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ON A SYSTEM OF INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATIONS ARISING IN OPTICAL MEDIA WITH A $\chi^{(2)}$ NONLINEARITY

VAN DUONG DINH AND AMIN ESFAHANI

ABSTRACT. We study a system of inhomogeneous nonlinear Schrödinger equations arising in optical media with a $\chi^{(2)}$ nonlinearity whose local strength is subject to cusp-shaped spatial modulation, $\chi^{(2)} \sim |x|^{-\alpha}$ with $\alpha > 0$, which can be induced by spatially nonuniform poling. We first establish a vectorial Gagliardo–Nirenberg type inequality related to the system, by which we determine sufficient conditions on initial data leading to the existence of globally in time solutions of the system. We also provide criteria for the existence of non-radial blow-up solutions with mass-critical and mass-supercritical nonlinearities. By exploiting the spatial decay of the nonlinearity at infinity, we establish the non-radial energy scattering in the mass-supercritical regime for global solutions with initial data lying below a mass-energy threshold. We prove, by variational analysis, the existence and qualitative properties of ground states related to the system. The limiting behavior of positive radial ground states once α approaches zero is studied by means of the mountain-pass energy. The stability and instability of ground state standing waves are also investigated.

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1. INTRODUCTION

In this paper, we are interested in a system of inhomogeneous nonlinear Schrödinger equations in optical media, namely

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u + |x|^{-\alpha}\bar{u}v = 0, \\ i\partial_t v + \frac{\kappa}{2}\Delta v - \gamma v + \frac{1}{2}|x|^{-\alpha}u^2 = 0, \end{cases} \quad (1.1)$$

where $u, v : \mathbb{R}_t \times \mathbb{R}_x^d \rightarrow \mathbb{C}$ are unknown wave functions, $d \geq 1$, $\alpha > 0$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. This system describes the two-wave (degenerate, alias type-I) quadratic interactions for the complex fundamental-frequency (FF) and second-harmonic (SH) amplitudes in the presence of the spatial singular modulation of the $\chi^{(2)}$ nonlinearity. The real coefficient γ represents the SH-FF mismatch and, by rescaling, can be $0, \pm 1$. We refer the readers to [31, 32] for a derivation of this system from the physical context.

Our first aim is to study the Cauchy problem associated with (1.1). In the nonsingular case $\alpha = 0$, where (1.1) was introduced as a non-relativistic version of some Klein-Gordon systems (see [24]), the well-posedness in $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ with $1 \leq d \leq 4$ and in $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ with $1 \leq d \leq 6$ were completely investigated in [24] by the contraction argument combined with Strichartz estimates. Moreover, the global well-posedness in $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ when $d = 4$ as well as the global existence in $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ when $1 \leq d \leq 3$ were obtained due to the conservation of mass and energy. In addition, for initial data in $H^1(\mathbb{R}^5) \times H^1(\mathbb{R}^5)$, a condition for global existence was found which depends on the size of the initial data when compared to the associated ground states ([24]). The global existence and energy scattering were studied in [22, 23, 45, 33], while the finite time blow-up were showed in [10, 35, 25]. See also [36, 37, 38, 13] for other results related to more general systems of NLS with quadratic interaction.

By applying the abstract argument due to Cazenave [5], we show that system (1.1) is locally well-posed in $\mathcal{H}^1 := H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ for $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$ (See Section 2). In addition, \mathcal{H}^1 -solutions to (1.1) satisfy conservation laws of mass and energy:

$$\mathbb{M}(u, v) := \|u\|_{L^2}^2 + 2\|v\|_{L^2}^2, \quad (\text{Mass})$$

$$\mathbb{E}(u, v) := \frac{1}{2}\mathbb{K}(u, v) + \gamma\|v\|_{L^2}^2 - \mathbb{P}(u, v), \quad (\text{Energy})$$

where

$$\begin{aligned} \mathbb{K}(u, v) &:= \|\nabla u\|_{L^2}^2 + \kappa\|\nabla v\|_{L^2}^2, \\ \mathbb{P}(u, v) &:= \Re \int_{\mathbb{R}^d} |x|^{-\alpha} u^2 \bar{v} \, dx. \end{aligned}$$

Notice when $\gamma = 0$ that system (1.1) has a scaling invariance

$$(u_\lambda(t, x), v_\lambda(t, x)) = (\lambda^{2-\alpha}u(\lambda^2 t, \lambda x), \lambda^{2-\alpha}v(\lambda^2 t, \lambda x)), \quad \lambda > 0.$$

That is, if $(u(t), v(t))$ is a solution to (1.1) with initial data (u_0, v_0) , then $(u_\lambda(t), v_\lambda(t))$ is also a solution to (1.1) with initial data $(u_\lambda(0), v_\lambda(0))$. This scaling leaves the $\dot{\mathcal{H}}^{s_c}$ -norm of initial data invariant, i.e.,

$$\|(u_\lambda(0), v_\lambda(0))\|_{\dot{\mathcal{H}}^{s_c}} := \|u_\lambda(0)\|_{\dot{H}^{s_c}} + \|v_\lambda(0)\|_{\dot{H}^{s_c}} = \|(u_0, v_0)\|_{\dot{\mathcal{H}}^{s_c}}, \quad \forall \lambda > 0,$$

where

$$s_c := \frac{d}{2} - 2 + \alpha \quad (1.2)$$

is the critical Sobolev exponent. When $s_c = 0$ or $\alpha = \frac{4-d}{2}$, system (1.1) is called mass-critical; while for $s_c = 1$ or $\alpha = \frac{6-d}{2}$, system (1.1) is known as energy-critical. According to this terminology, our local well-posedness result is available only for the energy-subcritical regime and it is summarized as follows:

Dimension	Mass-subcritical	Mass-critical	Mass-supercritical
1	$0 < \alpha < 1$	NA	NA
2	$0 < \alpha < 1$	$\alpha = 1$	$1 < \alpha < 2$
3	$0 < \alpha < \frac{1}{2}$	$\alpha = \frac{1}{2}$	$\frac{1}{2} < \alpha < \frac{3}{2}$
4	NA	NA	$0 < \alpha < 1$
5	NA	NA	$0 < \alpha < \frac{1}{2}$

TABLE 1. The local well-posedness of (1.1) in \mathcal{H}^1 .

The case of energy-critical nonlinearity $\alpha = \frac{6-d}{2}$ with $3 \leq d \leq 5$ is special and will be addressed in a forthcoming work.

Once the local theory is established, we aim to give necessary and sufficient conditions on initial data which clarify the global existence, energy scattering, and blowing-up behavior of solutions to (1.1). To obtain these results, we show, in Section 3, the following vectorial Gagliardo–Nirenberg type inequality

$$\mathbb{P}(u, v) \leq C_{\text{GN}} (\mathbb{K}(u, v))^{\frac{d+2\alpha}{4}} (\mathbb{M}(u, v))^{\frac{6-d-2\alpha}{4}}, \quad \forall (u, v) \in \mathcal{H}^1.$$

The optimal constant associated with this inequality is achieved by nonlinear ground states of (1.1) (see Definition 3.1). The proof of such an inequality relies on a compactness result (see Lemma 3.3) making use of the decay at infinity of the weighted term $|x|^{-\alpha}$.

Having the best constant of the vectorial Gagliardo–Nirenberg inequality in our hand, we derive, in Section 4, sufficient conditions on the initial data under which the solution of (1.1) exists globally in time for both mass-critical and mass-supercritical nonlinearities. By the aforementioned Gagliardo–Nirenberg inequality, the existence of global solutions in the mass-subcritical regime is guaranteed without any additional conditions. In the mass-critical regime, solutions exist globally in time provided that the mass of initial data is smaller than the mass of nonlinear ground states. While for the mass-supercritical nonlinearity, we found a mass-energy threshold for which solutions exist globally in time if initial data lies below this threshold.

We next turn our attention, in Section 5, to finding conditions under which the solutions of (1.1) blow up or grow up. To this end, we introduce

$$\mathbb{G}(u, v) = \mathbb{K}(u, v) - \frac{d+2\alpha}{2} \mathbb{P}(u, v) \quad (1.3)$$

which is nothing but the Pohozaev functional obtained by

$$\frac{d}{dt} \left(2 \Im \int_{\mathbb{R}^d} x \cdot (\nabla u(t) \bar{u}(t) + \nabla v(t) \bar{v}(t)) \, dx \right) = 2\mathbb{G}(u(t), v(t)).$$

We note that the left hand side is exactly the time derivative of the standard virial quantity

$$\int_{\mathbb{R}^d} |x|^2 (|u(t)|^2 + 2|v(t)|^2) \, dx$$

only if $\kappa = \frac{1}{2}$, which is usually referred to as the mass-resonance condition. In the mass-critical case, we show (see Theorem 5.1) that if $\mathbb{G}(u(t), v(t))$ is bounded uniformly from above by a negative constant, i.e.,

$$\sup_{t \in (-T_*, T^*)} \mathbb{G}(u(t), v(t)) \leq -\delta \quad (1.4)$$

for some $\delta > 0$, where $(-T_*, T^*)$ is the maximal time of existence, then the solution blows up in finite time in the mass-resonance case $\kappa = \frac{1}{2}$; while in the non mass-resonance case $\kappa \neq \frac{1}{2}$, the solution either blows up in finite time or it grows up along any diverging time equation with an explicit lower bound on its blow-up rate. The blow-up condition (1.4) is verified as long as initial data satisfies $\mathbb{H}(u_0, v_0) < 0$, where

$$\mathbb{H}(u, v) := \begin{cases} \mathbb{E}(u, v) & \text{if } \gamma \geq 0, \\ \mathbb{E}(u, v) + \frac{|\gamma|}{2} \mathbb{M}(u, v) & \text{if } \gamma < 0. \end{cases} \quad (1.5)$$

In the mass-supercritical case, we prove (see Theorem 5.2) that under the same condition (1.4), the solution blows up in finite time for both mass-resonance and non mass-resonance cases when $2 \leq d \leq 4$; while in five dimensions, the solution either blows up in finite time or it grows up along a single diverging time sequence with an explicit lower bound on its blow-up rate. The blow-up criterion (1.4) is satisfied provided that either $\mathbb{H}(u_0, v_0) < 0$ or, if $\mathbb{H}(u_0, v_0) \geq 0$, initial data lies above a mass-energy threshold. We emphasize that our blow-up results do not impose any symmetric conditions or finite variance on initial data, hence we obtain stronger results comparing to the ones for system of NLS with quadratic interaction in [9, 35, 25, 13] (see [9, 35, 25] for radial or finite variance blow-up solutions and [13] for cylindrical blow-up solutions). Our method is inspired by [39, 40] using localized virial estimates. However, instead of making use of the radial symmetry, we exploit the spatial decay of the nonlinearity at infinity. Our results also extend recent blow-up results for the single inhomogeneous nonlinear Schrödinger equation in [9, 1, 8].

We next aim to prove, in Section 6, the energy scattering for global solutions in the mass-supercritical regime. We establish (see Theorem 6.1) that for $3 \leq d \leq 5$, global solutions with initial data lying below

the mass-energy threshold scatters in \mathcal{H}^1 in both directions regardless the mass-resonance or non mass-resonance cases. To our knowledge, the energy scattering for quadratic system of NLS (i.e., $\alpha = 0$) with non-radial data was known only for a slight perturbation of the mass resonance case, i.e., $|\kappa - 1/2| \ll 1$ (see e.g., [22, 23, 33, 45, 37]). Here we prove the non-radial scattering for all $\kappa > 0$. Our strategy is based on a recent work of Murphy [34] which is a combination of a scattering criterion and a space-time estimate. Here we make an effective use of the spatial decay at infinity of $|x|^{-\alpha}$.

In Section 7, we show that (1.1) possesses a standing wave $(e^{i\omega t}\varphi(x), e^{2i\omega t}\psi(x))$ by minimizing the action functional $\mathbb{A}_\omega = \mathbb{E} + \omega\mathbb{M}$ on a Nehari submanifold of \mathcal{H}^1 . Our proof relies on a type of compact embedding (see Lemma 3.3) which is also a consequence of the spatial decay of $|x|^{-\alpha}$ at infinity. We also show that the minimizers obtained by this method are indeed the ground states of (1.1). Here by ground states of (1.1), we mean non-trivial critical points of \mathbb{A}_ω that minimizes the action functional \mathbb{A}_ω between all non-trivial critical points of \mathbb{A}_ω . In addition, we show that ground states of (1.1) decay exponentially at infinity. This property will be useful to study the strong instability of ground state standing waves.

It will be an interesting issue to go through the limit behavior of ground states when α decreases to zero. In this light, in Section 8, we apply the concentration-compactness principle and derive new variational property of ground states by proving that the minimum value obtained through the Nehari constrained minimization problem is indeed the mountain-pass level (Lemma 8.3). This enables us to find a uniform bound on the nonlinear term of the energy functional and show that positive radial ground states of (1.1) (which depends on α) converges to a positive radial ground state of (1.1) with $\alpha = 0$ when $\alpha \rightarrow 0$.

In the last section—Section 9, we show that the set of ground states is stable with respect to the flow of the Cauchy problem associated with (1.1) if $L''(\omega) > 0$, where $L(\omega)$ is the so called Lyapunov function (see Theorem 9.1). In the case $\gamma = 0$, this is equivalent to the mass-subcritical regime $d + 2\alpha < 4$. To study the instability in the mass-critical and mass-supercritical cases, we define a new variational problem. This allows us to define a subset of \mathcal{H}^1 which is invariant under the flow of (1.1). Additionally, for any initial data belonging to this set, we prove that the corresponding solution enjoys a uniform upper bound by a negative constant. This, combined with our previous blow-up results, yields the strong instability of ground state standing waves for (1.1). We highlight that our strong instability results are much stronger comparing to the ones for the system of NLS (see [11, 25]). In fact, in [11], the strong instability was proved only for $\kappa = \frac{1}{2}$; while in [25], the strong instability was showed only for radial ground state standing waves. Here we have the strong instability for any $\kappa > 0$ and general (not necessarily radial) ground state standing waves.

Finally, in Appendix A, we give some related results concerning ground state solutions. First, by using the variational characteristics of ground states and the uniqueness of positive radial solution of

$$\frac{1}{2}\Delta\varphi - \omega\varphi + |x|^{-\alpha}\varphi^2 = 0,$$

we prove the uniqueness (up to dilations) of positive ground states of (1.1) in the particular case $\kappa = 2 + \frac{\gamma}{\omega}$ (see Theorem A.2). Second, we show by means of the Mountain-pass theorem (with symmetry) that there are infinitely many distinct radially symmetric solutions to the ground state equation that grows up the action functional \mathbb{A}_ω (see Theorem A.3).

We finish the introduction by listing some notations which will be used throughout the sequel.

Notations. Denote

$$\begin{aligned} \mathcal{H}^1 &:= H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d), & \dot{\mathcal{H}}^1 &:= \dot{H}^1(\mathbb{R}^d) \times \dot{H}^1(\mathbb{R}^d), \\ \mathcal{L}^p &:= L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d), & \mathcal{W}^{1,p} &:= W^{1,p}(\mathbb{R}^d) \times W^{1,p}(\mathbb{R}^d) \end{aligned}$$

with the norms

$$\begin{aligned} \|(u, v)\|_{\mathcal{H}^1} &:= \|u\|_{H^1} + \|v\|_{H^1}, & \|(u, v)\|_{\dot{\mathcal{H}}^1} &:= \|u\|_{\dot{H}^1} + \|v\|_{\dot{H}^1}, \\ \|(u, v)\|_{\mathcal{L}^p} &:= \|u\|_{L^p} + \|v\|_{L^p}, & \|(u, v)\|_{\mathcal{W}^{1,p}} &:= \|u\|_{W^{1,p}} + \|v\|_{W^{1,p}}. \end{aligned}$$

When $p = 2$, we use \mathcal{H}^1 instead of $\mathcal{W}^{1,2}$. Let $I \subset \mathbb{R}$ be an interval. We denote

$$\mathcal{L}^p(I, \mathcal{L}^q) := L^p(I, L^q(\mathbb{R}^d)) \times L^p(I, L^q(\mathbb{R}^d)), \quad \mathcal{L}^p(I, \mathcal{W}^{1,q}) := L^p(I, W^{1,q}(\mathbb{R}^d)) \times L^p(I, W^{1,q}(\mathbb{R}^d))$$

with norms

$$\|(u, v)\|_{\mathcal{L}^p(I, \mathcal{L}^q)} := \|u\|_{L^p(I, L^q)} + \|v\|_{L^p(I, L^q)}, \quad \|(u, v)\|_{\mathcal{L}^p(I, \mathcal{W}^{1,q})} := \|u\|_{L^p(I, W^{1,q})} + \|v\|_{L^p(I, W^{1,q})}.$$

2. LOCAL WELL-POSEDNESS

In this section, we show the local well-posedness for (1.1) in the energy-subcritical regime and our result reads as follows.

Proposition 2.1. *Let $1 \leq d \leq 5$, $\kappa > 0$, $\gamma \in \mathbb{R}$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. For any $(u_0, v_0) \in \mathcal{H}^1$, there exists a unique maximal solution*

$$(u, v) \in C((-T_*, T^*), \mathcal{H}^1) \cap C^1((-T_*, T^*), \mathcal{H}^{-1})$$

to (1.1) with initial data $(u, v)|_{t=0} = (u_0, v_0)$, where $\mathcal{H}^{-1} := H^{-1}(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ is the dual space of \mathcal{H}^1 . The maximal time satisfies the blow-up alternative: if $T^* < \infty$ (resp. $T_* < \infty$), then

$$\lim_{t \nearrow T^*} \|(u(t), v(t))\|_{\mathcal{H}^1} = \infty \quad \left(\text{resp.} \quad \lim_{t \searrow -T_*} \|(u(t), v(t))\|_{\mathcal{H}^1} = \infty \right).$$

In addition, there are conservation laws of mass and energy, i.e.,

$$\mathbb{M}(u(t), v(t)) = \|u(t)\|_{L^2}^2 + 2\|v(t)\|_{L^2}^2 = \mathbb{M}(u_0, v_0),$$

$$\mathbb{E}(u(t), v(t)) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\kappa}{2}\|\nabla v(t)\|_{L^2}^2 + \gamma\|v(t)\|_{L^2}^2 - \Re \int_{\mathbb{R}^d} |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx = \mathbb{E}(u_0, v_0),$$

for all $t \in (-T_*, T^*)$.

Remark 2.1. The condition $\alpha < \frac{6-d}{2}$ for $3 \leq d \leq 5$ ensures that our problem is energy-subcritical, i.e., $s_c < 1$.

The proof of Proposition 2.1 relies on an abstract theory of Cazenave [5, Theorem 3.3.9, Remark 3.3.12, and Theorem 4.3.1]. More precisely, let

$$\vec{u} := (u_1, u_2), \quad \mathcal{A}\vec{u} := (\alpha_1 \Delta u_1, \alpha_2 \Delta u_2)$$

and consider

$$i\partial_t \vec{u} + \mathcal{A}\vec{u} + g(\vec{u}) = 0. \quad (2.1)$$

Proposition 2.2 ([5]). *Let $g = g_1 + g_2 + g_3$ satisfy the following conditions:*

- (1) For $j = 1, 2, 3$, $g_j \in C(\mathcal{H}^1, \mathcal{H}^{-1})$, $g_j(\vec{0}) = \vec{0}$, and there exists $G_j \in C^1(\mathcal{H}^1, \mathbb{R})$ such that $g_{jk} = \partial_k G_j$ for $k = 1, 2$, where $g_j = (g_{j1}, g_{j2})$ and ∂_k stands for the Fréchet derivative with respect to the k -th variable.
- (2) For $j = 1, 2, 3$, there exist $r_j, \rho_j \in [2, \frac{2d}{d-2}]$ if $d \geq 2$ or $r_j, \rho_j \in [2, \infty]$ if $d = 1$ such that for every $M > 0$, there exists $C(M) > 0$ so that

$$\|g_j(\vec{u}) - g_j(\vec{v})\|_{\mathcal{L}^{\rho'_j}} \leq C(M) \|\vec{u} - \vec{v}\|_{\mathcal{L}^{r_j}}$$

for all $\vec{u}, \vec{v} \in \mathcal{H}^1$ satisfying $\|\vec{u}\|_{\mathcal{H}^1} + \|\vec{v}\|_{\mathcal{H}^1} \leq M$. Here (ρ_j, ρ'_j) is a Hölder conjugate pair.

- (3) There exists $(\beta_1, \beta_2) \in \mathbb{R}^2$ such that for $j = 1, 2, 3$ and every $\vec{u} \in \mathcal{H}^1$,

$$\Im(\beta_1 g_{j1}(\vec{u}) \bar{u}_1 + \beta_2 g_{j2}(\vec{u}) \bar{u}_2) = 0 \text{ a.e. in } \mathbb{R}^d.$$

Then for any $\vec{u}_0 \in \mathcal{H}^1$, there exists a unique maximal solution

$$\vec{u} \in C((-T_*, T^*), \mathcal{H}^1) \cap C^1((-T_*, T^*), \mathcal{H}^{-1})$$

to (2.1) with initial data $u|_{t=0} = \vec{u}_0$. The maximal time satisfies the blow-up alternative: if $T^* < \infty$ (resp. $T_* < \infty$), then $\lim_{t \nearrow T^*} \|\vec{u}(t)\|_{\mathcal{H}^1} = \infty$ (resp. $\lim_{t \searrow -T_*} \|\vec{u}(t)\|_{\mathcal{H}^1} = \infty$). In addition, there are conservation laws of mass and energy, i.e.,

$$\mathbb{M}(\vec{u}(t)) = \beta_1 \|u_1(t)\|_{L^2}^2 + \beta_2 \|u_2(t)\|_{L^2}^2 = \mathbb{M}(\vec{u}_0), \quad (\text{Mass})$$

$$\mathbb{E}(\vec{u}(t)) = \alpha_1 \|\nabla u_1(t)\|_{L^2}^2 + \alpha_2 \|\nabla u_2(t)\|_{L^2}^2 - 2 \sum_{j=1}^3 G_j(\vec{u}(t)) = \mathbb{E}(\vec{u}_0), \quad (\text{Energy})$$

for all $t \in (-T_*, T^*)$.

Proof of Proposition 2.1. We apply Proposition 2.2 with

$$(\alpha_1, \alpha_2) = \left(\frac{1}{2}, \frac{\kappa}{2} \right), \quad (\beta_1, \beta_2) = (1, 2)$$

and

$$\begin{aligned} g_1(u, v) &= (0, -\gamma v), \\ g_2(u, v) &= \left(\mathbf{1}_{B_1} |x|^{-\alpha} \bar{u} v, \frac{1}{2} \mathbf{1}_{B_1} |x|^{-\alpha} u^2 \right), \\ g_3(u, v) &= \left(\mathbf{1}_{B_1^c} |x|^{-\alpha} \bar{u} v, \frac{1}{2} \mathbf{1}_{B_1^c} |x|^{-\alpha} u^2 \right), \end{aligned}$$

where

$$B_1 := \{x \in \mathbb{R}^d : |x| < 1\}, \quad B_1^c := \mathbb{R}^d \setminus B_1.$$

We have

$$\begin{aligned} G_1(u, v) &= -\frac{\gamma}{2} \|v\|_{L^2}^2, \\ G_2(u, v) &= \frac{1}{2} \Re \int_{B_1} |x|^{-\alpha} u^2 \bar{v} \, dx, \\ G_3(u, v) &= \frac{1}{2} \Re \int_{B_1^c} |x|^{-\alpha} u^2 \bar{v} \, dx. \end{aligned}$$

One readily check that the conditions (1) and (3) are fulfilled. It remains to check the condition (2).

- For g_1 , we simply take $r_1 = \rho_1 = 2$ and get

$$\|g_1(u_1, v_1) - g_1(u_2, v_2)\|_{\mathcal{L}^2} = \gamma \|v_1 - v_2\|_{L^2} \leq \gamma \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{L}^2}.$$

- For g_2 , we write

$$\|g_2(u_1, v_1) - g_2(u_2, v_2)\|_{\mathcal{L}^{\rho_2'}} = \|\mathbf{1}_{B_1} |x|^{-\alpha} (\bar{u}_1 v_1 - \bar{u}_2 v_2)\|_{L^{\rho_2'}} + \|\mathbf{1}_{B_1} |x|^{-\alpha} (u_1^2 - u_2^2)\|_{L^{\rho_2'}} =: (\text{I}) + (\text{II}).$$

We only estimate (I) since the one for (II) is treated in a similar manner. We have

$$(\text{I}) \leq \|\mathbf{1}_{B_1} |x|^{-\alpha} (\bar{u}_1 - \bar{u}_2) v_1\|_{L^{\rho_2'}} + \|\mathbf{1}_{B_1} |x|^{-\alpha} \bar{u}_2 (v_1 - v_2)\|_{L^{\rho_2'}} =: (\text{I}_1) + (\text{I}_2).$$

By Hölder's inequality,

$$(\text{I}_1) \leq \|\mathbf{1}_{B_1} |x|^{-\alpha}\|_{L^{\gamma_2}} \|v_1\|_{L^{\rho_2}} \|u_1 - u_2\|_{L^{\rho_2}}$$

with $\gamma_2, \rho_2 \in [1, \infty]$ satisfying

$$\frac{1}{\rho_2} = \frac{1}{\gamma_2} + \frac{2}{\rho_2} \quad \text{or} \quad 1 - \frac{1}{\gamma_2} = \frac{3}{\rho_2}.$$

We first choose $\frac{1}{\gamma_2} = \frac{\alpha}{d} + \varepsilon$ with $\varepsilon > 0$ to be chosen later. This ensures $\|\mathbf{1}_{B_1} |x|^{-\alpha}\|_{L^{\gamma_2}} < \infty$. It follows that

$$\rho_2 = \frac{3d}{d - \alpha - d\varepsilon} > 2$$

for any $\varepsilon > 0$.

For $d = 1, 2$, we use the Sobolev embedding $H^1(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for all $2 \leq q < \infty$ to get

$$(\text{I}_1) \leq C \|v_1\|_{H^1} \|u_1 - u_2\|_{L^{\rho_2}} \leq C(M) \|u_1 - u_2\|_{L^{\rho_2}} \leq C(M) \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{L}^{\rho_2}}$$

provided that $\|(u_1, v_1)\|_{\mathcal{H}^1} + \|(u_2, v_2)\|_{\mathcal{H}^1} \leq M$.

For $3 \leq d \leq 5$, we observe that $\rho_2 < \frac{2d}{d-2}$ for $\varepsilon > 0$ small enough due to $\alpha < \frac{6-d}{2}$. The Sobolev embedding $H^1(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for all $2 \leq q \leq \frac{2d}{d-2}$ yields

$$(\text{I}_1) \leq C(M) \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{L}^{\rho_2}}$$

provided that $\|(u_1, v_1)\|_{\mathcal{H}^1} + \|(u_2, v_2)\|_{\mathcal{H}^1} \leq M$. The term (I₂) is treated similarly, and thus the condition (2) is satisfied by g_2 with $\rho_2 = r_2$.

- For g_3 , we choose $\rho_3 = r_3 = 3$ and we estimate

$$\|g_3(u_1, v_1) - g_3(u_2, v_2)\|_{\mathcal{L}^{\frac{3}{2}}} = \|\mathbf{1}_{B_1^c} |x|^{-\alpha} (\bar{u}_1 v_1 - \bar{u}_2 v_2)\|_{L^{\frac{3}{2}}} + \|\mathbf{1}_{B_1^c} |x|^{-\alpha} (u_1^2 - u_2^2)\|_{L^{\frac{3}{2}}} =: (\text{III}) + (\text{IV}).$$

As above, we only consider (III). We write

$$(\text{III}) \leq \|\mathbf{1}_{B_1^c} |x|^{-\alpha} (\bar{u}_1 - \bar{u}_2) v_1\|_{L^{\frac{3}{2}}} + \|\mathbf{1}_{B_1^c} |x|^{-\alpha} \bar{u}_2 (v_1 - v_2)\|_{L^{\frac{3}{2}}} =: (\text{III}_1) + (\text{III}_2).$$

By Hölder's inequality, we have

$$(\text{III}_1) \leq \|\mathbf{1}_{B_1^c} |x|^{-\alpha}\|_{L^\infty} \|v_1\|_{L^3} \|u_1 - u_2\|_{L^3}.$$

As $H^1(\mathbb{R}^d) \subset L^3(\mathbb{R}^d)$ for $1 \leq d \leq 5$, we get

$$(\text{III}_1) \leq C \|v_1\|_{H^1} \|u_1 - u_2\|_{L^3} \leq C(M) \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{L}^3}$$

provided that $\|(u_1, v_1)\|_{\mathcal{H}^1} + \|(u_2, v_2)\|_{\mathcal{H}^1} \leq M$. Thus the condition (2) is also satisfied by g_3 . The proof is complete. \square

3. A VECTORIAL GAGLIARDO–NIRENBERG INEQUALITY

In this section, we prove the following Gagliardo–Nirenberg type inequality. This plays a key role in obtaining a threshold for global existence and finite time blow-up.

Proposition 3.1 (Gagliardo–Nirenberg inequality). *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. Then the following inequality holds for all $(u, v) \in \mathcal{H}^1$:*

$$\mathbb{P}(u, v) \leq C_{\text{GN}} (\mathbb{K}(u, v))^{\frac{d+2\alpha}{4}} (\mathbb{M}(u, v))^{\frac{6-d-2\alpha}{4}}. \quad (3.1)$$

The sharp constant C_{GN} is attained by a pair of functions (φ, ψ) which is a non-trivial solution to

$$\begin{cases} \frac{1}{2}\Delta\varphi - \varphi + |x|^{-\alpha}\varphi\psi = 0, \\ \frac{\kappa}{2}\Delta\psi - 2\psi + \frac{1}{2}|x|^{-\alpha}\varphi^2 = 0, \end{cases} \quad x \in \mathbb{R}^d. \quad (3.2)$$

In particular, we can take (φ, ψ) to be positive, radially symmetric, and radially decreasing.

Before giving the proof of Proposition 3.1, let us recall the corresponding scalar weighted Gagliardo–Nirenberg inequality.

Proposition 3.2 ([4, 15, 44]). *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. Then we have*

$$\int_{\mathbb{R}^d} |x|^{-\alpha} |f(x)|^3 dx \leq C_{\text{gn}} \|\nabla f\|_{L^2}^{\frac{d+2\alpha}{2}} \|f\|_{L^2}^{\frac{6-d-2\alpha}{2}}, \quad f \in H^1(\mathbb{R}^d). \quad (3.3)$$

The optimal constant C_{gn} is attained by a unique positive radial solution to

$$\Delta Q - Q + |\alpha|^{-\alpha} Q^2 = 0, \quad x \in \mathbb{R}^d. \quad (3.4)$$

Remark 3.1. The inequality (3.3) (without optimal constants) is a special case of the so-called Caffarelli–Kohn–Nirenberg inequalities [4]. The sharp constant in (3.3) was proved in [15]. The uniqueness of positive radial solutions to (3.4) is due to [44].

To prove Proposition 3.1, we need the following compactness result.

Lemma 3.3. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. Let $\{(u_n, v_n)\}_n$ be a bounded sequence in \mathcal{H}^1 . Then there exist $(u, v) \in \mathcal{H}^1$ and a subsequence still denoted by $\{(u_n, v_n)\}_n$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in \mathcal{H}^1 and*

$$\mathbb{P}(u_n, v_n) \rightarrow \mathbb{P}(u, v) \text{ as } n \rightarrow \infty. \quad (3.5)$$

Proof. Let $\{(u_n, v_n)\}_n$ be a bounded sequence in \mathcal{H}^1 . Then $\{u_n\}_n$ and $\{v_n\}_n$ are bounded sequences in $H^1(\mathbb{R}^d)$. Thus there exist $u, v \in H^1(\mathbb{R}^d)$ and subsequences still denoted by $\{u_n\}_n$ and $\{v_n\}_n$ such that

- $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^d)$.
- $u_n \rightarrow u$ and $v_n \rightarrow v$ strongly in $L_{\text{loc}}^q(\mathbb{R}^d)$ for all $q \geq 1$ and $q < \frac{2d}{d-2}$ if $d \geq 3$.

Let $\varepsilon > 0$. We first write

$$\mathbb{P}(u_n, v_n) - \mathbb{P}(u, v) = \Re \int_{\mathbb{R}^d} |x|^{-\alpha} (u_n^2 - u^2) \bar{v}_n dx + \Re \int_{\mathbb{R}^d} |x|^{-\alpha} u^2 (\bar{v}_n - \bar{v}) dx =: (\text{I}) + (\text{II}).$$

For $R > 0$ depending on ε to be chosen shortly, we write

$$(\text{I}) = \Re \int_{|x| \leq R} |x|^{-\alpha} (u_n^2 - u^2) \bar{v}_n dx + \Re \int_{|x| \geq R} |x|^{-\alpha} (u_n^2 - u^2) \bar{v}_n dx = (\text{I}_1) + (\text{I}_2).$$

- For (I_2) , we have

$$|(\text{I}_2)| \leq R^{-\alpha} \|u_n + u\|_{L^3} \|u_n - u\|_{L^3} \|v_n\|_{L^3} \leq CR^{-\alpha} = \frac{\varepsilon}{4}$$

provided that $R = \left(\frac{\varepsilon}{4C}\right)^{-\frac{1}{\alpha}}$, where we have used the Sobolev embedding $H^1(\mathbb{R}^d) \subset L^3(\mathbb{R}^d)$ for $1 \leq d \leq 5$.

- For (I_1) , we estimate

$$|(\text{I}_1)| \leq \| |x|^{-\alpha} \|_{L^\gamma(|x| \leq R)} \|u_n + u\|_{L^\rho} \|u_n - u\|_{L^\rho(|x| \leq R)} \|v_n\|_{L^\rho},$$

where $1 \leq \gamma, \rho \leq \infty$ are such that

$$1 = \frac{1}{\gamma} + \frac{3}{\rho}.$$

We first choose $\frac{1}{\gamma} = \frac{\alpha}{d} + \nu$ for some $\nu > 0$ to be determined shortly, hence

$$\rho = \frac{3d}{d - \alpha - d\nu} > 2$$

for any $\nu > 0$. With this choice, we have

$$\| |x|^{-\alpha} \|_{L^\gamma(|x| \leq R)} \leq CR^{d\nu}$$

for some universal constant $C > 0$.

If $d = 1, 2$, we use the Sobolev embedding $H^1(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for $q \geq 2$ and the fact that $u_n \rightarrow u$ strongly in $L^q(|x| \leq R)$ for all $q \geq 1$ to have for n large enough depending on ε ,

$$|(I_1)| \leq CR^{d\nu} \|u_n - u\|_{L^\rho(|x| \leq R)} < \frac{\varepsilon}{4}.$$

If $3 \leq d \leq 5$, we see that $\rho < \frac{2d}{d-2}$ for $\nu > 0$ sufficiently small due to $\alpha < \frac{6-d}{2}$. The Sobolev embedding $H^1(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for all $2 \leq q \leq \frac{2d}{d-2}$ and the convergence $u_n \rightarrow u$ strongly in $L^q(|x| \leq R)$ for all $1 \leq q < \frac{2d}{d-2}$ imply for n large enough depending on ε ,

$$(I_1) \leq CR^{d\nu} \|u_n - u\|_{L^\rho(|x| \leq R)} < \frac{\varepsilon}{4}.$$

Collecting the above estimates, we have for n sufficiently large depending on ε , $|(I)| < \frac{\varepsilon}{2}$. A similar argument goes for (II) and we obtain for n large enough depending on ε ,

$$|\mathbb{P}(u_n, v_n) - \mathbb{P}(u, v)| < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we conclude the proof. \square

Proof of Proposition 3.1. We only consider the case where $(u, v) \neq (0, 0)$ otherwise the result holds trivially. We consider the Weinstein functional

$$\mathbb{W}(u, v) := \frac{\mathbb{P}(u, v)}{(\mathbb{K}(u, v))^{\frac{d+2\alpha}{4}} (\mathbb{M}(u, v))^{\frac{6-d-2\alpha}{4}}}.$$

Thus

$$C_{\text{GN}} := \sup \{ \mathbb{W}(u, v) : (u, v) \in \mathcal{H}^1 \setminus \{(0, 0)\} \}.$$

We first observe that $0 < C_{\text{GN}} < \infty$. Indeed, to see $C_{\text{GN}} > 0$, we simply take u, v strictly positive functions in $H^1(\mathbb{R}^d)$. To see $C_{\text{GN}} < \infty$, we use the Hölder inequality and (3.3) to get

$$\begin{aligned} \mathbb{P}(u, v) &= \Re \int_{\mathbb{R}^d} (|x|^{-\frac{\alpha}{3}} u)^2 (|x|^{-\frac{\alpha}{3}} \bar{v}) \, dx \\ &\leq \left(\int_{\mathbb{R}^d} |x|^{-\alpha} |u|^3 \, dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^d} |x|^{-\alpha} |v|^3 \, dx \right)^{\frac{1}{3}} \\ &\leq C_{\text{gn}} \left(\|\nabla u\|_{L^2}^{\frac{d+2\alpha}{2}} \|u\|_{L^2}^{\frac{6-d-2\alpha}{2}} \right)^{\frac{2}{3}} \left(\|\nabla v\|_{L^2}^{\frac{d+2\alpha}{2}} \|v\|_{L^2}^{\frac{6-d-2\alpha}{2}} \right)^{\frac{1}{3}} \\ &\leq C(\kappa) C_{\text{gn}} (\mathbb{K}(u, v))^{\frac{d+2\alpha}{4}} (\mathbb{M}(u, v))^{\frac{6-d-2\alpha}{4}} \end{aligned}$$

for some constant $C(\kappa) > 0$ depending on κ . Hence $C_{\text{GN}} \leq C(\kappa) C_{\text{gn}}$.

Now let $\{(u_n, v_n)\}_n$ be an optimizing sequence for C_{GN} . As the Weinstein functional is invariant under the scaling

$$\mu u(\lambda x), \quad \mu v(\lambda x), \quad \mu, \lambda > 0,$$

we can use a suitable choice of scaling to assume (without loss of generality) that $\mathbb{M}(u_n, v_n) = \mathbb{K}(u_n, v_n) = 1$ for all n and $\mathbb{P}(u_n, v_n) \rightarrow C_{\text{GN}}$. Thus $\{(u_n, v_n)\}_n$ is a bounded sequence in \mathcal{H}^1 . By Lemma 3.3, there exist (u, v) and a subsequence still denoted by $\{(u_n, v_n)\}_n$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in \mathcal{H}^1 and $\mathbb{P}(u_n, v_n) \rightarrow \mathbb{P}(u, v)$ as $n \rightarrow \infty$. By the weak convergence in \mathcal{H}^1 , we have

$$\mathbb{K}(u, v) \leq \liminf_{n \rightarrow \infty} \mathbb{K}(u_n, v_n) = 1, \quad \mathbb{M}(u, v) \leq \liminf_{n \rightarrow \infty} \mathbb{M}(u_n, v_n) = 1.$$

We infer that

$$C_{\text{GN}} = \lim_{n \rightarrow \infty} \mathbb{P}(u_n, v_n) = \mathbb{P}(u, v) \leq \mathbb{W}(u, v) \leq C_{\text{GN}}$$

hence

$$\mathbb{K}(u, v) = \mathbb{M}(u, v) = 1, \quad \mathbb{P}(u, v) = C_{\text{GN}}$$

or (u, v) is an optimizer for C_{GN} . It follows that

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathbb{W}(u + \varepsilon\chi, v + \varepsilon\vartheta) = 0, \quad \forall \chi, \vartheta \in C_0^\infty(\mathbb{R}^d). \quad (3.6)$$

A direct computation yields

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathbb{P}(u + \varepsilon\chi, v + \varepsilon\vartheta) &= \frac{1}{2} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{\mathbb{R}^d} |x|^{-\alpha} [(u + \varepsilon\chi)^2(\bar{v} + \varepsilon\bar{\vartheta}) + (\bar{u} + \varepsilon\bar{\chi})^2(v + \varepsilon\vartheta)] \, dx \\ &= 2 \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \bar{u}v\bar{\chi} \, dx + \Re \int_{\mathbb{R}^d} |x|^{-\alpha} u^2\bar{\vartheta} \, dx \end{aligned}$$

and

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathbb{K}(u + \varepsilon\chi, v + \varepsilon\vartheta) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{\mathbb{R}^d} |\nabla u + \varepsilon\nabla\chi|^2 + \kappa|\nabla v + \varepsilon\nabla\vartheta|^2 \, dx \\ &= -2 \Re \int_{\mathbb{R}^d} \Delta u \bar{\chi} \, dx - 2\kappa \Re \int_{\mathbb{R}^d} \Delta v \bar{\vartheta} \, dx \end{aligned}$$

and

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathbb{M}(u + \varepsilon\chi, v + \varepsilon\vartheta) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{\mathbb{R}^d} |u + \varepsilon\chi|^2 + 2|v + \varepsilon\vartheta|^2 \, dx \\ &= 2 \Re \int_{\mathbb{R}^d} u \bar{\chi} \, dx + 4 \Re \int_{\mathbb{R}^d} v \bar{\vartheta} \, dx. \end{aligned}$$

From (3.6), we infer that

$$\frac{\mathbb{P}'(u, v)}{\mathbb{P}(u, v)} - \frac{d + 2\alpha}{4} \frac{\mathbb{K}'(u, v)}{\mathbb{K}(u, v)} - \frac{6 - d - 2\alpha}{4} \frac{\mathbb{M}'(u, v)}{\mathbb{M}(u, v)} = 0,$$

which is

$$\begin{aligned} &\frac{1}{\mathbb{P}(u, v)} \Re \int_{\mathbb{R}^d} |x|^{-\alpha} (2\bar{u}v\bar{\chi} + u^2\bar{\vartheta}) \, dx \\ &- \frac{d + 2\alpha}{4\mathbb{K}(u, v)} \Re \int_{\mathbb{R}^d} (-2\Delta u \bar{\chi} - 2\kappa\Delta v \bar{\vartheta}) \, dx - \frac{6 - d - 2\alpha}{4\mathbb{M}(u, v)} \Re \int_{\mathbb{R}^d} (2u\bar{\chi} + 4v\bar{\vartheta}) \, dx = 0. \end{aligned}$$

Testing again with $(i\chi, i\vartheta)$ instead of (χ, ϑ) , we obtain the same equality but with the imaginary part instead of the real one. In particular, we get

$$\begin{cases} \frac{d+2\alpha}{2\mathbb{K}(u, v)} \Delta u - \frac{6-d-2\alpha}{2\mathbb{M}(u, v)} u + \frac{2}{\mathbb{P}(u, v)} |x|^{-\alpha} \bar{u}v = 0, \\ \frac{d+2\alpha}{2\mathbb{K}(u, v)} \kappa \Delta v - \frac{6-d-2\alpha}{\mathbb{M}(u, v)} v + \frac{1}{\mathbb{P}(u, v)} |x|^{-\alpha} u^2 = 0, \end{cases}$$

in the weak sense. By setting

$$u(x) = \mu\varphi(\lambda x), \quad v(x) = \mu\psi(\lambda x)$$

with

$$\lambda^2 = \frac{6 - d - 2\alpha}{d + 2\alpha} \frac{\mathbb{K}(u, v)}{\mathbb{M}(u, v)}, \quad \mu = \frac{6 - d - 2\alpha}{2\lambda^\alpha} \frac{\mathbb{P}(u, v)}{\mathbb{M}(u, v)},$$

we see that $\mathbb{W}(\varphi, \psi) = \mathbb{W}(u, v) = C_{\text{GN}}$ or (φ, ψ) is an optimizer for C_{GN} , and (φ, ψ) is a non-trivial solution to (3.2).

Now let $|\varphi|^*$ and $|\psi|^*$ be the Schwarz symmetric rearrangement of $|\varphi|$ and $|\psi|$ respectively. We have $\| |\varphi|^* \|_{L^2} = \| \varphi \|_{L^2}$ and, by Polya-Szegö's inequality,

$$\| \nabla |\varphi|^* \|_{L^2} \leq \| \nabla |\varphi| \|_{L^2} \leq \| \nabla \varphi \|_{L^2}.$$

By the extended Hardy-Littlewood inequality (see [3, Theorem 1]), we also have

$$\mathbb{P}(\varphi, \psi) \leq \int_{\mathbb{R}^d} |x|^{-\alpha} |\varphi|^2 |\psi| \, dx \leq \int_{\mathbb{R}^d} |x|^{-\alpha} (|\varphi|^*)^2 |\psi|^* \, dx = \mathbb{P}(|\varphi|^*, |\psi|^*).$$

In particular, $\mathbb{W}(\varphi, \psi) \leq \mathbb{W}(|\varphi|^*, |\psi|^*)$ or $(|\varphi|^*, |\psi|^*)$ is also an optimizer for C_{GN} . Thus we can assume, without loss of generality, that φ and ψ are non-negative, radially symmetric, and radially decreasing. It follows that

$$\begin{cases} \frac{1}{2} \Delta \varphi - \varphi = -|x|^{-\alpha} \bar{\varphi} \psi \leq 0, \\ \frac{\kappa}{2} \Delta \psi - 2\psi = -\frac{1}{2} |x|^{-\alpha} \varphi^2 \leq 0, \end{cases} \quad \text{on } \mathbb{R}^d.$$

By the maximum principle (see e.g., [20, Theorem 3.5]), both φ and ψ are positive. The proof is complete. \square

Definition 3.1 (Nonlinear ground state). A non-trivial solution (φ, ψ) to (3.2) is called a nonlinear ground state related to (3.2) if it optimizes the Gagliardo–Nirenberg type inequality (3.1).

The existence of nonlinear ground states is proved in Proposition 3.1. In the following lemma, we collect some properties of solutions of (3.2).

Lemma 3.4. *Let (φ, ψ) be a non-trivial solution to (3.2). Then the following identities hold:*

$$\frac{1}{2}\mathbb{K}(\varphi, \psi) + \mathbb{M}(\varphi, \psi) - \frac{3}{2}\mathbb{P}(\varphi, \psi) = 0, \quad (3.7)$$

$$\frac{d-2}{4}\mathbb{K}(\varphi, \psi) + \frac{d}{2}\mathbb{M}(\varphi, \psi) - \frac{d-\alpha}{2}\mathbb{P}(\varphi, \psi) = 0. \quad (3.8)$$

In particular, we have

$$\mathbb{K}(\varphi, \psi) = \frac{d+2\alpha}{2}\mathbb{P}(\varphi, \psi) = \frac{2(d+2\alpha)}{6-d-2\alpha}\mathbb{M}(\varphi, \psi), \quad (3.9)$$

Proof. Multiplying the first equation with $\bar{\varphi}$ and the second one with $\bar{\psi}$, integrating over \mathbb{R}^d , and taking the real part, we get (3.7). Multiplying the first equation with $x \cdot \nabla \bar{\varphi}$ and the second one with $x \cdot \nabla \bar{\psi}$, integrating over \mathbb{R}^d , and taking the real part, we obtain (3.8). From (3.7) and (3.8), we infer (3.9). Here we have used the following identities:

$$\begin{aligned} \Re \int_{\mathbb{R}^d} \Delta \varphi x \cdot \nabla \bar{\varphi} \, dx &= \frac{d-2}{2} \|\nabla \varphi\|_{L^2}^2, \\ \Re \int_{\mathbb{R}^d} \varphi x \cdot \nabla \bar{\varphi} \, dx &= -\frac{d}{2} \|\varphi\|_{L^2}^2, \\ \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \bar{\varphi} \psi x \cdot \nabla \bar{\varphi} \, dx &= 0, \\ \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 x \cdot \nabla \bar{\psi} \, dx &= -(d-\alpha) \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 \bar{\psi} \, dx. \end{aligned} \quad (3.10)$$

The first two identities are standard. Let us check the last two ones. We have

$$\begin{aligned} \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \bar{\varphi} \psi x \cdot \nabla \bar{\varphi} \, dx &= \sum_{j=1}^d \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \bar{\varphi} \psi x_j \partial_j \bar{\varphi} \, dx \\ &= -\sum_{j=1}^d \Re \int_{\mathbb{R}^d} \partial_j (|x|^{-\alpha} \bar{\varphi} \psi x_j) \bar{\varphi} \, dx \\ &= -\sum_{j=1}^d \Re \int_{\mathbb{R}^d} (|x|^{-\alpha} \bar{\varphi} \psi + |x|^{-\alpha} \bar{\varphi} \partial_j \psi x_j - \alpha x_j^2 |x|^{-\alpha-2} \bar{\varphi} \psi) \bar{\varphi} \, dx \\ &= -(d-\alpha) \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 \bar{\psi} \, dx - \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 x \cdot \nabla \bar{\psi} \, dx. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 x \cdot \nabla \bar{\psi} \, dx &= \sum_{j=1}^d \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 x_j \partial_j \bar{\psi} \, dx \\ &= -\sum_{j=1}^d \Re \int_{\mathbb{R}^d} \partial_j (|x|^{-\alpha} \varphi^2 x_j) \bar{\psi} \, dx \\ &= -\sum_{j=1}^d \Re \int_{\mathbb{R}^d} (|x|^{-\alpha} \varphi^2 + 2|x|^{-\alpha} \varphi \partial_j \varphi x_j - \alpha x_j^2 |x|^{-\alpha-2} \varphi^2) \bar{\psi} \, dx \\ &= -(d-\alpha) \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 \bar{\psi} \, dx - 2 \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \bar{\varphi} \psi x \cdot \nabla \bar{\varphi} \, dx. \end{aligned}$$

From these two identities, we get the desired equalities. \square

4. GLOBAL EXISTENCE

In this section, we prove some global existence results for (1.1) in the mass-subcritical, mass-critical, and mass-supercritical cases.

Proposition 4.1. *Let $1 \leq d \leq 3$, $0 < \alpha < \min\{d, \frac{4-d}{2}\}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. For every $(u_0, v_0) \in \mathcal{H}^1$, the corresponding solution to (1.1) with initial data $(u, v)|_{t=0} = (u_0, v_0)$ exists globally in time.*

Proof. By the Gagliardo–Nirenberg type inequality (3.1) and the conservation of mass, we have

$$\begin{aligned} \mathbb{P}(u(t), v(t)) &\leq C_{\text{GN}} (\mathbb{K}(u(t), v(t)))^{\frac{d+2\alpha}{4}} (\mathbb{M}(u(t), v(t)))^{\frac{6-d-2\alpha}{4}} \\ &= C_{\text{GN}} (\mathbb{M}(u_0, v_0))^{\frac{6-d-2\alpha}{4}} (\mathbb{K}(u(t), v(t)))^{\frac{d+2\alpha}{4}}, \quad \forall t \in (-T_*, T^*). \end{aligned}$$

As $\alpha < \frac{4-d}{2}$, the Young inequality yields

$$\mathbb{P}(u(t), v(t)) \leq \frac{1}{4} \mathbb{K}(u(t), v(t)) + C(C_{\text{GN}}, \mathbb{M}(u_0, v_0)), \quad \forall t \in (-T_*, T^*).$$

We also have

$$-\gamma \|v(t)\|_{L^2}^2 \leq \frac{|\gamma|}{2} \mathbb{M}(u(t), v(t)) = \frac{|\gamma|}{2} \mathbb{M}(u_0, v_0), \quad \forall t \in (-T_*, T^*). \quad (4.1)$$

It follows that

$$\mathbb{E}(u(t), v(t)) - \gamma \|v(t)\|_{L^2}^2 \geq \frac{1}{4} \mathbb{K}(u(t), v(t)) - C(C_{\text{GN}}, \mathbb{M}(u_0, v_0))$$

hence

$$\begin{aligned} \frac{1}{4} \mathbb{K}(u(t), v(t)) &\leq \mathbb{E}(u(t), v(t)) - \gamma \|v(t)\|_{L^2}^2 + C(C_{\text{GN}}, \mathbb{M}(u_0, v_0)) \\ &\leq \mathbb{E}(u_0, v_0) + \frac{|\gamma|}{2} \mathbb{M}(u_0, v_0) + C(C_{\text{GN}}, \mathbb{M}(u_0, v_0)), \quad \forall t \in (-T_*, T^*). \end{aligned}$$

The blow-up alternative implies that $T_*, T^* = \infty$. \square

Proposition 4.2. *Let $2 \leq d \leq 3$, $\alpha = \frac{4-d}{2}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Let $(u_0, v_0) \in \mathcal{H}^1$ be such that*

$$\mathbb{M}(u_0, v_0) < \mathbb{M}(\varphi, \psi), \quad (4.2)$$

where (φ, ψ) is a nonlinear ground state related to (3.2). Then the corresponding solution to (1.1) with initial data $(u, v)|_{t=0} = (u_0, v_0)$ exists globally in time.

Proof. We first observe that in the mass-critical case $\alpha = \frac{4-d}{2}$, by using (3.9), we have

$$C_{\text{GN}} = \frac{1}{2 (\mathbb{M}(\varphi, \psi))^{1/2}}.$$

Thus the Gagliardo–Nirenberg inequality (3.1) becomes

$$\mathbb{P}(u(t), v(t)) \leq \frac{1}{2} \left(\frac{\mathbb{M}(u(t), v(t))}{\mathbb{M}(\varphi, \psi)} \right)^{1/2} \mathbb{K}(u(t), v(t)) = \frac{1}{2} \left(\frac{\mathbb{M}(u_0, v_0)}{\mathbb{M}(\varphi, \psi)} \right)^{1/2} \mathbb{K}(u(t), v(t)), \quad \forall t \in (-T_*, T^*).$$

This together with (4.1) yield

$$\frac{1}{2} \left(1 - \left(\frac{\mathbb{M}(u_0, v_0)}{\mathbb{M}(\varphi, \psi)} \right)^{1/2} \right) \mathbb{K}(u(t), v(t)) \leq \mathbb{E}(u_0, v_0) + \frac{|\gamma|}{2} \mathbb{M}(u_0, v_0), \quad \forall t \in (-T_*, T^*).$$

By (4.2), we infer that $\mathbb{K}(u(t), v(t))$ is bounded uniformly in $t \in (-T_*, T^*)$. The blow-up alternative implies that the solution exists globally in time. \square

Proposition 4.3. *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Let (φ, ψ) be a nonlinear ground state related to (3.2). Let $(u_0, v_0) \in \mathcal{H}^1$ be such that*

$$\mathbb{H}(u_0, v_0) (\mathbb{M}(u_0, v_0))^\sigma < \mathbb{E}_0(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad (4.3)$$

$$\mathbb{K}(u_0, v_0) (\mathbb{M}(u_0, v_0))^\sigma < \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad (4.4)$$

where $\sigma = \frac{6-d-2\alpha}{d+2\alpha-4}$, $\mathbb{H}(u_0, v_0)$ is as in (1.5), and

$$\mathbb{E}_0(\varphi, \psi) := \frac{1}{2} \mathbb{K}(\varphi, \psi) - \mathbb{P}(\varphi, \psi).$$

Then the corresponding solution to (1.1) with initial data $(u, v)|_{t=0} = (u_0, v_0)$ exists globally in time. In addition, we have

$$\sup_{t \in \mathbb{R}} \|(u(t), v(t))\|_{\mathcal{H}^1} \leq C(u_0, v_0, \varphi, \psi). \quad (4.5)$$

Remark 4.1. By (4.10) and (4.11), the quantities appeared in the right hand sides of (4.3) and (4.4) do not depend on the choice of nonlinear ground states (φ, ψ) .

Proof of Proposition 4.3. By the Gagliardo–Nirenberg inequality (3.1), we have

$$\begin{aligned}
& (\mathbb{E}(u(t), v(t)) - \gamma \|v(t)\|_{L^2}^2) (\mathbb{M}(u(t), v(t)))^\sigma \\
&= \frac{1}{2} \mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma - \mathbb{P}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma \\
&\geq \frac{1}{2} \mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma - C_{\text{GN}} (\mathbb{K}(u(t), v(t)))^{\frac{d+2\alpha}{4}} (\mathbb{M}(u(t), v(t)))^{\frac{6-d-2\alpha}{4} + \sigma} \\
&= g (\mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma), \quad \forall t \in (-T_*, T^*),
\end{aligned}$$

where

$$g(\lambda) := \frac{1}{2} \lambda - C_{\text{GN}} \lambda^{\frac{n+2\alpha}{4}}. \quad (4.6)$$

On the other hand, using (4.1), we have

$$\mathbb{E}(u(t), v(t)) - \gamma \|v(t)\|_{L^2}^2 \leq \begin{cases} \mathbb{E}(u(t), v(t)) & \text{if } \gamma \geq 0, \\ \mathbb{E}(u(t), v(t)) + \frac{|\gamma|}{2} \mathbb{M}(u(t), v(t)) & \text{if } \gamma < 0, \end{cases}$$

or

$$\mathbb{E}(u(t), v(t)) - \gamma \|v(t)\|_{L^2}^2 \leq \mathbb{H}(u(t), v(t)), \quad \forall t \in (-T_*, T^*). \quad (4.7)$$

By the conservation of mass and energy, we get

$$g (\mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma) \leq \mathbb{H}(u_0, v_0) (\mathbb{M}(u_0, v_0))^\sigma,$$

which together with (4.3) imply

$$g (\mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma) < \mathbb{E}_0(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall t \in (-T_*, T^*). \quad (4.8)$$

From (4.4), we claim that

$$\mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma < \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall t \in (-T_*, T^*). \quad (4.9)$$

In fact, in one hand, we have from a direct computation using (3.9) that

$$C_{\text{GN}} = \frac{\mathbb{P}(\varphi, \psi)}{(\mathbb{K}(\varphi, \psi))^{\frac{d+2\alpha}{4}} (\mathbb{M}(\varphi, \psi))^{\frac{6-d-2\alpha}{4}}} = \frac{2}{d+2\alpha} (\mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma)^{-\frac{d+2\alpha-4}{4}} \quad (4.10)$$

which implies

$$g (\mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma) = \frac{d+2\alpha-4}{2(d+2\alpha)} \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma = \mathbb{E}_0(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma. \quad (4.11)$$

Assume by contradiction that there exists $t_0 \in (-T_*, T^*)$ such that

$$\mathbb{K}(u(t_0), v(t_0)) (\mathbb{M}(u(t_0), v(t_0)))^\sigma \geq \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma.$$

Since the map $t \mapsto \mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma$ is continuous (as $(u, v) \in C((-T_*, T^*), \mathcal{H}^1)$), we infer from (4.4) that there exists $t_1 \in (-T_*, T^*)$ so that

$$\mathbb{K}(u(t_1), v(t_1)) (\mathbb{M}(u(t_1), v(t_1)))^\sigma = \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma.$$

Thus

$$g (\mathbb{K}(u(t_1), v(t_1)) (\mathbb{M}(u(t_1), v(t_1)))^\sigma) = \mathbb{E}_0(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma$$

which contradicts (4.8), and the claim 4.9 follows. From (4.9), the conservation of mass gives

$$\mathbb{K}(u(t), v(t)) < \mathbb{K}(\varphi, \psi) \left(\frac{\mathbb{M}(\varphi, \psi)}{\mathbb{M}(u_0, v_0)} \right)^\sigma, \quad \forall t \in (-T_*, T^*)$$

which shows that $T_* = T^* = \infty$. The bound (4.5) follows from (4.9) and the mass conservation. The proof is complete. \square

5. BLOW-UP

This section is devoted to the existence of blow-up solutions to (1.1). Our bow-up results are as follows.

Theorem 5.1 (Mass-critical case). *Let $2 \leq d \leq 3$, $\alpha = \frac{4-d}{2}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Let $(u_0, v_0) \in \mathcal{H}^1$ and $(u, v) \in C((-T_*, T^*), \mathcal{H}^1)$ be the corresponding maximal solution to (1.1). Assume that*

$$\sup_{t \in (-T_*, T^*)} \mathbb{G}(u(t), v(t)) \leq -\delta \quad (5.1)$$

for some constant $\delta > 0$, where $\mathbb{G}(u, v)$ is as in (1.3).

- If $\kappa = \frac{1}{2}$, then the solution blows up in finite time, i.e., $T_*, T^* < \infty$ and

$$\lim_{t \nearrow T^*} \|(u(t), v(t))\|_{\mathcal{H}^1} = \infty, \quad \lim_{t \searrow -T_*} \|(u(t), v(t))\|_{\mathcal{H}^1} = \infty.$$

- If $\kappa \neq \frac{1}{2}$, then the solution either blows up in finite time or it blows up in infinite time in the sense that $T^* = \infty$ and there exists $C > 0$ such that

$$\mathbb{K}(u(t), v(t)) \geq Ct^2, \quad \forall t \geq t_0$$

for some $t_0 > 0$ sufficiently large. A similar result holds for negative times.

In particular, if $\mathbb{H}(u_0, v_0) < 0$, where $\mathbb{H}(u_0, v_0)$ is as in (1.5), then the above blow-up results hold.

Theorem 5.2 (Mass-supercritical case). *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha > \frac{4-d}{2}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Let $(u_0, v_0) \in \mathcal{H}^1$ and $(u, v) \in C((-T_*, T^*), \mathcal{H}^1)$ be the corresponding maximal solution to (1.1). Assume that (5.1) holds.*

- If $2 \leq d \leq 4$, then the solution blows up in finite time.
- If $d = 5$, then the solution either blows up in finite time or it blows up in infinite time in the sense that $T^* = \infty$ and

$$\sup_{t \in [0, \infty)} \mathbb{K}(u(t), v(t)) = \infty. \quad (5.2)$$

Moreover, we have for all $T > 0$,

$$\sup_{t \in [0, T]} \mathbb{K}(u(t), v(t)) \geq \begin{cases} CT^4 & \text{if } \kappa = \frac{1}{2}, \\ CT^{4/3} & \text{if } \kappa \neq \frac{1}{2}. \end{cases} \quad (5.3)$$

A similar statement holds for negative times.

In particular, if $(u_0, v_0) \in \mathcal{H}^1$ satisfies either $\mathbb{H}(u_0, v_0) < 0$ or, if $\mathbb{H}(u_0, v_0) \geq 0$, we assume that

$$\mathbb{H}(u_0, v_0) (\mathbb{M}(u_0, v_0))^\sigma < \mathbb{E}_0(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad (5.4)$$

$$\mathbb{K}(u_0, v_0) (\mathbb{M}(u_0, v_0))^\sigma > \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad (5.5)$$

where (φ, ψ) is a nonlinear ground state related to (3.2) and $\mathbb{H}(u_0, v_0)$ is as in (1.5), then the above blow-up results hold.

In our blow-up results, we do not assume any symmetric assumption or finite variance of initial data. The proofs rely on localized virial estimates which was proposed by Ogawa and Tsutsumi [39, 40]. Here instead of making use of the radial assumption, we exploit the spatial decay of the nonlinearity at infinity. This decay property has been used for the single inhomogeneous nonlinear Schrödinger equation (see [1, 8]). In our context of system of inhomogeneous NLS, there are additional difficulties appeared in the non mass-resonance case $\kappa \neq \frac{1}{2}$, namely the lack of second derivative of the virial quantity. We overcome it by using an ODE technique which is inspired by Boulenger, Himmelsbach, and Lenzmann's work [2].

5.1. Virial identities.

Lemma 5.3. *Let (u, v) be a \mathcal{H}^1 -solution to (1.1) defined on the maximal time interval $(-T_*, T^*)$. Let $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a sufficiently smooth and decaying function. Define*

$$\mathcal{V}_\chi(t) := \int_{\mathbb{R}^d} \chi(|u(t)|^2 + 2|v(t)|^2) dx$$

and

$$\mathcal{M}_\chi(t) := \Im \int_{\mathbb{R}^d} \nabla \chi \cdot (\nabla u(t) \bar{u}(t) + \nabla v(t) \bar{v}(t)) dx.$$

Then for all $t \in (-T_*, T^*)$,

$$\frac{d}{dt} \mathcal{V}_\chi(t) = \Im \int_{\mathbb{R}^d} \nabla \chi \cdot (\nabla u(t) \bar{u}(t) + 2\kappa \nabla v(t) \bar{v}(t)) \, dx$$

and

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_\chi(t) &= \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 \chi (|u(t)|^2 + \kappa |v(t)|^2) \, dx + \sum_{j,k=1}^d \Re \int_{\mathbb{R}^d} \partial_{jk}^2 \chi (\partial_j \bar{u}(t) \partial_k u(t) + \kappa \partial_j \bar{v}(t) \partial_k v(t)) \, dx \\ &\quad - \frac{1}{2} \Re \int_{\mathbb{R}^d} \Delta \chi |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx + \Re \int_{\mathbb{R}^d} \nabla \chi \cdot \nabla (|x|^{-\alpha} u^2(t) \bar{v}(t)) \, dx. \end{aligned}$$

In particular, if $\kappa = \frac{1}{2}$, then we have

$$\frac{d}{dt} \mathcal{V}_\chi(t) = \mathcal{M}_\chi(t). \quad (5.6)$$

Remark 5.1. If $\chi(x) = |x|^2$, we have

$$\frac{d}{dt} \mathcal{M}_{|x|^2}(t) = 2\mathbb{G}(u(t), v(t)),$$

where $G(u, v)$ is as in (1.3).

Remark 5.2. If χ is radial, then we have

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_\chi(t) &= \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 \chi (|u(t)|^2 + \kappa |v(t)|^2) \, dx + \int_{\mathbb{R}^d} \frac{\chi'}{r} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) \, dx \\ &\quad + \int_{\mathbb{R}^d} \left(\frac{\chi''}{r^2} - \frac{\chi'}{r^3} \right) (|x \cdot \nabla u(t)|^2 + \kappa |x \cdot \nabla v(t)|^2) \, dx \\ &\quad - \frac{1}{2} \Re \int_{\mathbb{R}^d} \left(\chi'' + (d-1+2\alpha) \frac{\chi'}{r} \right) |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx. \end{aligned}$$

This follows from a direct computation using the fact that

$$\begin{aligned} \sum_{j,k}^n \partial_{jk}^2 \chi \partial_j \bar{u} \partial_k u &= \frac{\chi'}{r} |\nabla u|^2 + \left(\frac{\chi''}{r^2} - \frac{\chi'}{r^3} \right) |x \cdot \nabla u|^2, \\ \Delta \chi &= \chi'' + (n-1) \frac{\chi'}{r}, \\ \nabla \chi \cdot \nabla (|x|^{-\alpha}) &= -\alpha |x|^{-\alpha} \frac{\chi'}{r}. \end{aligned}$$

Proof of Lemma 5.3. It follows directly from the following observation: for any $a \in \mathbb{R}$,

$$\begin{aligned} \partial_t (|u|^2 + a|v|^2) &= -\nabla \cdot \Im(\bar{u} \nabla u) - a\kappa \nabla \cdot \Im(\bar{v} \nabla v) - (a-2) \Im(|x|^{-\alpha} u^2 \bar{v}), \\ \partial_t (\Im(\bar{u} \partial_k u) + \Im(\bar{v} \partial_k v)) &= \frac{1}{4} \partial_k \Delta (|u|^2) + \frac{\kappa}{4} \partial_k \Delta (|v|^2) - \sum_{j=1}^d \partial_j \Re(\partial_j \bar{u} \partial_k u) - \kappa \sum_{j=1}^d \partial_j \Re(\partial_j \bar{v} \partial_k v) \\ &\quad + \frac{1}{2} \partial_k \Re(|x|^{-\alpha} u^2 \bar{v}) + \Re(\partial_k (|x|^{-\alpha} u^2 \bar{v})), \quad \forall k = 1, \dots, d. \end{aligned}$$

To see these identities, we apply Lemma A.1 with $\beta = \frac{1}{2}$ and $H = -|x|^{-\alpha} \bar{u} v$ to get

$$\begin{aligned} \partial_t |u|^2 &= -\nabla \cdot \Im(\bar{u} \nabla u) - 2 \Im(|x|^{-\alpha} \bar{u}^2 v) = -\nabla \cdot \Im(\bar{u} \nabla u) + 2 \Im(|x|^{-\alpha} u^2 \bar{v}), \\ \partial_t \Im(\bar{u} \partial_k u) &= \frac{1}{4} \partial_k \Delta (|u|^2) - \sum_{j=1}^d \partial_j \Re(\partial_j \bar{u} \partial_k u) - 2 \Re(|x|^{-\alpha} u \partial_k u \bar{v}) + \partial_k \Re(|x|^{-\alpha} u^2 \bar{v}). \end{aligned}$$

Applying Lemma A.1 for $\beta = \frac{\kappa}{2}$ and $H = \gamma v - \frac{1}{2}|x|^{-\alpha}u^2$, we get

$$\begin{aligned}\partial_t |v|^2 &= -\kappa \nabla \cdot \Im(\bar{v} \nabla v) + 2 \Im \left[\bar{v} \left(\gamma v - \frac{1}{2}|x|^{-\alpha}u^2 \right) \right] = -\kappa \nabla \cdot \Im(\bar{v} \nabla v) - \Im(|x|^{-\alpha}u^2 \bar{v}), \\ \partial_t \Im(\bar{v} \partial_k v) &= \frac{\kappa}{4} \partial_k \Delta(|v|^2) - \kappa \sum_{j=1}^d \partial_j \Re(\partial_j \bar{v} \partial_k v) \\ &\quad + 2 \Re \left[\left(\gamma \bar{v} - \frac{1}{2}|x|^{-\alpha}u^2 \right) \partial_k v \right] - \partial_k \Re \left[\left(\gamma v - \frac{1}{2}|x|^{-\alpha}u^2 \right) \bar{v} \right] \\ &= \frac{\kappa}{4} \partial_k \Delta(|v|^2) - \kappa \sum_{j=1}^d \partial_j \Re(\partial_j \bar{v} \partial_k v) - \Re(|x|^{-\alpha}u^2 \partial_k \bar{v}) + \frac{1}{2} \partial_k \Re(|x|^{-\alpha}u^2 \bar{v}).\end{aligned}$$

Collecting the above identities and using the fact that

$$\partial_k \Re(|x|^{-\alpha}u^2 \bar{v}) = \Re(\partial_k(|x|^{-\alpha})u^2 \bar{v} + 2|x|^{-\alpha}u \partial_k u \bar{v} + |x|^{-\alpha}u^2 \partial_k \bar{v}),$$

we obtain the desired identities. \square

5.2. A cutoff function. Let $R > 0$. We define the radial function

$$\chi_R(x) = \chi_R(r) = R^2 \chi(r/R), \quad r = |x|. \quad (5.7)$$

with

$$\chi(r) = \int_0^r \zeta(s) \, ds,$$

where $\zeta : [0, \infty) \rightarrow [0, \infty)$ satisfies

$$\zeta(r) = \begin{cases} 2r & \text{if } 0 \leq r \leq 1, \\ 2r - 2(r-1)^4 & \text{if } 1 < r \leq 1 + \frac{1}{\sqrt[3]{4}}, \\ \text{smooth and } \zeta'(r) < 0 & \text{if } 1 + \frac{1}{\sqrt[3]{4}} < r \leq 2, \\ 0 & \text{if } r > 2. \end{cases}$$

We collect some properties of χ_R in the following lemma.

Lemma 5.4. *We have*

$$\|\nabla^j \chi_R\|_{L^\infty} \lesssim R^{2-j}, \quad 0 \leq j \leq 4, \quad (5.8)$$

and

$$\text{supp}(\nabla^j \chi_R) \subset \begin{cases} \{|x| \leq 2R\} & \text{if } j = 1, 2, \\ \{R \leq |x| \leq 2R\} & \text{if } j = 3, 4. \end{cases} \quad (5.9)$$

and

$$\frac{\chi'_R(r)}{r} \leq 2, \quad \chi''_R(r) \leq 2, \quad \forall r \geq 0. \quad (5.10)$$

In addition, we have

$$\frac{\chi'_R(r)}{r} - \chi''_R(r) \geq 0, \quad \forall r \geq 0. \quad (5.11)$$

Proof. The estimates (5.8)–(5.10) follow directly from the choice of χ . Let us check (5.11). We have

$$\frac{\chi'_R(r)}{r} = \frac{\zeta(r/R)}{r/R} = \begin{cases} 2 & \text{if } 0 \leq r/R \leq 1, \\ 2 - 2 \frac{(r/R-1)^4}{r/R} & \text{if } 1 < r/R \leq 1 + \frac{1}{\sqrt[3]{4}}, \\ \geq 0 & \text{if } 1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2, \\ 0 & \text{if } r/R > 2. \end{cases}$$

We also have

$$\chi''_R(r) = \zeta'(r/R) = \begin{cases} 2 & \text{if } 0 \leq r/R \leq 1, \\ 2 - 8(r/R-1)^3 & \text{if } 1 < r/R \leq 1 + \frac{1}{\sqrt[3]{4}}, \\ < 0 & \text{if } 1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2, \\ 0 & \text{if } r/R > 2. \end{cases}$$

From this, we deduce the result. Note that

$$8(r/R - 1)^3 - 2\frac{(r/R - 1)^4}{r/R} = 2(r/R - 1)^3 \left(4 - \frac{r/R - 1}{r/R}\right) = 2(r/R - 1)^3 \frac{3r/R + 1}{r/R} \geq 0.$$

□

5.3. Localized virial identity. Let χ_R be as in (5.7). We define the localized virial quantity

$$\mathcal{M}_{\chi_R}(t) = \Im \int_{\mathbb{R}^d} \nabla \chi_R \cdot (\nabla u(t) \bar{u}(t) + \nabla v(t) \bar{v}(t)) \, dx. \quad (5.12)$$

Then we have (using Remark 5.2)

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\chi_R}(t) &= \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 \chi_R (|u(t)|^2 + \kappa |v(t)|^2) \, dx + \int_{\mathbb{R}^d} \frac{\chi'_R}{r} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) \, dx \\ &\quad + \int_{\mathbb{R}^d} \left(\frac{\chi''_R}{r^2} - \frac{\chi'_R}{r^3} \right) (|x \cdot \nabla u(t)|^2 + \kappa |x \cdot \nabla v(t)|^2) \, dx \\ &\quad - \frac{1}{2} \Re \int_{\mathbb{R}^d} \left(\chi''_R + (d-1+2\alpha) \frac{\chi'_R}{r} \right) |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx. \end{aligned}$$

We can rewrite it as

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\chi_R}(t) &= 2\mathbb{G}(u(t), v(t)) + \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 \chi_R (|u(t)|^2 + \kappa |v(t)|^2) \, dx \\ &\quad - \int_{\mathbb{R}^d} \left(2 - \frac{\chi'_R}{r} \right) (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) \, dx \\ &\quad + \int_{\mathbb{R}^d} \left(\frac{\chi''_R}{r^2} - \frac{\chi'_R}{r^3} \right) (|x \cdot \nabla u(t)|^2 + \kappa |x \cdot \nabla v(t)|^2) \, dx \\ &\quad + \frac{1}{2} \Re \int_{\mathbb{R}^d} \left((2 - \chi''_R) + (d-1+2\alpha) \left(2 - \frac{\chi'_R}{r} \right) \right) |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx \\ &= 2\mathbb{G}(u(t), v(t)) + E_1(u(t), v(t)) + E_2(u(t), v(t)) + E_3(u(t), v(t)), \end{aligned}$$

where

$$\begin{aligned} E_1(u, v) &= \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 \chi_R (|u|^2 + \kappa |v|^2) \, dx, \\ E_2(u, v) &= - \int_{\mathbb{R}^d} \left(2 - \frac{\chi'_R}{r} \right) (|\nabla u|^2 + \kappa |\nabla v|^2) \, dx + \int_{\mathbb{R}^d} \left(\frac{\chi''_R}{r^2} - \frac{\chi'_R}{r^3} \right) (|x \cdot \nabla u|^2 + \kappa |x \cdot \nabla v|^2) \, dx, \quad (5.13) \\ E_3(u, v) &= \frac{1}{2} \Re \int_{\mathbb{R}^d} \left((2 - \chi''_R) + (d-1+2\alpha) \left(2 - \frac{\chi'_R}{r} \right) \right) |x|^{-\alpha} u^2 \bar{v} \, dx. \end{aligned}$$

5.4. Mass-critical blow-up solutions. Before giving the proof of Theorem 5.1, we need the following preliminary lemma.

Lemma 5.5. *Define*

$$\chi_{1R}(r) := 2 - \frac{\chi'_R(r)}{r}, \quad \chi_{2R}(r) := 2 - \chi''_R(r) + 3 \left(2 - \frac{\chi'_R(r)}{r} \right). \quad (5.14)$$

The following properties hold:

$$|\chi_{2R}(r)| \lesssim 1, \quad \forall r \geq 0 \quad (5.15)$$

$$|\nabla(\chi_{2R}^{\frac{2}{3}}(r))| \lesssim R^{-1}, \quad \forall r \geq 0, \quad (5.16)$$

and for $R > 0$ sufficiently large,

$$\chi_{1R}(r) - CR^{-\alpha} \chi_{2R}^{\frac{4}{3}}(r) \geq 0, \quad \forall r \geq 0. \quad (5.17)$$

Proof. We have

$$2 - \frac{\chi'_R(r)}{r} = \begin{cases} 0 & \text{if } 0 \leq r/R \leq 1, \\ 2\frac{(r/R-1)^4}{r/R} & \text{if } 1 < r/R \leq 1 + \frac{1}{\sqrt[3]{4}}, \\ 2 - \frac{\zeta(r/R)}{r/R} & \text{if } 1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2, \\ 2 & \text{if } r/R > 2, \end{cases}$$

and

$$2 - \chi_R''(r) = \begin{cases} 0 & \text{if } 0 \leq r/R \leq 1, \\ 8(r/R - 1)^3 & \text{if } 1 < r/R \leq 1 + \frac{1}{\sqrt[3]{4}}, \\ 2 - \zeta'(r/R) & \text{if } 1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2, \\ 2 & \text{if } r/R > 2. \end{cases}$$

Note that $\zeta'(r/R) < 0$ for all $1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2$. From this, we have (5.15).

To verify (5.16), we consider several cases. If $0 \leq r/R \leq 1$ or $r/R > 2$, then $\chi_{2R}(r)$ is constant. So (5.16) holds trivially.

If $1 < r/R \leq 1 + \frac{1}{\sqrt[3]{4}}$, we have

$$\chi_{2R}(r) = 8(r/R - 1)^3 + 6 \frac{(r/R - 1)^4}{r/R} = (r/R - 1)^3 \left(14 - \frac{6}{r/R} \right).$$

Thus

$$\chi_{2R}^{\frac{2}{3}}(r) = (r/R - 1)^2 \left(14 - \frac{6}{r/R} \right)^{\frac{2}{3}} =: h(r/R),$$

where

$$h(\lambda) = (\lambda - 1)^2 \left(14 - \frac{6}{\lambda} \right)^{\frac{2}{3}}.$$

It follows that

$$|\nabla(\chi_{2R}^{\frac{2}{3}}(r))| = |\partial_r(\chi_{2R}^{\frac{2}{3}}(r))| = \frac{1}{r} |h'(r/R)|,$$

where

$$h'(\lambda) = 2(\lambda - 1) \left(14 - \frac{6}{\lambda} \right)^{-\frac{1}{3}} \left(14 - \frac{4}{\lambda} - \frac{2}{\lambda^2} \right).$$

We readily see that $|h'(\lambda)| \leq C$ for all $1 < \lambda \leq 1 + \frac{1}{\sqrt[3]{4}}$. Thus (5.16) holds in this range.

If $1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2$, then $\chi_R''(r) = \zeta'(r/R) < 0$ and

$$\frac{\chi_R'(r)}{r} = \frac{\zeta(r/R)}{r/R} \in \left[\frac{\zeta(2)}{2}, \frac{\zeta(1 + 1/\sqrt[3]{4})}{1 + 1/\sqrt[3]{4}} \right] = \left[0, \frac{3 + 4\sqrt[3]{4}}{2 + 2\sqrt[3]{4}} \right].$$

Hence

$$\chi_{2R}(r) \geq 2 + \frac{3}{2 + 2\sqrt[3]{4}}, \quad \forall 1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2.$$

As one can readily check that $|\nabla\chi_{2R}(r)| \lesssim R^{-1}$ and $\nabla(\chi_{2R}^{\frac{2}{3}}(r)) = \frac{2}{3} \frac{\nabla\chi_{2R}(r)}{\chi_{2R}^{\frac{1}{3}}(r)}$, we deduce that (5.16) holds for $1 + \frac{1}{\sqrt[3]{4}} < r/R \leq 2$. This finishes the proof of (5.16).

Finally, let us show (5.17). As above, we consider three cases. If $0 \leq r/R \leq 1$, then (5.17) is obvious as $\chi_{1R}(r) = \chi_{2R}(r) = 0$.

If $1 < r/R \leq 1 + \frac{1}{\sqrt[3]{4}}$, we have

$$\chi_{1R}(r) = 8 \frac{(r/R - 1)^4}{r/R}$$

and

$$\chi_{2R}(r) = (r/R - 1)^3 \left(14 - \frac{6}{r/R} \right) < (r/R - 1)^3 \left(14 - \frac{6\sqrt[3]{4}}{1 + \sqrt[3]{4}} \right).$$

Thus we get

$$\begin{aligned} \chi_{1R}(r) - CR^{-\alpha} \chi_{2R}^{\frac{4}{3}}(r) &> 8 \frac{(r/R - 1)^4}{r/R} - CR^{-\alpha} (r/R - 1)^4 \left(14 - \frac{6\sqrt[3]{4}}{1 + \sqrt[3]{4}} \right)^{\frac{4}{3}} \\ &= (r/R - 1)^4 \left(\frac{8}{r/R} - CR^{-\alpha} \left(14 - \frac{6\sqrt[3]{4}}{1 + \sqrt[3]{4}} \right)^{\frac{4}{3}} \right). \end{aligned}$$

As $1 < r/R \leq 1 + \frac{1}{\sqrt[3]{4}}$, by taking $R > 0$ sufficiently large, we get

$$\chi_{1R}(r) - CR^{-\alpha} \chi_{2R}^{\frac{4}{3}}(r) \geq (r/R - 1)^4.$$

If $r/R > 1 + \frac{1}{\sqrt[3]{4}}$, we have $\zeta'(r/R) \leq 0$, hence

$$\chi_{1R}(r) = 2 - \frac{\chi'_R(r)}{r} = 2 - \frac{\zeta(r/R)}{r/R} \geq 2 - \frac{\zeta(1 + 1/\sqrt[3]{4})}{1 + 1/\sqrt[3]{4}} = \frac{1}{2 + 2\sqrt[3]{4}}.$$

On the other hand, we have $|\chi_{2R}(r)| \lesssim 1$ (by (5.15)). Therefore, (5.17) holds provided $R > 0$ is taken sufficiently large. The proof is complete. \square

Proof of Theorem 5.1. Let $\mathcal{M}_{\chi_R}(t)$ be as in (5.12). We have

$$\frac{d}{dt} \mathcal{M}_{\chi_R}(t) = 2\mathbb{G}(u(t), v(t)) + E_1(u(t), v(t)) + E_2(u(t), v(t)) + E_3(u(t), v(t)), \quad \forall t \in (-T_*, T^*),$$

where E_1, E_2, E_3 are as in (5.13). By the conservation of mass, we infer from (5.8) that

$$E_1(u(t), v(t)) \leq C(\kappa, \mathbb{M}(u_0, v_0))R^{-2}.$$

Using (5.11), we have

$$E_2(u(t), v(t)) \leq - \int_{\mathbb{R}^d} \chi_{1R} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) dx,$$

where χ_{1R} is as in (5.14). From (5.14), we also have

$$E_3(u(t), v(t)) = \frac{1}{2} \Re \int_{\mathbb{R}^d} \chi_{2R} |x|^{-\alpha} u^2(t) \bar{v}(t) dx.$$

As $\text{supp}(\chi_{2R}) \subset \{|x| \geq R\}$, we see that

$$\begin{aligned} E_3(u(t), v(t)) &\lesssim R^{-\alpha} \left(\int_{\mathbb{R}^d} \chi_{2R} |u(t)|^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^d} \chi_{2R} |v(t)|^3 dx \right)^{\frac{1}{3}} \\ &\lesssim R^{-\alpha} \left(\int_{\mathbb{R}^d} \chi_{2R} |u(t)|^3 dx + \int_{\mathbb{R}^d} \chi_{2R} |v(t)|^3 dx \right), \end{aligned}$$

where we have used the inequality $a^2b \leq \frac{1}{3}(2a^3 + b^3)$ for all $a, b \geq 0$.

Let us first consider the case $d = 3$ which corresponds to $\alpha = \frac{1}{2}$. We estimate

$$\begin{aligned} \int_{\mathbb{R}^d} \chi_{2R} |u(t)|^3 dx &\leq \|\chi_{2R} |u(t)|^{\frac{3}{2}}\|_{L^4} \| |u(t)|^{\frac{3}{2}} \|_{L^{\frac{4}{3}}} \\ &\leq \|\chi_{2R}^{\frac{2}{3}} u(t)\|_{L^6}^{\frac{3}{2}} \|u(t)\|_{L^2}^{\frac{3}{2}} \\ &\leq C(\mathbb{M}(u_0, v_0)) \|\nabla(\chi_{2R}^{\frac{2}{3}} u(t))\|_{L^2}^{\frac{3}{2}} \\ &\leq C(\mathbb{M}(u_0, v_0)) \left(\|\nabla(\chi_{2R}^{\frac{2}{3}} u(t))\|_{L^2}^2 + 1 \right) \\ &\leq C(\mathbb{M}(u_0, v_0)) \left(\|\chi_{2R}^{\frac{2}{3}} \nabla u(t)\|_{L^2}^2 + \|\nabla(\chi_{2R}^{\frac{2}{3}}) u(t)\|_{L^2}^2 + 1 \right). \end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^d} \chi_{2R} |v(t)|^3 dx \leq C(\kappa, \mathbb{M}(u_0, v_0)) \left(\kappa \|\chi_{2R}^{\frac{2}{3}} \nabla v(t)\|_{L^2}^2 + \kappa \|\nabla(\chi_{2R}^{\frac{2}{3}}) v(t)\|_{L^2}^2 + 1 \right).$$

Thus we get

$$\begin{aligned} E_3(u(t), v(t)) &\leq CR^{-\alpha} \left(\|\chi_{2R}^{\frac{2}{3}} \nabla u(t)\|_{L^2}^2 + \kappa \|\chi_{2R}^{\frac{2}{3}} \nabla v(t)\|_{L^2}^2 \right) \\ &\quad + CR^{-\alpha} \left(\|\nabla(\chi_{2R}^{\frac{2}{3}}) u(t)\|_{L^2}^2 + \kappa \|\nabla(\chi_{2R}^{\frac{2}{3}}) v(t)\|_{L^2}^2 \right) + CR^{-\alpha} \end{aligned}$$

for some constant $C = C(\kappa, \mathbb{M}(u_0, v_0)) > 0$. It follows that

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\chi_R}(t) &\leq 2\mathbb{G}(u(t), v(t)) - \int_{\mathbb{R}^d} \left(\chi_{1R} - CR^{-\alpha} \chi_{2R}^{\frac{4}{3}} \right) (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) dx \\ &\quad + CR^{-\alpha} \int_{\mathbb{R}^d} |\nabla(\chi_{2R}^{\frac{2}{3}})|^2 (|u(t)|^2 + \kappa |v(t)|^2) dx + CR^{-\alpha}, \quad \forall t \in (-T_*, T^*). \end{aligned}$$

Thanks to (5.1), (5.16), (5.17), and the conservation of mass, we take $R > 0$ sufficiently large to get

$$\frac{d}{dt} \mathcal{M}_{\chi_R}(t) \leq -\delta, \quad \forall t \in (-T_*, T^*).$$

- In the mass-resonance case, i.e., $\kappa = \frac{1}{2}$, we have

$$\frac{d^2}{dt^2} \mathcal{V}_{\chi_R}(t) = \frac{d}{dt} \mathcal{M}_{\chi_R}(t) \leq -\delta, \quad \forall t \in (-T_*, T^*),$$

where

$$\mathcal{V}_{\chi_R}(t) = \int_{\mathbb{R}^d} \chi_R(|u(t)|^2 + 2|v(t)|^2) dx. \quad (5.18)$$

Taking the integration over $[0, t]$, we obtain

$$\mathcal{V}_{\chi_R}(t) \leq \mathcal{V}_{\chi_R}(0) + t \frac{d}{dt} \mathcal{V}_{\chi_R}(0) - \frac{\delta}{2} t^2, \quad \forall t \in [0, T^*].$$

Assume by contradiction that $T^* = \infty$. As $\delta > 0$, there exists $t_* > 0$ sufficiently large such that $\mathcal{V}_{\chi_R}(t_*) < 0$ which is not possible, hence $T^* < \infty$.

- In the non mass-resonance case, i.e., $\kappa \neq \frac{1}{2}$, if $T^* < \infty$, we are done. Otherwise, if $T^* = \infty$, then we have

$$\frac{d}{dt} \mathcal{M}_{\chi_R}(t) \leq -\delta, \quad \forall t \in [0, \infty).$$

Integrating over $[0, t]$, we get

$$\mathcal{M}_{\chi_R}(t) \leq \mathcal{M}_{\chi_R}(0) - \delta t \leq -\frac{\delta}{2} t, \quad \forall t \geq t_0 := \frac{|\mathcal{M}_{\chi_R}(0)|}{\delta}.$$

By Hölder's inequality, we infer that

$$\begin{aligned} \frac{\delta}{2} t &\leq |\mathcal{M}_{\chi_R}(t)| \leq \|\nabla \chi_R\|_{L^\infty} (\|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2} + \|\nabla v(t)\|_{L^2} \|v(t)\|_{L^2}) \\ &\leq C(R, \kappa, \mathbb{M}(u_0, v_0)) \sqrt{\mathbb{K}(u(t), v(t))}, \quad \forall t \geq t_0 \end{aligned}$$

which yields $\mathbb{K}(u(t), v(t)) \geq Ct^2$ for some constant $C = C(R, \kappa, \delta, \mathbb{M}(u_0, v_0)) > 0$.

This completes the proof for positive times in three dimensions. The one for negative times is treated in a similar manner.

The proof is similar when $d = 2$ and $\alpha = 1$. We need to use the Sobolev embedding $H^1(\mathbb{R}^2) \subset L^6(\mathbb{R}^2)$ to get

$$\begin{aligned} \int_{\mathbb{R}^d} \chi_{2R} |u(t)|^3 dx &\leq \|\chi_{2R} |u(t)|^{\frac{3}{2}}\|_{L^4} \| |u(t)|^{\frac{3}{2}} \|_{L^{\frac{4}{3}}} \\ &\leq \|\chi_{2R}^{\frac{2}{3}} u(t)\|_{L^6}^{\frac{3}{2}} \|u(t)\|_{L^2}^{\frac{3}{2}} \\ &\leq C(\mathbb{M}(u_0, v_0)) \|\chi_{2R}^{\frac{2}{3}} u(t)\|_{H^1}^{\frac{3}{2}} \\ &\leq C(\mathbb{M}(u_0, v_0)) \left(\|\chi_{2R}^{\frac{2}{3}} u(t)\|_{H^1}^2 + 1 \right) \\ &\leq C(\mathbb{M}(u_0, v_0)) \left(\|\nabla(\chi_{2R}^{\frac{2}{3}} u(t))\|_{L^2}^2 + \|\chi_{2R}^{\frac{2}{3}} u(t)\|_{L^2}^2 + 1 \right) \\ &\leq C(\mathbb{M}(u_0, v_0)) \left(\|\chi_{2R}^{\frac{2}{3}} \nabla u(t)\|_{L^2}^2 + \|\nabla(\chi_{2R}^{\frac{2}{3}} u(t))\|_{L^2}^2 + \|\chi_{2R}^{\frac{2}{3}} u(t)\|_{L^2}^2 + 1 \right). \end{aligned}$$

Similarly, we also have

$$\int_{\mathbb{R}^d} \chi_{2R} |v(t)|^3 dx \leq C(\kappa, \mathbb{M}(u_0, v_0)) \left(\kappa \|\chi_{2R}^{\frac{2}{3}} \nabla v(t)\|_{L^2}^2 + \kappa \|\nabla(\chi_{2R}^{\frac{2}{3}} v(t))\|_{L^2}^2 + \kappa \|\chi_{2R}^{\frac{2}{3}} v(t)\|_{L^2}^2 + 1 \right).$$

The remaining argument is exactly the same as above as the additional term can be estimated, using (5.15) and the mass conservation, as

$$\|\chi_{2R}^{\frac{2}{3}} u(t)\|_{L^2}^2 + \kappa \|\chi_{2R}^{\frac{2}{3}} v(t)\|_{L^2}^2 \leq C(\kappa, \mathbb{M}(u_0, v_0)).$$

Finally, we will verify that (5.1) is satisfied if $\mathbb{H}(u_0, v_0) < 0$. It follows directly from the conservation of mass and energy, and the fact (see (4.7)) that

$$\mathbb{G}(u(t), v(t)) = \mathbb{K}(u(t), v(t)) - 2\mathbb{P}(u(t), v(t)) = 2 \left(\mathbb{E}(u(t), v(t)) - \gamma \|v(t)\|_{L^2}^2 \right) \leq 2\mathbb{H}(u(t), v(t)).$$

The proof is complete. \square

5.5. Mass-supercritical blow-up solutions. Before proving Theorem 5.2, we have the following observation.

Lemma 5.6. *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha > \frac{4-d}{2}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Let $(u_0, v_0) \in \mathcal{H}^1$ and $(u, v) \in C((-T_*, T^*), \mathcal{H}^1)$ be the corresponding maximal solution to (1.1). Assume that (5.1) holds. Then there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that*

$$\mathbb{G}(u(t), v(t)) + \varepsilon \mathbb{K}(u(t), v(t)) \leq -\frac{\delta}{2}, \quad \forall t \in (-T_*, T^*) \quad (5.19)$$

for all $0 < \varepsilon \leq \varepsilon_0$. In addition,

$$\inf_{t \in (-T_*, T^*)} \mathbb{K}(u(t), v(t)) \geq C \quad (5.20)$$

for some constant $C > 0$.

Proof. By (4.7), and the conservation laws of mass and energy, we have for all $t \in (-T_*, T^*)$,

$$\begin{aligned} \mathbb{G}(u(t), v(t)) &= \mathbb{K}(u(t), v(t)) - \frac{d+2\alpha}{2} \mathbb{P}(u(t), v(t)) \\ &= \frac{d+2\alpha}{2} (\mathbb{E}(u(t), v(t)) - \gamma \|v(t)\|_{L^2}^2) - \frac{d+2\alpha-4}{4} \mathbb{K}(u(t), v(t)) \\ &\leq \frac{d+2\alpha}{2} \mathbb{H}(u(t), v(t)) - \frac{d+2\alpha-4}{4} \mathbb{K}(u(t), v(t)). \end{aligned} \quad (5.21)$$

For $\varepsilon > 0$ small, we infer from (5.21) that

$$\begin{aligned} \mathbb{G}(u(t), v(t)) + \varepsilon \mathbb{K}(u(t), v(t)) &\leq \mathbb{G}(u(t), v(t)) + \varepsilon \left(\frac{2(d+2\alpha)}{d+2\alpha-4} \mathbb{H}(u(t), v(t)) - \frac{4}{d+2\alpha-4} \mathbb{G}(u(t), v(t)) \right) \\ &= \left(1 - \frac{4\varepsilon}{d+2\alpha-4} \right) \mathbb{G}(u(t), v(t)) + \frac{2\varepsilon(d+2\alpha)}{d+2\alpha-4} \mathbb{H}(u(t), v(t)) \end{aligned}$$

which together with (5.1), and the conservation laws of mass and energy imply

$$\mathbb{G}(u(t), v(t)) + \varepsilon \mathbb{K}(u(t), v(t)) \leq -\left(1 - \frac{4\varepsilon}{d+2\alpha-4} \right) \delta + \frac{2\varepsilon(d+2\alpha)}{d+2\alpha-4} \mathbb{H}(u_0, v_0) \leq -\frac{\delta}{2}$$

provided that $0 < \varepsilon \leq \varepsilon_0$ with some $\varepsilon_0 = \varepsilon_0(\delta) > 0$. This proves (5.19).

To see (5.19), we argue by contradiction. Assume that there exists a time sequence $(t_n)_n \subset (-T_*, T^*)$ such that $\mathbb{K}(u(t_n), v(t_n)) \rightarrow 0$. By (3.1) and the conservation of mass, we readily see that $\mathbb{G}(u(t_n), v(t_n)) \rightarrow 0$. This contradicts (5.1). \square

Proof of Theorem 5.2. We will consider separately two cases: $2 \leq d \leq 4$ and $d = 5$.

Case 1. $2 \leq d \leq 4$. Let $\mathcal{M}_{\chi_R}(t)$ be as in (5.12). We have

$$\frac{d}{dt} \mathcal{M}_{\chi_R}(t) = 2\mathbb{G}(u(t), v(t)) + E_1(u(t), v(t)) + E_2(u(t), v(t)) + E_3(u(t), v(t)), \quad \forall t \in (-T_*, T^*),$$

with E_1, E_2, E_3 as in (5.13). By (5.8), the conservation of mass implies

$$E_1(u(t), v(t)) \leq C(\kappa, \mathbb{M}(u_0, v_0))R^{-2}.$$

Thanks to (5.10) and (5.11), we have $E_2(u(t), v(t)) \leq 0$. On the other hand, from (5.9) and (5.10), we have

$$\begin{aligned} E_3(u(t), v(t)) &\leq C \int_{|x| \geq R} |x|^{-\alpha} |u(t)|^2 |v(t)| \, dx \\ &\leq CR^{-\alpha} \|u(t)\|_{L^3}^2 \|v(t)\|_{L^3}. \end{aligned}$$

Using the standard Gagliardo–Nirenberg inequality

$$\|f\|_{L^3}^3 \leq C \|\nabla f\|_{L^2}^{\frac{d}{2}} \|f\|_{L^2}^{\frac{6-d}{2}},$$

we get

$$\begin{aligned} E_3(u(t), v(t)) &\leq CR^{-\alpha} \|\nabla u(t)\|_{L^2}^{\frac{d}{3}} \|u(t)\|_{L^2}^{\frac{6-d}{3}} \|\nabla v(t)\|_{L^2}^{\frac{d}{6}} \|v(t)\|_{L^2}^{\frac{6-d}{6}} \\ &\leq C(\kappa, \mathbb{M}(u_0, v_0))R^{-\alpha} (\mathbb{K}(u(t), v(t)))^{\frac{d}{4}} \\ &\leq \begin{cases} C(\kappa, \mathbb{M}(u_0, v_0))R^{-\alpha} (\mathbb{K}(u(t), v(t)) + 1) & \text{if } d = 2, 3, \\ C(\kappa, \mathbb{M}(u_0, v_0))R^{-\alpha} \mathbb{K}(u(t), v(t)) & \text{if } d = 4. \end{cases} \end{aligned}$$

Collecting the above estimates, we obtain for all $t \in (-T_*, T^*)$,

$$\frac{d}{dt}\mathcal{M}_{\chi_R}(t) \leq 2\mathbb{G}(u(t), v(t)) + \begin{cases} CR^{-\alpha}\mathbb{K}(u(t), v(t)) + CR^{-\alpha} & \text{if } d = 2, 3, \\ CR^{-\alpha}\mathbb{K}(u(t), v(t)) + CR^{-2} & \text{if } d = 4, \end{cases}$$

for some constant C depending on κ and $\mathbb{M}(u_0, v_0)$.

Using (5.19), there exists $\varepsilon_0 > 0$ such that for all $t \in (-T_*, T^*)$,

$$\frac{d}{dt}\mathcal{M}_{\chi_R}(t) \leq -(2\varepsilon_0 - CR^{-\alpha})\mathbb{K}(u(t), v(t)) - \delta + \begin{cases} CR^{-\alpha} & \text{if } d = 2, 3, \\ CR^{-2} & \text{if } d = 4. \end{cases}$$

Taking $R > 0$ sufficiently large, we get

$$\frac{d}{dt}\mathcal{M}_{\chi_R}(t) \leq -\varepsilon_0\mathbb{K}(u(t), v(t)) - \frac{\delta}{2}, \quad \forall t \in (-T_*, T^*).$$

From this, we deduce the blow-up as follows.

- In the mass-resonance case, i.e., $\kappa = \frac{1}{2}$, we have

$$\frac{d^2}{dt^2}\mathcal{V}_{\chi_R}(t) = \frac{d}{dt}\mathcal{M}_{\chi_R}(t) \leq -\frac{\delta}{2}, \quad \forall t \in (-T_*, T^*),$$

where \mathcal{V}_{χ_R} is as in (5.18). The same argument as in the proof of Theorem 5.1 yields $T_*, T^* < \infty$.

- In the non mass-resonance, i.e., $\kappa \neq \frac{1}{2}$, we argue by contradiction. Assume that $T^* = \infty$. We have

$$\frac{d}{dt}\mathcal{M}_{\chi_R}(t) \leq -\varepsilon_0\mathbb{K}(u(t), v(t)) - \frac{\delta}{2}, \quad \forall t \in [0, \infty). \quad (5.22)$$

Integrating this inequality over $[0, t]$, we obtain

$$\mathcal{M}_{\chi_R}(t) \leq \mathcal{M}_{\chi_R}(0) - \frac{\delta}{2}t \leq 0, \quad \forall t \geq t_0 := \frac{2|\mathcal{M}_{\chi_R}(0)|}{\delta}.$$

Integrating (5.22) over $[t_0, t]$, we get

$$\mathcal{M}_{\chi_R}(t) \leq -\varepsilon_0 \int_{t_0}^t \mathbb{K}(u(s), v(s)) ds, \quad \forall t \geq t_0.$$

On the other hand, we have

$$\begin{aligned} |\mathcal{M}_{\chi_R}(t)| &\leq \|\nabla\chi_R\|_{L^\infty} (\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2} + \|\nabla v(t)\|_{L^2}\|v(t)\|_{L^2}) \\ &\leq C(R, \kappa, \mathbb{M}(u_0, v_0))\sqrt{\mathbb{K}(u(t), v(t))} \end{aligned}$$

which implies

$$\mathcal{M}_{\chi_R}(t) \leq -A \int_{t_0}^t |\mathcal{M}_{\chi_R}(s)|^2 ds, \quad \forall t \geq t_0,$$

where $A = A(\varepsilon_0, R, \kappa, \delta, \mathbb{M}(u_0, v_0)) > 0$. Set

$$y(t) = \int_{t_0}^t |\mathcal{M}_{\chi_R}(s)|^2 ds.$$

We see that y is strictly increasing and non-negative and

$$y'(t) = |\mathcal{M}_{\chi_R}(t)|^2 \geq A^2 y^2(t).$$

Fix some $t_1 > t_0$. We integrate this inequality over $[t_1, t]$ and get

$$y(t) \geq \frac{y(t_1)}{1 - A^2 y(t_1)(t - t_1)}, \quad \forall t \geq t_1.$$

Thus we have

$$y(t) \rightarrow +\infty \text{ as } t \nearrow t_* := t_1 + \frac{1}{A^2 y(t_1)} > t_1.$$

Hence $\mathcal{M}_{\chi_R}(t) \leq -Ay(t) \rightarrow -\infty$ as $t \nearrow t_*$. Therefore, the solution cannot exist for all time $t \geq 0$ and, consequently, we must have $T^* < \infty$. A similar argument goes for negative times. This finishes the proof for $2 \leq d \leq 4$.

Case 2. $d = 5$. We only consider the positive times since the one for negative times is treated in a similar manner. If $T^* < \infty$, we are done. Otherwise, if $T^* = \infty$, we first show (5.2). Assume by contradiction that

$$\sup_{t \in [0, \infty)} \mathbb{K}(u(t), v(t)) < \infty. \quad (5.23)$$

Arguing as above, we have

$$\frac{d}{dt} \mathcal{M}_{\chi_R}(t) \leq 2\mathbb{G}(u(t), v(t)) + CR^{-\alpha} (\mathbb{K}(u(t), v(t)))^{\frac{5}{4}} + CR^{-2}, \quad \forall t \in [0, \infty).$$

By (5.19), there exists $\varepsilon_0 > 0$ such that

$$\frac{d}{dt} \mathcal{M}_{\chi_R}(t) \leq -2\varepsilon_0 \mathbb{K}(u(t), v(t)) - \delta + CR^{-\alpha} (\mathbb{K}(u(t), v(t)))^{\frac{5}{4}} + CR^{-2}, \quad \forall t \in [0, \infty). \quad (5.24)$$

By (5.23), we take $R > 0$ sufficiently large and deduce

$$\frac{d}{dt} \mathcal{M}_{\chi_R}(t) \leq -\varepsilon_0 \mathbb{K}(u(t), v(t)) - \frac{\delta}{2}, \quad \forall t \in [0, \infty).$$

From this, we can argue exactly as in Case 1 to show that there exists a finite time $t_* > 0$ such that $\mathcal{M}_{\chi_R}(t) \rightarrow -\infty$ as $t \nearrow t_*$. This is a contradiction. Thus we have proved (5.2).

Let us now prove (5.3). To this end, we fix $T > 0$ and set

$$R(T) := B \sup_{t \in [0, T]} (\mathbb{K}(u(t), v(t)))^{\frac{1}{4}}$$

for some $B > 0$ to be chosen later. Applying (5.2) with $R = R(T)$, we have

$$\frac{d}{dt} \mathcal{M}_{\chi_{R(T)}}(t) \leq -2\varepsilon_0 \mathbb{K}(u(t), v(t)) + C(R(T))^{-\alpha} (\mathbb{K}(u(t), v(t)))^{\frac{5}{4}} + C(R(T))^{-2}, \quad \forall t \in [0, \infty).$$

We observe that

$$\begin{aligned} C(R(T))^{-\alpha} (\mathbb{K}(u(t), v(t)))^{\frac{5}{4}} &= \frac{CB^{-\alpha} (\mathbb{K}(u(t), v(t)))^{\frac{1}{4}}}{\sup_{t \in [0, T]} (\mathbb{K}(u(t), v(t)))^{\frac{1}{4}}} \\ &\leq CB^{-\alpha}, \quad \forall t \in [0, T] \end{aligned}$$

and

$$\begin{aligned} \frac{C(R(T))^{-2}}{\mathbb{K}(u(t), v(t))} &= \frac{CB^{-2}}{\mathbb{K}(u(t), v(t)) \sup_{t \in [0, T]} (\mathbb{K}(u(t), v(t)))^{\frac{1}{2}}} \\ &\leq \frac{CB^{-2}}{C_0^{3/2}}, \quad \forall t \in [0, T], \end{aligned}$$

where C_0 is as in (5.20). Thus we get

$$\frac{d}{dt} \mathcal{M}_{\chi_{R(T)}}(t) \leq - \left(2\varepsilon_0 - CB^{-\alpha} - \frac{CB^{-2}}{C_0^{3/4}} \right) \mathbb{K}(u(t), v(t)), \quad \forall t \in [0, T].$$

Taking $B > 0$ sufficiently large, we obtain

$$\frac{d}{dt} \mathcal{M}_{\chi_{R(T)}}(t) \leq -\varepsilon_0 \mathbb{K}(u(t), v(t)), \quad \forall t \in [0, T]. \quad (5.25)$$

As above, we consider two cases: mass-resonance and non mass-resonance.

- In the mass-resonance case, we have

$$\frac{d^2}{dt^2} \mathcal{V}_{\chi_{R(T)}}(t) = \frac{d}{dt} \mathcal{M}_{\chi_{R(T)}}(t) \leq -\varepsilon_0 \mathbb{K}(u(t), v(t)), \quad \forall t \in [0, T].$$

It follows that

$$0 \leq \mathcal{V}_{\chi_{R(T)}}(T) \leq \mathcal{V}_{\chi_{R(T)}}(0) + T \frac{d}{dt} \mathcal{V}_{\chi_{R(T)}}(0) - \varepsilon_0 \int_0^T \int_0^s \mathbb{K}(u(\tau), v(\tau)) \, d\tau \, ds.$$

Observe that

$$\mathcal{V}_{\chi_{R(T)}}(0) = \int_{\mathbb{R}^d} \chi_{R(T)} (|u_0|^2 + 2|v_0|^2) \, dx \leq C(u_0, v_0)(R(T))^2$$

and

$$\frac{d}{dt} \mathcal{V}_{\chi_{R(T)}}(0) = \Im \int_{\mathbb{R}^d} \nabla \chi_{R(T)} \cdot (\nabla u_0 \bar{u}_0 + 2\kappa \nabla v_0 \bar{v}_0) \, dx \leq C(\kappa, u_0, v_0)R(T).$$

Thanks to (5.20), we deduce

$$\varepsilon_0 \frac{C_0}{2} T^2 \leq \varepsilon_0 \int_0^T \int_0^s \mathbb{K}(u(\tau), v(\tau)) \, d\tau \, ds \leq CTR(T) + C(R(T))^2$$

which implies $R(T) \geq CT$. In particular, we obtain $\sup_{t \in [0, T]} \mathbb{K}(u(t), v(t)) \geq CT^4$.

- In the non mass-resonance case, we have from (5.25) that

$$\mathcal{M}_{\chi_{R(T)}}(T) \leq \mathcal{M}_{\chi_{R(T)}}(0) - \varepsilon_0 \int_0^T \mathbb{K}(u(s), v(s)) \, ds.$$

which together with (5.20) imply

$$\varepsilon_0 T \leq \varepsilon_0 \int_0^T \mathbb{K}(u(s), v(s)) \, ds \leq |\mathcal{M}_{\chi_{R(T)}}(0)| + |\mathcal{M}_{\chi_{R(T)}}(T)|.$$

Using $|\mathcal{M}_{\chi_{R(T)}}(0)| \leq C(\kappa, u_0, v_0)R(T)$ and

$$\begin{aligned} |\mathcal{M}_{\chi_{R(T)}}(T)| &\leq \|\nabla \chi_{R(T)}\|_{L^\infty} (\|\nabla u(T)\|_{L^2} \|u(T)\|_{L^2} + \|\nabla v(T)\|_{L^2} \|v(T)\|_{L^2}) \\ &\leq C(\kappa, \mathbb{M}(u_0, v_0))R(T) \sqrt{\mathbb{K}(u(T), v(T))} \\ &\leq C(\kappa, \mathbb{M}(u_0, v_0))(R(T))^3, \end{aligned}$$

we obtain

$$\varepsilon_0 T \leq CR(T) + C(R(T))^3.$$

This shows that $R(T) \geq CT^{1/3}$, hence $\inf_{t \in [0, T]} \mathbb{K}(u(t), v(t)) \geq CT^{\frac{4}{3}}$.

Finally, we prove that (5.1) is fulfilled provided that either $\mathbb{H}(u_0, v_0) < 0$, or if $\mathbb{H}(u_0, v_0) \geq 0$, we assume (5.4) and (5.5).

- The case $\mathbb{H}(u_0, v_0) < 0$. From (5.21), we have

$$\mathbb{G}(u(t), v(t)) \leq \frac{d+2\alpha}{2} \mathbb{H}(u_0, v_0)$$

which proves (5.1) with $\delta = -\frac{d+2\alpha}{2} \mathbb{H}(u_0, v_0) > 0$.

- The case $\mathbb{H}(u_0, v_0) \geq 0$ in which (5.4) and (5.5) are assumed. By the same argument as in the proof of Proposition 4.3, we can prove that

$$\mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma > \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall t \in (-T_*, T^*). \quad (5.26)$$

On the other hand, by (5.4), we can take $\rho \in (0, 1)$ such that

$$\mathbb{H}(u_0, v_0) (\mathbb{M}(u_0, v_0))^\sigma \leq (1-\rho) \mathbb{E}_0(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma.$$

which together with the conservation of mass and energy yield

$$\begin{aligned} \mathbb{H}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma &\leq (1-\rho) \mathbb{E}_0(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma \\ &= (1-\rho) \frac{d+2\alpha-4}{2(d+2\alpha)} \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall t \in (-T_*, T^*). \end{aligned}$$

It follows from (5.21) and (5.26) that

$$\begin{aligned} &\mathbb{G}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma \\ &\leq \frac{d+2\alpha}{2} \mathbb{H}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma - \frac{d+2\alpha-4}{4} \mathbb{K}(u(t), v(t)) (\mathbb{M}(u(t), v(t)))^\sigma \\ &\leq (1-\rho) \frac{d+2\alpha-4}{4} \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma - \frac{d+2\alpha-4}{4} \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma \\ &= -\rho \frac{d+2\alpha-4}{4} \mathbb{K}(\varphi, \psi) (\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall t \in (-T_*, T^*). \end{aligned}$$

This shows (5.1) with

$$\delta = \rho \frac{d+2\alpha-4}{4} \mathbb{K}(\varphi, \psi) \left(\frac{\mathbb{M}(\varphi, \psi)}{\mathbb{M}(u_0, v_0)} \right)^\sigma > 0.$$

The proof of Theorem 5.2 is now complete. \square

6. ENERGY SCATTERING

The purpose of this section is to prove the following asymptotic behavior (or energy scattering) of \mathcal{H}^1 -solutions to (1.1) in the mass-supercritical regime.

Theorem 6.1 (Energy scattering). *Let $3 \leq d \leq 5$, $0 < \alpha < \min\{2, \frac{d}{2}\}$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Let (φ, ψ) be a nonlinear ground state related to (3.2). Let $(u_0, v_0) \in \mathcal{H}^1$ satisfy (4.3) and (4.4). Then the corresponding solution to (1.1) exists globally in time and scatters in \mathcal{H}^1 in both directions.*

Remark 6.1. The restriction $\alpha < \frac{d}{2}$ is technical due to our nonlinear estimates (see Lemma 6.4). This prevents us to show the energy scattering in two dimensions.

6.1. Dispersive and Strichartz estimates. Let $\beta_1, \beta_2 \in \mathbb{R}$ and denote $\mathcal{S}(t) := e^{it(\beta_1 \Delta - \beta_2)}$ the Schrödinger operator. We have the following dispersive estimates (see e.g., [5]): for $2 \leq r \leq \infty$,

$$\|\mathcal{S}(t)f\|_{L_x^r} \lesssim |t|^{-\left(\frac{d}{2} - \frac{d}{r}\right)} \|f\|_{L_x^{r'}}, \quad \forall t \neq 0 \quad (6.1)$$

for all $f \in L_x^{r'}(\mathbb{R}^d)$.

Let $0 \leq s < \min\{1, \frac{d}{2}\}$. A pair (q, r) is called \dot{H}^s -admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s$$

and

$$\begin{cases} \frac{2d}{d-2s} < r < \frac{2d}{d-2} & \text{if } d \geq 3, \\ \frac{2}{1-s} < r < \infty & \text{if } d = 2, \\ \frac{2}{1-2s} < r < \infty & \text{if } d = 1. \end{cases} \quad (6.2)$$

The set of all \dot{H}^s -admissible pairs is denoted by \mathcal{A}_s . Similarly, a pair (q, r) is called \dot{H}^{-s} -admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} + s$$

and r satisfies (6.2). We denote by \mathcal{A}_{-s} the set of all \dot{H}^{-s} -admissible pairs.

Proposition 6.2 (Strichartz estimates [5, 26, 18, 21]). *Then for $d \geq 1$ and $0 \leq s < \min\{1, \frac{d}{2}\}$, we have*

$$\|\mathcal{S}(t)f\|_{L_t^q(\mathbb{R}, L_x^r)} \lesssim \|f\|_{\dot{H}_x^s}$$

for any $f \in \dot{H}^s(\mathbb{R}^d)$ and any $(q, r) \in \mathcal{A}_s$. Moreover, for any interval $I \subset \mathbb{R}$ containing 0, there exists $C > 0$ independent of I such that

$$\left\| \int_0^t \mathcal{S}(t-\tau)F(\tau) \, d\tau \right\|_{L_t^q(I, L_x^r)} \lesssim \|F\|_{L_t^{q'}(I, L_x^{b'})}$$

for any $F \in L_t^{q'}(I, L_x^{b'}(\mathbb{R}^d))$, any $(q, r) \in \mathcal{A}_s$, and any $(a, b) \in \mathcal{A}_{-s}$.

6.2. Nonlinear estimates. Let $\theta > 0$ be a small parameter. We introduce the following exponents

$$\begin{aligned} q &= \frac{4}{2-\theta}, & r &= \frac{2d}{d-2+\theta}, \\ \bar{q} &= \frac{4(2-\theta)}{d-2+2\alpha-\theta(d-5+2\alpha)}, & \bar{r} &= \frac{2d(2-\theta)}{d+2-2\alpha-\theta(5-2\alpha)}, \\ \bar{a} &= \frac{4(2-\theta)}{6-d-2\alpha+\theta}, & \tilde{a} &= \frac{2(3-\theta)}{2d-6+4\alpha-\theta(d-4+2\alpha)}, \\ \hat{a} &= \frac{2(3-\theta)}{6-d-2\alpha}, & \hat{r} &= \frac{d(3-\theta)}{d-\alpha-\theta(2-\alpha)}, \\ \bar{m}_\pm &= \frac{d}{2-\alpha \mp d\theta}. \end{aligned}$$

Lemma 6.3. *Let $2 \leq d \leq 5$, $0 < \alpha < 2$, and $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$. Then there exists $\theta > 0$ sufficiently small so that*

$$\begin{cases} (q, r), (\bar{q}, \bar{r}) \in \mathcal{A}_0, \\ (\bar{a}, \bar{r}), (\hat{a}, \hat{r}) \in \mathcal{A}_{s_c}, \\ (\tilde{a}, \hat{r}) \in \mathcal{A}_{-s_c}, \\ 2 < \bar{m}_\pm < \frac{2d}{d-2}, \end{cases}$$

where s_c is the critical Sobolev exponent given in (1.2).

Proof. The proof follows from straightforward computations. Note that the condition (6.2) follows by taking $\theta > 0$ sufficiently small and using $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$. \square

Lemma 6.4. *Let $3 \leq d \leq 5$, $0 < \alpha < \min\{2, \frac{d}{2}\}$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, and $I \subset \mathbb{R}$ be an interval. Let $\theta > 0$ be a small parameter as in Lemma 6.3. Then we have*

$$\| |x|^{-\alpha} \bar{u} v \|_{L_t^{\hat{a}'}(I, L_x^{\hat{r}'})} \lesssim \| u \|_{L_t^\infty(I, H_x^1)} \| u \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}})}^{1-\theta} \| v \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}})}, \quad (6.3)$$

$$\| |x|^{-\alpha} \bar{u} v \|_{L_t^{\hat{a}'}(I, L_x^{\hat{r}'})} \lesssim \| u \|_{L_t^\infty(I, H_x^1)} \| u \|_{L_t^{\bar{a}}(I, L_x^{\bar{r}})}^{1-\theta} \| v \|_{L_t^{\bar{a}}(I, L_x^{\bar{r}})}, \quad (6.4)$$

$$\| \nabla (|x|^{-\alpha} \bar{u} v) \|_{L_t^{\hat{a}'}(I, L_x^{\hat{r}'})} \lesssim \| u \|_{L_t^\infty(I, H_x^1)} \| u \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}})}^{1-\theta} \| \nabla v \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}})} + \| v \|_{L_t^\infty(I, H_x^1)} \| v \|_{L_t^{\bar{a}}(I, L_x^{\bar{r}})}^{1-\theta} \| \nabla u \|_{L_t^{\bar{a}}(I, L_x^{\bar{r}})}. \quad (6.5)$$

Proof. We follow an argument of [6, 12]. By Hölder's inequality, we have

$$\| |x|^{-\alpha} \bar{u} v \|_{L_x^{\hat{r}'}} \leq \| |x|^{-\alpha} \|_{L_x^\gamma(A)} \| \bar{u} v \|_{L_x^\rho}$$

provided that $\gamma, \rho > 1$ is such that $\frac{1}{\hat{r}'} = \frac{1}{\gamma} + \frac{1}{\rho}$, where A is either $B_1 = B(0, 1)$ the unit ball or $B_1^c = \mathbb{R}^d \setminus B_1$. To ensure $\| |x|^{-\alpha} \|_{L_x^\gamma(A)} < \infty$, we take

$$\frac{1}{\gamma} = \frac{\alpha}{d} \pm \theta^2$$

with the plus sign for $A = B_1$ and the minus sign for $A = B_1^c$. Using the fact that

$$\frac{1}{\rho} = \frac{1}{\hat{r}'} - \frac{1}{\gamma} = \frac{2d - 2\alpha - (d-2)\theta}{d(3-\theta)} \mp \theta^2 = \frac{\theta}{\bar{m}_\pm} + \frac{2-\theta}{\hat{r}'},$$

we have

$$\| |x|^{-\alpha} \bar{u} v \|_{L_x^{\hat{r}'}} \lesssim \| u \|_{L_x^{\bar{m}_\pm}}^\theta \| u \|_{L_x^{\hat{r}'}}^{1-\theta} \| v \|_{L_x^{\hat{r}'}}.$$

As $\frac{1}{\hat{a}'} = \frac{2-\theta}{\hat{a}}$, Hölder's inequality and Sobolev embedding with $2 < \bar{m}_\pm < \frac{2d}{d-2}$ yield

$$\begin{aligned} \| |x|^{-\alpha} \bar{u} v \|_{L_t^{\hat{a}'}(I, L_x^{\hat{r}'})} &\lesssim \| u \|_{L_t^\infty(I, L_x^{\bar{m}_\pm})}^\theta \| u \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}'})}^{1-\theta} \| v \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}'})} \\ &\lesssim \| u \|_{L_t^\infty(I, H_x^1)}^\theta \| u \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}'})}^{1-\theta} \| v \|_{L_t^{\hat{a}}(I, L_x^{\hat{r}'})}. \end{aligned}$$

This shows (6.3). We also have the following estimate which will be useful later

$$\| |x|^{-\alpha} \bar{u} v \|_{L_x^{\hat{r}'}} \leq \| |x|^{-\alpha} \|_{L_x^\gamma(A)} \| u \|_{L_x^{\bar{m}_\pm}}^\theta \| u \|_{L_x^{\hat{r}'}}^{1-\theta} \| v \|_{L_x^{\hat{r}'}}. \quad (6.6)$$

We next prove (6.5). We have

$$| \nabla (|x|^{-\alpha} \bar{u} v) | \leq |x|^{-\alpha} | \nabla (\bar{u} v) | + \alpha |x|^{-\alpha} |x|^{-1} (\bar{u} v),$$

hence

$$\| \nabla (|x|^{-\alpha} \bar{u} v) \|_{L_x^{\hat{r}'}} \leq \| |x|^{-\alpha} \nabla (\bar{u} v) \|_{L_x^{\hat{r}'}} + \alpha \| |x|^{-\alpha} |x|^{-1} (\bar{u} v) \|_{L_x^{\hat{r}'}}.$$

We estimate

$$\| |x|^{-\alpha} \nabla (\bar{u} v) \|_{L_x^{\hat{r}'}} \leq \| |x|^{-\alpha} \|_{L_x^\gamma(A)} \| \nabla (\bar{u} v) \|_{L_x^\rho}$$

provided that $\gamma, \rho > 1$ and $\frac{1}{\hat{r}'} = \frac{1}{\gamma} + \frac{1}{\rho}$. To make $\| |x|^{-\alpha} \|_{L_x^\gamma(A)} < \infty$, we take $\frac{1}{\gamma} = \frac{\alpha}{d} \pm \theta^2$ as before. In particular, we have

$$\frac{1}{\rho} = \frac{1}{\hat{r}'} - \frac{1}{\gamma} = \frac{d+2-2\alpha-\theta}{2d} \mp \theta^2.$$

Similarly, we have

$$\| |x|^{-\alpha} |x|^{-1} (\bar{u} v) \|_{L_x^{\hat{r}'}} \leq \| |x|^{-\alpha} \|_{L_x^\gamma(A)} \| |x|^{-1} (\bar{u} v) \|_{L_x^\rho} \lesssim \| |x|^{-1} (\bar{u} v) \|_{L_x^\rho}.$$

As $\alpha < \frac{d}{2}$, we see that $1 < \rho < d$ by taking $\theta > 0$ sufficiently small. Applying the following Hardy inequality (see e.g., [41]): for $1 < \rho < d$,

$$\| |x|^{-1} f \|_{L_x^\rho} \leq \frac{\rho}{d-\rho} \| \nabla f \|_{L_x^\rho},$$

we get

$$\| |x|^{-\alpha} |x|^{-1} (\bar{u} v) \|_{L_x^{\hat{r}'}} \lesssim \| \nabla (\bar{u} v) \|_{L_x^\rho}.$$

In particular, we obtain

$$\begin{aligned} \| \nabla (|x|^{-\alpha} \bar{u} v) \|_{L_x^{\hat{r}'}} &\lesssim \| \nabla (\bar{u} v) \|_{L_x^\rho} \\ &\lesssim \| \bar{u} \nabla v \|_{L_x^\rho} + \| v \nabla \bar{u} \|_{L_x^\rho} \\ &\lesssim \| u \|_{L_x^{\bar{m}_\pm}}^\theta \| u \|_{L_x^{\bar{r}'}}^{1-\theta} \| \nabla v \|_{L_x^{\bar{r}'}} + \| v \|_{L_x^{\bar{m}_\pm}}^\theta \| v \|_{L_x^{\bar{r}'}}^{1-\theta} \| \nabla u \|_{L_x^{\bar{r}'}}. \end{aligned}$$

where we have used the fact that $\frac{1}{\rho} = \frac{\theta}{\overline{m}_{\pm}} + \frac{2-\theta}{\overline{r}}$. Another application of the Hölder inequality with $\frac{1}{q'} = \frac{1-\theta}{\overline{a}} + \frac{1}{\overline{q}}$ and Sobolev embedding yields

$$\begin{aligned} \|\nabla(|x|^{-\alpha}\overline{u}v)\|_{L_t^{q'}(I, L_x^{r'})} &\lesssim \|u\|_{L_t^\infty(I, L_x^{\overline{m}_{\pm}})}^\theta \|u\|_{L_t^{\overline{a}}(I, L_x^{\overline{r}})}^{1-\theta} \|\nabla v\|_{L_t^{\overline{q}}(I, L_x^{\overline{r}})} + \|v\|_{L_t^\infty(I, L_x^{\overline{m}_{\pm}})}^\theta \|v\|_{L_t^{\overline{a}}(I, L_x^{\overline{r}})}^{1-\theta} \|\nabla u\|_{L_t^{\overline{q}}(I, L_x^{\overline{r}})} \\ &\lesssim \|u\|_{L_t^\infty(I, H_x^1)}^\theta \|u\|_{L_t^{\overline{a}}(I, L_x^{\overline{r}})}^{1-\theta} \|\nabla v\|_{L_t^{\overline{q}}(I, L_x^{\overline{r}})} + \|v\|_{L_t^\infty(I, H_x^1)}^\theta \|v\|_{L_t^{\overline{a}}(I, L_x^{\overline{r}})}^{1-\theta} \|\nabla u\|_{L_t^{\overline{q}}(I, L_x^{\overline{r}})} \end{aligned}$$

which is (6.5). The estimate (6.4) is treated similarly (even simpler) as for (6.5). The proof is complete. \square

Lemma 6.5. *Let $3 \leq d \leq 5$, $0 < \alpha < 2$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, and $\frac{2d}{d+4} < m < \frac{2d}{d+2}$, we have*

$$\| |x|^{-\alpha}\overline{u}v \|_{L_x^m} \lesssim \|u\|_{H_x^1} \|v\|_{H_x^1}. \quad (6.7)$$

Proof. We estimate

$$\| |x|^{-\alpha}\overline{u}v \|_{L_x^m} \leq \| |x|^{-\alpha} \|_{L_x^\gamma(A)} \|\overline{u}v\|_{L_x^\rho} \lesssim \|u\|_{L_x^{2\rho}} \|v\|_{L_x^{2\rho}} \lesssim \|u\|_{H_x^1} \|v\|_{H_x^1} \quad (6.8)$$

provided that $\gamma, \rho > 1$, $\frac{1}{m} = \frac{1}{\gamma} + \frac{1}{\rho}$, $\| |x|^{-\alpha} \|_{L_x^\gamma(A)} < \infty$, and $2\rho \in \left[2, \frac{2d}{d-2}\right]$, where $A = B_1$ or B_1^c . To make $\| |x|^{-\alpha} \|_{L_x^\gamma(A)} < \infty$, we take $\gamma > 1$ so that

$$\frac{1}{\gamma} = \frac{\alpha}{d} \pm \theta^2$$

with the plus sign for $A = B_1$ and the minus sign for $A = B_1^c$. It follows that

$$\rho = \frac{dm}{d - \alpha m \mp \theta^2 dm}.$$

For $\theta > 0$ sufficiently small, the condition $2\rho \in \left[2, \frac{2d}{d-2}\right]$ is fulfilled provided that $\frac{d}{d+\alpha} < m < \frac{d}{d-2+\alpha}$. As $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, we infer that

$$\frac{d}{d+\alpha} < \frac{2d}{d+4}, \quad \frac{2d}{d+2} < \frac{d}{d-2+\alpha}.$$

Thus for $\frac{2d}{d+4} < m < \frac{2d}{d+2}$, we can choose $\theta > 0$ sufficiently small so that (6.8) holds. \square

6.3. Scattering criterion.

Lemma 6.6 (Small data scattering). *Let $3 \leq d \leq 5$, $0 < \alpha < \min\{2, \frac{d}{2}\}$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Suppose that (u, v) is a global \mathcal{H}^1 -solution to (1.1) satisfying*

$$\sup_{t \in \mathbb{R}} \|(u(t), v(t))\|_{\mathcal{H}^1} \leq E \quad (6.9)$$

for some constant $E > 0$. Then there exists $\delta = \delta(E) > 0$ sufficiently small such that if

$$\|(\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T))\|_{\mathcal{L}^{\overline{a}}([T, +\infty), \mathcal{L}^{\overline{r}}) \cap \mathcal{L}^{\overline{a}}([T, +\infty), \mathcal{L}^{\overline{r}})} < \delta \quad (6.10)$$

for some $T > 0$, then the solution scatters in \mathcal{H}^1 forward in time, i.e., there exists $(u_+, v_+) \in \mathcal{H}^1$ such that

$$\lim_{t \rightarrow +\infty} \|(u(t), v(t)) - (\mathcal{S}_1(t)u_+, \mathcal{S}_2(t)v_+)\|_{\mathcal{H}^1} = 0.$$

A similar statement holds for the negative time direction.

Proof. Using the Duhamel formula

$$(u(t), v(t)) = (\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T)) + i \int_T^t (\mathcal{S}_1(t-\tau)|x|^{-\alpha}\overline{u}(\tau)v(\tau), \mathcal{S}_2(t-\tau)|x|^{-\alpha}u^2(\tau)) \, d\tau,$$

Strichartz estimates, and (6.3), we have

$$\begin{aligned} \|(u, v)\|_{\mathcal{L}^{\overline{a}}(I, \mathcal{L}^{\overline{r}})} &\leq \|(\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T))\|_{\mathcal{L}^{\overline{a}}(I, \mathcal{L}^{\overline{r}})} + C\|(|x|^{-\alpha}\overline{u}v, |x|^{-\alpha}u^2)\|_{L^{\overline{a}'}(I, \mathcal{L}^{\overline{r}'})} \\ &\leq \|(\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T))\|_{\mathcal{L}^{\overline{a}}(I, \mathcal{L}^{\overline{r}})} + C\|u\|_{L_t^\infty(I, H_x^1)}^\theta \|u\|_{L_t^{\overline{a}}(I, L_x^{\overline{r}})}^{1-\theta} \|v\|_{L_t^{\overline{a}}(I, L_x^{\overline{r}})} \\ &\quad + C\|u\|_{L_t^\infty(I, H_x^1)}^\theta \|u\|_{L_t^{\overline{a}}(I, L_x^{\overline{r}})}^{2-\theta} \\ &\leq \|(\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T))\|_{\mathcal{L}^{\overline{a}}(I, \mathcal{L}^{\overline{r}})} + C\|(u, v)\|_{\mathcal{L}^\infty(I, \mathcal{H}^1)}^\theta \|(u, v)\|_{\mathcal{L}^{\overline{a}}(I, \mathcal{L}^{\overline{r}})}^{2-\theta}, \end{aligned}$$

where $I = [T, +\infty)$. A similar estimate goes for $\|(u, v)\|_{\mathcal{L}^{\dot{a}}(I, \mathcal{L}^{\dot{r}})}$ and we get

$$\begin{aligned} \|(u, v)\|_{\mathcal{L}^{\bar{\alpha}}(I, \mathcal{L}^{\bar{r}}) \cap \mathcal{L}^{\dot{a}}(I, \mathcal{L}^{\dot{r}})} &\leq \|(\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T))\|_{\mathcal{L}^{\bar{\alpha}}(I, \mathcal{L}^{\bar{r}}) \cap \mathcal{L}^{\dot{a}}(I, \mathcal{L}^{\dot{r}})} \\ &\quad + CE^\theta \|(u, v)\|_{\mathcal{L}^{\bar{\alpha}}(I, \mathcal{L}^{\bar{r}}) \cap \mathcal{L}^{\dot{a}}(I, \mathcal{L}^{\dot{r}})}^{2-\theta} \\ &\leq \delta + CE^\theta \|(u, v)\|_{\mathcal{L}^{\bar{\alpha}}(I, \mathcal{L}^{\bar{r}}) \cap \mathcal{L}^{\dot{a}}(I, \mathcal{L}^{\dot{r}})}^{2-\theta}. \end{aligned}$$

Taking $\delta > 0$ small depending on E , the continuity argument yields

$$\|(u, v)\|_{\mathcal{L}^{\bar{\alpha}}(I, \mathcal{L}^{\bar{r}}) \cap \mathcal{L}^{\dot{a}}(I, \mathcal{L}^{\dot{r}})} \lesssim \delta. \quad (6.11)$$

Next using (6.5), we have

$$\begin{aligned} \|(\langle \nabla \rangle u, \langle \nabla \rangle v)\|_{\mathcal{L}^{\bar{q}}(I, \mathcal{L}^{\bar{r}})} &\leq \|(\langle \nabla \rangle \mathcal{S}_1(t-T)u(T), \langle \nabla \rangle \mathcal{S}_2(t-T)v(T))\|_{\mathcal{L}^{\bar{q}}(I, \mathcal{L}^{\bar{r}})} \\ &\quad + \left\| \int_T^t (\langle \nabla \rangle \mathcal{S}_1(t-\tau)|x|^{-\alpha}\bar{u}(\tau)v(\tau), \langle \nabla \rangle \mathcal{S}_2(t-\tau)|x|^{-\alpha}u^2(\tau)) \, d\tau \right\|_{\mathcal{L}^{\bar{q}}(I, \mathcal{L}^{\bar{r}})} \\ &\leq C\|(\langle \nabla \rangle u(T), \langle \nabla \rangle v(T))\|_{\mathcal{L}^2} + C\|(\langle \nabla \rangle (|x|^{-\alpha}\bar{u}v), \langle \nabla \rangle (|x|^{-\alpha}u^2))\|_{\mathcal{L}^{q'}(I, \mathcal{L}^{r'})} \\ &\leq C\|(u(T), u(T))\|_{\mathcal{H}^1} + C\|u\|_{L_t^\infty(I, H_x^1)}^\theta \|u\|_{L_t^{\frac{1-\theta}{\bar{q}}}(I, L_x^{\bar{r}})}^{1-\theta} \|\langle \nabla \rangle v\|_{L_t^{\bar{q}}(I, L_x^{\bar{r}})} \\ &\quad + C\|v\|_{L_t^\infty(I, H_x^1)}^\theta \|v\|_{L_t^{\frac{1-\theta}{\bar{q}}}(I, L_x^{\bar{r}})}^{1-\theta} \|\langle \nabla \rangle u\|_{L_t^{\bar{q}}(I, L_x^{\bar{r}})} \\ &\quad + C\|u\|_{L_t^\infty(I, H_x^1)}^\theta \|u\|_{L_t^{\frac{1-\theta}{\bar{q}}}(I, L_x^{\bar{r}})}^{1-\theta} \|\langle \nabla \rangle u\|_{L_t^{\bar{q}}(I, L_x^{\bar{r}})} \\ &\leq C\|(u(T), u(T))\|_{\mathcal{H}^1} + C\|(u, v)\|_{\mathcal{L}^\infty(I, \mathcal{H}^1)}^\theta \|(u, v)\|_{\mathcal{L}^{\bar{\alpha}}(I, \mathcal{L}^{\bar{r}})}^{1-\theta} \|(\langle \nabla \rangle u, \langle \nabla \rangle v)\|_{\mathcal{L}^{\bar{q}}(I, \mathcal{L}^{\bar{r}})} \\ &\leq CE + CE^\theta \delta^{1-\theta} \|(\langle \nabla \rangle u, \langle \nabla \rangle v)\|_{\mathcal{L}^{\bar{q}}(I, \mathcal{L}^{\bar{r}})}. \end{aligned}$$

Taking $\delta > 0$ sufficiently small depending on E , we get

$$\|(\langle \nabla \rangle u, \langle \nabla \rangle v)\|_{\mathcal{L}^{\bar{q}}(I, \mathcal{L}^{\bar{r}})} \lesssim E. \quad (6.12)$$

Now let $t_2 > t_1 > T$. By Strichartz estimates and (6.5), we have

$$\begin{aligned} \|(\mathcal{S}_1(-t_2)u(t_2), \mathcal{S}_2(-t_2)v(t_2)) - (\mathcal{S}_1(-t_1)u(t_1), \mathcal{S}_2(-t_1)v(t_1))\|_{\mathcal{H}^1} \\ = \left\| \int_{t_1}^{t_2} (\mathcal{S}_1(-\tau)|x|^{-\alpha}\bar{u}(\tau)v(\tau), \mathcal{S}_2(-\tau)|x|^{-\alpha}u^2(\tau)) \, d\tau \right\|_{\mathcal{H}^1} \\ \lesssim \|(\langle \nabla \rangle (|x|^{-\alpha}\bar{u}v), \langle \nabla \rangle (|x|^{-\alpha}u^2))\|_{L^{q'}((t_1, t_2), \mathcal{L}^{r'})} \\ \lesssim \|(u, v)\|_{\mathcal{L}^\infty((t_1, t_2), \mathcal{H}^1)}^\theta \|(u, v)\|_{\mathcal{L}^{\bar{\alpha}}((t_1, t_2), \mathcal{L}^{\bar{r}})}^{1-\theta} \|(\langle \nabla \rangle u, \langle \nabla \rangle v)\|_{\mathcal{L}^{\bar{q}}((t_1, t_2), \mathcal{L}^{\bar{r}})}. \end{aligned}$$

Thanks to (6.9), (6.11), and (6.12), we see that

$$\|(\mathcal{S}_1(-t_2)u(t_2), \mathcal{S}_2(-t_2)v(t_2)) - (\mathcal{S}_1(-t_1)u(t_1), \mathcal{S}_2(-t_1)v(t_1))\|_{\mathcal{H}^1} \rightarrow 0 \text{ as } t_1, t_2 \rightarrow +\infty$$

hence $(\mathcal{S}_1(-t)u(t), \mathcal{S}_2(-t)v(t))_{t \rightarrow +\infty}$ is a Cauchy sequence in \mathcal{H}^1 . Thus there exists $(u_+, v_+) \in \mathcal{H}^1$ such that

$$(\mathcal{S}_1(-t)u(t), \mathcal{S}_2(-t)v(t)) \rightarrow (u_+, v_+) \text{ strongly in } \mathcal{H}^1.$$

In particular, we have

$$(u_+(t), v_+(t)) = (\mathcal{S}_1(-T)u(T), \mathcal{S}_2(-T)v(T)) + i \int_T^{+\infty} (\mathcal{S}_1(-\tau)|x|^{-\alpha}\bar{u}(\tau)v(\tau), \mathcal{S}_2(-\tau)|x|^{-\alpha}u^2(\tau)) \, d\tau.$$

Repeating the same argument as above, we prove as well that

$$\|(u(t), v(t)) - (\mathcal{S}_1(t)u_+, \mathcal{S}_2(t)v_+)\|_{\mathcal{H}^1} \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

The proof is complete. \square

Lemma 6.7 (Scattering criterion). *Let $3 \leq d \leq 5$, $0 < \alpha < \min\{2, \frac{d}{2}\}$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Suppose that (u, v) is a global \mathcal{H}^1 -solution to (1.1) satisfying (6.9) for some constant $E > 0$. Then there exist $\varepsilon = \varepsilon(E) > 0$ and $R = R(E) > 0$ such that if*

$$\liminf_{t \rightarrow +\infty} \int_{|x| \leq R} |u(t, x)|^2 + 2|v(t, x)|^2 \, dx \leq \varepsilon^2, \quad (6.13)$$

then the solution scatters in \mathcal{H}^1 forward in time. A similar statement holds for the negative time direction.

Proof. Using Lemma 6.6, it suffices to check that the smallness condition (6.10) holds. We prove this by following the argument of Murphy [34] (see also [7]) which is inspired by earlier works [14, 43]. Let $\varepsilon > 0$ be a small constant. Let $T > 0$ be a large time to be chosen later depending on ε . We will show that

$$\|(\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T))\|_{\mathcal{L}^{\bar{\alpha}}([T,+\infty), \mathcal{L}^{\bar{\tau}}) \cap \mathcal{L}^{\hat{\alpha}}([T,+\infty), \mathcal{L}^{\hat{\tau}})} \lesssim \varepsilon^\rho \quad (6.14)$$

for some constant $\rho > 0$. To this end, we write, using the Duhamel formula, for $T > \varepsilon^{-\beta}$ with some $\beta > 0$ to be determined later,

$$\begin{aligned} & (\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T)) \\ &= (\mathcal{S}_1(t)u_0, \mathcal{S}_2(t)v_0) + i \int_0^T (\mathcal{S}_1(t-\tau)|x|^{-\alpha}\bar{u}(\tau)v(\tau), \mathcal{S}_2(t-\tau)|x|^{-\alpha}u^2(\tau)) \, d\tau \\ &= \mathcal{F}_0(t) + \mathcal{F}_1(t) + \mathcal{F}_2(t), \end{aligned} \quad (6.15)$$

where (u_0, v_0) is initial data at time $t = 0$ and

$$\begin{aligned} \mathcal{F}_0(t) &:= (\mathcal{S}_1(t)u_0, \mathcal{S}_2(t)v_0), \\ \mathcal{F}_1(t) &:= i \int_{I_1} (\mathcal{S}_1(t-\tau)|x|^{-\alpha}\bar{u}(\tau)v(\tau), \mathcal{S}_2(t-\tau)|x|^{-\alpha}u^2(\tau)) \, d\tau, \\ \mathcal{F}_2(t) &:= i \int_{I_2} (\mathcal{S}_1(t-\tau)|x|^{-\alpha}\bar{u}(\tau)v(\tau), \mathcal{S}_2(t-\tau)|x|^{-\alpha}u^2(\tau)) \, d\tau, \end{aligned}$$

with $I_1 = [0, T - \varepsilon^{-\beta}]$ and $I_2 = [T - \varepsilon^{-\beta}, T]$. The terms \mathcal{F}_0 , \mathcal{F}_1 , and \mathcal{F}_2 are referred as the linear, the distance past, and the recent past parts respectively.

Step 1. The linear part. To estimate the linear part, we use Strichartz estimates to have

$$\|\mathcal{F}_0\|_{\mathcal{L}^{\bar{\alpha}}([0,+\infty), \mathcal{L}^{\bar{\tau}}) \cap \mathcal{L}^{\hat{\alpha}}([0,+\infty), \mathcal{L}^{\hat{\tau}})} \lesssim \|(u_0, v_0)\|_{\mathcal{H}^1}.$$

By the monotone convergence theorem, there exists $T > \varepsilon^{-\beta}$ large so that

$$\|\mathcal{F}_0\|_{\mathcal{L}^{\bar{\alpha}}([T,+\infty), \mathcal{L}^{\bar{\tau}}) \cap \mathcal{L}^{\hat{\alpha}}([T,+\infty), \mathcal{L}^{\hat{\tau}})} \lesssim \varepsilon. \quad (6.16)$$

Step 2. The distance past part. We first observe that there exists $(\bar{b}, \bar{\varepsilon}) \in \mathcal{A}_0$ such that

$$\frac{1}{\bar{a}} = \frac{1-s_c}{\bar{b}} + \theta s_c, \quad \frac{1}{\bar{r}} = \frac{1-s_c}{\bar{\varepsilon}} + \frac{d-2-4\theta}{2d} s_c.$$

In fact, as $(\bar{a}, \bar{r}) \in \mathcal{A}_{s_c}$, we readily see that $\frac{2}{\bar{b}} + \frac{d}{\bar{\varepsilon}} = \frac{d}{2}$. To ensure $(\bar{b}, \bar{\varepsilon}) \in \mathcal{A}_0$, it remains to check that $2 < \bar{\varepsilon} < \frac{2d}{d-2}$ or equivalently $\bar{b} > 2$. The later is equivalent to

$$\frac{1-s_c}{\frac{1}{\bar{a}} - \theta s_c} > 2 \iff \frac{6-d-2\alpha}{\frac{6-d-2\alpha+\theta}{2(3-\theta)} - \theta(d-4+2\alpha)} > 2.$$

The limit as $\theta \rightarrow 0$ of the right hand side is strictly larger than 2, so the above condition is satisfied by taking $\theta > 0$ small. By the Hölder inequality, we have

$$\|\mathcal{F}_1\|_{\mathcal{L}^{\bar{\alpha}}([T,+\infty), \mathcal{L}^{\bar{\tau}})} \leq \|\mathcal{F}_1\|_{\mathcal{L}^{\bar{b}}([T,+\infty), \mathcal{L}^{\bar{\varepsilon}})}^{1-s_c} \|\mathcal{F}_1\|_{\mathcal{L}^{\frac{1}{\bar{\theta}}}([T,+\infty), \mathcal{L}^{\frac{2d}{d-2-4\theta}})}^{s_c}.$$

Using the fact that

$$\mathcal{F}_1(t) = (\mathcal{S}_1(t-T+\varepsilon^{-\beta})u(T-\varepsilon^{-\beta}) - \mathcal{S}_1(t)u_0, \mathcal{S}_2(t-T+\varepsilon^{-\beta})v(T-\varepsilon^{-\beta}) - \mathcal{S}_2(t)v_0),$$

Strichartz estimates and (6.9) imply

$$\|\mathcal{F}_1\|_{\mathcal{L}^{\bar{b}}([T,+\infty), \mathcal{L}^{\bar{\varepsilon}})} \lesssim 1.$$

On the other hand, by dispersive estimates (6.1), (6.7), and (6.9), we have for $t \geq T$,

$$\begin{aligned} \|\mathcal{F}_1(t)\|_{L_x^{\frac{2d}{d-2-4\theta}}} &\lesssim \int_0^{T-\varepsilon^{-\beta}} (t-\tau)^{-1-2\theta} \|(|x|^{-\alpha}\bar{u}v, |x|^{-\alpha}u^2)\|_{\mathcal{L}^{\frac{2d}{d+2+4\theta}}} \, d\tau \\ &\lesssim \int_0^{T-\varepsilon^{-\beta}} (t-s)^{-1-2\theta} \|(u(\tau), v(\tau))\|_{\mathcal{H}^1}^2 \, d\tau \\ &\lesssim (t-T+\varepsilon^{-\beta})^{-2\theta}. \end{aligned}$$

It follows that

$$\|\mathcal{F}_1\|_{\mathcal{L}^{\frac{1}{\bar{\theta}}}([T,+\infty), \mathcal{L}^{\frac{2d}{d-2-4\theta}})} \lesssim \|(t-T+\varepsilon^{-\beta})^{-2\theta}\|_{L_t^{\frac{1}{\bar{\theta}}}([T,+\infty))} \lesssim \varepsilon^{\beta\theta}.$$

In particular, we get

$$\|\mathcal{F}_1\|_{\mathcal{L}^{\bar{\alpha}}([T,+\infty), \mathcal{L}^{\bar{\tau}})} \lesssim \varepsilon^{\beta\theta s_c}.$$

The estimate for $\|\mathcal{F}_1\|_{\mathcal{L}^{\hat{a}}([T,+\infty),\mathcal{L}^{\hat{r}})}$ is treated in a similar manner. In fact, we also observe that there exists $(\hat{b}, \hat{e}) \in \mathcal{A}_0$ such that

$$\frac{1}{\hat{a}} = \frac{1-s_c}{\hat{b}} + \theta s_c, \quad \frac{1}{\hat{r}} = \frac{1-s_c}{\hat{e}} + \frac{d-2-4\theta}{2d} s_c.$$

Note that the condition $\hat{b} > 2$ is equivalent to

$$\frac{1-s_c}{\frac{1}{\hat{a}} - \theta s_c} > 2 \iff \frac{6-d-2\alpha}{\frac{6-d-2\alpha}{3-\theta} - \theta(d-4+2\alpha)} > 2$$

which is clearly fulfilled for $\theta > 0$ small. Estimating as above, we arrive at

$$\|\mathcal{F}_1\|_{\mathcal{L}^{\hat{a}}([T,+\infty),\mathcal{L}^{\hat{r}})} \lesssim \varepsilon^{\beta\theta s_c}$$

hence

$$\|\mathcal{F}_1\|_{\mathcal{L}^{\bar{a}}([T,+\infty),\mathcal{L}^{\bar{r}}) \cap \mathcal{L}^{\hat{a}}([T,+\infty),\mathcal{L}^{\hat{r}})} \lesssim \varepsilon^{\beta\theta s_c}. \quad (6.17)$$

Step 3. The recent past part. Let $R > 0$ be a large parameter depending on ε to be determined shortly. Using (6.13) and enlarging T if necessary, we have

$$\int_{|x| \leq R} (|u(T, x)|^2 + 2|v(T, x)|^2) dx \leq \varepsilon^2,$$

hence

$$\int_{\mathbb{R}^d} \varrho_R(x) (|u(T, x)|^2 + 2|v(T, x)|^2) dx \leq \varepsilon^2,$$

where $\varrho_R(x) = \varrho(x/R)$ with $C_0^\infty(\mathbb{R}^d)$ satisfying $0 \leq \varrho \leq 1$ and

$$\varrho(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| \geq 1. \end{cases} \quad (6.18)$$

By Lemma 5.3, we have from (6.9) that

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{R}^d} \varrho_R (|u(t)|^2 + 2|v(t)|^2) dx \right| &= \left| \Im \int_{\mathbb{R}^d} \nabla \varrho_R \cdot (\nabla u(t) \bar{u}(t) + \kappa \nabla v(t) \bar{v}(t)) dx \right| \\ &\leq \|\nabla \varrho_R\|_{L_x^\infty} (\|\nabla u(t)\|_{L_x^2} \|u(t)\|_{L_x^2} + \kappa \|\nabla v(t)\|_{L_x^2} \|v(t)\|_{L_x^2}) \\ &\lesssim R^{-1}. \end{aligned}$$

Thus we have for all $t \in I_2 = [T - \varepsilon^{-\beta}, T]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varrho_R(x) (|u(t, x)|^2 + 2|v(t, x)|^2) dx &= \int_{\mathbb{R}^d} \varrho_R(x) (|u(T, x)|^2 + 2|v(T, x)|^2) dx \\ &\quad + \int_t^T \left(\frac{d}{d\tau} \int_{\mathbb{R}^d} \varrho_R(x) (|u(\tau, x)|^2 + 2|v(\tau, x)|^2) dx \right) d\tau \\ &\leq \varepsilon^2 + CR^{-1}(T-t) \\ &\leq \varepsilon^2 + CR^{-1}\varepsilon^{-\beta} \\ &\lesssim \varepsilon^2 \end{aligned}$$

provided that $R \sim \varepsilon^{-2-\beta}$. In particular, we have

$$\sup_{t \in I_2} \int_{\mathbb{R}^d} \varrho_R(x) (|u(t, x)|^2 + 2|v(t, x)|^2) dx \lesssim \varepsilon^2$$

hence

$$\sup_{t \in I_2} \|(\varrho_R u(t), \varrho_R v(t))\|_{\mathcal{L}^2} \lesssim \varepsilon^2$$

as $\varrho_R^2 \leq \varrho_R$.

On the other hand, we have from (6.6) and (6.9) that

$$\begin{aligned}
\|\varrho_R|x|^{-\alpha}\bar{u}(t)v(t)\|_{L_x^{\hat{r}'}} &\leq \| |x|^{-\alpha} \|_{L_x^{\gamma}(|x|\leq R)} \|u(t)\|_{L_x^{\frac{\theta}{m_+}}}^\theta \|u(t)\|_{L_x^{\hat{r}}}^{1-\theta} \|\varrho_R v(t)\|_{L_x^{\hat{r}}} \\
&\lesssim R^{d\theta^2} \|u(t)\|_{H_x^1} \|\varrho_R v(t)\|_{L_x^{\hat{r}}} \\
&\lesssim \varepsilon^{-d(2+\beta)\theta^2} \|u(t)\|_{H_x^1} \|\varrho_R v(t)\|_{L_x^2}^{\frac{2d-(d-2)\hat{r}}{2\hat{r}}} \|\varrho_R v(t)\|_{L_x^{\frac{d(\hat{r}-2)}{2\hat{r}}}}^{\frac{2d}{L_x^{\frac{d-2}{2\hat{r}}}}} \\
&\lesssim \varepsilon^{-d(2+\beta)\theta^2} \|u(t)\|_{H_x^1} \|v(t)\|_{H_x^1}^{\frac{d(\hat{r}-2)}{2\hat{r}}} \|\varrho_R v(t)\|_{L_x^2}^{\frac{2d-(d-2)\hat{r}}{2\hat{r}}} \\
&\lesssim \varepsilon^{\frac{2d-(d-2)\hat{r}}{2\hat{r}}-d(2+\beta)\theta^2}, \quad \forall t \in I_2,
\end{aligned}$$

where we recall that $\frac{1}{\gamma} = \frac{\alpha}{d} + \theta^2$ inside the ball. We also have from (6.6) and (6.9) that

$$\begin{aligned}
\|(1 - \varrho_R)|x|^{-\alpha}\bar{u}(t)v(t)\|_{L_x^{\hat{r}'}} &\lesssim \| |x|^{-\alpha}\bar{u}(t)v(t) \|_{L_x^{\hat{r}'}(|x|\geq R/2)} \\
&\leq \| |x|^{-\alpha} \|_{L_x^{\gamma}(|x|\geq R/2)} \|u(t)\|_{L_x^{\frac{\theta}{m_-}}}^\theta \|u(t)\|_{L_x^{\hat{r}}}^{1-\theta} \|v(t)\|_{L_x^{\hat{r}}} \\
&\lesssim R^{-d\theta^2} \|u(t)\|_{H_x^1} \|v(t)\|_{H_x^1} \\
&\lesssim R^{-d\theta^2} \\
&\lesssim \varepsilon^{d(2+\beta)\theta^2}, \quad \forall t \in I_2,
\end{aligned}$$

where $\frac{1}{\gamma} = \frac{\alpha}{d} - \theta^2$ outside the ball. Thus we get

$$\| |x|^{-\alpha}\bar{u}(t)v(t) \|_{L_x^{\hat{r}'}} \lesssim \varepsilon^{\frac{2d-(d-2)\hat{r}}{2\hat{r}}-d(2+\beta)\theta^2} + \varepsilon^{d(2+\beta)\theta^2}, \quad \forall t \in I_2.$$

A similar estimate goes for $\| |x|^{-\alpha}u^2(t) \|_{L_x^{\hat{r}'}}$ and we obtain

$$\| (|x|^{-\alpha}\bar{u}v, |x|^{-\alpha}u^2) \|_{\mathcal{L}^\infty(I_2, \mathcal{L}^{\hat{r}'})} \lesssim \varepsilon^{\frac{2d-(d-2)\hat{r}}{2\hat{r}}-d(2+\beta)\theta^2} + \varepsilon^{d(2+\beta)\theta^2}.$$

By Strichartz estimates, we have

$$\begin{aligned}
\|\mathcal{F}_2\|_{\mathcal{L}^{\hat{a}}([T,+\infty), \mathcal{L}^{\hat{r}}) \cap \mathcal{L}^{\bar{a}}([T,+\infty), \mathcal{L}^{\bar{r}})} &\lesssim \| (|x|^{-\alpha}\bar{u}v, |x|^{-\alpha}u^2) \|_{\mathcal{L}^{\hat{a}'}(I_2, \mathcal{L}^{\hat{r}'})} \\
&\lesssim |I_2|^{\frac{1}{\hat{a}'}} \| (|x|^{-\alpha}\bar{u}v, |x|^{-\alpha}u^2) \|_{\mathcal{L}^\infty(I_2, \mathcal{L}^{\hat{r}'})} \\
&\lesssim \varepsilon^{\frac{2d-(d-2)\hat{r}}{2\hat{r}}-d(2+\beta)\theta^2 - \frac{\hat{\beta}}{\hat{a}'}} + \varepsilon^{d(2+\beta)\theta^2 - \frac{\hat{\beta}}{\hat{a}'}}.
\end{aligned}$$

Now taking $\beta = \frac{2d\theta^2\hat{a}'}{2-d\theta^2\hat{a}'}$, we get

$$\|\mathcal{F}_2\|_{\mathcal{L}^{\hat{a}}([T,+\infty), \mathcal{L}^{\hat{r}}) \cap \mathcal{L}^{\bar{a}}([T,+\infty), \mathcal{L}^{\bar{r}})} \lesssim \varepsilon^{\frac{2d-(d-2)\hat{r}}{2\hat{r}} - \frac{6d\theta^2}{2-d\theta^2\hat{a}'}} + \varepsilon^{\frac{2d\theta^2\hat{a}'}{2-d\theta^2\hat{a}'}}. \quad (6.19)$$

Collecting (6.15), (6.16), (6.17), and (6.19), we obtain

$$\| (\mathcal{S}_1(t-T)u(T), \mathcal{S}_2(t-T)v(T)) \|_{\mathcal{L}^{\hat{a}}([T,+\infty), \mathcal{L}^{\hat{r}}) \cap \mathcal{L}^{\bar{a}}([T,+\infty), \mathcal{L}^{\bar{r}})} \lesssim \varepsilon + \varepsilon^{\frac{2d\theta^3\hat{a}'s_c}{2-d\theta^2\hat{a}'}} + \varepsilon^{\frac{2d-(d-2)\hat{r}}{2\hat{r}} - \frac{6d\theta^2}{2-d\theta^2\hat{a}'}} + \varepsilon^{\frac{2d\theta^2\hat{a}'}{2-d\theta^2\hat{a}'}}$$

for some $T > \varepsilon^{-\frac{2d\theta^2\hat{a}'}{2-d\theta^2\hat{a}'}}$ which proves (6.14) by choosing $\theta > 0$ sufficiently small. The proof is complete. \square

6.4. Energy scattering.

Proposition 6.8 (Coercivity property). *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. Let (φ, ψ) be a nonlinear ground state related to (3.2). Let $(u_0, v_0) \in \mathcal{H}^1$ satisfy (4.3) and (4.4). Then there exists $R_0 = R_0(\kappa, u_0, v_0, \varphi, \psi) > 0$ such that the corresponding solution to (1.1) satisfies for all $R \geq R_0$,*

$$\mathbb{G}(\varrho_R u(t), \varrho_R v(t)) \geq \delta \int_{\mathbb{R}^d} |x|^{-\alpha} (|\varrho_R(x)u(t, x)|^3 + |\varrho_R(x)v(t, x)|^3) dx, \quad t \in \mathbb{R} \quad (6.20)$$

for some constant $\delta > 0$ depending on $\kappa, u_0, v_0, \varphi$, and ψ , where ϱ_R is as in (6.18).

Proof. The proof is divided into several steps.

Step 1. A uniform bound. We first show that there exists $\rho = \rho(u_0, v_0, \varphi, \psi) \in (0, 1)$ such that

$$\mathbb{K}(u(t), v(t))(\mathbb{M}(u(t), v(t)))^\sigma \leq (1 - \rho)\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall t \in \mathbb{R}. \quad (6.21)$$

We first observe that for $(u_0, v_0) \neq (0, 0)$ satisfying (4.3) and (4.4), we have $\mathbb{H}(u_0, v_0) > 0$. To see this, let g be as in (4.6). Thanks to (4.10), we readily see that $g'(\lambda_0) = 0$ with

$$\lambda_0 = \left(\frac{2}{(d+2\alpha)C_{\text{GN}}} \right)^{\frac{4}{d+2\alpha-4}} = \mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma$$

and g is strictly increasing on $(0, \lambda_0)$. This together with $(u_0, v_0) \neq (0, 0)$ and (4.4) yield

$$\mathbb{H}(u_0, v_0)(\mathbb{M}(u_0, v_0))^\sigma \geq g(\mathbb{K}(u_0, v_0)(\mathbb{M}(u_0, v_0))^\sigma) > g(0) = 0$$

which implies $\mathbb{H}(u_0, v_0) > 0$.

Now, using (4.3), we take $\vartheta = \vartheta(u_0, v_0, \varphi, \psi) \in (0, 1)$ so that

$$\mathbb{H}(u_0, v_0)(\mathbb{M}(u_0, v_0))^\sigma \leq (1 - \vartheta)\mathbb{E}_0(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma.$$

As (see the proof of Proposition 4.3)

$$g(\mathbb{K}(u(t), v(t))(\mathbb{M}(u(t), v(t))))^\sigma \leq \mathbb{H}(u_0, v_0)(\mathbb{M}(u_0, v_0))^\sigma, \quad \forall t \in \mathbb{R},$$

we use the fact that

$$\mathbb{E}_0(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma = \frac{d+2\alpha-4}{2(d+2\alpha)}\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma = \frac{d+2\alpha-4}{4}C_{\text{GN}}(\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma)^{\frac{d+2\alpha}{4}}$$

to get

$$\begin{aligned} & \frac{d+2\alpha}{d+2\alpha-4} \frac{\mathbb{K}(u(t), v(t))(\mathbb{M}(u(t), v(t)))^\sigma}{\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma} \\ & - \frac{4}{d+2\alpha-4} \left(\frac{\mathbb{K}(u(t), v(t))(\mathbb{M}(u(t), v(t)))^\sigma}{\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma} \right)^{\frac{d+2\alpha}{4}} \leq 1 - \vartheta, \quad \forall t \in \mathbb{R}. \end{aligned}$$

Denote

$$\lambda(t) := \frac{\mathbb{K}(u(t), v(t))(\mathbb{M}(u(t), v(t)))^\sigma}{\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma}.$$

We see that $\lambda(t) \in (0, 1)$ for all $t \in \mathbb{R}$ due to (4.9) and

$$h(\lambda(t)) \leq 1 - \vartheta, \quad \forall t \in \mathbb{R},$$

where

$$h(\lambda) := \frac{d+2\alpha}{d+2\alpha-4}\lambda - \frac{4}{d+2\alpha-4}\lambda^{\frac{d+2\alpha}{4}}.$$

As h is continuous on $[0, 1]$, $h(0) = 0$, and $h(1) = 1$, there exists $\rho \in (0, 1)$ such that $h(1 - \rho) = 1 - \vartheta$. In particular, we have

$$h(\lambda(t)) \leq h(1 - \rho), \quad \forall t \in \mathbb{R}.$$

Since h is strictly increasing on $(0, 1)$, we infer that $\lambda(t) \leq 1 - \rho$ for all $t \in \mathbb{R}$ and (6.21) follows.

Step 2. A truncated uniform bound. There exists $R_0 = R_0(\kappa, u_0, v_0, \varphi, \psi) > 0$ such that for all $R \geq R_0$,

$$\mathbb{K}(\varrho_R u(t), \varrho_R v(t))(\mathbb{M}(\varrho_R u(t), \varrho_R v(t)))^\sigma \leq \left(1 - \frac{\rho}{2}\right) \mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall t \in \mathbb{R}. \quad (6.22)$$

In fact, as $0 \leq \varrho_R \leq 1$, we have $\mathbb{M}(\varrho_R u(t), \varrho_R v(t)) \leq \mathbb{M}(u(t), v(t))$ for all $R > 0$ and all $t \in \mathbb{R}$. On the other hand, by integration by parts, we have

$$\int_{\mathbb{R}^d} |\nabla(\varrho f)|^2 dx = \int_{\mathbb{R}^d} \varrho^2 |\nabla f|^2 dx - \int_{\mathbb{R}^d} \varrho \Delta \varrho |f|^2 dx$$

which implies

$$\begin{aligned} \mathbb{K}(\varrho_R u(t), \varrho_R v(t)) &= \int_{\mathbb{R}^d} \varrho_R^2 (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) dx - \int_{\mathbb{R}^d} \varrho_R \Delta \varrho_R (|u(t)|^2 + \kappa |v(t)|^2) dx \\ &\leq \mathbb{K}(u(t), v(t)) + C(\kappa)R^{-2}\mathbb{M}(u(t), v(t)). \end{aligned}$$

By the conservation of mass and (6.21), we get

$$\begin{aligned} \mathbb{K}(\varrho_R u(t), \varrho_R v(t))(\mathbb{M}(\varrho_R u(t), \varrho_R v(t)))^\sigma &\leq \mathbb{K}(u(t), v(t))(\mathbb{M}(u(t), v(t)))^\sigma + C(\kappa)R^{-2}(\mathbb{M}(u(t), v(t)))^{\sigma+1} \\ &\leq (1 - \rho)\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma + C(\kappa)R^{-2}(\mathbb{M}(u_0, v_0))^{\sigma+1} \\ &\leq \left(1 - \frac{\rho}{2}\right) \mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma, \quad \forall R \geq R_0, \quad \forall t \in \mathbb{R} \end{aligned}$$

provided that $R_0 > 0$ is taken sufficiently large depending on ρ, κ , and $\mathbb{M}(u_0, v_0)$. This shows (6.22).

Step 3. A coercivity estimate. We claim that if

$$\mathbb{K}(f, g)(\mathbb{M}(f, g))^\sigma \leq (1 - \nu)\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma$$

for some $0 < \nu < 1$, then there exists $\delta = \delta(\nu, \kappa, \varphi, \psi) > 0$ such that

$$\mathbb{G}(f, g) \geq \delta \int_{\mathbb{R}^d} |x|^{-\alpha} (|f|^3 + |g|^3) dx.$$

Thanks to this claim, the desired estimate (6.20) follows immediately from (6.22). To prove the claim, we write

$$\mathbb{G}(f, g) = \mathbb{K}(f, g) - \frac{d+2\alpha}{2}\mathbb{P}(f, g) = \frac{d+2\alpha}{2}\mathbb{E}_0(f, g) - \frac{d+2\alpha-4}{4}\mathbb{K}(f, g).$$

Using the Gagliardo–Nirenberg inequality (3.1) and (4.10), we have

$$\begin{aligned} \mathbb{E}_0(f, g) &\geq \frac{1}{2}\mathbb{K}(f, g) - C_{\text{GN}}(\mathbb{K}(f, g))^{\frac{d+2\alpha}{4}}(\mathbb{M}(f, g))^{\frac{6-d-2\alpha}{4}} \\ &= \frac{1}{2}\mathbb{K}(f, g) \left(1 - C_{\text{GN}}(\mathbb{K}(f, g)(\mathbb{M}(f, g))^\sigma)^{\frac{d+2\alpha-4}{4}}\right) \\ &\geq \frac{1}{2}\mathbb{K}(f, g) \left(1 - C_{\text{GN}}((1-\nu)\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma)^{\frac{d+2\alpha-4}{4}}\right) \\ &= \frac{1}{2}\mathbb{K}(f, g) \left(1 - \frac{2}{d+2\alpha}(1-\nu)^{\frac{d+2\alpha-4}{4}}\right). \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{G}(f, g) &\geq \frac{d+2\alpha}{4}\mathbb{K}(f, g) \left(1 - \frac{2}{d+2\alpha}(1-\nu)^{\frac{d+2\alpha-4}{4}}\right) - \frac{d+2\alpha-4}{4}\mathbb{K}(f, g) \\ &= \left(1 - (1-\nu)^{\frac{d+2\alpha-4}{4}}\right)\mathbb{K}(f, g). \end{aligned} \tag{6.23}$$

On the other hand, we have from the standard Gagliardo–Nirenberg inequality (3.3) that

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^{-\alpha} (|f|^3 + |g|^3) dx &\leq C_{\text{gn}} \left(\|\nabla f\|_{L^2}^{\frac{d+2\alpha}{2}} \|f\|_{L^2}^{\frac{6-d-2\alpha}{2}} + \|\nabla g\|_{L^2}^{\frac{d+2\alpha}{2}} \|g\|_{L^2}^{\frac{6-d-2\alpha}{2}} \right) \\ &\leq C(\kappa)(\mathbb{K}(f, g))^{\frac{d+2\alpha}{4}}(\mathbb{M}(f, g))^{\frac{6-d-2\alpha}{4}} \\ &= C(\kappa)\mathbb{K}(f, g)(\mathbb{K}(f, g)(\mathbb{M}(f, g))^\sigma)^{\frac{d+2\alpha-4}{4}} \\ &\leq C(\kappa)((1-\nu)\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma)^{\frac{d+2\alpha-4}{4}}\mathbb{K}(f, g) \end{aligned}$$

which together with (6.23) yields

$$\int_{\mathbb{R}^d} |x|^{-\alpha} (|f|^3 + |g|^3) dx \leq \frac{C(\kappa)((1-\nu)\mathbb{K}(\varphi, \psi)(\mathbb{M}(\varphi, \psi))^\sigma)^{\frac{d+2\alpha-4}{4}}}{\left(1 - (1-\nu)^{\frac{d+2\alpha-4}{4}}\right)} \mathbb{G}(f, g).$$

This proves the claim and the proof is complete. \square

Proof of Theorem 6.1. The solution exists globally in time due to Proposition 4.3. It remains to show the energy scattering. It suffices to consider the positive time direction since the one in the negative time direction is treated in a similar manner. The proof is divided into two steps.

Step 1. A space-time estimate. We first show that there exists $C = C(\kappa, u_0, v_0, \varphi, \psi) > 0$ such that for any time interval $J \subset \mathbb{R}$,

$$\int_J \int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t, x)|^3 + |v(t, x)|^3) dx dt \leq C|J|^{\frac{1}{1+\alpha}}. \tag{6.24}$$

To see this, we introduce a cutoff function $\eta : [0, \infty) \rightarrow [0, 2]$ satisfying

$$\eta(r) = \begin{cases} 2 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

Define the function $\phi : [0, \infty) \rightarrow [0, \infty)$ by

$$\phi(r) := \int_0^r \int_0^s \eta(\tau) d\tau.$$

Let $R > 0$. We define the radial function

$$\phi_R(x) = \phi_R(r) = R^2 \phi(r/R), \quad r = |x| \tag{6.25}$$

and the localized virial quantity

$$\mathcal{M}_{\phi_R}(t) = \Im \int_{\mathbb{R}^d} \nabla \phi_R \cdot (\nabla u(t) \bar{v}(t) + \nabla v(t) \bar{u}(t)) \, dx. \quad (6.26)$$

Using Remark 5.2, we have

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\phi_R}(t) &= \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 \phi_R (|u(t)|^2 + \kappa |v(t)|^2) \, dx + \int_{\mathbb{R}^d} \frac{\phi'_R}{r} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) \, dx \\ &\quad + \int_{\mathbb{R}^d} \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) (|x \cdot \nabla u(t)|^2 + \kappa |x \cdot \nabla v(t)|^2) \, dx \\ &\quad - \frac{1}{2} \Re \int_{\mathbb{R}^d} \left(\phi''_R + (d-1+2\alpha) \frac{\phi'_R}{r} \right) |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx. \end{aligned}$$

As $\phi_R(r) = r^2$ for $0 \leq r \leq R$, we rewrite the above identity as

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\phi_R}(t) &= 2 \int_{|x| \leq R} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) \, dx - (d+2\alpha) \Re \int_{|x| \leq R} |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx \\ &\quad + E_1(u(t), v(t)) + E_2(u(t), v(t)) + E_3(u(t), v(t)), \end{aligned}$$

where

$$\begin{aligned} E_1(u, v) &= \frac{1}{4} \int_{\mathbb{R}^d} \Delta^2 \phi_R (|u|^2 + \kappa |v|^2) \, dx, \\ E_2(u, v) &= \int_{|x| > R} \frac{\phi'_R}{r} (|\nabla u|^2 + \kappa |\nabla v|^2) \, dx + \int_{|x| > R} \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) (|x \cdot \nabla u|^2 + \kappa |x \cdot \nabla v|^2) \, dx, \\ E_3(u, v) &= -\frac{1}{2} \Re \int_{|x| > R} \left(\phi''_R + (d-1+2\alpha) \frac{\phi'_R}{r} \right) |x|^{-\alpha} u^2 \bar{v} \, dx. \end{aligned}$$

By the conservation of mass, we have

$$|E_1(u(t), v(t))| \leq C(\kappa, \mathbb{M}(u_0, v_0)) R^{-2}.$$

As $0 \leq \phi''_R, \frac{\phi'_R}{r} \leq 2$, we have

$$\begin{aligned} \frac{\phi'_R}{r} |\nabla u|^2 + \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) |x \cdot \nabla u|^2 &= \frac{\phi''_R}{r^2} |x \cdot \nabla u|^2 + \frac{\phi'_R}{r} \left(|\nabla u|^2 - \frac{1}{r^2} |x \cdot \nabla u|^2 \right) \\ &\geq \frac{\phi''_R}{r^2} |x \cdot \nabla u|^2 \geq 0. \end{aligned}$$

A similar estimate holds for v and we get $E_2(u(t), v(t)) \geq 0$. We also have

$$\begin{aligned} |E_3(u(t), v(t))| &\leq C \int_{|x| > R} |x|^{-\alpha} |u(t)|^2 |v(t)| \, dx \\ &\leq CR^{-\alpha} \|u(t)\|_{L^3}^2 \|v(t)\|_{L^3} \\ &\leq CR^{-\alpha} \|u(t)\|_{H^1}^2 \|v(t)\|_{H^1} \\ &\leq CR^{-\alpha} \end{aligned}$$

for some constant $C > 0$ depending on $\kappa, u_0, v_0, \varphi$, and ψ . Here we have used the Sobolev embedding $H^1(\mathbb{R}^d) \subset L^3(\mathbb{R}^d)$ for $3 \leq d \leq 5$ and (4.5).

It follows that

$$\frac{d}{dt} \mathcal{M}_{\phi_R}(t) \geq 2 \int_{|x| \leq R} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) \, dx - (d+2\alpha) \Re \int_{|x| \leq R} |x|^{-\alpha} u^2(t) \bar{v}(t) \, dx + CR^{-\alpha}, \quad \forall t \in \mathbb{R}$$

as $\alpha < 2$. Now let ϱ_R be as in (6.18). Observe that

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla(\varrho_R u(t))|^2 \, dx &= \int_{\mathbb{R}^d} \varrho_R^2 |\nabla u(t)|^2 \, dx - \int_{\mathbb{R}^d} \varrho_R \Delta \varrho_R |u(t)|^2 \, dx \\ &= \int_{|x| \leq R} |\nabla u(t)|^2 \, dx - \int_{R/2 < |x| \leq R} (1 - \varrho_R^2) |\nabla u(t)|^2 \, dx - \int_{\mathbb{R}^d} \varrho_R \Delta \varrho_R |u(t)|^2 \, dx \\ &\leq \int_{|x| \leq R} |\nabla u(t)|^2 \, dx - \int_{\mathbb{R}^d} \varrho_R \Delta \varrho_R |u(t)|^2 \, dx \end{aligned}$$

due to $0 \leq \varrho_R \leq 1$. The same estimate goes for v and we get

$$\begin{aligned} \int_{|x| \leq R} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) dx &\geq \int_{\mathbb{R}^d} (|\nabla(\varrho_R u(t))|^2 + \kappa |\nabla(\varrho_R v(t))|^2) dx + \int_{\mathbb{R}^d} \varrho_R \Delta \varrho_R (|u(t)|^2 + \kappa |v(t)|^2) dx \\ &= \mathbb{K}(\varrho_R u(t), \varrho_R v(t)) + O(R^{-2}), \end{aligned}$$

where we have used $\|\Delta \varrho_R\|_{L^\infty} \lesssim R^{-2}$ and the mass conservation to get the second line. In addition, we have

$$\Re \int_{\mathbb{R}^d} |x|^{-\alpha} (\varrho_R u(t))^2 \overline{\varrho_R v(t)} dx = \Re \int_{|x| \leq R} |x|^{-\alpha} u^2(t) \overline{v(t)} dx - \Re \int_{R/2 < |x| \leq R} |x|^{-\alpha} (1 - \varrho_R^3) u^2(t) \overline{v(t)} dx.$$

The last term is estimated as for $E_3(u, v)$ and we obtain

$$\Re \int_{|x| \leq R} |x|^{-\alpha} u^2(t) \overline{v(t)} dx = \mathbb{P}(\varrho_R u(t), \varrho_R v(t)) + O(R^{-\alpha}).$$

In particular, using (6.20), there exists $R_0 = R_0(\kappa, u_0, v_0, \varphi, \psi) > 0$ such that for all $R \geq R_0$,

$$\begin{aligned} 2 \int_{|x| \leq R} (|\nabla u(t)|^2 + \kappa |\nabla v(t)|^2) dx - (d+2\alpha) \Re \int_{|x| \leq R} |x|^{-2} u^2(t) \overline{v(t)} dx \\ \geq 2\mathbb{K}(\varrho_R u(t), \varrho_R v(t)) - (d+2\alpha) \mathbb{P}(\varrho_R u(t), \varrho_R v(t)) + O(R^{-\alpha}) \\ = 2\mathbb{G}(\varrho_R u(t), \varrho_R v(t)) + O(R^{-\alpha}) \\ \geq 2\delta \int_{\mathbb{R}^d} |x|^{-\alpha} (|\varrho_R u(t)|^3 + |\varrho_R v(t)|^3) dx + O(R^{-\alpha}), \quad \forall t \in \mathbb{R}. \end{aligned}$$

We have proved that for all $R \geq R_0$,

$$\frac{d}{dt} \mathcal{M}_{\phi_R}(t) \geq 2\delta \int_{\mathbb{R}^d} |x|^{-\alpha} (|\varrho_R u(t)|^3 + |\varrho_R v(t)|^3) dx + O(R^{-\alpha}), \quad \forall t \in \mathbb{R}.$$

Integrating in time, we get for any time interval $J \subset \mathbb{R}$,

$$\int_J \int_{\mathbb{R}^d} |x|^{-\alpha} (|\varrho_R u(t)|^3 + |\varrho_R v(t)|^3) dx dt \lesssim \sup_{t \in J} |\mathcal{M}_{\phi_R}(t)| + R^{-\alpha} |J|.$$

By the definition of ϱ_R , we get

$$\int_J \int_{|x| \leq R/2} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx dt \lesssim \sup_{t \in J} |\mathcal{M}_{\phi_R}(t)| + R^{-2} |J| \lesssim R + R^{-\alpha} |J|,$$

where we use that

$$|\mathcal{M}_{\phi_R}(t)| \leq \|\nabla \phi_R\|_{L^\infty} (\|\nabla u(t)\|_{L^2} \|v(t)\|_{L^2} + \|\nabla v(t)\|_{L^2} \|u(t)\|_{L^2}) \lesssim R.$$

On the other hand, estimating as for $E_3(u, v)$, we have

$$\int_{|x| > R/2} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx \lesssim R^{-\alpha}$$

which implies

$$\int_J \int_{|x| > R/2} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx dt \lesssim R^{-\alpha} |J|.$$

Collecting these estimates, we get for all $R \geq R_0$,

$$\int_J \int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx dt \lesssim R + R^{-\alpha} |J|.$$

For $|J| \geq R_0^{1+\alpha}$, we take $R = |J|^{\frac{1}{1+\alpha}} \geq R_0$ to get

$$\int_J \int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx dt \lesssim |J|^{\frac{1}{1+\alpha}}.$$

If $|J| \leq R_0^{1+\alpha}$, we use the standard Gagliardo–Nirenberg inequality (3.3) and (4.5) to get

$$\begin{aligned} \int_J \int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx dt &\leq C(\kappa) \int_J (\mathbb{K}(u(t), v(t))^{\frac{d+2\alpha}{4}} (\mathbb{M}(u(t), v(t)))^{\frac{6-d-2\alpha}{4}}) dt \\ &\lesssim |J| \lesssim R_0^\alpha |J|^{\frac{1}{1+\alpha}}. \end{aligned}$$

In all cases, we have proved (6.24).

Step 2. A L^2 -limit. From (6.24), we infer that there exists $t_n \rightarrow +\infty$ such that for any $R > 0$,

$$\lim_{n \rightarrow \infty} \int_{|x| \leq R} |u(t_n, x)|^2 + 2|v(t_n, x)|^2 dx = 0. \quad (6.27)$$

This condition, combined with the scattering criterion given in Lemma 6.7, yields the energy scattering in the positive time direction. To see (6.27), we infer from (6.24) that

$$\liminf_{t \rightarrow +\infty} \int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx = 0.$$

Indeed, if it is not true, then there exist $t_0 > 0$ and $\rho_0 > 0$ such that

$$\int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx \geq \rho_0, \quad \forall t \geq t_0.$$

Take an interval $J \subset [t_0, +\infty)$. We have

$$\int_J \int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t)|^3 + |v(t)|^3) dx dt \geq \rho_0 |J|$$

which contradicts (6.24) for $|J|$ sufficiently large. Thus there exists $t_n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t_n)|^3 + |v(t_n)|^3) dx = 0.$$

Now let $R > 0$. We have

$$\begin{aligned} \int_{|x| \leq R} |u(t_n)|^2 dx &\leq \left(\int_{|x| \leq R} |x|^{2\alpha} dx \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^d} |x|^{-\alpha} |u(t_n)|^3 dx \right)^{\frac{2}{3}} \\ &\leq CR^{\frac{d+2\alpha}{3}} \left(\int_{\mathbb{R}^d} |x|^{-\alpha} |u(t_n)|^3 dx \right)^{\frac{2}{3}}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{|x| \leq R} |u(t_n)|^2 + 2|v(t_n)|^2 dx &\leq CR^{\frac{d+2\alpha}{3}} \left[\left(\int_{\mathbb{R}^d} |x|^{-\alpha} |u(t_n)|^3 dx \right)^{\frac{2}{3}} + \left(\int_{\mathbb{R}^d} |x|^{-\alpha} |v(t_n)|^3 dx \right)^{\frac{2}{3}} \right] \\ &\leq CR^{\frac{d+2\alpha}{3}} \left(\int_{\mathbb{R}^d} |x|^{-\alpha} (|u(t_n)|^3 + |v(t_n)|^3) dx \right)^{\frac{2}{3}} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

which proves (6.27). The proof is complete. \square

7. GROUND STATE STANDING WAVES

In this section, we study the existence of standing waves associated to (1.1) and their properties. Here by standing waves, we mean solutions to (1.1) of the form $(u(t, x), v(t, x)) = (e^{i\omega t} \varphi(x), e^{2i\omega t} \psi(x))$, where $\omega \in \mathbb{R}$ is a frequency and (φ, ψ) is a solution to the system of elliptic equations

$$\begin{cases} \frac{1}{2} \Delta \varphi - \omega \varphi + |x|^{-\alpha} \overline{\varphi} \psi = 0, \\ \frac{\kappa}{2} \Delta \psi - 2\omega \psi - \gamma \psi + \frac{1}{2} |x|^{-\alpha} \varphi^2 = 0, \end{cases} \quad x \in \mathbb{R}^d. \quad (7.1)$$

We look for ground states of (7.1), i.e., solutions to (7.1) which minimize the action functional

$$\mathbb{A}_\omega(\varphi, \psi) = \mathbb{E}(\varphi, \psi) + \omega \mathbb{M}(\varphi, \psi)$$

over all non-trivial solutions to (7.1). The set of ground states of (7.1) is denoted by

$$\mathcal{G}(\omega) := \{(\varphi, \psi) \in \mathcal{H}^1 \setminus \{(0, 0)\} : \mathbb{A}'_\omega(\varphi, \psi) = 0, \mathbb{A}_\omega(\varphi, \psi) \leq \mathbb{A}_\omega(f, g), \forall (f, g) \in \mathcal{H}^1 \setminus \{(0, 0)\} : \mathbb{A}'_\omega(f, g) = 0\}.$$

Theorem 7.1. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. If $\omega, \gamma \in \mathbb{R}$ satisfies $\omega > 0$ and $\gamma + 2\omega > 0$, then there exists at least a ground state of (7.1). Moreover, the set of all ground states of (7.1) is characterized by*

$$\mathcal{G}(\omega) = \{(\varphi, \psi) \in \mathcal{H}^1 \setminus \{0\}, \mathbb{A}_\omega(\varphi, \psi) = d_\omega, \mathbb{B}_\omega(\varphi, \psi) = 0\},$$

where

$$\mathbb{B}_\omega(\varphi, \psi) = \langle \mathbb{A}'_\omega(\varphi, \psi), (\varphi, \psi) \rangle = \mathbb{K}(\varphi, \psi) + 2\omega \mathbb{M}(\varphi, \psi) + 2\gamma \|\psi\|_{L^2}^2 - 3\mathbb{P}(\varphi, \psi)$$

and

$$d_\omega = \inf_{(\varphi, \psi) \in \mathcal{N}_\omega} \mathbb{A}_\omega(\varphi, \psi) \quad (7.2)$$

with

$$\mathcal{N}_\omega = \{(\varphi, \psi) \in \mathcal{H}^1 \setminus \{0\}, \mathbb{B}_\omega(\varphi, \psi) = 0\}.$$

Before proving Theorem 7.1, we have the following observation.

Lemma 7.2. *If (φ, ψ) is a nontrivial solution of (7.1), then it satisfies*

$$\begin{aligned} \frac{1}{2}\mathbb{K}(\varphi, \psi) + \omega\mathbb{M}(\varphi, \psi) + \gamma\|\psi\|_{L^2}^2 - \frac{3}{2}\mathbb{P}(\varphi, \psi) &= 0, \\ \frac{d-2}{4}\mathbb{M}(\varphi, \psi) + \frac{d\omega}{2}\mathbb{M}(\varphi, \psi) + \frac{d\gamma}{2}\|\psi\|_{L^2}^2 - \frac{d-\alpha}{2}\mathbb{P}(\varphi, \psi) &= 0. \end{aligned} \quad (7.3)$$

In particular, we have

$$\mathbb{K}(\varphi, \psi) = \frac{d+2\alpha}{2}\mathbb{P}(\varphi, \psi) = \frac{2(d+2\alpha)}{6-d-2\alpha}(\omega\mathbb{M}(\varphi, \psi) + \gamma\|\psi\|_{L^2}^2). \quad (7.4)$$

Moreover, if $\alpha < \frac{4-d}{2}$, then a necessary condition on the frequency parameter for the existence of non-trivial solutions for (7.1) is $\omega > 0$. In addition, if $\alpha < \frac{2-d}{2}$, then we need $2\omega + \gamma > 0$.

Proof. The identities (7.3) and (7.4) follow from a direct computation using (3.10). Let us show the necessary conditions for the existence of non-trivial solutions to (7.1). Multiplying the first equation with $\overline{\varphi}$ and the second one with $\overline{\psi}$, integrating over \mathbb{R}^d , and taking the real part, we get

$$\begin{aligned} -\frac{1}{2}\|\nabla\varphi\|_{L^2}^2 - \omega\|\varphi\|_{L^2}^2 + \mathbb{P}(\varphi, \psi) &= 0, \\ -\frac{\kappa}{2}\|\nabla\psi\|_{L^2}^2 - (2\omega + \gamma)\|\psi\|_{L^2}^2 + \frac{1}{2}\mathbb{P}(\varphi, \psi) &= 0. \end{aligned}$$

As $\mathbb{P}(\varphi, \psi) = \frac{2}{d+2\alpha}\mathbb{K}(\varphi, \psi)$, we obtain

$$\begin{aligned} \omega\|\varphi\|_{L^2}^2 &= \frac{2\kappa}{d+2\alpha}\|\nabla\psi\|_{L^2}^2 + \frac{4-d-2\alpha}{2(d+2\alpha)}\|\nabla\varphi\|_{L^2}^2, \\ (2\omega + \gamma)\|\psi\|_{L^2}^2 &= \frac{1}{d+2\alpha}\|\nabla\varphi\|_{L^2}^2 + \frac{(2-d-2\alpha)\kappa}{2(d+2\alpha)}\|\nabla\psi\|_{L^2}^2. \end{aligned}$$

Since $\kappa > 0$, we have the above necessary conditions. \square

Proposition 7.3. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. If $\omega, \gamma \in \mathbb{R}$ satisfies $\omega > 0$ and $\gamma + 2\omega > 0$, then there exists at least a minimizer for d_ω , says (φ, ψ) . In addition, (φ, ψ) is a non-trivial solution to (7.1). Furthermore, we can take (φ, ψ) to be positive, radially symmetric, and radially decreasing.*

Proof. The proof is done in several steps.

Step 1. We first show that $d_\omega > 0$. It is easy to see (using the mean value theorem for the function $\lambda \mapsto \mathbb{A}_\omega(\lambda\varphi, \lambda\psi)$) that $\mathcal{N}_\omega \neq \emptyset$. We notice that for any $(\varphi, \psi) \in \mathcal{N}_\omega$,

$$3\mathbb{P}(\varphi, \psi) = \mathbb{K}(\varphi, \psi) + 2\omega\mathbb{M}(\varphi, \psi) + 2\gamma\|\psi\|_{L^2}^2, \quad (7.5)$$

hence $\mathbb{P}(\varphi, \psi) > 0$ due to $\omega, \gamma + 2\omega > 0$. Using (3.1), we have

$$\mathbb{P}(\varphi, \psi) \leq C(\mathbb{K}(\varphi, \psi))^{\frac{d+2\alpha}{4}}(\mathbb{M}(\varphi, \psi))^{\frac{6-d-2\alpha}{4}} \leq C(\mathbb{K}(\varphi, \psi) + \mathbb{M}(\varphi, \psi))^{\frac{3}{2}} \quad (7.6)$$

for some universal constant $C > 0$. As $\omega, \gamma + 2\omega > 0$, we observe that

$$\begin{aligned} 2\omega\mathbb{M}(\varphi, \psi) + 2\gamma\|\psi\|_{L^2}^2 &= 2\omega\|\varphi\|_{L^2}^2 + 2(2\omega + \gamma)\|\psi\|_{L^2}^2 \\ &\geq 2\min\{\omega, \gamma + 2\omega\}(\|\varphi\|_{L^2}^2 + \|\psi\|_{L^2}^2) \\ &\geq \min\{\omega, \gamma + 2\omega\}\mathbb{M}(\varphi, \psi). \end{aligned}$$

It follows that

$$\mathbb{P}(\varphi, \psi) \leq C(\omega, \gamma)(\mathbb{K}(\varphi, \psi) + 2\omega\mathbb{M}(\varphi, \psi) + 2\gamma\|\psi\|_{L^2}^2)^{\frac{3}{2}}.$$

Hence we get from (7.5) that

$$\mathbb{K}(\varphi, \psi) + 2\omega\mathbb{M}(\varphi, \psi) + 2\gamma\|\psi\|_{L^2}^2 \geq C(\omega, \gamma) > 0$$

for some constant $C(\omega, \gamma)$ depending on ω, γ . Using

$$\mathbb{A}_\omega(\varphi, \psi) = \frac{1}{3}\mathbb{B}_\omega(\varphi, \psi) + \frac{1}{6}(\mathbb{K}(\varphi, \psi) + 2\omega\mathbb{M}(\varphi, \psi) + 2\gamma\|\psi\|_{L^2}^2), \quad (7.7)$$

we obtain

$$\mathbb{A}_\omega(\varphi, \psi) \geq \frac{1}{6}C(\omega, \gamma) > 0, \quad \forall (\varphi, \psi) \in \mathcal{N}_\omega.$$

Consequently, $d_\omega > 0$.

Step 2. We next show the existence of a minimizer for d_ω . Let $\{(\varphi_n, \psi_n)\}_n$ be a minimizing sequence for d_ω , i.e., $\mathbb{B}_\omega(\varphi_n, \psi_n) = 0$ and $\mathbb{A}_\omega(\varphi_n, \psi_n) \rightarrow d_\omega$ as $n \rightarrow \infty$. From Step 1, we have

$$\mathbb{K}(\varphi_n, \psi_n) + 2\omega\mathbb{M}(\varphi_n, \psi_n) + 2\gamma\|\psi_n\|_{L^2}^2 = 6\mathbb{A}(\varphi_n, \psi_n) \leq 6d_\omega + C, \quad \forall n.$$

This means that $\{(\varphi_n, \psi_n)\}_n$ is bounded in \mathcal{H}^1 . By Lemma 3.3, there exist $(\varphi, \psi) \in \mathcal{H}^1$ and a subsequence still denoted by $\{(\varphi_n, \psi_n)\}_n$ such that $(\varphi_n, \psi_n) \rightharpoonup (\varphi, \psi)$ weakly in \mathcal{H}^1 and $\mathbb{P}(\varphi_n, \psi_n) \rightarrow \mathbb{P}(\varphi, \psi)$ as $n \rightarrow \infty$. Observe from (7.5) and (7.7) that

$$\mathbb{A}_\omega(\varphi, \psi) = \frac{1}{3}\mathbb{B}_\omega(\varphi, \psi) + \frac{1}{2}\mathbb{P}(\varphi, \psi), \quad (7.8)$$

hence

$$\mathbb{P}(\varphi, \psi) = \lim_{n \rightarrow \infty} \mathbb{P}(\varphi_n, \psi_n) = 2 \lim_{n \rightarrow \infty} \mathbb{A}(\varphi_n, \psi_n) = 2d_\omega > 0.$$

This shows that $(\varphi, \psi) \neq (0, 0)$. On the other hand, the lower semi-continuity implies

$$\mathbb{K}(\varphi, \psi) + 2\omega\mathbb{M}(\varphi, \psi) + 2\gamma\|\psi\|_{L^2}^2 \leq \liminf_{n \rightarrow \infty} (\mathbb{K}(\varphi_n, \psi_n) + 2\omega\mathbb{M}(\varphi_n, \psi_n) + 2\gamma\|\psi_n\|_{L^2}^2).$$

Thus we have

$$\mathbb{B}_\omega(\varphi, \psi) \leq \liminf_{n \rightarrow \infty} \mathbb{B}_\omega(\varphi_n, \psi_n) = 0$$

and

$$\mathbb{A}_\omega(\varphi, \psi) \leq \lim_{n \rightarrow \infty} \mathbb{A}_\omega(\varphi_n, \psi_n) = d_\omega.$$

We infer that $\mathbb{B}_\omega(\varphi, \psi) = 0$, hence (φ, ψ) is a minimizer for d_ω . In fact, if $\mathbb{B}(\varphi, \psi) < 0$, then there exists $\lambda_0 \in (0, 1)$ such that $\mathbb{B}_\omega(\lambda_0\varphi, \lambda_0\psi) = 0$. We deduce from the definition of d_ω that

$$d_\omega \leq \mathbb{A}_\omega(\lambda_0\varphi, \lambda_0\psi) = \frac{1}{2}\mathbb{P}(\lambda_0\varphi, \lambda_0\psi) = \frac{\lambda_0^3}{2}\mathbb{P}(\varphi, \psi) < \mathbb{P}(\varphi, \psi) = d_\omega$$

which is a contradiction.

Step 3. We now show that (φ, ψ) is a solution to (7.1). Since (φ, ψ) is a minimizer of (7.2), there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that $\mathbb{A}'_\omega(\varphi, \psi) = \eta\mathbb{B}'_\omega(\varphi, \psi)$. It follows that

$$0 = \mathbb{B}_\omega(\varphi, \psi) = \langle \mathbb{A}'_\omega(\varphi, \psi), (\varphi, \psi) \rangle = \eta \langle \mathbb{B}'_\omega(\varphi, \psi), (\varphi, \psi) \rangle.$$

A direct computation (see the proof of Proposition 3.1) yields

$$\mathbb{B}'_\omega(\varphi, \psi) = (-2\Delta\varphi + 4\omega\varphi - 6|x|^{-\alpha}\overline{\varphi}\psi, -2\kappa\Delta\psi + (8\omega + 4\gamma)\psi - 3|x|^{-\alpha}\varphi^2).$$

Thus we get

$$\langle \mathbb{B}'_\omega(\varphi, \psi), (\varphi, \psi) \rangle = 2\mathbb{K}(\varphi, \psi) + 4\omega\mathbb{M}(\varphi, \psi) + 4\gamma\|\psi\|_{L^2}^2 - 9\mathbb{P}(\varphi, \psi) = -3\mathbb{P}(\varphi, \psi) = -6d_\omega < 0,$$

where we have used (7.5) to get the second equality. This shows that $\eta = 0$, that is (φ, ψ) is a solution of (7.1).

Step 4. Let (φ, ψ) be a minimizer for d_ω . Denote $|\varphi|^*$ and $|\psi|^*$ the Schwarz symmetric rearrangement of $|\varphi|$ and $|\psi|$ respectively. We have (see the proof of Proposition 3.1)

$$\begin{aligned} \||\varphi|^*\|_{L^2} &= \|\varphi\|_{L^2}, \\ \|\nabla|\varphi|^*\|_{L^2} &\leq \|\nabla\varphi\|_{L^2}, \\ \mathbb{P}(\varphi, \psi) &\leq \mathbb{P}(|\varphi|^*, |\psi|^*). \end{aligned}$$

In particular, we get

$$\mathbb{B}_\omega(|\varphi|^*, |\psi|^*) \leq \mathbb{B}_\omega(\varphi, \psi) = 0, \quad \mathbb{A}_\omega(|\varphi|^*, |\psi|^*) \leq \mathbb{A}_\omega(\varphi, \psi) = d_\omega.$$

Arguing as in Step 2, we see that $(|\varphi|^*, |\psi|^*)$ is also a minimizer for d_ω . From Step 3, we infer that there exists a non-trivial solution to (7.1) which is non-negative, radially symmetric, and radially decreasing. Finally, the maximal principle (see again the proof of Proposition 3.1) shows that this solution is indeed positive. The proof is complete. \square

Proof of Theorem 7.1. It suffices to show that $(\varphi, \psi) \in \mathcal{G}(\omega)$ if and only if (φ, ψ) is a minimizer of d_ω .

Let (φ, ψ) is a minimizer for d_ω . We will show that $(\varphi, \psi) \in \mathcal{G}(\omega)$. Take any $(f, g) \in \mathcal{H}^1 \setminus \{(0, 0)\}$ satisfying $\mathbb{A}'_\omega(f, g) = 0$. It follows that $\mathbb{B}_\omega(f, g) = \langle \mathbb{A}'_\omega(f, g), (f, g) \rangle = 0$ or $(f, g) \in \mathcal{N}_\omega$. It follows that $\mathbb{A}_\omega(\varphi, \psi) \leq \mathbb{A}_\omega(f, g)$. In particular, we have $(\varphi, \psi) \in \mathcal{G}(\omega)$.

Let $(\varphi, \psi) \in \mathcal{G}(\omega)$. We will show that (φ, ψ) is a minimizer for d_ω . As $\mathbb{A}'_\omega(\varphi, \psi) = 0$, we have $\mathbb{B}_\omega(\varphi, \psi) = 0$. Take (φ_0, ψ_0) a minimizer for d_ω (it exists due to Proposition 7.3). As above, we have $(\varphi_0, \psi_0) \in \mathcal{G}(\omega)$. Since $(\varphi, \psi) \in \mathcal{G}(\omega)$, we deduce that $\mathbb{A}_\omega(\varphi, \psi) = \mathbb{A}_\omega(\varphi_0, \psi_0) = d_\omega$. Thus (φ, ψ) is a minimizer for d_ω . \square

Notice that by following the arguments of [19, Theorem 2.9] (see also [16]), one can show that any radial minimizer $(\varphi, \psi) \in \mathcal{H}^1$ of (7.1) satisfies $(\varphi, \psi) \in (C(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \{0\})) \times (C(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \{0\}))$.

We also have the following exponential decay of general (not necessarily radial) ground states of (7.1) which is needed for the instability result.

Lemma 7.4. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. Let $\omega, \gamma \in \mathbb{R}$ satisfy $\omega > 0$, $\gamma + 2\omega > 0$, and (φ, ψ) be a ground state of (7.1). Then (φ, ψ) decays exponentially at infinity.*

Proof. We proceed in two steps.

Step 1. We first prove that φ, ψ are bounded functions. The proof relies on the following so-called Cafferalli-Kohn-Nirenberg inequality (see e.g., [30]):

$$\| |x|^{-a} u \|_{L^r} \lesssim \| D^m u \|_{L^p}^\theta \| u \|_{L^p}^{1-\theta} \quad (7.9)$$

provided that $p \geq 1, a, r > 0$ real-numbers and $m \geq 1$ an integer satisfy

$$\frac{1}{r} = \frac{a}{d} + \frac{1}{p} - \frac{m\theta}{d}, \quad \frac{1}{r} > \frac{a}{d}, \quad \theta \in [0, 1].$$

From (7.1), we have

$$\begin{aligned} \|\varphi\|_{W^{2,q_1}} &\lesssim \| |x|^{-\alpha} \bar{\varphi} \psi \|_{L^{q_1}} \lesssim \| |x|^{-\frac{\alpha}{2}} \varphi \|_{L^{2q_1}} \| |x|^{-\frac{\alpha}{2}} \psi \|_{L^{2q_1}} \lesssim \|\varphi\|_{H^1} \|\psi\|_{H^1}, \\ \|\psi\|_{W^{2,q_1}} &\lesssim \| |x|^{-\alpha} \varphi^2 \|_{L^{q_1}} = \| |x|^{-\frac{\alpha}{2}} \varphi \|_{L^{2q_1}}^2 \lesssim \|\varphi\|_{H^1}^2, \end{aligned}$$

provided that

$$\frac{1}{2q_1} = \frac{\alpha}{2d} + \frac{1}{2} - \frac{\theta_1}{d}, \quad \frac{1}{2q_1} > \frac{\alpha}{2d}, \quad \theta_1 \in [0, 1].$$

This is equivalent to

$$\frac{1}{q_1} = 1 - \frac{2\theta_1 - \alpha}{d}, \quad d > 2\theta_1, \quad \theta_1 \in [0, 1].$$

- When $d = 1$, we have $0 < \alpha < 1$. We can take $\theta_1 = \frac{1}{2} - \varepsilon_1$ with $0 < \varepsilon_1 < \frac{1-\alpha}{2}$. With this choice, we have $q_1 > 1$. In particular, we have $\varphi, \psi \in W^{2,q_1}(\mathbb{R})$.

- When $d = 2$, we have $0 < \alpha < 2$. We take $\theta_1 = 1 - \varepsilon_1$ with $0 < \varepsilon_1 < \frac{2-\alpha}{2}$. We have $\varphi, \psi \in W^{2,q_1}(\mathbb{R}^2)$ with $q_1 > 1$.

- Let us consider the case $d = 3$ and $0 < \alpha < 1$. We take $\theta_1 = 1$ and get $q_1 = \frac{3}{1+\alpha} > \frac{3}{2}$. Hence $\varphi, \psi \in W^{2,q_1}(\mathbb{R}^3)$ with $q_1 > \frac{3}{2}$.

- Finally, we consider the case $d = 3$ and $1 \leq \alpha < \frac{3}{2}$ or $4 \leq d \leq 5$ and $0 < \alpha < \frac{6-d}{2}$. We can take $\theta_1 = 1$ and get $q_1 = \frac{d}{d+\alpha-2} < \frac{d}{2}$. We reapply (7.9) to have

$$\begin{aligned} \|\varphi\|_{W^{2,q_2}} &\lesssim \| |x|^{-\alpha} \bar{\varphi} \psi \|_{L^{q_2}} \lesssim \| |x|^{-\frac{\alpha}{2}} \varphi \|_{L^{2q_2}} \| |x|^{-\frac{\alpha}{2}} \psi \|_{L^{2q_2}} \lesssim \|\varphi\|_{W^{2,q_1}} \|\psi\|_{W^{2,q_1}}, \\ \|\psi\|_{W^{2,q_2}} &\lesssim \| |x|^{-\alpha} \varphi^2 \|_{L^{q_2}} = \| |x|^{-\frac{\alpha}{2}} \varphi \|_{L^{2q_2}}^2 \lesssim \|\varphi\|_{W^{2,q_1}}^2, \end{aligned}$$

provided that

$$\frac{1}{2q_2} = \frac{\alpha}{2d} + \frac{1}{q_1} - \frac{2\theta_2}{d}, \quad \frac{1}{2q_2} > \frac{\alpha}{2d}, \quad \theta_2 \in [0, 1]$$

or

$$\frac{1}{q_2} = \frac{2}{q_1} - \frac{4\theta_2 - \alpha}{d}, \quad d > 2q_1\theta_2, \quad \theta_2 \in [0, 1].$$

We take $\theta_2 = 1$ and get $q_2 = \frac{d}{2d+3\alpha-8}$. Observe that $q_2 > q_1$ as $d + 2\alpha < 6$. If $q_2 > \frac{d}{2}$, then we stop. Otherwise, we repeat the same argument until we get $q_n > \frac{d}{2}$.

In all cases, we prove that $\varphi, \psi \in W^{2,q_n}(\mathbb{R}^d)$ with some $q_n > \frac{d}{2}$. Thus Morrey's inequality implies that $\varphi, \psi \in C^{0,\delta}(\mathbb{R}^d)$ with some $\delta > 0$. In particular, φ, ψ are bounded functions.

Step 2. We now prove the exponential decay. Let $\varepsilon > 0$ and define $\theta_\varepsilon(x) := e^{\frac{a|x|}{1+\varepsilon|x|}}$ with some constant $a > 0$ to be chosen later. We readily check that

$$|\nabla \theta_\varepsilon(x)| \leq a\theta_\varepsilon(x) \leq ae^{a|x|}, \quad \forall \varepsilon > 0, \quad \forall x \in \mathbb{R}^d. \quad (7.10)$$

From (7.1), multiplying both sides of the first equation to $\theta_\varepsilon \bar{\varphi}$, integrating over \mathbb{R}^d , and taking the real part, we get

$$\frac{1}{2} \Re \int \nabla \varphi \cdot \nabla (\theta_\varepsilon \bar{\varphi}) \, dx + \omega \int \theta_\varepsilon |\varphi|^2 \, dx = \Re \int |x|^{-\alpha} \theta_\varepsilon \bar{\varphi}^2 \psi \, dx.$$

Using $\nabla(\theta_\varepsilon \bar{\varphi}) = \bar{\varphi} \nabla \theta_\varepsilon + \theta_\varepsilon \nabla \bar{\varphi}$ and (7.10), we have

$$\Re(\nabla \varphi \cdot \nabla (\theta_\varepsilon \bar{\varphi})) \geq \theta_\varepsilon |\nabla \varphi|^2 - a\theta_\varepsilon |\varphi| |\nabla \varphi|$$

which, by Cauchy-Schwarz' inequality, yields

$$\Re(\nabla\varphi \cdot \nabla(\theta_\varepsilon\bar{\varphi})) \geq \frac{1}{2}\theta_\varepsilon|\nabla\varphi|^2 - \frac{a^2}{2}\theta_\varepsilon|\varphi|^2.$$

Taking $a = \sqrt{3\omega}$, we obtain

$$\frac{1}{4} \int \theta_\varepsilon (|\nabla\varphi|^2 + \omega|\varphi|^2) dx \leq \int |x|^{-\alpha}\theta_\varepsilon|\varphi|^2|\psi| dx.$$

As $\psi \in L^\infty(\mathbb{R}^d)$, we take $R > 0$ sufficiently large so that $|x|^{-\alpha}|\psi(x)| \leq \frac{\omega}{8}$ for all $|x| \geq R$. It follows that

$$\begin{aligned} \frac{1}{4} \int \theta_\varepsilon (|\nabla\varphi|^2 + \omega|\varphi|^2) dx &\leq \int_{|x| \leq R} |x|^{-\alpha}\theta_\varepsilon|\varphi|^2|\psi| dx + \int_{|x| \geq R} |x|^{-\alpha}\theta_\varepsilon|\varphi|^2|\psi| dx \\ &\leq \int_{|x| \leq R} |x|^{-\alpha}\theta_\varepsilon|\varphi|^2|\psi| dx + \frac{\omega}{8} \int_{|x| \geq R} \theta_\varepsilon|\varphi|^2 dx. \end{aligned}$$

In particular, we get

$$\int \theta_\varepsilon (|\nabla\varphi|^2 + \omega|\varphi|^2) dx \lesssim \int_{|x| \leq R} |x|^{-\alpha}\theta_\varepsilon|\varphi|^2|\psi| dx.$$

Now let $1 < \gamma < \frac{d}{\alpha}$ and $\rho > 1$ satisfy $\frac{1}{\gamma} + \frac{1}{\rho} = 1$. By Hölder's inequality, we have

$$\int_{|x| \leq R} |x|^{-\alpha}\theta_\varepsilon|\varphi|^2|\psi| dx \leq \| |x|^{-\alpha} \|_{L^\gamma(|x| \leq R)} \| \theta_\varepsilon|\varphi|^2|\psi| \|_{L^\rho(|x| \leq R)} = CR^{d-\alpha\gamma} \| \theta_\varepsilon|\varphi|^2|\psi| \|_{L^\rho(|x| \leq R)}.$$

As φ, ψ are bounded, we infer from (7.10) that the right hand side is bounded uniformly in $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$, we obtain

$$\int e^{\sqrt{3\omega}|x|} (|\nabla\varphi|^2 + \omega|\varphi|^2) dx < \infty.$$

This shows the exponential decay of φ . The same argument goes for ψ . The proof is complete. \square

8. α -LIMIT

In this section, we study the behavior of positive radial ground states of (7.1) when α approaches zero.

Theorem 8.1. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\omega > 0$, and $\gamma + 2\omega > 0$. Let $\{(\varphi_\alpha, \psi_\alpha)\}_\alpha \subset \mathcal{H}^1$ be a family of positive radial ground states of (7.1). Then there exists a positive radial ground state $(\varphi, \psi) \in \mathcal{H}^1$ of*

$$\begin{cases} \frac{1}{2}\Delta\varphi - \omega\varphi + \varphi\psi = 0, \\ \frac{\kappa}{2}\Delta\psi - 2\omega\psi - \gamma\psi + \frac{1}{2}\varphi^2 = 0, \end{cases} \quad x \in \mathbb{R}^d, \quad (8.1)$$

such that up to a subsequence,

$$\lim_{\alpha \rightarrow 0} \|(\varphi_\alpha, \psi_\alpha) - (\varphi, \psi)\|_{\mathcal{H}^1} = 0.$$

To this end, we first observe that the action functional has a mountain-pass geometry. Thus we define the mountain pass value by

$$c_\omega = \inf_{\zeta \in \Gamma} \max_{t \in [0, 1]} \mathbb{A}_\omega(\zeta(t)), \quad (8.2)$$

where

$$\Gamma = \{\zeta \in C([0, 1], \mathcal{H}^1), \zeta(0) = 0, \mathbb{A}_\omega(\zeta(1)) < 0\}.$$

Lemma 8.2. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\omega > 0$, and $\gamma + 2\omega > 0$. Then \mathbb{A}_ω has a non-trivial critical point $(\varphi, \psi) \in \mathcal{H}^1$ such that $\mathbb{A}_\omega(\varphi, \psi) = c_\omega$.*

Proof. We proceed in two steps.

Step 1. We first show that there exists $\nu > 0$ such that $c_\omega \geq \nu > 0$. Thanks to (7.6), there are $C_1, C_2 > 0$ such that for any $(\varphi, \psi) \in \mathcal{H}^1$,

$$\mathbb{A}_\omega(\varphi, \psi) \geq C_1 \|(\varphi, \psi)\|_{\mathcal{H}^1}^2 - C_2 \|(\varphi, \psi)\|_{\mathcal{H}^1}^3.$$

Thus there exist $C, \nu > 0$ such that $\mathbb{A}_\omega(\varphi, \psi) > 0$ for all $0 < \|(\varphi, \psi)\|_{\mathcal{H}^1} < C$ and $\mathbb{A}_\omega(\varphi, \psi) \geq \nu$ for all $\|(\varphi, \psi)\|_{\mathcal{H}^1} = C$. Now take any curve $\zeta \in \Gamma$, since $\mathbb{A}_\omega(\zeta(1)) < 0$, we must have $\|\zeta(1)\|_{\mathcal{H}^1} > C$. As $\zeta(0) = 0$, the continuity of $t \mapsto \zeta(t)$ implies that there exists $t_0 \in (0, 1)$ such that $\|\zeta(t_0)\|_{\mathcal{H}^1} = C$. Thus

$$\max_{t \in [0, 1]} \mathbb{A}_\omega(\zeta(t)) \geq \mathbb{A}_\omega(\zeta(t_0)) \geq \nu.$$

Taking the infimum over all $\zeta \in \Gamma$, we obtain $c_\omega \geq \nu > 0$.

Step 2. Compactness of mountain-pass sequences. Applying the mountain-pass theorem (see e.g. [46, Theorem 5.1]), there exists a sequence $\{(\varphi_n, \psi_n)\}_n \subset \mathcal{H}^1$ such that $\mathbb{A}_\omega(\varphi_n, \psi_n) \rightarrow c_\omega$ and $\mathbb{A}'_\omega(\varphi_n, \psi_n) \rightarrow 0$ in \mathcal{H}^{-1} . Using (7.7), we have

$$\begin{aligned} C\|(\varphi_n, \psi_n)\|_{\mathcal{H}^1}^2 &\leq \frac{1}{6}(\mathbb{K}(\varphi_n, \psi_n) + 2\omega\mathbb{M}(\varphi_n, \psi_n) + 2\gamma\|\psi_n\|_{L^2}^2) \\ &= \mathbb{A}_\omega(\varphi_n, \psi_n) - \frac{1}{3}\mathbb{B}_\omega(\varphi_n, \psi_n) \\ &\leq c_\omega + 1, \quad \forall n \end{aligned}$$

where $\mathbb{B}_\omega(\varphi_n, \psi_n) = \langle \mathbb{A}'_\omega(\varphi_n, \psi_n), (\varphi_n, \psi_n) \rangle \rightarrow 0$ as $n \rightarrow \infty$. This means that $\{(\varphi_n, \psi_n)\}_n$ is bounded in \mathcal{H}^1 . By Lemma 3.3, there exist $(\varphi, \psi) \in \mathcal{H}^1$ and a subsequence still denoted by $\{(\varphi_n, \psi_n)\}_n$ such that $(\varphi_n, \psi_n) \rightharpoonup (\varphi, \psi)$ weakly in \mathcal{H}^1 and $\mathbb{P}(\varphi_n, \psi_n) \rightarrow \mathbb{P}(\varphi, \psi)$ as $n \rightarrow \infty$. Using (7.8), we have

$$\mathbb{P}(\varphi, \psi) = \lim_{n \rightarrow \infty} \mathbb{P}(\varphi_n, \psi_n) = \lim_{n \rightarrow \infty} \left(2\mathbb{A}_\omega(\varphi_n, \psi_n) - \frac{2}{3}\mathbb{B}_\omega(\varphi_n, \psi_n) \right) = 2c_\omega > 0$$

which implies that $(\varphi, \psi) \neq (0, 0)$. We next show that (φ, ψ) is a critical point of \mathbb{A}_ω . To see this, it is enough to prove that for any $(\chi, \vartheta) \in (C_0^\infty(\mathbb{R}^d))^2$,

$$\langle \mathbb{P}'(\varphi_n, \psi_n), (\chi, \vartheta) \rangle \rightarrow \langle \mathbb{P}'(\varphi, \psi), (\chi, \vartheta) \rangle$$

as $n \rightarrow \infty$, that is,

$$\Re \int_{\mathbb{R}^d} |x|^{-\alpha} \overline{\varphi}_n \psi_n \overline{\chi} \, dx \rightarrow \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \overline{\varphi} \psi \overline{\chi} \, dx, \quad \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi_n^2 \overline{\vartheta} \, dx \rightarrow \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 \overline{\vartheta} \, dx. \quad (8.3)$$

Let $R > 0$ be such that $\text{supp}(\chi) \subset \{|x| \leq R\}$. We estimate

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} |x|^{-\alpha} (\overline{\varphi}_n \psi_n - \overline{\varphi} \psi) \overline{\chi} \, dx \right| \\ &= \left| \int_{\mathbb{R}^d} |x|^{-\alpha} \overline{\varphi}_n (\psi_n - \psi) \overline{\chi} \, dx \right| + \left| \int_{\mathbb{R}^d} |x|^{-\alpha} (\overline{\varphi}_n - \overline{\varphi}) \psi \overline{\chi} \, dx \right| \\ &\leq \| |x|^{-\alpha} \|_{L^\gamma(|x| \leq R)} \|\chi\|_{L^\rho} (\|\varphi_n\|_{L^\rho} \|\psi_n - \psi\|_{L^\rho(|x| \leq R)} + \|\varphi_n - \varphi\|_{L^\rho(|x| \leq R)} \|\psi\|_{L^\rho}), \end{aligned}$$

where $1 \leq \gamma, \rho \leq \infty$ are such that

$$1 = \frac{1}{\gamma} + \frac{3}{\rho}.$$

Arguing as in the proof of Lemma 3.3 and using the compact embedding $H^1(\mathbb{R}^d) \subset L^q(|x| \leq R)$ for all $q \geq 1$ and all $q < \frac{2d}{d-2}$ if $d \geq 3$, we prove that

$$\Re \int_{\mathbb{R}^d} |x|^{-\alpha} \overline{\varphi}_n \psi_n \overline{\chi} \, dx \rightarrow \Re \int_{\mathbb{R}^d} |x|^{-\alpha} \overline{\varphi} \psi \overline{\chi} \, dx \text{ as } n \rightarrow \infty.$$

The second convergence in (8.3) is proved in the same fashion. Therefore, $\mathbb{A}'_\omega(\varphi, \psi) = 0$. We also have from (7.8) that

$$\begin{aligned} \mathbb{A}_\omega(\varphi, \psi) &= \mathbb{A}_\omega(\varphi, \psi) - \frac{1}{3}\mathbb{B}_\omega(\varphi, \psi) = \frac{1}{2}\mathbb{P}(\varphi, \psi) \\ &= \frac{1}{2} \liminf_{n \rightarrow \infty} \mathbb{P}(\varphi_n, \psi_n) \\ &= \liminf_{n \rightarrow \infty} \left(\mathbb{A}_\omega(\varphi_n, \psi_n) - \frac{1}{3}\mathbb{B}_\omega(\varphi_n, \psi_n) \right) \\ &= c_\omega. \end{aligned}$$

The proof is complete. \square

Lemma 8.3. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\omega > 0$, and $\gamma + 2\omega > 0$. There holds that $c_\omega = d_\omega$, where d_ω is as in (7.2).*

Proof. It is easily to see from Lemma 8.2 that $c_\omega \geq d_\omega$. To see the reverse inequality, we take a ground state (φ, ψ) of (7.1), i.e., $d_\omega = \mathbb{A}_\omega(\varphi, \psi)$ and $\mathbb{B}_\omega(\varphi, \psi) = 0$. For $t > 0$, we define $\zeta(t) = (t\varphi, t\psi)$. We have

$$\begin{aligned} \mathbb{A}_\omega(\zeta(t)) &= \mathbb{E}_\omega(\zeta(t)) + \omega\mathbb{M}(\zeta(t)) \\ &= \frac{1}{2}\mathbb{K}(t\varphi, t\psi) + \gamma\|t\psi\|_{L^2}^2 - \mathbb{P}(t\varphi, t\psi) + \omega\mathbb{M}(t\varphi, t\psi) \\ &= t^2 \left(\frac{1}{2}\mathbb{K}(\varphi, \psi) + \gamma\|\psi\|_{L^2}^2 + \omega\mathbb{M}(\varphi, \psi) \right) - t^3\mathbb{P}(\varphi, \psi) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \end{aligned}$$

Thus there exists $L > 0$ such that $\mathbb{A}_\omega(\zeta(L)) < 0$. Moreover, as $\mathbb{B}_\omega(\varphi, \psi) = 0$, we see that

$$\max_{t \geq 0} \mathbb{A}_\omega(\zeta(t)) = \mathbb{A}_\omega(\zeta(1)) = \mathbb{A}_\omega(\varphi, \psi) = d_\omega.$$

Now we define a curve $\tilde{\zeta} : [0, 1] \rightarrow \mathcal{H}^1$ by $\tilde{\zeta}(t) := \zeta(Lt)$. We have $\tilde{\zeta} \in \Gamma$ and

$$c_\omega \leq \max_{t \in [0, 1]} \mathbb{A}_\omega(\tilde{\zeta}(t)) = \max_{t \in [0, L]} \mathbb{A}_\omega(\zeta(t)) \leq \max_{t \geq 0} \mathbb{A}_\omega(\zeta(t)) = d_\omega.$$

□

To emphasize the role of α , we denote

$$\begin{aligned} \mathbb{P}_\alpha(f, g) &:= \Re \int_{\mathbb{R}^d} |x|^{-\alpha} f^2 \bar{g} \, dx, \\ \mathbb{E}_\alpha(f, g) &:= \frac{1}{2} \mathbb{K}(f, g) + \gamma \|g\|_{L^2}^2 - \mathbb{P}_\alpha(f, g), \\ \mathbb{A}_{\alpha, \omega}(f, g) &:= \mathbb{E}_\alpha(f, g) + \omega \mathbb{M}(f, g), \\ \mathbb{B}_{\alpha, \omega}(f, g) &:= \langle \mathbb{A}'_{\alpha, \omega}(f, g), (f, g) \rangle = \mathbb{K}(f, g) + 2\omega \mathbb{M}(f, g) + 2\gamma \|g\|_{L^2}^2 - 3\mathbb{P}_\alpha(f, g), \end{aligned}$$

and

$$d_{\alpha, \omega} := \inf \{ \mathbb{A}_{\alpha, \omega}(f, g) : (f, g) \in \mathcal{H}^1 \setminus \{(0, 0)\}, \mathbb{B}_{\alpha, \omega}(f, g) = 0 \}.$$

Lemma 8.4. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. Let $(\varphi, \psi) \in \mathcal{H}^1$. Then*

$$\lim_{\alpha \rightarrow 0} \mathbb{P}_\alpha(\varphi, \psi) = \mathbb{P}_0(\varphi, \psi).$$

Proof. By Hölder's inequality, we have

$$\begin{aligned} |\mathbb{P}_\alpha(\varphi, \psi) - \mathbb{P}_0(\varphi, \psi)| &= \left| \int_{\mathbb{R}^d} \varphi^2 \bar{\psi} (|x|^{-\alpha} - 1) \, dx \right| \\ &\leq \int_{\mathbb{R}^d} |\varphi|^2 |\psi| \left| |x|^{-\alpha} - 1 \right| \, dx \\ &\leq \int_{|x| \geq 1} \left| |x|^{-\alpha} - 1 \right| |\varphi|^2 |\psi| \, dx + \left\| |x|^{-\alpha} - 1 \right\|_{L^p(|x| \leq 1)} \|\varphi^2 \psi\|_{L^{p'}} \\ &\leq \int_{|x| \geq 1} \left| |x|^{-\alpha} - 1 \right| |\varphi|^2 |\psi| \, dx + C_p \|\varphi\|_{L^{3p'}}^3 \|\psi\|_{L^{3p'}} \left(\int_0^1 |1 - r^\alpha|^{p} r^{d-1-\alpha p} \, dr \right)^{\frac{1}{p}}, \end{aligned}$$

where $\frac{2d}{6-d} < p < \frac{d}{\alpha}$. The first term tends to zero as $\alpha \rightarrow 0$ due to the dominated convergence theorem. For the second term, the first condition on p ensures that $H^1(\mathbb{R}^d) \subset L^{3p'}(\mathbb{R}^d)$ while the second condition ensure that

$$\int_0^1 |1 - r^\alpha|^{p} r^{d-1-\alpha p} \, dr \rightarrow 0 \text{ as } \alpha \rightarrow 0$$

due to the dominated convergence theorem. □

Lemma 8.5. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, and $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$. Let $\{\alpha_n\}_n > 0$ be a sequence converging to zero. Let $\{(\varphi_n, \psi_n)\}_n \subset \mathcal{H}^1$ be a sequence of radial functions that converges weakly in \mathcal{H}^1 to $(\varphi, \psi) \in \mathcal{H}^1$. Then up to a subsequence,*

$$\lim_{n \rightarrow \infty} P_{\alpha_n}(\varphi_n, \psi_n) = \mathbb{P}_0(\varphi, \psi).$$

Proof. By the compact embedding $H_{\text{rad}}^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ and passing to a subsequence if necessary, we may assume that $(\varphi_n, \psi_n) \rightarrow (\varphi, \psi)$ in \mathcal{L}^q for any $q > 2$ and $q < \frac{2d}{d-2}$ if $d \geq 3$. We have for $V_n(x) = |x|^{-\alpha_n} - 1$ from the Lemma 8.4 that

$$\begin{aligned} |\mathbb{P}_{\alpha_n}(\varphi_n, \psi_n) - \mathbb{P}_0(\varphi, \psi)| &= \left| \int_{\mathbb{R}^d} |x|^{-\alpha_n} (\varphi_n^2 \psi_n - \varphi^2 \psi) \, dx \right| + \left| \int_{\mathbb{R}^d} \varphi^2 \psi V_n \, dx \right| \\ &\leq \int_{\mathbb{R}^d} |x|^{-\alpha_n} (|\varphi_n^2 - \varphi^2| |\psi_n - \psi| + |\varphi|^2 |\psi_n - \psi| + |\varphi_n^2 - \varphi^2| |\psi|) \, dx + o_n(1) \\ &= A_n + B_n + C_n + o_n(1). \end{aligned}$$

From the Hölder inequality we have

$$\begin{aligned}
A_n &= \int_{|x| \geq 1} |x|^{-\alpha_n} |\varphi_n - \varphi| |\varphi_n + \varphi| |\psi_n - \psi| \, dx + \int_{|x| \leq 1} |x|^{-\alpha_n} |\varphi_n - \varphi| |\varphi_n + \varphi| |\psi_n - \psi| \, dx \\
&\leq \| |x|^{-\alpha_n} \|_{L^\infty(|x| \geq 1)} \|\varphi_n - \psi\|_{L^3} \|\varphi_n + \varphi\|_{L^3} \|\psi_n - \psi\|_{L^3} \\
&\quad + \| |x|^{-\alpha_n} \|_{L^p(|x| \leq 1)} \|\varphi_n - \varphi\|_{L^{3p'}} \|\varphi_n + \varphi\|_{L^{3p'}} \|\psi_n - \psi\|_{L^{3p'}} \\
&\leq o_n(1) + C \left(\frac{1}{d - \alpha_n p} \right)^{\frac{1}{p}} \|\varphi_n - \varphi\|_{L^{3p'}} \|\psi_n - \psi\|_{L^{3p'}} = o_n(1)
\end{aligned}$$

as $n \rightarrow \infty$, where $p \in (\frac{2d}{6-d}, \frac{d}{\alpha_n})$. Similarly, one can show that $B_n, C_n \rightarrow 0$ as $n \rightarrow \infty$. \square

Lemma 8.6. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\omega > 0$, and $\gamma + 2\omega > 0$. There holds $\lim_{\alpha \rightarrow 0} d_{\alpha, \omega} = d_{0, \omega}$.*

Proof. Let \mathcal{H}_+^1 be the set of positive functions $(\varphi, \psi) \in \mathcal{H}^1$. We observe that

$$P_\alpha(\varphi, \psi) = \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 \psi \, dx > 0.$$

After a straightforward calculation, for any $(\varphi, \psi) \in \mathcal{H}_+^1$, there exists $\lambda_0 > 0$ such that $B_{\alpha, \omega}(\lambda_0 \varphi, \lambda_0 \psi) = 0$ and

$$\mathbb{A}_{\alpha, \omega}(\lambda_0 \varphi, \lambda_0 \psi) = \max_{\lambda \geq 0} \mathbb{A}_{\alpha, \omega}(\lambda \varphi, \lambda \psi).$$

By Lemma 8.3 and Lemma 8.5, we infer that

$$\begin{aligned}
c_{\alpha, \omega} &= d_{\alpha, \omega} \leq \mathbb{A}_{\alpha, \omega}(\lambda_0 \varphi, \lambda_0 \psi) \\
&= \max_{\lambda \geq 0} \mathbb{A}_{\alpha, \omega}(\lambda \varphi, \lambda \psi) \\
&\rightarrow \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi, \lambda \psi) \text{ as } \alpha \rightarrow 0.
\end{aligned}$$

Here we have used the fact that $F(x, y) := \max_{\lambda \geq 0} (\lambda^2 x - \lambda^3 y)$ is continuous on $(0, \infty) \times (0, \infty)$. Taking the infimum over all $(\varphi, \psi) \in \mathcal{H}_+^1$, we get

$$\limsup_{\alpha \rightarrow 0} c_{\alpha, \omega} \leq \inf_{(\varphi, \psi) \in \mathcal{H}_+^1} \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi, \lambda \psi).$$

We next claim that

$$\inf_{(\varphi, \psi) \in \mathcal{H}_+^1} \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi, \lambda \psi) = c_{0, \omega} \tag{8.4}$$

which yields

$$\limsup_{\alpha \rightarrow 0} c_{\alpha, \omega} \leq c_{0, \omega}. \tag{8.5}$$

To see (8.4), we observe that for $(\varphi, \psi) \in \mathcal{H}_+^1$, there exists $\lambda_1 > 0$ such that $B_{0, \omega}(\lambda_1 \varphi, \lambda_1 \psi) = 0$ and

$$\mathbb{A}_{0, \omega}(\lambda_1 \varphi, \lambda_1 \psi) = \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi, \lambda \psi).$$

By the definition of $d_{0, \omega}$ and $c_{0, \omega} = d_{0, \omega}$, we have

$$c_{0, \omega} \leq \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi, \lambda \psi),$$

hence

$$c_{0, \omega} \leq \inf_{(\varphi, \psi) \in \mathcal{H}_+^1} \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi, \lambda \psi).$$

On the other hand, we take (φ_0, ψ_0) be a positive minimizer for $d_{0, \omega}$. Since $B_{0, \omega}(\varphi_0, \psi_0) = 0$, we have

$$\mathbb{A}_{0, \omega}(\varphi_0, \psi_0) = \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi_0, \lambda \psi_0).$$

As $(\varphi_0, \psi_0) \in \mathcal{H}_+^1$, we have

$$c_{0, \omega} = d_{0, \omega} = \mathbb{A}_{0, \omega}(\varphi_0, \psi_0) = \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi_0, \lambda \psi_0) \geq \inf_{(\varphi, \psi) \in \mathcal{H}_+^1} \max_{\lambda \geq 0} \mathbb{A}_{0, \omega}(\lambda \varphi, \lambda \psi).$$

Collecting the above inequalities, we prove the claim.

Now taking $(\varphi_\alpha, \psi_\alpha)$ a positive ground state of (7.1). From (7.7), we have

$$c_{\alpha, \omega} \simeq \frac{1}{6} \|(\varphi_\alpha, \psi_\alpha)\|_{\mathcal{H}^1}^2, \tag{8.6}$$

hence $\{(\varphi_\alpha, \psi_\alpha)\}_\alpha$ remains bounded in \mathcal{H}^1 as $\alpha \rightarrow 0$. By Lemma 8.5 again, we have also that

$$c_{\alpha,\omega} = \mathbb{A}_{\alpha,\omega}(\varphi_\alpha, \psi_\alpha) = \max_{\lambda \geq 0} \mathbb{A}_{\alpha,\omega}(\lambda\varphi_\alpha, \lambda\psi_\alpha) = \max_{\lambda \geq 0} \mathbb{A}_{0,\omega}(\lambda\varphi_\alpha, \lambda\psi_\alpha) + o(\alpha)$$

as $\alpha \rightarrow 0$. This implies

$$\liminf_{\alpha \rightarrow 0} c_{\alpha,\omega} \geq c_{0,\omega}. \quad (8.7)$$

Combining (8.5) and (8.7), we obtain the desired result. \square

Proof of Theorem 8.1. It is clear from (8.6) and $c_{\alpha,\omega} \rightarrow c_{0,\omega}$ as $\alpha \rightarrow 0$ that $\{(\varphi_n, \psi_n)\}_n$ is uniformly bounded in \mathcal{H}^1 . Then $\{(\varphi_n, \psi_n)\}_n$ converges, up to a subsequence, weakly in \mathcal{H}^1 to some non-negative radial functions $(\varphi, \psi) \in \mathcal{H}^1$. From Lemma 8.6 and the weak convergence of $\{(\varphi_n, \psi_n)\}_n$, we deduce that (φ, ψ) is a weak solution of (8.1). Moreover, using Lemma 8.6 and (8.6), we get $\|(\varphi_n, \psi_n)\|_{\mathcal{H}^1}^2 \rightarrow \|(\varphi, \psi)\|_{\mathcal{H}^1}^2$ as $n \rightarrow \infty$. This together with the weak convergence gives the strong convergence of $\{(\varphi_n, \psi_n)\}_n$ to (φ, ψ) in \mathcal{H}^1 . By (8.6) and the fact that $c_{0,\omega} > 0$, we infer that $\|(\varphi_n, \psi_n)\|_{\mathcal{H}^1}$ is uniformly bounded from below. In particular, (φ, ψ) is non-trivial solution due to the strong convergence of $\{(\varphi_n, \psi_n)\}_n$. Finally, the positivity of (φ, ψ) comes from the maximum principle. \square

9. STABILITY AND INSTABILITY OF STANDING WAVES

9.1. Stability. We study the stability of standing waves of (1.1). For the reader's convenience, we recall the notion of stability as follows.

Definition 9.1. A set $X \subset \mathcal{H}^1$ is called \mathcal{H}^1 -stable with respect to (1.1) if for any $\epsilon > 0$, there exists some $\delta > 0$ such that, for any (u_0, v_0) in $U_\delta(X)$, the δ -neighborhood of X in \mathcal{H}^1 , the solution $(u(t), v(t))$ of (1.1) with initial data $(u, v)|_{t=0} = (u_0, v_0)$ satisfies $(u(t), v(t)) \in U_\epsilon(X)$ for all $t > 0$. Otherwise we say X is \mathcal{H}^1 -unstable.

Let $\mathcal{G}(\omega)$ be the set of all ground states of (7.1). Denote $L(\omega) := \mathbb{A}_\omega(\varphi_\omega, \psi_\omega)$ the Lyapunov function for $(\varphi_\omega, \psi_\omega) \in \mathcal{G}(\omega)$. It is not hard to see that L is well-defined, positive, continuous, and strictly increasing in ω (see e.g., [17, Lemma 2.7]). We sometime use the notation $\mathcal{G}(\omega, \gamma, \alpha)$ to mention the dependence on ω, γ , and α . Notice that if $\gamma = 0$ and $(\varphi_\omega, \psi_\omega) \in \mathcal{G}(\omega, 0, \alpha)$, then $\omega^{\frac{\alpha}{2}-1}(\varphi_\omega, \psi_\omega)(\omega^{-\frac{1}{2}}x) \in \mathcal{G}(1, 0, \alpha)$.

Theorem 9.1. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\omega > 0$, and $\gamma + 2\omega > 0$. Then $\mathcal{G}(\omega, \gamma, \alpha)$ is \mathcal{H}^1 -stable with respect to (1.1) if $L''(\omega) > 0$. In particular, $\mathcal{G}(\omega, 0, \alpha)$ is \mathcal{H}^1 -stable if $d + 2\alpha < 4$.*

Lemma 9.2. *There exist $\epsilon > 0$ and a continuous function $\omega : U_\epsilon(\mathcal{G}(\omega, \gamma, \alpha)) \rightarrow \mathbb{R}$ such that $L(\omega(\varphi_\omega, \psi_\omega)) = \mathbb{A}_\omega(\varphi_\omega, \psi_\omega)$ for all $(\varphi_\omega, \psi_\omega) \in \mathcal{G}(\omega, \gamma, \alpha)$.*

Proof. It follows from the continuity and the strictly monotonicity of L that L^{-1} is continuous and strictly increasing. The fact $L(\omega) = \mathbb{A}_\omega(\varphi_\omega, \psi_\omega)$ for all $(\varphi_\omega, \psi_\omega) \in \mathcal{G}(\omega, \gamma, \alpha)$ and the continuity of \mathbb{A}_ω imply that there is $\epsilon > 0$ such that \mathbb{A}_ω is in the range of L for all $(\varphi_\omega, \psi_\omega) \in \mathcal{G}(\omega, \gamma, \alpha)$. So $\omega(\varphi_\omega, \psi_\omega) = L^{-1}(\mathbb{A}_\omega(\varphi_\omega, \psi_\omega))$ presents the desired function. \square

Lemma 9.3. *Assume that $L''(\omega) > 0$. Then there exists $\epsilon > 0$ such that for any $(\varphi, \psi) \in \mathcal{G}(\omega, \gamma, \alpha)$ and any $(f, g) \in U_\epsilon(\mathcal{G}(\omega, \gamma, \alpha))$,*

$$\mathbb{E}(f, g) - \mathbb{E}(\varphi, \psi) + \omega(f, g)(\mathbb{M}(f, g) - \mathbb{M}(\varphi, \psi)) \geq \frac{1}{4}L''(\omega)(\omega(f, g) - \omega)^2.$$

Proof. The proof is similar to [17, Lemma 4.3] with natural modifications, so we omit the details. \square

Proof of Theorem 9.1. Assume that $L''(\omega) > 0$ and that $\mathcal{G}(\omega, \gamma, \alpha)$ is not stable. There exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, there exist $(\varphi_n, \psi_n) \in U_{\frac{1}{n}}(\mathcal{G}(\omega, \gamma, \alpha))$ and $t_n > 0$ such that the solution $(u_n(t), v_n(t))$ of (1.1) with the initial data $(u_n(0), v_n(0)) = (\varphi_n, \psi_n)$ satisfies

$$\inf_{(\varphi, \psi) \in \mathcal{G}(\omega, \gamma, \alpha)} \|(u_n(t_n), v_n(t_n)) - (\varphi, \psi)\|_{\mathcal{H}^1} = \epsilon. \quad (9.1)$$

Since $(\varphi_n, \psi_n) \in U_{\frac{1}{n}}(\mathcal{G}(\omega, \gamma, \alpha))$, we can find $(\tilde{\varphi}_n, \tilde{\psi}_n) \in \mathcal{G}(\omega, \gamma, \alpha)$ such that $\|(\varphi_n, \psi_n) - (\tilde{\varphi}_n, \tilde{\psi}_n)\|_{\mathcal{H}^1} < \frac{1}{n}$ for all $n \geq 1$. By the conservation of mass and energy, we have $\mathbb{E}(u_n(t_n), v_n(t_n)) - \mathbb{E}(\tilde{\varphi}_n, \tilde{\psi}_n) \rightarrow 0$ and $\mathbb{M}(u_n(t_n), v_n(t_n)) - \mathbb{M}(\tilde{\varphi}_n, \tilde{\psi}_n) \rightarrow 0$ as $n \rightarrow \infty$. If we choose ϵ sufficiently small, then we get from Lemma 9.3 that

$$\begin{aligned} \mathbb{E}(u_n(t_n), v_n(t_n)) - \mathbb{E}(\tilde{\varphi}_n, \tilde{\psi}_n) + \omega(u_n(t_n), v_n(t_n))(\mathbb{M}(u_n(t_n), v_n(t_n)) - \mathbb{M}(\tilde{\varphi}_n, \tilde{\psi}_n)) \\ \geq \frac{1}{4}L''(\omega)(\omega(u_n(t_n), v_n(t_n)) - \omega)^2; \end{aligned}$$

so that $\omega(u_n(t_n), v_n(t_n)) \rightarrow \omega$ as $n \rightarrow \infty$. Hence we obtain from the continuity of L that

$$\mathbb{A}_\omega(u_n(t_n), v_n(t_n)) \rightarrow L(\omega) \quad \text{as } n \rightarrow \infty.$$

Moreover, it is easy to see that $\mathbb{B}_\omega(u_n(t_n), v_n(t_n)) \rightarrow 0$ as $n \rightarrow \infty$. Hence, $\{(u_n(t_n), v_n(t_n))\}_n$ is a minimizing sequence of d_ω . By Proposition 7.3, there exist a subsequence of $\{(u_n(t_n), v_n(t_n))\}_n$, still denoted by the same and $(\varphi, \psi) \in \mathcal{G}(\omega, \gamma, \alpha)$ such that

$$\|(u_n(t_n), v_n(t_n)) - (\varphi, \psi)\|_{\mathcal{H}^1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This contradicts (9.1).

In the case $\gamma = 0$, if $(\varphi_\omega, \psi_\omega)$ satisfies (7.1), then it is seen that the scaled function $(\varphi, \psi)(x) = \omega^{\frac{\alpha}{2}-1}(\varphi_\omega, \psi_\omega)(\omega^{-\frac{1}{2}}x)$ satisfies (7.1) with $\omega = 1$. Using Lemma 7.2, we see that

$$L(\omega) = \frac{2\omega}{6-d-2\alpha} \mathbb{M}(\varphi_\omega, \psi_\omega) = \frac{2}{6-d-2\alpha} \omega^{3-\alpha-\frac{d}{2}} \mathbb{M}(\varphi, \psi).$$

Finally we deduce that $L''(\omega) > 0$ iff $d + 2\alpha < 4$. \square

9.2. Strong instability. Concerning the strong instability of standing waves of (1.1), we have the following results.

Theorem 9.4 (Mass-critical case). *Let $2 \leq d \leq 3$, $\alpha = \frac{4-d}{2}$, $\kappa > 0$, $\gamma = 0$, $\omega > 0$, and $\gamma + 2\omega > 0$. Let (φ, ψ) be a ground state of (7.1).*

- If $\kappa = \frac{1}{2}$, then the standing wave $(e^{i\omega t}\varphi(x), e^{2i\omega t}\psi(x))$ is strongly unstable by finite-time blow-up under the flow of (1.1) in the sense that for any $\epsilon > 0$, there exists $(u_0, v_0) \in \mathcal{H}^1$ such that $\|(u_0, v_0) - (\varphi, \psi)\|_{\mathcal{H}^1} < \epsilon$ and the solution $(u(t), v(t))$ of (1.1) with $(u, v)|_{t=0} = (u_0, v_0)$ blows up in finite time.
- If $\kappa \neq \frac{1}{2}$, then the standing wave $(e^{i\omega t}\varphi(x), e^{2i\omega t}\psi(x))$ is strongly unstable by finite-time or infinite-time blow-up under the flow of (1.1) in the sense that for any $\epsilon > 0$, there exists $(u_0, v_0) \in \mathcal{H}^1$ such that $\|(u_0, v_0) - (\varphi, \psi)\|_{\mathcal{H}^1} < \epsilon$ and the solution $(u(t), v(t))$ of (1.1) with $(u, v)|_{t=0} = (u_0, v_0)$ either blows up in finite time or it blows up in infinite time and satisfies

$$\mathbb{K}(u(t), v(t)) \geq Ct^2, \quad \forall |t| \geq t_0$$

for some $t_0 > 0$ sufficiently large.

Theorem 9.5 (Mass-supercritical case). *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$, $\kappa > 0$, $\omega > 0$, and $\gamma + 2\omega > 0$. Let (φ, ψ) be a ground state of (7.1).*

- If $2 \leq d \leq 4$, or $d = 5$ and $\kappa = \frac{1}{2}$, then the standing wave $(e^{i\omega t}\varphi(x), e^{2i\omega t}\psi(x))$ is strongly unstable by finite time blow-up under the flow of (1.1).
- If $d = 5$ and $\kappa \neq \frac{1}{2}$, then the standing wave $(e^{i\omega t}\varphi(x), e^{2i\omega t}\psi(x))$ is strongly unstable by finite-time or infinite-time blow-up under the flow of (1.1) in the sense that for any $\epsilon > 0$, there exists $(u_0, v_0) \in \mathcal{H}^1$ such that $\|(u_0, v_0) - (\varphi, \psi)\|_{\mathcal{H}^1} < \epsilon$ and the solution $(u(t), v(t))$ of (1.1) with $(u, v)|_{t=0} = (u_0, v_0)$ either blows up in finite time or it blows up in infinite time and satisfies for any $T > 0$

$$\sup_{t \in [0, T]} \mathbb{K}(u(t), v(t)) \geq CT^{4/3}.$$

Let define the following variational problem

$$d_\omega^- := \inf_{(\varphi, \psi) \in \mathcal{N}_\omega^-} \mathbb{A}_\omega(\varphi, \psi), \quad (9.2)$$

where

$$\mathcal{N}_\omega^- = \{(\varphi, \psi) \in \mathcal{H}^1 : \mathbb{G}(\varphi, \psi) = 0, \mathbb{B}_\omega(\varphi, \psi) < 0\},$$

where \mathbb{G} is the Pohozaev functional given in (1.3).

Lemma 9.6. *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\frac{4-d}{2} \leq \alpha < \frac{6-d}{2}$, $\kappa > 0$, $\omega > 0$, and $\gamma + 2\omega > 0$. Then \mathcal{N}_ω^- is nonempty and $d_\omega^- > 0$.*

Proof. Given $(\varphi, \psi) \in \mathcal{H}^1 \setminus \{(0, 0)\}$, it is easy to see from the mean value theorem that there is $\lambda > 0$ such that $\mathbb{G}(\lambda\varphi, \lambda\psi) = 0$. Now assume that $(\varphi, \psi) \in \mathcal{H}^1 \setminus \{(0, 0)\}$ satisfying $\mathbb{G}(\varphi, \psi) = 0$. As $\mathbb{G}(\varphi, \psi) = 0$, we deduce that

$$\mathbb{B}_\omega(\lambda^{2-\alpha}\varphi(\lambda \cdot), \lambda^{2-\alpha}\psi(\lambda \cdot)) = \lambda^{4-2\alpha-d} (\omega \mathbb{M}(\varphi, \psi) + \gamma \|\psi\|_{L^2}^2) + \frac{d+2\alpha-6}{2(d+2\alpha)} \lambda^{6-2\alpha-d} \mathbb{K}(\varphi, \psi) \rightarrow -\infty$$

as $\lambda \rightarrow +\infty$. This shows that \mathcal{N}_ω^- is nonempty set.

For the second part, first we note from $\mathbb{G}(\varphi, \psi) = 0$ that

$$\mathbb{A}_\omega(\varphi, \psi) = \frac{d + 2\alpha - 4}{2(d + 2\alpha)} \mathbb{K}(\varphi, \psi) + \omega \mathbb{M}(\varphi, \psi) + \gamma \|\psi\|_{L^2}^2 > 0 \quad (9.3)$$

for all $(\varphi, \psi) \in \mathcal{N}_\omega^-$, hence $d_\omega^- \geq 0$.

• If $d > 4 - 2\alpha$, it follows from $\mathbb{B}_\omega(\varphi, \psi) < 0$ and (3.1) that

$$\frac{1}{2} \mathbb{K}(\varphi, \psi) + \omega \mathbb{M}(\varphi, \psi) + \gamma \|\psi\|_{L^2}^2 < \frac{3}{2} \mathbb{P}(\varphi, \psi) \lesssim \left(\frac{1}{2} \mathbb{K}(\varphi, \psi) + \omega \mathbb{M}(\varphi, \psi) + \gamma \|\psi\|_{L^2}^2 \right)^3$$

which together with (9.3) imply $\mathbb{A}_\omega(\varphi, \psi) \gtrsim 1$ and then $d_\omega^- > 0$.

• If $d = 4 - 2\alpha$, then $\mathbb{A}_\omega(\varphi, \psi) = \omega \mathbb{M}(\varphi, \psi) + \gamma \|\psi\|_{L^2}^2$. Assume that there exists a sequence $\{(\varphi_n, \psi_n)\}_n \subset \mathcal{N}_\omega^-$ such that $\mathbb{A}_\omega(\varphi_n, \psi_n) \rightarrow 0$ as $n \rightarrow \infty$. Then,

$$\omega \mathbb{M}(\varphi_n, \psi_n) + \gamma \|\psi_n\|_{L^2}^2 \rightarrow 0 \quad (9.4)$$

as $n \rightarrow \infty$. Since $\mathbb{B}_\omega(\varphi_n, \psi_n) < 0$, then (3.1) shows that

$$\frac{1}{2} \mathbb{K}(\varphi_n, \psi_n) + \omega \mathbb{M}(\varphi_n, \psi_n) + \gamma \|\psi_n\|_{L^2}^2 < C \mathbb{K}(\varphi_n, \psi_n) (\omega \mathbb{M}(\varphi_n, \psi_n) + \gamma \|\psi_n\|_{L^2}^2)^{\frac{1}{2}}$$

for some constant $C > 0$. With the same constant, we infer from (9.4) that

$$\frac{1}{2} \mathbb{K}(\varphi_n, \psi_n) + \omega \mathbb{M}(\varphi_n, \psi_n) + \gamma \|\psi_n\|_{L^2}^2 \geq C \mathbb{K}(\varphi_n, \psi_n) (\omega \mathbb{M}(\varphi_n, \psi_n) + \gamma \|\psi_n\|_{L^2}^2)^{\frac{1}{2}},$$

for n sufficiently large. This is a contradiction, hence $d_\omega^- > 0$ in this case. \square

We set $\varphi = \min\{d_\omega, d_\omega^-\}$ and define the following set:

$$K^- = \{(\varphi, \psi) \in \mathcal{H}^1 : \mathbb{A}_\omega(\varphi, \psi) < \varphi, \mathbb{B}_\omega(\varphi, \psi) < 0, \mathbb{G}(\varphi, \psi) < 0\}.$$

Lemma 9.7. *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\frac{4-d}{2} \leq \alpha < \frac{6-d}{2}$, $\kappa > 0$, $\omega > 0$, and $\gamma + 2\omega > 0$. Let $(u_0, v_0) \in K^-$. Then the corresponding solution to (1.1) with initial data $(u, v)|_{t=0} = (u_0, v_0)$ satisfies $u(t) \in K^-$ for all $t \in (-T_*, T^*)$. In addition, we have*

$$\sup_{t \in (-T_*, T^*)} \mathbb{G}(u(t), v(t)) \leq -\delta \quad (9.5)$$

for some $\delta > 0$.

Proof. We first show that K^- is invariant under the flow of (1.1). Let $(u_0, v_0) \in K^-$ and $(u(t), v(t))$ with $t \in (-T_*, T^*)$ be the unique maximal solution of (1.1) with initial data $(u, v)|_{t=0} = (u_0, v_0)$. The conservation of mass and energy shows that $\mathbb{A}_\omega((u(t), v(t))) = \mathbb{A}_\omega(u_0, v_0) < \varphi$ for $t \in (-T_*, T^*)$. We next show that $\mathbb{B}_\omega(u(t), v(t)) < 0$ for all $t \in (-T_*, T^*)$. If there exists $t_0 \in (-T_*, T^*)$ such that $\mathbb{B}_\omega(u(t_0), v(t_0)) \geq 0$, then by the continuity, there is $t_1 \in (-T_*, T^*)$ such that $\mathbb{B}_\omega(u(t_1), v(t_1)) = 0$. By the definition of d_ω , we have $\mathbb{A}_\omega(u(t_1), v(t_1)) \geq d_\omega \geq \varphi$ contradicting with $\mathbb{A}_\omega(u(t), v(t)) < \varphi$ for all $t \in (-T_*, T^*)$. Therefore, $\mathbb{B}_\omega(u(t), v(t)) < 0$ for all $t \in (-T_*, T^*)$. We now prove $\mathbb{G}(u(t), v(t)) < 0$ for all $t \in (-T_*, T^*)$. Assume that there exists $t_2 \in (0, T^*)$ such that $\mathbb{G}(u(t_2), v(t_2)) \geq 0$. Then there is, by the continuity, $t_3 \in (-T_*, T^*)$ such that $\mathbb{G}(u(t_3), v(t_3)) = 0$. This shows that $(u(t_3), v(t_3)) \in \mathcal{N}_\omega^-$, hence $\mathbb{A}_\omega(u(t_3), v(t_3)) \geq d_\omega^- \geq \varphi$ contradicting with $\mathbb{A}_\omega(u(t), v(t)) < \varphi$ for all $t \in (-T_*, T^*)$. Therefore, $\mathbb{G}(u(t), v(t)) < 0$ for all $t \in [0, T^*)$.

To prove (9.5), we denote $(u, v) = (u(t), v(t))$ and set $(u_\lambda(x), v_\lambda(x)) = (\lambda^{\frac{d}{2}} u(\lambda x), \lambda^{\frac{d}{2}} v(\lambda x))$ with $\lambda > 0$. Since $\mathbb{G}(u, v) < 0$, we have

$$\begin{aligned} \mathbb{B}_\omega(u_\lambda, v_\lambda) &= \lambda^2 \mathbb{K}(u, v) + 2\omega \mathbb{M}(u, v) + 2\gamma \|v\|_{L^2}^2 - 3\lambda^{\frac{d+2\alpha}{2}} \mathbb{P}(u, v) \\ &< \left(\lambda^2 - \frac{6}{d+2\alpha} \lambda^{\frac{d+2\alpha}{2}} \right) \mathbb{K}(u, v) + \omega \mathbb{M}(u, v) + \gamma \|v\|_{L^2}^2. \end{aligned}$$

As $\mathbb{B}_\omega(u, v) < 0$, then there is $\lambda_0 > 0$ such that $\mathbb{B}_\omega(u_{\lambda_0}, v_{\lambda_0}) = 0$. On the other hand, we observe that the function $\lambda \mapsto \mathbb{A}_\omega(u_\lambda, v_\lambda) - \frac{\lambda^2}{2} \mathbb{G}(u, v)$ attains its maximum at $\lambda = 1$ if $\frac{4-d}{2} < \alpha < \frac{6-d}{2}$ and it is constant if $\alpha = \frac{4-d}{2}$. The fact $\mathbb{G}(u, v) < 0$ shows that

$$\varphi \leq d_\omega \leq \mathbb{A}_\omega(u_{\lambda_0}, v_{\lambda_0}) \leq \mathbb{A}_\omega(u_{\lambda_0}, v_{\lambda_0}) - \frac{\lambda_0^2}{2} \mathbb{G}(u, v) \leq \mathbb{A}_\omega(u, v) - \frac{1}{2} \mathbb{G}(u, v).$$

Thus we get

$$\mathbb{G}(u, v) \leq 2(\mathbb{A}_\omega(u, v) - \varphi) = 2(\mathbb{A}_\omega(u_0, v_0) - \varphi) < 0.$$

This shows (9.5) with $\delta := 2(\varphi - \mathbb{A}_\omega(u_0, v_0)) > 0$ and the proof is complete. \square

Proof of Theorems 9.4 and 9.5. By Lemma 7.2, we have $\mathbb{B}_\omega(\varphi, \psi) = \mathbb{G}(\varphi, \psi) = 0$. Then $\mathbb{B}_\omega(\lambda\varphi, \lambda\psi) < 0$ and $\mathbb{G}(\lambda\varphi, \lambda\psi) < 0$ for $\lambda > 1$. Since $\lambda \mapsto \mathbb{A}_\omega(\lambda\varphi, \lambda\psi)$ attains its maximum at $\lambda = 1$, we have $\mathbb{A}_\lambda(\lambda\varphi, \lambda\psi) < \mathbb{A}_\omega(\varphi, \psi)$ for all $\lambda > 1$. Therefore, we obtain that $(\lambda\varphi, \lambda\psi) \in K^-$ for all $\lambda > 1$. Given $\epsilon > 0$, we can take $\lambda_\epsilon > 1$, sufficiently close to 1, such that

$$\|(u_0, v_0) - (\varphi, \psi)\|_{\mathcal{H}^1} < \epsilon,$$

where $(u_0, v_0) = (\lambda_\epsilon\varphi, \lambda_\epsilon\psi)$. By Lemma 9.7 and Theorems 5.1, 5.2, we have the desired result for $2 \leq d \leq 4, \kappa > 0$ and $d = 5, \kappa \neq \frac{1}{2}$. The case $d = 5$ and $\kappa = \frac{1}{2}$ is treated as follows. By the exponential decay (see Lemma 6.3), we infer that $(u_0, v_0) \in (L^2(\mathbb{R}^d, |x|^2 dx))^2$. By local theory and virial identity, we have $(u(t), v(t)) \in (L^2(\mathbb{R}^d, dx))^2$ for all $t \in (-T_*, T^*)$ and

$$\frac{d^2}{dt^2} \mathcal{V}_{|x|^2}(t) = 2\mathbb{G}(u(t), v(t)) \leq -2\delta, \quad \forall t \in (-T_*, T^*),$$

where

$$\mathcal{V}_{|x|^2}(t) := \int_{\mathbb{R}^d} |x|^2 (|u(t)|^2 + 2|v(t)|^2) dx.$$

The standard convexity argument shows that $T_*, T^* < \infty$. The proof is complete. \square

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APPENDIX A. SOME RELATED RESULTS

A.1. Virial identity.

Lemma A.1. *Let u is a solution to $i\partial_t u + \beta\Delta u = H$. Then the following identities hold:*

$$\begin{aligned} \partial_t |u|^2 &= -2\beta\nabla \cdot \Im(\bar{u}\nabla u) + 2\Im(\bar{u}H), \\ \partial_t \Im(\bar{u}\partial_k u) &= \frac{\beta}{2}\partial_k \Delta(|u|^2) - 2\beta \sum_{j=1}^d \partial_j \Re(\partial_j \bar{u}\partial_k u) + 2\Re(\bar{H}\partial_k u) - \partial_k \Re(H\bar{u}), \quad \forall k = 1, \dots, d. \end{aligned}$$

Proof. We have

$$\begin{aligned} \partial_t |u|^2 &= 2\Re(\bar{u}\partial_t u) \\ &= 2\Re[\bar{u}(i\beta\Delta u - iH)] \\ &= -2\beta\Im(\bar{u}\Delta u) + 2\Im(\bar{u}H) \\ &= -2\beta\nabla \cdot \Im(\bar{u}\nabla u) + 2\Im(\bar{u}H); \end{aligned}$$

which shows the first identity.

Next we have

$$\begin{aligned} \partial_t \Im(\bar{u}\partial_k u) &= \Im(\bar{\partial_t u}\partial_k u + \bar{u}\partial_k \partial_t u) \\ &= \Im[(-i\beta\Delta \bar{u} + i\bar{H})\partial_k u + \bar{u}\partial_k (i\beta\Delta u - iH)] \\ &= \Re(-\beta\Delta \bar{u}\partial_k u + \beta\bar{u}\partial_k \Delta u) + \Re(\bar{H}\partial_k u - \bar{u}\partial_k H). \end{aligned}$$

As $\partial_k \Re(H\bar{u}) = \Re(\partial_k H\bar{u} + H\partial_k \bar{u})$, we infer that

$$\partial_t \Im(\bar{u}\partial_k u) = \Re(-\beta\Delta \bar{u}\partial_k u + \beta\bar{u}\partial_k \Delta u) + 2\Re(\bar{H}\partial_k u) - \partial_k \Re(H\bar{u}). \quad (\text{A.1})$$

On the other hand, we have

$$\begin{aligned} \sum_{j=1}^d \partial_j \Re(\partial_j \bar{u}\partial_k u) &= \sum_{j=1}^d \Re(\partial_j^2 \bar{u}\partial_k u + \partial_j \bar{u}\partial_j^2 u) \\ &= \Re(\Delta \bar{u}\partial_k u) + \Re(\nabla \bar{u} \cdot \nabla \partial_k u) \end{aligned}$$

and

$$\begin{aligned} \partial_k \Delta(|u|^2) &= \partial_k (\Delta u \bar{u} + 2\nabla u \cdot \nabla \bar{u} + u \Delta \bar{u}) \\ &= \Delta \partial_k u \bar{u} + \Delta u \partial_k \bar{u} + 2\nabla \partial_k u \cdot \nabla \bar{u} + 2\nabla u \cdot \nabla \partial_k \bar{u} + \partial_k u \Delta \bar{u} + u \Delta \partial_k \bar{u} \\ &= 2\Re(\Delta \partial_k u \bar{u} + \Delta \bar{u} \partial_k u + 2\nabla u \cdot \nabla \partial_k \bar{u}). \end{aligned}$$

It follows that

$$\Re(-\beta\Delta\bar{u}\partial_k u + \beta\bar{u}\partial_k\Delta u) = \frac{\beta}{2}\partial_k\Delta(|u|^2) - 2\beta\sum_{j=1}^d\partial_j\Re(\partial_j\bar{u}\partial_k u)$$

which together with (A.1) imply the second identity. \square

A.2. Uniqueness of positive ground states. Denote $\mathcal{K}_\omega(\varphi, \psi) := \mathbb{K}(\varphi, \psi) + 2\omega\mathbb{M}(\varphi, \psi) + 2\gamma\|\psi\|_{L^2}^2$. By the homogeneity of \mathbb{P} and \mathcal{K}_ω , we deduce that $(\varphi, \psi) \in \mathcal{G}(\omega)$ also achieves the minimum

$$m(\omega) = \inf \left\{ \frac{\mathcal{K}_\omega(f, g)}{\mathbb{P}^{\frac{2}{3}}(f, g)} : (f, g) \in \mathcal{H}^1, \mathbb{P}(f, g) > 0 \right\}.$$

Notice when $\kappa = 2 + \frac{\gamma}{\omega}$ that $(\varphi, \psi) = (\pm\sqrt{2\kappa}Q, Q)$ satisfies (7.1), where Q is the unique positive radially symmetric solution (see [44]) of

$$\frac{1}{2}\Delta Q - \omega Q + |x|^{-\alpha}Q^2 = 0. \quad (\text{A.2})$$

Theorem A.2. *Let $1 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\omega > 0$, and $\gamma + 2\omega > 0$. Assume that $\kappa = 2 + \frac{\gamma}{\omega}$. Then the positive ground states of (7.1) are unique up to dilations.*

Proof. Let (φ, ψ) be a positive ground state of (7.1) (it exists due to Proposition 7.3). By the Young and Hölder inequalities, we have

$$m(\omega) = \frac{\mathcal{K}_\omega(\varphi, \psi)}{\mathbb{P}^{\frac{2}{3}}(\varphi, \psi)} \geq \frac{3\kappa^{\frac{1}{3}}}{2^{\frac{2}{3}}} \left(\frac{\mathbb{K}_0(\varphi)}{\|\varphi\|_{L_\alpha^3}^2} \right)^{\frac{2}{3}} \left(\frac{\mathbb{K}_0(\psi)}{\|\psi\|_{L_\alpha^3}^2} \right)^{\frac{1}{3}} \geq \frac{3\kappa^{\frac{1}{3}}}{2^{\frac{2}{3}}} m_0(\omega), \quad (\text{A.3})$$

where

$$\|\varphi\|_{L_\alpha^3}^3 := \int_{\mathbb{R}^d} |x|^{-\alpha} |\varphi(x)|^3 dx, \quad \mathbb{K}_0(\varphi) = \|\nabla\varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2$$

and

$$m_0(\omega) = \inf \left\{ \frac{\mathbb{K}_0(\varphi)}{\|\varphi\|_{L_\alpha^3}^2} : \varphi \in H^1(\mathbb{R}^d) \setminus \{0\} \right\}$$

and similarly for ψ . On the other hand, let Q be the unique positive radial solution of (A.2). It is known that Q is a minimizer for $m_0(\omega)$. We deduce that

$$m(\omega) \leq \frac{\mathcal{K}_\omega(\pm\sqrt{2\kappa}Q, Q)}{\mathbb{P}^{\frac{2}{3}}(\pm\sqrt{2\kappa}Q, Q)} = \frac{3\kappa^{\frac{1}{3}}}{2^{\frac{2}{3}}} \frac{\mathbb{K}_0(Q)}{\|Q\|_{L_\alpha^3}^2} = \frac{3\kappa^{\frac{1}{3}}}{2^{\frac{2}{3}}} m_0(\omega).$$

Combining the above inequalities, we get

$$m(\omega) = \frac{3\kappa^{\frac{1}{3}}}{2^{\frac{2}{3}}} m_0(\omega), \quad (\text{A.4})$$

hence the equalities occur for both Young and Hölder inequalities in (A.3). Consequently, we obtain $\varphi = a\psi$ for some constant $a > 0$. Actually, it is seen from (A.4) that $a = \sqrt{2\kappa}$. In particular, ψ is a minimizer for $m_0(\omega)$ and is a positive solution to (A.2).

Now let ψ^* be the symmetric decreasing rearrangement of ψ . By Polya-Szegö and Hardy-Littlewood inequalities, namely

$$\|\nabla\psi^*\|_{L^2} \leq \|\nabla\psi\|_{L^2}, \quad \int_{\mathbb{R}^d} |x|^{-\alpha}\psi^3 dx \leq \int_{\mathbb{R}^d} |x|^{-\alpha}(\psi^*)^3 dx,$$

we deduce that ψ^* is also a minimizer for $m_0(\omega)$. In addition, we have

$$\int_{\mathbb{R}^d} |x|^{-\alpha}\psi^3 dx = \int_{\mathbb{R}^d} |x|^{-\alpha}(\psi^*)^3 dx$$

which, by [28, Theorem 3.4] and the fact that $(|x|^{-\alpha})^* = |x|^{-\alpha}$, yields $\psi(x) = \psi^*(x)$. In particular, ψ is a positive radial solution to (A.2). By the uniqueness of positive radial solutions to (A.2), we infer that $\psi(x) = Q(x)$. Therefore, we get $(\varphi, \psi) = (\sqrt{2\kappa}Q, Q)$ and the proof is complete. \square

A.3. Infinite many radial solutions.

Theorem A.3. *Let $2 \leq d \leq 5$, $0 < \alpha < \min\{2, d\}$, $\alpha < \frac{6-d}{2}$ if $3 \leq d \leq 5$, $\omega > 0$, and $\gamma + 2\omega > 0$. Then (7.1) possesses infinitely many distinct radially symmetric real-valued solutions $\{(u_n, v_n)\}_n$ satisfying*

$$\mathbb{A}_\omega(u_n, v_n) \rightarrow +\infty$$

as $n \rightarrow \infty$.

Proof. We first notice that \mathbb{A}_ω is not even in the second variable. Given $u \in H_{\text{rad}}^1(\mathbb{R}^d)$, we denote the unique positive solution

$$v_u(x) = \frac{1}{2}K * (|x|^{-\alpha}u^2)$$

of

$$\frac{\kappa}{2}\Delta v - \gamma v - 2\omega v + \frac{1}{2}|x|^{-\alpha}u^2 = 0. \quad (\text{A.5})$$

Here K is the Yukawa potential defined via the Fourier transform of $\hat{K}(\xi) = \frac{1}{2\omega + \gamma + \frac{\kappa}{2}|\xi|^2}$. We also define the functional $S_\omega : H_{\text{rad}}^1(\mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$S_\omega(u) = \frac{1}{4}\|\nabla u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{4}\int_{\mathbb{R}^d}|x|^{-\alpha}u^2v_u \, dx.$$

Then (7.1) is equivalent to

$$-\omega u + \frac{1}{2}\Delta u + |x|^{-\alpha}uv_u = 0. \quad (\text{A.6})$$

Moreover, $S_\omega \in C^1(H_{\text{rad}}^1(\mathbb{R}^d))$ and $S'_\omega(u) = 0$ is equivalent to that (u, v) satisfies (7.1). On the other hand, for any $u \in H^1(\mathbb{R}^d)$, by multiplying (A.5) with v_u and integrating on \mathbb{R}^d , we get

$$\|v_u\|_{H^1}^2 \simeq \int_{\mathbb{R}^d}|x|^{-\alpha}u^2v_u \, dx \lesssim \left(\int_{\mathbb{R}^d}|x|^{-\alpha}|u(x)|^3 \, dx\right)^2 \int_{\mathbb{R}^d}|x|^{-\alpha}|v_u(x)|^3 \, dx \lesssim \|u\|_{H^1}^2 \|v_u\|_{H^1}$$

which yields

$$\|v_u\|_{H^1} \lesssim \|u\|_{H^1}^2$$

and

$$\int_{\mathbb{R}^d}|x|^{-\alpha}u^2v_u \, dx \lesssim \|u\|_{H^1}^4.$$

Thus

$$S_\omega(u) \gtrsim \frac{1}{4}\|u\|_{H^1}^2 - \|u\|_{H^1}^4$$

for any $u \in H_{\text{rad}}^1(\mathbb{R}^d)$. This implies that S_ω has a strict local minimum at the origin. In addition, it follows from (A.5), Hölder's inequality, and (A.6) that

$$\int_{\mathbb{R}^d}|x|^{-\alpha}u^3 \, dx = 2 \left\langle u, \omega v - \frac{\kappa}{2}\Delta v + \gamma v \right\rangle \lesssim \|(u, v)\|_{\mathcal{H}^1}^2 \lesssim \|u\|_{H^1}^2 + \int_{\mathbb{R}^d}|x|^{-\alpha}u^2v_u \, dx$$

resulting

$$S_\omega(u) \lesssim \|u\|_{H^1}^2 - \int_{\mathbb{R}^d}|x|^{-\alpha}|u|^3 \, dx$$

for any $u \in H_{\text{rad}}^1(\mathbb{R}^d)$. Then, for any finite dimensional subspace A of $H_{\text{rad}}^1(\mathbb{R}^d)$, there is $R > 0$ such that $S_\omega(u) < 0$ for any $u \in A \setminus B_R(0)$, due to the equivalency of norms in finite dimensional spaces. Therefore, since S_ω satisfies the Palais-Smale condition, all conditions of Theorem 9.12 in [42] hold, so the result is concluded. \square

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