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Extending the macsum aggregation to interval-valued inputs

Yassine Hmidy[1], Agnès Rico[2], and Olivier Strauss[1]

[1] LIRMM, Université de Montpellier, CNRS, France
[2] LIRIS, Université Claude Bernard Lyon 1, CNRS, France

Abstract. Due to a lack of information or access to the data, or simply due to the imprecise nature of the data, there are cases where a function that maps interval-valued inputs is more appropriate to model a system. With the concern to keep the advantageous properties of the Choquet integral, we propose here a model that maps interval-valued inputs onto interval-valued outputs whose upper and lower bounds are Choquet integrals with respect to a parametric set function. This model is an extension to interval-valued inputs of the macsum aggregation proposed in [16]. In this article, we show that this extension can be easily computed thanks to some properties of the macsum parametric set function.

Keywords: Interval system · Choquet integral · Macsum aggregation.

1 Introduction

As mentioned by Grabisch in [7], the linear model is the main tool for modeling dependencies among data. The Choquet integral can be viewed as a generalization of this modeling. It is widely used as a flexible aggregation function in many fields, such as multiple criteria decision making [6], classification [3], regression [10] or data fusion [1]. Its main advantage is that it allows to quantify the importance of each variable and the interaction between groups of variables. Therefore it is an aggregation function that offers interpretability on the top of flexibility.

In a recent paper, Strauss et al. [16] were interested in the use of the Choquet integral to model incoherent systems. An incoherent system is a system which, for the same inputs, can have different outputs. This kind of behavior can be due, for example, to unmeasured inputs, to a partially random behavior of the modeled system or to partial lack of knowledge on the system.

There are many works interested in this type of modeling [11, 15, 17, 16]. The originality of the approach proposed by Strauss et al. is that it aims at providing the convex envelope of all the values that should have been outputted by the considered incoherent system for the same inputs. The modeling they propose, under the name of macsum, is an aggregation function that maps a vector of precise inputs onto an interval-valued output. The upper and lower bounds of this interval is computed respectively by the Choquet integral w.r.t. a particular submodular set function, called the macsum operator, and its conjugate.

As stated in [16], this modeling can be thought of as an imprecise linear model, thus representing a lack of knowledge about the input-output relationship of the modeled system. It has the major advantage over most of the other Choquet integral based approaches in that it depends only on a small number of parameters leading to very simple computation.

However, the inaccuracy may not be due solely to scant knowledge of the system. It can also be due to an imprecise measurement of the inputs [13, 4, 14]. One can therefore be interested in modeling a system that is imprecisely known and whose inputs are imprecise too. For example, in [12], Lacerda and Crespo propose to extend their interval predictor model to the case where inputs are interval-valued.

In this article, we are interested, as Lacerda and Crespo, in extending the macsum model to the case where the inputs are interval-valued. One of the difficulties of this kind of extension is that it can lead to very complex computations, thus losing the advantage of the simplicity of the macsum model. We show in this paper that extending the macsum model to interval-valued inputs leads to computation whose complexity is equivalent to that of the macsum model with precise inputs. As mentioned by Dubois and Prade [2], a minimal requirement for those extensions should be to be monotonically increasing in the wide sense with respect to each argument. We propose here two extensions. The first extension is disjunctive. It aims at dealing with the case where some sets to be aggregated cannot be considered as reliable. It presents the advantage of containing all the information but can lead to too broad intervals. The second extension is conjunctive. It aims at dealing with consonant sets of information. It would lead to more narrow intervals, giving the minimal set of common information in the considered sets.

This article is organized as follows. Section 2 gives the background needed to understand the following. Section 3 is a reminder of the macsum aggregation. Section 4 introduces the disjunctive extension and Section 5 the conjunctive extension of the macsum aggregation to interval-valued inputs we propose. In section 6, an example is given to illustrate the low complexity calculation of the bounds of these extensions. We then conclude in Section 7.

2 Background

- $\Omega = \{1, \ldots, N\} \subset \mathbb{N}$.
- A vector is a function $\boldsymbol{x} : \Omega \to \mathbb{R}$ defined by an element of $\mathbb{R}^N$ denoted $\boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N$.
- Let $\boldsymbol{x} \in \mathbb{R}^{\mathbb{N}}$, we define $\boldsymbol{x}^+, \boldsymbol{x}^- \in \mathbb{R}^{\mathbb{N}}$ such that $\forall i \in \Omega$, $x_i^+ = \max(0, x_i)$ and $x_i^- = \min(0, x_i)$.
- $\underline{\overline{x}} = [\underline{x}, \overline{x}]$ is a real interval whose lower bound is $\underline{x}$ and upper bound is $\overline{x}$.
- $\mathbb{IR}$ is the set of real intervals.
- A vector of real intervals is an element of $\mathbb{IR}^N$ denoted $\underline{\overline{\boldsymbol{x}}} = (\underline{\overline{x}}_1, \underline{\overline{x}}_2, .., \underline{\overline{x}}_N)$.
- We say that $\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}$ if $\forall i \in \Omega$, $x_i \in \underline{\overline{x}}_i$.
- An aggregation function $f : \mathbb{R}^N \to \mathbb{R}$ maps several input values onto a single real value.

- An interval aggregation function $f : \mathbb{IR}^N \to \mathbb{IR}$ maps several interval-valued inputs onto a single interval-valued output.
- A set function is a function $\mu : 2^{\Omega} \to \mathbb{R}$ that maps any subset of Ω onto a real value. To a set function μ is associated a complementary set function μ^c: $\forall A \subseteq \Omega$, $\mu^c(A) = \mu(\Omega) - \mu(A^c)$, with $\mu(\varnothing) = 0$.
- A set function μ is said to be submodular if $\forall A, B \subseteq \Omega$, $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$.
- A set function μ is said to be supermodular if $\forall A, B \subseteq \Omega$, $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$.
- The asymmetric Choquet integral of $\boldsymbol{x} \in \mathbb{R}^N$ w.r.t. a set function μ, denoted $\check{\mathbb{C}}_{\mu}$ [9], can be defined by:

$$\check{\mathbb{C}}_{\mu}(\boldsymbol{x}) = \sum_{k=1}^{N} x_{(k)}.(\mu(A_{(k)}) - \mu(A_{(k+1)})) = \sum_{k=1}^{N} (x_{(k)} - x_{(k-1)}).\mu(A_{(k)}),$$

  where (.) is the permutation that sorts the element of $\boldsymbol{x}$ in increasing order: $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(N)}$ with $x_{(0)} = 0$ and $A_{(i)}$ $(i \in \Omega)$ being the coalition of Ω such that $A_{(i)} = \{(i), \ldots, (N)\}$ with $A_{(N+1)} = \varnothing$.

- $\forall \underline{\overline{x}}, \underline{\overline{y}} \in \mathbb{IR}$, the natural union between $\underline{\overline{x}}$ and $\underline{\overline{y}}$ and the natural intersection between $\underline{\overline{x}}$ and $\underline{\overline{y}}$ are respectively defined as: $\underline{\overline{x}} \cup \underline{\overline{y}} = \{z \in \mathbb{R} |\ z \in \underline{\overline{x}} \text{ or } z \in \underline{\overline{y}}\}$ and $\underline{\overline{x}} \cap \underline{\overline{y}} = \{z \in \mathbb{R} |\ z \in \underline{\overline{x}} \text{ and } z \in \underline{\overline{y}}\}$.
- $\forall \underline{\overline{x}}, \underline{\overline{y}} \in \mathbb{IR}$, the extensive union between $\underline{\overline{x}}$ and $\underline{\overline{y}}$ and the extensive intersection between $\underline{\overline{x}}$ and $\underline{\overline{y}}$ are respectively defined as: $\underline{\overline{x}} \tilde{\cup} \underline{\overline{y}} = [\min(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})]$ and $\underline{\overline{x}} \tilde{\cap} \underline{\overline{y}} = [\max(\underline{x}, \underline{y}), \min(\overline{x}, \overline{y})]$.

  Remark 1. The natural and extensive unions of two intervals $\underline{\overline{x}}$ and $\underline{\overline{y}}$ are equal if $\underline{\overline{x}} \cap \underline{\overline{y}} \neq \varnothing$.

  Remark 2. $\underline{\overline{x}} \tilde{\cup} \underline{\overline{y}}$ is the convex envelope of $\underline{\overline{x}} \cup \underline{\overline{y}}$.

  Remark 3. If $\underline{\overline{x}} \cap \underline{\overline{y}} \neq \varnothing$, then $\underline{\overline{x}} \cap \underline{\overline{y}} = \underline{\overline{x}} \tilde{\cap} \underline{\overline{y}}$.

3 The macsum aggregation

A set function is said to be parametric when its computing involves values of a predefined set of parameters. The macsum operator is a parametric set function denoted $\nu_{\boldsymbol{\varphi}}$ where $\boldsymbol{\varphi}$ stands for the vector of parameters. This set function was introduced in [16], and is defined as $\forall \boldsymbol{\varphi} \in \mathbb{R}^N$, $\forall A \subseteq \Omega$:

$$\nu_{\boldsymbol{\varphi}}(A) = \max_{i \in A} \varphi_i^+ + \min_{i \in \Omega} \varphi_i^- - \min_{i \in A^c} \varphi_i^-,$$
$$\nu_{\boldsymbol{\varphi}}^c(A) = \min_{i \in A} \varphi_i^- + \max_{i \in \Omega} \varphi_i^+ - \max_{i \in A^c} \varphi_i^+.$$

Let $\boldsymbol{\varphi} \in \mathbb{R}^N$, the macsum aggregation is defined by using the macsum operator $\nu_{\boldsymbol{\varphi}}$ as:

$$\mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = [\check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}^c}(\boldsymbol{x}), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})]. \tag{1}$$

Let $\boldsymbol{\psi} \in \mathbb{R}^N$, $\forall A \subseteq \Omega$, we denote $\lambda_{\boldsymbol{\psi}}$ the linear parametric set function defined by:

$$\lambda_{\boldsymbol{\psi}}(A) = \sum_{i \in A} \psi_i.$$

Let $\boldsymbol{\psi} \in \mathbb{R}^N$, the linear aggregation is defined by using the linear operator $\lambda_{\boldsymbol{\psi}}$ as:

$$\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \sum_{i \in \Omega} \psi_i . x_i.$$

Contrarily to the macsum aggregation, the linear aggregation is precise-valued. It can easily be extended to interval valued inputs by $\forall \underline{\overline{\boldsymbol{x}}} \in \mathbb{IR}$:

$$\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\underline{\overline{\boldsymbol{x}}}) = \{\check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) \ / \ \boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}\} = \left[\inf_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} \mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}), \sup_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} \mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})\right],$$

this set being convex due to the fact that $\underline{\overline{\boldsymbol{x}}}$ is convex and $\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}$ is linear.

The macsum core of a vector $\boldsymbol{\varphi}$ of $\mathbb{R}^N$ is a convex subset of $\mathbb{R}^N$ defined as:

$$\mathcal{M}(\boldsymbol{\varphi}) = \left\{\boldsymbol{\psi} \in \mathbb{R}^N \ / \ \forall A \subseteq \Omega, \nu^c_{\boldsymbol{\varphi}}(A) \leq \lambda_{\boldsymbol{\psi}}(A) \leq \nu_{\boldsymbol{\varphi}}(A)\right\}.$$

Since the macsum operator is submodular, as proven in [16], the macsum core of a vector of $\mathbb{R}^N$ is never empty. Moreover, Equation (1) can be interpreted as:

$$\mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \left\{\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) \ / \ \boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})\right\},$$

this set being convex [16]: $\forall \boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$, $\exists y \in \mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})$ such that $y = \mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})$ and $\forall y \in \mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})$, $\exists \boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$ such that $y = \mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})$.

4 Disjunctive extension to interval-valued inputs

Extending the macsum aggregation to interval valued inputs consists in considering aggregating the set of all $\mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})$ with $\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}$. In this section, we consider a very conservative aggregation consisting in keeping all the aggregated interval-valued outputs of the macsum aggregations for each $\boldsymbol{x}$ of $\underline{\overline{\boldsymbol{x}}}$. This can be defined,$\forall \underline{\overline{\boldsymbol{x}}} \in \mathbb{IR}^N$, by:

$$\mathcal{D}_{\nu_{\boldsymbol{\varphi}}}(\underline{\overline{\boldsymbol{x}}}) = \bigcup\nolimits_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} \mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \bigcup\nolimits_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} \left[\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})\right] = \left\{\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) \ / \ \boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}, \ \boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})\right\}. \tag{2}$$

Remark 4. Given the Choquet integral w.r.t. the macsum operator is continuous as proven in [8] (Propostion 5.39) and $\underline{\overline{\boldsymbol{x}}}$ is a compact subspace of $\mathbb{R}^N$, we have that $\bigcup_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} [\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})] = \tilde{\bigcup}_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} [\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})]$.

Thus Equation (2) can be rewritten, $\forall \underline{\overline{\boldsymbol{x}}} \in \mathbb{IR}^N$:

$$\mathcal{D}_{\nu_{\boldsymbol{\varphi}}}(\underline{\overline{\boldsymbol{x}}}) = \tilde{\bigcup}\nolimits_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} \left[\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})\right] = \left[\inf_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \sup_{\boldsymbol{x} \in \underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})\right]. \tag{3}$$

Due to the definition of the disjunctive extension given in Equation (2), and the fact that the $\mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})$ are convex sets of $\mathbb{R}$, the bounds of $\mathcal{D}_{\nu_{\boldsymbol{\varphi}}}(\underline{\overline{\boldsymbol{x}}})$ are reached, i.e. $\exists \boldsymbol{y} \in \underline{\overline{\boldsymbol{x}}}$ and $\exists \boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$ such that $\sup_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(y)$ and $\exists \boldsymbol{y}' \in \underline{\overline{\boldsymbol{x}}}$ and $\exists \boldsymbol{\psi}' \in \mathcal{M}(\boldsymbol{\varphi})$ such that $\inf_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \mathcal{A}_{\lambda_{\boldsymbol{\psi}'}}(y')$.

The two following propositions are needed to compute the disjunctive extension.

Proposition 1. *(Proposition 5.1 in [16]). Let $\boldsymbol{\varphi} \in \mathbb{R}^N$, let $\boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$, $\forall i \in \Omega$, $\varphi_i > 0 \Rightarrow \psi_i \geq 0$ and $\varphi_i < 0 \Rightarrow \psi_i \leq 0$.*

Proposition 2. *Let $\boldsymbol{\varphi} \in \mathbb{R}^N$, let $\boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$, $\forall i \in \Omega$, $\varphi_i = 0 \Rightarrow \psi_i = 0$.*

Proof. Let $j \in \Omega$ such that $\varphi_j = 0$. We have $\nu_\varphi(\{j\}) = 0 + \min_{i\in\Omega} \varphi_i^- - \min_{i\in\Omega\setminus j} \varphi_i^-$.
As $\varphi_j = 0$, either $\min_{i\in\Omega} \varphi_i^- < 0$ and then $\min_{i\in\Omega} \varphi_i^- = \min_{i\in\Omega\setminus j} \varphi_i^-$ which gives $\nu_\varphi(\{j\}) = 0$, or $\min_{i\in\Omega} \varphi_i^- = \varphi_j = \min_{i\in\Omega\setminus j} \varphi_i^- = 0$, which also gives $\nu_\varphi(\{j\}) = 0$.
Furthermore, we have $\nu^c_\varphi(\{j\}) = 0 + \max_{i\in\Omega} \varphi_i^+ - \max_{i\in\Omega\setminus j} \varphi_i^+$. Proving that $\max_{i\in\Omega} \varphi_i^+ = \max_{i\in\Omega\setminus j} \varphi_i^+$ and consequently $\nu^c_\varphi(\{j\}) = 0$ can be done in the same manner.
As $\boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$, we have $\forall A \subseteq \Omega, \nu^c_\varphi(A) \leq \lambda_\psi(A) \leq \nu_\varphi(A)$, therefore $\nu^c_\varphi(\{j\}) \leq \lambda_\psi(\{j\}) \leq \nu_\varphi(\{j\})$, thus $0 \leq \psi_j \leq 0$ i.e. $\psi_j = 0$.

Therefore the non-negativity (resp. non-positivity) of an element of a parameter entails the non-negativity (resp. non-positivity) of the corresponding element of any parameter of its macsum core.

Proposition 3. *(Proposition 2.2 in [16]) Let $\boldsymbol{\varphi} \in \mathbb{R}^N$, let $\boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$, $\forall \boldsymbol{x} \in \mathbb{R}^N$ we have*
$\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) \in \mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \left[\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})\right]$.

Considering that $\forall \boldsymbol{x} \in \mathbb{IR}$, $\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\boldsymbol{\psi}}(\boldsymbol{x})$ and $\check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\boldsymbol{\psi}}(\boldsymbol{x})$ Equation (3) can be rewritten:

$$\mathcal{D}_{\nu_{\boldsymbol{\varphi}}}(\underline{\overline{\boldsymbol{x}}}) = \left[\inf_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}), \sup_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})\right] \tag{4}$$

$$= \left[\inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \inf_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}), \sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \sup_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})\right]. \tag{5}$$

Now, let $\underline{\overline{\boldsymbol{x}}} \in \mathbb{IR}^N$ be a vector of real intervals and let $\boldsymbol{\varphi} \in \mathbb{R}^N$. Let us define $\boldsymbol{x}^* \in \mathbb{R}^N$ the vector such that $\forall i \in \Omega$, $x_i^* = \overline{x}_i$ if $\varphi_i \geq 0$ and $x_i^* = \underline{x}_i$ if $\varphi_i < 0$. Let us also define $\boldsymbol{x}_* \in \mathbb{R}^N$ the vector such that $\forall i \in \Omega$, $x_{*i} = \underline{x}_i$ if $\varphi_i \geq 0$ and $x_{*i} = \overline{x}_i$ if $\varphi_i < 0$.

Proposition 4. $\forall \underline{\overline{\boldsymbol{x}}} \in \mathbb{IR}^N$, $\forall \boldsymbol{\varphi} \in \mathbb{R}^N$, $\forall \boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$,
$\sup_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}^*)$ *and* $\inf_{\boldsymbol{x}\in\underline{\overline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}_*)$.

Proof. Given Propositions 1 and 2, we know that $\forall i \in \Omega$, if $\varphi_i \geq 0$ then $\psi_i \geq 0$ and if $\varphi_i < 0$ then $\psi_i \leq 0$. Furthermore, by construction $\check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \sum_{i\in\Omega} \psi_i^+.x_i + \sum_{i\in\Omega} \psi_i^-.x_i$. Thus $\forall \boldsymbol{x} \in \overline{\underline{\boldsymbol{x}}}$ we have: $\check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \sum_{i\in\Omega} \psi_i.x_i \leq \sum_{i\in\Omega} \psi_i^+.\overline{x}_i + \sum_{i\in\Omega} \psi_i^-.\underline{x}_i = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}^*)$. Consequently $\sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}^*)$. Proving that $\inf_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}_*)$ can be done by the same way. □

Proposition 5. $\forall \overline{\underline{\boldsymbol{x}}} \in \mathbb{IR}^N$, $\forall \boldsymbol{\varphi} \in \mathbb{R}^N$, $\mathcal{D}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = [\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*)]$.

Proof. We know from [16] that $\forall \boldsymbol{x} \in \mathbb{R}^N, \forall \boldsymbol{\varphi} \in \mathbb{R}^N, \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})$, and from Proposition 4 that $\sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}^*)$. Thus, $\sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*)$. We can prove that $\inf_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*)$ in the same way. □

Thus the relation between the parameter of the macsum operator and the vectors of its core allows to compute the macsum aggregation on interval-valued inputs with a linear complexity.

5 Conjunctive extension to interval-valued inputs

In this section, we consider the conjunctive aggregation that can be viewed as the antonymous version of the previously defined disjunctive aggregation. Where disjunctive aggregation is conservative and tries not to reject any information, conjunctive aggregation tries to reduce the set of values to those for which each set being aggregated agrees.

There are two possible ways to create this conjunctive aggregation: either by making the conjunction on the input vectors $\boldsymbol{x}$ belonging to the interval-valued input vector $\overline{\underline{\boldsymbol{x}}}$, thus taking the conjunctive counterpart of Equation (4), or by making the conjunction on the vectors $\boldsymbol{\psi}$ belonging to the macsum core of the macsum parameter $\boldsymbol{\varphi}$, thus taking the conjunctive counterpart of Equation (5).

The first approach leads to:

$$\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \bigcap_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \bigcap_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \{\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) \ / \ \boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})\}, \tag{6}$$

while the second approach leads to:

$$\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \bigcap_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\overline{\underline{\boldsymbol{x}}}) = \bigcap_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \{\mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) \ / \ \boldsymbol{x} \in \overline{\underline{\boldsymbol{x}}}\}. \tag{7}$$

Let us consider the computation of Equation (6). If $\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) \neq \varnothing$, then:

$$\begin{aligned}\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) &= \tilde{\bigcap}_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \mathcal{A}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \tilde{\bigcap}_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \Big[\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})\Big] \\ &= \Big[\sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}), \inf_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x})\Big] = \Big[\sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}), \inf_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})\Big].\end{aligned} \tag{8}$$

In the same way, computation of Equation (7), if $\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) \neq \varnothing$, leads to:

$$\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \tilde{\bigcap}_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \mathcal{A}_{\lambda_{\boldsymbol{\psi}}}(\overline{\underline{\boldsymbol{x}}}) = \Big[\sup_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \inf_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}), \inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})\Big]. \quad (9)$$

Proposition 6. *$\forall\overline{\underline{\boldsymbol{x}}} \in \mathbb{IR}^N$, $\forall\boldsymbol{\varphi} \in \mathbb{R}^N$, the upper bound of $\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$ equals the lower bound of $\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$ and vice versa.*

Proof. Lets remind that $\forall\boldsymbol{\varphi} \in \mathbb{R}^N$, $\forall\boldsymbol{\psi} \in \mathcal{M}(\boldsymbol{\varphi})$, we have that $\sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}^*)$ and that $\forall\boldsymbol{x} \in \mathbb{R}^N$, $\inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x})$.
Therefore, $\inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}^*) = \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*)$.
Moreover, $\sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}) = \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*)$.
Thus, $\inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x}) = \sup_{\boldsymbol{x}\in\overline{\underline{\boldsymbol{x}}}} \inf_{\boldsymbol{\psi}\in\mathcal{M}(\boldsymbol{\varphi})} \check{\mathbb{C}}_{\lambda_{\boldsymbol{\psi}}}(\boldsymbol{x})$, i.e. the upper bound of $\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$ equals the lower bound of $\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$. By the same reasoning we obtain that the lower bound of $\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$ equals the upper bound of $\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$. □

If we refer to the theory of generalized intervals [5], we can say that if $\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$ is proper then $\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$ is improper and conversely. In other words, $\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \varnothing \Rightarrow \mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) \neq \varnothing$ and $\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \varnothing \Rightarrow \mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) \neq \varnothing$. In that sense those two approaches can be considered as dual. We thus have:

$$\mathcal{C}^{\triangleleft}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \Big[\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*)\Big], \text{ if } \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*) \leq \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*), \text{ and}$$
$$\mathcal{C}^{\triangleright}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \Big[\check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*), \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*)\Big], \text{ if } \check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*) \geq \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*).$$

We thus propose to consider, for the conjunctive extension of the macsum aggregation to interval-valued inputs, the following expression:

$$\mathcal{C}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}}) = \Big[\min\big(\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*)\big), \max\big(\check{\mathbb{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*), \check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}_*)\big)\Big].$$

This formula gathers in the same expression the two conjunctive counterparts (i.e. Equations (8) and (9)) of the disjunctive extension. As in the disjunctive case, this extension only needs computation of two Choquet integrals.

6 Example

In this section, we give a simple example illustrating the simplicity of the computation of the macsum aggregation of interval-valued inputs. Let $\overline{\underline{\boldsymbol{x}}} = ([-2,3],[0,1],[-5,8]) \in \mathbb{IR}^3$ and let $\boldsymbol{\varphi} = (-2,3,-1) \in \mathbb{R}^3$. Let us compute $\mathcal{D}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$ and $\mathcal{C}_{\nu_{\boldsymbol{\varphi}}}(\overline{\underline{\boldsymbol{x}}})$. We have $\boldsymbol{x}^* = (-2,1,-5)$ and $\boldsymbol{x}_* = (3,0,8)$.
Now lets sort $\boldsymbol{x}^*$ in increasing order: $\boldsymbol{x}^*_{(.)} = (x^*_3, x^*_1, x^*_2) = (-5,-2,1)$.
Therefore $\boldsymbol{\varphi}_{(.)} = (\varphi_3, \varphi_1, \varphi_2) = (-1,-2,3)$.
Remember that $\nu_{\varphi}(A_{(k)}) = \max_{i\in A_{(k)}} \varphi^+_i - \min_{i\in A^c_{(k)}} \varphi^-_i + \min_{i\in\Omega} \varphi^-_i$, with $A_{(k)} = \{(k),\ldots,(N)\}$, thus $\nu_{\varphi}(A_{(k)}) = \max_{i=k}^{N} \varphi^+_{(i)} - \min_{i=1}^{k-1} \varphi^-_{(i)} + \min_{i\in\Omega} \varphi^-_i$

(by convention $\min_{i=1}^{0} \varphi_{(i)}^{-} = 0$).
Thus:

$$\begin{aligned}
\check{\mathbb{C}}_{\nu_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*) &= \sum_{k=1}^{3} (x^*_{(k)} - x^*_{(k-1)}).\nu_{\boldsymbol{\varphi}}(A_{(k)}), \\
&= \sum_{k=1}^{3} (x^*_{(k)} - x^*_{(k-1)}). \left(\max_{i=k}^{3} \varphi^{+}_{(i)} - \min_{i=1}^{k-1} \varphi^{-}_{(i)} + \min_{i\in\Omega} \varphi^{-}_{i} \right), \\
&= (x^*_{(1)} - x^*_{(0)}). \left(\max_{i=1}^{3} \varphi^{+}_{(i)} - \min_{i=1}^{0} \varphi^{-}_{(i)} + \min_{i\in\Omega} \varphi^{-}_{i} \right) \\
&+ (x^*_{(2)} - x^*_{(1)}). \left(\max_{i=2}^{3} \varphi^{+}_{(i)} - \min_{i=1}^{1} \varphi^{-}_{(i)} + \min_{i\in\Omega} \varphi^{-}_{i} \right) \\
&+ (x^*_{(3)} - x^*_{(2)}). \left(\max_{i=3}^{3} \varphi^{+}_{(i)} - \min_{i=1}^{2} \varphi^{-}_{(i)} + \min_{i\in\Omega} \varphi^{-}_{i} \right), \\
&= -5. (3-2) + (-2+5). (3+1-2) + (1+2). (3+2-2), \\
&= -5 + 6 + 9 = 10.
\end{aligned}$$

Lets now sort $\boldsymbol{x}_*$ in increasing order: $\boldsymbol{x}_{*(.)} = (x_{*2}, x_{*1}, x_{*3}) = (0, 3, 8)$.
Therefore $\boldsymbol{\varphi}_{(.)} = (\varphi_2, \varphi_1, \varphi_3) = (3, -2, -1)$.
Remember that $A^c_{(k)} = \{(1), \dots (k-1)\}$, we have $\nu^c_{\varphi}(A_{(k)}) = \min_{i=k}^{N} \varphi^{-}_{(i)} - \max_{i=1}^{k-1} \varphi^{+}_{(i)} + \max_{i\in\Omega} \varphi^{+}_{i}$
(by convention $\min_{i=1}^{0} \varphi^{-}_{(i)} = 0$).
Then:

$$\begin{aligned}
\mathbb{\check{C}}_{\nu^c_{\boldsymbol{\varphi}}}(\boldsymbol{x}^*) &= \sum_{k=1}^{3} (x_{*(k)} - x_{*(k-1)}).\nu^c_{\boldsymbol{\varphi}}(A_{(k)}), \\
&= \sum_{k=1}^{3} (x_{*(k)} - x_{*(k-1)}). \left(\min_{i=k}^{3} \varphi^{-}_{(i)} - \max_{i=1}^{k-1} \varphi^{+}_{(i)} + \max_{i\in\Omega} \varphi^{+}_{i} \right), \\
&= (x_{*(1)} - x_{*(0)}). \left(\min_{i=1}^{3} \varphi^{-}_{(i)} - \max_{i=1}^{0} \varphi^{+}_{(i)} + \max_{i\in\Omega} \varphi^{+}_{i} \right) \\
&+ (x_{*(2)} - x_{*(1)}). \left(\min_{i=2}^{3} \varphi^{-}_{(i)} - \max_{i=1}^{1} \varphi^{+}_{(i)} + \max_{i\in\Omega} \varphi^{+}_{i} \right) \\
&+ (x_{*(3)} - x_{*(2)}). \left(\min_{i=3}^{3} \varphi^{-}_{(i)} - \max_{i=1}^{2} \varphi^{+}_{(i)} + \max_{i\in\Omega} \varphi^{+}_{i} \right), \\
&= 0. (-2+3) + (3-0). (-2-3+3) + (8-3). (-1-3+3), \\
&= 0 - 6 - 5 = -11.
\end{aligned}$$

Finally, $\mathcal{D}_{\nu_{\boldsymbol{\varphi}}}(([-2,3],[0,1],[-5,8])) = [-11, 10]$.
By the same way we show that $\mathcal{C}_{\nu_{\boldsymbol{\varphi}}}(([-2,3],[0,1],[-5,8])) = [-6, 7]$.

7 Discussion

The macsum aggregation, as introduced in [16], allows to model incoherent systems. Aggregating precise inputs within the macsum approach leads to interval-valued outputs. In contrast to a simple linear model, it gives a convex set of linear models which makes it more flexible, while preserving the simplicity of use of linear model, as it depends only on N parameters, N being the number of inputs. A promising path would be to use this extension of the macsum aggregation as a simple and flexible learning model. This gives rise to other theoretical issues on its ability to approximate any functions that maps interval-valued inputs onto interval-valued outputs. Moreover, as the macsum operator needs the separation of the positive and negative elements of its parameter, further work could go towards the definition of a simpler submodular parametric set function in order to facilitate the computations during a learning process.

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