step-sizes are positive and  $(\eta^k)_{k \in \mathbb{N}}$  takes values in  $[0, 1)$  and has finite sum. As a consequence, each sequence  $(\tau^k)_{k \in \mathbb{N}}$  and  $(\sigma_i^k)_{k \in \mathbb{N}}$  is a.s. bounded by above by a deterministic constant  $\alpha^{-1} > 0$ . The equivalence with  $N_i^k \in \mathbb{S}_\alpha(X \times Y_i)$  for all  $i \in \llbracket 1, n \rrbracket$ , and with  $N^k \in \mathbb{S}_\alpha(X \times Y)$ , is straightforward.  $\square$

### 2.3 Convergence of random $C$ -stable sequences in random variable metrics

Let  $H$  be a Hilbert space and  $C \subset H$  a subset of  $H$ . Let  $(\Omega, \sigma(\Omega), \mathbb{P})$  be a probability space. All random variables in the following are assumed to be defined on  $\Omega$  and measurable with respect to  $\sigma(\Omega)$  unless stated otherwise. Let  $(Q^k)_{k \in \mathbb{N}}$  be a random sequence of  $\mathbb{S}(H)$ .

A random sequence  $(u^k)_{k \in \mathbb{N}}$  with values in  $H$  is said to be *stable with respect to the target  $C$  relative to  $(Q^k)_{k \in \mathbb{N}}$*  if for all  $u \in C$ , the sequence  $(\|u^k - u\|_{Q^k})_{k \in \mathbb{N}}$  converges almost surely. The following theorem then states sufficient conditions for the convergence of such sequences.

**Theorem 2.5** (Convergence of  $C$ -stable sequences). *Let  $H$  be a separable Hilbert space,  $C$  a closed non-empty subset of  $H$ ,  $(Q^k)_{k \in \mathbb{N}}$  a random sequence of  $\mathbb{S}(H)$ , and  $(u^k)_{k \in \mathbb{N}}$  a random sequence of  $H$ . If the following conditions are met:*

- (i)  $(Q^k)_{k \in \mathbb{N}}$  takes values in  $\mathbb{S}_\alpha(H)$  for a given  $\alpha > 0$  and is uniformly a.s. quasi-decreasing,
- (ii)  $(u^k)_{k \in \mathbb{N}}$  is stable with respect to the target  $C$  relative to  $(Q^k)_{k \in \mathbb{N}}$ ,
- (iii) every weak sequential cluster point of  $(u^k)_{k \in \mathbb{N}}$  is almost surely in  $C$ ,

then  $(u^k)_{k \in \mathbb{N}}$  converges almost surely weakly to a random variable in  $C$ .

Stability with respect to a target set  $C$  is implied by Féjer and quasi-Féjer monotonicity with respect to  $C$ , which have been studied either for random sequences [10] or in the framework of variable metrics [11], but to the best of our knowledge not both at the same time. The proof of Theorem 2.5 follows the same lines than [10, Proposition 2.3 (iii)] and uses two results from [11].

*Proof.* The set  $C$  is a closed subset of the separable Hilbert space  $H$ , hence is separable. Let  $\{c^n, n \in \mathbb{N}\}$  be a countable set whose closure is equal to  $C$ . Thanks to assumption (ii), there exists for all  $n \in \mathbb{N}$  a measurable subset  $\Omega_{(ii)}^n$  of  $\Omega$  with probability one such that the sequence  $(\|u^k(\omega) - c^n\|_{W^k(\omega)})_{k \in \mathbb{N}}$  converges for all  $\omega \in \Omega_{(ii)}^n$ . Furthermore, let  $\Omega_{(i)}$  and  $\Omega_{(iii)}$  be measurable subsets of  $\Omega$  of probability one corresponding to the almost sure property for assumptions (i) and (iii) respectively. Let

$$\tilde{\Omega} = \left( \bigcap_{n \geq 0} \Omega_{(ii)}^n \right) \cap \Omega_{(i)} \cap \Omega_{(iii)}.$$

As the intersection of a countable number of measurable subsets of probability one,  $\tilde{\Omega}$  is itself a measurable set of  $\Omega$  with  $\mathbb{P}(\tilde{\Omega}) = 1$ . Fix  $\omega \in \tilde{\Omega}$  for the rest of the proof.

The sequence  $(Q^k(\omega))_{k \in \mathbb{N}}$  takes values in  $\mathbb{S}_\alpha(H)$  for  $\alpha > 0$  and is quasi-decreasing with control  $(\eta^k(\omega))_{k \in \mathbb{N}}$ . Furthermore, for all  $k \in \mathbb{N}$ ,

$$\|Q^k(\omega)\| \leq \left( \prod_{j=0}^{k-1} (1 + \eta^j) \right) \|Q^0(\omega)\| \leq \left( \prod_{j=0}^{\infty} (1 + \eta^j) \right) \|Q^0(\omega)\|,$$

where the product  $\prod_{j=0}^{\infty} (1 + \eta^j)$  is finite because  $(\eta^k)_{k \in \mathbb{N}}$  is positive and summable. By [11, Lemma 2.3],  $(Q^k(\omega))_{k \in \mathbb{N}}$  converges pointwise strongly to some  $Q(\omega) \in \mathbb{S}_\alpha(H)$ .

Furthermore, for all  $x \in C$ , there exists a sequence  $(x^n)_{n \in \mathbb{N}}$  with values in  $\{c^n, n \in \mathbb{N}\}$  converging strongly to  $x$ . By assumption, for all  $n \in \mathbb{N}$ , the sequence  $(\|u^k(\omega) - x^n\|_{Q^k(\omega)})_{k \in \mathbb{N}}$  converges to a limit which shall be called  $l^n(\omega)$ . For all  $n \in \mathbb{N}$  and  $k \in \mathbb{N}$ , we can write thanks to the triangular inequality:

$$-\|x^n - x\|_{Q^k(\omega)} \leq \|u^k(\omega) - x\|_{Q^k(\omega)} - \|u^k(\omega) - x^n\|_{Q^k(\omega)} \leq \|x^n - x\|_{Q^k(\omega)}.$$

By taking the limit  $k \rightarrow +\infty$ , it follows that:

$$\begin{aligned} -\|x^n - x\|_{Q(\omega)} &\leq \liminf_{k \rightarrow \infty} \|u^k(\omega) - x\|_{Q^k(\omega)} - l^n(\omega) \\ &\leq \limsup_{k \rightarrow \infty} \|u^k(\omega) - x\|_{Q^k(\omega)} - l^n(\omega) \leq \|x^n - x\|_{Q(\omega)}. \end{aligned}$$

Taking now the limit  $n \rightarrow +\infty$  shows that the sequence  $(\|u^k(\omega) - x\|_{Q^k(\omega)})_{k \in \mathbb{N}}$  converges for all  $x \in C$ . On the other hand, because  $\omega \in \Omega_{(iii)}$ , the weak cluster points of  $(u^k(\omega))_{k \in \mathbb{N}}$  lie in  $C$ . Hence, by [11, Theorem 3.3], the sequence  $(u^k(\omega))_{k \in \mathbb{N}}$  converges almost surely to a point  $u(\omega) \in C$ .  $\square$

We are now equipped to prove Theorem 2.1. We show in Section 2.4 and Section 2.5 that A-SPDHG satisfies to points (ii) and (iii) of Theorem 2.5 respectively and conclude the proof in Section 2.6. Interestingly, the proofs of point (ii) and of point (iii) rely on two different ways of apprehending A-SPDHG. Point (ii) relies on a convex optimisation argument: by taking advantage of the measurability of the primal variable at step  $k + 1$  with respect to  $\mathcal{F}^k$ , one can write a contraction-type inequality relating the conditional expectation of the iterates' norm at step  $k + 1$  to the iterates' norm at step  $k$ . Point (iii) relies on monotone operator theory: we use the fact that the update from the half-shifted iterations  $(y^k, x^{k+1})$  to  $(y^{k+1}, x^{k+2})$  can be interpreted as a step of a proximal-point algorithm on  $X \times Y_i$  conditionally to  $i$  being the index randomly selected at step  $k$ .

## 2.4 A-SPDHG is stable with respect to the set of saddle-points

In this section, we show that  $(x^k, y^k)_{k \in \mathbb{N}}$  is stable with respect to  $\mathcal{C}$  relative to the variable metrics sequence  $(N^k)_{k \in \mathbb{N}}$  defined in equation (2.8) above. We introduce the operators  $P \in \mathbb{S}(Y)$  and  $\Sigma^k \in \mathbb{S}(Y)$  defined respectively by

$$(Py)_i = p_i y_i, \quad (\Sigma^k y)_i = \sigma_i^k y_i, \quad i \in \llbracket 1, n \rrbracket,$$

and the functionals  $(U^k)_{k \in \mathbb{N}}$ ,  $(V^k)_{k \in \mathbb{N}}$  defined for all  $(x, y) \in X \times Y$  as:

$$\begin{aligned} U^k(y) &= \|y\|_{(P\Sigma^k)^{-1}}^2, \\ V^k(x, y) &= \|x\|_{(\tau^k)^{-1}}^2 - 2\langle P^{-1}Ax, y \rangle + \|y\|_{(P\Sigma^k)^{-1}}^2. \end{aligned}$$

We begin by recalling the cornerstone inequality satisfied by the iterates of SPDHG stated first in [8] and reformulated in [1].

**Lemma 2.6** ([1], Lemma 4.1). *For every saddle-point  $(x^*, y^*)$ , it a.s. stands that for all  $k \in \mathbb{N} \setminus \{0\}$ ,*

$$\begin{aligned} & \mathbb{E} [V^{k+1}(x^{k+1} - x^*, y^{k+1} - y^k) + U^{k+1}(y^{k+1} - y^*) | \mathcal{F}^k] \\ & \leq V^{k+1}(x^k - x^*, y^k - y^{k-1}) + U^{k+1}(y^k - y^*) \\ & \quad - V^{k+1}(x^{k+1} - x^k, y^k - y^{k-1}). \end{aligned} \quad (2.11)$$

The second step is to relate the assumptions of Theorem 2.1 to properties of the functionals appearing in (2.11). Let us introduce  $Y_{\text{sparse}} \subset Y$  the set of elements  $(y_1, \dots, y_n)$  having at most one non-vanishing component.

**Lemma 2.7** (Properties of functionals of interest). *Under the assumptions of Theorem 2.1, there exists a non-negative, summable sequence  $(\eta^k)_{k \in \mathbb{N}}$  such that a.s. for every iterate  $k \in \mathbb{N}$  and  $x \in X, y \in Y, z \in Y_{\text{sparse}}$ :*

$$U^{k+1}(y) \leq (1 + \eta^k)U^k(y), \quad (2.12a)$$

$$V^{k+1}(x, z) \leq (1 + \eta^k)V^k(x, z), \quad (2.12b)$$

$$\|(x, z)\|_{N^k}^2 \geq \alpha \|(x, z)\|^2, \quad (2.12c)$$

$$V^k(x, z) \geq (1 - \beta) \|(x, z)\|_{N^k}^2, \quad (2.12d)$$

$$|\langle P^{-1}Ax, z \rangle| \leq \beta \|x\|_{(\tau^k)^{-1}}^2 \|z\|_{(P\Sigma^k)^{-1}}^2. \quad (2.12e)$$

*Proof.* Let  $(\eta_i^k)_{k \in \mathbb{N}}$  and  $(\tilde{\eta}_i^k)_{k \in \mathbb{N}}$  be the controls of  $(M_i^k)_{k \in \mathbb{N}}$  and  $(N_i^k)_{k \in \mathbb{N}}$  respectively for all  $i \in \llbracket 1, n \rrbracket$ . We define a common control  $(\eta^k)_{k \in \mathbb{N}}$  by:

$$\eta^k = \max \{ \max \{ \eta_i^k, \tilde{\eta}_i^k \}, i \in \llbracket 1, n \rrbracket \}, \quad k \in \mathbb{N}. \quad (2.13)$$

For all  $y \in Y$ , we can write

$$U^{k+1}(y) = \sum_{i=1}^n \|(0, y_i)\|_{N_i^{k+1}}^2 \leq (1 + \eta^k) \sum_{i=1}^n \|(0, y_i)\|_{N_i^k}^2 = (1 + \eta^k)U^k(y),$$

which proves (2.12a). Let us now fix  $x \in X, z \in Y_{\text{sparse}}$  and  $k \in \mathbb{N}$ . By definition, there exists  $i \in \llbracket 1, n \rrbracket$  such that  $z_j = 0$  for all  $j \neq i$ . We obtain the inequalities (2.12b)-(2.12d) by writing:

$$V^{k+1}(x, z) = \|(x, z_i)\|_{M_i^{k+1}}^2 \leq (1 + \eta^k) \|(x, z_i)\|_{M_i^k}^2 = (1 + \eta^k)V^k(x, z),$$

$$\|(x, z)\|_{N^k}^2 = \|(x, z_i)\|_{N_i^k}^2 \geq \alpha \|(x, z_i)\|^2 = \|(x, z)\|^2,$$

$$V^k(x, z) = \|(x, z_i)\|_{M_i^k}^2 \geq (1 - \beta) \|(x, z_i)\|_{N_i^k}^2 = (1 - \beta) \|(x, z)\|_{N^k}^2.$$

Finally, let us introduce

$$\widetilde{M}_i^k = \begin{pmatrix} \frac{1}{\tau^k} \text{Id} & \frac{1}{p_i} A_i \\ \frac{1}{p_i} A_i^* & \frac{1}{p_i \sigma_i^k} \text{Id} \end{pmatrix} \in \mathbb{S}(X \times Y_i), \quad i \in \llbracket 1, n \rrbracket.$$

Thanks to Lemma 6.1, we can write that for all  $i \in \llbracket 1, n \rrbracket$  and  $k \in \mathbb{N}$ ,

$$M_i^k \succcurlyeq (1 - \beta)N_i^k \quad \Leftrightarrow \quad \tau^k \sigma_i^k \frac{\|A_i\|^2}{p_i} \leq \beta \quad \Leftrightarrow \quad \widetilde{M}_i^k \succcurlyeq (1 - \beta)N_i^k.$$

By the same reasoning as above,

$$\|(x, z_i)\|_{\widetilde{M}_i^k}^2 \geq (1 - \beta)\|(x, z_i)\|_{N_i^k}^2 = (1 - \beta)\|(x, z)\|_{N^k}^2.$$

We observe that

$$\begin{aligned} \langle P^{-1}Ax, z \rangle &= \|(x, z_i)\|_{N_i^k}^2 - \|(x, z_i)\|_{M_i^k}^2 \leq \beta\|(x, z)\|_{N^k}^2, \\ \langle P^{-1}Ax, z \rangle &= \|(x, z_i)\|_{\widetilde{M}_i^k}^2 - \|(x, z_i)\|_{N_i^k}^2 \geq -\beta\|(x, z)\|_{N^k}^2, \end{aligned}$$

which proves (2.12e).  $\square$

**Lemma 2.8** (A-SPDHG is  $\mathcal{C}$ -stable). *Under the assumptions of Theorem 2.1, the sequence  $(x^k, y^k)_{k \in \mathbb{N}}$  of Algorithm 2.2 is stable with respect to  $\mathcal{C}$  relative to  $(N^k)_{k \in \mathbb{N}}$ .*

*Proof.* By definition of A-SPDHG with serial sampling, the difference between two consecutive dual iterates is almost surely sparse:

$$y^k - y^{k-1} \in Y_{\text{sparse}}, \quad k \in \mathbb{N} \setminus \{0\}.$$

Let us define the sequences

$$a^k = V^k(x^k - x^*, y^k - y^{k-1}) + U^k(y^k - y^*), \quad b^k = V^{k+1}(x^{k+1} - x^k, y^k - y^{k-1}),$$

which are a.s. non-negative thanks to (2.12c) and (2.12d). Furthermore, inequalities (2.11), (2.12a) and (2.12b) imply that almost surely for all  $k \in \mathbb{N} \setminus \{0\}$ ,

$$\mathbb{E}[a^{k+1} | \mathcal{F}^k] \leq (1 + \eta^k)a^k - b^k.$$

By Robbins-Siegmund lemma [23],  $(a^k)$  converges almost surely,  $\sup_k \mathbb{E}[a^k] < \infty$  and  $\sum_k \mathbb{E}[b^k] < \infty$ . From the last point in particular, we can write thanks to (2.12d) and the monotone convergence theorem:

$$\begin{aligned} \mathbb{E} \left[ \sum_k \|y^k - y^{k-1}\|_{(P\Sigma^{k+1})^{-1}}^2 \right] &\leq \mathbb{E} \left[ \sum_k \|(x^{k+1} - x^k, y^k - y^{k-1})\|_{N^{k+1}}^2 \right] \\ &\leq \mathbb{E} \left[ \sum_k b^k \right] = \sum_k \mathbb{E}[b^k] < \infty, \end{aligned}$$

hence  $\sum_k \|y^k - y^{k-1}\|_{(P\Sigma^{k+1})^{-1}}^2$  is almost surely finite, thus  $(\|y^k - y^{k-1}\|_{(P\Sigma^{k+1})^{-1}})$  converges almost surely to 0. Furthermore,  $\sup_k \mathbb{E}[a^k] < \infty$  hence  $\sup_k \|x^k - x^*\|_{(\tau^k)^{-1}}$  is finite, and by (2.12e), one can write that for  $k \in \mathbb{N} \setminus \{0\}$ ,

$$\begin{aligned} |\langle P^{-1}A(x^k - x^*), y^k - y^{k-1} \rangle| &\leq \beta \|x^k - x^*\|_{(\tau^{k+1})^{-1}}^2 \|y^k - y^{k-1}\|_{(P\Sigma^{k+1})^{-1}}^2 \\ &\leq \beta(1 + \eta^k) \|x^k - x^*\|_{(\tau^k)^{-1}}^2 \|y^k - y^{k-1}\|_{(P\Sigma^{k+1})^{-1}}^2. \end{aligned}$$

We know that  $(\eta^k)^{k \in \mathbb{N}}$  is summable hence converges to 0. As a consequence,

$$|\langle P^{-1}A(x^k - x^*), y^k - y^{k-1} \rangle| \rightarrow 0 \quad \text{almost surely.}$$

To conclude with, thanks to the identity

$$a^k = \|(x^k - x^*, y^k - y^*)\|_{N^k}^2 + \langle P^{-1}A(x^k - x^*), y^k - y^{k-1} \rangle, \quad k \in \mathbb{N} \setminus \{0\},$$

the almost sure convergence of  $(a^k)_{k \in \mathbb{N}}$  implies in turn that of  $(\|(x^k - x^*, y^k - y^*)\|_{N^k}^2)_{k \in \mathbb{N}}$ .  $\square$

As a by-product of the proof, one also obtains the following useful lemma.

**Lemma 2.9.** *Under the assumptions of Theorem 2.1,*

$$\mathbb{E} \left[ \sum_k \|(x^{k+1} - x^k, y^k - y^{k-1})\|^2 \right] < \infty \quad \text{and a.s.} \quad \|x^{k+1} - x^k\| \rightarrow 0.$$

*Proof.* The first assertion is a straightforward consequence of

$$\mathbb{E} \left[ \sum_k b^k \right] = \sum_k \mathbb{E} [b^k] < \infty$$

and bounds (2.12c) and (2.12d). Furthermore, it implies that  $\sum_k \|(x^{k+1} - x^k, y^k - y^{k-1})\|^2$  is a.s. finite, hence  $(\|(x^{k+1} - x^k, y^k - y^{k-1})\|)$  a.s. converges to 0, and so does  $(\|x^{k+1} - x^k\|)$ .  $\square$

## 2.5 Weak cluster points of A-SPDHG are saddle-points

The goal of this section is to prove that A-SPDHG satisfies to point (ii) of Theorem 2.5. For all  $i \in \llbracket 1, n \rrbracket$  and positive scalars  $\sigma_i$  and  $\tau$ , define

$$F_i = \begin{pmatrix} \partial g & A_i^* \\ -A_i & \partial f_i^* \end{pmatrix}, \quad O_i^{\sigma_i, \tau} = \begin{pmatrix} \frac{1}{\tau} \text{Id} & -\frac{1}{p_i} A_i^* \\ -A_i & \frac{1}{\sigma_i} \text{Id} \end{pmatrix}.$$

**Lemma 2.10.** *Under the assumptions of Theorem 2.1,*

- (i) *The operator  $F_i$  is maximally monotone for all  $i \in \llbracket 1, n \rrbracket$ .*
- (ii) *Let us call  $(I^k)_{k \in \mathbb{N}}$  the random index with value in  $\llbracket 1, n \rrbracket$  selected at iteration  $k$ . For all  $i \in \llbracket 1, n \rrbracket$  and  $\sigma_i, \tau > 0$ , there exists an operator  $T_i^{\sigma_i, \tau} : X \times Y_i \rightarrow X \times Y_i$  such that on the event  $\{I^k = i\}$ ,*

$$(x^{k+2}, y_i^{k+1}) = T_i^{\sigma_i^{k+1}, \tau^{k+2}}(x^{k+1}, y_i^k). \quad (2.14)$$

- (iii) *For all  $i \in \llbracket 1, n \rrbracket$ , scalars  $\sigma_i, \tau > 0$  and  $(x, y_i) \in X \times Y_i$ , the following identity is satisfied:*

$$O_i^{\sigma_i, \tau} (T_i^{\sigma_i, \tau}(x, y_i) - (x, y_i)) \in F_i (T_i^{\sigma_i, \tau}(x, y_i)). \quad (2.15)$$

**Remark 2.11** (A-SPDHG as a random asymmetric proximal point algorithm). *A sequence  $(u^k)_{k \in \mathbb{N}}$  of a Hilbert space  $H$  satisfying to the variational inequality*

$$\langle u - u^{k+1}, F(u^{k+1}) + O(u^{k+1} - u^k) \rangle \geq 0, \quad u \in H, \quad k \in \mathbb{N}, \quad (2.16)$$

can be interpreted as a proximal point algorithm in  $H$  equipped with the norm induced by  $O$  if  $F$  is a strongly monotone operator and a  $O$  a positive definite self-adjoint linear operator. PDHG satisfies to an identity of the type (2.16) as shown in [17]. For SPDHG, equation (2.15) implies that conditionally on the random index selection, the iterate  $u^k = (x^{k+1}, y_i^k)$  satisfies to the variational inequality (2.16) with  $F = F^i$  and  $O = O_i^{\sigma_i^{k+1}, \tau^{k+2}}$ . However the operator  $O_i^{\sigma_i, \tau}$  is not self-adjoint, because the extrapolation parameter is  $p_i^{-1} \neq 1$ . SPDHG can thus be seen as a random asymmetric proximal point algorithm.

*Proof of Lemma 2.10.* For point (i), observe that  $F_i$  is the sum of the two operators

$$\begin{pmatrix} \partial g & 0 \\ 0 & \partial f_i^* \end{pmatrix}, \quad \begin{pmatrix} 0 & A_i^* \\ -A_i & 0 \end{pmatrix}.$$

As by assumption the functionals  $g$  and  $f_i^*$  are convex, lower semi-continuous and proper,  $\partial g$  and  $\partial f_i^*$  are strongly monotone and so is the operator on the left. The skew-symmetric bounded linear operator on the right is strongly monotone as shown in [2, example 20.35]. Hence  $F_i$  is strongly monotone as the sum of strongly monotone operators.

Let us now prove point (ii). On the event  $\{I^k = i\}$ , A-SPDHG update procedure can be rewritten as

$$\begin{aligned} y_i^{k+1} &= \text{prox}_{\sigma_i^{k+1} f_i^*}(y_i^k + \sigma_i^{k+1} A_i x^{k+1}), \quad \bar{y}_i^{k+1} = y_i^{k+1} + \frac{1}{p_i} (y_i^{k+1} - y_i^k), \\ x^{k+2} &= \text{prox}_{\tau^{k+2} g}(x^{k+1} - \tau^{k+2} A^* \bar{y}^{k+1}). \end{aligned}$$

Hence identity (2.14) stands with  $T_i^{\sigma_i, \tau}(x, y_i) = (\hat{x}, \hat{y}_i)$  and

$$\hat{x} = \text{prox}_{\tau g} \left( x - \tau A^* \left( \hat{y}_i + \frac{1}{p_i} (\hat{y}_i - y_i) \right) \right), \quad \hat{y}_i = \text{prox}_{\sigma_i f_i^*}(y_i + \sigma_i A_i x).$$

Finally for point (iii), observe that the equations above can be rewritten as

$$\begin{aligned} & \begin{cases} x - \tau A^* \left( \hat{y}_i + \frac{1}{p_i} (\hat{y}_i - y_i) \right) - \hat{x} & \in \tau \partial g(\hat{x}) \\ y_i + \sigma_i A_i x - \hat{y}_i & \in \sigma_i \partial f_i^*(\hat{y}_i) \end{cases} \Leftrightarrow \begin{cases} 0 & \in \partial g(\hat{x}) - \frac{1}{\tau} (x - \hat{x}) + A^* \left( \hat{y}_i + \frac{1}{p_i} (\hat{y}_i - y_i) \right) \\ 0 & \in \partial f_i^*(\hat{y}_i) - \frac{1}{\sigma_i} (y_i - \hat{y}_i) - A_i x \end{cases} \\ \Leftrightarrow & \begin{cases} 0 & \in \partial g(\hat{x}) - \frac{1}{\tau} (x - \hat{x}) + \frac{1}{p_i} A^* (\hat{y}_i - y_i) + A^* \hat{y}_i \\ 0 & \in \partial f_i^*(\hat{y}_i) - \frac{1}{\sigma_i} (y_i - \hat{y}_i) - A_i (x - \hat{x}) - A_i \hat{x} \end{cases} \Leftrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in F_i \begin{pmatrix} \hat{x} \\ \hat{y}_i \end{pmatrix} - O_i^{\tau, \sigma_i} \begin{pmatrix} \hat{x} - x \\ \hat{y}_i - y_i \end{pmatrix}. \end{aligned}$$

□

Even though the operator  $O_i^{\sigma_i, \tau}$  is not auto-adjoint, we can leverage identities (2.14) and (2.15) to obtain the desired result.

**Lemma 2.12** (Cluster points of A-SPDHG are saddle points). *Let  $(\tilde{x}, \tilde{y})$  be a random variable almost surely taking its values in the set of weak cluster points of  $(x^k, y^k)_{k \in \mathbb{N}}$ , and assume that the assumptions of Theorem 2.1 hold. Then  $(\tilde{x}, \tilde{y})$  is a.s. in  $\mathcal{C}$ .*

*Proof.* Thanks to Lemma 2.9, Lemma 2.10 and the monotone convergence theorem,

$$\begin{aligned} \mathbb{E} \left[ \sum_k \|(x^{k+1} - x^k, y^k - y^{k-1})\|^2 \right] &= \sum_k \mathbb{E} \left[ \|(x^{k+1} - x^k, y^k - y^{k-1})\|^2 \right] \\ &= \sum_k \mathbb{E} \left[ \mathbb{E} [\|(x^{k+1} - x^k, y^k - y^{k-1})\|^2 | I^k] \right] \\ &= \sum_k \sum_{i=1}^n \mathbb{P}(I^{k-1} = i) \mathbb{E} \left[ \left\| T_i^{\sigma_i^k, \tau^{k+1}}(x^{k+1}, y_i^k) - (x^k, y_i^{k-1}) \right\|^2 \right] \\ &= \sum_{i=1}^n p_i \mathbb{E} \left[ \sum_k \left\| T_i^{\sigma_i^k, \tau^{k+1}}(x^{k+1}, y_i^k) - (x^k, y_i^{k-1}) \right\|^2 \right] \\ &< \infty. \end{aligned}$$

Let us fix an index  $i \in \llbracket 1, n \rrbracket$ . By assumption,  $p_i$  is positive, hence the quantity

$$\mathbb{E} \left[ \sum_{k \in \mathbb{N}} \left\| T_i^{\sigma_i^k, \tau^{k+1}}(x^{k+1}, y_i^k) - (x^k, y_i^{k-1}) \right\|^2 \right]$$

is finite, hence the series inside the expectation is a.s. finite, and in turn the summand converges almost surely to 0:

$$\left\| T_i^{\sigma_i^k, \tau^{k+1}}(x^{k+1}, y_i^k) - (x^k, y_i^{k-1}) \right\| \rightarrow 0 \quad \text{almost surely.}$$

Let us define

$$\delta_i^k = O_i^k \left( T_i^{\sigma_i^k, \tau^{k+1}}(x^{k+1}, y_i^k) - (x^k, y_i^{k-1}) \right), \quad k \in \mathbb{N} \setminus \{0\}.$$

For all  $k \in \mathbb{N}$ , one can see that

$$\left\| O_i^{\sigma_i^k, \tau^{k+1}} \right\|^2 \leq 2 \left( \max \left\{ \frac{1}{(\tau^{k+1})^2}, \frac{1}{(\sigma_i^k)^2} \right\} + \frac{\|A_i\|^2}{p_i^2} \right) \leq 2 \left( \alpha^2 + \frac{\|A_i\|^2}{p_i^2} \right) = M^2,$$

where the second bound comes from assumption (ii) of Theorem 2.1. As a consequence,

$$\|\delta_i^k\| \leq M \left\| T_i^{\sigma_i^k, \tau^{k+1}}(x^{k+1}, y_i^k) - (x^k, y_i^{k-1}) \right\| \rightarrow 0 \quad \text{almost surely.}$$

Let us now consider a sub-sequence  $(x^{k_m}, y^{k_m})_{m \in \mathbb{N}}$  which converges weakly a.s. to  $(\tilde{x}, \tilde{y})$ . By Lemma 2.9, the sequence  $(\|x^{k+1} - x^k\|)$  converges a.s. to 0, hence  $(x^{k_m+1})$  converges weakly a.s. to  $\tilde{x}$  and  $(x^{k_m+1}, y^{k_m})_{m \in \mathbb{N}}$  converges weakly a.s. to  $(\tilde{x}, \tilde{y})$ . By Lemma 2.10,  $\delta_i^{k_m} \in F^i(x^{k_m+1}, y^{k_m})$  for every  $k_m \in \mathbb{N} \setminus \{0\}$  and  $F^i$  is maximally monotone. This implies that a.s.  $0 \in F^i(\tilde{x}, \tilde{y})$  [2, Proposition 20.38], that is to say a.s.

$$A_i \tilde{x} \in \partial f_i^*(\tilde{y}_i), \quad -A_i^* \tilde{y} \in \partial g(\tilde{x}).$$

This being true for all  $i \in \llbracket 1, n \rrbracket$ ,  $(\tilde{x}, \tilde{y})$  a.s. satisfies to (2.3) hence is a.s. in  $\mathcal{C}$ .  $\square$



## 2.6 Proof of Theorem 2.1

Under the assumptions of Theorem 2.1, the set  $\mathcal{C}$  of saddle-points is closed and non-empty and  $X \times Y$  is a separable Hilbert space. By Lemma 2.3, the variable metrics sequence  $(N^k)_{k \in \mathbb{N}}$  defined in (2.8) satisfies to condition (i) of Theorem 2.5. Furthermore, the iterates of Algorithm 2.2 comply with condition (ii) and (iii) of Theorem 2.5 by Lemma 2.8 and Lemma 2.12 respectively, hence converge almost surely to a point in  $\mathcal{C}$ .

## 3 Algorithmic Design and Practical Implementations

In this section we present practical instances of our A-SPDHG algorithm, where we specify a step-size adjustment rule which satisfies our assumptions in convergence proof. We extend the adaptive step-size balancing rule for deterministic PDHG, which is proposed by [14], into our stochastic setting, with minibatch approximation to minimize the computational overhead.

We present our first practical implementation in Algorithm 3.1 as our rule-(a), where we estimate the progress of primal and dual via the two sequences  $v^k$  and  $d^k$  evaluated in each iteration, utilizing the fact that:

$$\mathbb{E} \left[ (x^k - x^{k+1})/\tau^{k+1} - \frac{1}{p_i} A_i^*(y_i^k - y_i^{k+1}) \right] \in \partial g(x^{k+1}) + A^* y^{k+1}, \quad (3.1)$$

meanwhile:

$$(y_i^k - y_i^{k+1})/\sigma_k - A_i(x^k - x^{k+1}) \in \partial f_i^*(y_i^{k+1}) - A_i x^{k+1}. \quad (3.2)$$

Hence our  $v^k$  and  $d^k$  estimate the lengths of the subgradients of the saddle-point objective  $g(x) + \sum_i \langle A_i x, y_i \rangle - f_i^*(y_i)$  w.r.t. the primal and dual updates at each iteration. By making them balanced on the fly, we can enforce the algorithm to achieve similar progress in both primal and dual steps, hence improve the convergence. Note that here we adopt the choice of  $\ell_1$ -norm as the length measure for  $v^k$  and  $d^k$  as done by Goldstein et al [14, 15], since we also observe numerically the benefit over the more intuitive choice of  $\ell_2$ -norm.

For full-batch case ( $n = 1$ ), it reduces to the adaptive PDHG proposed by [14, 15]. We adjust the ratio between primal and dual step sizes according to the ratio between  $v^k$  and  $d^k$ , and whenever the step-sizes changed, we shrink  $\alpha$  (which controls the amplitude of the changes) by a factor  $\eta \in (0, 1)$  – we typically choose  $\eta = 0.995$  in our experiments. For the choice of  $s$ , we choose  $s = \|A\|$  as our default.<sup>1</sup>

Noting that unlike the deterministic case which does not have the need of extra matrix-vector multiplication since  $A^* y^k$  and  $A x^k$  can be memorized, our stochastic extension will require the computation of  $A_i x^k$  since we will sample different subsets between back-to-back iterations with high probability. Assuming that the evaluations of these matrix-vector products dominate the computational cost, we will have a maximum 50% overhead in terms of FLOPs count which is acceptable. Moreover, we found numerically that very often we can further reduce this overhead significantly by approximation tricks such as subsampling:

$$d^{k+1} \approx \frac{\rho}{p_i} \|S^k(y_i^k - y_i^{k+1})/\sigma^{k+1} - (S^k A_i)(x^k - x^{k+1})\|_1 \quad (3.3)$$

<sup>1</sup>The choice of  $s$  is crucial for the convergence behavior of rule (a), and we found numerically that it is better to scale with the operator norm  $\|A\|$  instead of depending on the range of pixel values as suggested in [15].

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**Algorithm 3.1:** A-SPDHG, rule (a)

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Input: dual step-size  $\sigma^0$ , primal step-size  $\tau^0$ ,  $\alpha^0 \in (0, 1)$ ,  $\eta \in (0, 1)$ ,  $\delta > 1$ , probabilities  $(p_i)_{1 \leq i \leq n}$ ; primal variable  $x^0$ , dual variable  $y^0$   
 Initialize  $\bar{y}^0 = y^0$ ,  $v^0 = d^0 = 0$ ,  $s = \|A\|$   
**for**  $k \in \llbracket 0, K - 1 \rrbracket$  **do**  
   **If**  $v^k > sd^k \delta$  **then**  $\tau^{k+1} = \frac{\tau^k}{1 - \alpha^k}$ ,  $\sigma^{k+1} = \sigma^k(1 - \alpha^k)$ ,  $\alpha^{k+1} = \alpha^k \eta$   
   **If**  $v^k < sd^k / \delta$  **then**  $\tau^{k+1} = \tau^k(1 - \alpha^k)$ ,  $\sigma^{k+1} = \frac{\sigma^k}{1 - \alpha^k}$ ,  $\alpha^{k+1} = \alpha^k \eta$   
   **If**  $sd^k / \delta \leq v^k \leq sd^k \delta$  **then**  $\tau^{k+1} = \tau^k$ ,  $\sigma^{k+1} = \sigma^k$ ,  $\alpha^{k+1} = \alpha^k$   
    $x^{k+1} = \text{prox}_{\tau^{k+1}g}(x^k - \tau^{k+1}A^*\bar{y}^k)$   
   Randomly pick  $i \in \llbracket 1, n \rrbracket$  with probability  $p_i$   
    $y_j^{k+1} = \begin{cases} \text{prox}_{\sigma^{k+1}f_i^*}(y_i^k + \sigma^{k+1}A_i x^{k+1}) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
    $\bar{y}_j^{k+1} = \begin{cases} y_i^{k+1} + \frac{1}{p_i}(y_i^{k+1} - y_i^k) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
    $v^{k+1} = \|(x^k - x^{k+1})/\tau^{k+1} - \frac{1}{p_i}A_i^*(y_i^k - y_i^{k+1})\|_1$   
    $d^{k+1} = \frac{1}{p_i}\|(y_i^k - y_i^{k+1})/\sigma^{k+1} - A_i(x^k - x^{k+1})\|_1$  – or approximate this step by (3.3)  
**end for**  
**return**  $x^K$

---

with  $S^k$  being a random subsampling operator satisfying  $\mathbb{E}[(S^k)^T S^k] = \frac{1}{\rho} \text{Id}$ . In our experiments we choose 10% subsampling for this approximation hence the overhead is reduced from 50% to only 5% which is negligible.

More recently, Yokota and Hontani [26] propose a variant of adaptive step-size balancing scheme for PDHG, utilizing the angles between the subgradients  $\partial g(x^{k+1}) + A^*y^{k+1}$  and the difference of the updates  $x^k - x^{k+1}$ .

If these two directions are highly aligned, then the primal step size can be increased for bigger step. If these two directions have a large angle, then the primal step-size should be shrunken. By extending this scheme to stochastic setting we obtain another choice of adaptive scheme for SPDHG.

We present this scheme in Algorithm 3.2 as our rule (b). At iteration  $k$ , compute:

$$q^{k+1} = (x^k - x^{k+1})/\tau^{k+1} - \frac{1}{p_i}A_i^*(y_i^k - y_i^{k+1}), \quad (3.4)$$

as an unbiased estimate of  $\partial g(x^{k+1}) + A^*y^{k+1}$ , then measure the cosine of the angle between this and  $x^k - x^{k+1}$ :

$$w^{k+1} = \frac{\langle x^k - x^{k+1}, q^{k+1} \rangle}{(\|x^k - x^{k+1}\|_2 \|q^{k+1}\|_2)}. \quad (3.5)$$

The threshold  $c$  for the cosine value (which triggers the increase of the primal step-size) typically needs to be very close to 1 (we use  $c = 0.999$ ).

Recently Zdun et al [27] proposed a heuristic similar to our rule (b), but they choose  $q^{k+1}$  to be the approximation for an element of  $\partial g(x^{k+1})$  instead of  $\partial g(x^{k+1}) + A^*y^{k+1}$ . Our choice follows

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**Algorithm 3.2:** A-SPDHG, rule (b)

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Input: dual step-size  $\sigma^0$ , primal step-size  $\tau^0$ ,  $\eta \in (0, 1)$ , probabilities  $(p_i)_{1 \leq i \leq n}$ ; primal variable  $x^0$ , dual variable  $y^0$   
Initialize  $\bar{y}^0 = y^0$ ,  $w^0 = 0$ ,  $\alpha^0 = 1$   
**for**  $k \in \llbracket 0, K-1 \rrbracket$  **do**  
  **If**  $w^k < 0$  **then**  $\tau^{k+1} = \frac{\tau^k}{1+\alpha^k}$ ,  $\sigma^{k+1} = \sigma^k(1+\alpha^k)$ ,  $\alpha^{k+1} = \alpha^k \eta$   
  **If**  $w^k \geq c$  **then**  $\tau^{k+1} = \tau^k(1+\alpha^k)$ ,  $\sigma^{k+1} = \frac{\sigma^k}{1+\alpha^k}$ ,  $\alpha^{k+1} = \alpha^k \eta$   
  **If**  $0 \leq w^k < c$  **then**  $\tau^{k+1} = \tau^k$ ,  $\sigma^{k+1} = \sigma^k$ ,  $\alpha^{k+1} = \alpha^k$   
   $x^{k+1} = \text{prox}_{\tau^{k+1}g}(x^k - \tau^{k+1}A^*\bar{y}^k)$   
  Randomly pick  $i \in \llbracket 1, n \rrbracket$  with probability  $p_i$   
   $y_j^{k+1} = \begin{cases} \text{prox}_{\sigma^{k+1}f_i^*}(y_i^k + \sigma^{k+1}A_i x^{k+1}) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
   $\bar{y}_j^{k+1} = \begin{cases} y_i^{k+1} + \frac{1}{p_i}(y_i^{k+1} - y_i^k) & \text{if } j = i \\ y_j^k & \text{if } j \neq i \end{cases}$   
   $q^{k+1} = (x^k - x^{k+1})/\tau^{k+1} - \frac{1}{p_i}A_i^*(y_i^k - y_i^{k+1})$   
   $w^{k+1} = \langle x^k - x^{k+1}, q^{k+1} \rangle / (\|x^k - x^{k+1}\|_2 \|q^{k+1}\|_2)$   
  **end for**  
**return**  $x^K$ 

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more closely to the original scheme of Yokota and Hontani [26]. We numerically found that their scheme is not competitive in our settings and we hence do not recommend it.

## 4 Numerical Experiments

In this section we present our numerical studies of our proposed scheme in solving one of the most typical imaging inverse problems, the X-ray computed tomography. We compare our A-SPDHG algorithm with the original SPDHG, on different choices of starting ratio of the primal and dual step-sizes.

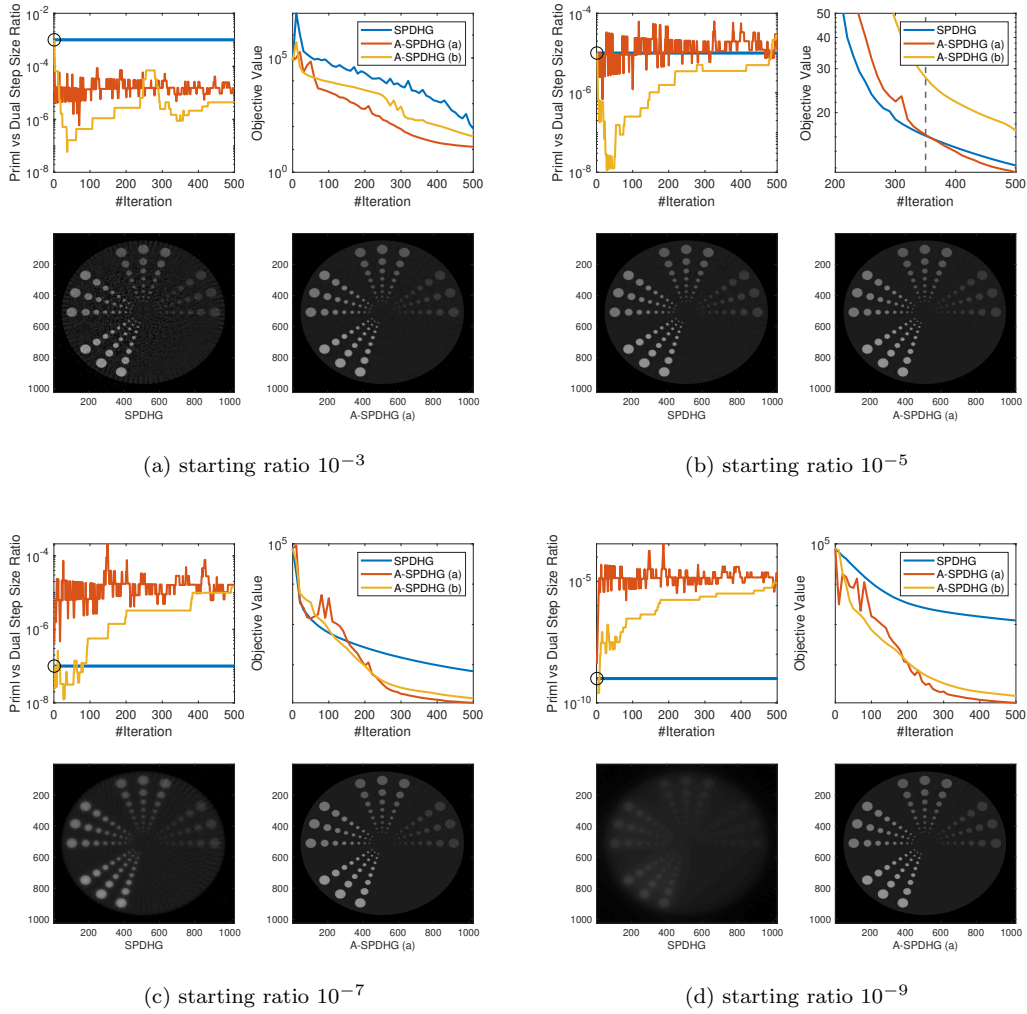


Figure 1: Comparison between SPDHG and our A-SPDHG on X-ray Sparse-View CT (Example 1), with a variety of starting primal-dual step-size ratios. Here the forward operator  $A \in \mathbb{R}^{m \times d}$  where the dimension  $m = 368640$ ,  $d = 1048576$ . We includes the images reconstructed by the algorithms at termination (50th epoch). In the first plot of each subfigure, the black circle indicates the starting step-size ratio for all the algorithms, same for the following figures.

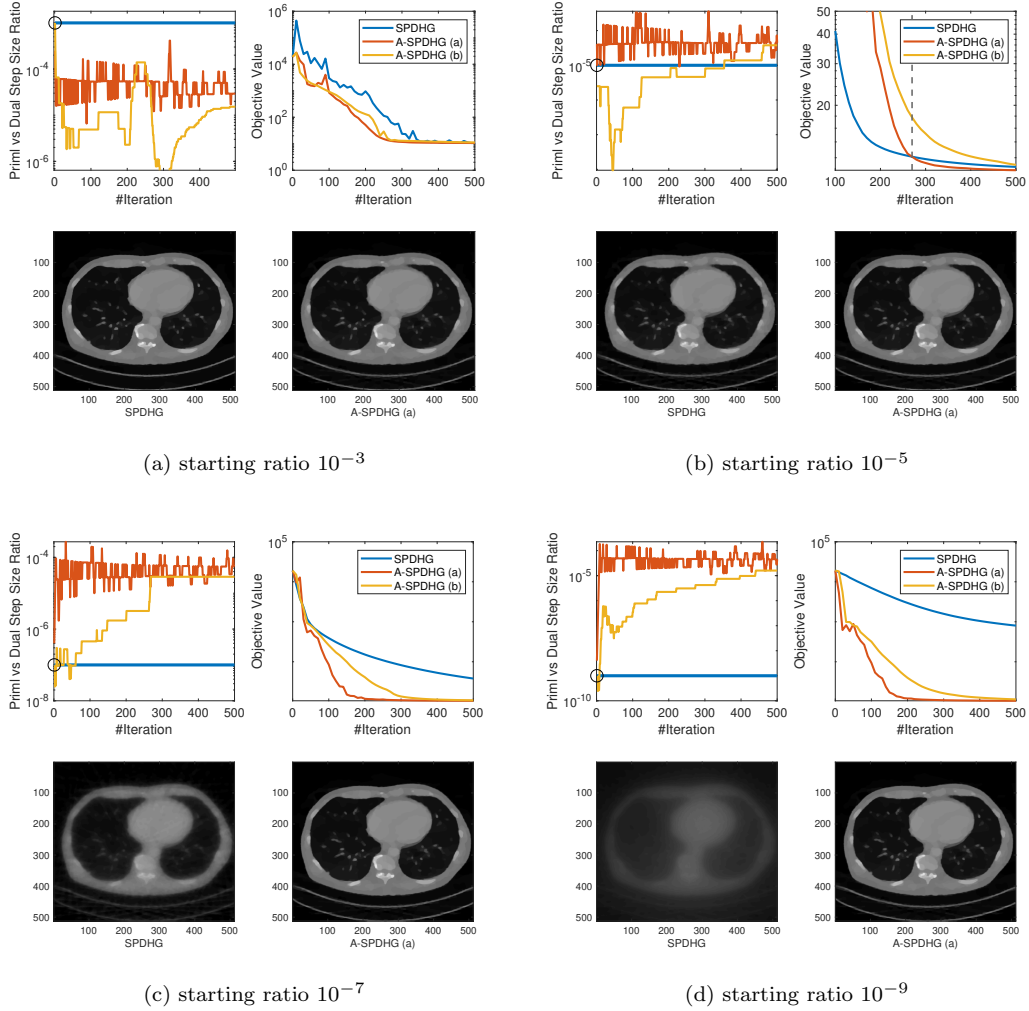


Figure 2: Comparison between SPDHG and our A-SPDHG on X-ray Sparse-View CT (Example 2), with a variety of starting primal-dual step-size ratios. Here the forward operator  $A \in \mathbb{R}^{m \times d}$  where the dimension  $m = 92160$ ,  $d = 262144$ . We includes the images reconstructed by the algorithms at termination (50th epoch).

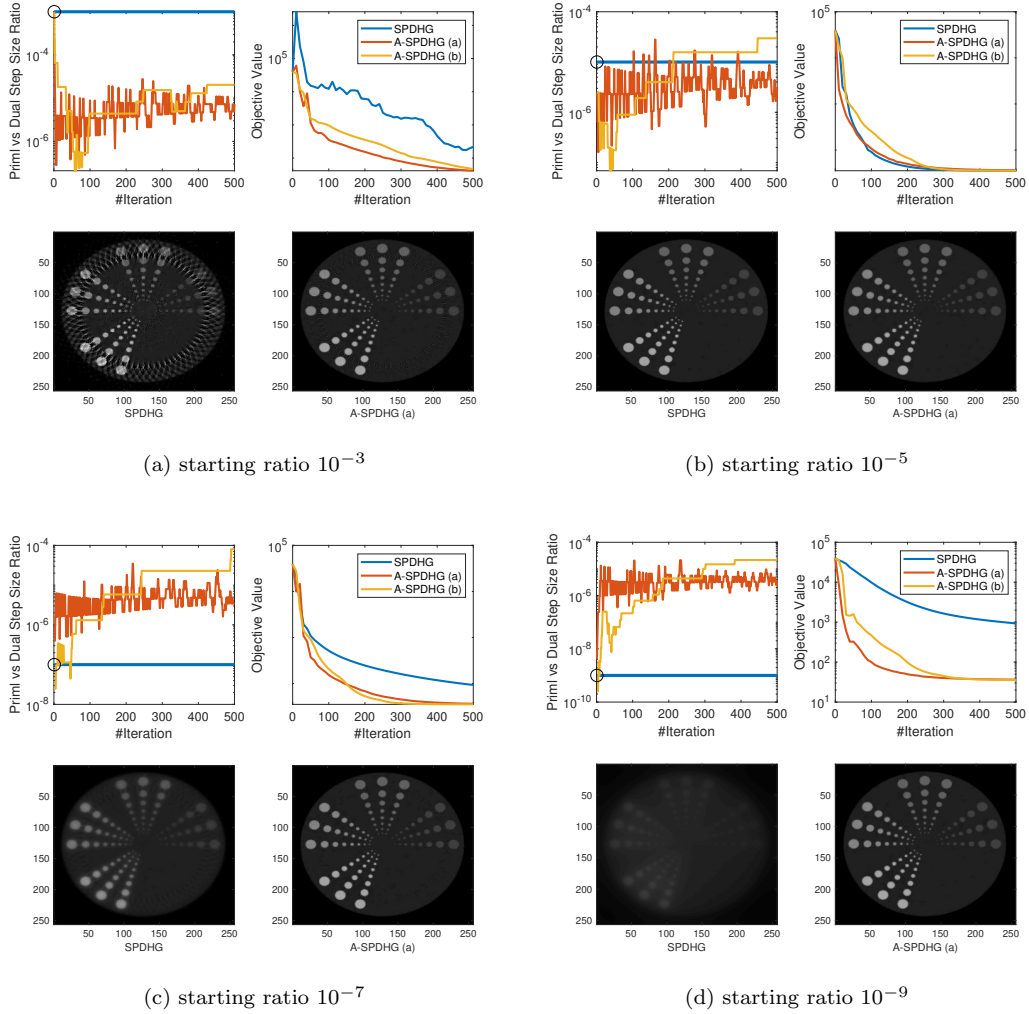


Figure 3: Comparison between SPDHG and our A-SPDHG on X-ray Low-Dose CT (where we use a large number of highly-noisy X-ray measurements), with a variety of starting primal-dual step-size ratios. Here the forward operator  $A \in \mathbb{R}^{m \times d}$  where the dimension  $m = 184320$ ,  $d = 65536$ . We resized the phantom image to 256 by 256. We included the images reconstructed by the algorithms at termination (50th epoch).

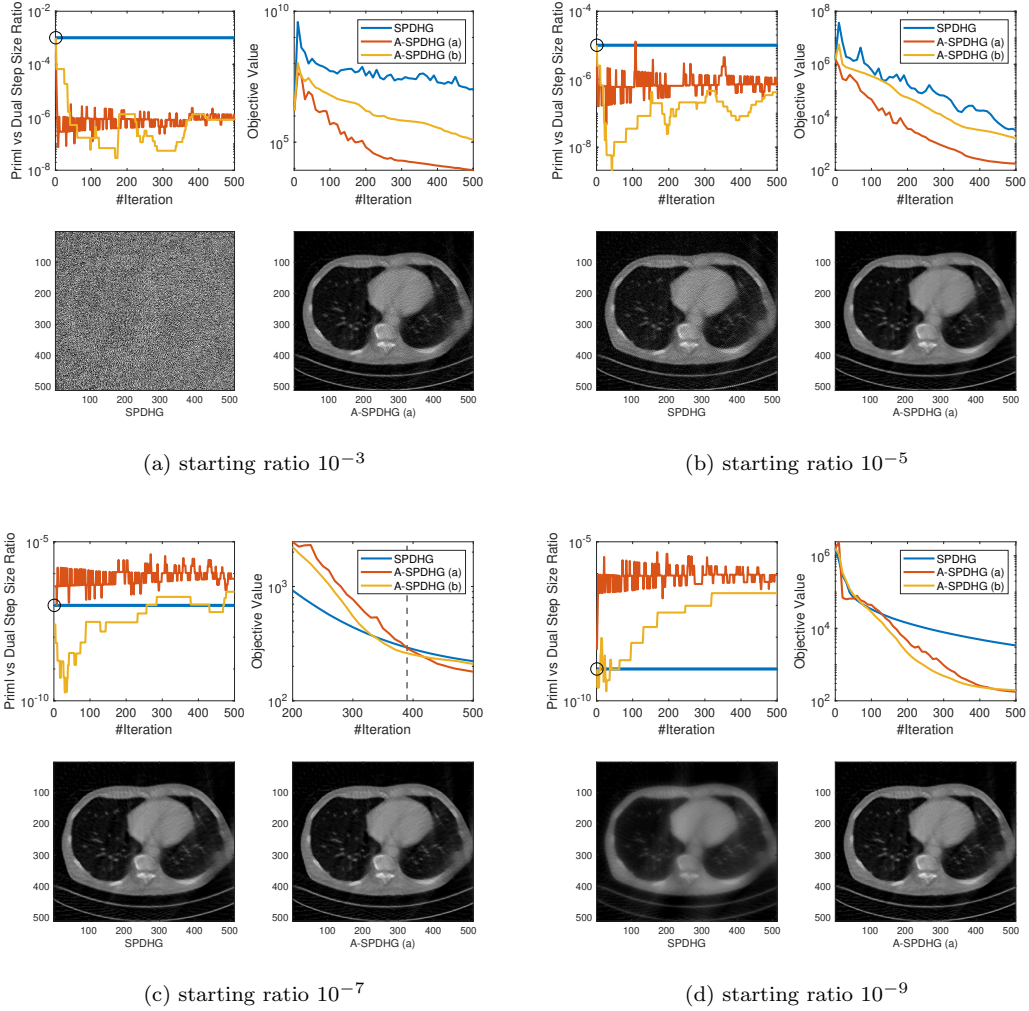


Figure 4: Comparison between SPDHG and our A-SPDHG on X-ray Limited-Angle CT (Example 2), with a variety of starting primal-dual step-size ratios. Here the forward operator  $A \in \mathbb{R}^{m \times d}$  where the dimension  $m = 92160$ ,  $d = 262144$ . We includes the images reconstructed by the algorithms at termination (50th epoch).

In our X-ray CT imaging example, we seek to reconstruct the tomography images from fan-beam X-ray measurement data, by solving the following TV-regularized objective:

$$x^* \in \arg \min_{x \in [0,1]^d} \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|Dx\|_1 \quad (4.1)$$

where  $D$  denotes the 2D differential operator,  $A \in \mathbb{R}^{m \times d}$  and  $x \in \mathbb{R}^d$ . We consider three fanbeam CT imaging modalities: Sparse-View CT, Low-Dose CT and Limited-Angle CT. We test the A-SPDHG and SPDHG on two images of different sizes (Example 1 on a phantom image sized  $1024 \times 1024$ , while Example 2 being real CT image sized  $512 \times 512$ ), on 4 different starting ratios ( $10^{-3}$ ,  $10^{-5}$ ,  $10^{-7}$  and  $10^{-9}$ ). We split the measurement data and operator into  $n = 10$  minibatches for both algorithms. For A-SPDHG we choose to use the approximation step for  $d^k$  presented in (3.3) with 10% subsampling hence the computational overhead is negligible in this experiment.

We present our numerical results in Figures 1, 2, 3 and 4. Among these, Figures 1 and 2 report the results for large-scale sparse-view CT experiments on a phantom image and a real CT image, while Figure 3 reports the results for low-dose CT experiments where we simulate a large number of measurements corrupted with a significant amount Poisson noise, and then, in Figure 4 we report the results for limited-angle CT where a range of 60-degree of measurement angle is missing. In all these examples we can consistently observe that no matter how we initialize the primal-dual step-size ratio, our A-SPDHG can automatically and consistently adjust the step size ratio to the optimal choice which is around either  $10^{-5}$  or  $10^{-7}$  for these four different CT problems, and significantly outperform the vanilla SPDHG for the cases where the starting ratio is away from the optimal range. Meanwhile, even for the cases where the starting ratio of SPDHG algorithm is near-optimal, we can observe consistently from most of these examples that our scheme outperforms the vanilla SPDHG algorithm locally after a certain number of iterations (highlighted by the vertical dash lines in relevant subfigures), which further indicates the benefit of adaptivity for this class of algorithms<sup>2</sup>. Note that throughout all these different examples, we use only one fixed set of parameters for A-SPDHG suggested in the previous section, which again indicates the strong practicality of our scheme.

We should also note that conceptually all the hyperparameters in our adaptive schemes are basically the controllers of the adaptivity of the algorithm (while for extreme choices we recover the vanilla SPDHG). In Figures 5 and 7, we present some numerical studies on the choices of hyperparameters of rule (a) and rule (b) of A-SPDHG algorithm. We choose the fixed starting ratio of  $10^{-7}$  for primal-dual step-sizes in these experiments. For rule (a), we found that it is robust to the choice of the starting shrinking rate  $\alpha_0$ , shrinking speed  $\eta$  and the gap  $\delta$ . Overall, we found that these parameters have weak impact of the convergence performance of our rule (a) and easy to choose.

For rule (b), we found that the performance is more sensitive to the choice of parameter  $c$  and  $\eta$  comparing to rule (a), although the dependence is still weak. Our numerical studies suggest that rule (a) is a better-performing choice than rule (b), but each of them have certain weaknesses, which require further studies and improvements. Nevertheless, we need to emphasis that all these parameters are essentially controlling the degree of adaptivity of the algorithms and fairly easy to choose, noting that for all these CT experiments with varying sizes/dimensions and modalities we

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<sup>2</sup>The most typical example here would be the Figure 1(b) where the optimal step-size ratio selected by the adaptive scheme at convergence is almost exactly  $10^{-5}$ , where we have set SPDHG to run with this ratio. We can still observe benefit of local convergence acceleration given by our adaptive scheme.



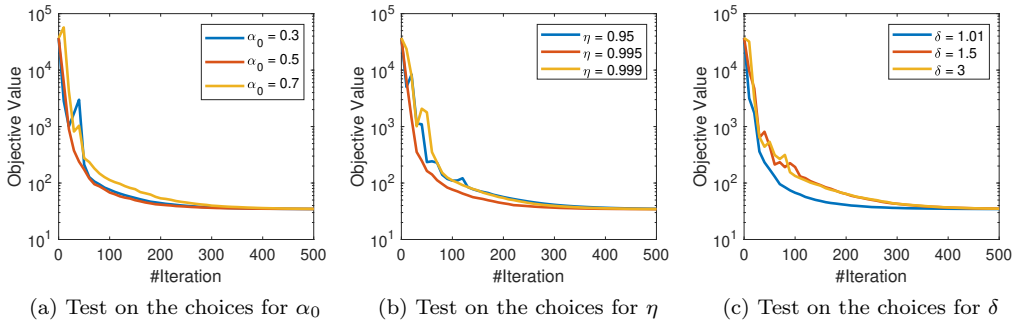


Figure 5: Test on different choices of parameters of A-SPDHG (rule-a) on X-ray Low-Dose fanbeam CT example, starting ratio of primal-dual step-sizes:  $10^{-7}$ . We can observe that the performance of ASPDHG has only minor dependence on these parameter choices.

only use one fixed set of the hyperparameters in A-SPDHG, and we are already able to consistently observe numerical improvements over vanilla SPDHG.

## 5 Conclusion

In this work we propose a new framework (A-SPDHG) for adaptive step-size balancing in stochastic primal-dual hybrid gradient methods. We first derive theoretically sufficient conditions on the adaptive primal and dual step-sizes for ensuring convergence in the stochastic setting. We then propose a number of practical schemes which satisfy the condition for convergence, and our numerical results on imaging inverse problems supports the effectiveness of our approach.

To our knowledge, this work constitutes the first theoretical analysis of adaptive step-sizes for a stochastic primal-dual algorithm. Our on-going work includes the theoretical analysis and algorithmic design of further accelerated stochastic primal-dual methods with line-search schemes for even faster convergence rates.

## 6 Complementary material for Section 2

We begin by a useful lemma.

**Lemma 6.1.** *Let  $a, b$  be positive scalars,  $\beta \in (0, 1)$ , and  $P$  a bounded linear operator from a Hilbert space  $X$  to a Hilbert space  $Y$ . Then,*

$$(ab)^{-1/2}\|P\| \leq 1 \Leftrightarrow \begin{pmatrix} a Id & P \\ P^* & b Id \end{pmatrix} \succcurlyeq 0. \quad (6.1)$$

$$(ab)^{-1/2}\|P\| \leq \beta \Leftrightarrow \begin{pmatrix} a Id & P \\ P^* & b Id \end{pmatrix} \succcurlyeq (1 - \beta) \begin{pmatrix} a Id & 0 \\ 0 & b Id \end{pmatrix}. \quad (6.2)$$

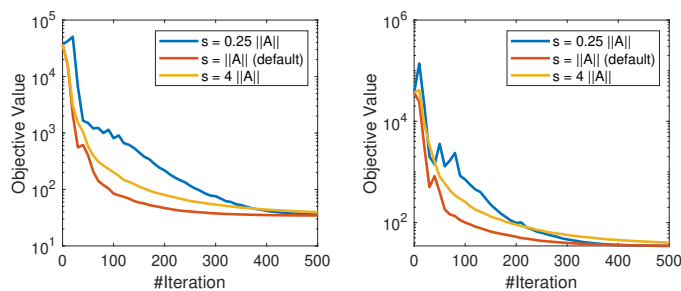
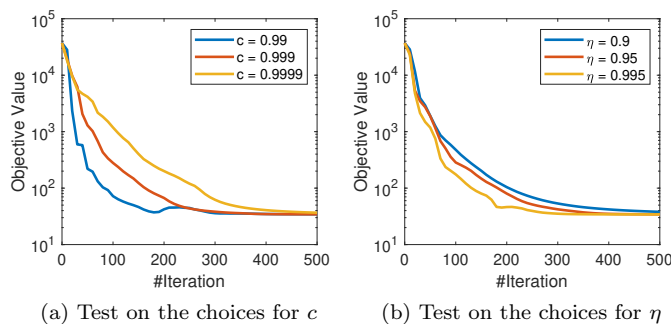


Figure 6: Test on the default choice  $s = \|A\|$  of A-SPDHG (rule-a) on X-ray Low-Dose fanbeam CT example. Left figure: starting ratio of primal-dual step-sizes:  $10^{-7}$ . Right figure: starting ratio of primal-dual step-sizes:  $10^{-5}$ . We can observe that our default choice of  $s$  is indeed a reasonable choice (at least near-optimal) in practice, and when deviating from it may lead to slower convergence.



(a) Test on the choices for  $c$  (b) Test on the choices for  $\eta$

Figure 7: Test on different choices of parameters of A-SPDHG (rule-b) on X-ray Low-Dose fanbeam CT example, starting ratio of primal-dual step-sizes:  $10^{-7}$ .

*Proof.* Let us call

$$M = \begin{pmatrix} a \text{Id} & P \\ P^* & b \text{Id} \end{pmatrix}.$$

For all  $(x, y) \in X \times Y$ ,

$$\|(x, y)\|_M^2 \geq a\|x\|^2 + b\|y\|^2 - 2\|P\|\|x\|\|y\| = \|x\|_a^2 + \|y\|_b^2 - 2(ab)^{-1/2}\|P\|\|x\|_a\|y\|_b,$$

which proves the direct implication of (6.1). For the converse implication, consider  $x \in X$  such that  $\|Px\| = \|P\|\|x\|$  and  $y = -\lambda Px$  for a scalar  $\lambda$ . Then, the non-negativity of the polynomial

$$\|(x, y)\|_M^2 = b\|P\|^2\lambda^2 - 2\|P\|^2\lambda + a$$

for all  $\lambda \in \mathbb{R}$  implies that  $\|P\|^4 - ab\|P\|^2 \geq 0$ , which is equivalent to the desired conclusion  $(ab)^{-1/2}\|P\| \leq 1$ .

Equivalence (6.2) is straightforward by noticing that

$$\begin{pmatrix} a\text{Id} & P \\ P^* & b\text{Id} \end{pmatrix} \succcurlyeq (1 - \beta) \begin{pmatrix} a\text{Id} & 0 \\ 0 & b\text{Id} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \beta a\text{Id} & P \\ P^* & \beta b\text{Id} \end{pmatrix} \succcurlyeq 0.$$

□

Let us now turn to the proof of Lemma 2.2.

*Proof of Lemma 2.2.* Let us assume that the step-sizes satisfy to the assumptions of the lemma. Then, Assumption (i) of Theorem 2.1 is straightforwardly satisfied. Moreover, for  $i \in \llbracket 1, n \rrbracket$ , the product sequence  $(\tau^k \sigma_i^k)_{k \in \mathbb{N}}$  is constant along the iterations by equation (2.6) and satisfies to equation (2.5) for iterate  $k = 0$ , thus satisfies to (2.5) for all  $k \in \mathbb{N}$ , which proves Assumption (ii). Finally, equation (2.7) implies that Assumption (iii) is satisfied. □

## 7 Declarations

### 7.1 Ethical approval

This declaration is not applicable.

### 7.2 Competing interests

There are no competing interests to declare.

### 7.3 Authors' contribution

CD, MJE and AC elaborated the proof strategy and CD wrote parts 1, 2 and 6. JT worked on the algorithmic design, performed the numerical experiments and wrote parts 3-5. All authors reviewed the manuscript.

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### 7.5 Availability of data and materials

The related implementation of the algorithms and the image data used in the experiment will be made available on the website <https://junqitang.com>. For the phantom image example we use the one in the experimental section of [8], while for the real CT image example we use an image from the Mayo Clinic Dataset [21] which is publicly available.

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